PROMOTING DISTANCE MATHEMATICAL DISCOURSE: ONLINE RESOURCES FOR EFFECTIVE COMMUNICATION IN GEOMETRY

Marina Cazzola
Università degli Studi di Milano-Bicocca
Milano Italy
marina.cazzola@unimib.it

Abstract

Communication in mathematics can be particularly difficult because multimodal registers are needed in order to develop mathematical ideas (usually a simultaneous use of written texts, formulas, schemas, diagrams, geometric figures, graphs, and so on is essential). Effective communication of mathematics needs to switch from one register to another, from one representation to another, as much as possible. One could go as far as to say that the true mathematical competence is knowing how to use the right representation at the right time. For this reason, it turns out that old style pen and paper techniques are the best suited for carrying out a mathematical discourse. Mathematicians are known to like to use blackboards, both when they work on their researches and when lecturing.
From a didactic point of view, students as well can take advantage of the possibility of writing by hand on paper both when they carry out their studying and, for example, when they are under examination. This is particularly true with written exams that contain open-ended questions and ask to provide justifications for the given answers, and in case of oral exams, which are the type of exam that best allows for verifying the capacity of building a mathematical discourse.
With the outbreak of the COVID-19 emergency, in spring 2020 all exams needed to be held remotely and pen and paper turned out to be not available. Some technical solutions available were in fact not usable due to the lack of technical equipment on the part of the students. E.g. to effectively use an online whiteboard, you need to have a digital pen: you simply cannot write mathematics with just the mouse.
By means of the WIMS (WWW Interactive Multipurpose Server) platform, that we introduced at ICERI19 (cfr. [1]), it was possible to build an online work environment that made it easier for students to work in mathematics, with particular attention to the task of conducting a mathematical discourse.
We will describe in detail the setup used for the geometry course for prospective primary school teachers, showing the effectiveness of WIMS:

- in complementing lectures by posing engaging problems, e.g. in order to let student experiment with the graphic properties of the geometric objects being studied;
- as a tool to replace the written exam, allowing for the conceiving of an open-ended questions test, with automatic verification of the answers;
- as a drawing board to be used during oral exams, so to let students easily provide drawings and representation to support their discourse.

Some videos of students' use of these tools are available, which also show the achievement of a good level of ICT skills.
Keywords: teacher training, mathematics, mathematical discourse

1. COMMUNICATION IN MATHEMATICS

Mathematics deals with abstract concepts that need multiple representations to be fully explored: easily a typical mathematical discourse keeps switching from textual registers (actual pieces of texts, most of the time full of formulas), to visual registers, such as diagrams and drawings varying from graphical representations to geometric figures. Effective communication in mathematics requires multiple communication registers, and this particularly applies to mathematics lectures: when explaining a mathematical topic, teachers should make use of the most appropriate representation for each of the facets of the concept being covered. Mathematicians are well known for their use of chalk and blackboards: they like to build their lectures "on the fly", trying to bring the audience to experience the flow of thoughts that

Citation information:
M. Cazzola (2020), Promoting distance mathematical discourse: online resources for effective communication in geometry, ICERI2020 Proceedings, pp. 8739–8748. IATED (ISBN: 978-84-09-24232-0)

