



Equivariant localization and holography

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Abstract

We discuss the theory of equivariant localization focussing on applications relevant for holography. We consider geometries comprising compact and non-compact toric orbifolds, as well as more general non-compact toric Calabi–Yau singularities. A key object in our constructions is the *equivariant volume*, for which we describe two methods of evaluation: the Berline–Vergne fixed point formula and the Molien–Weyl formula, supplemented by the Jeffrey–Kirwan prescription. We present two applications in supersymmetric field theories. Firstly, we describe a method for integrating the anomaly polynomial of SCFTs on compact toric orbifolds. Secondly, we discuss equivariant orbifold indices that are expected to play a key role in the computation of supersymmetric partition functions. In the context of supergravity, we propose that the equivariant volume can be used to characterize universally the geometry of a large class of supersymmetric solutions. As an illustration, we employ equivariant localization to prove the factorization in gravitational blocks of various supergravity free energies, recovering previous results as well as obtaining generalizations.

Keywords Supergravity · Anomalies · Toric orbifolds · Holography · Equivariant volume · Localization

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1 Introduction

The theory of equivariant co-homology is a rich mathematical subject, with numerous applications in diverse areas of geometry and mathematical physics. A remarkable consequence of this theory is that quite generally, on manifolds endowed with a Hamiltonian action of a Lie group, there exist *localization formulas* that express certain integrals in terms of contributions arising from the fixed point sets of the group action, thus simplifying enormously their evaluation. A prime example of this feature is the classic result of Duistermaat–Heckmann [1]. More generally, such integrals involve equivariant characteristic classes and may arise in the evaluation of equivariant indices of transversally elliptic operators [2, 3]. See, for instance, the review articles [4, 5], or [6] for a discussion perhaps more accessible to physicists. In this paper, we will

need an extension of these results to the setup of orbifolds [7], including spindles and other orbifolds with singularities in co-dimension less than two, that are not so often considered in the physics literature but have recently received attention in the context of holography.

A very general framework in which the ideas of equivariant localization are realized is that of symplectic toric geometry. In this case, one starts with a symplectic manifold (M_{2m}, ω) in dimension $2m$, equipped with an effective Hamiltonian action of the real torus \mathbb{T}^m . The image of the associated moment maps is a convex rational polytope $\mathcal{P} \subset \mathbb{Q}^m$ [8]. Together with the angular coordinates on the torus, $\phi_i \sim \phi_i + 2\pi$, the moment maps y_i can be used as “symplectic coordinates”, endowing the manifold with a natural coordinate system (y_i, ϕ_i) in which any metric can then be written in a canonical form in terms of combinatorial data of the polytope [9]. In the toric setting, the fixed points sets of the \mathbb{T}^m action are always isolated singularities, so that the localization formulas take the form of sums over fixed points. The applications of the localization theorems in this context range from the quantization of symplectic manifolds to algorithms for computing the volumes of polytopes and counting integral points. See, e.g., [10, 11]. The extensions to symplectic toric orbifolds was discussed in [12] and that to non-compact toric cones in [13].

In theoretical physics, equivariant localization came to the fore in the work of Nekrasov as a technique for calculating partition functions counting instantons in supersymmetric field theories [14]. See, e.g., [15] for a review. Applications of toric geometry motivated by the AdS/CFT correspondence were first discussed in [16] and further developed in [17], where the volume functional of toric Sasakian manifolds was shown to be extremized by Sasaki–Einstein metrics. The extension to the more general equivariant setting and the relation to fixed point theorems and the index-character of the associated Calabi–Yau cone singularities was explored in [18]. From the viewpoint of holography, these results can be used to compute the volume and other properties of Sasaki–Einstein manifolds, without explicit knowledge of the metric, from which in turn one can infer properties of the dual field theories. Subsequent developments in geometry include results about: toric Sasaki–Einstein metrics [19, 20], Kähler–Einstein metrics [21], extremal Sasaki metrics [22, 23], conformally Kähler Einstein–Maxwell metrics [24].

Following on the steps of [17, 18], we take holography as a motivation for uncovering precise mathematical relationships between geometry and supersymmetric field theories. In particular, we provide new evidence that toric geometry and equivariant localization are well-suited mathematical frameworks for this purpose. In the context of supergravity, our aim is to develop a universal approach to study the geometry underlying supersymmetric solutions based on *extremization problems* analogous to those formulated in [17, 25]. We will argue that the functionals to be extremized can be calculated in each case using the technique of equivariant localization, generalizing the results appeared in [18, 26–28].¹ We will consider, in particular, the novel supergravity constructions featuring compact spaces with conical singularities, like spindles and other orbifolds [33–53]. These solutions imply that we should work in

¹ For applications of localization to the calculation of the supergravity path integral, see, for example, [29–32].

the orbifold setting from the outset, and indeed, from our results we will obtain a direct localization proof of the factorization in gravitational blocks [54] recently discussed in the literature [35, 40, 51, 55].

In the context of supersymmetric field theory, our motivation is that of extending to the orbifold realm some tools that are well-established for field theories compactified on smooth manifolds. Specifically, working with orbifold equivariant co-homology, we wish to put on a sounder mathematical footing the technique of integration of the anomaly polynomials of even-dimensional SCFTs compactified on orbifolds. Furthermore, following [56], where a new index for three-dimensional $\mathcal{N} = 2$ theories was computed exploiting the spindle index-character, we present a discussion of equivariant orbifold indices, that we expect to be key building blocks for computing supersymmetric partition functions of SQFTs on orbifolds, extending Pestun's [57] approach to supersymmetric localization. When the underlying space is (the resolution of) a non-compact Calabi–Yau singularity the same objects have been employed previously to compute Hilbert series of the moduli spaces of supersymmetric field theories [58–60]. We expect that new insights can be gained from the study of equivariant indices and their relation to the equivariant volume. In this paper, however, we restrict our localization techniques to the evaluation of classical geometrical objects. We now summarize the structure of this paper.

In Sect. 2, we recall the symplectic geometry description of compact toric orbifolds, following [9, 12, 61, 62]. We introduce the *equivariant volume* of symplectic toric orbifolds, that is our main object of interest. We discuss two alternative methods for evaluating this, namely the Berline–Vergne fixed point formula and the Molien–Weyl integral formula, that exploits the presentations of the toric orbifolds as symplectic quotients, based on the Lerman and Tolman's generalization [12] of Delzant's construction [8]. Although the topics covered in this section are mainly not original, we consider an extension to non-compact toric orbifolds, which leads to localization formulas for odd-dimensional orbifolds, arising as the base of complex cones thus generalizing the results of [18, 63] for the Sasakian volume. Aspects of the equivariant volume of non-compact symplectic toric manifolds were recently studied in [64, 65]. In this section, we also discuss the *equivariant orbifold index* of the Dolbeault complex, twisted by a holomorphic line orbi-bundle. In the toric setting, the geometric interpretation of this object is that of counting integer lattice points inside a convex integral polytope (or polyhedral cone, in the non-compact case), corresponding to sections of the line bundle. It can then be thought of as the quantum (or K-theoretical) version of the equivariant volume, which is recovered in a limit in which the lattice spacing goes to zero. Besides this close relationship with the equivariant volume, equivariant orbifold indices are expected to provide fundamental building blocks for the construction of supersymmetric partition functions defined on orbifolds [56], with applications to black hole microstate counting. We therefore present some examples of these indices in Appendix C.

In Sect. 3, we discuss in detail a number of examples of toric orbifolds and their associated equivariant volumes. We start with the complex projective line $\mathbb{W}\mathbb{P}^1_{[n_1, n_2]}$, also known as the spindle. This is the simplest compact toric orbifold, which arises in complex dimension one. It has a prominent role in several recent supergravity

constructions corresponding to various wrapped branes [33–49]. Moving to complex dimension two, we consider generic compact toric orbifolds, described by a rational convex polyhedron with an arbitrary number of vertices. We then consider in more detail triangles, namely weighted projective spaces $\mathbb{W}\mathbb{P}_{[n_1, n_2, n_3]}^2$ and their quotients by discrete groups, as well as quadrilaterals. Examples of supergravity solutions comprising Hermitian (non-Kähler) metrics on quadrilateral toric orbifolds are discussed in [50–53]. As a warm-up for Sect. 5, we also describe the peculiarities of the non-compact case in a few explicit examples.

Section 4 concerns the application of equivariant localization to the calculation of the anomaly polynomial of 4d and 6d SCFTs compactified on various orbifolds. Using this approach we prove the localized form of the anomaly polynomial for theories compactified on the spindle [33, 35–37], as well as on general four-dimensional toric orbifolds, for which one example was considered in [50]. Our approach leads to a uniform derivation for SCFTs compactified on different orbifolds and explains the localized form of the integrated anomaly polynomials that was previously observed in examples.

Finally, in Sect. 5 we discuss the application of equivariant localization in the context of supergravity. Firstly, we show that the results of [18] on the localized form of the Sasakian volume are immediately recovered from the equivariant volume of the associated non-compact Calabi–Yau singularity. Furthermore, we prove that also the *master volume* introduced in [66], in the context of GK geometry [67], can be extracted from the equivariant volume and as a corollary we derive the factorization in gravitational blocks of the supergravity free energies for D3 and M5 branes wrapped on the spindle, reproducing the results of [55] in the toric case. We then propose analogous constructions for other branes wrapped on the spindle from which, in each case, we can extract the localized form of the gravitational blocks, which was anticipated in [40]. This leads us to propose that equivariant localization is the common thread of the geometry of supersymmetric supergravity solutions, at least in setups with a holographic interpretation. In particular, we believe that the equivariant volume should be a key object for setting up extremal problems characterizing supersymmetric geometries, in different supergravity theories.

In Sect. 6, we discuss our findings and indicate some directions for future work. The paper contains three appendices with some complementary material.

2 The equivariant volume

2.1 The symplectic geometry description of toric orbifolds

In this section, we review the geometry of symplectic toric orbifolds following the general formalism developed in [9, 12, 61, 62]. We emphasize that although it appears that we are relying on symplectic geometry, we will be interested in computing topological quantities that do not depend on the existence of an integrable symplectic structures and it should be possible to reformulate our computations entirely in terms of complex geometry. In particular, our results are applicable also to situations in which one is

interested in metrics that are not compatible with a symplectic structure, such as the Hermitian (non-Kähler) metrics on toric orbifolds constructed in [50–53].

We consider a symplectic toric orbifold $(\mathbb{M}_{2m}, \omega)$ in dimension $2m$ equipped with an effective Hamiltonian action of the real torus $\mathbb{T}^m = \mathbb{R}^m/2\pi\mathbb{Z}^m$. We introduce symplectic coordinates (y_i, ϕ_i) with $i = 1, \dots, m$ where ϕ_i are angular coordinates on the torus, $\phi_i \sim \phi_i + 2\pi$. In terms of these coordinates, the symplectic form is given by²

$$\omega = dy_i \wedge d\phi_i. \tag{2.1}$$

By a generalization of Delzant’s theorem [8], compact symplectic toric orbifolds are classified by labelled polytopes which are the image of \mathbb{M}_{2m} under the moment maps y_i associated with the toric action [12]. The image of \mathbb{M}_{2m} is simply obtained by forgetting the angular coordinates ϕ_i , and it is a rational simple convex polytope \mathcal{P} in \mathbb{R}^m . We can describe it by introducing a set of linear functions³

$$l_a(y) = y_i v_i^a - \lambda_a, \quad a = 1, \dots, d, \tag{2.2}$$

where v^a are vectors in \mathbb{R}^m . The convex polytope is the subset of \mathbb{R}^m defined by

$$\mathcal{P} = \{y \in \mathbb{R}^m : l_a(y) \geq 0\} \quad a = 1, \dots, d. \tag{2.3}$$

The linear equations

$$l_a(y) = 0 \tag{2.4}$$

define the facets \mathcal{F}_a of the polytope. We denote with d the number of facets of \mathcal{P} . The condition that \mathcal{P} be rational is equivalent to the fact that the vectors v^a have integer entries. The condition that \mathcal{P} be simple requires that each vertex p lies at the intersection of precisely m facets

$$l_{a_1}(p) = l_{a_2}(p) = \dots = l_{a_m}(p) = 0, \tag{2.5}$$

and the corresponding m vectors $\{v^{a_1}, \dots, v^{a_m}\}$ are a basis for \mathbb{R}^m . The polytope comes equipped with a label for each facet, a positive integer n_a such that the structure group of every point in the inverse image of \mathcal{F}_a is \mathbb{Z}_{n_a} .

This construction exhibits \mathbb{M}_{2m} as a torus fibration over the polytope \mathcal{P} and generalizes the familiar construction for toric varieties. As in the latter case, the torus fibration is non-degenerate in the interior of \mathcal{P} . A particular one-cycle in \mathbb{T}^m , determined by the vector v^a , collapses at the facet \mathcal{F}_a . Thus, each facet \mathcal{F}_a defines a symplectic subspace of \mathbb{M}_{2m} of real codimension two. In the complex case, this becomes a divisor⁴

² We adopt the Einstein summation convention for the indices (two repeated indices imply the sum). However, for the sake of clarity, in some formulas we will write explicitly the sums over the indices.

³ We use interchangeably the notation y and y_i to denote a point in \mathbb{R}^m .

⁴ They are called ramification divisors, while $\hat{D}_a = n_a D_a$ are called branched divisors, with multiplicity (or ramification index) n_a .

which we will denote D_a . Similarly, at the intersection of q facets, more cycles in \mathbb{T}^m degenerate and we have a symplectic subspace of \mathbb{M}_{2m} of real codimension $2q$. The intersection of m facets is a vertex of the polytope, and it corresponds to a fixed point of the \mathbb{T}^m action. We denote with n the number of such fixed points. In particular, we can give an alternative definition of the polytope \mathcal{P} as the convex hull of the images of the fixed points of \mathbb{M}_{2m} .

In the context of toric varieties, the vectors v^a define what is called the fan of \mathbb{M}_{2m} . We will use the same terminology but the reader should be aware that we are dealing with a generalization of the concept which allows for more general type of orbifold singularities. For toric varieties, the vectors v^a are primitive, while in the case of generic symplectic toric orbifolds they are not. We can always define for each v^a a primitive vector \hat{v}^a and a positive integer n_a such that $v^a = n_a \hat{v}^a$.⁵ The integer n_a is precisely the label of the facet \mathcal{F}_a defined above. In particular, each symplectic divisor D_a has a local \mathbb{Z}_{n_a} singularity. This cannot happen for toric varieties which are normal and have no singularities of complex co-dimension less than two. Notice also that the local singularity at the fixed point given by the intersection of the m facets \mathcal{F}_{a_i} , with $i = 1 \dots m$, has order $d = |\det(v^{a_1}, \dots, v^{a_m})|$. \mathbb{M}_{2m} is a smooth manifold if and only if all the labels n_a are one and for each vertex $|\det(v^{a_1}, \dots, v^{a_m})| = 1$.

We want to equip \mathbb{M}_{2m} with a compatible complex structure. Any \mathbb{T}^m -invariant Kähler metric on \mathbb{M}_{2m} is of the form [61]

$$ds^2 = G_{ij}(y)dy_i dy_j + G^{ij}(y)d\phi_i d\phi_j, \tag{2.6}$$

where G_{ij} is determined by a symplectic potential $G(y)$ as

$$G_{ij} = \frac{\partial^2 G}{\partial y_i \partial y_j} \tag{2.7}$$

and $G^{ij} = (G^{-1})_{ij}$ is the inverse matrix. Holomorphic coordinates are given by

$$z_i = x_i + i\phi_i, \quad x_i = \frac{\partial G}{\partial y_i}. \tag{2.8}$$

The existence of a symplectic potential is equivalent to the integrability of the complex structure.

The canonical metric on \mathbb{M}_{2m} is given by [9]

$$G(y) = \frac{1}{2} \sum_{a=1}^d l_a \log l_a, \tag{2.9}$$

so that

$$G_{ij} = \frac{\partial^2 G}{\partial y_i \partial y_j} = \frac{1}{2} \sum_{a=1}^d \frac{v_i^a v_j^a}{l_a}. \tag{2.10}$$

⁵ Notice that we are using opposite conventions with respect to [51]!

Notice that G_{ij} has poles at the facets but the metric is smooth up to orbifold singularities. The inverse matrix G^{ij} has rank $m - 1$ at the facets, indicating that a one cycle in \mathbb{T}^m is degenerate. With a change of coordinates, we obtain a smooth orbifold metric on \mathcal{F}_a . The structure of poles of (2.9) is compatible with the degeneration of \mathbb{T}^m at the faces of \mathcal{P} , and it is precisely designed to obtain an orbifold metric on \mathbb{M}_{2m} . The most general Kähler metric on \mathbb{M}_{2m} is discussed in [62], and it is obtained by replacing the symplectic potential $G(y)$ for the canonical metric with $G(y) + h(y)$, where $h(y)$ is smooth on the whole \mathcal{P} . The topological quantities we will discuss in this paper do not depend on the metric, and we can safely set $h(y) = 0$. In our applications to holography, we will encounter metrics that are not Kähler and not even symplectic, but the underlying spaces are in fact symplectic toric orbifolds and we can therefore use the symplectic coordinates and the canonical metric to compute topological quantities that ultimately will not depend on the metric.

Each facet \mathcal{F}_a defines a \mathbb{T}^m -invariant divisor D_a and an associated line bundle L_a . These objects will be important for our construction so we will spend some time discussing their properties. The first Chern class of L_a has been explicitly computed in [9, 62]⁶

$$c_1(L_a) = -\frac{i}{2\pi} [\partial\bar{\partial} \log l_a], \tag{2.11}$$

where with $[\alpha]$ we denote the co-homology class of the differential form α . An explicit representative is given by⁷

$$c_1(L_a) = d\left(\mu_a^i d\phi_i\right), \tag{2.13}$$

where

$$\mu_a^i = \mu_a^i(y) = -\frac{1}{4\pi} \frac{G^{ij} v_j^a}{l_a}. \tag{2.14}$$

Notice that μ_a^i can be seen as a moment map for the torus action

$$i_{\partial\phi_i} c_1(L_a) = -d\mu_a^i. \tag{2.15}$$

From $G^{ij} G_{jk} = \delta_k^i$, we find

$$\sum_{a=1}^d \mu_a^i v_k^a = -\frac{\delta_k^i}{2\pi}. \tag{2.16}$$

⁶ We believe that there is a minus sign error in equation (6.17) in [9], and we have therefore changed the sign here.

⁷ Writing as in (2.8), $z_i = x_i + i\phi_i$ where $x_i = \frac{\partial G}{\partial y_i}$, we have $\partial = \frac{1}{2} \sum_{i=1}^2 (dx_i + i d\phi_i) \left(\frac{\partial}{\partial x_i} - i \frac{\partial}{\partial \phi_i} \right)$ and, therefore, for a torus invariant function f that only depends on y

$$\partial\bar{\partial} f = -\frac{i}{2} \frac{\partial^2 f}{\partial x_i \partial x_j} dx_i \wedge d\phi_j = -\frac{i}{2} d\left(\frac{\partial f}{\partial x_j} d\phi_j\right) = -\frac{i}{2} d\left(G^{kj} \frac{\partial f}{\partial y_k} d\phi_j\right). \tag{2.12}$$

With some abuse of language, we will often denote with $c_1(L)$ an explicit representative of the corresponding co-homology class.

We can relate the Chern classes of the divisors to the Kähler form ω as follows. From

$$G_{ij}y_j = \frac{1}{2} \sum_a \frac{v_i^a v_j^a y_j}{l_a} = \frac{1}{2} \sum_a \frac{v_i^a (l_a + \lambda_a)}{l_a} = \frac{1}{2} \left(\sum_a \frac{\lambda_a v_i^a}{l_a} + \sum_a v_i^a \right), \tag{2.17}$$

we obtain

$$y_i = -2\pi \sum_a \lambda_a \mu_i^a + \frac{1}{2} \sum_a G^{ij} v_j^a, \tag{2.18}$$

so that

$$\omega = dy_i \wedge d\phi_i = -2\pi \sum_a \lambda_a d(\mu_i^a d\phi_i) + \frac{1}{2} d \left(\sum_a G^{ij} v_j^a d\phi_i \right). \tag{2.19}$$

Using the degeneration of G^{ij} at the facets, one can check that $\sum_a G^{ij} v_j^a d\phi_i$ is a well-defined one-form, so that the last term on the right-hand side of (2.19) is exact. Therefore, we obtain the important relation (see Theorem 6.3 in [9])

$$\frac{[\omega]}{2\pi} = - \sum_a \lambda_a c_1(L_a). \tag{2.20}$$

Notice that this equation holds only in co-homology. We see that the parameters λ_a defining the shape of the polytopes through (2.3) are parameterizing the Kähler moduli of the symplectic orbifold.

There are actually only $d - m$ independent line bundles. Indeed, using (2.13) and (2.16), we find m relations among the Chern classes

$$\sum_{a=1}^d v_i^a c_1(L_a) = 0, \tag{2.21}$$

that translate into the familiar m linear relations among the divisors

$$\sum_{a=1}^d v_i^a D_a = 0. \tag{2.22}$$

These relations imply that, if d is the number of vectors in the fan, there are $d - m$ independent $(2m - 2)$ -cycles in homology and therefore only $d - m$ independent Kähler parameters. The d parameters λ_a are associated with the \mathbb{T}^m -invariant divisors and provide an overparameterization of the Kähler moduli.

In this paper, we will also consider non-compact cases, in particular non-compact Calabi–Yau cones. In this case, the polytope is replaced by a rational convex polyhedral

cone. Calabi–Yau cones have typically singularities worse than orbifold. This happens when more than m facets intersect at a vertex. To use the general formalism of this section, we will perform a partial resolution to have only orbifold singularities. In applications to holography, we will also encounter polytopes and polyhedral cones that are *not convex*. We will obtain results by performing a suitable extrapolation from the convex case.

2.2 The definition of the equivariant volume

In this section, we define the equivariant volume. In order to simplify the exposition, in this and the next two sections we assume that \mathbb{M}_{2m} is compact. We will discuss the subtleties and necessary modifications for the non-compact case in Sect. 2.5.

We want to work equivariantly in the \mathbb{T}^m action on \mathbb{M}_{2m} , which is generated by the m vector fields $\partial\phi_i$. Correspondingly, we introduce m equivariant parameters ϵ_i , with $i = 1, \dots, m$ and the vector field $\xi = \epsilon_i \partial\phi_i$. We can introduce a Hamiltonian $H = \epsilon_i y_i$ for this vector field

$$i_\xi \omega = -dH, \tag{2.23}$$

and define an equivariant Kähler form

$$\omega^\mathbb{T} = \omega + 2\pi H = d(y_i d\phi_i) + 2\pi \epsilon_i y_i. \tag{2.24}$$

We similarly introduce equivariant Chern classes for the line bundles L_a

$$c_1^\mathbb{T}(L_a) = c_1(L_a) + 2\pi \epsilon_i \mu_i^a = d(\mu_i^a d\phi_i) + 2\pi \epsilon_i \mu_i^a, \tag{2.25}$$

using the moment maps μ^a . All these forms are equivariantly closed

$$(d + 2\pi i_\xi) \omega^\mathbb{T} = 0, \quad (d + 2\pi i_\xi) c_1^\mathbb{T}(L_a) = 0, \tag{2.26}$$

with respect to the \mathbb{T}^m action. Notice that these expressions are formal linear combinations of forms of different degrees.

Our main object of interest is the equivariant volume of \mathbb{M}_{2m} [1],

$$\mathbb{V}(\lambda_a, \epsilon_i) = \frac{1}{(2\pi)^m} \int_{\mathbb{M}_{2m}} e^{-H} \frac{\omega^m}{m!}, \tag{2.27}$$

which is sometimes referred in the literature to as “symplectic volume” or “equivariant symplectic volume”.

We can write the equivariant volume as

$$\mathbb{V}(\lambda_a, \epsilon_i) = (-1)^m \int_{\mathbb{M}_{2m}} e^{-H - \frac{\omega}{2\pi}} = (-1)^m \int_{\mathbb{M}_{2m}} e^{-\epsilon_i y_i - \frac{\omega}{2\pi}}, \tag{2.28}$$

since the integrand over \mathbb{M}_{2m} receives contribution only from the piece of degree $2m$ in the expansion of the exponential. In terms of the equivariant forms (2.24) and (2.25), we have

$$\mathbb{V}(\lambda_a, \epsilon_i) = (-1)^m \int_{\mathbb{M}_{2m}} e^{-\frac{\omega^\mathbb{T}}{2\pi}} = (-1)^m \int_{\mathbb{M}_{2m}} e^{\sum_a \lambda_a c_1^\mathbb{T}(L_a)}. \tag{2.29}$$

Notice also that the two integrands in (2.29) are not equal but they differ by an equivariantly exact form by (2.20) and the integrals are therefore equal. Notice that the last inequality strictly holds only if \mathbb{M}_{2m} is compact. We will return to this point in Sect. 2.5.

A geometrical interpretation of the equivariant volume of compact orbifolds is that it is the generating functional for the integrals of the equivariant Chern classes

$$\mathbb{V}(\lambda_a, \epsilon_i) = (-1)^m \sum_p \frac{1}{p!} \sum_{a_1, \dots, a_p=1}^d \lambda_{a_1} \dots \lambda_{a_p} \int_{\mathbb{M}_{2m}} c_1^\mathbb{T}(L_{a_1}) \dots c_1^\mathbb{T}(L_{a_p}). \tag{2.30}$$

The equivariant intersection numbers

$$D_{a_1 \dots a_p} = \int_{\mathbb{M}_{2m}} c_1^\mathbb{T}(L_{a_1}) \dots c_1^\mathbb{T}(L_{a_p}) = (-1)^m \frac{\partial^p \mathbb{V}(\lambda_a, \epsilon_i)}{\partial \lambda_{a_1} \dots \partial \lambda_{a_p}} \Big|_{\lambda_a=0} \tag{2.31}$$

are polynomials in ϵ_i and they are topological in nature. Notice that the integrand in (2.31) is a formal linear combination of forms of various degree and the integral selects the piece of degree $2m$. In particular, the equivariant intersection numbers are different from zero for all $p \geq m$.

The expression in (2.27) can be easily reduced to an integral over the polytope \mathcal{P} by performing the angular integrations

$$\mathbb{V}(\lambda_a, \epsilon_i) = \frac{1}{(2\pi)^m} \int_{\mathbb{M}_{2m}} e^{-H} \frac{\omega^m}{m!} = \int_{\mathcal{P}} e^{-H} dy_1 \dots dy_m = \int_{\mathcal{P}} e^{-\epsilon_i y_i} dy_1 \dots dy_m. \tag{2.32}$$

Thus, we have another interpretation of the equivariant volume as the volume of the polytope \mathcal{P} with measure e^{-H} . Integrals of this type and their relation to equivariant localization are discussed in [11].

The equivariant volume satisfies some interesting identities. From the expression (2.29)

$$\mathbb{V}(\lambda_a, \epsilon_i) = (-1)^m \int_{\mathbb{M}_{2m}} e^{\sum_a \lambda_a (c_1(L_a) + 2\pi \epsilon_i \mu_i^a)} \tag{2.33}$$

and the identities (2.21) and (2.16), we find that the identity

$$\mathbb{V}(\lambda_a + \beta_j v_j^a, \epsilon_i) = e^{-\beta_j \epsilon_j} \mathbb{V}(\lambda_a, \epsilon_i) \tag{2.34}$$

holds, for arbitrary $\beta_i \in \mathbb{R}^m$. This formula reflects the fact that only $d - m$ parameters λ_a are geometrically independent. A closely related and useful identity can be obtained by taking derivatives of (2.33) and using again (2.21) and (2.16)

$$\sum_{a=1}^d v_i^a \frac{\partial \mathbb{V}}{\partial \lambda_a} = -\epsilon_i \mathbb{V}. \tag{2.35}$$

Similarly, we have

$$\sum_{a_1, \dots, a_q=1}^d v_{i_1}^{a_1} \dots v_{i_q}^{a_q} \frac{\partial^q \mathbb{V}}{\partial \lambda_{a_1} \dots \partial \lambda_{a_q}} = (-1)^q \epsilon_{i_1} \dots \epsilon_{i_q} \mathbb{V}. \tag{2.36}$$

In the next sections, we review two different methods to evaluate $\mathbb{V}(\lambda_a, \epsilon_i)$, the fixed point and the Molien–Weyl formula. They are both discussed in the literature. Here we adapt them to our notations and we discuss the relations among them.

2.3 The fixed point formula

The equivariant volume can be computed using a fixed point formula. This can be obtained from (2.27) using the Duistermaat–Heckman theorem [1], the localization formula for equivariant co-homology [68, 69], or a direct evaluation of (2.32) [10, 11]. Here we use the localization approach and, for completeness, in ‘‘Appendix A’’ we report an explicit proof for $m = 2$.

