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Learning in a double phase cobweb model

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Abstract In this paper we study a class of markets, among which we can mention agricultural and energy markets, characterized by seasonality, i.e. in which demand and/or supply conditions cyclically alternate with a precise and known periodicity. We propose a new theoretical framework based on a cobweb model with adaptive expectations, accordingly modified to be consistent with market's seasonality. The model, consisting in a second order non-autonomous difference equation, is investigated with the aim of understanding how the periodical nature of the market together with the agents' expectation formation mechanism affect the resulting dynamics. We analytically prove the emergence of dynamical scenarios that are missing in the classic cobweb model for non-seasonal markets, such as quasi-periodic dynamics and an ambiguous role on stability of the expectation weight. Finally, we discuss their economic rationale with the help of numerical simulations. In such a peculiar economic framework, agents' learning plays a key role to explain the dynamical properties of economic observables.

Keywords: Double phase market, expectations, cobweb model, time dependent demand/supply functions, stability, complex dynamics.
JEL classification: C62, C63, Q41, Q02

The development of the present work benefits from the invaluable research carried on by Gian Italo Bischi over the years, which gave impulse to both research about economic modelling and dynamical systems. The link of the present work to his research activity is twofold. Firstly, as in his earlier contributions [7, 5] about dynamical analysis of an economic problem, we deal with the study of the emergence of complex phenomena in a cobweb model. Moreover, we aim at providing a feasible, boundedly rational mechanism that describes the agents' learning. This topic has been central in the research by Gian Italo, as remarked by his studies on gradient based rule of thumbs [6], on the role of expectations on dynamics [3], on oligopolies with firms having incomplete information about the demand function [8], on the role of memory [1, 2, 4].

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1 Introduction

The supply and/or consumption patterns characterizing some classes of goods are affected by particular time at which they are produced and/or used, with the consequent outcome of recurrent price fluctuations that follows a broadly predictable sequence. This peculiarity is referred as seasonality. The provision of agricultural goods is a clear example of such a situation. Crop production has to be planned in advance and its harvest usually takes place in a single, specific season, giving rise to the well-know "harvest lows" and "post-harvest rally" price behavior (see e.g. [27,29]). Another example consists of energy goods, in particular electricity, whose consumption changes depending on the year period, day of the week or even the hour of the day, and whose production can be in part affected by time as well (e.g. solar energy). In energy markets, being able to understand the seasonal effects is fundamental to forecast price behavior ([26,22,20]). Finally, among other relevant examples of goods whose supply and/or consumption is affected by seasonality, we can mention clothing, toys and food ([31,17,28]). It's worth mentioning that such markets are often characterized by peculiar distributions in economic observables[1].

In seasonal markets demand and/or supply curves change over time with an underlying deterministic pattern, according to the daily, weekly, monthly recurrence of consumption and production. Therefore, qualitatively identical couples of demand/supply curves arise with cyclical regularity, giving rise to a characteristic and to a large extent predictable seasonal pattern[2] in the resulting price series. Indeed, suppliers are aware of the market seasonality and they take their production decision accordingly.

Given the relevance of the previous class of markets, the aim of the present research is to develop a theoretical model for the study of the evolution of economic observables, like prices, quantities, in a prototypical market in which demand and supply functions are affected by seasonality. To this end, we focus on the simplest kind of cyclicity characterized by period 2. The general market under consideration is unique (as well as the exchanged good), but it is structured as a sequence of cyclically alternating *phases*, each one characterized by a couple of demand/supply functions. For such reason, the considered class of seasonal markets can be addressed as *double phase* markets, in opposition to the classic framework to which we refer as *single phase* market. The most suitable setting to describe the previous class of markets is the competitive one[3], so the methodological approach we pursue relies on the cobweb framework[4].

[1] Just as an example, times series of electricity prices are characterized by an elevated volatility, spikes, with returns' distributions that show a strong, leptokurtic deviation from normality (see e.g. [23,9]).

[2] Indeed, stochastic fluctuations may also superimpose to such a cyclical behavior, arising from nondeterministic shocks that affect the demand and/or supply side.

[3] Agricultural markets are the typical competitive market example provided in microeconomics courses, as well as energy markets have been liberalized in the last twenty years.

[4] For an introduction and survey on cobweb models we refer to [19]. A first attempt to describe seasonal markets through a cobweb model is proposed in [13], while the effect of demand seasonality in a monopolistic market model is studied in [12].

A key problem in the theoretical description of markets lies in modelling the way agents make production decisions on the basis of their information endowment about the economic environment, i.e. the way they form expectations about prices. The expectation formation mechanisms proposed in the literature are not suitable for a double phase market, as they are not shaped to take into account seasonality, so the first problem we address is that of agents' learning in a double phase framework. Perfect rationality assumption is not suitable for such frameworks, due to the intrinsic complexity of the economic environment and the inability of such a hypothesis to give explanation of the phenomena characterizing economic variables' dynamics. Grounding on the classic adaptive expectations, we introduce a modified adjustment mechanism in which the agents form their expectations learning from the last two periods, i.e., in a double phase framework, from a whole sequence of market phases. The adjustment of the previous expected price is then regulated by an expectation weight, representing, as in a single phase cobweb model, the relevance the agents give to the expectation error. From the mathematical point of view, the resulting model is essentially different from a cobweb model for a single phase market and consists of a non-autonomous difference equation[5].

Due to the novelty, a relevant part of this contribution is devoted to the study of analytical properties of the model, to compare it with the classic single phase cobweb model. The resulting framework exhibits much more elements of complexity and ambiguity than that classic, in which instability can just arise by means of a period doubling bifurcation and the expectation weight has a destabilizing effect. Conversely, in a double phase cobweb model, both periodic, chaotic and quasi-periodic dynamics can arise, even for a given market configuration, and the expectation weight can also have a stabilizing role. Moreover, we show that when agents form their expectations on the basis of errors related to both market phases, they can be able to learn how to correct erratic price dynamics characterizing each phase. The main drivers of the emergence of new phenomena are discussed, both from the dynamical and economic points of view, with the help of numerical simulations.

The remainder of the paper is organized as follows. In Section 2 we introduce the double phase cobweb model, which is then studied from the analytical point of view in Section 3. The dynamical and economic rationale of the results are discussed in Section 4. Conclusions and future perspectives are collected in Section 5. Proofs of Propositions can be found in Appendix.

[5] It is well known that if agents takes into account in their expectation formation mechanism several previously realized prices, then the resulting difference equation is non-autonomous [11,1], and this has been already applied to cobweb models [10,24,5,16], too. However, in such literature the non-autonomous nature of the resulting equation is due to a refinement of the expectation formation mechanism, while the economic framework under consideration is left unchanged. In the present contribution, it is a consequence of the market seasonality, and hence it is ascribable to peculiar structure of the economic environment, which in turn affects the expectation formation mechanism. Moreover, in the present model, the last two periods taken into account are characterized by a different couple of demand/supply functions.

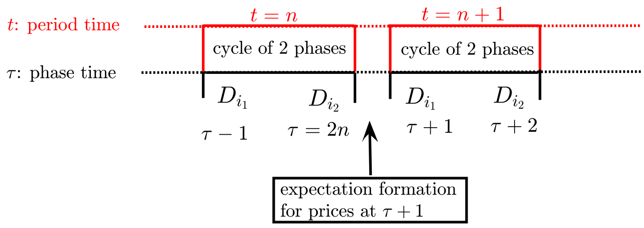

Fig. 1 Time levels of the double phase cobweb model. Black color: Phase-time level τ. At τ the market is characterized by different demand (D_i) and supply (S_i) functions. Red color: period-time t. Each period time t collects a whole cycle of 2 consecutive phase-times.

2 Double phase cobweb model

We consider a family of markets in which the unique traded good is characterized by consumption and/or production that vary depending on the particular time at which the good is exchanged, with a deterministic cyclical recurrence (seasonality) with period 2. We assume that exchanges occur at discrete times $\tau \in \mathbb{N}$. We then have a sequence of *market phases*, which, thanks to the assumption of deterministic cyclicity of period 2, can be represented through a sequence of couples of demand/supply functions (D_1, S_1) and (D_2, S_2), each related to a particular market phase. We assume that demand functions $D_i : I \to \mathbb{R}^+$, $i = 1, 2$, where I is a suitable interval, are smooth and decreasing functions. Similarly, we assume that supply functions $S_i : J \to \mathbb{R}^+$, $i = 1, 2$, where J is a suitable interval, are smooth increasing functions[6]. We remark that the shapes of both demand and supply functions, as well as their domains I and J, may depend on the institutional characteristics of the particular market under consideration. We only assume that $S_i(J) \subset D_i(I)$ and that each function D_i always has one intersection with the corresponding function S_i, for $i = 1, 2$.

We refer to the time level identified by τ as the *phase-time* level, as it consists of a sequence of market phases. The phase-time level is graphically sketched in Figure 1 using black color. Without loss of generality, we can assume that when τ is odd (respectively even), the market is characterized by demand/supply functions D_1 and S_1, (resp. D_2 and S_2). We then have that two phase-times $\tau_a > \tau_b$ are *in-phase* (i.e. share the same demand/supply functions couple) when $\tau_a = \tau_b + 2n$ (i.e. when a and b are both even or odd) for some $n \in \mathbb{N}$, while, otherwise, they are *out-of-phase* (i.e. they are characterized by different demand/supply functions couples).

In addition to the phase-time level, it is possible to introduce another time perspective at which study a double phase market, in which at each time

[6] For the sake of simplicity, in this section we assume that all the demand (resp. supply) functions share the same domain, but such assumption is not essential and can be easily removed. Moreover, note that the proposed setting also encompasses the situations in which either or both the demand function is constant in time (i.e. $D_1 \equiv D_2$) or/and the supply function is constant in time (i.e. $S_1 \equiv S_2$).

$t \in \mathbb{N}$ a sequence of two consecutive couples of demand/supply functions is simultaneously considered. We refer to this time level t as *period-time level*, in which each *period-time* t collects a whole period of phase-times[7]. The period-time level, superimposed to the phase time level, is sketched in Figure 1 in red color. To give a concrete example, if phase time represents the sequence of daytimes and nights (each daytime is followed by a night, which is in turn followed by a daytime and cyclically so on), each period time represents a whole day (which is followed by another whole day), which is characterized in terms of a daytime-night couple.

The market demand and supply functions for a double-phase market can be respectively described by introducing a unique couple of time periodic functions $D : I \times \mathbb{N} \to \mathbb{R}^+$ and $S : J \times \mathbb{N} \to \mathbb{R}^+$ defined by

$$D(p_\tau, \tau) = \begin{cases} D_1(p_\tau) & \text{if } \tau \text{ is odd,} \\ D_2(p_\tau) & \text{if } \tau \text{ is even,} \end{cases} \qquad S(p_\tau, \tau) = \begin{cases} S_1(p_\tau) & \text{if } \tau \text{ is odd,} \\ S_2(p_\tau) & \text{if } \tau \text{ is even,} \end{cases} \tag{1}$$

where p_τ is the market price at τ. At each phase-time τ, the demanded and supplied quantities are respectively $q^D_\tau = D(p_\tau, \tau)$ and $q^S_\tau = S(\pi_\tau, \tau)$, where π_τ is the price that agents expect for phase-time τ.

To complete the cobweb model we need to detail the expectation formation mechanism. The further complexity of double-phase markets with respect to those single-phase makes clear that it is almost impossible for the agents to know each aspect of the market and to perfectly foresee its future evolution, so it is more appropriate to consider a boundedly rational expectation formation mechanism for the agents. However, the agents are indeed aware of the demand and supply seasonality, and so they know that phase-time $\tau+1$ will be in phase with phase-time $\tau - 1$ (and not with τ).

The boundedly rational mechanism we propose is grounded on the adaptive expectations [25, 14, 18, 19] of the classic single phase cobweb framework, for which the next period expected price is adapted from the last period one on the basis of the last expectation error, i.e. the difference between the last expected and realized price. In a double-phase setting, information come from both in-phase and out-of-phase past market realizations. Even if last in-phase information is indeed the most significant (as demand/supply functions at time $\tau + 1$ will be the same as those at phase $\tau - 1$, and not as those at time τ), last out-of-phase prices provide the latest price information, which might signal particular demand/supply conditions that are expected to last for some times τ[8]. We then assume that expected price $\pi_{\tau+1}$ is formed anchoring to the previous in-phase expected price $\pi_{\tau-1}$ and adapting it on the basis of the past

[7] Phase-time and period-time levels are indeed linked and we can unambiguously move from τ to t and vice-versa. The ith phase of period-time t corresponds to phase-time $\tau = 2(t-1) + i$. Conversely, from phase-time τ, we can unequivocally obtain the corresponding period-time $t = \lfloor(\tau - 1)/2\rfloor + 1$ and phase $i = \tau - 2(t - 1)$, where $\lfloor z\rfloor$ stands for largest integer not greater than z.