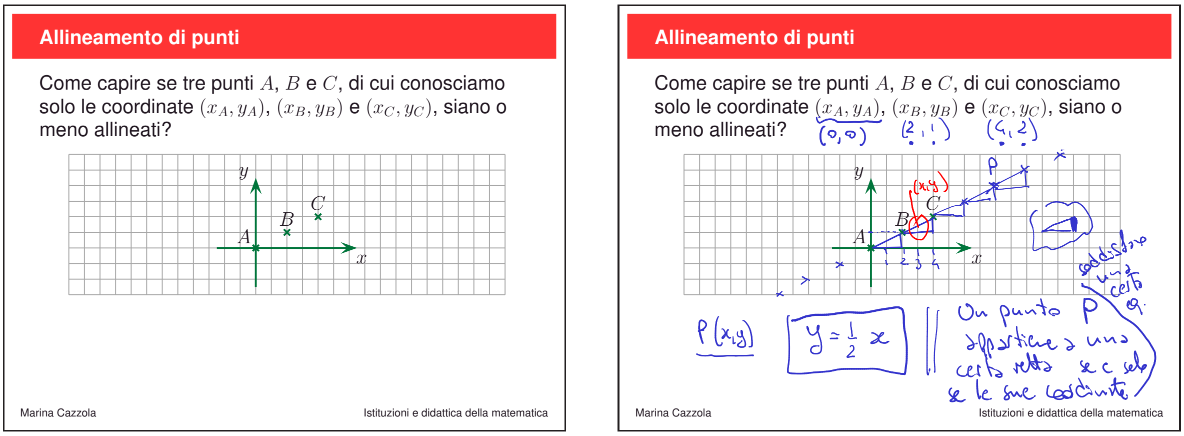


Fig. 1: Pen and paper

build up the idea. Even lectures using computer typed slides and a projector, should follow the old-style blackboard model, as this allows the teacher to interact with the students and better engage them. Too often mathematics is seen as an arid domain of definitions and formulas to be learned by heart: the live creation of the stream of arguments, with all the detours due to possible interaction with the audience, can be useful to give a taste of the real essence of mathematics. It is not too difficult to achieve this: mathematics aware typesetting systems for computer presentations are available. Teachers can prepare in advance most of the texts and graphics to be shown, to use them as a basis for the argumentations. Also, a tablet with a digital pen can be used to act on the slide "the mathematicians' way" (e.g. see Figure 1).

2. THE COVID-19 EMERGENCY

At the start of the spring semester 2020 whatever was planned had to be revised because of the spread of the Coronavirus. More precisely, on February 27th we were informed that on the following Monday, March 2nd, all lectures were supposed to start being held remotely: forget chalk and blackboard, all the teaching would have to take place through videos of the lectures prepared in advance and web video conferences. Moreover, as soon as it was understood that the emergency would last much longer than hoped for (a couple of weeks. . .), it became clear that all of the teaching related activities had to be held remotely. We were engaged in designing an environment to remotely held exams too.

2.1 Tools

We will describe our experience at the University of Milano-Bicocca (Italy). Since a few years our university started a program of technological innovation of didactics. A standard LMS (in our case Moodle) is now used as the main interface for communication between the faculty and the students, for all administrative and didactical matters. Moreover, almost all the classrooms are equipped with hardware and software to allow teachers to video-record their lectures. The system is integrated with the LMS above and recorded lectures can be immediately uploaded to Moodle and made available to the students. This equipment was going to be very useful during the emergency, as these classroom registration tools were available for teachers to record the lectures' videos. Nevertheless, the need of limiting staff access to the university premises as much as possible urged the university to invite teacher to work from their homes. Softwares were provided for Windows or Mac users, otherwise teachers could rely on any video recording tool available in the public domain (e.g. for Linux users). Our university has also made available to teachers and students both the Google suite (in particular Drive, Meet, Classroom) and the Microsoft suite (in particular OneDrive, Office and Teams).
As long as mathematics is concerned, the need arose to understand if and how those tools could be functional to the specificities of communication in mathematics. With respect to this aspect, what somehow characterizes our site compared to the Italian panorama is the availability of a WIMS server (WWW Interactive Multipurpose server) with specific features particularly suited for the communication of mathematics. We already introduced WIMS in [1] and refer to [2], [3], [4], [5], [6] for a detailed description of the system and for accounts of experiences of its use in teaching. In this paper we will focus on some

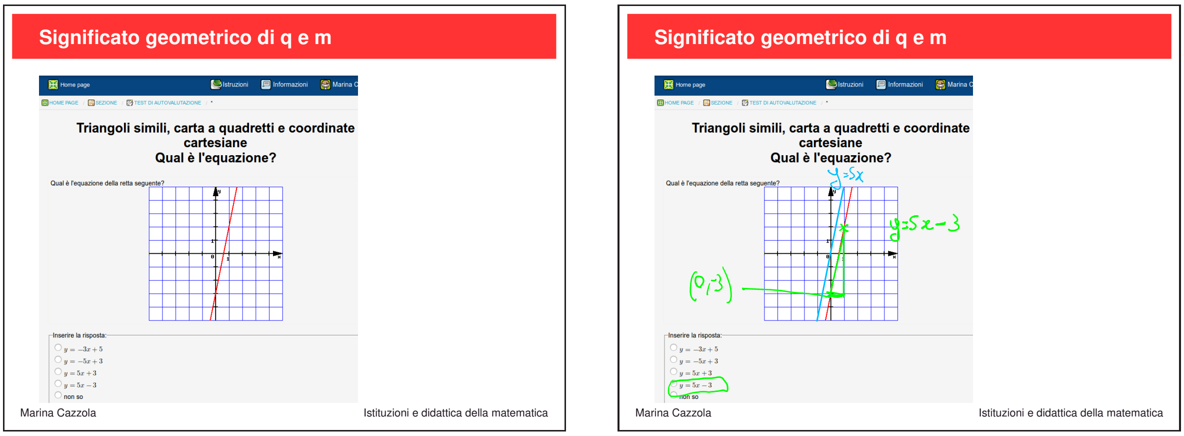


Fig. 2: Example of a WIMS activity (1)

examples in which the strengths of WIMS emerged as a support for remotely held lectures, exercise classes and exams.