Consider a symplectic toric orbifold \mathbb{M}_{2m} with the properties discussed in Sect. 2.1. The torus action \mathbb{T}^m acts on \mathbb{M}_{2m} with n isolated fixed points corresponding to the vertices of the polytope \mathcal{P} . Consider an equivariantly closed form $\alpha^{\mathbb{T}}$ on \mathbb{M}_{2m} . The equivariant localization theorem for orbifolds applied to our situation states that

$$\int_{\mathbb{M}_{2m}} \alpha^{\mathbb{T}} = \sum_{A=1}^n \frac{\alpha^{\mathbb{T}}|_{y_A}}{d_A e^{\mathbb{T}}|_{y_A}}, \tag{2.37}$$

where the sum is over the fixed points y_A of the \mathbb{T}^m action, $e^{\mathbb{T}}$ is the equivariant Euler class of the tangent bundle at y_A and d_A are the orders of the orbifold singularity at the fixed point y_A . In particular, applying this theorem to the computation of the equivariant volume (2.29) gives

$$\mathbb{V}(\lambda_a, \epsilon_i) = (-1)^m \sum_{A=1}^n \frac{e^{-\frac{1}{2\pi} \omega^{\mathbb{T}}|_{y_A}}}{d_A e^{\mathbb{T}}|_{y_A}} = (-1)^m \sum_{A=1}^n \frac{e^{\sum_{a=1}^d \lambda_a c_1^{\mathbb{T}}(L_a)|_{y_A}}}{d_A e^{\mathbb{T}}|_{y_A}}. \tag{2.38}$$

To evaluate the localization formula, we need to compute the restriction $c_1^{\mathbb{T}}(L_a)|_{y_A}$ of the equivariant Chern classes of L_a to the A -th fixed point. The fixed point y_A is defined by m linear equations

$$l_{a_1} = \dots = l_{a_m} = 0 \tag{2.39}$$

for a choice of m intersecting facets associated with the vectors $\{v^{a_1}, \dots, v^{a_m}\}$. The order of the orbifold singularity is given by

$$d_A = |\det(v^{a_1}, \dots, v^{a_m})|. \tag{2.40}$$

We will denote the A -th fixed point also with the m -tuples of indices $A = (a_1, \dots, a_m)$ defining the vertex. In the neighbourhood of the fixed point $A = (a_1, \dots, a_m)$

$$G^{ij} = \frac{1}{2} \frac{v_i^{a_1} v_j^{a_1}}{l_{a_1}} + \frac{1}{2} \frac{v_i^{a_2} v_j^{a_2}}{l_{a_2}} + \dots + \frac{1}{2} \frac{v_i^{a_m} v_j^{a_m}}{l_{a_m}} + \dots, \tag{2.41}$$

up to regular pieces. This can be inverted to give

$$G^{ij} = \frac{2}{d_A^2} \left((u_A^{a_1})_i (u_A^{a_1})_j l_{a_1} + (u_A^{a_2})_i (u_A^{a_2})_j l_{a_2} + \dots + (u_A^{a_m})_i (u_A^{a_m})_j l_{a_m} \right) + \dots, \tag{2.42}$$

where the vectors $u_A^{a_i}$ are constructed by taking the wedge product of the $m - 1$ vectors v^{a_j} with $j \neq i$. The sign ambiguity is determined by requiring $u_A^{a_i} \cdot v^{a_i} = d_A$. The vectors $u_A^{a_i}$ have integer entries and satisfy

$$u_A^{a_i} \cdot v^{a_j} = d_A \delta_{ij} \tag{2.43}$$

as well as

$$v_i^{a_1} (u_A^{a_1})_j + v_i^{a_2} (u_A^{a_2})_j + \dots + v_i^{a_m} (u_A^{a_m})_j = d_A \delta_{ij}. \tag{2.44}$$

They have a dual geometrical interpretation as the inward normal vectors to the facets of the cone $(v^{a_1}, \dots, v^{a_m})$ in the fan or, equivalently, as integer vectors along the m edges of the polytope \mathcal{P} meeting at y_A . Notice that the m -tuple of vectors $u_A^{a_i}$ depends on the choice of vertex A .

The restriction of the moment maps (2.14) to the fixed points then gives

$$\mu_i^a \Big|_{y_A} = \begin{cases} -\frac{1}{2\pi} \frac{u_i^a}{d_A} & \text{if } a \in A \\ 0 & \text{if } a \notin A \end{cases} \tag{2.45}$$

and therefore

$$c_1^{\mathbb{T}}(L_a) \Big|_{y_A} = \begin{cases} -\frac{\epsilon_i u_i^a}{d_A} & \text{if } a \in A \\ 0 & \text{if } a \notin A \end{cases}. \tag{2.46}$$

Notice that the restriction of $c_1^{\mathbb{T}}(L_a)$ to a fixed point is not zero only if the point belongs to the facet \mathcal{F}_a .

Finally the tangent bundle at a fixed point splits as a direct sum of the m line bundles L_{a_i} associated with $A = (a_1, \dots, a_m)$ and so we have [18]

$$e^{\mathbb{T}}|_{y_A} = \prod_{a_i \in A} c_1^{\mathbb{T}}(L_{a_i})|_{y_A} = (-1)^m \prod_{i=1}^m \frac{\epsilon \cdot u_A^{a_i}}{d_A}. \tag{2.47}$$

Now the fixed point formula gives

$$\mathbb{V}(\lambda_a, \epsilon_i) = \sum_{A=(a_1, \dots, a_m)} \frac{e^{-\sum_{i=1}^m \lambda_{a_i} (\frac{\epsilon \cdot u_A^{a_i}}{d_A})}}{d_A \prod_{i=1}^m \frac{\epsilon \cdot u_A^{a_i}}{d_A}}, \tag{2.48}$$

where A runs over the n vertices of the polytope. This formula can be also obtained by computing the volume of the polytope \mathcal{P} , as discussed, for example, in [11].

We note on passing that the identities (2.34), (2.35) and (2.36) can be also derived from the fixed point formula. For example,

$$\begin{aligned} \mathbb{V} \left(\lambda_a + \sum_{j=1}^m \beta_j v_j^a, \epsilon_i \right) &= \sum_{A=(a_1, \dots, a_m)} e^{-\sum_{i,j,k=1}^m \beta_j v_j^{a_i} (\frac{u_A^{a_i}}{d_A})_{k\epsilon_k}} \frac{e^{-\sum_{i=1}^m \lambda_{a_i} (\frac{\epsilon \cdot u_A^{a_i}}{d_A})}}{d_A \prod_{i=1}^m \frac{\epsilon \cdot u_A^{a_i}}{d_A}} \\ &= e^{-\sum_{i=1}^m \beta_i \epsilon_i} \mathbb{V}(\lambda_a, \epsilon_i), \end{aligned} \tag{2.49}$$

where we explicitly wrote all the sums to explain the derivation. In particular, we used (2.44) to perform the intermediate summation over i . The proof of (2.35) and (2.36) is similar.

2.4 The Molien–Weyl formula

The equivariant volume can be also computed using a Molien–Weyl integral formula following [64, 65].

The orbifold \mathbb{M}_{2m} can be realized as a symplectic quotient. Consider the space \mathbb{C}^d , where d is the number of vectors in the fan, and the subgroup K of \mathbb{T}^d of elements of the form

$$\left(e^{2\pi i Q_1}, \dots, e^{2\pi i Q_d} \right) \in \mathbb{T}^d, \tag{2.50}$$

where

$$\sum_{a=1}^d Q_a v_i^a \in \mathbb{Z}, \quad i = 1, \dots, d. \tag{2.51}$$

Then, \mathbb{M}_{2m} is the symplectic reduction [12, 62]

$$\mathbb{M}_{2m} = \mathbb{C}^d // K, \tag{2.52}$$

generalizing a familiar result in toric geometry. Notice that, in general, the group K has a continuous part $U(1)^{d-m}$

$$Q_a = \sum_{A=1}^{d-m} Q_a^A \alpha_A, \quad \alpha_A \in \mathbb{R} \tag{2.53}$$

which can be expressed in terms of the GLSM charges $Q_a^A \in \mathbb{Z}$, with $A = 1, \dots, d-m$, satisfying

$$\sum_{a=1}^d Q_a^A v_i^a = 0, \tag{2.54}$$

as well as a discrete part of more complicated characterization.

Consider first the case where there is no discrete part in the quotient and $\mathbb{M}_{2m} = \mathbb{C}^d // U(1)^{d-m}$. The authors of [64, 65] derived a formula for the equivariant volume as a function of Kähler parameters t_A of the symplectic reduction and the equivariant parameters of \mathbb{C}^d . This is a Molien–Weyl formula obtained by averaging the equivariant volume of \mathbb{C}^d with respect to the $U(1)^{d-m}$ action. The formula reads

$$\mathbb{V}_{MW}(t_A, \bar{\epsilon}_a) = \int \prod_{A=1}^{d-m} \frac{d\phi_A}{2\pi i} \frac{e^{\sum_A \phi_A t_A}}{\prod_{a=1}^d (\bar{\epsilon}_a + \sum_A \phi_A Q_a^A)}. \tag{2.55}$$

A particular contour of integration should be used depending on the direction of the vector of Kähler parameters t_A . A prescription based on the Jeffrey–Kirwan (JK) residue [70] has been proposed in [64, 65]. Formula (2.55) should be also divided by the order of the discrete group corresponding to the torsion part of K when this is present. We will discuss subtleties related to the choice of contour and discrete groups when illustrating examples.

Notice that \mathbb{V}_{MW} depends on $d-m$ Kähler parameters t_A , which is the right number of geometrically inequivalent parameters for \mathbb{M}_{2m} . It also depends on d equivariant parameters $\bar{\epsilon}_a$ associated with the \mathbb{T}^d action on the ambient space, which is larger than the m parameters that we expect for \mathbb{M}_{2m} . However, $d-m$ equivariant parameters can be eliminated by shifting the integration variables. Indeed, up to an exponential factor, the Molien–Weyl integral is invariant under the shift

$$\mathbb{V}_{MW} \left(t_A, \bar{\epsilon}_a + \sum_A Q_a^A \eta_A \right) = e^{-\sum_A \eta_A t_A} \mathbb{V}_{MW}(t_A, \bar{\epsilon}_a), \tag{2.56}$$

for $\eta_A \in \mathbb{R}$.

We thus have two expressions for the equivariant volume, $\mathbb{V}(\epsilon_i, \lambda_a)$, depending on m equivariant parameters ϵ_i and d Kähler parameters λ_a but with the "gauge" invariance (2.34), and $\mathbb{V}_{MW}(\bar{\epsilon}_a, t_A)$, depending on d equivariant parameters $\bar{\epsilon}_a$ and $d-m$ Kähler parameters t_A but with the "gauge" invariance (2.56). The two expressions

must agree with a suitable mapping between parameters. The two set of parameters can be related by comparing gauge-invariant quantities. Based on several examples, we find evidence that this mapping is given by

$$t_A = - \sum_a \lambda_a Q_a^A, \quad \epsilon_i = \sum_a v_i^a \bar{\epsilon}_a, \tag{2.57}$$

and we conjecture that the two expressions for the equivariant volume agree up to a multiplicative factor when expressed in terms of the over-redundant variables $(\lambda_a, \bar{\epsilon}_a)$, namely⁸

$$\mathbb{V}_{MW} \left(t_A = - \sum_a \lambda_a Q_a^A, \bar{\epsilon}_a \right) = e^{\sum_a \lambda_a \bar{\epsilon}_a} \mathbb{V} \left(\lambda_a, \epsilon_i = \sum_a v_i^a \bar{\epsilon}_a \right). \tag{2.58}$$

As a consistency check of this formula, we can apply to both sides the operator $\sum_a v_i^a \frac{\partial}{\partial \lambda_a}$. Since the GLSM charges satisfy (2.54) and \mathbb{V}_{MW} is a function of $t_A = - \sum_a \lambda_a Q_a^A$, we obtain zero on the left-hand side. On the right-hand side, we obtain

$$e^{\sum_a \lambda_a \bar{\epsilon}_a} \left(\sum_a v_i^a \bar{\epsilon}_a \mathbb{V} + \sum_a v_i^a \frac{\partial \mathbb{V}}{\partial \lambda_a} \right), \tag{2.59}$$

which is also zero because of the identity (2.35) and the identification (2.57).

2.5 Remarks on the non-compact case

Most of the previous results hold also for non-compact orbifolds, but with some important differences. We still define the equivariant volume as

$$\mathbb{V}(\lambda_a, \epsilon_i) = \frac{1}{(2\pi)^m} \int_{\mathbb{M}_{2m}} e^{-H} \frac{\omega^m}{m!} = (-1)^m \int_{\mathbb{M}_{2m}} e^{-\frac{\omega^T}{2\pi}}, \tag{2.60}$$

which we can also write as an integral over a polyhedron \mathcal{P}

$$\mathbb{V}(\lambda_a, \epsilon_i) = \int_{\mathcal{P}} e^{-\epsilon_i y_i} dy_1 \dots dy_m. \tag{2.61}$$

Since \mathbb{M}_{2m} and \mathcal{P} are non-compact, we need to assume that the integrals are convergent. This happens for the examples we will be interested in, where \mathcal{P} is asymptotically a cone. In this case the Hamiltonian $H = \epsilon_i y_i$ acts as a convergence factor, at least if the vector ϵ lies inside the cone.

The fixed point formula and the Molien formula hold under general conditions also for non-compact orbifolds, and we can even use them as an operative definition for

⁸ It would be interesting to give a formal proof of (2.58). This would require a careful analysis of all the residues in (2.55), which is a hard problem, and we do not attempt such a proof here.

the equivariant volume. For example, the fixed point formula only assumes that there are isolated fixed points in \mathcal{P} and that there are no contributions from infinity. Notice that the identities (2.34), (2.35) and (2.36), which follow from the fixed point formula, are still valid.

However, the co-homological interpretation (2.30) fails. In particular, the two integrals in (2.29) are not equal in general. The second term in (2.19)

$$\omega = dy_i \wedge d\phi_i = -2\pi \sum_a \lambda_a c_1(L_a) + \frac{1}{2} d \left(\sum_a G^{ij} v_j^a d\phi_i \right), \tag{2.62}$$

is still exact, but it cannot be ignored in an integral over non-compact orbifolds.⁹ As a result, the equivariant volume can be still formally expanded in power series of λ_a but the coefficients cannot be straightforwardly interpreted as generalized intersection numbers as in (2.30). In particular, while in the compact case, the power series of λ_a starts at order m and the coefficients are polynomials in ϵ_i , in the non-compact setting the power series has all coefficients different from zero and these are in general rational function of ϵ_i , as it follows from the fixed point formula. We will discuss explicit examples of non-compact orbifolds in Sects. 3.3 and 5.

2.6 Relation to the equivariant orbifold index

In this section, we briefly discuss how the equivariant volume arises as a limit of an index character, following the logic in [18, 64, 65]. Examples of calculations of characters are provided in ‘‘Appendix C’’ as this is not the main theme of this paper.

The equivariant index for the twisted Dolbeault complex on \mathbb{M}_{2m} is defined by

$$\mathbb{Z}(q_i, \Lambda_a) = \sum_{p=0}^m (-1)^p \text{Tr} \left\{ q | H^{(0,p)} \left(\mathbb{M}_{2m}, \mathcal{O} \left(\sum_a \Lambda_a D_a \right) \right) \right\} \tag{2.63}$$

where $q \in \mathbb{T}^m$ is an element of the torus and $\Lambda_a \in \mathbb{Z}$, $a = 1, \dots, d$ are integers specifying a choice of line bundle. The trace is taken on the induced action on the cohomology. The index can be computed with the Hirzebruch–Riemann–Roch theorem. We start with the smooth case that is considerably simpler, and we take all the labels to be one, the vectors v^a to be primitive and all the orders $d_A = 1$. The fixed point formula for the equivariant index is

$$\mathbb{Z}(\Lambda_a, q_i) = \sum_{A=(a_1, \dots, a_m)} \frac{q^{-\sum_{i=1}^m \Lambda_{a_i} u_A^{a_i}}}{\prod_{i=1}^m (1 - q^{u_A^{a_i}})}, \tag{2.64}$$

⁹ Notice that this does not affect the argument for the fixed point formula. The contribution of $\omega^{\mathbb{T}}$ to the fixed point is still $\omega^{\mathbb{T}} = -2\pi \sum_a \lambda_a c_1^{\mathbb{T}}(L_a)$, since the extra term $\frac{1}{2} \sum_a G^{ij} v_j^a$ vanishes at the fixed point by (2.42).

where the symbol \underline{q}^m is an abbreviation for $q_1^{n_1} \dots q_m^{n_m}$. The geometrical interpretation of the equivariant index is generically to count integer points

$$\mathbb{Z}(\Lambda_a, q_i) = \sum_{\underline{m} \in \Delta(\Lambda_a)} \underline{q}^m, \tag{2.65}$$

in the polytope

$$\Delta(\Lambda_a) = \{m \cdot v^a \geq -\Lambda_a\}, \quad m \in \mathbb{Z}^m, \tag{2.66}$$

as discussed, for example, in [11]. The equivariant volume can be obtained by setting $q_i = e^{-\hbar \epsilon_i}$ and taking the limit $\hbar \rightarrow 0$. The limit of (2.64) is singular but exhibits the equivariant volume as the coefficient of the leading pole

$$\mathbb{Z}(\Lambda_a, q_i) \underset{\hbar \rightarrow 0}{=} \frac{\mathbb{V}(\lambda_a, \epsilon_i)}{\hbar^m} + \dots \tag{2.67}$$

where we scale $\Lambda_a = -\lambda_a/\hbar$.

Analogously, the Molien–Weyl formula reads [64, 65]

$$\mathbb{Z}_{MW}(T_A, \bar{q}_a) = (-1)^{d-m} \int \prod_{A=1}^{d-m} \frac{dz_A}{2\pi i z_A} \frac{\prod_A z_A^{-T_A}}{\prod_{a=1}^d \left(1 - \prod_A z_A^{Q_a^A} \bar{q}_a\right)}, \tag{2.68}$$

where the contour is defined again through the JK prescription. Setting $\bar{q}_a = e^{-\hbar \bar{\epsilon}_a}$, $T_A = t_a/\hbar \in \mathbb{Z}$ and changing variables to $z_a = e^{-\hbar \phi_A}$, we find

$$\mathbb{Z}_{MW}(T_A, \bar{q}_a) \underset{\hbar \rightarrow 0}{=} \frac{\mathbb{V}_{MW}(t_A, \epsilon_i)}{\hbar^m} + \dots \tag{2.69}$$

The Molien–Weyl formula (2.68) can be interpreted as an average over the complexified group $U(1)^{d-m}$ acting on the coordinates of the ambient space \mathbb{C}^d . The continuous average in (2.68) should be supplemented by a discrete average if the group K appearing in the symplectic reduction $\mathbb{M}_{2m} = \mathbb{C}^d // K$ has a torsion part.

The formulas (2.64) and (2.68) are familiar in the context of non-compact conical Calabi–Yau singularities. For $\Lambda_a = T_A = 0$, they have been used to compute Hilbert series for the mesonic moduli space of the dual $\mathcal{N} = 1$ superconformal theories [58, 59]. For $\Lambda_a, T_A \neq 0$ they compute Hilbert series for the baryonic moduli space [60].

The relation between the two formulas is

$$\mathbb{Z}_{MW} \left(T_A = \sum_a Q_a^A \Lambda_a, \bar{q}_a \right) = \prod_{a=1}^d \bar{q}_a^{\Lambda_a} \mathbb{Z} \left(\Lambda_a, q_i = \prod_a \bar{q}_a^{v_i^a} \right). \tag{2.70}$$

The corresponding fixed point formulas for orbifolds are more complicated [7, 71]. Each fixed point contribution is replaced by a discrete Molien formula implementing the quotient \mathbb{Z}_{d_A}

$$\mathbb{Z}(\Lambda_a, q_i) = \sum_{A=(a_1, \dots, a_m)} \frac{1}{d_A} \sum_{k=0}^{d_A-1} \frac{e^{-2\pi i k \sum_{i=1}^m J_i \Lambda_{a_i} \underline{q}} q^{-\sum_{i=1}^m \Lambda_{a_i} \underline{u}_A^{a_i} / d_A}}{\prod_{i=1}^m (1 - e^{2\pi i k J_i \underline{q}} q^{\underline{u}_A^{a_i} / d_A})}, \tag{2.71}$$

where

$$(e^{2\pi i J_1}, \dots, e^{2\pi i J_m}) \in \mathbb{T}^m, \quad \sum_{i=1}^m v^{a_i} J_i \in \mathbb{Z}, \tag{2.72}$$

is a generator of the local orbifold singularity \mathbb{Z}_{d_A} . In the limit $\hbar \rightarrow 0$, only the term with $k = 0$ contributes at the leading order and we recover the fixed point formula (2.48) as the coefficient of the most singular term (see (2.67)).

In the special case of non-trivial labels but no extra orbifold singularities, we can use the formulas in Appendix C to resum (2.71) and we obtain

$$\mathbb{Z}(\Lambda_a, q_i) = \sum_A \frac{\prod_{i=1}^m q^{-\hat{u}_A^{a_i} \lfloor \frac{\Lambda_{a_i}}{n_{a_i}} \rfloor}}{\prod_{i=1}^m (1 - q^{\hat{u}_A^{a_i}})}, \tag{2.73}$$

where the floor function $\lfloor x \rfloor$ denotes the integer part of x , $v^a = n_a \hat{v}^a$ and $u^a = n_a \hat{u}^a$ have been decomposed into primitive vectors \hat{v}^a , \hat{u}^a and the label n_a and we assume that $\hat{d}_{a,a+1} = \det(\hat{v}^a, \hat{v}^{a+1}) = 1$. One can check with elementary methods that this formula computes indeed the number of integer points in the polytope (2.66).

3 Examples from geometry and holography

To illustrate the previous discussion, in this section we provide explicit examples of the equivariant volumes for some simple compact and non-compact orbifolds. The specific examples are chosen among the orbifolds that appear as building blocks of various supergravity solutions with a holographic dual. We also derive some results that will be used in Sects. 4 and 5, where we provide more physical applications.

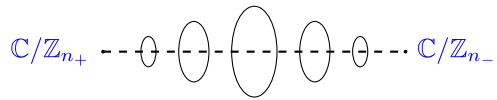
3.1 The spindle

The first example that we consider is the weighted projective line $\mathbb{W}\mathbb{P}^1_{[n_+, n_-]}$, namely the spindle Σ , which is at heart of the recent novel supergravity constructions. It can be defined as the set of pairs of complex numbers (z_1, z_2) under the identification

$$(z_1, z_2) \sim (\lambda^{n_+} z_1, \lambda^{n_-} z_2), \quad \lambda \in \mathbb{C}^*. \tag{3.1}$$

This is the most general toric symplectic orbifold in two real dimensions. It is the simplest example of toric orbifold that is not a toric variety, as its singular points occur in complex co-dimension one. In particular, the orbifold points, that are fixed under the $U(1)$ action, are also the divisors and therefore $d = n = 2$. The image of Σ under

Fig. 1 The spindle Σ as a circle fibration over a segment



the moment map is simply a segment $\mathcal{P} = I$, exactly as for $\mathbb{P}^1 \simeq S^2$ and the two facets \mathcal{F}_a are its end-points, defined by the linear equations

$$l_a(y) = v^a y - \lambda_a = 0, \quad a = 1, 2, \tag{3.2}$$

where the one-dimensional non-primitive vectors are $v^1 = n_1, v^2 \equiv -n_2$, where we will also denote $n_1 \equiv n_+$ and $n_2 \equiv n_-$.¹⁰ The two labels are $n_+, n_- \in \mathbb{N}$. Thus, Σ is a circle fibration over the segment I , with the circle collapsing at the end-points, where are the two $\mathbb{C}/\mathbb{Z}_{n_+}, \mathbb{C}/\mathbb{Z}_{n_-}$ singularities, see Fig. 1.

In the following, it will be convenient to define two rescaled Kähler parameters $\lambda_1 = n_+ \hat{\lambda}_1, \lambda_2 = n_- \hat{\lambda}_2$, so that

$$l_1(y) = n_+(y - \hat{\lambda}_1), \quad l_2(y) = -n_-(y + \hat{\lambda}_2). \tag{3.3}$$

We can then write explicitly the one-dimensional (canonical) metric (2.10)

$$G_{11} = \frac{1}{2} \left(\frac{(v^1)^2}{l_1} + \frac{(v^2)^2}{l_2} \right) = \frac{1}{2} \left(\frac{n_+}{y - \hat{\lambda}_1} - \frac{n_-}{y + \hat{\lambda}_2} \right), \tag{3.4}$$

and the related moment maps

$$\mu_1 = \frac{1}{2\pi} \frac{y + \hat{\lambda}_2}{n_-(y - \hat{\lambda}_1) - n_+(y + \hat{\lambda}_2)}, \quad \mu_2 = \frac{1}{2\pi} \frac{y - \hat{\lambda}_1}{n_-(y - \hat{\lambda}_1) - n_+(y + \hat{\lambda}_2)}. \tag{3.5}$$

In the conventions of Sect. 2.1, in which $l_a(y) \geq 0$, we have that¹¹ $y \in [y_1, y_2] = [\hat{\lambda}_1, -\hat{\lambda}_2]$, with the extrema of the interval corresponding to the north and south poles of Σ , respectively.

We can then write all the relations discussed in Sect. 2.1 explicitly in terms of the moment maps (3.5). The explicit representatives of the first Chern classes $c_1(L_a)$ associated with the line bundles L_a defined by the extrema of the interval (the two divisors) are

$$c_1(L_a) = d(\mu_a d\phi), \tag{3.6}$$

where $\phi \sim \phi + 2\pi$ is the azimuthal coordinate, and denoting by ϵ the equivariant parameter, their equivariant versions are

$$c_1^{\mathbb{T}}(L_a) = d(\mu_a d\phi) + 2\pi\epsilon\mu_a. \tag{3.7}$$

¹⁰ We use this notation to facilitate comparison with the spindle literature in supergravity.

¹¹ Thus, in these conventions, the fibration exists for $\hat{\lambda}_1 < -\hat{\lambda}_2$.

The equivariant Kähler form is:

$$\omega^{\mathbb{T}} = \omega + 2\pi H = dy \wedge d\phi + 2\pi \epsilon y, \tag{3.8}$$

where $H = \epsilon y$ is the Hamiltonian for the vector field $\xi = \epsilon \partial_\phi$. With this information, we can check various relations explicitly, without appealing to the fixed point theorem. For example, it is immediate to compute

$$\int_{\Sigma} c_1(L_a) = 2\pi \int_{y_1}^{y_2} d\mu_a = 2\pi(\mu_a(-\hat{\lambda}_2) - \mu_a(\hat{\lambda}_1)) = \frac{1}{n_a}. \tag{3.9}$$

The co-homological relation (2.20) can also be verified explicitly, noting that

$$y = -2\pi \sum_a \lambda_a \mu_a + \Phi(y), \tag{3.10}$$

where

$$\Phi(y) = \frac{(n_+ - n_-)(y + \hat{\lambda}_2)(y - \hat{\lambda}_1)}{n_+(\hat{\lambda}_2 + y) + n_-(\hat{\lambda}_1 - y)}, \tag{3.11}$$

is such that $\Phi(y_a) = 0$, implying that $\Phi(y)d\phi$ is an exact one-form on the spindle.

Let us now turn to the fixed point relations and the equivariant volume. From (3.5), it immediately follows that the restriction of the moment maps to the fixed points reads

$$\mu_a|_{y_b} = -\frac{1}{2\pi} \frac{u^a}{n_a} \delta_{ab}, \tag{3.12}$$

where $u^1 = 1, u^2 = -1$ and therefore

$$c_1^{\mathbb{T}}(L_a)|_{y_b} = -\frac{\epsilon u^a}{n_a} \delta_{ab}. \tag{3.13}$$

Using these and the fact that the equivariant Euler class of the tangent bundle at a fixed point y_a is

$$e^{\mathbb{T}}|_{y_a} = -\frac{\epsilon u^a}{n_a}, \tag{3.14}$$

one can reproduce the integrals (3.9) from the fixed point theorem (2.37). The equivariant volume can be easily computed either directly

$$\mathbb{V}(\lambda_a, \epsilon) = -\int_{\Sigma} e^{-\epsilon y - \frac{\omega}{2\pi}} = \int_{y_1}^{y_2} e^{-\epsilon y} dy = \frac{1}{\epsilon} \left(e^{-\frac{\epsilon \lambda_1}{n_+}} - e^{\frac{\epsilon \lambda_2}{n_-}} \right), \tag{3.15}$$

or using the fixed point formula

$$\mathbb{V}(\lambda_a, \epsilon) = - \sum_{a=1}^2 \frac{e^{-\frac{\omega_{\mathbb{T}}}{2\pi}|_{y_a}}}{n_a e^{\mathbb{T}}|_{y_a}} = \frac{1}{\epsilon} \left(e^{-\frac{\epsilon \lambda_1}{n_+}} - e^{\frac{\epsilon \lambda_2}{n_-}} \right), \tag{3.16}$$

which of course gives the same result. From this, it is immediate to compute the nonzero equivariant intersection numbers (2.31),

$$\int_{\Sigma} c_1^{\mathbb{T}}(L_1)^s = \frac{(-\epsilon)^{s-1}}{n_+^s}, \quad \int_{\Sigma} c_1^{\mathbb{T}}(L_2)^s = \frac{\epsilon^{s-1}}{n_-^s}, \tag{3.17}$$

generalizing (3.9) to $s > 1$.