[8] Such a possibility is not just a merely theoretical chance. Successfully attempts to provide, through econometric approaches, predictive techniques for the price dynamics in multiphase markets (like those pursued for instance in [20, 21]) make use of data coming from

expectation errors. In particular, agents can take into account both in-phase, $p_{\tau-1}-\pi_{\tau-1}$, and out-of-phase $p_\tau-\pi_\tau$ expectation errors. The resulting double phase adaptive expectation formation mechanism is then

$$\pi_{\tau+1}=\pi_{\tau-1}+\omega\left[\nu(p_{\tau-1}-\pi_{\tau-1})+(1-\nu)(p_\tau-\pi_\tau)\right], \tag{2}$$

where $\omega\in(0,1]$ is the expectation weight and $0\le\nu\le 1$ is the *phase-weight*, which specifies the relevance given by the agents to phase errors. We stress that as ω increases, more relevance is given to expectation errors, while, as it decreases, the anchoring bias to the previous in-phase expected price becomes more significant. Since in-phase error $p_{\tau+1}-\pi_{\tau+1}$ is the most significant for the determination of $\pi_{\tau+1}$, we assume that $\nu>1-\nu$, i.e. $\nu>1/2$. We underline that phase-weight ν is (inversely) related to the degree of coupling of the different market phases, which is null when $\nu=1$ (since no relevance is given to out-of-phase errors) and maximum when $\nu\approx 1/2$ (since in-phase and out-of-phase error approximatively have the same relevance). In the former case, phases are independent, in the sense that the expectation errors at odd phase times have no influence on expected prices for even times, and vice-versa. If $\nu<1$ but still close to 1 ($\nu\lesssim 1$), we have that out-of-phase errors have a small influence on π_{t+1}, and so in this case we can speak of a weak coupling of phases. As ν decreases, out-of-phase errors becomes more and more relevant, and the coupling degree increases. Finally, we note that for $\omega=0$, as in classic adaptive expectations, we would have no dynamical adjustment, as $\pi_{\tau+1}=\pi_{\tau-1}$.

Imposing temporary equilibrium condition $D(p_\tau,\tau)=S(\pi_\tau,\tau)$, we obtain $p_\tau=D^{-1}(S(\pi_\tau,\tau),\tau)$, where $D^{-1}(q_\tau,\tau)$ is

$$D^{-1}(q_\tau,\tau)=\begin{cases}D_1^{-1}(q_\tau) & \tau \text{ is odd,}\\ D_2^{-1}(q_\tau) & \tau \text{ is even.}\end{cases}$$

Combining (2) and temporary equilibrium condition we obtain the double phase cobweb model for the phase-time level, represented by the second order non-autonomous difference equation

$$\pi_{\tau+1}=\pi_{\tau-1}+\nu\omega\Big(D^{-1}(S(\pi_{\tau-1},\tau-1),\tau-1)-\pi_{\tau-1}\Big)+(1-\nu)\omega\Big(D^{-1}(S(\pi_\tau,\tau),\tau)-\pi_\tau\Big), \tag{3}$$

for given initial expected prices π_0 and π_{-1}. We stress that the non-autonomous nature of the present model is intrinsically connected with the seasonality of demand/supply functions characterizing the market itself, which in turn induces the peculiar form of adaptive expectations.

In general, the non-autonomous equation (3) does not possess a steady state, because of the cyclic nature of the demand and supply functions. In single phase markets, a temporary equilibrium is a steady state when it clears

both in-phase and out-of-phase market realizations. The effectiveness of such approaches is a hint of the fact that agents, in order to make their decisions, really take into account prices of different market phases.

the market and it is constant in time. In this sense, we can speak of a steady state equilibrium. For a double phase market, assuming that phase-time τ corresponds to the ith phase of the market, we can only require that p_τ is a market clearing price for phase i (i.e. $D_i(p_\tau) = S_i(p_\tau)$) and that $p_{\tau+2n} = p_\tau$ for any $n \in \mathbb{N}$. From a dynamical viewpoint, this corresponds to a steady cycle[9] of period 2.

To be able to introduce a definition of steady state equilibrium for double phase markets, we need to consider model (3) at the period-time level t (see the upper part of Figure 1), at which we study the evolution of vectors $\boldsymbol{\pi}_t \in \mathbb{R}^2$, consisting of a whole cycle of 2 expected prices. From $\boldsymbol{\pi}_t$ we can then obtain the corresponding vectors of realized prices and of traded quantities. From the previous considerations and assumptions on functions D_i and S_i, we have that there exists a unique vector $\mathbf{p}^* = (p^{1,*}, p^{2,*})$ that realizes

$$D(p^{i,*}, i) = S(p^{i,*}, i),\ i = 1, 2. \tag{4}$$

This allows introducing the following definition.

Definition 1 We say that $\mathbf{p}^*$ is a period steady state equilibrium if it fulfills (4). We define each $p^{i,*}$ as the ith phase steady state equilibrium.

The dual time representation of double phase markets in terms of phase-time and period-time is intimately related to the representation of the proposed model in terms of a non-autonomous and autonomous dynamical system. Non-autonomous dynamical equation (3) was obtained at the phase-time level τ; to rewrite it at the period-time level t, we consider functions $\varepsilon_i : J \to \mathbb{R}$ for $i = 1, 2$ defined by $\varepsilon_i(x) = \left(D_i^{-1}(S_i(x)) - x\right),\ i = 1, 2$, which allows introducing the autonomous dynamical system

$$\begin{cases} \pi^1_{t+1} = & F^1(\pi^1_t, \pi^2_t) = \pi^1_t + \nu\omega\varepsilon_1(\pi^1_t) + \omega(1-\nu)\varepsilon_2(\pi^2_t), \\ \pi^2_{t+1} = & F^2(\pi^1_t, \pi^2_t) = \pi^2_t + \nu\omega\varepsilon_2(\pi^2_t) + \omega(1-\nu)\varepsilon_1(F^1(\pi^1_t, \pi^2_t)), \end{cases} \tag{5}$$

where functions $F^i : J^2 \to \mathbb{R}, i = 1, 2$ are defined by the right hand sides of the previous equations. The autonomous system (5) can be rewritten in the compact vector form as $\boldsymbol{\pi}_{t+1} = \mathbf{F}(\boldsymbol{\pi}_t)$ where, setting $\boldsymbol{\pi}_t = (\pi^1_t, \pi^2_t)$ for $t \geq 0$, function $\mathbf{F} : J^2 \to \mathbb{R}^2$ is defined by $F(\boldsymbol{\pi}_t) = (F_1(\pi^1_t, \pi^2_t), F_2(\pi^1_t, \pi^2_t))$. The way the autonomous system (5) is linked to the non-autonomous equation (3) is clarified by the following Proposition.

Proposition 1 *If π_{-1}, π_0 are initial data of the non-autonomous equation* (3) *and we take $(\pi^1_0, \pi^2_0) = (\pi_{-1}, \pi_0)$ as the initial datum of the autonomous system* (5)*, then $\pi^i_t = \pi_{2(t-1)+1}$ for any $t \geq 0$.*

We recall that the existence of two different time levels is a consequence of the double phase framework. However, it is also significantly connected the mathematical characterization of the model. In fact, the model is introduced

[9] We highlight that, due to the double phase nature of the market, classic adaptive expectations $\pi_{\tau+1} = \pi_\tau + \omega(p_\tau - \pi_\tau)$ are not consistent with such a "cyclical" steady state.

in a more straightforward way at the phase-time level but it can be more easily studied at the period-time level (e.g. its stability can be suitably investigated by means of the wide literature concerning autonomous systems). Expectation formation mechanism (2), which is grounded on the sequence of market phases, is more naturally introduced at the phase-time level. In (5), each element π_t^i of vector $\boldsymbol{\pi}_{t+1}$ represents the expected price of a distinct phase, and consequently, each equation of System (5) describes the evolution of single phase's prices.

We note that System (5) consists of 2 coupled equations, in general depending on some or all the components π^i of vector $\boldsymbol{\pi}$. However, in the extreme case of $\nu = 1$, expectation mechanism (2) simplifies into

$$\pi_{\tau+1} = \pi_{\tau-1} + \omega(p_{\tau-1} - \pi_{\tau-1}), \tag{6}$$

and model (5) reduces to the diagonal system

$$\pi_{t+1}^i = \pi_t^i + \omega\left(D_i^{-1}(S_i(\pi_t^i)) - \pi_t^i\right), \qquad i = 1, 2. \tag{7}$$

In this case (6) is close to the classic adaptive expectation formation mechanism and each $\pi_{\tau+1}^i$ in (7) only depends on the in-phase previous expected price, so we actually have 2 distinct, independent equations (which provide independent dynamics). In this case the model can then be assimilated to 2 independent classic single phase cobweb models with adaptive expectations. For this reason, in what follows we consider the case of $\nu = 1$ as modeled by 2 independent equations (and not by a single diagonal system), while we consider all the remaining cases of $\nu \neq 1$ as represented by a single two-dimensional system. We will respectively refer to (5) for $\nu < 1$ and to (7) as *coupled* and *uncoupled* model and by saying uncoupled (respectively coupled) phases we will refer to each phase of the coupled (respectively uncoupled) model. We will be mainly interested in studying the behavior of the coupled model, while the uncoupled model will be treated as an intermediate situation between the classic single phase and the coupled double phase cobweb models.

3 Analysis of the model

We start studying the possible steady states of system (5) in the following proposition, which guarantees that the expectation formation mechanism preserves the steady state equilibrium.

Proposition 2 *The only*[10] *steady state of system* (5) *is the period steady state equilibrium* $\mathbf{p}^* = (p^{1,*}, p^{2,*})$.

Since the expectation mechanism (2) is a generalization of classic adaptive expectations, we can compare the resulting dynamics with those of the single phase setting and understand what effects are introduced when agents act in

[10] We note that for the previous result we need $\nu \neq 1/2$. If we allowed for $\nu = 1/2$ (namely, if we consider the arithmetic mean of errors), system (5) would have additional, spurious, steady states.

a double phase setting. From the stability results of the single phase cobweb model with adaptive expectations[11] we have that a steady state equilibrium p^* is locally asymptotically stable provided that

$$1-\frac{2}{\omega}<\frac{S'(p^*)}{D'(p^*)}=s\,(<1)\;\Leftrightarrow\;\omega>\bar{\omega}=\frac{2}{1-s}, \tag{8}$$

namely if, at p^*, the slope of the supply function is sufficiently small with respect to the absolute value of the slope of the demand function (the rightmost inequality in (8) is automatically fulfilled since S'/D' is negative). We recall that $|s|$ corresponds to the ratio between the elasticities of supply and of demand, while $\bar{\omega}$, when belonging to $(0,1)$, is the stability threshold on ω above which the equilibrium becomes unstable. Along the lines of (8), we can introduce the relative slopes of functions S_i with respect to D_i at $p^{i,*}$, defined by $s_i = S_i'(p^{i,*})/D_i'(p^{i,*}),\ i=1,2$, and thresholds $\bar{\omega}_i = 2/(1-s_i)\ i=1,2$. In what follows we will simply refer to each s_i as to relative slope or relative elasticity.

The remainder of this section is devoted to the study of the local stability of equilibrium with respect to expectation weight, phase weight and relative slopes. To describe the possible dynamical behaviors on varying a parameter ξ (which will be either ω,ν or s_i) inside an interval $I=(\xi_a,\xi_b)$[12] we introduce the following scenarios, which are sketched in Figure 2:
• flip (FD) and Neimark-Sacker (NSD) destabilizing scenarios, if there exists $\xi_1\in(\xi_a,\xi_b)$ such that $\mathbf{p}^*$ is stable on (ξ_a,ξ_1) and unstable for $\xi\in(\xi_1,\xi_b)$ and for $\xi=\xi_1$ stability is respectively lost through a flip and a Neimark-Sacker bifurcation;
• mixed scenario (M), if there exist $\xi_1<\xi_2$, with $\xi_i\in(\xi_a,\xi_b)$ such that $\mathbf{p}^*$ is stable on $(\xi_a,\xi_1)\cup(\xi_2,\xi_b)$ and unstable on (ξ_1,ξ_2);
• mixed-destabilizing scenario (MD), if there exist $\xi_1<\xi_2<\xi_3$, with $\xi_i\in(\xi_a,\xi_b), i=1,2,3$, such that $\mathbf{p}^*$ is stable on $(\xi_a,\xi_1)\cup(\xi_2,\xi_3)$ and unstable on $(\xi_1,\xi_2)\cup(\xi_3,\xi_b)$;
• unconditionally stable scenario (US), if $\mathbf{p}^*$ is stable on (ξ_a,ξ_b). Finally, we remark that, in the following propositions, we avoid to detail situations in which stability or instability occur only at the boundary of parameter sets, as they can be qualitatively encompassed into the other scenarios.