3. THE EMERGENCY SETUP

In this talk we wish to describe our experience with a course in mathematics, more precisely the course "Basic mathematics for teaching" for prospective primary school teachers at the University of Milano-Bicocca (e.g. see [7]). The topic covered in the course are: elements of Euclidean geometry, measure and proportionality, elements of the geometry of transformations (in particular similarities and isometries), geometrical constructions on graph paper. In the following sections we will briefly present the pre-COVID organization of the course and the changes that had to be introduced due to the new setup. Before the emergency, the course included lectures, exercise classes and laboratories. This organization could not be changed and so each of the different types of activities had to be adapted to remote learning. Each of them presents peculiarities, that requires ad hoc solution. In particular, the labs required a major redesign work and we refer to [8] for the description of how they were implemented. In the following sections we will focus on lectures, exercise classes and exams, that share a certain extent of problems and solutions. The description below follows the lectures-exercises-exams order because it is actually the temporal order in which the activities took place, and therefore it is possible to highlight the evolution in the use of the tools due to experience. This path also implies a growing involvement of students: during the lectures, the main character is the teacher, but at the final exam this role shifts to the student.

3.1 Lectures

Before the emergency, lectures followed a standard scheme, taking into account the point of view described in section 1.: the environment is a large classroom where the teacher projects a presentation prepared in advance, more or less using it as a basis for entering details in order to allow for a minimum level of interaction with the students. This schema was repurposed almost verbatim to a remote learning setup. Video recording of the teacher's talks were periodically made available to the students, according to the course timetable, through Moodle (the example in Figure 1 is actually taken from one of the videos made available to the students). A special effort was made to save a minimum level of teacher-student interaction that face-to-face lectures allowed, but that was completely lost in the asynchronous recorded videos. In order to achieve this, we followed a flipped-classroom model, and meetings for synchronous video conferencing were then scheduled. During these meetings, teachers and students discussed various aspects of the subject topics that emerged in the video lectures, and the teachers answered any of the students' questions. To strongly link these meetings with the lectures, we decided to record the synchronous chats and to make the videos available to the entire class. Just to give an idea, the ratio between asynchronous video recordings and synchronous video chats was slightly higher than 1:1.
The time dedicated to lectures (in particular to the synchronous chats) was also used to test the usability of the interactive tools available: both Google Meet and Microsoft Teams were extensively tested in order to explore the technical strengths and weaknesses of each of them (e.g. according to various

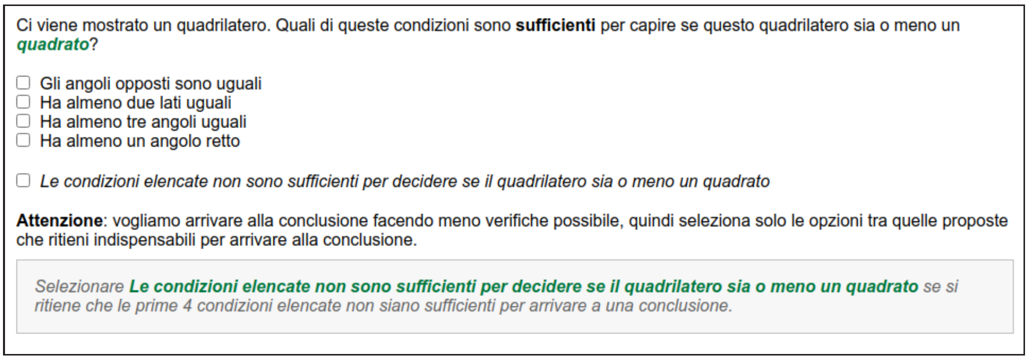


Fig. 3: Example of a WIMS activity (2)