We end this subsection noting that it is straightforward to check directly that the equivariant volume depends only on the co-homology class of $[\omega]$. Specifically, if we use a different representative of $[\omega]$, such as

$$\omega' = dy \wedge d\phi - d(\Phi(y)d\phi), \tag{3.18}$$

then we have

$$\begin{aligned} \mathbb{V}(\lambda_a, \epsilon) &= - \int_{\Sigma} e^{-\frac{\omega'}{2\pi}} = \frac{1}{2\pi} \int_{\Sigma} e^{-\epsilon y + \epsilon \Phi(y)} \omega' \\ &= \int_{y_1}^{y_2} e^{-\epsilon y + \epsilon \Phi(y)} d(y - \Phi(y)) = \int_{y_1}^{y_2} e^{-\epsilon y} dy, \end{aligned} \tag{3.19}$$

where the last equality follows trivially from a change of variables and the fact that $\Phi(y_a) = 0$.

3.2 Four-dimensional toric orbifolds

We now move to compact four-dimensional orbifolds ($m = 2$), that have applications in holography [50–52], as well as in geometry [72–74]. The image of \mathbb{M}_4 under the moment map is a rational simple convex polygon \mathcal{P} in \mathbb{R}^2 and its facets \mathcal{F}_a , defined by the linear equations $l_a(y) = 0$, are the edges of the polygon, as shown in Fig. 2. Of course each vertex p_a of the polygon lies at the intersection of precisely 2 edges, so that the simplicity condition is automatic. Moreover, the number of facets/edges d of \mathcal{P} coincides with the number of vertices n , that are also the fixed points of the \mathbb{T}^2 action. The condition that \mathcal{P} be rational is equivalent to the fact that the d normals to the edges, v^a , have integer entries; however, they do not need to be primitive. The highest common factor n_a of the two entries of each of the v^a is precisely the label associated with an edge/facet \mathcal{F}_a .

Each facet defines a symplectic subspace of \mathbb{M}_4 of real dimension two, namely a divisor D_a of complex dimension one, with a local $\mathbb{C}/\mathbb{Z}_{n_a}$ singularity. The vertices p_a of \mathcal{P} correspond precisely to points where two divisors D_a, D_{a+1} intersect, with

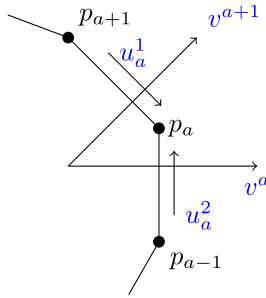


Fig. 2 Fixed points are associated with vertices p_a of the polytope. The vectors in the fan are orthogonal to the facets. For each vertex, there is a corresponding cone (v^a, v^{a+1}) in the fan. The two inwards normals $u_a^i, 1 = 1, 2$ to the cone, which lie along the edges of the polytope, enter in the fixed point formula through the quantities $\epsilon_i^a = \frac{\epsilon \cdot u_a^i}{d_{a,a+1}}$

local singularity $\mathbb{C}^2 / \mathbb{Z}_{d_{a,a+1}}$, where $d_{a,a+1} = |\det(v^a, v^{a+1})|$. The linear dependence of the d vectors v^a in \mathbb{R}^2 is expressed by the relations

$$\sum_{a=1}^d Q_a^A v^a = 0, \tag{3.20}$$

defining the GLSM charges $Q_a^A \in \mathbb{Z}$, with $A = 1, \dots, d-2$, that are used to present \mathbb{M}_4 as the symplectic quotient $\mathbb{M}_4 = \mathbb{C}^d // U(1)^{d-2}$, up to torsion factors. Correspondingly there are two linear relations among the divisors

$$\sum_{a=1}^d v_i^a D_a = 0, \quad i = 1, 2. \tag{3.21}$$

The equivariant volume and equivariant intersection numbers can be conveniently written in terms of the quantities

$$\epsilon_i^a = \frac{\epsilon \cdot u_a^i}{d_{a,a+1}}, \quad i = 1, 2, \tag{3.22}$$

where u_a^i are the inward normals to the cones (v^a, v^{a+1}) as discussed in Sect. 2.3 and pictured in Fig. 2. We can rewrite these as

$$\epsilon_1^a = -\frac{\det(v^{a+1}, \epsilon)}{\det(v^a, v^{a+1})}, \quad \epsilon_2^a = \frac{\det(v^a, \epsilon)}{\det(v^a, v^{a+1})}, \tag{3.23}$$

where $\epsilon \equiv (\epsilon_1, \epsilon_2)$. In writing these formulas, we have assumed that the vectors v^a lie on the plane in anticlockwise order and we have identified cyclically $v^{a+d} \equiv v^a$. In particular, the equivariant Euler class of the tangent bundle at a fixed point y_a reads

$$e^{\mathbb{T}}|_{y_a} = \epsilon_1^a \epsilon_2^a. \tag{3.24}$$

The restriction to the fixed points of the equivariant Chern classes $c_1^{\mathbb{T}}(L_a)$ (2.46) can be written as

$$c_1^{\mathbb{T}}(L_a)|_{y_b} = -\left(\delta_{a,b} \epsilon_1^b + \delta_{a,b+1} \epsilon_2^b\right), \tag{3.25}$$

so that the general formula for the equivariant intersection numbers reads

$$\int_{\mathbb{M}_4} c_1^{\mathbb{T}}(L_{a_1}) \dots c_1^{\mathbb{T}}(L_{a_p}) = \sum_a \frac{c_1^{\mathbb{T}}(L_{a_1})|_{y_a} \dots c_1^{\mathbb{T}}(L_{a_p})|_{y_a}}{d_{a,a+1} \epsilon_1^a \epsilon_2^a}. \tag{3.26}$$

Below we report explicit formulas for intersection numbers up to $p = 4$. First of all, for $p = 0, 1$ the integrals vanish automatically, giving rise to identities that may also be verified with elementary algebra. Specifically, we have

$$0 = \int_{\mathbb{M}_4} 1 = \sum_a \frac{1}{d_{a,a+1} \epsilon_1^a \epsilon_2^a}, \tag{3.27}$$

$$0 = \int_{\mathbb{M}_4} c_1^{\mathbb{T}}(L_a) = -\frac{1}{d_{a,a+1} \epsilon_2^a} - \frac{1}{d_{a-1,a} \epsilon_1^{a-1}}. \tag{3.28}$$

For the double intersection numbers, we reproduce the intersections matrix of divisors (see, e.g. [51]), which is independent of the equivariant parameters ϵ_1, ϵ_2 :

$$D_a \cdot D_b = D_{ab} = \int_{\mathbb{M}_4} c_1^{\mathbb{T}}(L_a) c_1^{\mathbb{T}}(L_b) = \begin{cases} \frac{1}{d_{a-1,a}} & \text{if } b = a - 1, \\ \frac{1}{d_{a,a+1}} & \text{if } b = a + 1, \\ -\frac{d_{a-1,a+1}}{d_{a-1,a} d_{a,a+1}} & \text{if } b = a, \\ 0 & \text{otherwise,} \end{cases} \tag{3.29}$$

where we used the identity

$$\frac{\epsilon_1^a}{d_{a,a+1} \epsilon_2^a} + \frac{\epsilon_2^{a-1}}{d_{a-1,a} \epsilon_1^{a-1}} = -\frac{d_{a-1,a+1}}{d_{a-1,a} d_{a,a+1}} \tag{3.30}$$

to deal with the terms occurring in the self-intersection $D_a \cdot D_a$. Similarly, we can compute the integral of three equivariant first Chern classes, that read:

$$D_{a_1 a_2 a_3} = \int_{\mathbb{M}_4} c_1^{\mathbb{T}}(L_{a_1}) c_1^{\mathbb{T}}(L_{a_2}) c_1^{\mathbb{T}}(L_{a_3})$$

$$= - \begin{cases} \frac{\epsilon_2^{a-1}}{d_{a-1,a}} & \text{if } a_i = a_j = a_k + 1 \equiv a, \\ \frac{\epsilon_1^a}{d_{a,a+1}} & \text{if } a_i = a_j = a_k - 1 \equiv a, \\ \frac{(\epsilon_1^a)^2}{d_{a,a+1}\epsilon_2^a} + \frac{(\epsilon_2^{a-1})^2}{d_{a-1,a}\epsilon_1^{a-1}} & \text{if } a_1 = a_2 = a_3 \equiv a, \\ 0 & \text{otherwise,} \end{cases} \tag{3.31}$$

where the diagonal term can be shown to be linear in ϵ_i , using the identity

$$\frac{(\epsilon_1^a)^2}{d_{a,a+1}\epsilon_2^a} + \frac{(\epsilon_2^{a-1})^2}{d_{a-1,a}\epsilon_1^{a-1}} = -\frac{d_{a-1,a+1}}{d_{a-1,a}^2 d_{a,a+1}} (2d_{a-1,a}\epsilon_1^a + d_{a-1,a+1}\epsilon_2^a). \tag{3.32}$$

For the integrals of four equivariant first Chern classes, we have

$$D_{a_1 a_2 a_3 a_4} = \int_{\mathbb{M}_4} c_1^{\mathbb{T}}(L_{a_1})c_1^{\mathbb{T}}(L_{a_2})c_1^{\mathbb{T}}(L_{a_3})c_1^{\mathbb{T}}(L_{a_4})$$

$$= \begin{cases} \frac{\epsilon_2^{a-1}\epsilon_1^{a-1}}{d_{a-1,a}} & \text{if } a_i = a_j = a_k + 1 = a_l + 1 \equiv a, \\ \frac{\epsilon_1^a\epsilon_2^a}{d_{a,a+1}} & \text{if } a_i = a_j = a_k - 1 = a_l - 1 \equiv a, \\ \frac{(\epsilon_2^{a-1})^2}{d_{a-1,a}} & \text{if } a_i = a_j = a_k = a_l + 1 \equiv a, \\ \frac{(\epsilon_1^a)^2}{d_{a,a+1}^3} & \text{if } a_i = a_j = a_k = a_l - 1 \equiv a, \\ \frac{(\epsilon_1^a)^3}{d_{a,a+1}\epsilon_2^a} + \frac{(\epsilon_2^{a-1})^3}{d_{a-1,a}\epsilon_1^{a-1}} & \text{if } a_1 = a_2 = a_3 = a_4 \equiv a, \\ 0 & \text{otherwise,} \end{cases} \tag{3.33}$$

where the diagonal term can be shown to be quadratic in ϵ_i , using the identity

$$\frac{(\epsilon_1^a)^3}{d_{a,a+1}\epsilon_2^a} + \frac{(\epsilon_2^{a-1})^3}{d_{a-1,a}\epsilon_1^{a-1}}$$

$$= -\frac{d_{a-1,a+1}}{d_{a-1,a}^3 d_{a,a+1}} (3d_{a-1,a}^2(\epsilon_1^a)^2 + 3d_{a-1,a}d_{a-1,a+1}\epsilon_1^a\epsilon_2^a + d_{a-1,a+1}^2(\epsilon_2^a)^2). \tag{3.34}$$

The fixed point formula (2.48) for the equivariant volume specializes to the expression¹²

$$\mathbb{V}(\lambda_a, \epsilon_i) = \sum_{a=1}^d \frac{1}{d_{a,a+1}\epsilon_1^a\epsilon_2^a} e^{-\lambda_a\epsilon_1^a - \lambda_{a+1}\epsilon_2^a}, \tag{3.35}$$

¹² Notice that there is no summation on a in the exponent.

which is easily evaluated in examples. On the other hand, starting from the definition (2.27), the equivariant volume can also be written as an integral over the polygon \mathcal{P} as

$$V(\lambda_a, \epsilon_i) = \int_{\mathcal{P}} e^{-y_i \epsilon_i} dy_1 dy_2, \tag{3.36}$$

that can be evaluated explicitly using Stokes’ theorem. Notice that the dependence on the Kähler parameters λ_a arises from the shape of $\mathcal{P} = \mathcal{P}(\lambda_a)$. The details of this calculation are given in “Appendix A”.

3.2.1 The weighted projective space $\mathbb{W}\mathbb{P}^2_{[N_1, N_2, N_3]}$

As a first example of equivariant volume for four-dimensional toric orbifolds, we consider the weighted projective space $\mathbb{W}\mathbb{P}^2_{[N_1, N_2, N_3]}$ defined as the set of triples of complex numbers (z_1, z_2, z_3) under the identification

$$(z_1, z_2, z_3) \sim (\lambda^{N_1} z_1, \lambda^{N_2} z_2, \lambda^{N_3} z_3), \quad \lambda \in \mathbb{C}^*. \tag{3.37}$$

There are various presentations in terms of labelled polytopes [62], and it is interesting to consider some of them in order to understand better the role of the labels.

Consider first the fan

$$v^1 = (n_3, n_3), \quad v^2 = (-n_1, 0), \quad v^3 = (0, -n_2), \tag{3.38}$$

where each of the three facets has non-trivial labels. We take the labels n_a to be mutually coprime. The GLSM charges are given by

$$Q \equiv (N_1, N_2, N_3) = (n_1 n_2, n_2 n_3, n_1 n_3), \tag{3.39}$$

and the symplectic quotient description $\mathbb{W}\mathbb{P}^2_{[N_1, N_2, N_3]} = \mathbb{C}^3 // U(1)$, where $U(1)$ acts with charge Q , is just the definition (3.37) of the weighted projective space. Notice, however, that our choice of fan corresponds to “non-minimal” $Q_a = N_a$ that are products of integers.

The equivariant volume can be computed with the fixed point formula (3.35)

$$V(\lambda_a, \epsilon_i) = \frac{e^{-\epsilon_2 \lambda_1 / n_3 - (\epsilon_2 - \epsilon_1) \lambda_2 / n_1}}{\epsilon_2 (\epsilon_2 - \epsilon_1)} + \frac{e^{\epsilon_1 \lambda_2 / n_1 + \epsilon_2 \lambda_3 / n_2}}{\epsilon_1 \epsilon_2} + \frac{e^{-\epsilon_1 \lambda_1 / n_3 - (\epsilon_1 - \epsilon_2) \lambda_3 / n_2}}{\epsilon_1 (\epsilon_1 - \epsilon_2)}. \tag{3.40}$$

This expression is a rational function of ϵ_i but when expanded in power series in λ_a as

$$V(\lambda_a, \epsilon_i) = \sum_{k=0}^{\infty} V^{(k)}(\lambda_a, \epsilon_i), \tag{3.41}$$

where $\mathbb{V}^{(k)}$ is the component of degree k in λ_a , all singular terms cancel. The constant and linear terms vanish, and the other $\mathbb{V}^{(k)}(\lambda_a, \epsilon_i)$ are homogeneous polynomials of degree $k - 2$ in ϵ that encode the equivariant intersection numbers of the line bundles L_a . In particular, $\mathbb{V}^{(2)}(\lambda_a, \epsilon_i)$ coincides with the non-equivariant limit

$$\mathbb{V}(\lambda_a, 0) = \frac{1}{2} \left(\frac{n_1 n_2 \lambda_1 + n_2 n_3 \lambda_2 + n_1 n_3 \lambda_3}{n_1 n_2 n_3} \right)^2 = \frac{1}{2} \frac{(N_1 \lambda_1 + N_2 \lambda_2 + N_3 \lambda_3)^2}{N_1 N_2 N_3}, \tag{3.42}$$

encoding the classical intersections numbers of the divisors D_a . Since only one divisor is geometrically independent because of (2.22), this is a quadratic form of rank one.

We can compare the result with the Molien–Weyl formula (2.55),

$$\mathbb{V}_{MW}(t, \bar{\epsilon}) = \int \frac{d\phi}{2\pi} \frac{e^{t\phi}}{(\bar{\epsilon}_1 + n_1 n_2 \phi)(\bar{\epsilon}_2 + n_3 n_2 \phi)(\bar{\epsilon}_3 + n_1 n_3 \phi)}, \tag{3.43}$$

where we use the charges Q . The JK prescription for a single integration is very simple.¹³ It prescribes to take all residues associated with ϕ with positive charge for $t > 0$, and minus the residues for ϕ with negative charge for $t < 0$. Taking $t > 0$ and performing the residue computation, we obtain

$$\begin{aligned} \mathbb{V}_{MW}(t, \bar{\epsilon}) &= \frac{e^{-\bar{\epsilon}_1 t / (n_1 n_2)}}{(n_3 \bar{\epsilon}_1 - n_1 \bar{\epsilon}_2)(n_3 \bar{\epsilon}_1 - n_2 \bar{\epsilon}_3)} + \frac{e^{-\bar{\epsilon}_2 t / (n_2 n_3)}}{(n_3 \bar{\epsilon}_1 - n_1 \bar{\epsilon}_2)(n_2 \bar{\epsilon}_3 - n_1 \bar{\epsilon}_2)} \\ &\quad + \frac{e^{-\bar{\epsilon}_3 t / (n_1 n_3)}}{(n_3 \bar{\epsilon}_1 - n_2 \bar{\epsilon}_3)(n_1 \bar{\epsilon}_2 - n_2 \bar{\epsilon}_3)}, \end{aligned} \tag{3.44}$$

with non-equivariant limit

$$\frac{1}{2} \frac{t^2}{(n_1 n_2 n_3)^2} = \frac{1}{2} \frac{t^2}{N_1 N_2 N_3}. \tag{3.45}$$

We see that each residue corresponds to a fixed point. It is easy to see that the two master volumes coincide under the identification (2.58)

$$\mathbb{V}_{MW} \left(t = - \sum_{a=1}^3 N_a \lambda_a, \bar{\epsilon}_a \right) = e^{\sum_{a=1}^3 \lambda_a \bar{\epsilon}_a} \mathbb{V}(\lambda_a, \epsilon_1 = \bar{\epsilon}_1 n_3 - \bar{\epsilon}_2 n_1, \epsilon_2 = \bar{\epsilon}_1 n_3 - \bar{\epsilon}_3 n_2). \tag{3.46}$$

It is interesting to consider also the fan

$$v^1 = (-n_3, 0), \quad v^2 = (0, -n_3), \quad v^3 = (n_1, n_2), \tag{3.47}$$

¹³ See [75] for a physics-oriented review.

where now the GLSM charges are the “minimal” ones

$$Q \equiv (N_1, N_2, N_3) = (n_1, n_2, n_3). \tag{3.48}$$

However, in this case there is also a torsion part in the symplectic quotient description. Indeed, the symplectic action on \mathbb{C}^3 is given by

$$(e^{2\pi i Q_1}, e^{2\pi i Q_2}, e^{2\pi i Q_3}), \tag{3.49}$$

where

$$\sum_a Q_a v_i^a \in \mathbb{Z}. \tag{3.50}$$

For the minimal fan (3.47) with relatively prime n_a , the relations

$$-n_3 Q_1 + n_1 Q_3 \in \mathbb{Z}, \quad -n_3 Q_2 + n_2 Q_3 \in \mathbb{Z}, \tag{3.51}$$

have solution

$$Q_1 = n_1 \alpha + \frac{k_1}{n_3}, \quad Q_2 = n_2 \alpha + \frac{k_2}{n_3}, \quad Q_3 = n_3 \alpha, \quad k_i = 0, \dots, n_3 - 1, \quad \alpha \in \mathbb{R}. \tag{3.52}$$

One k_i can be further gauged away by choosing $\alpha = \text{integer}/n_3$ (assuming again that the n_a are relatively prime). So we are left with a continuous $U(1)$ action generated by α and an extra discrete group¹⁴

$$\Gamma = \frac{\mathbb{Z}_{N_3} \times \mathbb{Z}_{N_3}}{\mathbb{Z}_{N_3}}, \tag{3.53}$$

so that

$$\mathbb{M}_4 = \mathbb{W}\mathbb{P}^2_{[N_1, N_2, N_3]} / \Gamma. \tag{3.54}$$

As an exercise, the interested reader can check that the relation

$$\mathbb{V}_{MW} \left(t = - \sum_a \lambda_a N_a, \bar{\epsilon}_a \right) \equiv e^{\sum_a \lambda_a \bar{\epsilon}_a} \mathbb{V} \left(\lambda_a, \epsilon_i = \sum_a v_i^a \bar{\epsilon}_a \right), \tag{3.55}$$

still holds. The right-hand side can be computed as before with the fixed point formula (3.35), while the left-hand side with the Molien–Weyl formula. However, since the Molien–Weyl integral is blind to the torsion part, the formula (2.55) should be further

¹⁴ For the non-minimal case (3.38), we have $-n_3 Q_1 + n_1 Q_2, -n_3 Q_1 + n_2 Q_3 \in \mathbb{Z}$ which implies $Q_1 = n_1 n_2 \alpha + k_1/n_3, Q_2 = n_3 n_2 \alpha + k_2/n_1$ and $Q_3 = n_1 n_3 \alpha + k_3/n_2$ but all the k_i can be absorbed by taking $\alpha = \text{integer}/n_i$ if the n_i are coprime and we are left with the $U(1)$ action with charges $(n_1 n_2, n_2 n_3, n_1 n_3)$.

divided by the order N_3 of the discrete group Γ . For example, the non-equivariant volume is now

$$\frac{1}{2} \frac{(N_1\lambda_1 + N_2\lambda_2 + N_3\lambda_3)^2}{N_1N_2N_3^2} = \frac{1}{2} \frac{t^2}{N_1N_2N_3^2}, \tag{3.56}$$

with an extra factor of N_3 .

Finally, we can consider an example taken from [62].¹⁵ The fan is

$$v^1 = (-n_2n_3, 0), \quad v^2 = (0, -n_1n_3), \quad v^3 = (n_1n_2, n_1n_2), \tag{3.57}$$

and the GLSM charges are the “minimal” ones

$$Q \equiv (N_1, N_2, N_3) = (n_1, n_2, n_3). \tag{3.58}$$

However, the equations $\sum_a Q_a v_i^a \in \mathbb{Z}$

$$-n_3n_2Q_1 + n_1n_2Q_3 \in \mathbb{Z}, \quad -n_3n_1Q_2 + n_1n_2Q_3 \in \mathbb{Z}, \tag{3.59}$$

have solution

$$Q_1 = n_1\alpha + \frac{k_1}{n_2n_3}, \quad Q_2 = n_2\alpha + \frac{k_2}{n_1n_3}, \quad Q_3 = n_3\alpha + \frac{k_3}{n_1n_2}, \quad \alpha \in \mathbb{R}, \tag{3.60}$$

and one k_i can be further gauged away by choosing $\alpha = \text{integer}/(n_1n_2n_3)$. So we are left with a continuous $U(1)$ action generated by α and an extra discrete group so that

$$\mathbb{M}_4 = \mathbb{W}\mathbb{P}^2_{[N_1, N_2, N_3]} / \Gamma, \tag{3.61}$$

where

$$\Gamma = \frac{\mathbb{Z}_{N_2N_3} \times \mathbb{Z}_{N_1N_3} \times \mathbb{Z}_{N_2N_3}}{\mathbb{Z}_{N_1N_2N_3}}, \tag{3.62}$$

is a discrete group of order $N_1N_2N_3$. We still find

$$\mathbb{V}_{MW} \left(t = - \sum_a \lambda_a N_a, \bar{\epsilon}_a \right) \equiv e^{\sum_a \lambda_a \bar{\epsilon}_a} \mathbb{V} \left(\lambda_a, \epsilon_i = \sum_a v_i^a \bar{\epsilon}_a \right), \tag{3.63}$$

¹⁵ The reader should be aware that, in [62], the weighted projective space is denoted with $\mathbb{W}\mathbb{P}^2_{(N_1, N_2, N_3)}$ and the example we are discussing with $\mathbb{W}\mathbb{P}^2_{[N_1, N_2, N_3]}$.

where the Molien–Weyl integral should be further divided by $N_1 N_2 N_3$. The non-equivariant volume is

$$\frac{1}{2} \frac{(N_1 \lambda_1 + N_2 \lambda_2 + N_3 \lambda_3)^2}{N_1^2 N_2^2 N_3^2} = \frac{1}{2} \frac{t^2}{N_1^2 N_2^2 N_3^2}. \tag{3.64}$$

3.2.2 Quadrilaterals

Let us now discuss the most general four-dimensional toric orbifold with four fixed points, that we refer to as *quadrilaterals*, following [73]. A sub-class of this family of orbifolds has been discussed in [51], and below we will discuss how one can retrieve those results. It is convenient to parameterize the data of this orbifold in terms of the six vector products

$$d_{a,b} \equiv \det(v^a, v^b) \in \mathbb{Z}, \tag{3.65}$$

where v^a , $a = 1, \dots, 4$ is the set of toric data, as in Fig. 3. Then, a vector identity implies that these satisfy the following relation

$$d_{1,2} d_{3,4} - d_{2,3} d_{4,1} = d_{1,3} d_{2,4}, \tag{3.66}$$

showing that there are five independent integers characterizing a quadrilateral. One can show that the charges of the $U(1)^2$ action on \mathbb{C}^4 can be written in a particularly symmetric form using this parameterization, and read

$$Q^1 = (d_{2,4}, d_{4,1}, 0, d_{1,2}), \quad Q^2 = (d_{2,3}, -d_{1,3}, d_{1,2}, 0). \tag{3.67}$$

Without loss of generality, one can use an $SL(2, \mathbb{Z})$ transformation to set $v^1 = (n_-, 0)$, where $n_- \in \mathbb{Z}$. The remaining vectors solving the constraint

$$\sum_{a=1}^4 Q_a^A v^a = 0, \tag{3.68}$$

are then easily determined and the full set reads

$$\begin{aligned} v^1 &= (n_-, 0), & v^2 &= (a_+ d_{2,4}, d_{1,2}/n_-), \\ v^3 &= (a_- d_{2,3} + a_+ d_{3,4}, d_{1,3}/n_-), & v^4 &= (a_- d_{2,4}, -d_{4,1}/n_-), \end{aligned} \tag{3.69}$$

where $d_{1,2}, d_{1,3}, d_{4,1}$ are integer multiples of n_- and $a_{\pm} \in \mathbb{Z}$ such that

$$a_- d_{1,2} + a_+ d_{4,1} = -n_-, \tag{3.70}$$

which always exist by Bezout’s lemma.

The equivariant volume $\mathbb{V}(\lambda_a, \epsilon_i)$ is easily computed using the general fixed point formula (3.35), but we shall not write it explicitly as it does not have a compact

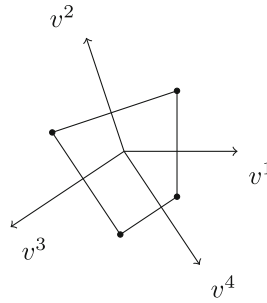


Fig. 3 The fan and the polytope for a generic quarilateral

expression. Expanding it in Taylor series in powers of λ_a as in (3.41), one can check that $V^{(0)}(\epsilon_i) = V^{(1)}(\lambda_a, \epsilon_i) = 0$, whereas the quadratic part is independent of ϵ_i and reads

$$V^{(2)}(\lambda_a) = \frac{1}{2} \sum_{a,b} \lambda_a \lambda_b D_{ab}, \tag{3.71}$$

where D_{ab} is the intersection matrix of divisors (3.29), which reads

$$D_{ab} = \begin{pmatrix} \frac{d_{2,4}}{d_{1,2}d_{4,1}} & \frac{1}{d_{1,2}} & 0 & \frac{1}{d_{4,1}} \\ \frac{1}{d_{1,2}} & \frac{-d_{1,3}}{d_{1,2}d_{2,3}} & \frac{1}{d_{2,3}} & 0 \\ 0 & \frac{1}{d_{2,3}} & -\frac{d_{2,4}}{d_{2,3}d_{3,4}} & \frac{1}{d_{3,4}} \\ \frac{1}{d_{4,1}} & 0 & \frac{1}{d_{3,4}} & \frac{d_{1,3}}{d_{3,4}d_{4,1}} \end{pmatrix}. \tag{3.72}$$

It may be useful to compare with the orbifold geometry studied in [51] (see also [50]), which may be viewed as spindle fibered over another spindle, and interpreted as a natural orbifold generalization of Hirzebruch surfaces. This is obtained setting

$$\begin{aligned} d_{1,3} &= 0, & d_{2,4} &= -t, & d_{1,2} &= m_- n_-, \\ d_{2,3} &= m_- n_+, & d_{3,4} &= m_+ n_+, & d_{4,1} &= n_- m_+. \end{aligned} \tag{3.73}$$

and $r_+ = -ta_+, r_- = -ta_-$. For these values, the vectors of the fan reduce to those written in eq. (4.27) of [51] and the GLSM charges become

$$Q^1 = (-t, m_+ n_-, 0, m_- n_-), \quad Q^2 = (n_+, 0, n_-, 0). \tag{3.74}$$

We can then perform the $U(1)^2$ quotient of \mathbb{C}^4 in two stages. The first quotient using weights Q^1 gives the line bundle $O(-t) \rightarrow \Sigma_{m_+, m_-}$. The second quotient using weights Q^2 projectivize this bundle, giving a $\Sigma_{n_+, n_-} \rightarrow \Sigma_{m_+, m_-}$ bundle. Indeed for $n_+ = n_- = m_+ = m_- = 1$, this reduces exactly to the Hirzebruch surface \mathbb{F}_t , which is the most general family of toric manifolds with four fixed points.