The role of ω and ν on the local asymptotic stability of $\mathbf{p}^*$ is studied in the following proposition.

Proposition 3 *Period steady state equilibrium* $\mathbf{p}^*$ *is locally asymptotically stable provided that*

$$2\omega\nu-(\bar{\omega}_1+\bar{\omega}_2)<0, \tag{9a}$$

$$(2\nu^2-2\nu+1)\omega^2-2\nu\frac{\bar{\omega}_1+\bar{\omega}_2}{2}\omega+\bar{\omega}_1\bar{\omega}_2>0. \tag{9b}$$

[11] For seminal results about the possible dynamical behaviors of linear and nonlinear cobweb models, we refer to [15,25,14,18,19].

[12] We describe scenarios for the case of an open interval (ξ_a,ξ_b); the same definitions can be easily adapted to include one or both extrema, too.

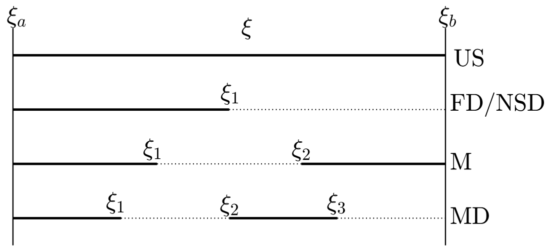

Fig. 2 Possible stability scenarios with respect to a parameter ξ. A solid (respectively dotted) line is used for stability (respectively instability) intervals.

If the steady state loses its stability through a Neimark-Sacker (resp. period doubling) bifurcation (see [30]), then condition (9a) *(respectively* (9b)*) is violated.*

The previous proposition is actually the generalization of the stability condition (8) to double phase markets. In agreement with the classic cobweb model, local stability depends on the expectation weight ω and on the relative slopes at the steady state. We note that stability conditions (9) are symmetric with respect to $\bar{\omega}_i$, i.e. with respect to the relative slopes s_i, so we can assume $|s_1| \geq |s_2|$.

Due to the linear (9a) and quadratic (9b) conditions in both ω and ν, we can have up to three stability thresholds. To obtain a relevant characterization of local stability on varying ω and ν, we proceed as follows. Assuming a fixed economic setting at the equilibrium (described by s_1, s_2), we investigate how increasing the degree of coupling between phases (namely, decreasing ν from $\nu = 1$ to $\nu \to 1/2$) affects the possible scenarios on varying the expectation weight. To foster understanding of the results, it's worth focusing on the very simple situation in which the two phases are uncoupled ($\nu = 1$). As we are going to show, the dynamical behavior of the coupled model is strongly influenced by that of the uncoupled one, so in the next Proposition we summarize the possible stability scenarios for the uncoupled model. The proof is omitted, since it can be easily inferred by (8).

Proposition 4 *Let* $\nu = 1$. *Then*

(I) *if* $\max_{i=1,2} |s_i| < 1$, *both phase steady state equilibria* $p^{i,*}$ *are stable for any* $\omega \in (0, 1]$*;*
(II) *if* $|s_1| = |s_2| > 1$, *both phase steady state equilibria* $p^{i,*}$ *are stable for* $\omega < \bar{\omega}_1 = \bar{\omega}_2$ *and unstable for* $\omega > \bar{\omega}_1 = \bar{\omega}_2$.
(III) *if* $|s_1| > 1 > |s_2|$, *both phase steady state equilibria* $p^{i,*}$ *are stable for* $\omega < \bar{\omega}_1$, *while for* $\omega > \bar{\omega}_1$ *we have that* $p^{1,*}$ *is unstable and* $p^{2,*}$ *is stable;*
(IV) *if* $|s_1| > |s_2| > 1$, *both phase steady state equilibria* $p^{i,*}$ *are stable for* $\omega < \bar{\omega}_1$, *while for* $\bar{\omega}_1 < \omega < \bar{\omega}_2$ *we have that* $p^{1,*}$ *is unstable and* $p^{2,*}$ *is stable and for* $\omega > \bar{\omega}_2$ *both* $p^{i,*}$ *are unstable.*

Stability is always lost through a flip bifurcation.

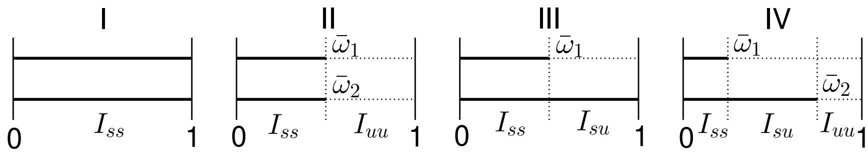

Fig. 3 Stability scenarios of the uncoupled model ($\nu = 1$). A solid (respectively dotted) horizontal line is used for stability (respectively instability) intervals. Vertical dotted lines subdivide $(0, 1)$ into intervals in which, for any ω, $p^{i,*}$ are both stable ($\omega \in I_{ss}$), $p^{1,*}$ is unstable and $p^{2,*}$ is stable ($\omega \in I_{su}$) and $p^{i,*}$ are both unstable ($\omega \in I_{uu}$).

The four situations described in Propositions 4 are depicted in Figure 3. Only when both $|s_i| < 1$ (case I) or $|s_1| = |s_2| > 1$ (case II) phase steady state equilibria $p^{1,*}$ and $p^{2,*}$ are either both locally stable or both unstable for each expectation weight. Conversely, if we have different relative slopes at each $p^{i,*}$ and $|s_1| > 1$, for some values of ω only one phase steady state equilibrium is stable.

What happens when agents, through their expectation mechanism, introduce a coupling between the dynamics of different phases? Firstly, due to the strongly coupled nature of the dynamical system (5), a whatever weak coupling ($\nu \lesssim 1$) causes $p^{i,*}$ to be necessarily either both locally asymptotically stable or both unstable. However, the phase coupling does not completely cancel out the dynamics of the uncoupled phases, from which, as we are going to show, it is still possible to infer and understand the behavior of the coupled model. Observing Figure 3, we can always subdivide interval $(0, 1]$ into three (possibly empty) subintervals:

- $I_{ss} = (0, \min\{\bar{\omega}_1, 1\})$, in which $p^{i,*}$ are both stable;
- $I_{su} = (\min\{\bar{\omega}_1, 1\}, \min\{\bar{\omega}_2, 1\})$, in which $p^{2,*}$ is stable while $p^{1,*}$ is unstable;
- $I_{uu} = (\min\{\bar{\omega}_2, 1\}, 1)$, in which $p^{i,*}$ are both unstable.

In the next Propositions we study the possible scenarios as ω increases on either $\omega \in I_{ss}, \omega \in I_{su}$ or $\omega \in I_{uu}$. We start considering the simplest situation, in which both phase steady stable equilibria are stable for $\nu = 1$.

Proposition 5 *Let $\omega \in I_{ss}$, then $\mathbf{p}^*$ is stable for any $\nu \in (1/2, 1)$.*

The previous proposition predictably says that coupling stable uncoupled dynamics we always obtain stable dynamics. When instead only one phase steady stable equilibrium is stable for $\nu = 1$, results become more articulated.

Proposition 6 *Let $\bar{\omega}_1 < 1$ and $\omega \in I_{su}$, then there exist $\nu_2 \leq \nu_1$, with $\nu_i \in [1/2, 1)$ depending on s_i, such that*

- *if $\nu \in (\nu_1, 1)$, we have a flip destabilizing scenario for $\omega \in I_{su}$;*
- *if $\nu \in (\nu_2, \nu_1)$, we have mixed scenario for $\omega \in I_{su}$;*
- *if $\nu \in (1/2, \nu_2)$, we have an unconditionally stable scenario for $\omega \in I_{su}$.*

Moreover, the set of values of ω for which the period steady state is locally asymptotically stable becomes increasingly large as ν decreases.

Before commenting Proposition 6, we investigate what happens when both uncoupled phase steady state equilibria are unstable.

Proposition 7 *Let $\bar{\omega}_2 < 1$ and $\omega \in I_{uu}$, then there exist $\nu_2 < \nu_1$, with $\nu_1 \in (1/2, 1]$ and $\nu_2 \in [1/2, 1)$ depending on s_i, such that*

- *if $\nu \in (\nu_1, 1)$, we have an unconditionally unstable scenario for $\omega \in I_{uu}$;*
- *if $\nu \in (\nu_2, \nu_1)$, a Neimark-Sacker destabilizing scenario occurs for $\omega \in I_{uu}$;*
- *if $\nu \in (1/2, \nu_2)$, we have an unconditionally stable scenario for $\omega \in I_{uu}$.*

In particular, we have that $\nu_1 = 1$ if and only if $|s_1| = |s_2|$. Moreover, the set of values of ω for which the period steady state is locally asymptotically stable grows as the coupling become increasingly stronger.

Propositions 6 and 7 deserve several comments. The possible stability loss through either a flip or a Neimark-Sacker bifurcation is uniquely determined by the dynamical behavior of the uncoupled model. Period steady state equilibrium loses stability at some $\bar{\omega}$ through a period doubling bifurcation only if, for that $\bar{\omega}$, $p^{i,*}$ are one stable and the other unstable for $\nu = 1$, while Neimark-Sacker bifurcation can occur at some $\bar{\omega}$ only if both $p^{i,*}$ are simultaneously unstable for that $\bar{\omega}$ in the uncoupled model. The other remarkable result is that the expectation weight can have a stabilizing effect, in the sense that increasing ω may lead dynamics from instability to stability. This is possible only when $\mathbf{p}^*$ loses stability through a period doubling bifurcation. Conversely, when stability is lost at some $\bar{\omega}$ through a Neimark-Sacker bifurcation, Proposition 7 shows that $\mathbf{p}^*$ can not be locally asymptotically stable for any $\omega > \bar{\omega}$. The emergence of different kinds of unstable dynamics and the possibly ambiguous role of the expectation weight are two of the most significant dynamical novelties introduced by the double phase framework. We will come back on the interpretation of such important results in Section 4.

Focusing on Proposition 6, we have that if the coupling degree is sufficiently small and $p^{1,*}$ and $p^{2,*}$ are respectively locally asymptotically unstable and stable in the uncoupled model, a flip bifurcation always occurs as ω varies in I_{su}. Accordingly to Propositions 6, as ν decreases, the flip destabilizing scenario can evolve in different ways, depending on the relative slopes $|s_i|$. Increasing the coupling degree we may have that $\mathbf{p}^*$ becomes unconditionally stable for any $\omega \in I_{su}$ or a mixed scenario can occur, with the consequent return to stability as ω increases. In this last case, suitably decreasing ν we may also have an unconditionally stable scenario as ω varies in I_{su}.

Concerning Proposition 7, the remarkable aspect is that when neither $p^{1,*}$ nor $p^{2,*}$ are locally asymptotically stable for $\nu = 1$, on varying $\omega \in I_{uu}$ a Neimark-Sacker bifurcation always occurs for a suitable coupling degree. In general, we also have that instability is preserved under too weak couplings, while, depending on $|s_i|$ and if the coupling is sufficiently strong, we may have an unconditionally stable scenario as ω varies in I_{uu}.