tasks: user management, built-in recording facilities, usability, ...). From our point of view, the two video conferencing systems turned out to be completely equivalent, and in the end we opted for Google Meet only because that is the one to which the students responded better (fewer difficulty of use by the students emerged, and, as the aim of the course was not to train students in the use of this kind of media, we decided to meet their requests).
As we already pointed out, in this first phase the burden of finding effective means of communication is still on the teacher: even when the students ask questions, it is always the teacher who manages the presentation (and therefore have the pen and paper problem). Gradually, towards the end of the course, studentes were gradually asked to activate presentations themselves.
Lectures were also complemented by exercises, some of which were provided in a web-based form, with automatic scoring, through WIMS. A complete set of the selection of the WIMS exercises used for the course is available in the international distribution of WIMS (e.g. see [9]). For almost all the topics covered, basilar competencies can be tested through automated exercises. For example, the exercise shown in Figure 2 (e.g. see [10]) was used by the teacher in a video lecture, to show how to apply the rules previously explained (cfr. Figure 1).
It turned out that WIMS exercises nicely complemented lectures and actually engaged the students: many of the discussions held in the live sessions were actually stimulated by some WIMS exercise. It seems that the automatic marking and feedback that students receive for each submitted exercise induce them to rethink and refine their study of the subject.
Another strength of WIMS turned out to be the availability of many different exercises, which allowed to cover many different facets of each topic. Also, the possibility of graphically manipulating figures turned out to be a winning feature, especially for a course in geometry.
As pointed out in [3], the real issue with our setup is the fact that the use of automatic exercises can lead the students to shift focus from a real understanding of mathematics: "students might get the idea that the goal of mathematics is to find the *fastest* strategy to in order to get a computer-approved answer, a strategy that does not always correspond to a real understanding of the subject" [3]. An effort was thus made, both in developing new exercises, and in using the synchronous chats to explicitly ask the students for explanations of every detail of their findings. An exercise as the one shown in Figure 3 (see [11]) was developed, in order to have students experience the hypothetico-deductive method in geometry. The use of this exercise was very effective, because it did engage students, who kept asking questions about it in the live chats.

3.2 Exercise classes

Before the emergency, exercise classes were organized in small groups and dealt with worksheets given to the students in advance (7 meetings, each devoted to one of the 7 worksheets). In our pre-COVID model, students were supposed to work on the exercises before attending the class, so to discuss with the tutors their findings and their doubts. Most of the exercises in the worksheets had a WIMS analogous, with automatic marking, and extra online support was available for students who could not attend or had further questions.
During the emergency, students were split in groups of about 40 students and tutors were given the freedom to organize their work to their own pace. Two different models were implemented. Some of the tutors gathered their group in synchronous video conferences and held the class as it would have been in presence. Each group had a Moodle forum available for organization matters and for extra questions.

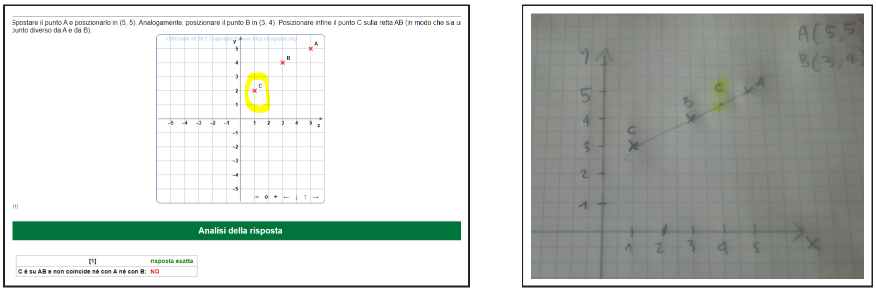


Fig. 4: Student examples: working on a WIMS exercise

Some tutors chose to use the forum and to hold their exercise classes asynchronously. Students had to work on the worksheet, then ask their questions in the forum. Subsequently, either the tutor would post stimulus questions on some aspects of the exercises, or the students were asked to post in the forum a full solution of some of the exercises. This setup did engage the students but dealing with all the asynchronous conversations in the forum turned out to be too demanding on the tutors, so they gradually switched from an interaction entirely based on the forum, to live meetings.
An extra support Moodle forum was active during the whole semester, so to allow any student of any group to post any unresolved question they had. Tutors monitoring this forum were particularly instructed to strongly promote problem solving: whenever a student posed a question, they had to avoid to give immediately a complete solution, but rather try to start a conversation so to lead the student to build an answer to the question almost by himself.
From the technical point of view, we made a big effort in order to monitor all the Moodle forums so to understand and prevent the possible difficulties students might have due to the remote learning setup with respect to the difficulties inherent in communication of mathematics. We have been able to observe some critical issues due to the technologies in use and the students' ability of making use of such.
Many of the discussions that took place in the forums started from WIMS exercises, as the automatic correction forced the students to face an answer marked "wrong". On the left of Figure 4 you can see the student's resolution of a WIMS exercise asking to make alignments of points using graph paper (recalling the topic seen in lectures, and shown in Figure 1 and Figure 2). The answer that the student submitted is marked "wrong" by WIMS, so she posts the screenshot of her resolution of the exercise, marking her answer in yellow with an image editor, and asking for an explanation. Tutors respond to the student asking to better explain her resolution, so she switch from the computer screenshot to pen and paper, returning an alternative solution that was not possible to give via WIMS (because in fact it was not consistent with the hypotheses of the given problem). She then takes a photo of her work and post it back to the forum (see Figure 4 on the right). That is to say, the student needed to revert to pen and paper to better revise her reasoning, and thus the misunderstanding of the requests of the problem emerged. From the digital point of view, she needed some ingenuity for effectively communicate her thoughts. Instead of using actual paper and upload a photo of the drawing, the effect could have been achieved by computer via a software allowing for some kind of "handwriting" input. Many softwares of this kind are available (the ones included in the Google and Microsoft suites, that is Jamboard and Notes, were actually shown at lectures), nevertheless the use of such tools is hard if you do not have suitable hardwares (e.g. a digital pen: handwriting is almost impossible with a mouse and not always really comfortable with a touchsceen). This evident difficulty made clear that the eventual realization of remote exams would not have an easy solution. Exams environments will have to rely not only on the softwares provided by the university but also on the hardwares owned by the students.
Another aspect that we have been able to monitor concerns the fact that one of the peculiarities of mathematics is the large use of formulas, that are not really easy to typeset in a textual environment like a Moodle forum. A LaTeX plugin is available, but students need to know how to code the formula (and this is not the case for prospective primary school teachers). In Figure 5 on the left you can see an example of an exchange in the forum using a textual register (with a figure attached), and the difficulty of inserting a short mathematical text is successfully resolved by using its verbatim form

```
H = radice quadrata (AB^2-AB/2^2)
```