It is instructive to compare with the Molien–Weyl formula. We do it explicitly for the case of the $\Sigma_{n_+, n_-} \rightarrow \Sigma_{m_+, m_-}$ bundle. The Molien–Weyl integral reads

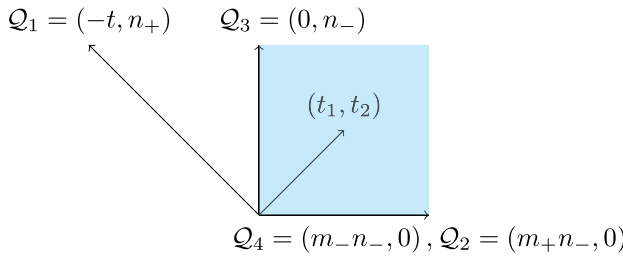


Fig. 4 The JK prescription for the $\Sigma_{n_+, n_-} \rightarrow \Sigma_{m_+, m_-}$ bundle

$$V_{MW} = n_- \int \frac{d\phi_1}{2\pi} \frac{d\phi_2}{2\pi} \frac{e^{t_1\phi_1 + t_2\phi_2}}{(\bar{\epsilon}_1 - t\phi_1 + n_+\phi_2)(\bar{\epsilon}_2 + n_-m_+\phi_1)(\bar{\epsilon}_3 + n_-\phi_2)(\bar{\epsilon}_4 + m_-n_-\phi_1)}. \tag{3.75}$$

The multiplicative factor n_- takes into account that the $U(1)^2$ action is not effective since

$$g_1 = e^{2\pi i \frac{n_+}{n_-} Q_1}, \quad g_2 = e^{-2\pi i \frac{1}{n_-} Q_2}, \tag{3.76}$$

acts in the same way on all points in \mathbb{C}^4 . The Molien formula mods out twice by a discrete subgroup \mathbb{Z}_{n_-} and we need to multiply by n_- to obtain the right result. The poles in the integrand are associated with two-dimensional vectors $Q^a = (Q_a^1, Q_a^2)$, $a = 1, 2, 3, 4$. The JK prescription instructs us to take the (simultaneous) residues at the poles Q_{a_1} and Q_{a_2} if the vector of Kähler parameters (t_1, t_2) is contained in the cone (Q_{a_1}, Q_{a_2}) . When (t_1, t_2) lies in the first quadrant, it is contained in four different cones (see Fig. 4).

In the Kähler chamber $t_1 > 0$ and $t_2 > 0$, we then add four contributions. One can check that

$$V_{MW} \left(t_A = -\sum_a \lambda_a Q_a^A, \bar{\epsilon}_a \right) = e^{\sum_a \lambda_a \bar{\epsilon}_a} V \left(\lambda_a, \epsilon_i = \sum_a v_i^a \bar{\epsilon}_a \right), \tag{3.77}$$

where the gauge-invariant variables (2.57) are

$$\begin{aligned} t_1 &= t\lambda_1 - m_+n_-\lambda_2 - m_-n_-\lambda_4 & t_2 &= -n_+\lambda_1 - n_-\lambda_3 \\ \epsilon_1 &= n_-\bar{\epsilon}_1 + r_+\bar{\epsilon}_2 - n_+\bar{\epsilon}_3 + r_-\bar{\epsilon}_4 & \epsilon_2 &= m_-\bar{\epsilon}_2 - m_+\bar{\epsilon}_4. \end{aligned} \tag{3.78}$$

The non-equivariant volume is

$$V_{MW}(t_A, 0) = \frac{2n_+t_1t_2 + m_-r_-t_1^2 + m_+r_+t_2^2}{2m_-m_+n_-^2n_+^2}. \tag{3.79}$$

3.3 Non-compact examples

We consider now some non-compact examples where the polytope \mathcal{P} is asymptotically a polyhedral cone. We compute the volume using the fixed point formula, assuming that there is a finite number of fixed points and no contributions from infinity. As anticipated in Sect. 2.5, the equivariant volume is a *rational* function of ϵ_i . As we will see, two of the singular terms for the Calabi–Yau examples can be identified with the Sasakian volume of [17, 18] and the master volume introduced in [66]. This identification will be discussed in more detail in Sect. 5.

3.3.1 $\mathbb{C}^2/\mathbb{Z}_p$

Our first non-compact example is $\mathbb{C}^2/\mathbb{Z}_p$, which is a conical orbifold singularity. The most general toric action is obtained using as fan the vectors

$$v^1 = (1, 0), \quad v^2 = (q, p). \tag{3.80}$$

The corresponding \mathbb{Z}_p action on $(z_1, z_2) \in \mathbb{C}^2$ is given by

$$(z_1, z_2) \rightarrow \left(e^{2\pi i Q_1} z_1, e^{2\pi i Q_2} z_2 \right), \tag{3.81}$$

where the GLSM charges are pure torsion, namely

$$\sum_a Q_a v^a \in \mathbb{Z}^2, \tag{3.82}$$

whose solution is, for p and q coprime, given by

$$Q_1 = k \frac{q}{p}, \quad Q_2 = -\frac{k}{p}, \quad k = 0, \dots, p - 1. \tag{3.83}$$

This is then the $\mathbb{C}^2/\mathbb{Z}_p$ quotient

$$(z_1, z_2) \rightarrow \left(\omega_p^q z_1, \omega_p^{-1} z_2 \right), \tag{3.84}$$

with $\omega_p = e^{\frac{2\pi i}{p}}$, which for $|z_1|^2 + |z_2|^2 = 1$ gives the Lens space $L(p, q)$. The equivariant volume is obtained from the general formula (3.35), and is entirely encoded by the contribution of the single fixed point with \mathbb{Z}_p singularity, namely

$$\mathbb{V}_{\mathbb{C}^2/\mathbb{Z}_p}(\lambda_a, \epsilon_i) = p \frac{e^{\frac{1}{p}(\lambda_1(q\epsilon_2 - p\epsilon_1) - \lambda_2\epsilon_2)}}{(p\epsilon_1 - q\epsilon_2)\epsilon_2}. \tag{3.85}$$

Expanding (3.85) in powers of λ_a , we obtain

$$\mathbb{V}_{\mathbb{C}^2/\mathbb{Z}_p}(\lambda_a, \epsilon_i) = \frac{p}{\epsilon_2(p\epsilon_1 - q\epsilon_2)} - \left(\frac{\lambda_1}{\epsilon_2} + \frac{\lambda_2}{p\epsilon_1 - q\epsilon_2} \right) + \mathcal{O}(\lambda_a^2), \tag{3.86}$$

where the constant and linear terms are nonzero, as a difference with the compact case. They are also singular functions of ϵ . The constant piece is actually the Sasakian volume of the Lens space $L(p, q)$, and the linear term is the corresponding master volume introduced in [66]. In particular, for $q = 1$ that corresponds to the ‘‘supersymmetric’’ Lens space $L(p, 1)$, the linear term coincides precisely with (B.6) of [76].

Since there are no continuous GLSM charges, the Molien–Weyl formula is trivial

$$V_{\mathbb{C}^2/\mathbb{Z}_p}^{MW}(\bar{\epsilon}_a) = \frac{1}{p\bar{\epsilon}_1\bar{\epsilon}_2}, \tag{3.87}$$

where there are no Kähler parameters t and we divide by p for the order of the discrete group \mathbb{Z}_p . Nevertheless, it is still true that

$$V_{\mathbb{C}^2/\mathbb{Z}_p}^{MW}(\bar{\epsilon}_a) = e^{\sum_a \lambda_a \bar{\epsilon}_a} V_{\mathbb{C}^2/\mathbb{Z}_p} \left(\lambda_a, \epsilon_i = \sum_a v_i^a \bar{\epsilon}_a \right), \tag{3.88}$$

where the gauge-invariant combinations are

$$\epsilon_1 = \bar{\epsilon}_1 + q\bar{\epsilon}_2, \quad \epsilon_2 = p\bar{\epsilon}_2. \tag{3.89}$$

3.3.2 $\mathcal{O}(-p) \rightarrow \Sigma$

Let us now consider the asymptotically conical non-compact orbifold¹⁶ $\mathcal{O}(-p) \rightarrow \Sigma$, where $\Sigma = \Sigma_{n_+, n_-}$ is a spindle with $\mathbb{Z}_{n_+}, \mathbb{Z}_{n_-}$ singularities at its poles. This can be thought of as a ‘‘blow-up’’ of the $\mathbb{C}^2/\mathbb{Z}_p$ of the previous section, where the apex of the cone is replaced with the spindle Σ . In general, this is a partial resolution, but in the special case that $n_+ = n_- = 1$, it is a complete resolution.

The GLSM charges are given by

$$Q = (n_+, -p, n_-), \tag{3.90}$$

where notice that the space is a Calabi–Yau if and only if $p = n_+ + n_-$. In particular, only for $p = 2$ we have a complete resolution of the CY singularity $\mathbb{C}^2/\mathbb{Z}_2$. The vectors of the fan solving the condition

$$\sum_{a=1}^3 Q_a v^a = 0, \tag{3.91}$$

can be taken to be

$$v^1 = (1, 0), \quad v^2 = (t, n_-) \quad v^3 = (q, p), \tag{3.92}$$

¹⁶ The non-compact examples of this section are the total space of some vector bundles over a base orbifold, but we denote them using the more schematic notation ‘‘fibre \rightarrow base’’.

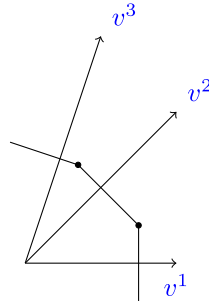


Fig. 5 The fan and the polytope for $\mathcal{O}(-p) \rightarrow \Sigma$. The resolution introduces a compact facet in the polytope that is orthogonal to v^2 . The corresponding segment represents the toric polytope of a spindle Σ

with $t, q \in \mathbb{Z}$ satisfying

$$tp - qn_- = n_+, \tag{3.93}$$

which always exist by Bezout’s lemma. Indeed, we see that the basis (3.92) is obtained from (3.80) by adding the vector (t, n_-) , which is normal to the edge representing the spindle, as illustrated in Fig. 5.

Alternatively, the basis (3.92) can be obtained from (3.69) of the compact example, by removing v^3 and setting $n_- = 1, d_{4,1} = -p, d_{1,2} = n_-, d_{2,4} = -n_+$ and $a_+d_{2,4} = t, a_-d_{2,4} = q$. The condition (3.70) obeyed by a_+, a_- becoming the condition (3.93) for t, q .

The equivariant volume is obtained from the general formula (3.35), and now it receives contributions from the two orbifold fixed points, with $\mathbb{Z}_{n_-}, \mathbb{Z}_{n_+}$ singularities, namely

$$\mathbb{V}_{\mathcal{O}(-p) \rightarrow \Sigma}(\lambda_a, \epsilon_i) = n_- \frac{e^{\frac{1}{n_-}(\lambda_1(t\epsilon_2 - n_- \epsilon_1) - \epsilon_2 \lambda_2)}}{\epsilon_2(n_- \epsilon_1 - t\epsilon_2)} + n_+ \frac{e^{\frac{1}{n_+}(\lambda_2(q\epsilon_2 - p\epsilon_1) + \lambda_3(n_- \epsilon_1 - t\epsilon_2))}}{(q\epsilon_2 - p\epsilon_1)(n_- \epsilon_1 - t\epsilon_2)}, \tag{3.94}$$

and expanding this in λ_a , we get

$$\mathbb{V}_{\mathcal{O}(-p) \rightarrow \Sigma}(\lambda_a, \epsilon_i) = \frac{p}{\epsilon_2(p\epsilon_1 - q\epsilon_2)} - \left(\frac{\lambda_1}{\epsilon_2} + \frac{\lambda_3}{p\epsilon_1 - q\epsilon_2} \right) + \mathcal{O}(\lambda_a^2), \tag{3.95}$$

which coincides with (3.86). Thus, we see that up to linear order, the information about the (partial) resolution is washed out as there is no dependence on n_- and n_+ , nor on the Kähler parameter λ_2 , corresponding to the compact divisor Σ . Specifically, the leading and sub-leading terms reproduce precisely the Sasakian and GMS master volume of the Lens space $L(p, q)$. Notice that this behaviour is independent of the Calabi–Yau condition $p = n_+ + n_-$. Another notable case is given by $n_+ = n_- = p = 1$, which

corresponds to the smooth non-compact space $\mathcal{O}(-1) \rightarrow \mathbb{P}^1$, namely the blow-up of \mathbb{C}^2 at one point¹⁷. This is not a Calabi–Yau and its link is the round S^3 .

We can also compare with the Molien–Weyl formula

$$\mathbb{V}_{\mathcal{O}(-p) \rightarrow \Sigma}(t, \bar{\epsilon}_a) = \int \frac{d\phi}{2\pi} \frac{e^{t\phi}}{(\bar{\epsilon}_1 + n_+\phi)(\bar{\epsilon}_2 - p\phi)(\bar{\epsilon}_3 + n_-\phi)}, \tag{3.96}$$

where, according to the JK prescription, for $t > 0$ we take the residues for the terms with positive charge Q_i , $\phi = -\bar{\epsilon}_1/n_+$ and $\phi = -\bar{\epsilon}_3/n_-$. We find, as usual

$$\mathbb{V}_{\mathcal{O}(-p) \rightarrow \Sigma} \left(t = -\sum_a Q_a \lambda_a, \bar{\epsilon}_a \right) = e^{\sum_a \lambda_a \bar{\epsilon}_a} \mathbb{V}_{\mathcal{O}(-p) \rightarrow \Sigma} \left(\lambda_a, \epsilon_i = \sum_a v_i^a \bar{\epsilon}_a \right), \tag{3.97}$$

where the gauge-invariant combinations are

$$t = -n_+\lambda_1 + p\lambda_2 - n_-\lambda_3, \quad \epsilon_1 = \bar{\epsilon}_1 + t\bar{\epsilon}_2 + q\bar{\epsilon}_3, \quad \epsilon_2 = n_-\bar{\epsilon}_2 + p\bar{\epsilon}_3. \tag{3.98}$$

3.3.3 $\mathcal{O}(-1) \oplus \mathcal{O}(-1) \rightarrow \mathbb{P}^1$

We now consider the two small resolutions of the conifold singularity $\{z_1 z_2 = z_3 z_4 | z_i \in \mathbb{C}^4\}$, corresponding to the asymptotically conical non-compact manifold $\mathcal{O}(-1) \oplus \mathcal{O}(-1) \rightarrow \mathbb{P}^1$, known in the physics literature as resolved conifold. The toric data consist in the fan

$$v^1 = (1, 0, 0), \quad v^2 = (1, 1, 0), \quad v^3 = (1, 1, 1), \quad v^4 = (1, 0, 1), \tag{3.99}$$

with GLSM charges

$$Q = (1, -1, 1, -1). \tag{3.100}$$

The corresponding polytope is a conical non-compact polyhedron with four facets all intersecting at the tip of the cone. Since the vectors v^a lie on a plane, or, equivalently $\sum_a Q_a = 0$, this defines a conical Calabi–Yau threefold. In order to use the results of Sect. 2, we need to resolve the conifold singularity. This can be done in a standard way by triangulating the fan as in Fig. 6. The fan is now the union of the two cones (v^1, v^2, v^4) and (v^2, v^3, v^4) , each associated with a vertex of the polytope and therefore with a fixed point of the torus action. The original conical singularity has been replaced by a compact two cycle, which can be visualized as a circle fibration over the segment connecting the two vertices of the polytope. There are actually two small resolutions related by a flop. The second one is obtained by triangulating the fan in a different

¹⁷ In this case, we have $t - q = 1$ and therefore we can take $t = 1, q = 0$, reproducing the obvious fan $v^1 = (1, 0), v^2 = (1, 1), v^3 = (0, 1)$.

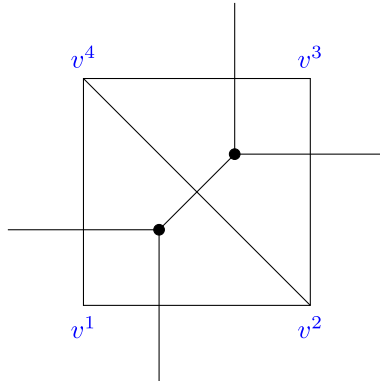


Fig. 6 One of the two small resolutions of the conifold singularity and the corresponding non-compact polytope projected on the plane where the vectors v^a live

way, by adding the line $v^1 - v^3$ and considering the fan which is the union of the two cones (v^1, v^2, v^3) and (v^1, v^3, v^4) .

The fixed point formula (2.48) reads

$$\mathbb{V}(\lambda_a, \epsilon_i) = \sum_{A=(a_1, a_2, a_3)} \frac{e^{-\epsilon \cdot (\lambda_{a_1} u_A^{a_1} + \lambda_{a_2} u_A^{a_2} + \lambda_{a_3} u_A^{a_3}) / d_{a_1, a_2, a_3}}}{d_{a_1, a_2, a_3} \left(\frac{\epsilon \cdot u_A^{a_1}}{d_{a_1, a_2, a_3}} \right) \left(\frac{\epsilon \cdot u_A^{a_2}}{d_{a_1, a_2, a_3}} \right) \left(\frac{\epsilon \cdot u_A^{a_3}}{d_{a_1, a_2, a_3}} \right)}, \quad (3.101)$$

where A runs over the triangular cones of the resolution, $d_{a_1, a_2, a_3} = |\det(v^{a_1}, v^{a_2}, v^{a_3})|$ is the order of the orbifold singularity at the fixed point and u_A^a are the inward normals to the faces of the cones as in Fig. 7.

Consider first the resolution in Fig. 6. The necessary data are

$$(a_1, a_2, a_3) = (1, 2, 4), \quad u_A^{a_1} = (1, -1, -1), \quad u_A^{a_2} = (0, 1, 0), \quad u_A^{a_3} = (0, 0, 1),$$

$$(a_1, a_2, a_3) = (2, 3, 4), \quad u_A^{a_1} = (1, 0, -1), \quad u_A^{a_2} = (-1, 1, 1), \quad u_A^{a_3} = (1, -1, 0),$$

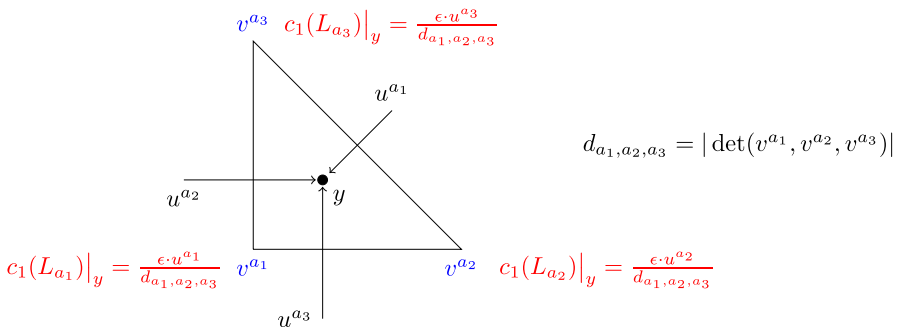


Fig. 7 The contribution of a single vertex of the polytope to the fixed point formula for $m = 3$

and the equivariant volume is

$$\mathbb{V}(\lambda_a, \epsilon_i) = \frac{e^{-(\epsilon_1 - \epsilon_2 - \epsilon_3)\lambda_1 - \epsilon_2\lambda_2 - \epsilon_3\lambda_4}}{\epsilon_2\epsilon_3(\epsilon_1 - \epsilon_2 - \epsilon_3)} + \frac{e^{-(\epsilon_1 - \epsilon_3)\lambda_2 - (-\epsilon_1 + \epsilon_2 + \epsilon_3)\lambda_3 - (\epsilon_1 - \epsilon_2)\lambda_4}}{(\epsilon_1 - \epsilon_2)(\epsilon_1 - \epsilon_3)(-\epsilon_1 + \epsilon_2 + \epsilon_3)}. \tag{3.102}$$

At $\lambda_a = 0$, we recover the Sasakian volume of the 5-dimensional base of the singular cone (cfr (7.30) in [18])

$$\mathbb{V}^{(0)}(\epsilon_i) = \mathbb{V}(\lambda_a = 0, \epsilon_i) = \frac{\epsilon_1}{\epsilon_2\epsilon_3(\epsilon_1 - \epsilon_2)(\epsilon_1 - \epsilon_3)}, \tag{3.103}$$

and the quadratic part is, up to a normalization, the master volume introduced in [66]¹⁸

$$\begin{aligned} &2\mathbb{V}^{(2)}(\epsilon_i) \\ &= \sum_{a=1}^4 \frac{\lambda_a (\lambda_{a-1} \det(v^a, v^{a+1}, \epsilon) - \lambda_a \det(v^{a-1}, v^{a+1}, \epsilon) + \lambda_{a+1} \det(v^{a-1}, v^a, \epsilon))}{\det(v^{a-1}, v^a, \epsilon) \det(v^a, v^{a+1}, \epsilon)}. \end{aligned} \tag{3.104}$$

Things work similarly for the second resolution

$$\begin{aligned} (a_1, a_2, a_3) = (1, 2, 3), & \quad u_A^{a_1} = (1, -1, 0), & \quad u_A^{a_2} = (0, 1, -1), & \quad u_A^{a_3} = (0, 0, 1), \\ (a_1, a_2, a_3) = (1, 3, 4), & \quad u_A^{a_1} = (1, 0, -1), & \quad u_A^{a_2} = (0, 1, 0), & \quad u_A^{a_3} = (0, -1, 1). \end{aligned} \tag{3.105}$$

The equivariant volume now reads

$$\mathbb{V}(\lambda_a, \epsilon_i) = \frac{e^{-(\epsilon_1 - \epsilon_2)\lambda_1 - (\epsilon_2 - \epsilon_3)\lambda_2 - \epsilon_3\lambda_3}}{\epsilon_3(\epsilon_1 - \epsilon_2)(\epsilon_2 - \epsilon_3)} + \frac{e^{-(\epsilon_1 - \epsilon_3)\lambda_1 - \epsilon_2\lambda_3 - (\epsilon_3 - \epsilon_2)\lambda_4}}{\epsilon_2(\epsilon_1 - \epsilon_3)(\epsilon_3 - \epsilon_2)}, \tag{3.106}$$

which is different from (3.102). The difference between the equivariant volumes for the two resolutions starts at order three in the Kähler parameters

$$\mathbb{V}\Big|_{\text{res2}} - \mathbb{V}\Big|_{\text{res1}} = \frac{(\lambda_1 - \lambda_2 + \lambda_3 - \lambda_4)^3}{6} + O(\lambda_a^4). \tag{3.107}$$

In particular, Eqs. (3.103) and (3.104) are valid for both resolutions. We see that the first terms in the λ_a expansion of \mathbb{V} , which are singular in ϵ_i , capture the geometry of the asymptotic singular cone and are therefore independent of the resolution. The difference (3.107) is instead a power series in λ_a with coefficients which are regular polynomials in ϵ_i , and this reflects the fact that the different flops differ by a compact cycle.

¹⁸ In this formula, we identify cyclically $v^5 = v^1$.

It is also interesting to compare with the result of the Molien–Weyl formula (2.55) which, using $Q = (1, -1, 1, -1)$, reads

$$V_{MW}(t, \bar{\epsilon}) = \int \frac{d\phi}{2\pi i} \frac{e^{t\phi}}{(\bar{\epsilon}_1 + \phi)(-\bar{\epsilon}_2 + \phi)(\bar{\epsilon}_3 + \phi)(-\bar{\epsilon}_4 + \phi)}. \tag{3.108}$$

We need to specify a prescription for the contour and the residues to take, which depends on the sign of t and the choice of resolution. The JK prescription requires to take the residues where the charge Q is positive for $t > 0$ (first resolution) and minus the residues where the charge Q is negative for $t < 0$ (second resolution). For example, for $t > 0$ we take the two residues $\phi = -\bar{\epsilon}_1$ and $\phi = -\bar{\epsilon}_3$, obtaining

$$V_{MW}(t, \bar{\epsilon}) = \frac{e^{-t\bar{\epsilon}_3}}{(\bar{\epsilon}_1 - \bar{\epsilon}_3)(\bar{\epsilon}_2 + \bar{\epsilon}_3)(\bar{\epsilon}_3 + \bar{\epsilon}_4)} - \frac{e^{-t\bar{\epsilon}_1}}{(\bar{\epsilon}_1 - \bar{\epsilon}_3)(\bar{\epsilon}_1 + \bar{\epsilon}_2)(\bar{\epsilon}_1 + \bar{\epsilon}_4)}, \tag{3.109}$$

while for $t < 0$ we take (minus) the residues associated with negative charge, $\phi = \bar{\epsilon}_2$ and $\phi = \bar{\epsilon}_4$. It is easy to check that in both cases

$$V_{MW} \left(t = - \sum_a \lambda_a Q_a, \bar{\epsilon}_a \right) \Big|_{t < 0}^{t > 0} = e^{\sum_a \lambda_a \bar{\epsilon}_a} \mathbb{V} \left(\lambda_a, \epsilon_i = \sum_a v_i^a \bar{\epsilon}_a \right) \Big|_{\text{res2}}^{\text{res1}}, \tag{3.110}$$

where the gauge-invariant variables explicitly read

$$t = -\lambda_1 + \lambda_2 - \lambda_3 + \lambda_4, \quad \epsilon_1 = \bar{\epsilon}_1 + \bar{\epsilon}_2 + \bar{\epsilon}_3 + \bar{\epsilon}_4, \quad \epsilon_2 = \bar{\epsilon}_2 + \bar{\epsilon}_3, \quad \epsilon_3 = \bar{\epsilon}_3 + \bar{\epsilon}_4. \tag{3.111}$$

4 Integrating the anomaly polynomial on orbifolds

In this section, we will consider field theories associated with D3 and M5 branes and we will show how the formalism of equivariant integration can be employed to determine the anomaly polynomials of lower-dimensional theories, obtained compactifying the original theories on orbifolds. For theories compactified on the spindle, these results were obtained in [33, 35–37], but the equivariant formalism allows for uniform derivations for various branes wrapped on different orbifolds. In the case of theories compactified on manifolds, the method of integration of the anomaly polynomial is well known and is reviewed in [77], to which we refer for more details. In this reference is also discussed the extension in which background gauge fields for the isometries of the compactification manifolds are turned on, and the relation to equivariant integration is spelled out. In a SCFT in (even) dimension d , the anomaly polynomial can be

thought of as a formal $(d+2)$ -form on an auxiliary $(d+2)$ -dimensional space Z_{d+2} that is the total space of a \mathbb{M}_p fibration over a $(d+2-p)$ -dimensional base space B_{d+2-p} ,

$$\mathbb{M}_p \hookrightarrow Z_{d+2} \rightarrow B_{d+2-p}, \tag{4.1}$$

and integrating it on \mathbb{M}_p gives a $(d+2-p)$ -form, that is the anomaly polynomial of a $(d-p)$ -dimensional SCFT. In [33], an extension of this construction to the case that $\mathbb{M}_2 = \mathbb{Z}$ is a spindle has been proposed and successfully matched to the dual supergravity solution. More generally, here we will assume that \mathbb{M}_p (and hence, generically, also Z_{d+2}) is an orbifold.