Juxtaposing the cases studied in Propositions 5, 6 and 7, we are able to obtain stability regions in (ω, ν) planes, for given slopes s_1 and s_2. We stress that as s_1 and s_2 change, stability regions change as well and some of the

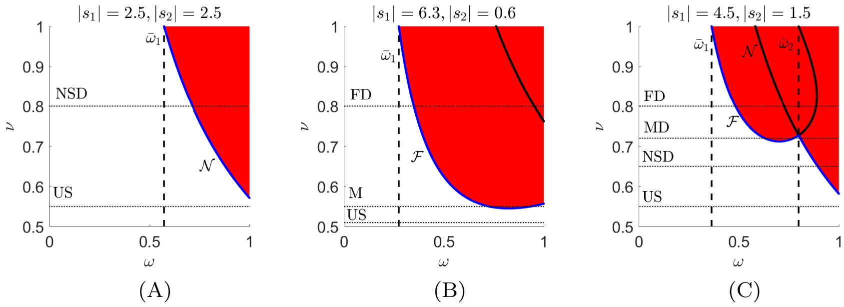

Fig. 4 Possible stability regions in the (ω, ν)-plane when $\bar{\omega}_1 = \bar{\omega}_2 < 1$ (panel A), when $\bar{\omega}_1 < 1 < \bar{\omega}_2$ (panel B) and when $\bar{\omega}_1 < \bar{\omega}_2 < 1$ (panel C). Red color is used for instability regions. Blue line $\mathcal{N}$ is the stability threshold curve, crossing which $\mathbf{p}^*$ loses stability through a Neimark-Sacker bifurcation. As a comparison, we plot a vertical dashed line representing the stability thresholds of the uncoupled ($\nu = 1$) model. Horizontal dotted lines show the possible scenarios on varying ω for fixed ν.

scenarios depicted in the next figures can disappear. In Figure 4 we always focus on slopes configurations that provide the maximum possible number of scenarios simultaneously occurring. For synthetic exposition of the results, we limit to a graphical representation, plotting in the (ω, ν)-plane different stability regions corresponding to cases (II), (III) and (IV), avoiding to depict the unconditionally stable case (I)[13].

In Figure 4 (A) we consider case (II), namely the very special situation of $|s_1| = |s_2| > 1$. In this case, for each $\omega \in (0, 1]$, $p^{i,*}$ are either both stable or both unstable for $\nu = 1$. Destabilization can only occur though a Neimark-Sacker bifurcation, and, for a suitably strong phase coupling, dynamics become unconditionally stable. In Figure 4 (B) we consider case (III), in which $|s_1| > 1 > |s_2|$. When $s_1 \neq s_2$, for a weak coupling the dynamics inherit instability of p^*_1 in the uncoupled model. As the coupling strength increases, the stability interval becomes larger, and can eventually coincide with $(0, 1]$. Finally, in Figure 4 (C) we consider case (IV), in which both $p^{i,*}$ become unstable but for different expectation weights ($\bar{\omega}_1 < \bar{\omega}_2$). For a sufficiently weak coupling, instability always occurs through a flip bifurcation, with the stability threshold that is increasingly close to $\bar{\omega}_1$ as $\nu \to 1$. Increasing the coupling strength, the flip destabilizing scenario is replaced by that mixed destabilizing, which is obtained putting side by side the mixed scenario for $\omega \in I_{su}$ and the Neimark-Sacker destabilizing scenario for $\omega \in I_{uu}$. In this case both stability and instability regions are unconnected. Further increasing the coupling degree, the mixed destabilizing scenario reduces to a Neimark-Sacker desta-

[13] We stress that for case (II) the possible scenarios are obtained juxtaposing the results of Propositions 5 and 7 (this last restricted to the particular case $|s_1| = |s_2|$), for case (III) possible scenarios come from Propositions 5 and Proposition 6, while those of case (IV) can be obtained juxtaposing "matching" scenarios resulting from Propositions 6 and 7 (for example it is not possible to have a flip destabilizing scenario for $\omega \in I_{su}$ followed by an unconditionally stable for $\omega \in I_{uu}$).

bilizing one. Finally we have a complete stabilization with respect to ω for sufficiently strong coupling degrees.

Before focusing on the role of the phase weight and the relative elasticity, we simply stress how the range of possible scenarios obtained on varying the expectation weight in a double phase cobweb model (i.e. unconditionally stable, flip destabilizing, Neimark-Sacker destabilizing, mixed and mixed-destabilizing scenarios) is by far wider than those obtained in a single phase cobweb model (i.e. unconditionally stable and flip destabilizing).

From the previous analysis, it is possible to infer the local stability behavior with respect to the phase weight. We can have up to three thresholds which can affect stability, two of them ($\nu_{F,i}, i = 1, 2$) coming from the solution of (9b) and the last one (ν_{NS}) from (9a). Depending on ω and s_i, by investigating the reciprocal positions of $\nu_{F,i}$ and ν_{NS} as well as their belonging to $(1/2, 1)$, it is possible to show the scenarios arising on varying ν, which can be inferred considering vertical sections of the stability regions reported in Figure 4. We stress that it is possible to prove that the portrayed situations cover all the possible behaviors with respect to ν. We just briefly summarize the possible stability scenarios in the following proposition, omitting the proof.

Proposition 8 *For suitable values of expectation weight $\omega \in (0, 1]$ and relative slopes at the phase steady state equilibria, on varying ν we can have an unconditionally stable, flip destabilizing, Neimark-Sacker destabilizing, unconditionally unstable scenarios.*

Proposition 8 says that, keeping the remaining parameters fixed, increasing ν has in general a destabilizing effect. This is in agreement with what suggested by the previous comments from which, recalling the results of Propositions 6 and 7, the stability regions with respect to ω becomes larger as ν decreases.

Apart from unconditional scenarios, increasing ν has the sole effect of introducing instability in the dynamics, which can occur by either flip or Neimark-Sacker bifurcations. In particular, on varying the phase-weight, no mixed scenario is possible. The dual possible route toward instability is determined by the joint effect of ω and s_i. Firstly, as noticeable looking at vertical sections of Figure 4 (B), when we are in case (III) of Proposition 4, only unconditionally stable/unstable and flip destabilizing scenarios with respect to ν can occur. Conversely, when we are in case (IV), all the scenarios predicted by Proposition 8 can arise (see Figure 4 (C)). However, in both cases, keeping s_i fixed but considering different the expectation weights, stability with respect to ν can exhibit a quite ambiguous behavior. To this end, we can focus on the stability regions reported in Figure 4 (C), considering a phase-weight ν which is slightly smaller than that at the intersection between stability curves $\mathcal{N}$ and $\mathcal{F}$ (e.g., in the first plot, for $\nu \approx 0.72$). If the expectation weight is close to $\bar{\omega}_2$, a slight perturbation on it can lead instability to either occur through either a flip (for $\omega \lesssim \bar{\omega}_2$) or a Neimark-Sacker ($\omega \gtrsim \bar{\omega}_2$) bifurcation.

In the remainder of this Section we briefly turn our attention to the role of the relative slopes. Firstly, we note that, for a single phase cobweb with adaptive expectations, we may rewrite the stability condition as $-1/\omega <$

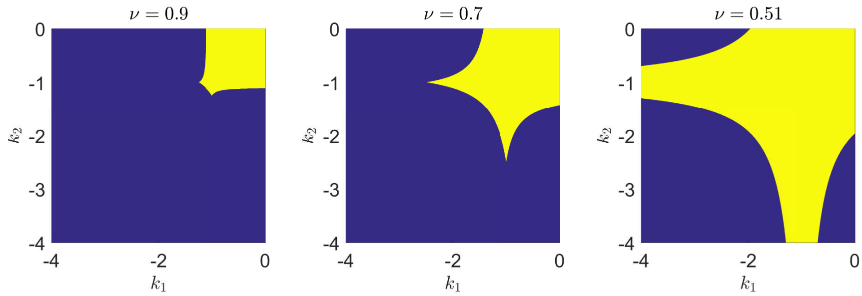

Fig. 5 Stability regions (yellow) with respect to k_i for different values of ν.

$1/2\left(S'(p^*)/D'(p^*)-1\right)$. Mimicking the right hand side of the previous inequality, we can introduce $k_i = 1/(2(s_i-1)), i=1,2$ and rewrite stability conditions (9a) and (9b) as

$$\begin{cases}(k_1+k_2)+2\nu\omega k_1k_2<0,\\ \omega k_1k_2(2\nu^2-2\nu+1)+\nu(k_1+k_2)+1>0,\end{cases} \tag{10}$$

whose graphical solution in (k_1,k_2)-plane for $\omega=1$ is reported in Figure 5 for some values of ν[14] . As we can see, the stability region becomes larger as ν decreases, in agreement with the previous results about ν.

4 Discussion of the results

In this section we investigate the conclusions of the analytical investigations of Section 3 from dynamical and economic perspectives, focusing on the most relevant deviations from the results obtained in the classical cobweb framework. In particular, with the help of numerical experiments, we aim at providing an explanation of the following new facts:

a) stabilization can be possible if the agents form their expectations suitably taking into account out-of-phase price dynamics;
b) dynamics arising when phases are uncoupled can significantly change when the agents form their expectations learning from both phases; both periodic, chaotic and quasi-periodic dynamics can emerge, even for the same given market configuration;
c) a more cautious updating of expected prices by the agents can be the source of instability in the dynamics.

Accordingly to the theoretical analysis of Section 3, all the previous evidences occur for different market's configuration and agents' behavior settings. In what follows, we focus on just a few scenarios, in order to put in evidence the economic driving forces that are responsible of the new dynamical phenomena.

[14] We remark that it is possible to show that the region defined by (10) is bounded for any $\nu \in (1/2,1)$, even if it becomes increasingly large as $\nu \to 1/2$.

All the remaining situations can be explained adopting similar arguments.

We recall that the emergence of instability in the double phase setting can be ascribed to two sources, being related either to the market configuration (encompassed in relative slopes s_i) or to the agents' behavior (encompassed in expectation weight ω and phase weight ν). The former one is the unique possible source of instability in a classic cobweb model with static expectations, as adaptive expectations can just lead unstable dynamics to become eventually stable as the expectation weight decreases[15].

Concerning the role of the market constituents, we stress that the market outcome p_τ, i.e. the price determined by temporary equilibrium condition, lies above (resp. below) the equilibrium price $p^{i,*}$ of the corresponding phase if the expected price for phase time τ is below (resp. above) $p^{i,*}$. If the relative elasticity of demand with respect to supply function is small, p_τ is closer to $p^{i,*}$ than the expected price for phase time τ, while, conversely, in the presence of a large relative elasticity, imposing temporary equilibrium condition, price p_τ would be farther to $p^{i,*}$ than expected price for phase time τ. In a single phase cobweb model, if the agents adopt static expectations, p_τ is assumed as the next period expected price, while under adaptive expectations, the previous expected price is just partially adjusted toward p_τ, and its relevance is softened as the expectation weight decreases. In a double phase framework, such two mechanisms act exactly in the same way in each phase, but the agents, making use of information coming from different phases, foster the emergence of scenarios that are completely different from those obtained in uncoupled phases.

In what follows, we focus on a specific example in which, without loss of generality, we encompass seasonality only in the demand function, setting $S(\pi) = S_1(\pi) = S_2(\pi)$. We consider the same demand and supply function shapes used in [18,19], namely $S(\pi) = b + \tanh(\lambda(\pi - c))$, where $b \geq 1, \lambda > 0$ and $c \geq 0$, and $D_i(p) = a_i - d_i p$, $i = 1, 2$, where both a_i and d_i are strictly positive. Moreover, in all simulations we set $b = 1, c = 6$ and $\lambda = 10$.

Even if the results are more effectively represented at period time level t, their explanation is more evident when investigated at the corresponding phase time level τ. To help in this, we report the expected price adjustment equations for a couple of consecutive phase times, assuming, without loss of generality, that $\tau + 1$ is odd. Setting $g_i(x) = D_i^{-1}(S_i(x))$ we have

$$\begin{aligned} \pi_{\tau+1} = \pi_t^1 &= \pi_{\tau-1} + \nu\omega\Big(g_1(\pi_{\tau-1}) - \pi_{\tau-1}\Big) + (1-\nu)\omega\Big(g_2(\pi_\tau) - \pi_\tau\Big) \\ \pi_{\tau+2} = \pi_t^2 &= \pi_\tau + \nu\omega\Big(g_2(\pi_\tau) - \pi_\tau\Big) + (1-\nu)\omega\Big(g_1(\pi_{\tau+1}) - \pi_{\tau+1}\Big). \end{aligned} \tag{11}$$

In Figure 6 we report a first family of simulations that are closely related to the stability scenario of Figure 4 (B). We obtained them setting $a_1 = 8.15, a_2 =$

[15] In [18] it was shown that in a nonlinear (single phase) framework, chaos can emerge as ω decreases, in a framework in which static expectations lead dynamics toward a period-2 cycle. However, the equilibrium is constantly unstable for the involved values of ω and such "qualitative" stabilization is strictly related to the particular shape of demand and supply functions, and not just to the cobweb model with adaptive expectations itself.