instead of the pretty printed one

$$\sqrt{AB^2 - \left(\frac{AB}{2}\right)^2}.$$

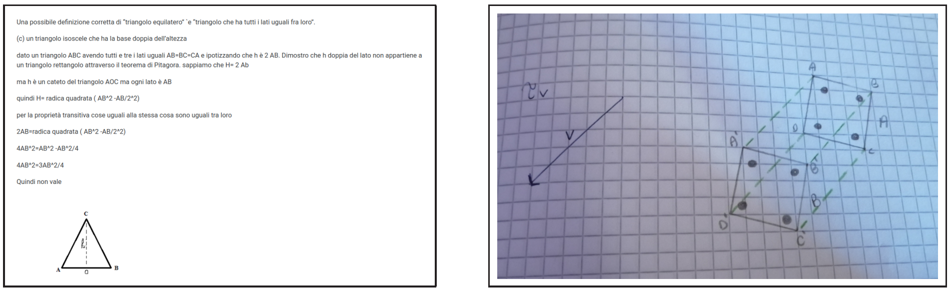


Fig. 5: Student examples: textual and graphic registers in the forum

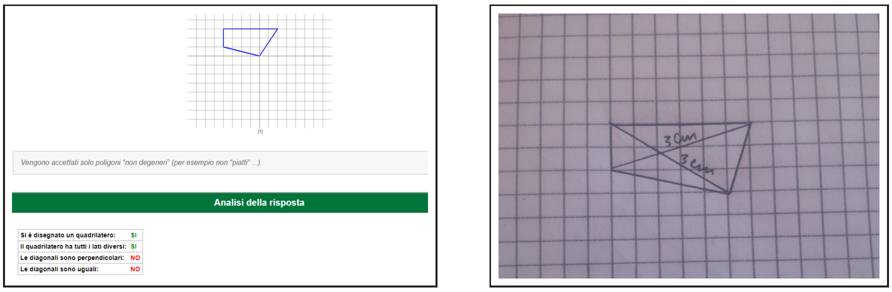
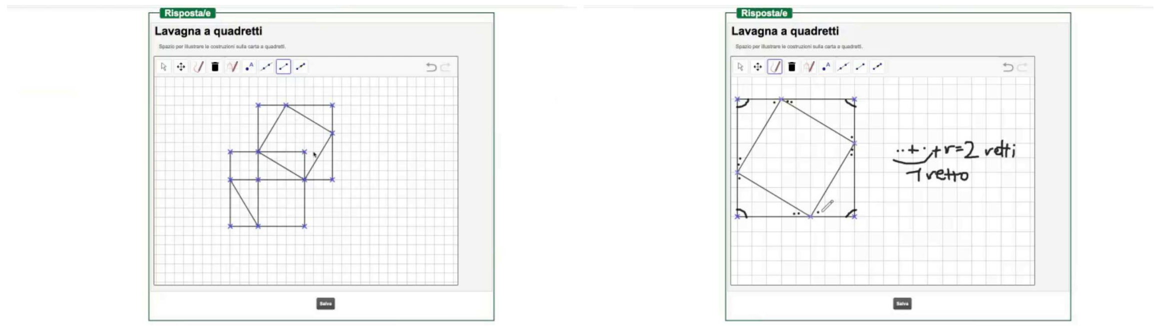


