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## Abstract

Quantitative risk management techniques should prove their efficacy when financially turbulent periods are about to occur. Along the common saying "who needs an umbrella on a sunny day?", a theoretical model is really helpful when it carries the right suggestion at the proper time, that is when markets start behaving hecticly. The beginning of the third decade of the 21st century carried along a turmoil that severely affected worldwide economy and changed it, probably for good. A consequent and plausible research question could be this: which financial quantitative approaches can still be considered reliable? This article tries to partially answer this question by testing if the mean-variance selection model (Markowitz [16], [17]) and some of his refinements can provide some useful hints in terms of portfolio management.

**Key Words:** Mean-Variance Portfolio Selection Models; Minimum Variance Portfolios; Risk Parity Approach, Black-Litterman model.

## 1) Introduction and motivation

It is well known that the mean-variance portfolio selection model proposed by Markowitz ([16], [17]) is one of the stepping-stones on which modern finance relies on. It is, unfortunately, also common knowledge that, when applied to real market data, optimal mean-variance portfolios suffer of a number of drawbacks. A relevant one, the extreme sensitiveness of portfolios' weights to small variations of market data, has been clearly pinpointed, amongst many others, by Chopra and Ziemba [5]; these authors state: "small changes in the input parameters can result in large changes in composition of the optimal portfolio". Further, in their article Chopra and Ziemba show that estimation errors in expected returns can be ten times more relevant, and therefore harmful, than an estimation error in variances.

To overcome this and other issues, for instance the non-normal distribution of historically observed stock returns and the fact that some optimal portfolios have stocks with negative weights, especially when large expected returns are imposed, a relevant number of approaches have been presented. Roughly speaking, these enhancements can be divided in two groups. The first encompasses 'naïve' procedures such as the equally weighted (EW) (tested, amongst others, by DeMiguel et al. [7]), minimum variance (MV) (see, for instance, Coqueret [6]), and equal risk contribution (ERC) (a detailed explanation on this topic can be found in Roncalli [19]) ones. If the EW approach is an elementary translation of 'common sense' portfolio diversification ("don't put all your eggs in one basket"), the MV and ERC ones are based on a mathematical approach; here, optimal portfolios do not deal directly with returns, allowing to bypass, up to some extent, the negative feature identified by Chopra and Ziemba.

The second family of models exploits a range of quantitative sophisticated concepts. A key contribution was proposed by Black and Litterman [4] (BL) where the effect on optimal portfolios of market observed data (the prior) is updated in Bayesian terms introducing the so-called `views' (the posterior), i.e. opinions on the future behaviour of the whole market or on one or more stocks, expressed by the investor or a financial expert. This blending procedure between estimated parameters obtained from observed data, that are inefficient but with a low bias, and a constant estimator that, conversely, is more efficient but with a large bias, is the base for the theory of shrinkage estimators (Meucci [18]). The crucial point in this approach is the determination of the influence to be attributed to the two classes of estimators on the resulting portfolio.

Another approach deals with robustness of portfolios (see, for instance, Goldfarb and Iyengar, [10] and [11]); loosely speaking, a portfolio is robust when its weights are scarcely affected by changes in input parameters. To achieve this, actual values of the input parameters are assumed to belong to an uncertainty set whose shape allow to perform a 'robust', two-step optimization. Schöttle and Werner ([20], [21]) "robustify" the standard Markowitz model letting parameters abide in what these author name 'confidence ellipsoids'. A different vein considers inserting in the original model some additional constraints, either 'wrong' (Jagannathan and Ma [13]) or 'right' (Behr et al. [3]). For instance, DeMiguel et al. [8] show that reliable optimal portfolios are identified imposing an additional constraint on the norm of the vector of portfolio weights. Finally, Fliege and Werner [9] revisit the Markowitz model in terms of multi-objective optimization solving a bi-objective problem.

Needless to say, all these improvements come with a cost and are of difficult implementation on a practical basis. Managing a portfolio under their terms requires capabilities that could be either unfeasible or not economical. Still, the correct handling of financial portfolios is a relevant issue in risk management. Can an investor still safely trust non 'highly sophisticated' techniques? As an attempt to provide a partial answer to this question, in this article results obtained using EW, MV and ERC methodologies are discussed at first. Secondly, a BL version of these portfolios is obtained with the aim of testing how views can encompass highly stressed financial periods and if a more complex approach is worth applying.

As said above, any risk-oriented portfolio management strategy should be helpful when needed the most, that is when financial markets face a strong turbulence. A perfect example of this is year 2020, when worldwide economy has been severely tackled. The Euro Stoxx 50 stock index, whose value encompasses stocks prices of 50 European companies chosen according to their size, dropped from a value of 3793.24 on January 2nd, 2020 to the yearly minimum level of 2385.82 on March 18th, 2020[1], right after the beginning of the spread of the first wave of the infamous COVID-19 pandemic, with a loss of 26.5%.