7.55 and $d_1 = d_2 = 1$, from which, by numerical estimation, the period steady state equilibrium results $(\pi^{1,*}, \pi^{2,*}) \approx (6.1920, 6.0544)$, with equilibrium relative elasticities given by $|s_1| \approx 7.5$ and $|s_2| \approx 0.82$. The two-dimensional bifurcation diagram[16] casts a first glance on the dynamical behavior as the expectation weight and the coupling among phases increase. Since in the present setting we have $|s_2| < 1 < |s_1|$, just the elements characterizing the market at odd phase times are possible sources of instability. This is evident looking at the couple of bifurcation diagrams in Figure 6 (B), obtained on increasing the expectation weight and in which phases are uncoupled ($\nu = 1$). The black diagram shows the unconditional stability of $\pi^{2,*}$, while the blue bifurcation diagram resembles that in [18], with a stability loss occurring through a flip bifurcation, leading to chaotic dynamics that qualitatively simplify into a period-2 cycle for sufficiently large values of ω, when adaptive expectations become close to those static ($\omega = 1$). We note that when the agents form their expectation on the basis of information coming from both market phases, the period-2 cycle for $\omega \approx 1$ can be replaced by more complex dynamical behaviors[17]. As ν decreases, we have a firstly partial and then complete stabilization of dynamics, with respectively a mixed and unconditionally stable scenarios on varying ω.

To address the element of novelty we reported at point (a) at the beginning of this section, we focus on the role of ν, so we set $\omega = 0.5$ and we look at the bifurcation diagrams reported in Figure 6 (C), studying them as ν decreases. In this scenario, the relevance given by the agents to expectation errors is kept fixed, while they form expectations taking into account more and more out-of-phase price information as ν decreases. We can observe a progressive stabilization of price dynamics at odd phase times (blue diagram), while prices π_t^2 (black diagram) undergo an initial increase in oscillations amplitude, which is then replaced by a decrease of them and finally by a gradual stabilization. This can be understood with the help of the sequence of time series reported in Figures 6 (D)-(I), obtained for decreasing values of ν. We recall that, at odd phase times, the relative elasticity s_1 is large, and this is the source of chaotic behavior of price π_t^1 when the two phases are uncoupled (blue line in Figure 6 (D)). Conversely, the small relative elasticity s_2 allows for quickly convergent price dynamics (black line). In the latter case, taking into account the information encompassed in the expectation error, the agents are able to gradually correct wrong price forecasts, while in the former situation erratic price trajectories last, sustained by market outcomes that are far from $\pi^{1,*}$ when the expected price is close to it.

Now we focus on what happens when agents try to learn from both in-phase

[16] In two-dimensional bifurcation diagrams, for each different combination of parameters, we ran a simulation with initial datum suitably close to the equilibrium values and we depicted the corresponding point on (ω, ν) plane using a color that represents the number of points of which the reached attractor consists, for variable π_t^1 (e.g. white color points out convergence toward $\pi^{1,*}$, red color toward a period-2 cycle, while cyan color toward an attractor consisting of more than 32 points).

[17] From Figure 6 (A), when $\nu \approx 0.85$, for suitably large values of ω the attractor consists of more than 32 points (cyan region), pointing out possible chaotic or quasi-periodic dynamics.

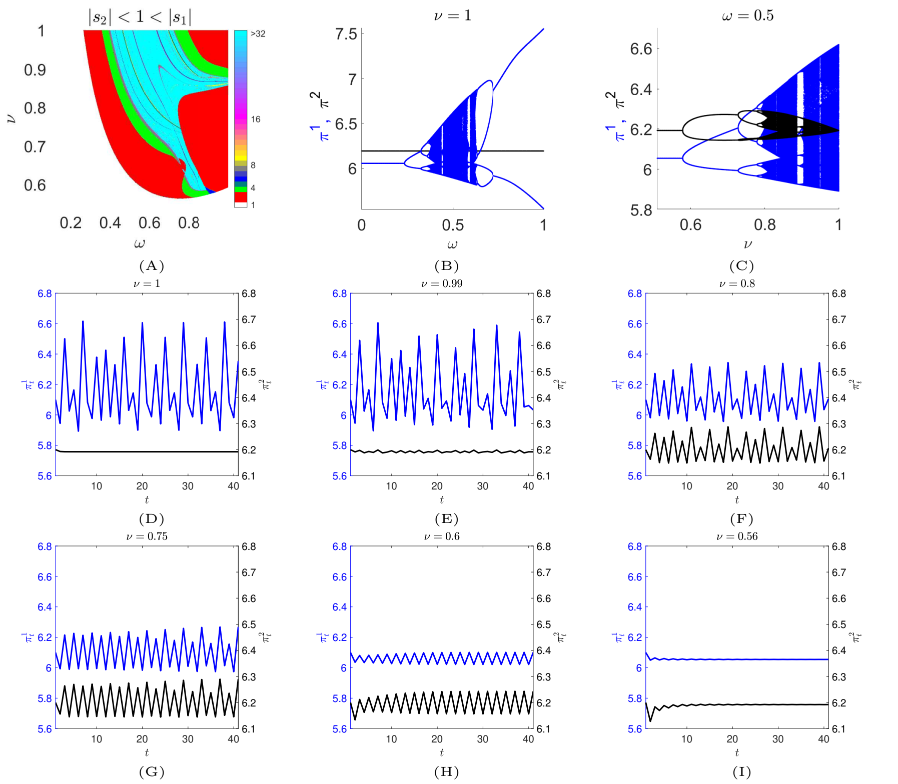

Fig. 6 (A) Two-dimensional bifurcation diagram in (ω, ν) plane. (B) Bifurcation diagrams as ω increases for the uncoupled. (C) Bifurcation diagrams as ν increases. (D)-(I): Time series for π_t^1 (blue) and π_t^2 (black) for different values of ν and $\omega = 0.5$.

and out-of-phase price information (Figures 6 (E)-(I)). If the agents base their learning process mostly on in-phase information (i.e. when the phase weight is close to 1), at odd phase times $\tau + 1$ (see also the the former equation in (11)) they will give a great relevance to turbulent price dynamics characterizing the previous odd phase times $\tau - 1$. Out-of-phase information is marginally used to form expected price at $\tau + 1$, so the overall effect is narrow and the turbulent trajectories of π_t^1 are essentially the same both for $\nu = 1$ and for $\nu = 0.99$ (blue lines in Figures 6 (D-E)).

At an even phase time $\tau + 2$ (see also the the latter equation in (11)) agents mostly rely on price information related to previous even phase time τ. In this case, the market outcome would be closer to the corresponding phase equilibrium price than the previous expected price, allowing the agents to learn the correct equilibrium price. However, this does not occur as the effect of learning from out-of-phase expectation error leads to a spread of the turbulence from π_t^1 to π_t^2. The consequence of this is evident in the time series related to $\nu = 0.99$, in which π_t^2 exhibits small, endogenous and non-periodic oscillations around $\pi^{2,*}$ (black line in Figure 6 (E)).

When the agents start giving more relevance to out-of-phase information ($\nu = 0.8$, Figure 6 (F)), the role of in-phase and out-of-phase expectation errors is more balanced being the former still the dominant ones. Expected price $\pi_{\tau+1}$ is mainly affected by the expectation error at $\tau - 1$, but the agents give to it a reduced relevance with respect to the previous cases. The consequence is a decrease in the price oscillations (blue line in Figure 6 (F)). Conversely, dynamics of π_t^2 (black line) now exhibit evident chaotic oscillations around $\pi^{2,*}$, since, as the coupling degree increases, the agents form expected prices more and more relying on turbulent out-of-phase expectation errors.

Up to now, the most evident effect is the instability transmission. The agents at even phase times try to learn from odd phase times, and their capability to correct odd phases expectation errors is impeded by the transmission of the errors they make at even phase times. However, price volatility at odd phase times reduces as well, so we could say that the agents "transfer" also the stability characterizing a market phase to the order. Such effects are even more evident if we observe the "parallel" price dynamics in time series of Figure 6, namely price growths, peaks and falls simultaneously occur in both phases, so that phase synchronization does not just consist in the occurrence of the same (stable/unstable) dynamics. More precisely, the intuition of this is that when agents form their expectations on the basis of both out-of-phase and in-phase information, they actually alter price dynamics of a phase introducing elements related to the dynamics of the other phase. In this sense, as the coupling degree increases, the dynamics of a given phase more and more bear information about what is going on *in the whole market*, and not only inside that phase. It could seem paradoxical that to correct the forecasting errors in a phase, the agents should give more relevance to the information related to the "wrong" phase, but actually the expectation errors related to such out-of-phase times bear information about *both* market phases. For this reason, if the coupling degree further increases, oscillations amplitude and/or complexity in the dynamics of π_t^1 declines, while the opposite occurs at even phase times. It is easy to understand that we come to a situation in which the price volatility is approximatively comparable in both phases (Figure 6 (G)). From here on, oscillations at odd phase times decrease (Figure 6 (I)) as well as time series at both phases start exhibiting smaller oscillations with a consequent dynamical simplification leading to convergence (Figure 6 (I)).

As the coupling degree increases, the couple of expectation errors are less and less a couple of in-phase and out-of-phase information, and increasingly become a couple of whole market related knowledge. It is then evident that the agents, when this happens, giving relevance to both expectations errors, have more chances to correct a wrong expectation about price grounding on two reliable information than on just one. The relevant economic result is that, by mixing information from both phases, agents can be able to learn how to correct a turbulent market outcome taking into account information coming from the other market phase. In this sense, since the expectation formation mechanism at even phase times allows for a correction of expectation errors, the more such errors bears "memory" of what happened at the other phase

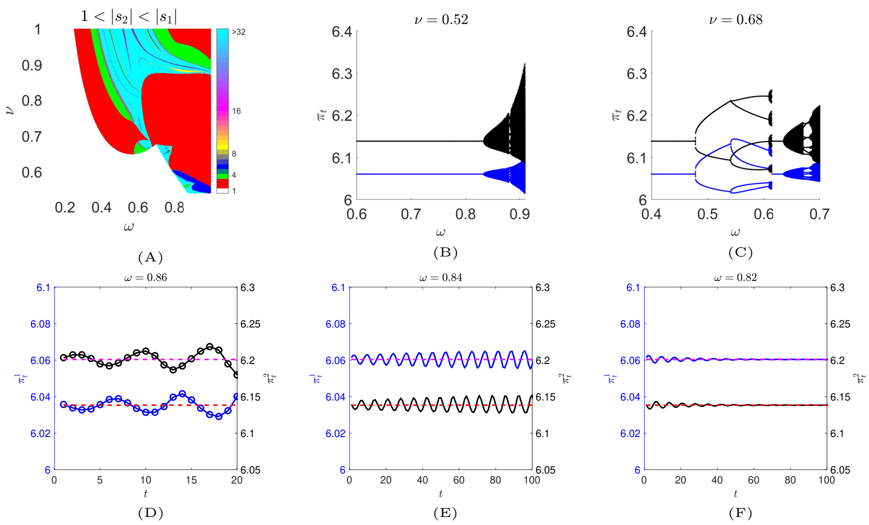

Fig. 7 (A) Two-dimensional bifurcation diagram in (ω, ν) plane. Bifurcation diagrams as ω increases when $\nu = 0.52$ (panel B) and $\nu = 0.68$ (panel C). (D)-(F): Time series for π_t^1 (blue) and π_t^2 (black) for different values of ω and $\nu = 0.52$, related to bifurcation diagrams in panel (B).

(as remarked also by qualitative synchronization of price dynamics), the more the agents will be able to learn how to adjust expectation errors in such other phase, and this will lead to an overall and gradual reduction of errors.

The previous rationale explains the stabilization phenomenon occurring when a "stable" phase is coupled to an "unstable" one. However, accordingly to Figure 4 (C), stabilization is possible even when two unstable phases are coupled. The phenomenon can be explained again with similar arguments, even if some of the underlying mechanisms basically change, leading to the emergence of quasi-periodic trajectories. To this end, we consider in Figure 7 a second family of simulations related to the stability scenario of Figure 4 (C), in which we set $a_1 = 7.6, a_2 = 7.2$ and $d_1 = d_2 = 1$. In this case, by numerical estimation, the period steady state equilibrium results $(\pi^{1,*}, \pi^{2,*}) \approx (6.06, 6.13)$. and we have $1 < 2.22 \approx |s_2| < |s_1| \approx 7.08$, so both phases are unstable when uncoupled and $\omega = 1$. The two-dimensional bifurcation diagram reports a scenario characterized by the highest level of ambiguity, among those analytically proved in Section 3, and instability can occur by means of either a flip bifurcation (when entering a red region from a white one) or a Neimark-Sacker bifurcation (when entering a cyan region from a white one, see also Figure 7 (C)). Increasing the expectation weight can give rise to a mixed-destabilizing scenario (see also see also Figure 7 (B)), as well as, when the equilibrium loses instability, quasi-periodic dynamics can occur.