Fig. 6: Student example: exercise revision in the forum

(Maybe a structured language as LaTeX would have helped the student spot the error of writing `AB/2^2` instead of `(AB/2)^2`, but, as much as communication is concerned, we consider the overall level achieved satisfactory enough.)
For more complex geometrical tasks, as the one you can see in Figure 5 on the right, the choice falls again to the use of pen and paper and a camera to take a photo.
We end this section with a further example focusing on the added value given by the possibility of interacting with WIMS exercises. We developed the module “OEF Polygons on graph paper” (see [12]), dealing with lattice polygons (e.g. see [3]), which engages the students with the task of dealing with non-stereotyped examples of polygons. In Figure 6 on the left you can see the reply given by the student to the request of drawing “a quadrilateral with sizes all of different lengths and diagonals which are congruent and perpendicular”. This kind of exercises accepts as answer polygons drawn by the students via a user-friendly interface (based on the capabilities of HTML5 canvas, nicely embedded in the WIMS system), and analyzes them according to various requests. It is in fact an open-ended question with a large variety of possible requests and corresponding possible correct answers. Often, exercises of these kinds were discussed in the support forum, and students were lead to revise their answers with the use of drawing on graph paper (as you can see in Figure 6 on the right), so to spot and correct their mistakes.

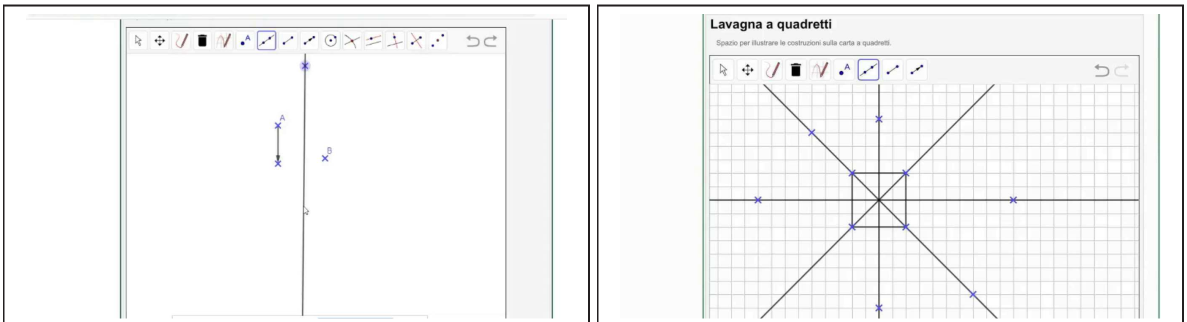
3.3 Exams

As expected, the real challenge for technologies applied to mathematics communication turned out to be the field of exams. Before the emergency, exams were split into three different steps. First, a computer-based evaluation of basic competencies, via the task of solving standard exercises. Secondly, a written exam with the aim of having the student carry out argumentations, that is explain how to solve slightly more difficult exercises than the ones solved previously. Thirdly and finally, an oral exam in which students are asked to answer any kind of question, both related to the previous part of the exam, and on any theoretical topic covered by the lectures.
During the emergency, exams had to be held remotely, thus implying a big security concern. Proctoring systems were acquired by our university, to prevent massive cheating, but the requirement of relying on student-owned hardwares made things difficult especially for the written part. The experience of monitoring students’ discussions about the exercises in the forums clearly showed that most of the students do need pen and paper if asked to carry out a discourse with that level of argumentation we usually required for the written exam. The large number of students (over 300) forced us to strongly rely on computerized tests and automatic proctoring tools. Luckily, we had had a lot of previous experience with the WIMS exam system and we had a significant battery of non-trivial exercises, many of which