The anomaly polynomial is a linear combination of characteristic classes of vector bundles on Z_{d+2} , which (using the splitting principle) can always be decomposed in terms of first Chern classes of holomorphic line (orbi-)bundles on Z_{d+2} . Specifically, for any $U(1)$ symmetry acting on \mathbb{M}_p , the latter can be fibered over B_{d+2-p} by gauging this $U(1)$ with a connection on a line bundle \mathcal{J} over B_{d+2-p} , denoted $A_{\mathcal{J}}$. When \mathbb{M}_{2m} is a toric orbifold, this consists in replacing the occurrences of each term $d\phi_i$ with $d\phi_i + A_{\mathcal{J}_i}$, where \mathcal{J}_i are auxiliary line bundles with $c_1(\mathcal{J}_i) = [F_{\mathcal{J}_i}]/2\pi \in H^2(B_{d+2-p}, \mathbb{Z})$ and $F_{\mathcal{J}_i} = dA_{\mathcal{J}_i}$. This recipe is formally equivalent to consider *equivariant first Chern classes*, as follows. Starting from the line bundles L_a with equivariant first Chern classes $c_1^{\mathbb{T}}(L_a)$ discussed in Sect. 2.1, one can make the replacement

$$c_1^{\mathbb{T}}(L_a) = c_1(L_a) + 2\pi\mu_a^i \epsilon_i \quad \mapsto \quad c_1(\mathcal{L}_a) = c_1(L_a) + 2\pi\mu_a^i c_1(\mathcal{J}_i). \tag{4.2}$$

One can then use these $c_1(\mathcal{L}_a)$ as a basis to parameterize the characteristic classes appearing in the anomaly polynomials.

4.1 D3 branes on the spindle

For 4-dimensional SCFTs, the anomaly polynomial is a 6-form defined on an six-dimensional orbifold Z_6 that we take to be the total space of a \mathbb{Z} fibration over an auxiliary space B_4 ,

$$\mathbb{Z} \hookrightarrow Z_6 \rightarrow B_4. \tag{4.3}$$

Neglecting terms that are sub-leading in the large N limit the six-form anomaly polynomial is given by

$$\mathcal{A}_{4d} = \frac{1}{6} \sum_{I,J,K} c_{IJK} c_1(F_I) c_1(F_J) c_1(F_K), \tag{4.4}$$

where c_{IJK} are the cubic 't Hooft anomaly coefficients $c_{IJK} = \text{Tr}(F_I F_J F_K)$ and F_I denote the generators of global $U(1)$ symmetries. The $c_1(F_I)$ are formal first Chern classes associated with these $U(1)_I$ symmetries that can be decomposed as¹⁹

¹⁹ To avoid clumsiness in the formulas, we systematically use the Einstein's notation for the indices a, b, \dots in this section.

$$c_1(F_I) = \Delta_I c_1(F_R^{2d}) - p_I^a c_1(\mathcal{L}_a), \quad c_1(\mathcal{L}_a) = c_1(L_a) + 2\pi\mu_a c_1(\mathcal{J}), \quad (4.5)$$

where $p_I^a \in \mathbb{Z}$ are “fluxes stuck at the fixed points”, L_a are the line bundles discussed in Sect. 3.2 and $c_1(F_R^{2d}) \in H^2(B_4, \mathbb{Z})$ is the first Chern class of the 2d R-symmetry line bundle.

The anomaly polynomial of the two-dimensional theory is obtained integrating the anomaly polynomial of the four-dimensional theory on the spindle,

$$\mathcal{A}_{2d} = \int_{\Sigma} \mathcal{A}_{4d}, \quad (4.6)$$

and is a four-form on B_4 . Substituting (4.5) in (4.4), it is immediate to see that this is exactly equivalent to the equivariant integral on the spindle

$$F(\Delta_I, \epsilon) = \frac{1}{6} \sum_{I,J,K} c_{IJK} \int_{\Sigma} (\Delta_I - p_I^a c_1^{\mathbb{T}}(L_a)) (\Delta_J - p_J^a c_1^{\mathbb{T}}(L_a)) (\Delta_K - p_K^a c_1^{\mathbb{T}}(L_a)), \quad (4.7)$$

where we have expressed the resulting four-form as a function of the variables Δ_I and the equivariant parameter ϵ . This can also be understood as allowing the trial 2d R-symmetry to mix with the $U(1)$ symmetry of the spindle, formally setting $c_1(\mathcal{J}) = \epsilon c_1(F_R^{2d})$ (thus “undoing” the replacement (4.2)), and extracting the coefficient of the four-form $\mathcal{A}_{2d} = F(\Delta_I, \epsilon) c_1(F_R^{2d})^2$. Expanding the integrand we obtain²⁰

$$F(\Delta_I, \epsilon) = \frac{1}{6} \sum_{I,J,K} c_{IJK} p_I^a \left[-3\Delta_J \Delta_K D_a + 3p_J^b \Delta_K D_{ab} - p_J^b p_K^c D_{abc} \right], \quad (4.8)$$

with the equivariant intersection numbers

$$D_a = \int_{\Sigma} c_1^{\mathbb{T}}(L_a), \quad D_{ab} = \int_{\Sigma} c_1^{\mathbb{T}}(L_a) c_1^{\mathbb{T}}(L_b), \quad D_{abc} = \int_{\Sigma} c_1^{\mathbb{T}}(L_a) c_1^{\mathbb{T}}(L_b) c_1^{\mathbb{T}}(L_c), \quad (4.9)$$

whose nonzero values are given in (3.17). Alternatively, the same result can be obtained by evaluating the equivariant integral (4.7) using the fixed point theorem (2.37), explaining the observation made in [35]. Specifically, we get

$$F(\Delta_I, \epsilon) = F^1(\Delta_I, \epsilon) + F^2(\Delta_I, \epsilon), \quad (4.10)$$

where, in terms of the trial 4d central charge

$$a_{4d}(\Delta_I) = \frac{9}{32} \sum_{I,J,K} c_{IJK} \Delta_I \Delta_J \Delta_K, \quad (4.11)$$

²⁰ We use that c_{IJK} are totally symmetric to write (4.8).

we have

$$F^a(\Delta_I, \epsilon) = (-1)^a \frac{16}{27\epsilon} a_{4d}(\Delta_I^a), \quad \Delta_I^a \equiv \Delta_I - (-1)^a p_I^a \frac{\epsilon}{n_a}. \quad (4.12)$$

Notice that the terms singular in ϵ (cubic in the Δ_I) cancels out, as expected.

The expression above depends on the p_I^a , but it can be rewritten in terms of “physical fluxes” n_I , defined as the integrals of the flavour line bundles [77]

$$c_1(E_I) \equiv -p_I^a c_1(L_a), \quad (4.13)$$

namely

$$n_I \equiv - \int_{\Sigma} c_1(E_I) = \frac{p_I^1}{n_1} + \frac{p_I^2}{n_2}. \quad (4.14)$$

Recall that in general supersymmetry can be preserved on the spindle by coupling with a background R-symmetry gauge field with first Chern class

$$c_1(E_R) = -\sigma^1 c_1(L_1) - \sigma^2 c_1(L_2), \quad (4.15)$$

where $\sigma^a = \pm 1$, corresponding to either twist or anti-twist [41]. Since the supercharge should couple to $2c_1(F_R^{2d}) + c_1(E_R)$ we must have

$$\sum_I \left(\Delta_I c_1(F_R^{2d}) + c_1(E_I) \right) = 2c_1(F_R^{2d}) + c_1(E_R), \quad (4.16)$$

implying the constraints

$$\sum_I \Delta_I = 2, \quad \sum_I p_I^a = \sigma^a. \quad (4.17)$$

As a consequence, the physical fluxes obey

$$\sum_I n_I = \frac{\sigma^1}{n_1} + \frac{\sigma^2}{n_2} \quad (4.18)$$

and we can introduce new variables

$$\varphi_I \equiv \Delta_I + \frac{1}{2} \left(\frac{p_I^1}{n_1} - \frac{p_I^2}{n_2} \right) \epsilon, \quad \sum_I \varphi_I - \frac{1}{2} \left(\frac{\sigma^1}{n_1} - \frac{\sigma^2}{n_2} \right) \epsilon = 2. \quad (4.19)$$

In terms of these variables, the gravitational blocks depend only on the physical fluxes, namely

$$F^a(\varphi_I, \epsilon) = (-1)^a \frac{16}{27\epsilon} a_{4d} \left(\varphi_I - (-1)^a \frac{n_I}{2} \epsilon \right), \quad (4.20)$$

yielding

$$F(\varphi_I, \epsilon) = -\frac{1}{24} \sum_{I,J,K} c_{IJK} n_I \left[2\varphi_J \varphi_K + n_J n_K \epsilon^2 \right], \tag{4.21}$$

in agreement with [40].

Notice that (4.21) is much simpler than the analogous expression in eq. (5.9) of [35] and is therefore the most convenient form to be used in the extremization. The reason is that in the variables φ_I the expression is manifestly independent of the redundant parameters r_0^I used in the construction in [35], as we now explain. In this reference, one starts from the background gauge fields $A_I = \rho_I(y) d\phi$ inherited from the supergravity solution, where y, ϕ are coordinates on the spindle. This is lifted to Z_6 by the gauging procedure $d\varphi \mapsto d\varphi + A_{\mathcal{J}}$, which leads to the connection one-forms $\mathcal{A}_I = \rho_I(y)(d\phi + A_{\mathcal{J}})$, where $dA_{\mathcal{J}} = 2\pi c_1(\mathcal{J})$, so that

$$d\mathcal{A}_I = \rho'_I(y)(d\phi + A_{\mathcal{J}}) + 2\pi \rho_I(y) c_1(\mathcal{J}). \tag{4.22}$$

Comparing with (4.5) leads us to identify $\rho_I(y)$ precisely with our moment maps, namely

$$\rho_I(y) = -2\pi \mathfrak{p}_I^a \mu_a. \tag{4.23}$$

In our notation,²¹ the functions $\rho_I(y)$ satisfy [35]

$$\rho_I(y_1) = \frac{1}{2} n_I - \frac{1}{4} r_0^I \left(\frac{1}{n_1} + \frac{1}{n_2} \right), \quad \rho_I(y_2) = -\frac{1}{2} n_I - \frac{1}{4} r_0^I \left(\frac{1}{n_1} + \frac{1}{n_2} \right), \tag{4.24}$$

where $\sum_I r_0^I = 2$, while using (3.12) we have

$$2\pi \mathfrak{p}_I^a \mu_a|_{y_b} = (-1)^b \frac{\mathfrak{p}_I^b}{n_b}, \tag{4.25}$$

implying that we must identify

$$\frac{r_0^I}{2} \left(\frac{1}{n_1} + \frac{1}{n_2} \right) = -\frac{\mathfrak{p}_I^1}{n_1} + \frac{\mathfrak{p}_I^2}{n_2}. \tag{4.26}$$

Summing over I , we then get

$$-\frac{\sigma^1}{n_1} + \frac{\sigma^2}{n_2} = \sum_I \left(-\frac{\mathfrak{p}_I^1}{n_1} + \frac{\mathfrak{p}_I^2}{n_2} \right) = \frac{1}{n_1} + \frac{1}{n_2}, \tag{4.27}$$

²¹ We identify their \mathfrak{p}^i with our $-n_I$.

consistently with $\sigma^1 = -1, \sigma^2 = 1$, corresponding to the anti-twist [35]. This shows that the constants r_0^I parameterize the redundancy in the relation between the physical fluxes n_I and the p_I^a . However, notice that (4.26) implies that $r_0^I \in \mathbb{Q}$.

4.2 M5 branes on the spindle

The anomaly polynomial of M5 brane 6-dimensional SCFTs is an eight-form defined on an eight-dimensional orbifold Z_8 that we take to be the total space of a Σ fibration over an auxiliary space B_6 ,

$$\Sigma \hookrightarrow Z_8 \rightarrow B_6. \tag{4.28}$$

Neglecting terms that are sub-leading in the large N limit, the eight-form anomaly polynomial is given by:

$$\mathcal{A}_{6d} = \frac{N^3}{24} p_2(R) = \frac{N^3}{24} c_1(F_1)^2 c_1(F_2)^2, \tag{4.29}$$

where $p_2(R)$ is the second Pontryagin class of the $SO(5)_R$ normal bundle to the M5 brane in the eleven-dimensional spacetime. The $F_I, I = 1, 2$ are the generators of $U(1)_1 \times U(1)_2 \subset SO(5)_R$ global symmetries that are preserved when the M5 brane is compactified on the spindle [37]. The $c_1(F_I)$ are the first Chern classes of the line bundles on Z_8 associated with these $U(1)_I$ symmetries and are decomposed as

$$c_1(F_I) = \Delta_I c_1(F_R^{4d}) - p_I^a c_1(\mathcal{L}_a). \tag{4.30}$$

The anomaly polynomial of the four-dimensional theory is obtained integrating the anomaly polynomial the six-dimensional theory on the spindle,

$$\mathcal{A}_{4d} = \int_{\Sigma} \mathcal{A}_{6d}, \tag{4.31}$$

and is a six-form on B_6 . We now substitute (4.30) in (4.29), where $c_1(\mathcal{L}_a)$ is as in Eq. (4.5) and $c_1(F_R^{4d}) \in H^2(B_6, \mathbb{Z})$ is the first Chern class of the 4d R-symmetry line bundle. Formally setting $c_1(\mathcal{J}) = \epsilon c_1(F_R^{4d})$ and extracting the coefficient of the six-form $\mathcal{A}_{4d} = F(\Delta_I, \epsilon) c_1(F_R^{4d})^3$ lead to the equivariant integral

$$F(\Delta_I, \epsilon) = \frac{N^3}{24} \int_{\Sigma} (\Delta_1 - p_1^a c_1^{\mathbb{T}}(L_a))^2 (\Delta_2 - p_2^a c_1^{\mathbb{T}}(L_a))^2, \tag{4.32}$$

which may be expanded as

$$\begin{aligned} F(\Delta_I, \epsilon) = \frac{N^3}{24} [& -2\Delta_1 \Delta_2 (\Delta_1 p_2^a + \Delta_2 p_1^a) D_a + (\Delta_1^2 p_2^a p_2^b + \Delta_2^2 p_1^a p_1^b + 4\Delta_1 \Delta_2 p_1^a p_2^b) D_{ab} \\ & - 2p_1^a (\Delta_1 p_2^b + \Delta_2 p_1^b) p_2^c D_{abc} + p_1^a p_1^b p_2^c p_2^d D_{abcd}], \end{aligned} \tag{4.33}$$

where the nonzero equivariant intersection numbers are given in (3.17). Alternatively, the same result can be obtained evaluating the equivariant integral (4.32) using the fixed point theorem (2.37). Specifically, we get

$$F(\Delta_I, \epsilon) = F^1(\Delta_I, \epsilon) + F^2(\Delta_I, \epsilon), \tag{4.34}$$

where, in terms of the “trial 6d central charge”²²

$$a_{6d}(\Delta_I) \equiv \frac{N^3}{24} \Delta_1^2 \Delta_2^2, \tag{4.35}$$

we have

$$F^a(\Delta_I, \epsilon) = (-1)^a \frac{1}{\epsilon} a_{6d}(\Delta_I^a), \quad \Delta_I^a \equiv \Delta_I - (-1)^a p_I^a \frac{\epsilon}{n_a}. \tag{4.36}$$

Notice that the terms singular in ϵ (quartic in the Δ_I) cancel out, as expected. The rest of the discussion proceeds exactly as for the D3 branes in the previous subsection. In particular, supersymmetry implies that the constraints (4.17) hold and the physical fluxes n_I obey (4.18). In terms of the variables φ_I (4.19), the fixed point contributions read:

$$F^a(\varphi_I, \epsilon) = (-1)^a \frac{1}{\epsilon} a_{6d}(\varphi_I - (-1)^a \frac{n_I}{2} \epsilon), \tag{4.37}$$

which give the very simple expression

$$F(\varphi_I, \epsilon) = -\frac{N^3}{48} (\varphi_1 n_2 + \varphi_2 n_1) (4\varphi_1 \varphi_2 + n_1 n_2 \epsilon^2), \tag{4.38}$$

in agreement with [40].

Let us briefly compare the results above with the corresponding calculation in [37]. Again, the functions $\rho_I(y)$ used in [37] should be identified with our moment maps as $\rho_I(y) = -2\pi p_I^a \mu_a$, and the resulting p_I^a are easily obtained from the expressions in eq. (A.2) of [37]. However, due to the supergravity coordinates and the specific gauge used in [37], these are quadratic irrational functions of the spindle parameters and the physical fluxes and do not depend on any free parameter. One can check that the constraint (4.17) on the p_I^a is respected, with $\sigma^1 = -1$ and $\sigma^2 = -1$, consistently with the fact that the discussion in [37] concerns the twist case.²³ Notice that the form of (4.38) is much simpler than the corresponding function computed [37]. In particular, in (4.38) there are no linear and cubic terms in ϵ .

²² There is no notion of trial central charge in a 6d SCFT, and however, we adopt this conventional definition, in analogy with the 4d case.

²³ We identify their P_i with our $-n_I$.

4.3 M5 branes on 4d orbifolds

We now consider M5 branes compactified on a four-dimensional toric orbifold²⁴ as discussed in Sect. 3.2. The eight-form anomaly polynomial (4.29) is defined on an eight-dimensional orbifold Z_8 that we take to be the total space of a \mathbb{M}_4 fibration over an auxiliary space B_4 ,

$$\mathbb{M}_4 \hookrightarrow Z_8 \rightarrow B_4, \tag{4.39}$$

and integrating it on \mathbb{M}_4 gives a four-form, that is the anomaly polynomial of a 2d SCFT:

$$\mathcal{A}_{2d} = \int_{\mathbb{M}_4} \mathcal{A}_{6d}. \tag{4.40}$$

The $c_1(F_I)$ now decompose as

$$c_1(F_I) = \Delta_I c_1(F_R^{2d}) - \mathfrak{p}_I^a c_1(\mathcal{L}_a), \quad c_1(\mathcal{L}_a) = c_1(L_a) + 2\pi \mu_a^i c_1(\mathcal{J}_i), \tag{4.41}$$

where $\mathcal{J}_1, \mathcal{J}_2$ are the auxiliary line bundles with $c_1(\mathcal{J}_i) \in H^2(Z_4, \mathbb{Z})$. The $L_a, a = 1, \dots, d$ are the line bundles discussed in Sect. 3.2 and $c_1(F_R^{2d}) \in H^2(Z_4, \mathbb{Z})$ is the first Chern class of the 2d R-symmetry line bundle.

Substituting (4.41) in (4.29) and setting $c_1(\mathcal{J}_i) = \epsilon_i c_1(F_R^{2d})$ lead to the equivariant integral

$$F(\Delta_I, \epsilon_i) = \frac{N^3}{24} \int_{\mathbb{M}_4} (\Delta_1 - \mathfrak{p}_1^a c_1^{\mathbb{T}}(L_a))^2 (\Delta_2 - \mathfrak{p}_2^a c_1^{\mathbb{T}}(L_a))^2, \tag{4.42}$$

which has exactly the same form of (4.32)! Indeed, expanding it, this gives again (4.33), where now $D_a = 0$ and the nonzero equivariant intersection numbers can be read off from (3.29), (3.31), and (3.33), respectively. On the other hand, employing the fixed point theorem, we can write (4.42) as a sum of contributions over the fixed points, namely²⁵

$$F(\Delta_I, \epsilon_i) = \frac{N^3}{24} \sum_{a=1}^n \frac{1}{d_{a,a+1} \epsilon_1^a \epsilon_2^a} \left(\Delta_1 - \mathfrak{p}_1^a \epsilon_1^a - \mathfrak{p}_1^{a+1} \epsilon_2^a \right)^2 \left(\Delta_2 - \mathfrak{p}_2^a \epsilon_1^a - \mathfrak{p}_2^{a+1} \epsilon_2^a \right)^2, \tag{4.43}$$

in agreement with the formula conjectured in [51]. In terms of the gravitational blocks, we have

$$F(\Delta_I, \epsilon_i) = \sum_{a=1}^n F^a(\Delta_I, \epsilon_i), \quad F^a(\Delta_I, \epsilon_i) = \frac{1}{d_{a,a+1} \epsilon_1^a \epsilon_2^a} a_{6d}(\Delta_I^a), \tag{4.44}$$

²⁴ The M5 brane anomaly polynomial in an example of this type of compactification was studied in [50].

²⁵ Notice that there is no summation on a in the expressions inside the parenthesis.

with

$$\Delta_I^a = \Delta_I - \mathfrak{p}_I^a \epsilon_1^a - \mathfrak{p}_I^{a+1} \epsilon_2^a. \tag{4.45}$$

The physical fluxes are now defined by integrating the first Chern classes of the flavour line bundles, $c_1(E_I) = -\mathfrak{p}_I^a c_1(L_a)$, over the various divisors, namely

$$\mathfrak{q}_I^a \equiv - \int_{D_a} c_1(E_I) = \mathfrak{p}_I^b D_{ab}. \tag{4.46}$$

However, recall that this relation cannot be inverted and therefore it is not manifest that (4.43) depends only on the physical fluxes. That this is true was proved in [51], as we now recall, translating the arguments to our current notation.²⁶ From (3.21), it immediately follows that

$$\sum_{a=1}^d v^a D_{ab} = 0 \quad \Rightarrow \quad \sum_{a=1}^d v^a \mathfrak{q}_I^a = 0, \tag{4.47}$$

therefore, only $d - 2$ physical fluxes are linearly independent. In particular, the ‘‘gauge’’ transformation

$$\mathfrak{p}_I^a \rightarrow \tilde{\mathfrak{p}}_I^a = \mathfrak{p}_I^a + \det(\lambda_I, v^a), \tag{4.48}$$

where $\lambda_I \in \mathbb{R}^2$ are arbitrary two-dimensional constant vectors,²⁷ leaves the physical fluxes \mathfrak{q}_I^a invariant. In order to discuss how this transformation affects (4.43), we will assume,²⁸ following [51], that supersymmetry requires the background R-symmetry gauge field to have first Chern class

$$c_1(E_R) = - \sum_{a=1}^d \sigma^a c_1(L_a), \tag{4.49}$$

where $\sigma^a = \pm 1$, generalizing the twist and anti-twist for the spindle [41]. This corresponds to the following constraints

$$\Delta_1 + \Delta_2 = 2 + \det(W, \epsilon), \quad \mathfrak{p}_1^a + \mathfrak{p}_2^a = \sigma^a + \det(W, v^a), \tag{4.50}$$

²⁶ Recall that in this paper we denote the non-primitive, ‘‘long’’, vectors of the fan by v^a , the corresponding divisors as D_a , and their intersection matrix as D_{ab} . While in [51] these were denoted by \hat{v}^a , \hat{D}_a , and $\hat{D}_{a,b}$, respectively. Similarly, $(\mathfrak{p}_I^a)_{\text{here}} = (m_a \mathfrak{p}_I^a)_{\text{there}}$ and $(\mathfrak{q}_I^a)_{\text{here}} = (\mathfrak{q}_I^a / m_a)_{\text{there}}$.

²⁷ Notice that these λ_I have nothing to do with the Kähler parameters λ_a appearing elsewhere.

²⁸ This holds in all known examples of supergravity solution corresponding to M5 branes wrapped on four-dimensional toric orbifolds [50–53]. It should be possible to prove this, along the lines of the analysis in [41].

where $\epsilon = (\epsilon_1, \epsilon_2)$ and $W \in \mathbb{R}^2$ is a two-dimensional constant vector transforming as

$$W \rightarrow \tilde{W} = W + \lambda_1 + \lambda_2 \tag{4.51}$$

under (4.48). Performing this transformation in (4.43) implies that the variables Δ_I^a change as

$$\Delta_I^a \rightarrow \Delta_I - \tilde{p}_I^a \epsilon_1^a - \tilde{p}_I^{a+1} \epsilon_2^a = \Delta_I - p_I^a \epsilon_1^a - p_I^{a+1} \epsilon_2^a - \det(\lambda_I, \epsilon), \tag{4.52}$$

thus

$$F(\Delta_I, \epsilon_i) \rightarrow F(\tilde{\Delta}_I, \epsilon_i), \quad \tilde{\Delta}_I \equiv \Delta_I - \det(\lambda_I, \epsilon), \tag{4.53}$$

where $\tilde{\Delta}_I$ obey the same constraint as the Δ_I , namely

$$\tilde{\Delta}_1 + \tilde{\Delta}_2 = 2 + \det(\tilde{W}, \epsilon) - \det(\lambda_1, \epsilon) - \det(\lambda_2, \epsilon) = 2 + \det(W, \epsilon). \tag{4.54}$$

This is exactly the same function as the initial one, completing the proof.

5 Non-compact Calabi–Yau singularities

The geometry of many type II and M theory solutions arising from branes can be modelled on singular Calabi–Yau cones. In this section, we consider the equivariant volume of partial resolutions of non-compact (asymptotically conical) Calabi–Yau singularities and provide many applications to holography.

Consider a non-compact toric Calabi–Yau X of complex dimension m defined by a fan with primitive vectors $v^a, a = 1, \dots, d$. The Calabi–Yau condition requires the vectors v^a to lie on a plane. We choose an $SL(m; \mathbb{Z})$ basis where the first component of all the vectors v^a is one

$$v^a = (1, w^a) \quad w^a \in \mathbb{Z}^{m-1}. \tag{5.1}$$

Notice that (2.54) implies that the GLSM charges Q_a^A are traceless

$$\sum_{a=1}^d Q_a^A = 0. \tag{5.2}$$

We assume that X is the (partial) resolution of a Calabi–Yau conical singularity, as in some of the examples in Sect. 3.3. This means that the convex rational polyhedron

$$\mathcal{P} = \{l_a(y) = v_i^a y_i - \lambda_a \geq 0 \quad a = 1, \dots, d\}, \tag{5.3}$$

is asymptotically a cone. More precisely, we assume that the large y_i approximation of the polyhedron

$$\mathcal{P}' = \{l_{a_k}(y) = v_i^{a_k} y_i \geq 0 \quad k = 1, \dots, d'\}, \tag{5.4}$$

is a non-compact cone with a single vertex in $y = 0$ and d' facets. Notice that only the non-compact facets of \mathcal{P} are relevant for the asymptotic behaviour and, in general $d' \leq d$. The vectors v^{a_k} , $k = 1, \dots, d'$, define the fan of the singular cone X_{sing} of which X is a resolution. The fan of X_{sing} consists of a single cone, and it corresponds to a singularity which is, in general, not of orbifold type.

The resolution replaces the singularity of \mathcal{P}' with compact cycles of real dimension $2m - 2$, corresponding to the bounded facets of \mathcal{P} , as well as sub-cycles of smaller dimensions.²⁹ We assume that \mathcal{P} has only orbifold singularities. From the dual point of view, the fan of X is the union of m -dimensional cones, each corresponding to a vertex of \mathcal{P} and therefore to a fixed point. Each cone is specified by a choice of m adjacent vectors $(v^{a_1}, \dots, v^{a_m})$. We use the notation $A = (a_1, \dots, a_m)$ to identify the set of such cones and we assume that there are n of them.

Using the results of Sect. 2.3, the equivariant volume is given by

$$\mathbb{V}(\lambda_a, \epsilon_i) = \sum_{A=(a_1, \dots, a_m)} \frac{e^{-\epsilon \cdot (\lambda_{a_1} u_A^{a_1} + \dots + \lambda_{a_m} u_A^{a_m}) / d_{a_1, \dots, a_m}}}{d_{a_1, \dots, a_m} \left(\frac{\epsilon \cdot u_A^{a_1}}{d_{a_1, \dots, a_m}} \right) \dots \left(\frac{\epsilon \cdot u_A^{a_m}}{d_{a_1, \dots, a_m}} \right)}, \tag{5.5}$$

where A runs over the m -dimensional cones of the resolution, $d_{a_1, \dots, a_m} = |\det(v^{a_1}, \dots, v^{a_m})|$, and u_A^a are the inward normal to the facets of A defined in Sect. 2.3.

5.1 GK geometry and the GMS master volume

In this section, we explore some general properties of the equivariant volume for non-compact Calabi–Yau singularities and its relation with other *volumes* appearing in the literature in similar contexts, like the Sasakian volume of [17, 18] and the master volume introduced in [66].

Consider the formal expansion

$$\mathbb{V}(\lambda_a, \epsilon_i) = \sum_{k=0}^{\infty} \mathbb{V}^{(k)}(\lambda_a, \epsilon_i), \tag{5.6}$$

where $\mathbb{V}^{(k)}(\lambda_a, \epsilon_i)$ is homogeneous of degree k in λ_a . We start by observing some general properties of the homogeneous quantities $\mathbb{V}^{(k)}(\lambda_a, \epsilon_i)$. From (2.35), by matching degrees in λ_a , we have

$$\sum_{a=1}^d v_i^a \frac{\partial \mathbb{V}^{(k)}}{\partial \lambda_a} = -\epsilon_i \mathbb{V}^{(k-1)}. \tag{5.7}$$

²⁹ The example of the conifold discussed in Sect. 3.3.3 is special in that there are no compact four-cycles. This is due to the fact we just subdivide the fan without introducing new vectors.