In the present setting, when $\omega = 1$, both the uncoupled phases exhibit dynamics that do not converge toward the equilibrium and, hence, temporary equilibrium condition has an effect that is partially different from that occurring in the scenario reported in Figure 6. In the present scenario both relative elasticities s_1 and s_2 are large, so also at even phase times the temporary equilibrium condition drives an expected price that is close to the equilibrium away from it. The overall outcome of coupling phases is that in this case prices dynamics are just synchronized with respect to the kind of dynamics (being both convergent, periodic, quasi-periodic and so on), but when at a phase time agents overestimate the equilibrium price, in the subsequent phase time they underestimate it. Accordingly to the literature about oscillators, in what follows we refer to this phenomenon as anti-phase synchronization. To explain this, let us assume for example that agents, in forming their expectations, give a nearly uniform relevance[18] to both in-phase and out-of-phase expectation errors and consider, for both phases, an initial datum that is slightly above the corresponding phase equilibrium (first couple of black and blue circles in Figure 6 (D)). Since agents overestimate both in-phase and out-of-phase equilibrium prices, the corresponding expectation errors are both negative and price π_1^1 decreases below the steady state value (second blue circle in Figure 6 (D)). For $\tau = 2$, agents form their expectation also on the basis of this new price information, so they get opposing information from in-phase and out-of-phase expectation errors. The actual strongest effect is that corresponding to the market outcome at out-of-phase time $\tau = 1$, so the out-of-phase expectation error is larger in absolute value, and π_2^1 is further pushed above the steady state value (second black circle in Figure 6 (D)). The consequence is that at period time $t = 1$, expected prices π_1^1 and π_1^2 are respectively an underestimation and an overestimation of the correspondent component of the period steady state equilibrium. The anti-phase synchronization of expected price immediately occurred after just one period, even starting from a qualitatively synchronized initial datum. At this point, two subsequent temporary equilibrium conditions produce opposite price mechanisms: if two subsequent expected prices are respectively below and above the corresponding phase equilibrium, the market outcomes are respectively above and below the corresponding phase equilibrium. Due to the effect of the nonlinearity in demand and supply functions, the deviation from the equilibrium price $\pi^{i,*}$ is strong when the expected price is close to $\pi^{i,*}$, while it becomes weaker and weaker as the expected price significantly departs from it. Assume for example that at phase times $\tau - 1$ and τ the expectation error are respectively negative and positive, as a consequence of a slight overestimation and underestimation of the equilibrium price, respectively. In this case, the most recent expectation error is then that largest in absolute value and this means that the agents' will be mostly influenced by the last out-of-phase market outcome, so they correct the previous in phase expected price pushing it further up. The opposite occurs at the next phase time, so expected prices increasingly deviate from the corresponding equilibria, until it becomes no more sustainable. In this case, the agents' will be mostly influenced by the last in-phase market outcome, and

[18] We assume this just for the sake of simplicity of the subsequent explanation. Using similar arguments it is possible to explain the occurrence of the same phenomenon also for non-uniform phase weight distributions.

this slows down or reverses the expected price movement. The above described interaction provides the explanation of the emergence of a Neimark-Sacker bifurcation in double phases cobweb model, as shown in Figure 7 (D). In such scenario, the expectation weight acts as in a single phase market. Decreasing ω, the possible overreactive price variations due to extreme deviation between expected and realized price can be reduced (Figure 7 (E)) and even canceled (Figure 7 (F)) by a suitably cautious agents' behavior.

We stress that also in the present market configuration, the agents can be able to progressively correct their expectations errors by taking into account both in-phase and out-of-phase information. To explain this, we note that the expectation formation mechanism described by (11) can be rewritten as

$$\begin{aligned}
\pi_{\tau+1} &= \nu\left[\pi_{\tau-1}+\omega\Big(g_1(\pi_{\tau-1})-\pi_{\tau-1}\Big)\right]+(1-\nu)\left[\pi_{\tau-1}+\omega\Big(g_2(\pi_\tau)-\pi_\tau\Big)\right]=\nu\tilde{\pi}^1_{\tau+1}+(1-\nu)\tilde{\pi}^2_{\tau+1},\\
\pi_{\tau+2} &= \nu\left[\pi_{\tau}+\omega\Big(g_2(\pi_{\tau})-\pi_{\tau}\Big)\right]+(1-\nu)\left[\pi_{\tau}+\omega\Big(g_1(\pi_{\tau+1})-\pi_{\tau+1}\Big)\right]=\nu\tilde{\pi}^2_{\tau+2}+(1-\nu)\tilde{\pi}^1_{\tau+2},
\end{aligned} \tag{12}$$

from which the double phase mechanism can be seen as the average of two single phase adaptive expectation mechanisms[19] $\tilde{\pi}^i_\tau$ and $\tilde{\pi}^j_\tau$. For instance, to form expectations about price at $\tau+1$, the agents obtain two guesses $\tilde{\pi}^1_{\tau+1}$ and $\tilde{\pi}^2_{\tau+1}$, both based on a correction of the previous in-phase expected price $\pi_{\tau-1}$, but with the former that is based on the last in-phase expectation error and the latter that is based on the last out-of–phase expectation error. Then, they consider a weighted average of them. If the expectation weight is suitably large, on the basis of the previous explanation of the mechanism leading to the anti-phase synchronization of prices, it is easy to show that such two guesses are an overestimation and an underestimation of the corresponding phase equilibrium prices. We already noted that, as ν decreases, the expectation error at a given phase encompasses an increasingly large amount of information related to both odd and even phases, spread by the agents expectation formation mechanisms, so that they become reliable for learning about the whole market behavior, and not just about a single phase of it. When this occur, the agents, looking at two wrong opposite guesses, are able to progressively correct the expected price by averaging out to a middling estimate which is increasingly precise. In this situation, we can again say that as the phases are more and more coupled, the errors of a phase increasingly bears "memory" of what happened at the other phase (in this case, in the form of a anti-phase synchronized error), and this again allows for an overall correction of errors.

Up to now, we gave evidence of the new facts reported at points (a) and (b) at the beginning of this section. Concerning point (c), ω can have a counterintuitive role on stability, as evident looking at the bifurcation diagrams reported in Figure 7 (C), in which, as ω decreases, a stable equilibrium can become un-

[19] To be precise, the expectation formation mechanism related to out-of-phase times is phase shifted, but, even if unusual, it is equally an adaptive expectation mechanism.

stable[20]. This is the most ambiguous result arising in a double phase cobweb model, and it is the effect of a quite complicated superimposition of market outcomes and agents' behavior. Reasoning as before, it would be easy to see that the two single phase adaptive mechanisms produce expected prices that are oscillating around the equilibrium price and that stay anti-phase synchronized[21]. We have explained how such anti-phase synchronization allows the agents to learn how to correct the in-phase expected price from two consecutive wrong (respectively underestimated and overestimated) expected prices. If ω is too large, both single phase adaptive mechanisms provide too erratic prices, and this prevents error correction. If ω is reduced, the agents average out two opposite estimations of equilibrium price, and errors cancel out. But for such an outcome, it is necessary that the two single phase adaptive mechanisms are "strong enough" to provide conflicting estimations. Since the two markets are characterized by different relative elasticities, as ω decreases, one of the two single phase adaptive mechanisms will start stabilizing, while the other will again exhibit persistent oscillations. If ω further decreases, the agents will average out $\tilde{\pi_t}^i$, which is still characterized by erratic dynamics, and $\tilde{\pi_t}^j$, which is now slowly converging and no more counter balancing the over/underestimation of previous phase price, so that the dominating behavior is that of $\tilde{\pi_t}^i$, and oscillations are now persistent[22]. If then ω is again reduced, also non convergent phase will enter a stabilization process and once again trajectories start to converge.

5 Conclusions and future perspectives

In this paper we have introduced and studied a cobweb model for double phase markets. The resulting model exhibit an high degree of complexity in price dynamics, which grounds on the intrinsic peculiarity of the exchanged good, whose market curves are time-varying, and on the consequent possibility for the agents to use and mix information coming from different past market phases in order to form their expectations. This aspect introduces mutual interdependence between dynamics of different market phases, fostering the emergence of a vast variety of dynamical scenarios. In contrast to the classic single-phase framework, we can have multiple stability/instability thresholds, and dynamics can substantially change (begin periodic, chaotic or quasi-periodic) depending on both the relevance the agents give to the expectation errors and to each phase prices. The key element to understand the new

[20] A similar situation occurs also in the simulation reported in Figure 6 (A), when we have a mixed scenario. The explanation is in line with the intuition provided in what follows for the mixed-destabilizing scenario.

[21] We stress that due to the phase shift in the single phase adaptive mechanisms within brackets in (12), this actually happens in at least one phase also when just one relative elasticity is large, even if expected prices are synchronized.

[22] As a confirmation of the proposed intuition, if the two relative elasticities are identical, the return to instability is not possible, while the instability interval is increasingly large as s_1 and s_2 are more and more heterogeneous.

phenomena is the learning mechanism of the agents, which is strictly related to the double phase nature of the market. If agents form their expectations about future prices observing both in-phase and out-of-phase price dynamics, they inherently convey dynamical elements from one phase to the other, so that, as the coupling degree increases, the observed price trajectories of each single phase are more and more explanatory of what happens in both phases, characterizing the market as a whole. Against this background, it is possible to understand the mechanisms leading to the emergence of quasi-periodic dynamics, or the possible ambiguous role of the expectation weight. In particular, it allows explaining why the agents are more able to correct expected prices if they take into account both in-phase and out-of-phase price dynamics. The double phase approach has very interesting policy implications for markets characterized by known periodicities. We refer in particular to power exchanges, where demand and supply are collected by market operators on hourly basis. Electricity prices are formed independently for each hour, but the whole series of prices is influenced by firms' bidding strategies that are based on expectations formed on the whole price realizations, not only on a single hour. The multiphase expectation mechanism introduced in this paper may explain this phenomenon and can become a tool for regulators and market operators, especially when reasoning on a market reform suitable for accommodating a high share of renewable energy sources, which are known to increase variability of equilibrium prices.
The extent of the foundational elements of the double phase cobweb model required comprehensive analytical and interpretative investigations, which left no space for dealing with other facets of multiphase markets modelling. In future researches we aim to extend the investigations in several directions. Firstly, we want to deepen the mathematical analysis of the global dynamical properties the model, which, due to the length of the local stability analysis and to the need of economic explanation of the many phenomena arising, we decided to postpone to a future work. A second improvement can be the endogenization of the phase weight choice, in order to understand how the agents can refine their learning strategy by choosing to what extent take into account the observed price dynamics of each phase. Moreover, we want to generalize the proposed modelling approach to multiphase markets, taking into account high period cyclicity. Finally, we aim at applying the modelling approach to the description of real markets affected by seasonality, as energy and electricity markets, in order to check if the elements of complexity and ambiguity arising in a double phase framework are able to explain the occurrence of stylized facts characterizing economic observables of such markets.

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Appendix

Proof (Proof of Proposition 1) Without loss of generality, let us assume that τ is odd. Recalling the definition of functions ε_i we can rewrite (3) as $\pi_{\tau+1} = \pi_{\tau-1} + \nu\omega\varepsilon_1(\pi_{\tau-1}) + (1-\nu)\omega\varepsilon_2(\pi_\tau)$ while considering (3) at the even phase time $\tau+1$ we have

$$\begin{aligned}\pi_{\tau+2} &= \pi_\tau + \nu\omega(D^{-1}(S(\pi_\tau,\tau),\tau) - \pi_\tau) + (1-\nu)\omega(D^{-1}(S(\pi_{\tau+1},\tau+1),\tau+1) - \pi_{\tau+1})\\ &= \pi_\tau + \nu\omega\varepsilon_2(\pi_\tau) + (1-\nu)\omega\varepsilon_1(\pi_{\tau+1}))\\ &= \pi_\tau + \nu\omega\varepsilon_2(\pi_\tau) + (1-\nu)\omega\varepsilon_1(\pi_{\tau-1} + \nu\omega\varepsilon_1(\pi_{\tau-1}) + (1-\nu)\omega\varepsilon_2(\pi_\tau))\end{aligned}$$

Recalling the definition of functions F^i we have $(\pi^1_{t+1}, \pi^2_{t+1}) = (\pi_{\tau+1}, \pi_{\tau+2})$, for any t, τ, which allows us to conclude the proof.