www.youtube.com/watch?v=okq5EOKNNRc

Fig. 7: Video examples: 1



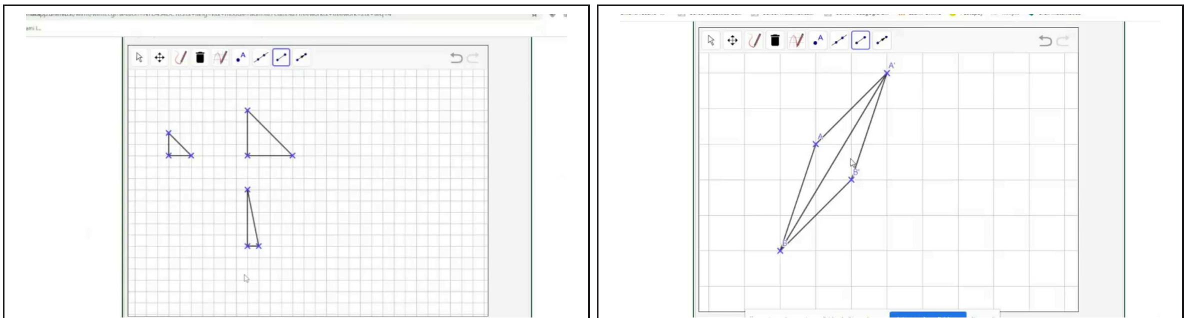
www.youtube.com/watch?v=UCTcrX_-EFA www.youtube.com/watch?v=IdECCgCNnZg

Fig. 8: Video examples: 2 and 3

open-ended, as some of the examples described above. This fact allowed us to transfer the assessment of many skills from the written exam to the computerized test. So we just needed to embed a WIMS exam into one of the available proctoring system (and we got a satisfactory setup as far as security was concerned).

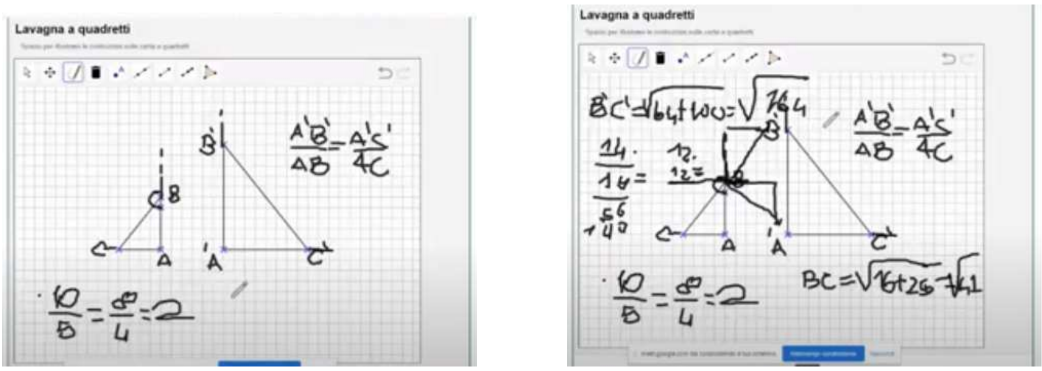
The real burden of verifying more advanced mathematical competences, such as reasoning, communication, being able to carry out a discourse, devising strategies, and using mathematical language, was left to the oral exam, that had to be held remotely by mean of video calls.

The evolution of WIMS allowed us to device "whiteboards" based on GeoGebra and embedded in the system, so that during the oral exams students were able to manipulate images and figure, and to carry out a discourse, just with a standard PC equipped with a standard mouse. Typically, each oral colloque started from a WIMS exercise the student had solved before, and whose resolution the student had to fully explain with references to theory. A series of videos is available documenting the process, and the effective interaction, through the WIMS system, between automated exercises, and the tasks of



www.youtube.com/watch?v=JZoI2I3Hj9o www.youtube.com/watch?v=Y1B4BdpFZEY

Fig. 9: Video examples: 4 and 5



To appear on www.youtube.com

Fig. 10: Video examples: 6

reasoning and arguing. These examples include:

- a student giving a demonstration of Pythagoras theorem https://www.youtube.com/watch?v=okq5EOKNNRc (e.g. see Figure 7), in this example the board is set so to force the snapping of the points on the grid, in the screenshot you can spot an attempt to write some text (on an iPad);
- a student, given a pair of points, listing all the isometries that send the first point into the second https://www.youtube.com/watch?v=UCTcrX_-EFA (e.g. see Figure 8 on the left), in this case the board is set so that the points can be drawn with no constraints, and only a selection of GeoGebra tools is made available to the student;
- a student exploring the idea of symmetry group by describing the symmetries of a square https://www.youtube.com/watch?v=IdECCgCNnZg (e.g. see Figure 8 on the right);
- a student explaining the idea of similarity of triangles https://www.youtube.com/watch?v=JZoI2l3Hj9o (see Figure 9 on the left);
- a student giving the definition of translation and a full argument to show that translations are indeed isometries, by means of the properties of parallelograms https://www.youtube.com/watch?v=Y1B4BdpFZEY (see Figure 9 on the right).