By considering the case $i = 1$ and the fact that, for Calabi–Yaus, $v_1^a = 1$, we also have

$$\sum_{a=1}^d \frac{\partial \mathbb{V}^{(k)}}{\partial \lambda_a} = -\epsilon_1 \mathbb{V}^{(k-1)}. \tag{5.8}$$

We can combine (5.7) and (5.8) into

$$\sum_{a=1}^d \frac{\partial \mathbb{V}^{(k)}}{\partial \lambda_a} (\epsilon_1 v_i^a - \epsilon_i) = 0. \tag{5.9}$$

It follows from this equation that $\mathbb{V}^{(k)}$ are invariant under the gauge transformation

$$\lambda_a \rightarrow \lambda_a + \sum_{i=1}^m \gamma_i (\epsilon_1 v_i^a - \epsilon_i), \tag{5.10}$$

which allows to eliminate $m - 1$ ³⁰ unphysical λ_a that do not correspond to non-trivial co-homology classes.

In the compact case, the terms with $k < m$ in the expansion (5.6) are identically zero. Indeed they can be written as in (2.31) and they are integrals of forms of degree less than $2m$. As we already discussed in Sect. 2.5, in the non-compact case this is not true and also the terms with $k < m$ are not vanishing. They are rational functions of ϵ encoding some interesting geometrical information that we will now elucidate.

It is important to observe that $\mathbb{V}^{(k)}(\lambda_a, \epsilon_i)$ with $k < m$ only depends on the d' Kähler parameters λ_a associated with the singular fan and the non-compact directions. The Kähler parameter λ_a associated with the compact directions starts contributing at order m in the expansion. This can be understood as follows. The compact Kähler parameters are associated with the bounded facets of the original polytope. We can always modify the polytope by adding new facets and making it compact. We can also assume that the bounded facets of the original polytope are not modified by this operation. The compact Kähler parameter enters in the fixed point formula only in the terms associated with the vertices of the bounded facets and, therefore, their contribution to \mathbb{V} for the old and new polytope is the same. Since the new polytope is compact, this contribution must start at order m .

The term of zero degree is simply

$$\mathbb{V}(0, \epsilon_i) = \frac{1}{(2\pi)^m} \int_X e^{-H} \frac{\omega^m}{m!} = \int_{\mathcal{P}'} e^{-\epsilon_i y_i} dy_1 \dots dy_m \tag{5.11}$$

and it computes the equivariant volume of the singular Calabi–Yau cone X_{sing} , or equivalently, the regularized volume of the polyhedron \mathcal{P}' . At $\lambda_a = 0$, the metric of X_{sing} becomes conical

$$ds^2(X) = dr^2 + r^2 ds^2(Y). \tag{5.12}$$

³⁰ $i = 1$ is trivial.

A choice of Kähler metric exhibits X_{sing} as a cone over a Sasakian manifold Y of real dimension $2m - 1$. As shows in [17, 18] there is family of Sasakian metrics parameterized by the Reeb vector $\xi = \epsilon_i \partial\phi_i$. They correspond to different choices of the radial coordinate $\frac{r^2}{2} = H$, where $H = \epsilon_i y_i$ is the Hamiltonian for ξ . Expression (5.11) then reduces, up to a numerical factor, to the Sasakian volume of Y [17, 18]

$$\text{vol}[Y](\epsilon_i) = \frac{2\pi^m}{(m - 1)!} \mathbb{V}(0, \epsilon_i). \tag{5.13}$$

The specialization of (5.5) to $\lambda_a = 0$ in the smooth case

$$\mathbb{V}(0, \epsilon_i) = \sum_{A=(a_1, \dots, a_m)} \frac{1}{d_{a_1, \dots, a_m} \left(\frac{\epsilon \cdot u_A^{a_1}}{d_{a_1, \dots, a_m}} \right) \dots \left(\frac{\epsilon \cdot u_A^{a_m}}{d_{a_1, \dots, a_m}} \right)}, \tag{5.14}$$

was derived indeed in [18].

The term of degree $m - 1$ is also interesting. It coincides, up to a numerical factor, with the master volume introduced in [66]. To define the master volume, one foliates the $2m - 1$ base Y as

$$ds^2(Y) = \eta^2 + ds_{2m-2}^2, \tag{5.15}$$

where η is the dual one form to ξ ($i_\xi \eta = 1$) and the metric ds_{2m-2}^2 is conformally Kähler with Kähler form ω_B . We turn on the d' Kähler parameters λ_a associated with the non-compact facets of the polyhedron by letting ω_B vary [66]

$$\frac{[\omega_B]}{2\pi} = - \sum_a \lambda_a c_a, \tag{5.16}$$

where c_a are the co-homology classes that uplift to $c_1(L_a)$ on the Calabi–Yau X_{sing} . Notice that $[d\eta] = 2\pi \sum_a c_a$ [66]. The master volume is then defined as

$$\begin{aligned} \mathcal{V}(\lambda_a, \epsilon_i) &= \int_Y \eta \wedge \frac{\omega_B^{m-1}}{(m - 1)!} = \frac{(-2\pi)^{m-1}}{(m - 1)!} \\ &\sum_{a_1, \dots, a_{m-1}} \lambda_{a_1} \dots \lambda_{a_{m-1}} \int_Y \eta \wedge c_{a_1} \wedge \dots \wedge c_{a_{m-1}}. \end{aligned} \tag{5.17}$$

For example, for $m = 3$, the master volume reads [66]

$$\begin{aligned} \mathcal{V}(\lambda_a, \epsilon_i) &= \frac{(2\pi)^3}{2!} \\ &\sum_{a=1}^{d'} \frac{\lambda_a (\lambda_{a-1} \det(v^a, v^{a+1}, \epsilon) - \lambda_a \det(v^{a-1}, v^{a+1}, \epsilon) + \lambda_{a+1} \det(v^{a-1}, v^a, \epsilon))}{\det(v^{a-1}, v^a, \epsilon) \det(v^a, v^{a+1}, \epsilon)} \end{aligned} \tag{5.18}$$

where the vectors v^a with $a = 1, \dots, d'$ run over the fan of the singular cone X_{sing} .

Notice that, in the original approach of [66], the master volume is defined by considering conical non-Kähler metrics on X . In our approach instead, we give up the conical condition on X and we use a Kähler metric. Despite the difference of approaches, we can recover the master volume as the term of degree $m - 1$ in the equivariant volume

$$\mathcal{V}(\lambda_a, \epsilon_i) = (2\pi)^m \mathbb{V}^{(m-1)}(\lambda_a, \epsilon_i), \tag{5.19}$$

where we stress that

$$\mathbb{V}^{(m-1)} = \frac{1}{(m-1)!} \sum_{a_1, \dots, a_{m-1}=1}^{d'} \frac{\partial^{m-1} \mathbb{V}}{\partial \lambda_{a_1} \dots \partial \lambda_{a_{m-1}}} \Big|_{\lambda=0} \lambda_{a_1} \dots \lambda_{a_{m-1}}, \tag{5.20}$$

is a function only of the Kähler parameters λ_a , $a = 1, \dots, d'$, associated with the fan of the singular cone. The identity (5.19) can be checked by direct computation. The case $m = 3$ is explicitly worked out in section B. By differentiating $m - 1$ times equation (2.35) for $i = 1$

$$\sum_{a_1=1}^d \frac{\partial \mathbb{V}}{\partial \lambda_{a_1}} = -\epsilon_1 \mathbb{V}, \tag{5.21}$$

setting $\lambda_a = 0$ and multiplying by $\lambda_{a_2} \dots \lambda_{a_m}$, we can also rewrite (5.19) as

$$\mathcal{V}(\lambda_a, \epsilon_i) = -\frac{(2\pi)^m}{\epsilon_1(m-1)!} \sum_{a_1, \dots, a_m=1}^{d'} \frac{\partial^m \mathbb{V}}{\partial \lambda_{a_1} \dots \partial \lambda_{a_m}} \Big|_{\lambda=0} \lambda_{a_1} \dots \lambda_{a_{m-1}}, \tag{5.22}$$

where the analogy with (5.17) is manifest.

As observed in [66], the master volume reduces to the Sasakian volume when all the Kähler parameters are equal. We can understand this statement in our formalism as follows

$$\begin{aligned} \mathbb{V}^{(0)}(\epsilon_i) &= \frac{(-1)^{m-1}}{\epsilon_1^{m-1}} \sum_{a_1, \dots, a_{m-1}=1}^{d'} \frac{\partial^n \mathbb{V}}{\partial \lambda_{a_1} \dots \partial \lambda_{a_{m-1}}} \Big|_{\lambda=0} \\ &= \frac{(m-1)!}{(-\epsilon_1)^{m-1} (2\pi)^m} \mathcal{V}(\lambda_a = 1, \epsilon_i), \end{aligned} \tag{5.23}$$

where we used (2.36) with all $i_k = 1$ and $\lambda_a = 0$ and Eq. (5.19).

The master volume plays an important role in supergravity solutions based on GK geometry [25, 66]. In this context, the so-called *supersymmetric action*

$$S_{\text{SUSY}} = -\sum_{a=1}^d \frac{\partial \mathcal{V}}{\partial \lambda_a}, \tag{5.24}$$

plays an even more important role. It is the object that needs to be extremized in order to find a solution of the equation of motions. Using (5.8) we find

$$S_{\text{SUSY}} = \epsilon_1 (2\pi)^m \mathbb{V}^{(m-2)}(\lambda_a, \epsilon_i). \tag{5.25}$$

5.2 Gravitational blocks from the equivariant volume

In this section, we use the equivariant volume to study type II and M theory branes compactified on a spindle. Supersymmetry is preserved either with a topological twist or an antitwist [41]. From the field theory point of view, we consider the case where a superconformal field theory compactified on a spindle flows in the IR to a superconformal quantum mechanics or to a two-dimensional superconformal field theory. The corresponding supergravity solutions describing the IR limit have a geometry $\text{AdS}_2 \times Z$ or $\text{AdS}_3 \times Z$, respectively, and can be interpreted as the near-horizon geometry of black holes or black strings.

The local geometry of such brane systems can be modelled in terms of CY m -folds. We can describe many D-branes and M-branes configurations in terms of a formal fibration

$$\text{CY}_m \hookrightarrow \text{CY}_{m+1} \rightarrow \Sigma, \tag{5.26}$$

where the CY_m encode the geometry and the information about the original higher-dimensional CFT.

It is often useful to define an off-shell free energy $F(\Delta_I, \epsilon)$, or *extremal function*,³¹ depending on chemical potentials Δ_I for the continuous global symmetries of the higher-dimensional CFT and an equivariant parameter ϵ for the rotation along the spindle, whose extremization gives the entropy of the black hole or the central charge of the two-dimensional CFT. All these extremal functions should arise by evaluating the supergravity action on supersymmetric solutions that obey a subset of the equations of motion. The extremization with respect to Δ_I and ϵ is then equivalent to imposing the remaining equations of motion. Explicit examples of this construction are given in [17, 66]. Extremal functions of known black holes and black strings can be expressed in terms of gravitational blocks [54]. The general form of the extremal functions for branes compactified on a spindle in this context was proposed in [40], covering also cases where the explicit computation of the supergravity action from first principles is still missing. The characteristic form of the off-shell free energy F is given by a gluing

$$F(\Delta_I, \epsilon) = \frac{1}{\epsilon} (\mathcal{F}(\Delta_I^+) \pm \mathcal{F}(\Delta_I^-)) \tag{5.27}$$

where the block \mathcal{F} encodes some universal properties of the higher-dimensional SCFT and it is related to the geometry of CY_m , and the gluing depends on the details of the

³¹ Often also called entropy functions, even if the related physical observable is not the entropy of a black hole.

fibration (5.26). For D3 and M5 branes, \mathcal{F} is related to the central charge of the higher-dimensional CFT. For other types of branes, \mathcal{F} is the sphere-free energy of the higher-dimensional CFT at large N [40, 54].

In this section, we focus on the gravitational interpretation of (5.27). In the case of M2 and D3 branes, the off-shell free energy can be expressed in terms of the supersymmetric action in the formalism of GK geometry and the decomposition (5.27) was explicitly proved in [55] from the gravitational point of view. We will show how to recover and reinterpret this results in terms of the equivariant volume. In the case of D2, D4 and M5 brane systems, we cannot apply the GK formalism but we will propose a possible and intriguing extension.

5.2.1 The geometry of CY_m fibred over the spindle

As a preliminary, in this section we derive a general expression for the equivariant volume of the fibration (5.26). Consider a CY cone m -fold defined by m -dimensional vectors v^a , with $v_1^a = 1$, and the $(m + 1)$ -dimensional toric geometry specified by the fan [55]

$$V^a = (0, v^a), \quad V^+ = (n_+, w_+), \quad V^- = (-\sigma n_-, w_-), \tag{5.28}$$

where $\sigma = \pm 1$. This is a fibration over a spindle $\mathbb{W}\mathbb{P}^1_{[n_+, n_-]}$ specified by the vectors w_{\pm} . Notice that the $(m + 1)$ -dimensional geometry is still a Calabi–Yau if the first component of the vectors w_{\pm} is one, $w_{\pm 1} = 1$, which we will assume. As shown in [55], this geometry explicitly appears in the gravity solutions of M2 and D3-branes compactified on a spindle with

$$w_+ = (1, -a_+ \vec{p}), \quad w_- = (1, -\sigma a_- \vec{p}), \tag{5.29}$$

where $a_- n_+ + a_+ n_- = 1$ and $\sigma = \pm 1$ and \vec{p} is a $(m - 1)$ -dimensional vector. In this context, supersymmetry is preserved with a twist ($\sigma = 1$) or an anti-twist ($\sigma = -1$). Notice also that, in the anti-twist case, the toric diagram is not convex and it does not strictly define a toric geometry. We will nevertheless proceed also in this case, considering it as an extrapolation from the twist case.

The fixed point formula is

$$\mathbb{V}_{CY_{m+1}}(\lambda_a, \epsilon_i) = \sum_{(a_1, \dots, a_{m+1}) \in A} \frac{e^{-\epsilon_{(m+1)} \cdot (\sum_{i=1}^{m+1} \lambda_{a_i} U^{a_i}) / d_{a_1, \dots, a_{m+1}}}}{d_{a_1, \dots, a_{m+1}} \prod_{i=1}^{m+1} \left(\frac{\epsilon_{(m+1)} \cdot U^{a_i}}{d_{a_1, \dots, a_{m+1}}} \right)}, \tag{5.30}$$

where A runs over the polyhedral cones of a resolution and $\epsilon_{(m+1)} = (\epsilon_0, \epsilon_1, \dots, \epsilon_m)$. For ease of notation, we drop the label A from the normal vectors U^a . We can choose a resolution for the CY_m by subdividing the m -dimensional fan. We then obtain a resolution of the CY_{m+1} by considering polyhedra where V^+ and V^- are added to the

m -dimensional cones $(V^{a_1}, \dots, V^{a_m})$. Let's assume also $d_{a_1, \dots, a_m} = 1$ for the CY_m . The inward normals to the tetrahedra are

$$\begin{aligned} (V^{a_1}, \dots, V^{a_m}, V^+) &\rightarrow U^{a_i} = (-u^{a_i} \cdot w_+, n_+ u^{a_i}), & U^+ &= (1, 0, \dots, 0), \\ (V^{a_1}, \dots, V^{a_m}, V^-) &\rightarrow U^{a_i} = -(-u^{a_i} \cdot w_-, -\sigma n_- u^{a_i}), & U^- &= (-1, 0, \dots, 0), \end{aligned} \tag{5.31}$$

where u^{a_i} are the inward normals to the CY_m cones, we identified $a_{m+1} = \pm$ and $\sigma = -1$ is obtained by analytic continuation.

The contribution of the tetrahedra with vertex V^+ is

$$\begin{aligned} \sum_{(a_1, \dots, a_3) \in A} &\frac{e^{-(\epsilon_{(m)} - \epsilon_0 w_+ / n_+) \cdot (\sum_{i=1}^m \lambda_{a_i} u^{a_i}) - \epsilon_0 \lambda_+ / n_+}}{\epsilon_0 \prod_{i=1}^m \left(\epsilon_{(m)} - \frac{\epsilon_0 w_+}{n_+} \right) \cdot u^{a_i}} \\ &= \frac{1}{\epsilon_0} \mathbb{V}_{CY_m} \left(\lambda_a + \frac{\epsilon_0}{n_+ \epsilon_1 - \epsilon_0} \lambda_+, \epsilon_{(m)} - \frac{\epsilon_0 w_+}{n_+} \right), \end{aligned} \tag{5.32}$$

where $\epsilon_{(m)} = (\epsilon_1, \dots, \epsilon_m)$, and we used $d_{a_1, \dots, a_{m+1}} = n_+$ and the identity among normals $\sum_i u^{a_i} = (1, 0, \dots, 0)$.³²

The contribution of the tetrahedra of V^- is obtained by replacing w_+ with w_- , n_+ with $-\sigma n_-$ and $\lambda_+ \equiv \lambda_a$ with a new variable λ_- . The signs in U^{a_i} are compensated by $d_{a_1, a_2, a_3, a_4} = -(-\sigma n_-)$ (we work for positive σ and analytically continue the result), which also brings an overall extra sign.

The final result is

$$\mathbb{V}_{CY_{m+1}} = \frac{1}{\epsilon_0} \mathbb{V}_{CY_m} (\lambda_a^+, \epsilon_i^+) - \frac{1}{\epsilon_0} \mathbb{V}_{CY_m} (\lambda_a^-, \epsilon_i^-) \tag{5.33}$$

where

$$\begin{aligned} \epsilon^+ &= \epsilon_{(m)} - \frac{\epsilon_0 w_+}{n_+}, & \lambda_a^+ &= \lambda_a + \frac{\epsilon_0}{n_+ \epsilon_1 - \epsilon_0} \lambda_+, \\ \epsilon^- &= \epsilon_{(m)} + \frac{\epsilon_0 w_-}{\sigma n_-}, & \lambda_a^- &= \lambda_a - \frac{\epsilon_0}{\sigma n_- \epsilon_1 + \epsilon_0} \lambda_-, \end{aligned} \tag{5.34}$$

and we see that $\mathbb{V}_{CY_{m+1}}$ can be obtained by gluing two copies of \mathbb{V}_{CY_m} .

³² Consider for simplicity $m = 3$. Assuming an order such that $\det(v^1, v^2, v^3) = 1$, $\sum_i u^{a_i} = v^2 \wedge v^3 + v^3 \wedge v^1 + v^1 \wedge v^2$. Let e_i be the canonical basis in \mathbb{R}^3 . Then $\sum_i u^{a_i} \cdot e_{2,3} = 0$. For example, $\sum_i u^{a_i} \cdot e_2 = \det(e_2, v^2, v^3) + \det(e_2, v^3, v^1) + \det(e_2, v^1, v^2) = v_3^3 - v_2^3 + v_1^3 - v_3^3 + v_2^3 - v_1^3 = 0$ where we used $v_1^1 = 1$. On the other hand, $\sum_i u^{a_i} \cdot e_1 = \det(e_1, v^2, v^3) + \det(e_1, v^3, v^1) + \det(e_1, v^1, v^2) = 1$ since it is the sum of the areas of three triangular cones obtained by triangulating (v^1, v^2, v^3) with the insertion of e_1 (it lies in the same plane). The sum of the three areas is the area of the original cone $d_{1,2,3} = 1$.

5.2.2 The case of D3 branes

The case of D3 and M2 branes can be described in terms of GK geometry [66]. We consider a system of branes sitting at the tip of a conical toric Calabi–Yau threefold singularity (CY_{*m*}) with Sasaki–Einstein base Y_{2m-1} and further compactified on a spindle. The dual supergravity solution has an AdS₂ × Z₉ and AdS₃ × Z₇ near-horizon geometry, for M2 and D3 branes, respectively. The internal manifolds Z_{2*m*+1} are obtained by fibering the Sasaki–Einstein base Y_{2m-1} over the spindle. Supersymmetry requires that the cone $C(Z_{2m+1})$ is topologically a CY_{*m*+1}, although the supergravity metric is not Ricci-flat.

The supergravity solution can be described in terms of a family of backgrounds that depends on the equivariant parameters ϵ_i and the Kähler parameters λ_α of the CY_{*m*+1} [66]. To avoid confusion, we will use Greek letters to label the vectors in the $m + 1$ -dimensional fan and the associated Kähler parameters. The conditions for supersymmetry can be compactly written as:

$$\sum_{\alpha} \frac{\partial S_{\text{SUSY}}}{\partial \lambda_{\alpha}} = 0, \quad v_m M_{\alpha} = -\frac{\partial S_{\text{SUSY}}}{\partial \lambda_{\alpha}}, \tag{5.35}$$

where S_{SUSY} is the supersymmetric action (5.24) of the CY_{*m*+1}

$$S_{\text{SUSY}} = \epsilon_1 (2\pi)^{m+1} \mathbb{V}_{CY_{m+1}}^{(m-1)}(\lambda_{\alpha}, \epsilon_i), \tag{5.36}$$

and M_{α} are integer fluxes, encoding the flux quantization conditions of the M-theory four-form or the type IIB RR five-form. v_m is a normalization constant that depends on the dimension. The fluxes M_{α} contain the information about the number of branes N and the topological details of the Sasaki–Einstein fibration over the spindle. Notice that for consistency of (5.35) with (5.9), we must have

$$\sum_{\alpha} V^{\alpha} M_{\alpha} = 0, \tag{5.37}$$

where V^{α} are the vectors in the fan of the CY_{*m*+1}.³³ Given (5.9) and (5.37), only $d - m + 1$ equations in (5.35) are actually independent. Combining Eq. (5.35) with the $m - 1$ gauge invariances (5.10), we can eliminate all the λ_{α} . We are left with a functional of the equivariant parameters ϵ_i that needs to be extremized in order to find the solution of the equations of motion.

Let us now specialize to the case of D3-branes where $m = 3$. The solution AdS₃ × Z₇ is dual to a two-dimensional CFT. In this context, the Killing vector $\xi = \sum_{i=1}^m \epsilon_i \partial_{\phi_i}$ is interpreted as the R-symmetry of the dual CFT and the extremization of the supersymmetric action is the geometrical dual of c -extremization [78] in two-dimensional CFTs [25, 66].

³³ In our examples, the index α will split into an index a associated with the CY_{*m*} fan and the two indices \pm associated with the spindle.

More explicitly, the on-shell value of the supersymmetric action, up to a normalization coefficient, is the exact central charge of the two-dimensional CFT, while the off-shell value of S_{SUSY} as a function of ϵ_i after imposing (5.35) equals the charge c as a function of a trial R-symmetry. This has been proved explicitly in [79] for the case of a compactification on S^2 , or on a Riemann surface, and in [55] for a spindle. More precisely, S_{SUSY} as a function of ϵ_i coincides with the trial c -function after the baryonic directions in the trial R-symmetry have been extremized.³⁴ It is also shown in [55] how to write the supersymmetric action as the gluing of two gravitational blocks [54]. We now recover this result in our formalism.

We can extract the supersymmetric action from the gluing formula (5.33)

$$\mathbb{V}_{CY_4} = \frac{1}{\epsilon_0} \mathbb{V}_{CY_3}(\lambda_a^+, \epsilon_i^+) - \frac{1}{\epsilon_0} \mathbb{V}_{CY_3}(\lambda_a^-, \epsilon_i^-), \tag{5.38}$$

where

$$\begin{aligned} \epsilon^+ &= \epsilon_{(3)} - \frac{\epsilon_0 w_+}{n_+}, & \lambda_a^+ &= \lambda_a + \frac{\epsilon_0}{n_+ \epsilon_1 - \epsilon_0} \lambda_+, \\ \epsilon^- &= \epsilon_{(3)} + \frac{\epsilon_0 w_-}{\sigma n_-}, & \lambda_a^- &= \lambda_a - \frac{\epsilon_0}{\sigma n_- \epsilon_1 + \epsilon_0} \lambda_-. \end{aligned} \tag{5.39}$$

Notice that the redefinitions are homogeneous in λ_α . This means that the previous identity can be easily truncated at a given order in λ_α . Taking the quadratic piece and using (5.19) and (5.25), we obtain

$$S_{\text{SUSY}}|_{CY_4} = 2\pi \frac{\epsilon_1}{\epsilon_0} (\mathcal{V}_{CY_3}(\lambda_a^+, \epsilon_i^+) - \mathcal{V}_{CY_3}(\lambda_a^-, \epsilon_i^-)), \tag{5.40}$$

since the supersymmetric action is the quadratic part of \mathbb{V}_{CY_4} and the 3d master volume is the quadratic part of \mathbb{V}_{CY_3} . We thus recover the factorization in gravitational blocks derived in [55] (see for example (7.27) and (7.31) in that paper).

We can be more explicit in the special case of S^5 and the dual $\mathcal{N} = 4$ SYM. This example is discussed in [55] in a different gauge. The vectors are

$$v^1 = (1, 0, 0), \quad v^2 = (1, 1, 0), \quad v^3 = (1, 0, 1), \tag{5.41}$$

and the equivariant and master volume read

$$\begin{aligned} \mathbb{V}_{\mathbb{C}^3} &= \frac{e^{-(\epsilon_1 - \epsilon_2 - \epsilon_3)\lambda_1 - \epsilon_2\lambda_2 - \epsilon_3\lambda_3}}{(\epsilon_1 - \epsilon_2 - \epsilon_3)\epsilon_2\epsilon_3}, \\ \mathcal{V}_{\mathbb{C}^3} &= \frac{(2\pi)^3 ((\epsilon_1 - \epsilon_2 - \epsilon_3)\lambda_1 + \epsilon_2\lambda_2 + \epsilon_3\lambda_3)^2}{2(\epsilon_1 - \epsilon_2 - \epsilon_3)\epsilon_2\epsilon_3}. \end{aligned} \tag{5.42}$$

³⁴ This is similar to what happens for the Sasakian volume and the trial a -charge for D3 branes sitting at the tip of a conical toric Calabi–Yau three-fold singularity [17, 80].

We parameterize the fluxes M_α as

$$(M_1, M_2, M_3, M_+, M_-) \equiv N(n_1, n_2, n_3, \frac{1}{n_+}, \frac{1}{\sigma n_-}), \tag{5.43}$$

where the constraint (5.37) further requires

$$\sum_{I=1}^3 n_I = -\frac{1}{n_+} - \frac{1}{\sigma n_-}, \quad n_2 = -\frac{w_{+2}}{n_+} - \frac{w_{-2}}{\sigma n_-}, \quad n_3 = -\frac{w_{+3}}{n_+} - \frac{w_{-3}}{\sigma n_-}. \tag{5.44}$$

We will be cavalier about the normalization of the fluxes in the following, but it is clear than N should be proportional to the number of colours of the dual theory. We can use the gauge freedom (5.10) to set $\lambda_1 = \lambda_2 = \lambda_3 = 0$. The master volume evaluated at the two poles in this gauge reads

$$\mathcal{V}_{\mathbb{C}^3}(\lambda^\pm, \epsilon_i^\pm) = \frac{(2\pi)^3 \epsilon_0^2 \lambda_\pm^2}{2n_\pm^2 (\epsilon_1^\pm - \epsilon_2^\pm - \epsilon_3^\pm) \epsilon_2^\pm \epsilon_3^\pm}, \tag{5.45}$$

and we can find λ_\pm by solving the $\alpha = \pm$ components of the second equation in (5.35) (recall that only two of them are independent)

$$\begin{aligned} -v_3 N &= \frac{(2\pi)^4 \epsilon_1 \epsilon_0 \lambda_+}{n_+ (\epsilon_1^+ - \epsilon_2^+ - \epsilon_3^+) \epsilon_2^+ \epsilon_3^+}, \\ -v_3 \sigma N &= -\frac{(2\pi)^4 \epsilon_1 \epsilon_0 \lambda_-}{n_- (\epsilon_1^- - \epsilon_2^- - \epsilon_3^-) \epsilon_2^- \epsilon_3^-}. \end{aligned} \tag{5.46}$$

Defining

$$\epsilon_0 = \epsilon, \quad \epsilon_1 = \Delta_1 + \Delta_2 + \Delta_3, \quad \epsilon_2 = \Delta_2, \quad \epsilon_3 = \Delta_3, \tag{5.47}$$

the extremization in [66] must be done under the condition $\epsilon_1 = 2/(m-3) = 2$ which corresponds to

$$\Delta_1 + \Delta_2 + \Delta_3 = 2. \tag{5.48}$$

We then see that Δ_I , with $I = 1, 2, 3$, parameterize the R-charges of the three chiral fields of $\mathcal{N} = 4$ SYM. The supersymmetric action (5.40) is then obtained by gluing blocks

$$S_{\text{SUSY}|CY_4} = \frac{\mathcal{F}(\Delta_1^+, \Delta_2^+, \Delta_3^+)}{\epsilon} - \frac{\mathcal{F}(\Delta_1^-, \Delta_2^-, \Delta_3^-)}{\epsilon}, \tag{5.49}$$

with the appropriate function for $\mathcal{N} = 4$ SYM [54]

$$\mathcal{F}(\Delta_1, \Delta_2, \Delta_3) = \frac{v_3^2}{64\pi^4} N^2 \Delta_1 \Delta_2 \Delta_3, \tag{5.50}$$

and

$$\begin{aligned} \Delta_1^+ &= \Delta_1 - \frac{\epsilon}{n_+} \left(1 - \sum_{I=2}^3 w_{+I} \right), & \Delta_2^+ &= \Delta_2 - \frac{\epsilon}{n_+} w_{+2}, & \Delta_3^+ &= \Delta_3 - \frac{\epsilon}{n_+} w_{+3}, \\ \Delta_1^- &= \Delta_1 + \frac{\epsilon}{\sigma n_-} \left(1 - \sum_{I=2}^3 w_{-I} \right), & \Delta_2^- &= \Delta_2 + \frac{\epsilon}{\sigma n_-} w_{-2}, & \Delta_3^- &= \Delta_3 + \frac{\epsilon}{\sigma n_-} w_{-3}. \end{aligned} \tag{5.51}$$

We can see that this result reproduces the anomaly polynomial computed in Sect. 4. In doing so, we need to be careful that there is some ambiguity in identifying the chemical potentials associated with the R-symmetry in (5.47). Any redefinition $\Delta_I \rightarrow \Delta_I + \delta_I \epsilon$ with $\sum_{I=1}^3 \delta_I = 0$ would respect the constraint (5.48) and would lead to a potentially good choice of R-symmetry chemical potential. To avoid this ambiguity, we can compare the objects that are invariant under this redefinition

$$\sum_{I=1}^3 \Delta_I^+ = 2 - \frac{\epsilon}{n_+}, \quad \sum_{I=1}^3 \Delta_I^- = 2 + \frac{\epsilon}{\sigma n_-}, \quad \Delta_I^+ - \Delta_I^- = \epsilon n_I. \tag{5.52}$$

It is then easy to see that the same relations hold for the anomaly as written in (4.10) with $\sigma^1 = -1$ and $\sigma^2 = -\sigma$.