Proof (Proof of Proposition 2) Identity $\mathbf{p}^* = \mathbf{F}(\mathbf{p}^*)$ is a straightforward consequence of (4). Now we need to show that if $p = \mathbf{F}(p)$, then $p = \mathbf{p}^*$. From (5) we immediately obtain

$$\begin{cases}\nu\omega\varepsilon_1(p^1) + \omega(1-\nu)\varepsilon_2(p^2) = 0,\\ \nu\omega\varepsilon_2(p^2) + \omega(1-\nu)\varepsilon_1(F^1(p^1,p^2)) = \nu\omega\varepsilon_2(p^2) + \omega(1-\nu)\varepsilon_1(p^1) = 0,\end{cases}$$

which is which an homogeneous square linear system, whose unknown vector is $(\varepsilon_1(p^1), \varepsilon_2(p^2))^T$ and the coefficient matrix is $A = (1-\nu, \nu; \nu, 1-\nu)$. We have $\det(A) \neq 0$, since $\nu \neq 1/2$. This means that its unique solution is the null vector. Recalling (4), we have that the unique solutions of $\varepsilon_i(p^i) = 0$ are $p^i = p^{i,*}$, for $i = 1, 2$.

Proof (Proof of Proposition 3) Let $J_F(\boldsymbol{\pi})$ be the Jacobian matrix of map $\mathbf{F}$ defined by (5) and let $J^*_F = J_F(\mathbf{p}^*)$. By direct computation we have

$$J^* = \begin{pmatrix}\nu\omega(s_1-1) + \omega^2(s_1-1)(s_2-1)(\nu-1)^2 + 1 & -\omega(s_2-1)(\nu-1)\\ -\omega(s_1-1)(\nu-1)(s_2\nu\omega - \nu\omega + 1) & \nu\omega(s_2-1)+1\end{pmatrix}.$$

In a two-dimensional difference equation a steady state is locally asymptotically stable provided that

$$\begin{cases}1 - \mathrm{Tr}(J^*) + \det(J^*) > 0,\\ 1 - \det(J^*) > 0,\\ 1 + \mathrm{Tr}(J^*) + \det(J^*) > 0.\end{cases} \tag{13}$$

Using $\mathrm{Tr}(J^*) = \nu\omega(s_1+s_2-2) + \omega^2(s_1-1)(s_2-1)(\nu-1)^2 + 2$ and $\det(J^*) = (\nu\omega(s_1-1)+1)(\nu\omega(s_2-1)+1)$ in system (13) and recalling the definition of $\bar{\omega}_1$ and $\bar{\omega}_2$, it is easy to see that the first condition in (13) is unconditionally fulfilled, while the second and the third one provide (9).

Let us introduce

$$\varphi_2(\omega,\nu)=\omega\nu-\frac{\bar\omega_1+\bar\omega_2}{2},\quad \varphi_3(\omega,\nu)=(2\nu^2-2\nu+1)\omega^2-2\nu\frac{\bar\omega_1+\bar\omega_2}{2}\omega+\bar\omega_1\bar\omega_2.$$

Proof (Proof of Proposition 5) We need to show that if $0<\omega\le\bar\omega_1$, then $\varphi_3(\omega,\nu)>0$ for any $\nu\in(1/2,1]$. We indeed have $\varphi_3(0,\nu)>0$, while

$$\varphi_3(\bar\omega_1,\nu)\ge(2\nu^2-3\nu+1)\bar\omega_1^2+(1-\nu)\bar\omega_2\bar\omega_1\ge 2(\nu-1)^2\bar\omega_1^2>0$$

where we used $\bar\omega_2\ge\bar\omega_1$. From $\partial_\omega\varphi_3(\omega,\nu)=2\omega(2\nu^2-2\nu+1)-\nu(\bar\omega_1+\bar\omega_2)$ we have $\partial_\omega\varphi_3(\bar\omega_1,\nu)=(4\nu^2-5\nu+2)\bar\omega_1-\nu\bar\omega_2\le 2\bar\omega_2(2\nu^2-3\nu+1)<0$, where we used $\bar\omega_2\ge\bar\omega_1$ and that $2\nu^2-3\nu+1<0$ for $\nu\in(1/2,1)$. The previous considerations prove that $\varphi_3(\omega,\nu)$ is positive at the ending points of $[0,\bar\omega_1]$ and strictly decreasing, independently of ν. Noting that $\varphi_2(\omega,\nu)>0$ for $\omega\in I_{ss}$ allows concluding.

The (possible) solutions of $\varphi_3(\omega,\nu)=0$ with respect to ω are given by

$$\omega_{F,i}=\frac{\nu}{2\nu^2-2\nu+1}\frac{\bar\omega_1+\bar\omega_2}{2}\pm\frac{\sqrt{\nu^2\bar\omega_1^2-6\nu^2\bar\omega_1\bar\omega_2+\nu^2\bar\omega_2^2+8\nu\bar\omega_1\bar\omega_2-4\bar\omega_1\bar\omega_2}}{2(2\nu^2-2\nu+1)},\tag{14}$$

for $i=1,2$, as well as the (possible) solution of $\varphi_2(\omega,\nu)=0$, defined by

$$\omega_{NS}=\frac{1}{\nu}\cdot\frac{\bar\omega_1+\bar\omega_2}{2}.\tag{15}$$

To prove Propositions 6 and 7, we start considering conditions (9) on larger sets than I_{su} and I_{uu}. Firstly we prove some preliminary Lemmas, in which we assume $\bar\omega_1<\bar\omega_2$.

Lemma 1 *Let us consider the curve defined by $\Gamma=\{(\omega,\nu)\in[\bar\omega_1,+\infty)\times[0,1]:\varphi_3(\omega,\nu)=0\}$. For $(\omega,\nu)\in[\bar\omega_1,\bar\omega_2)\times[0,1]$, we then have that*
1) Γ can be explicitly represented a through function $g:[\bar\omega_1,\bar\omega_2)\to[0,1]$, which is strictly decreasing in $[\bar\omega_1,\omega_m)$ and strictly increasing in $(\omega_m,\bar\omega_2)$, where $\omega_m\in(\bar\omega_1,\bar\omega_2)$;
2) the solution(s) of $g(\omega)=\hat\nu$ is $\omega_{F,1}(\nu)$ for $\hat\nu\ge(\bar\omega_1+\bar\omega_2)/(2\bar\omega_2)$ and are $\omega_{F,1}(\nu)\le\omega_{F,2}(\nu)$ for $g(\omega_m)\le\hat\nu<(\bar\omega_1+\bar\omega_2)/(2\bar\omega_2)$, while for $\hat\nu<g(\omega_m)$ we have no solutions.
Conversely, for $(\omega,\nu)\in[\bar\omega_2,+\infty)\times[(\bar\omega_1+\bar\omega_2)/(2\bar\omega_2),1]$, we have that
3) Γ can be explicitly represented through a function $h:[(\bar\omega_1+\bar\omega_2)/(2\bar\omega_2),1]\to[\bar\omega_2,+\infty)$ which is strictly increasing in $[(\bar\omega_1+\bar\omega_2)/(2\bar\omega_2),\nu_M)$ and strictly decreasing in $(\nu_M,1]$, where $\nu_M\in((\bar\omega_1+\bar\omega_2)/(2\bar\omega_2),1)$.

Proof Firstly we note that since $\varphi_3(\omega,\nu)$ is a second degree polynomial with respect to both ω and ν, we have that intersecting Γ with either $\omega=k$ or $\nu=k$ we find at most two solutions. Now we prove each point.

1) We start considering Γ on the subset $(\omega,\nu)\in[\bar\omega_1,\bar\omega_2]\times[0,1]$. In particular, solving $\varphi_3(\omega,\nu)=0$ with respect to ν we find

$$\nu_\pm(\omega)=\frac{2\omega+\bar\omega_1+\bar\omega_2\pm\sqrt{\Delta(\omega)}}{4\omega}$$

provided that $\Delta(\omega) = -4\omega^2+4\omega(\bar{\omega}_1+\bar{\omega}_2)+\bar{\omega}_1^2-6\bar{\omega}_1\bar{\omega}_2+\bar{\omega}_2^2 \geq 0$. Since $\Delta(\omega)$ is a concave parabola and $\Delta(\bar{\omega}_1) = \Delta(\bar{\omega}_2) = (\bar{\omega}_1 - \bar{\omega}_2)^2$, we have that $\Delta(\omega) > 0$ for $\omega \in [\bar{\omega}_1, \bar{\omega}_2]$ and both $\nu_+(\omega)$ and $\nu_-(\omega)$ are well-defined. Moreover, a simple direct check shows that $\nu_-(\omega) > 0$ for each $\omega \in [\bar{\omega}_1, \bar{\omega}_2]$.

We have that $\nu_+(\omega) \geq 1$ for $\omega \in [\bar{\omega}_1, \bar{\omega}_2]$. In fact, noting that $\nu_+(\bar{\omega}_1) = (\bar{\omega}_1+\bar{\omega}_2)/(2\bar{\omega}_1) > 1$, $\nu_-(\bar{\omega}_1) = 1$, $\nu_+(\bar{\omega}_2) = 1$ and $\nu_-(\bar{\omega}_2) = (\bar{\omega}_1+\bar{\omega}_2)/(2\bar{\omega}_2) < 1$, if there existed some $\tilde{\omega} \in (\bar{\omega}_1, \bar{\omega}_2)$ such that $\nu_+(\tilde{\omega}) \leq 1$, then $\varphi_3(\omega, 1) = 0$ would necessarily have more than two solutions. Similar arguments show that if $\omega \in [\bar{\omega}_1, \bar{\omega}_2]$ we have $\nu_-(\omega) \in [0, 1]$. This means on $[\bar{\omega}_1, \bar{\omega}_2)$ curve Γ coincides with function $g = \nu_-|_{[\bar{\omega}_1, \bar{\omega}_2)}$.

Since $\nu'_-(\bar{\omega}_1) = -1/\bar{\omega}_1 < 0$ and $\nu'_-(\bar{\omega}_2) = (\bar{\omega}_2 - \bar{\omega}_1)/(2\bar{\omega}_2^2) > 0$, thanks to the regularity of ν_-, from the intermediate value theorem we have at least an $\omega_m \in (\bar{\omega}_1, \bar{\omega}_2)$ such that $\nu'_-(\omega_m) = 0$. Moreover, ω_m is the unique stationary point of ν_-, as, otherwise, $\varphi_3(\omega, k) = 0$ would have more than two solutions (possibly considered with their own multiplicity) for some $k \in [0, 1]$. This means that $\nu_-(\omega)$ is strictly decreasing for $\omega \in [\bar{\omega}_1, \omega_m)$ and strictly increasing for $\omega \in (\omega_m, \bar{\omega}_2]$, and attains its minimum at $(\omega_m, \nu_m = \nu_-(\omega_m))$. Recalling that $g = \nu_-|_{[\bar{\omega}_1, \bar{\omega}_2)}$ concludes the proof.

2) If we set $\hat{\nu} \in [0, 1]$ and we solve $\varphi_3(\omega, \hat{\nu}) = 0$ for $\omega \in \mathbb{R}$ we either find two solutions $\omega_{F,1}(\nu) \leq \omega_{F,2}(\nu)$ or no solutions. If $\omega_{F,i}(\nu) \in [\bar{\omega}_1, \bar{\omega}_2)$, we indeed must have $g(\omega_{F,i}(\nu)) = \hat{\nu}$. Noting that $\nu_-(\bar{\omega}_2) < 1$, we can conclude the proof of this point using the monotonicity properties of ν_- shown at point 2).