In the videos, the students show the acquisition of a good level of ICT skills, with respect to the use of these whiteboards. We observed some attempts (not yet documented in a video) to include more and more written texts (e.g. see Figure 10).
This solution, however, has the flaw that only one of the users (typically the student) can operate on the blackboard and the other (typically the teacher) must interact verbally. We chose this solution anyway, rather than one of the many sharable whiteboards available on the web, mostly for security reasons: during an exam, the teacher has to be able to clearly spot who is actually writing, and that no intruder is interfering. It turned out that the extra task to translate the teacher's verbal instruction into graphical representation was a good way to test a further mathematical competency.

4. CONCLUSIONS

The COVID-19 emergency compelled many of us to set up a remote learning environment in a very short time. Luckily, our University provided a good variety of ICT tools. As far as mathematics is concerned, the availability of WIMS proved to be a valid complement to the lectures and a very useful tool for setting an exam environment.
Lecturers who had previously worked with WIMS had batteries of tests that allowed them to replace the written exam with a somehow equivalent remote computer-based solution. Some colleagues who did not have this background simply gave up at the idea of using WIMS for exams.
We found out that the critical point in the switch from a face-to-face model to a remote one is on the students' side. They can develop ICT competencies, but some tasks need a special hardware. For mathematics, the use of pen and paper techniques is essential (and we discovered that we share this problem with other subjects, e.g. teaching Chinese writing). Without a digital pen, the most viable solution for written test the request to the student to write their resolution on paper, scan the text and send either by mail or through facilities for file deposit (with all the relevant security concerns). Similarly, with

respect to the oral part, some colleagues reported about maths students who put together improvised paper boards and took the exam standing in front of the webcam while writing on these boards. You can probably expect such inventiveness in small classes and from maths students, we did not find it in our big class of prospective primary school teachers.
Our solution with digital whiteboards usable with a standard mouse turned out to be quite satisfactory, but we are aware that it was convenient because the exam focused on geometry. E.g., an exam about arithmetic would have required more handwriting.
The main disadvantage in our setup is the fact these WIMS-GeoGebra whiteboards are not sharable. Only the student can operate on them, and thus they do not provide an environment fully suitable for collaborative work. Time is required to design a better whiteboard facility embeddable in one of our digital environments.
Overall, at the end of the course we could observe a very good level of ICT competencies in the students.

References

[1] M. Cazzola, “Play with maths: Mathematical games through WIMS,” in *ICERI2019 Proceedings*, 12th International Conference of Education, Research and Innovation, pp. 2297–2305, IATED, 11th-13th November 2019.

[2] G. Xiao, “WIMS: An Interactive Mathematics Server,” *Journal of Online Mathematics and its Applications*, vol. 1, no. 1, 2001. Retrieved December 9, 2019, from https://www.maa.org/press/periodicals/loci/joma/wims-an-interactive-mathematics-server.

[3] B. Perrin-Riou, S. Lemaire, and M. Cazzola, “WIMS, a community of teachers, developers and users,” *Notices of the AMS*, vol. to appear, 2020.

[4] M. Cazzola, “WIMS all’Università di Milano-Bicocca,” *TD Tecnologie Didattiche*, vol. 19, no. 3, pp. 170–175, 2011.

[5] F. Guerimand, *WIMS Guide de l’utilisateur*. Les Editions ARCHIMEDE, 2004.

[6] M. Kobylanski, “WIMS: Innovative pedagogy with 21 year old interactive exercise software,” in *Technology in Mathematics Teaching* (A. G. and T. J., eds.), pp. 123–144, Springer, 2019.

[7] M. Cazzola, “Problem-based learning and teacher training in mathematics: How to design a math laboratory,” in *INTED2018 Proceedings*, 12th International Technology, Education and Development Conference, pp. 9038–9043, IATED, 5th-7th March 2018.

[8] M. Cazzola, “Problem-based learning and mathematics: rethinking laboratory at the times of crisis,” in *ICERI2020 Proceedings*, 13th International Conference of Education, Research and Innovation, IATED, 9th-10th November 2020.

[9] M. Cazzola, “Matematica per scienze della formazione primaria: esercizi e complementi.” https://wims.matapp.unimib.it/wims/wims.cgi?lang=it&+module=home&+user=anonymous,1368,anonymous, 2018. WIMS Open Class.

[10] F. Guerimand, “OEF fonctions affines et droites.” https://wims.matapp.unimib.it/wims/wims.cgi?module=H3/analysis/oefaffif.fr, 2004. WIMS learning object.

[11] M. Cazzola, “OEF definizioni.” https://wims.matapp.unimib.it/wims/wims.cgi?module=devel/marina/U2~geometry~oefdefin.it, 2020. WIMS learning object (in development).

[12] M. Cazzola, “OEF polygons on graph paper.” https://wims.matapp.unimib.it/wims/wims.cgi?module=E4/geometry/oefpolyqq.it, 2019. WIMS learning object.