Notice that the supersymmetric action scales like N^2 as expected for $\mathcal{N} = 4$ SYM. This can be understood from the fact that $S_{\text{SUSY}|CY_4}$ is quadratic in λ_α and, from (5.35), λ_α scales linearly with N .

5.2.3 The case of M2 branes

We consider now the case of M2 sitting at the tip of a conical toric CY_4 with Sasaki–Einstein base Y_7 and further compactified on a spindle. The dual M theory solutions correspond to 4d black holes and have an $AdS_2 \times Z_9$ near-horizon geometry. We can again describe the system in terms of GK geometry. The supersymmetric action provides an entropy functional for the black hole. The construction is the gravitational dual of \mathcal{I} -extremization [81] and has been applied to the case of a compactification on S^2 in [82–84]³⁵ and in the case of a compactification on the spindle in [55, 85].

From the gluing formula (5.33), we find

$$\mathbb{V}_{CY_5} = \frac{1}{\epsilon_0} \mathbb{V}_{CY_4} (\lambda_a^+, \epsilon_i^+) - \frac{1}{\epsilon_0} \mathbb{V}_{CY_4} (\lambda_a^-, \epsilon_i^-), \tag{5.53}$$

³⁵ Or, more generally, in the case of a compactification on a Riemann surface.

where

$$\begin{aligned} \epsilon^+ &= \epsilon_{(4)} - \frac{\epsilon_0 w_+}{n_+}, & \lambda_a^+ &= \lambda_a + \frac{\epsilon_0}{n_+ \epsilon_1 - \epsilon_0} \lambda_+, \\ \epsilon^- &= \epsilon_{(4)} + \frac{\epsilon_0 w_-}{\sigma n_-}, & \lambda_a^- &= \lambda_a - \frac{\epsilon_0}{\sigma n_- \epsilon_1 + \epsilon_0} \lambda_-. \end{aligned} \tag{5.54}$$

Taking the piece cubic in λ , we recover the result in [55]

$$S_{\text{SUSY}|CY_5} = 2\pi \frac{\epsilon_1}{\epsilon_0} (\mathcal{V}_{CY_4}(\lambda_a^+, \epsilon_i^+) - \mathcal{V}_{CY_4}(\lambda_a^-, \epsilon_i^-)). \tag{5.55}$$

To simplify the discussion, we restrict to the case $Z_7 = S^7$ which is dual to the ABJM theory with $k = 1$ compactified on the spindle. The vectors are

$$v^1 = (1, 0, 0, 0), \quad v^2 = (1, 1, 0, 0), \quad v^3 = (1, 0, 1, 0), \quad v^4 = (1, 0, 0, 1), \tag{5.56}$$

and the equivariant volume is

$$V_{\mathbb{C}^4} = \frac{e^{-(\epsilon_1 - \epsilon_2 - \epsilon_3 - \epsilon_4)\lambda_1 - \epsilon_2 \lambda_2 - \epsilon_3 \lambda_3 - \epsilon_4 \lambda_4}}{(\epsilon_1 - \epsilon_2 - \epsilon_3 - \epsilon_4)\epsilon_2 \epsilon_3 \epsilon_4}. \tag{5.57}$$

We parameterize the fluxes M_α as

$$(M_1, M_2, M_3, M_4, M_+, M_-) \equiv N \left(n_1, n_2, n_3, n_4, \frac{1}{n_+}, \frac{1}{\sigma n_-} \right), \tag{5.58}$$

where the constraint (5.37) further requires

$$\sum_{I=1}^4 n_I = -\frac{1}{n_+} - \frac{1}{\sigma n_-}, \quad n_I = -\frac{w_{+I}}{n_+} - \frac{w_{-I}}{\sigma n_-}, \quad I = 2, 3, 4, \tag{5.59}$$

and N will be related to the number of colours of the dual theory. We use again the gauge freedom (5.10) to set $\lambda_1 = \lambda_2 = \lambda_3 = \lambda_4 = 0$ and we determine λ_\pm solving (5.35). This time, the master volume evaluated at the two poles in this gauge reads

$$\begin{aligned} \mathcal{V}_{\mathbb{C}^4}(\lambda^+, \epsilon_i^+) &= \frac{(2\pi)^4 \epsilon_0^3 (-\lambda_+)^3}{6n_+^3 (\epsilon_1^+ - \epsilon_2^+ - \epsilon_3^+ - \epsilon_4^+) \epsilon_2^+ \epsilon_3^+ \epsilon_4^+}, \\ \mathcal{V}_{\mathbb{C}^4}(\lambda^-, \epsilon_i^-) &= \frac{(2\pi)^4 \epsilon_0^3 (\lambda_-)^3}{6(\sigma n_-)^3 (\epsilon_1^- - \epsilon_2^- - \epsilon_3^- - \epsilon_4^-) \epsilon_2^- \epsilon_3^- \epsilon_4^-}, \end{aligned} \tag{5.60}$$

and we can find λ_\pm by solving the $\alpha = \pm$ components of Eq. (5.35) (recall that only two of them are independent)

$$\begin{aligned} v_4 N &= \frac{(2\pi)^5 \epsilon_1 \epsilon_0^2 \lambda_+^2}{2n_+^2 (\epsilon_1^+ - \epsilon_2^+ - \epsilon_3^+ - \epsilon_4^+) \epsilon_2^+ \epsilon_3^+ \epsilon_4^+}, \\ v_4 N &= \frac{(2\pi)^5 \epsilon_1 \epsilon_0^2 \lambda_-^2}{2n_-^2 (\epsilon_1^- - \epsilon_2^- - \epsilon_3^- - \epsilon_4^-) \epsilon_2^- \epsilon_3^- \epsilon_4^-}. \end{aligned} \tag{5.61}$$

There is a sign ambiguity in solving (5.61) that we fix to match the gravitational entropy function result and the explicit geometric analysis in [55].³⁶ Define

$$\epsilon_0 = \frac{\epsilon}{2}, \quad \epsilon_1 = \frac{\Delta_1 + \Delta_2 + \Delta_3 + \Delta_4}{2}, \quad \epsilon_2 = \frac{\Delta_2}{2}, \quad \epsilon_3 = \frac{\Delta_3}{2}, \quad \epsilon_4 = \frac{\Delta_4}{2}. \tag{5.62}$$

The extremization in [66] must be done under the condition $\epsilon_1 = 2/(m - 3) = 1$, which corresponds to

$$\Delta_1 + \Delta_2 + \Delta_3 + \Delta_4 = 2. \tag{5.63}$$

We then see that Δ_I , with $I = 1, 2, 3, 4$, parameterize the R-charges of the four chiral fields of ABJM. The supersymmetric action (5.40) is then given by gluing blocks:

$$S_{\text{SUSY}}|_{CY_5} = \frac{\mathcal{F}(\Delta_1^+, \Delta_2^+, \Delta_3^+, \Delta_4^+)}{\epsilon} - \sigma \frac{\mathcal{F}(\Delta_1^-, \Delta_2^-, \Delta_3^-, \Delta_4^-)}{\epsilon}, \tag{5.64}$$

with the appropriate function for the ABJM theory [54]

$$\mathcal{F}(\Delta_1, \Delta_2, \Delta_3, \Delta_4) = \frac{v_4^{3/2}}{24\pi^{5/2}} N^{3/2} \sqrt{\Delta_1 \Delta_2 \Delta_3 \Delta_4}, \tag{5.65}$$

and

$$\begin{aligned} \Delta_1^+ &= \Delta_1 - \frac{\epsilon}{n_+} \left(1 - \sum_{I=2}^4 w_{+I} \right), & \Delta_I^+ &= \Delta_I - \frac{\epsilon}{n_+} w_{+I}, \quad I = 2, 3, 4, \\ \Delta_1^- &= \Delta_1 + \frac{\epsilon}{\sigma n_-} \left(1 - \sum_{I=2}^4 w_{+I} \right), & \Delta_I^- &= \Delta_I + \frac{\epsilon}{\sigma n_-} w_{-I}, \quad I = 2, 3, 4. \end{aligned} \tag{5.66}$$

Notice that

$$\sum_{I=1}^4 \Delta_I^+ = 2 - \frac{\epsilon}{n_+}, \quad \sum_{I=1}^4 \Delta_I^- = 2 + \frac{\epsilon}{\sigma n_-}, \quad \Delta_I^+ - \Delta_I^- = \epsilon n_I, \quad I = 1, 2, 3, 4. \tag{5.67}$$

³⁶ We see from (5.61) that $(\epsilon_1^\pm - \epsilon_2^\pm - \epsilon_3^\pm - \epsilon_4^\pm) \epsilon_2^\pm \epsilon_3^\pm \epsilon_4^\pm$ should be positive. We then choose the solution $\lambda_+ < 0, \lambda_- > 0$ which is consistent with [55]—for example, one can compare with (7.16) and (7.26) in that reference.

This formula matches the entropy function found in [40, 55].³⁷

Notice that the entropy function scales like $N^{3/2}$ as expected for the ABJM theory. The reason is that $S_{\text{SUSY}}|_{CY_3}$ is cubic in λ_α and, from (5.35), λ_α scales as \sqrt{N} . Notice also that the gluing (5.64) involves a different relative sign for the twist and anti-twist. This is typical of M2 and D4 branes and it is not present for D3 and M5 branes [40].

5.2.4 The case of M5 branes

The case of M5 branes compactified on a spindle has no known description in terms of GK geometry. However, a stack of M5 branes in flat space in M theory probes a transverse geometry that is $\mathbb{C}^2 \times \mathbb{R}$, suggesting that the relevant CY_2 is simply \mathbb{C}^2 . In the associated supergravity solution $AdS_7 \times S^4$, the base S^3 of the cone \mathbb{C}^2 is fibred over a real direction to give the S^4 . Notice that there is a supersymmetric generalization where the M5 branes probe a $\mathbb{C}^2/\mathbb{Z}_p \times \mathbb{R}$ and the CY_2 could be replaced by $\mathbb{C}^2/\mathbb{Z}_p$.

We now consider the CY_3 consisting of \mathbb{C}^2 fibred over the spindle with fan

$$V^1 = (0, 1, 0), \quad V^2 = (0, 1, 1), \quad V^+ = (n_+, w_+), \quad V^- = (-\sigma n_-, w_-), \tag{5.68}$$

with

$$w_+ = (1, p_+), \quad w_- = (1, p_-). \tag{5.69}$$

From the gluing formula (5.33), we have

$$V_{CY_3} = \frac{1}{\epsilon_0} V_{\mathbb{C}^2}(\lambda_a^+, \epsilon_i^+) - \frac{1}{\epsilon_0} V_{\mathbb{C}^2}(\lambda_a^-, \epsilon_i^-), \tag{5.70}$$

where

$$\begin{aligned} \epsilon^+ &= \epsilon_{(2)} - \frac{\epsilon_0 w_+}{n_+}, & \lambda_a^+ &= \lambda_a + \frac{\epsilon_0}{n_+ \epsilon_1 - \epsilon_0} \lambda_+, \\ \epsilon^- &= \epsilon_{(2)} + \frac{\epsilon_0 w_-}{\sigma n_-}, & \lambda_a^- &= \lambda_a - \frac{\epsilon_0}{\sigma n_- \epsilon_1 + \epsilon_0} \lambda_-, \end{aligned} \tag{5.71}$$

and the equivariant volume of \mathbb{C}^2 can be read from (3.85) with $p = q = 1$

$$V_{\mathbb{C}^2} = \frac{e^{-\lambda_1(\epsilon_1 - \epsilon_2) - \lambda_2 \epsilon_2}}{\epsilon_2(\epsilon_1 - \epsilon_2)}. \tag{5.72}$$

In analogy with S_{SUSY} in the context of GK geometry, we expect to be able to extract an off-shell free energy F , for the spindly M5 branes AdS_5 solutions found in [37] from the equivariant volume of \mathbb{C}^2 fibred over the spindle. We now show that this

³⁷ To compare with (5.22) in [40], one needs to identify $n_I^{\text{here}} = -n_i^{\text{there}}$, $n_{\pm}^{\text{here}} = n_{\mp}^{\text{there}}$ and $2\epsilon^{\text{there}} = -\epsilon$. The chemical potentials are related by the redefinition $\varphi_I = \Delta_I - \frac{\epsilon}{n_+} w_{+I} - \frac{\epsilon}{2} n_I$ for $I = 2, 3, 4$ and $\varphi_1 = \Delta_1 - \frac{\epsilon}{n_+} (1 - w_{+2} - w_{+3} - w_{+4}) - \frac{\epsilon}{2} n_1$.

is the case by an appropriate generalization of Eq. (5.35). A scaling argument for N suggests that the correct generalization is

$$v_{M5} M_\alpha = -\frac{\partial \mathbb{V}_{CY_3}^{(2)}}{\partial \lambda_\alpha}, \quad F = \mathbb{V}_{CY_3}^{(3)}, \quad \sum_\alpha M_\alpha V^\alpha = 0, \quad (5.73)$$

where the flux equation uses the quadratic piece, but the extremal function is the cubic piece of the equivariant volume. In this way, λ_α is linear in N and the extremal function F cubic in N , as expected for M5 branes. Here and in the following subsections we will not be concerned with precise normalizations.

Following the same logic as in previous sections, we solve the constraint on the fluxes

$$M_\alpha = N \left(n_1, n_2, \frac{1}{n_+}, \frac{1}{\sigma n_-} \right), \quad (5.74)$$

with $n_1 + n_2 = -\frac{1}{n_+} - \frac{1}{\sigma n_-}$ and define

$$\epsilon_0 = \frac{\epsilon}{2}, \quad \epsilon_1 = \frac{\Delta_1 + \Delta_2}{2}, \quad \epsilon_2 = \frac{\Delta_2}{2}, \quad (5.75)$$

with $\Delta_1 + \Delta_2 = 2$. The chemical potentials Δ_1 and Δ_2 can be associated with the Cartan subgroup of the $SO(5)$ R-symmetry of the $(2, 0)$ theory. We find that, after solving for λ_\pm in the gauge $\lambda_1 = \lambda_2 = 0$, the extremal function is given by gluing two blocks

$$F = \frac{\mathcal{F}(\Delta_1^+, \Delta_2^+)}{\epsilon} - \frac{\mathcal{F}(\Delta_1^-, \Delta_2^-)}{\epsilon}, \quad (5.76)$$

with the appropriate function for the $(2, 0)$ theory [54]

$$\mathcal{F}(\Delta_1, \Delta_2) = \frac{v_{M5}^3}{48} N^3 (\Delta_1 \Delta_2)^2, \quad (5.77)$$

and

$$\begin{aligned} \Delta_1^+ &= \Delta_1 - \frac{\epsilon}{n_+} (1 - p_+), & \Delta_2^+ &= \Delta_2 - \frac{\epsilon}{n_+} p_+, \\ \Delta_1^- &= \Delta_1 + \frac{\epsilon}{\sigma n_-} (1 - p_-), & \Delta_2^- &= \Delta_2 + \frac{\epsilon}{\sigma n_-} p_-. \end{aligned} \quad (5.78)$$

Notice that

$$\sum_{l=1}^2 \Delta_l^+ = 2 - \frac{\epsilon}{n_+}, \quad \sum_{l=1}^2 \Delta_l^- = 2 + \frac{\epsilon}{\sigma n_-}, \quad \Delta_l^+ - \Delta_l^- = \epsilon n_l. \quad (5.79)$$

We see that we have reproduced the $(2, 0)$ anomaly integrated on the spindle (4.34) with $\sigma^1 = -1$ and $\sigma^2 = -\sigma$.

5.2.5 The case of D4 branes

We now turn to the case of D4 branes compactified on a spindle. The massive type IIA supergravity solution associated with a system of D4 and D8 branes is topologically AdS_6 times an hemisphere S^4 [86], suggesting that the relevant CY_2 is again \mathbb{C}^2 . Supergravity solutions with an AdS_4 factor corresponding to compactifications of the dual 5d SCFT on a spindle have been found in [40, 42].

It is intriguing to observe that we can again extract an off-shell free energy for these spindly solutions from a generalization of Eq. (5.35) valid for GK geometry. We consider as before the equivariant volume (5.70) of \mathbb{C}^2 fibred over the spindle. This time we take

$$v_{D4}M_\alpha = -\frac{\partial V_{CY_3}^{(3)}}{\partial \lambda_\alpha}, \quad F = V_{CY_3}^{(5)}, \quad \sum_\alpha M_\alpha V^\alpha = 0, \quad (5.80)$$

so that $\lambda_\alpha \sim N^{1/2}$ and $F \sim N^{5/2}$ as expected for D4 branes.

We use the same definitions (5.74) and (5.75) of the previous section. After solving for λ_\pm in the gauge $\lambda_1 = \lambda_2 = 0$, the extremal function is given by³⁸

$$F = \frac{\mathcal{F}(\Delta_1^+, \Delta_2^+)}{\epsilon} - \sigma \frac{\mathcal{F}(\Delta_1^-, \Delta_2^-)}{\epsilon}, \quad (5.81)$$

where for the 5d SCFT [54]

$$\mathcal{F}(\Delta_1, \Delta_2) = \frac{v_{D4}^{5/2}}{60\sqrt{2}} N^{5/2} (\Delta_1 \Delta_2)^{3/2}, \quad (5.82)$$

and

$$\begin{aligned} \Delta_1^+ &= \Delta_1 - \frac{\epsilon}{n_+}(1 - p_+), & \Delta_2^+ &= \Delta_2 - \frac{\epsilon}{n_+}p_+, \\ \Delta_1^- &= \Delta_1 + \frac{\epsilon}{\sigma n_-}(1 - p_-), & \Delta_2^- &= \Delta_2 + \frac{\epsilon}{\sigma n_-}p_-. \end{aligned} \quad (5.83)$$

Notice that

$$\sum_{l=1}^2 \Delta_l^+ = 2 - \frac{\epsilon}{n_+}, \quad \sum_{l=1}^2 \Delta_l^- = 2 + \frac{\epsilon}{\sigma n_-}, \quad \Delta_l^+ - \Delta_l^- = \epsilon n_l. \quad (5.84)$$

We have reproduced the entropy function in [40, 42].³⁹

³⁸ We made a choice of determination for the fractional power that is similar to the M2 brane case and correctly reproduce the supergravity result.

³⁹ To compare with (5.33) in [40], one needs to identify $n_l^{\text{here}} = -n_l^{\text{there}}$, $n_\pm^{\text{here}} = n_\mp^{\text{there}}$ and $2\epsilon^{\text{there}} = -\epsilon$. The chemical potentials are related by the redefinition $\varphi_1 = \Delta_1 - \frac{\epsilon}{n_+}(1 - p_+) - \frac{\epsilon}{2}n_1$ and $\varphi_2 = \Delta_2 - \frac{\epsilon}{n_+}p_+ - \frac{\epsilon}{2}n_2$.

5.2.6 The case of D2 branes

We finally consider the case of D2 branes compactified on a spindle. The massive type IIA supergravity solution associated with a system of D2-branes is topologically $\text{AdS}_4 \times X_6$, where the internal manifold is a Sasaki–Einstein five-manifold foliated over a segment [87, 88]. Supergravity solutions with an AdS_2 factor corresponding to compactifications of the dual 3d SCFT on a spindle have been found in [46].

We expect that the corresponding extremal function is related to the equivariant volume of a CY_3 fibred over the spindle. A scaling argument suggests to use:

$$v_{D2} M_\alpha = -\frac{\partial \mathbb{V}_{\text{CY}_4}^{(4)}}{\partial \lambda_\alpha}, \quad F = \mathbb{V}_{\text{CY}_4}^{(5)}, \quad \sum_\alpha M_\alpha V^\alpha = 0, \quad (5.85)$$

so that $\lambda_\alpha \sim N^{1/3}$ and $F \sim N^{5/3}$ as expected for D2 branes.

We can verify that this is the case for the solution associated with \mathbb{C}^3 where the three-dimensional SCFT is a pure $\mathcal{N} = 2$ Chern–Simons theory with the same matter content and superpotential of the maximal supersymmetric Yang–Mills theory [87] that we indicate as D2_k . We have

$$\mathbb{V}_{\text{CY}_4} = \frac{1}{\epsilon_0} \mathbb{V}_{\mathbb{C}^3}(\lambda_a^+, \epsilon_i^+) - \frac{1}{\epsilon_0} \mathbb{V}_{\mathbb{C}^3}(\lambda_a^-, \epsilon_i^-), \quad (5.86)$$

where $\mathbb{V}_{\mathbb{C}^3}$ is given in (5.42). The fluxes can be written as in (5.43). Since the matter content of the theory is the same as for $\mathcal{N} = 4$ SYM we can still use the parameterization

$$\epsilon_0 = \epsilon, \quad \epsilon_1 = \Delta_1 + \Delta_2 + \Delta_3, \quad \epsilon_2 = \Delta_2, \quad \epsilon_3 = \Delta_3, \quad (5.87)$$

with $\Delta_1 + \Delta_2 + \Delta_3 = 2$, and interpret the Δ_I , with $I = 1, 2, 3$, as the R-charges of the three chiral fields of the theory.

We can use the gauge freedom (5.10) to set $\lambda_1 = \lambda_2 = \lambda_3 = 0$, and we can find λ_\pm by solving the first equation in (5.85). The extremal function (5.85) is then obtained by gluing two blocks

$$F = \frac{\mathcal{F}(\Delta_1^+, \Delta_2^+, \Delta_3^+)}{\epsilon} - \frac{\mathcal{F}(\Delta_1^-, \Delta_2^-, \Delta_3^-)}{\epsilon}, \quad (5.88)$$

where for the D2_k theory

$$\mathcal{F}(\Delta_1, \Delta_2, \Delta_3) \propto N^{5/3} (\Delta_1 \Delta_2 \Delta_3)^{2/3}, \quad (5.89)$$

and

$$\Delta_1^+ = \Delta_1 - \frac{\epsilon}{n_+} \left(1 - \sum_{l=2}^3 w_{+l} \right), \quad \Delta_2^+ = \Delta_2 - \frac{\epsilon}{n_+} w_{+2}, \quad \Delta_3^+ = \Delta_3 - \frac{\epsilon}{n_+} w_{+3},$$

$$\Delta_1^- = \Delta_1 + \frac{\epsilon}{\sigma n_-} \left(1 - \sum_{l=2}^3 w_{-l} \right), \quad \Delta_2^- = \Delta_2 + \frac{\epsilon}{\sigma n_-} w_{-2}, \quad \Delta_3^- = \Delta_3 + \frac{\epsilon}{\sigma n_-} w_{-3}. \quad (5.90)$$

Notice that

$$\sum_{l=1}^3 \Delta_l^+ = 2 - \frac{\epsilon}{n_+}, \quad \sum_{l=1}^3 \Delta_l^- = 2 + \frac{\epsilon}{\sigma n_-}, \quad \Delta_l^+ - \Delta_l^- = \epsilon n_l. \quad (5.91)$$

This is the natural expectation for the extremal function for a D2 brane theory. It would be interesting to have a more complete analysis in supergravity to compare with.

6 Discussion

In this paper, we provided several applications of equivariant localization to quantum field theory and holography. In particular, we have shown that the equivariant volume, a basic and well-studied geometrical object in symplectic geometry, is at the heart of many constructions characterizing supersymmetric geometries. It generalizes at once the Sasakian volume of [17, 18] and the master volume introduced in [66] that have been proven very useful in the study of superconformal field theories dual to branes sitting at Calabi–Yau singularities, and generalizations thereof. The corresponding quantum (or K-theoretical) version, the equivariant index-character, is analogously crucial for studying the Hilbert series of the moduli spaces of the corresponding SCFTs and we expect that it still has many surprises to unveil.

More specifically, we proposed that the equivariant volume should be the key object for all the extremal problems characterizing supersymmetric geometries with a holographic interpretation, in different supergravity theories. We showed in this paper that all the extremization problems associated with compactifications on the spindle of M2 and D3 brane at Calabi–Yau singularities as well as M5, D4 and D2 brane configurations in flat space, can be re-expressed in terms of the equivariant volume. In the case of M2 and D3 branes, it was known before that the extremization problem can be expressed in terms of the master volume [66] and we showed how this object is encoded in the equivariant volume of the associated fibered Calabi–Yaus. In the M5, D4 and D2 branes there is not yet an analogous of the construction in [66], but we showed that the relevant extremal functions can be written in terms of the equivariant volume. We leave for future work the analysis of more complicated situations, like M5 and D4 branes compactified on a four-dimensional orbifolds \mathbb{M}_4 or D2 branes in massive type IIA associated with a generic Sasaki–Einstein five-manifold foliated over a segment [88]. These examples are more complicated but we are confident that the corresponding extremal functions and extremization problems can be written and formulated solely in terms of the equivariant volume.

In this paper, we discussed the factorization (5.27) of the extremal functions associated with spindly black objects in terms of gravitational blocks both from the gravitational and the field theory point of views. On the geometry side, in *all* cases

the factorization is an immediate corollary of the fixed point localization formula for the equivariant volume, applied to the relevant non-compact geometry—see Eq. (5.33). For D3 and M5 branes, F and \mathcal{F}_m are related to the central charge of the lower and higher-dimensional SCFT, respectively, and the gluing (5.27) is nothing else than equivariant localization applied to the computation of the higher-dimensional anomaly polynomial that we discussed in Sect. 4. For other types of branes, \mathcal{F}_m is the sphere-free energy of the higher-dimensional SCFT at large N [40, 54], and (5.27) is typically related to the large N factorization properties of SCFT partition functions of theories with a holographic dual [89–91]. In this context, the factorization becomes visible after taking the large N limit of the SCFT free energy, which is usually expressed in terms of a matrix model. The partition function of three-dimensional $\mathcal{N} = 2$ SQFTs compactified on a spindle times a circle was derived in [56], generalizing at once the superconformal index and the topologically twisted index. The large N limit of this spindle index is currently under investigation [92] and we expect to reproduce the factorization (5.27), thus closing the circle.

The derivation of partition functions of SQFTs compactified on orbifolds is another arena where the results of this paper could be useful. Indeed, quantum field theory localization relies on the use of the equivariant index theorem [93], and the building blocks of this construction will involve generalizations of the index-character discussed in Sect. 2.6 and Appendix C. More precisely, the index-character enters as a basic building block for partition functions on $\mathbb{M} \times S^1$ where supersymmetry is preserved with a topological twist on \mathbb{M} , while it needs to be generalized in the case of general σ^a as in (4.49). In particular, it would be interesting to generalize the five-dimensional indices of [90, 94, 95] to the case of a five-dimensional SCFT defined on $\mathbb{M}_4 \times S^1$ and apply the result to reproduce the entropy function for the supergravity solutions associated with D4 branes compactified on a four-dimensional orbifold [51], as well as recover the anomaly polynomial results for M5 branes compactified on a four-dimensional orbifold discussed in Sect. 4.3.⁴⁰

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⁴⁰ This can be done by considering the five-dimensional $\mathcal{N} = 2$ SYM theory that decompactifies to the $(2, 0)$ theory in the UV.