3) We consider Γ on the subset $(\omega, \nu) \in [\bar{\omega}_2, +\infty) \times [0, 1]$. In this case, we could show that $\varphi_3(\omega, \nu) = 0$ does not implicitly define a function $\nu = f(\omega)$, so we solve $\varphi_3(\omega, \nu) = 0$ with respect to ω. We start noting that $\varphi_3(\omega, \nu) = 0$ does not have any solution in $[\bar{\omega}_2, +\infty) \times [0, \nu_-(\bar{\omega}_2) = (\bar{\omega}_1 + \bar{\omega}_2)/(2\bar{\omega}_2))$. If fact, from the monotonicity considerations on $\nu_-(\omega)$, $\varphi_3(\omega, \nu) = 0$ already has two solutions in $[\bar{\omega}_1, \bar{\omega}_2) \times [\nu_m, (\bar{\omega}_1 + \bar{\omega}_2)/(2\bar{\omega}_2))$, while if $(\omega, \nu) \in [\bar{\omega}_1, +\infty) \times [0, \nu_m)$ we have no solutions, as otherwise, from the intermediate value theorem, we should have at least another solution of $\varphi_3(\omega, \nu) = 0$ in $(\bar{\omega}_2, +\infty) \times [\nu_m, (\bar{\omega}_1 + \bar{\omega}_2)/(2\bar{\omega}_2))$. So we restrict to $[\bar{\omega}_2, +\infty) \times [(\bar{\omega}_1 + \bar{\omega}_2)/(2\bar{\omega}_2), 1]$. From the monotonicity considerations on $\nu_-(\omega)$, $\varphi_3(\omega, \nu) = 0$ has just one solution in $[\bar{\omega}_1, \bar{\omega}_2) \times [(\bar{\omega}_1 + \bar{\omega}_2)/(2\bar{\omega}_2), 1]$, so the other one must be in $[\bar{\omega}_2, +\infty) \times [(\bar{\omega}_1 + \bar{\omega}_2)/(2\bar{\omega}_2), 1]$. Such solution, recalling (14), coincides with $\omega = \omega_{F,2}(\nu)$. So we define $h = \omega_{F,2}(\nu)$.

It is sufficient to note that $h'(1) = -\bar{\omega}_2 < 0$ and $h'((\bar{\omega}_1 + \bar{\omega}_2)/(2\bar{\omega}_2)) = (2\bar{\omega}_2^2)/(\bar{\omega}_2 - \bar{\omega}_1) > 0$. Thanks to the regularity of h, proceeding as in the proof of 2), from the intermediate value theorem we have a unique $\nu_M \in ((\bar{\omega}_1 + \bar{\omega}_2)/(2\bar{\omega}_2), 1)$ such that $h'(\nu_M) = 0$. This concludes the proof.

Lemma 2 *Let* $(\omega, \nu) \in [\bar{\omega}_1, \bar{\omega}_2) \times [0, 1)$. *Then conditions* (9) *are fulfilled if*

$$
\begin{cases}
\bar{\omega}_1 \leq \omega < \omega_{F,1}(\nu) & \text{for } \nu \geq \dfrac{\bar{\omega}_1 + \bar{\omega}_2}{2\bar{\omega}_2} \quad (16a) \\
\bar{\omega}_1 \leq \omega < \omega_{F,1}(\nu) \vee \omega_{F,2}(\nu) < \omega < \bar{\omega}_2 & \text{for } g(\omega_m) \leq \nu < \dfrac{\bar{\omega}_1 + \bar{\omega}_2}{2\bar{\omega}_2} \quad (16b) \\
\bar{\omega}_1 \leq \omega < \bar{\omega}_2 & \text{for } 0 \leq \nu < g(\omega_m) \quad (16c)
\end{cases}
$$

Let $(\omega,\nu) \in [\bar{\omega}_2,+\infty) \times [0,1]$. *Then conditions* (9) *are no fulfilled if* $\nu \in [(\bar{\omega}_1+\bar{\omega}_2)/2\bar{\omega}_2,1]$ *and are satisfied if* $\omega < \omega_{NS}(\nu)$ *when* $\nu \in [0,(\bar{\omega}_1+\bar{\omega}_2)/2\bar{\omega}_2)$.

Proof We start proving that for $(\omega,\nu) \in [\bar{\omega}_1,\bar{\omega}_2) \times [0,1]$ conditions (9) reduce to $\varphi_3(\omega,\nu) > 0$. It is evident that $\varphi_3(\omega,\nu) > 0$ is necessary. To prove that it is also sufficient, we start noting that if $(\omega,\nu) \in A_1 = [\bar{\omega}_1,\bar{\omega}_2)\times[0,(\bar{\omega}_1+\bar{\omega}_2)/2\bar{\omega}_2)$, then $\varphi_2(\omega,\nu) < 0$ and condition (9a) is fulfilled, so (9) reduces to $\varphi_3(\omega,\nu) > 0$. Conversely, if $(\omega,\nu) \in s_2 = [\bar{\omega}_1,\bar{\omega}_2) \times [(\bar{\omega}_1+\bar{\omega}_2)/2\bar{\omega}_2,1]$, from point 2) of Lemma 1, we have that $\varphi_3(\omega,\nu) > 0$ is equivalent to $\omega < \omega_{F,1}(\nu)$. Condition $\varphi_2(\omega,\nu) < 0$ is equivalent to $\omega < \omega_{NS}(\nu)$, so conditions (9) reduce to $\omega < \min\{\omega_{F,1}(\nu),\omega_{NS}(\nu)\}$. To conclude, it is sufficient to show that $\omega_{F,1}(\nu) < \omega_{NS}(\nu)$, which is proved by noting that, from $\nu/(2\nu^2-2\nu+1) < 1/\nu$, we can write $\omega_{F,1}(\nu) < \frac{\nu}{2\nu^2-2\nu+1}\frac{\bar{\omega}_1+\bar{\omega}_2}{2} < \frac{1}{\nu}\frac{\bar{\omega}_1+\bar{\omega}_2}{2} < \omega_{NS}(\nu)$. Recalling that $\varphi_3(\omega,\nu)$ is a second degree polynomial with respect to ω and that the coefficient of ω^2 is strictly positive, point 2) of Lemma 1 straightforwardly leads to conclusion.

For the last part of the Lemma, we note that if $\nu \in [(\bar{\omega}_1+\bar{\omega}_2)/2\bar{\omega}_2,1)$ and $\omega > \bar{\omega}_2$, then condition (9b) is not satisfied. Conversely, if $\nu \in (0,(\bar{\omega}_1+\bar{\omega}_2)/2\bar{\omega}_2)$, from the first part of this Lemma, we have that condition (9b) is satisfied. Solving $\varphi_2(\omega,\nu) < 0$ concludes the proof.

Proof (Proof of Proposition 6) Since $\bar{\omega}_1 < 1$, interval I_{su} is not empty. To specify ν_1 and ν_2, we need to restrict the results of Lemma 2 to $I_{su}\times(1/2,1)$. In particular, we consider the four possible situations depending on whether $\omega_m < \min\{\bar{\omega}_2,1\}$ and on whether $\nu_m > 1/2$. In what follows we set $\omega_{SU} = \min\{\bar{\omega}_2,1\}$ so that we have $I_{su} = (\bar{\omega}_1,\omega_{SU})$. Similarly, we define $\nu_{SU} = \lim_{\omega\to\omega_{SU}} g(\omega)$. We remark that when $\bar{\omega}_2 > 1$, the previous limit can be replaced by the function evaluation $g(\omega_{SU}) = g(1)$. Finally, both ω_m and $\nu_m = g(\omega_m)$ are those defined in Lemmas 1 and 2.

Case (1): $\omega_m < \omega_{SU}$ and $\nu_m > 1/2$. In this case, recalling point 2) of Lemma 1, g is strictly decreasing in $[\bar{\omega}_1,\omega_m)$ and strictly increasing in (ω_m,ω_{SU}) and we take $\nu_1 = \nu_{SU}$ and $\nu_2 = \nu_m$. If $\nu \in (\nu_1,1)$, the flip destabilizing scenario is a consequence of (16a), if $\nu \in (\nu_2,\nu_1)$ the mixed scenario is a consequence of (16b) and if $\nu \in (1/2,\nu_2)$ the unconditionally stable scenario is a consequence of (16c).

Case (2): $\omega_m \geq \omega_{SU}$ and $\nu_m > 1/2$. In this case g is decreasing in I_{su} and we take $\nu_1 = \nu_2 = \nu_{SU}$. If $\nu \in (\nu_1,1) = (g(\omega_{SU}),1)$, the flip destabilizing scenario is a consequence of (16a), interval (ν_2,ν_1) is empty and if $\nu \in (1/2,\nu_2)$ the unconditionally stable scenario is a consequence of (16c).

Case (3): $\omega_m < \omega_{SU}$ and $\nu_m \leq 1/2$. This means that g is strictly decreasing in $[\bar{\omega}_1,\omega_m)$ and strictly increasing in (ω_m,ω_{SU}). We need to distinguish between two situations. If $\nu_{SU} > 1/2$, there exist $\omega_A,\omega_B \in I_{su}$ such that for $\omega \in [\omega_A,\omega_B]$ we have $g(\omega) \leq 1/2$. In this case we choose $\nu_1 = \nu_{SU}$ and $\nu_2 = 1/2$, so that if $\nu \in (\nu_1,1)$, the flip destabilizing scenario is a consequence of (16a), if $\nu \in (\nu_2,\nu_1)$ the mixed scenario is a consequence of (16b) and interval $(1/2,\nu_2)$ is empty. Conversely, if $g(1) \leq 1/2$, there exists $\omega_A \in I_{su}$ such that for $\omega \in [\omega_A,\omega_{SU})$ we have $g(\omega) \leq 1/2$. In this case we choose $\nu_1 = \nu_2 = 1/2$, so if $\nu \in (\nu_1, 1) = (1/2, 1)$, the flip destabilizing scenario is a consequence of (16a) while intervals (ν_2, ν_1) and $(1/2, \nu_2)$ are both empty.

Case (4): $\omega_m \geq 1$ and $\nu_m \leq 1/2$. In this case g is strictly decreasing in I_{su}. We need to distinguish between two situations. If $\nu_{SU} > 1/2$, then $g(\omega) > 1/2$ for any $\omega \in I_{su}$, so we choose $\nu_1 = \nu_2 = \nu_{SU}$ as in Case (2). Conversely, if $\nu_{SU} \leq 1/2$, there exists $\omega_A \in I_{su}$ such that for $\omega \in [\omega_A, \omega_{SU})$ we have $g(\omega) \leq 1/2$. In this case we choose $\nu_1 = \nu_2 = 1/2$ as in the latter situation of Case (3). The actual occurrence of each situation is proved by Figures 4 (B)-(C). Finally, the last part of the Proposition is a consequence of point 1) of Lemma 1. In fact since $\nu = g(\omega)$ is strictly decreasing on $[\bar{\omega}_1, \omega_m)$, we have that $\omega = g^{-1}(\nu) = \omega_{F,1}(\nu)$ is strictly decreasing on $(\nu_m, 1]$, too, and then $\omega_{F,1}(\nu)$ increases as ν decreases. Similarly proceeding, we obtain that $\omega_{F,2}(\nu)$ is strictly decreasing. Hence interval $(\omega_{F,1}(\nu), \min\{\omega_{F,1}(\nu), 1\})$ reduces as ν decreases.

Proof (Proof of Proposition 7) Since $\bar{\omega}_2 < 1$, interval I_{uu} is not empty. As in the Proof of Proposition 6, we need to restrict the results of Lemma 2 to $I_{uu} \times (1/2, 1)$ in order to provide explicit expressions for ν_1 and ν_2. From the second part of Lemma 2, we have that the only significant stability condition is $\varphi_2(\omega, \nu) < 0$. Moreover, since from Lemma 2 stability conditions are not satisfied if and only if $\nu \geq (\bar{\omega}_1 + \bar{\omega}_2)/2\bar{\omega}_2$, we can take $\nu_1 = (\bar{\omega}_1 + \bar{\omega}_2)/2\bar{\omega}_2$. We stress that $\nu_1 = 1$ if and only if $\bar{\omega}_1 = \bar{\omega}_2$, in which case we do not have unconditional instability.

For ν_2, we consider two possible situations depending on whether $(\bar{\omega}_1 + \bar{\omega}_2)/2 \leq 1/2$ or not.

Case (1): $(\bar{\omega}_1 + \bar{\omega}_2)/2 \leq 1/2$. In this case we have $\omega_{NS}(\nu) < 1$ for any $\nu \in (1/2, (\bar{\omega}_1 + \bar{\omega}_2)/(2\bar{\omega}_2))$ and we set $\nu_2 = 1/2$.

Case (2): $(\bar{\omega}_1 + \bar{\omega}_2)/2 > 1/2$. In this case we have $\omega_{NS}(\nu) \leq 1$ if $\nu \in [(\bar{\omega}_1 + \bar{\omega}_2)/2, (\bar{\omega}_1 + \bar{\omega}_2)/(2\bar{\omega}_2))$ and $\omega_{NS}(\nu) > 1$ if $\nu \in (1/2, (\bar{\omega}_1 + \bar{\omega}_2)/2)$, so we set $\nu_2 = (\bar{\omega}_1 + \bar{\omega}_2)/2$.

Finally, the last part of the Proposition can be proved using monotonicity considerations, as in the proof of Proposition 6.

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