A Direct proof of the fixed point formula for \mathbb{M}_4

In this section, we derive the fixed point formula (3.35) for four-dimensional toric orbifolds. We use the notations introduced in Sect. 3.2. We start by writing the equivariant volume as an integral over the polytope, as in (2.32)

$$\mathbb{V}(\lambda_a, \epsilon_i) = \frac{1}{(2\pi)^2} \int_{\mathbb{M}_4} e^{-H} \frac{\omega^2}{2} = \int_{\mathcal{P}} e^{-y_i \epsilon_i} dy_1 dy_2, \tag{A.1}$$

which can be evaluated by elementary methods. Defining the one-form

$$v \equiv \frac{e^{-y_i \epsilon_i}}{\epsilon \cdot \epsilon} (\epsilon_2 dy_1 - \epsilon_1 dy_2) \tag{A.2}$$

and, using Stoke’s theorem, we have

$$\mathbb{V}(\lambda_a, \epsilon_i) = \int_{\mathcal{P}} dv = \int_{\partial \mathcal{P}} v = \sum_a \int_{\mathcal{F}_a} v, \tag{A.3}$$

where a facet \mathcal{F}_a of the polytope is defined by the linear equation

$$\mathcal{F}_a = \{l_a \equiv v_i^a y_i - \lambda_a = 0\}. \tag{A.4}$$

We can introduce a coordinate⁴¹ $s \in [s_a^{\min}, s_a^{\max}]$ on each facet \mathcal{F}_a , writing

$$y_i|_{\mathcal{F}_a} = \tilde{v}_i^a s + \frac{\lambda_a}{v^a \cdot v^a} v_i^a, \tag{A.5}$$

where $\tilde{v}_i^a \equiv \epsilon_{ij} v_j^a$. The extrema of the interval can be determined by intersecting \mathcal{F}_a with \mathcal{F}_{a-1} and \mathcal{F}_{a+1} , respectively, and read

$$\begin{aligned} s_a^{\min} &= \frac{1}{\langle v^{a-1}, v^a \rangle} \left(\lambda_{a-1} - \frac{v^{a-1} \cdot v^a}{v^a \cdot v^a} \lambda_a \right), \\ s_a^{\max} &= \frac{1}{\langle v^{a+1}, v^a \rangle} \left(\lambda_{a+1} - \frac{v^{a+1} \cdot v^a}{v^a \cdot v^a} \lambda_a \right), \end{aligned} \tag{A.6}$$

where $\langle v, w \rangle \equiv \det(v, w)$. Plugging (A.5) into v and integrating we have

$$\int_{\mathcal{F}_a} v = \frac{v^a \cdot \epsilon}{\epsilon \cdot \epsilon \langle v^a, \epsilon \rangle} \exp \left[-\frac{\epsilon \cdot v^a}{v^a \cdot v^a} \lambda_a \right] \left(\exp \left[\langle v^a, \epsilon \rangle s_a^{\max} \right] - \exp \left[\langle v^a, \epsilon \rangle s_a^{\min} \right] \right). \tag{A.7}$$

⁴¹ With a slight abuse of notation we do not denote this with an index s_a .

Recalling that

$$\epsilon_1^a = -\frac{\langle v^{a+1}, \epsilon \rangle}{\langle v^a, v^{a+1} \rangle}, \quad \epsilon_2^a = \frac{\langle v^a, \epsilon \rangle}{\langle v^a, v^{a+1} \rangle}, \tag{A.8}$$

and using the vector identities

$$\begin{aligned} \langle v^{a+1}, v^a \rangle \epsilon \cdot v^a + \langle v^a, \epsilon \rangle v^a \cdot v^{a+1} &= -\langle \epsilon, v^{a+1} \rangle v^a \cdot v^a, \\ \langle v^{a-1}, v^a \rangle \epsilon \cdot v^a + \langle v^a, \epsilon \rangle v^{a-1} \cdot v^a &= -\langle \epsilon, v^{a-1} \rangle v^a \cdot v^a, \end{aligned} \tag{A.9}$$

we obtain the compact expression

$$\int_{\mathcal{F}_a} v = \frac{v^a \cdot \epsilon}{\epsilon \cdot \epsilon \langle v^a, \epsilon \rangle} \left(e^{-(\epsilon_1^a \lambda_a + \epsilon_2^a \lambda_{a+1})} - e^{-(\epsilon_1^{a-1} \lambda_{a-1} + \epsilon_2^{a-1} \lambda_a)} \right). \tag{A.10}$$

Now, for a four-dimensional orbifold

$$c_1^{\mathbb{T}}(L_a)|_{y_b} = -\left(\delta_{b,a} \epsilon_1^b + \delta_{b,a-1} \epsilon_2^b \right), \tag{A.11}$$

so that

$$\int_{\mathcal{F}_a} v = \frac{v^a \cdot \epsilon}{\epsilon \cdot \epsilon \langle v^a, \epsilon \rangle} \left(e^{\sum_b \lambda_b c_1^{\mathbb{T}}(L_b)|_{y_a}} - e^{\sum_b \lambda_b c_1^{\mathbb{T}}(L_b)|_{y_{a-1}}} \right). \tag{A.12}$$

Rearranging the contributions of the vertices in the sum

$$\begin{aligned} \mathbb{V}(\lambda_a, \epsilon_i) &= \sum_a \int_{\mathcal{F}_a} v = \sum_a \left(\frac{v^a \cdot \epsilon}{\epsilon \cdot \epsilon \langle v^a, \epsilon \rangle} - \frac{v^{a+1} \cdot \epsilon}{\epsilon \cdot \epsilon \langle v^{a+1}, \epsilon \rangle} \right) e^{\sum_b \lambda_b c_1(L_b)|_{y_a}} \\ &= \sum_a \frac{1}{d_{a,a+1} \epsilon_1^a \epsilon_2^a} e^{\sum_b \lambda_b c_1(L_b)|_a}, \end{aligned} \tag{A.13}$$

where we used another other standard vector identity

$$\langle v^{a+1}, \epsilon \rangle \epsilon \cdot v^a - \langle v^a, \epsilon \rangle v^{a+1} \cdot \epsilon = -\langle v^a, v^{a+1} \rangle \epsilon \cdot \epsilon, \tag{A.14}$$

we obtain the localization formula (3.35).

B Direct proof of the master volume formula for CY₃

We consider a singular Calabi–Yau cone with fan specified by the vectors $v^a = (1, w^a)$, with $a = 1, \dots, d'$ and $w^a \in \mathbb{Z}^2$. The projection on the plane with first coordinate equal to one is a convex polytope called toric diagram in the physics literature. We label the vectors v^a along the toric diagram in anticlockwise order and we identify vectors cyclically, $v^{d'+1} \equiv v^1$. In order to apply the fixed point formula we need to

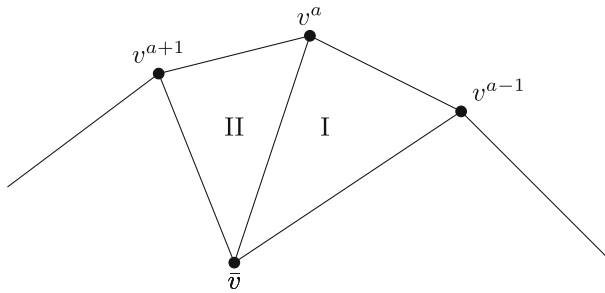


Fig. 8 The triangles in the toric diagram contributing a λ_a dependence in the fixed point formula for \mathbb{V}

resolve the singularity. This can be done by triangulating the toric diagram. We add new vectors $v^\alpha = (1, \bar{w}^\alpha)$, $\alpha = d' + 1, \dots, d$ with \bar{w}^α lying inside the toric diagram until it becomes the union of triangles.

The master volume \mathcal{V} is a function of the λ^α associated with the external vectors $v^\alpha = (1, w^\alpha)$ only. The equivariant volume \mathbb{V} is a function of all λ and can be obtained from the fixed point formula (5.5), where the sum is extended to all triangles in which the toric diagram has been partitioned. The computation is fortunately local. Every particular external λ_a enters in just two fixed point contributions, associated with the two triangles in Fig. 8, where \bar{v} is an internal point.

Denoting $\langle v, w, u \rangle \equiv \det(v, w, u)$, and taking into account the orientations of the inward normals u_A^a , we can write

$$\begin{aligned} \mathbb{V} = & \frac{d_I^2 e^{-\bar{\lambda} \langle v^{a-1}, v^a, \epsilon \rangle + \lambda_{a-1} \langle v^a, \bar{v}, \epsilon \rangle + \lambda_a \langle \bar{v}, v^{a-1}, \epsilon \rangle} / d_I}{\langle v^{a-1}, v^a, \epsilon \rangle \langle v^a, \bar{v}, \epsilon \rangle \langle \bar{v}, v^{a-1}, \epsilon \rangle} \\ & + \frac{d_{II}^2 e^{-(\lambda_a \langle v^{a+1}, \bar{v}, \epsilon \rangle + \bar{\lambda} \langle v^a, v^{a+1}, \epsilon \rangle + \lambda_{a+1} \langle \bar{v}, v^a, \epsilon \rangle) / d_{II}}}{\langle v^{a+1}, \bar{v}, \epsilon \rangle \langle v^a, v^{a+1}, \epsilon \rangle \langle \bar{v}, v^a, \epsilon \rangle} + \dots \end{aligned} \tag{B.1}$$

where $\bar{\lambda}$ is the Kähler parameter associated with the internal point \bar{v} , $d_I = |\langle v^{a-1}, v^a, \bar{v} \rangle|$ and $d_{II} = |\langle v^{a+1}, v^a, \bar{v} \rangle|$ are the orders of the local singularities and the dots refer to terms independent of λ_a . We can then compute

$$\begin{aligned} \frac{\partial^2 \mathbb{V}}{\partial \lambda_a^2} \Big|_{\lambda=0} &= \frac{\langle \bar{v}, v^{a-1}, \epsilon \rangle}{\langle v^{a-1}, v^a, \epsilon \rangle \langle v^a, \bar{v}, \epsilon \rangle} + \frac{\langle v^{a+1}, \bar{v}, \epsilon \rangle}{\langle v^a, v^{a+1}, \epsilon \rangle \langle \bar{v}, v^a, \epsilon \rangle} \\ &= -\frac{\langle v^{a-1}, v^{a+1}, \epsilon \rangle}{\langle v^{a-1}, v^a, \epsilon \rangle \langle v^a, v^{a+1}, \epsilon \rangle} \\ \frac{\partial^2 \mathbb{V}}{\partial \lambda_a \partial \lambda_{a-1}} \Big|_{\lambda=0} &= \frac{1}{\langle v^{a-1}, v^a, \epsilon \rangle} \\ \frac{\partial^2 \mathbb{V}}{\partial \lambda_a \partial \lambda_{a+1}} \Big|_{\lambda=0} &= \frac{1}{\langle v^a, v^{a+1}, \epsilon \rangle}, \end{aligned} \tag{B.2}$$

thus recovering the master volume expression (5.18).

C The equivariant index

In this appendix, we present some simple examples of character indices. It is an interesting topic, with many applications for example to localization computations [56], but, since it is not the main theme of this paper we will be brief.

C.1 The equivariant index for the spindle

We consider the spindle $\Sigma = \mathbb{W}\mathbb{P}^1_{[n_+, n_-]}$. As in Sect. 3.1 we take $v^1 = n_+, v^2 = -n_-$ and GLSM charges $Q = (n_-, n_+)$. We take n_+ and n_- relatively prime.

The character computes the equivariant index

$$\mathbb{Z}(q, \Lambda_a) = \sum_{p=0}^1 (-1)^p \text{Tr}\{q|H^{(0,p)}(\Sigma, \mathcal{O}(\Lambda_+ D_+ + \Lambda_- D_-))\}, \tag{C.1}$$

where q is the weight for the \mathbb{T} action. We recall that not all the \mathbb{T} -invariant divisors are inequivalent. In particular, for the spindle, (2.22) implies $n_+ D_+ = n_- D_-$.

The fixed point formula (2.71) has contribution from two poles of the spindle. In each contribution, we need to add an average over the local singularity \mathbb{Z}_{d_A}

$$\mathbb{Z}(\Lambda_a, q) = \frac{1}{n_+} \sum_{k=0}^{n_+-1} \frac{e^{-2\pi i k \Lambda_+ / n_+} q^{-\Lambda_+ / n_+}}{1 - e^{2\pi i k / n_+} q^{1/n_+}} + \frac{1}{n_-} \sum_{k=0}^{n_--1} \frac{e^{-2\pi i k \Lambda_- / n_-} q^{\Lambda_- / n_-}}{1 - e^{2\pi i k / n_-} q^{-1/n_-}}. \tag{C.2}$$

Setting $q = e^{-\epsilon \hbar}$ and $\Lambda_{\pm} = -\lambda_{1,2} / \hbar$ and taking the limit $\hbar \rightarrow 0$ we recover the equivariant volume (3.16) as the coefficient of the leading pole as in (2.67). Only the terms with $k = 0$ contribute to the limit. The index can be resummed using [56]

$$\frac{1}{k} \sum_{\ell=0}^{k-1} \frac{\omega_k^{-\alpha \ell}}{1 - \omega_k^{\ell} u} = \frac{u^{\alpha - k \lfloor \frac{\alpha}{k} \rfloor}}{1 - u^k}, \tag{C.3}$$

where $\omega_k = e^{\frac{2\pi i}{k}}$, u is a complex number, α is an integer number and the floor function denotes the integer part. This formula can be proved by expanding both sides in power series of u . The result is

$$\mathbb{Z}(\Lambda_a, q) = \frac{q^{-\lfloor \frac{\Lambda_+}{n_+} \rfloor}}{1 - q} + \frac{q^{\lfloor \frac{\Lambda_-}{n_-} \rfloor}}{1 - q^{-1}} = \frac{q^{-\lfloor \frac{\Lambda_+}{n_+} \rfloor} - q^{1 + \lfloor \frac{\Lambda_-}{n_-} \rfloor}}{1 - q}. \tag{C.4}$$

Notice that, although there were fractional powers of q in (C.2), the final formula contains only integer powers. We see that, for $\Lambda_{\pm} > 0$,

$$\mathbb{Z}(\Lambda_a, q) = q^{-\lfloor \frac{\Lambda_+}{n_+} \rfloor} + \dots + q^{\lfloor \frac{\Lambda_-}{n_-} \rfloor}, \tag{C.5}$$

the character has only positive signs and counts the sections of

$$H^0(\mathbb{W}\mathbb{P}^1_{[n_+, n_-]}, \mathcal{O}(\Lambda_+ D_+ + \Lambda_- D_-)), \tag{C.6}$$

refined with respect to the \mathbb{T} action. Geometrically, we see that the exponents are the integers

$$-\frac{\Lambda_+}{n_+} \leq m \leq \frac{\Lambda_-}{n_-}, \quad m \in \mathbb{Z}, \tag{C.7}$$

corresponding to the integer points in the polytope (2.66)

$$\Delta(\Lambda_a) = \{m \in \mathbb{Z} | v^1 m \geq -\Lambda_+, v^2 m \geq -\Lambda_-\}. \tag{C.8}$$

The Molien–Weyl formula (2.68) reads:

$$\mathbb{Z}_{MW}(T, \bar{q}_a) = - \int \frac{dz}{2\pi i z^{1+T}} \frac{1}{(1 - z^{n_-} \bar{q}_1)(1 - z^{n_+} \bar{q}_2)}. \tag{C.9}$$

The relation between the two formulas is as in (2.70)

$$\mathbb{Z}_{MW}(T = n_+ \Lambda_- + n_- \Lambda_+, \bar{q}_a) = \prod_{a=\pm} \bar{q}_a^{\Lambda_a} \mathbb{Z}(\Lambda_{\pm}, q = \bar{q}_+^{n_+} \bar{q}_-^{n_-}). \tag{C.10}$$

This can be checked by computing explicitly the residue related to \bar{q}_i . One obtain two sums over $z_{k_-} = \bar{q}_1^{-1/n_-} e^{2\pi i k_- / n_-}$ and $z_{k_+} = \bar{q}_2^{-1/n_+} e^{2\pi i k_+ / n_+}$ for $k_{\pm} = 0, \dots, n_{\pm} - 1$ that match the fixed point formula.⁴² On the other hand, we can also evaluate the integral in a simpler way. We define the grand-canonical partition function

$$\begin{aligned} \sum_{T=0}^{\infty} \mathbb{Z}_{MW}(T, \bar{q}_a) w^T &= - \int \frac{dz}{2\pi i} \frac{1}{(1 - z^{n_-} \bar{q}_1)(1 - z^{n_+} \bar{q}_2)(z - w)} \\ &= \frac{1}{(1 - w^{n_-} \bar{q}_1)(1 - w^{n_+} \bar{q}_2)}, \end{aligned} \tag{C.11}$$

where we evaluated the integral by deforming the contour circling $z_{k_{\pm}}$ into a contour around $z = w$, which is the only other singularity of the integrand. From this expression we see that $\mathbb{Z}_{MW}(T, \bar{q}_a)$ for $T > 0$ counts the monomials in (\bar{q}_1, \bar{q}_2) of charge T under the rescaling with weights $Q = (n_-, n_+)$. These are precisely the holomorphic sections

$$H^0(\mathbb{W}\mathbb{P}^1_{[n_+, n_-]}, \mathcal{O}(T)), \tag{C.12}$$

⁴² We assume that n_+ and n_+ are relatively prime.

where the line bundle $O(T)$ corresponds to Chern class $c_1 = [\frac{F}{2\pi}]$ with Chern number $\frac{1}{2\pi} \int F = \frac{T}{n_+n_-}$.

C.2 The equivariant index for $\mathbb{W}\mathbb{P}^2_{[N_1, N_2, N_3]}$

For simplicity, we only consider the case of the “non-minimal” fan

$$v^1 = (n_3, n_3), \quad v^2 = (-n_1, 0), \quad v^3 = (0, -n_2), \tag{C.13}$$

and $Q = (N_1, N_2, N_3) = (n_1n_2, n_2n_3, n_1n_3)$. The orbifold is $\mathbb{M}_4 = \mathbb{W}\mathbb{P}^2_{[N_1, N_2, N_3]}$ if the n_a are coprime. There are only labels and not extra orbifold singularities in the sense that the corresponding primitive vectors \hat{v}^a , obtained by dividing v^a by the label n_a , satisfy $\hat{d}_{a,a+1} = \det(\hat{v}^a, \hat{v}^{a+1}) = 1$.

The fixed point formula (2.71) requires averaging over the local orbifold singularity. For example, we can consider the contribution from the fixed point associated with the cone (v^2, v^3) . The local data are $d_{2,3} = n_1n_2$ and $u_1 = (-n_2, 0)$ and $u^2 = (0, -n_1)$. The local orbifold action is determined by $J_2n_1 = s, J_3n_2 = p$ where s, p are integers:

$$(e^{2\pi i J_2}, e^{2\pi i J_3}) = (e^{2\pi i \frac{s}{n_1}}, e^{2\pi i \frac{p}{n_2}}). \tag{C.14}$$

The fixed point contribution is

$$\begin{aligned} \mathbb{V} &= \frac{1}{n_1n_2} \sum_{s=0}^{n_1-1} \sum_{p=0}^{n_2-1} \frac{(e^{2\pi i \frac{s}{n_1}} q_1^{-1/n_1})^{-\Lambda_2} (e^{2\pi i \frac{p}{n_2}} q_2^{-1/n_2})^{-\Lambda_3}}{(1 - e^{2\pi i \frac{s}{n_1}} q_1^{-1/n_1})(1 - e^{2\pi i \frac{p}{n_2}} q_2^{-1/n_2})} \\ &= \frac{q_1^{\lfloor \frac{\Lambda_2}{n_1} \rfloor} q_2^{\lfloor \frac{\Lambda_3}{n_2} \rfloor}}{(1 - 1/q_1)(1 - 1/q_2)}, \end{aligned} \tag{C.15}$$

where we use the identity (C.3). The contributions of the other fixed points can be computed similarly. Combining the three contributions, we find

$$\mathbb{V}(\Lambda_a, q_i) = \frac{q_1^{\lfloor \frac{\Lambda_2}{n_1} \rfloor} q_2^{\lfloor \frac{\Lambda_3}{n_2} \rfloor}}{(1 - 1/q_1)(1 - 1/q_2)} + \frac{\left(\frac{q_1}{q_2}\right)^{-\lfloor \frac{\Lambda_3}{n_2} \rfloor} q_1^{-\lfloor \frac{\Lambda_1}{n_3} \rfloor}}{\left(1 - \frac{q_1}{q_2}\right)(1 - q_1)} + \frac{\left(\frac{q_2}{q_1}\right)^{-\lfloor \frac{\Lambda_2}{n_1} \rfloor} q_2^{-\lfloor \frac{\Lambda_1}{n_3} \rfloor}}{\left(1 - \frac{q_2}{q_1}\right)(1 - q_2)}, \tag{C.16}$$

in agreement with the general formula (2.73). One can check that

$$\mathbb{V}(\Lambda_a, q_i) = \sum_{m \in \Delta(\Lambda_a)} q^m \tag{C.17}$$

counts (with a \mathbb{T}^2 weight) the integer points in the polytope

$$\Delta(\Lambda_a) = \{m \cdot v^a \geq -\Lambda_a\}, \quad m \in \mathbb{Z}^2. \tag{C.18}$$

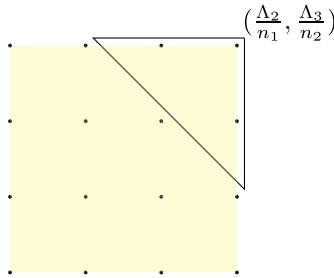


Fig. 9 The integer points counted by (C.15)

This can be proved with elementary methods. By expanding in power series (C.15) in $1/q_1$ and $1/q_2$, we obtain all the integer points in an infinite cone, as in Fig. 9.

By similarly expanding the other power series in (C.16) for $|q_i| > 1$ and $|q_2/q_1| > 1$, we find three competing contributions, which cancel except for the points inside the polytope as depicted in Fig. 10.

The Molien–Weyl formula gives instead

$$Z_{MW}(T, \bar{q}_a) = - \int \frac{dz}{2\pi i z^{1+T}} \frac{1}{(1 - z^{N_1} \bar{q}_1)(1 - z^{N_2} \bar{q}_2)(1 - z^{N_3} \bar{q}_3)}. \quad (C.19)$$

This can be explicitly evaluated by taking the residues at $z^{N_a} \bar{q}_a = 1$, and one recovers formula (2.70):

$$Z_{MW} \left(T_A = \sum_a Q_a^A \Lambda_a, \bar{q}_a \right) = \prod_{s=1}^d \bar{q}_a^{\Lambda_a} \mathbb{Z} \left(\Lambda_a, q_i = \prod_a \bar{q}_a^{v_i^a} \right), \quad (C.20)$$

where the gauge-invariant quantities are explicitly given by

$$q_1 = \bar{q}_1^{n_3} \bar{q}_2^{-n_1}, \quad q_2 = \bar{q}_1^{n_3} \bar{q}_3^{-n_2}, \quad T = n_1 n_2 \Lambda_1 + n_2 n_3 \Lambda_2 + n_3 n_1 \Lambda_3. \quad (C.21)$$

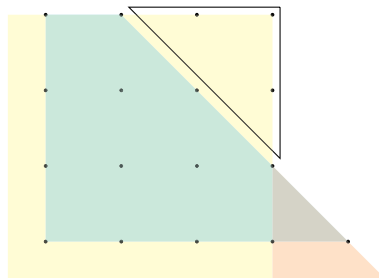


Fig. 10 The power series contributing to (C.16). The yellow and orange power series enter with a plus sign, while the green with a minus sign. The result is just the set of integer points in the triangle (colour figure online)

One can check that the three set of residues at $z^{N_a} \bar{q}_a = 1$ precisely correspond to the three fixed point contributions. For example, the residue associated with \bar{q}_1 reads

$$\begin{aligned}
 & \frac{1}{n_1 n_2} \sum_{k=0}^{n_1 n_2 - 1} \frac{\bar{q}_1^{T/N_1} \omega_{N_1}^{-kT}}{(1 - \omega_{N_1}^{kN_2} \bar{q}_2 \bar{q}_1^{-N_2/N_1})(1 - \omega_{N_1}^{kN_3} \bar{q}_3 \bar{q}_1^{-N_3/N_1})} \\
 &= \frac{1}{n_1 n_2} \sum_{k=0}^{n_1 n_2 - 1} \frac{\bar{q}_1^{T/n_1 n_2} \omega_{n_1 n_2}^{-kT}}{(1 - \omega_{n_1}^{kn_3} \bar{q}_2 \bar{q}_1^{-n_3/n_1})(1 - \omega_{n_2}^{kn_3} \bar{q}_3 \bar{q}_1^{-n_3/n_2})} \\
 &= \frac{1}{n_1 n_2} \sum_{k=0}^{n_1 n_2 - 1} \frac{\bar{q}_1^{\Lambda_1 + \frac{n_3 \Lambda_2}{n_1} + \frac{n_3 \Lambda_3}{n_2}} \omega_{n_1}^{-kn_3 \Lambda_2} \omega_{n_2}^{-kn_3 \Lambda_3}}{(1 - \omega_{n_1}^{kn_3} \bar{q}_2 \bar{q}_1^{-n_3/n_1})(1 - \omega_{n_2}^{kn_3} \bar{q}_3 \bar{q}_1^{-n_3/n_2})} \\
 &= \bar{q}_1^{\Lambda_1} \bar{q}_2^{\Lambda_2} \bar{q}_3^{\Lambda_3} \frac{(\bar{q}_2^{n_1} \bar{q}_1^{-n_3})^{-\lfloor \frac{\Lambda_2}{n_1} \rfloor} (\bar{q}_3^{n_2} \bar{q}_1^{-n_3})^{-\lfloor \frac{\Lambda_3}{n_2} \rfloor}}{(1 - \bar{q}_2^{n_1} \bar{q}_1^{-n_3})(1 - \bar{q}_3^{n_2} \bar{q}_1^{-n_3})} \\
 &= \bar{q}_1^{\Lambda_1} \bar{q}_2^{\Lambda_2} \bar{q}_3^{\Lambda_3} \frac{q_1^{\lfloor \frac{\Lambda_2}{n_1} \rfloor} q_2^{\lfloor \frac{\Lambda_3}{n_2} \rfloor}}{(1 - 1/q_1)(1 - 1/q_2)}, \tag{C.22}
 \end{aligned}$$

reproducing the contribution (C.15). In the sum, we replaced k with $n_3 k$ using the fact that the n_i are coprime. We also used the identity

$$\frac{1}{n_1 n_2} \sum_{k=0}^{n_1 n_2 - 1} \frac{\omega_{n_1}^{-k\Theta_1} \omega_{n_2}^{-k\Theta_2}}{(1 - \omega_{n_1}^k u_1)(1 - \omega_{n_2}^k u_2)} = \frac{u_1^{\Theta_1 - n_1 \lfloor \frac{\Theta_1}{n_1} \rfloor} u_2^{\Theta_2 - n_2 \lfloor \frac{\Theta_2}{n_2} \rfloor}}{(1 - u_1^{n_1})(1 - u_2^{n_2})}, \tag{C.23}$$

which can be proved by expanding both sides in power series of u .

The grand-canonical partition function

$$\sum_{T=0}^{\infty} \mathbb{Z}_{MW}(T, \bar{q}_a) w^T = - \int \frac{dz}{2\pi i} \frac{1}{(1 - z^{N_1} \bar{q}_1)(1 - z^{N_2} \bar{q}_2)(1 - z^{N_3} \bar{q}_3)(z - w)} \tag{C.24}$$

can be also computed deforming the contour to encircle $z = w$ which gives

$$\sum_{T=0}^{\infty} \mathbb{Z}_{MW}(T, \bar{q}_a) w^T = \frac{1}{(1 - w^{N_1} \bar{q}_1)(1 - w^{N_2} \bar{q}_2)(1 - w^{N_3} \bar{q}_3)}. \tag{C.25}$$

We see that $\mathbb{Z}_{MW}(T, \bar{q}_a)$ for $T > 0$ counts the monomials in $(\bar{q}_1, \bar{q}_2, \bar{q}_3)$ of charge T under the rescaling with weights $Q = (N_1, N_2, N_3)$. These are precisely the holomorphic sections

$$H^0(\mathbb{W}\mathbb{P}^2_{[N_1, N_2, N_3]}, \mathcal{O}(T)), \tag{C.26}$$

of the line bundle $O(T)$. In the familiar case of $\mathbb{W}\mathbb{P}^2$, where $N_i = 1$, these are just the homogeneous polynomials of degree T .

In the presence of $\hat{d}_{a,a+1} = \det(\hat{v}^a, \hat{v}^{a+1}) \neq 1$, we would need more general formulas for resumming the averages over the orbifold action and we will not discuss this case.

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