Numerical results presented below show how supposedly risk reducing approaches might perform poorly when applied to heavily bearish financial markets. One of the reasons for this debacle can be attributed to the fact that when financial markets plummet, correlation between random returns increase, limiting or almost totally obliterating benefits of diversification.

[1] Data retrieved from https://www.wsj.com/market-data/quotes/index/XX/SX5E/historical-prices website

A similar, negative result is obtained applying the BL approach. Pessimistic views reduce the values for expected returns and inflate variances and covariance. Correlations between random returns, though, do not change, leading MV and ERC portfolios to end up with the same weights.

The structure of this article is as follows: Section 2 presents the theory and some naïve improvements of the standard portfolio selection model that will be used in Section 3 to verify the performance of the strategies under scrutiny. Section 4 eventually concludes.

## 2) Markowitz's model and some refinements

The original portfolio selection model developed by Markowitz encompasses $n$ stocks whose random return $\tilde{R}_i$, $i = 1, \dots, n$, are assumed to be fully represented in terms of the vector of expected returns $\mathbf{r} = [\bar{R}_i] \in \mathbb{R}^n$, and of the variance/covariance matrix $\Sigma = [\sigma_{i,j}] \in \boldsymbol{\mathcal{M}}(n,n)$, $j = 1, \dots, n$, where $\sigma_{i,j} = \sigma_{j,i}$ is the covariance between random returns $\tilde{R}_i$ and $\tilde{R}_j$ and $\sigma_{i,i} = \sigma_i^2$ the variance of $\tilde{R}_i$. Letting $\mathbf{x} \in \mathbb{R}^{\boldsymbol{n}}$ be the vector portfolio weights, the optimal portfolio $\mathbf{x}^*$ according to Markowitz results solving the constrained optimization problem

$$
\begin{array}{lllll}
\min\limits_{\mathbf{x}} & \sigma_P^2 & = & \mathbf{x}^T\Sigma\mathbf{x} & \\
\text{subject to} & \bar{r}_P & = & \mathbf{r}^T\mathbf{x} = r_P & \text{expected return constraint} \\
 & & & \mathbf{1}^T\mathbf{x} = 1 & \text{budget constraint}
\end{array}
\qquad (1)
$$

with $\mathbf{1} = [1] \in \mathbb{R}^n$ the unity vector and where the required expected portfolio return $r_P$ is the only value an agent can choose (Szego [23]). Markowitz's model assumes that the expected return of a portfolio plays the role of a measure of its performance while its variance (or standard deviation) can be considered as a risk measure so that problem (1) seeks the 'best' (in the sense of the less risky) portfolio amongst those with the same expected return.
Additional constraints can be plugged into this model. For instance, vector $\mathbf{x}$ may only hold only non-negative values, that is $x_i \geq 0, i = 1, \dots, n$. This imposition forbids to solve problem (1) by means of the usual Lagrange's multipliers approach. No explicit solution for the optimal portfolio is available in this and similar cases. Jagannathan and Ma introduce an upper bound $x_{MAX}$ to weights so that $x_i \leq x_{MAX}$, $i = 1, \dots, n$. This forces portfolio to be sufficiently diversified as no weight can be excessively large. Behr et al. substitute the non-negativity constraint with $x_i \geq x_{MIN}$, $i = 1, \dots, n$, $x_{MIN} > 0$. Schöttle and Werner, instead, include in the variance minimization a worst-case scenario so that the problem can be stated as

$$
\begin{array}{lllll}
\min\limits_{\mathbf{x}} \max\limits_{(\mathbf{r},\Sigma)\in U} & \sigma_P^2 & = & \mathbf{x}^T\Sigma\mathbf{x} & \\
\text{subject to} & \bar{r}_P & = & \mathbf{r}^T\mathbf{x} = r_P & \text{expected return constraint} \\
 & & & \mathbf{1}^T\mathbf{x} = 1 & \text{budget constraint}
\end{array}
$$

where set $U$ is "the (joint) uncertainty set for the unknown parameters $(\mathbf{r}, \Sigma)$" (see [20]) and whose shape relates to statistics confidence intervals (Meucci [18]).
Going back to problem (1), the explicit formula for the optimal portfolio weights[2], as a function of $r_P$ is

$$
\mathbf{x}^*(r_P) = \frac{(c\Sigma^{-1}\mathbf{r} - b\Sigma^{-1}\mathbf{1})r_P + (a\Sigma^{-1}\mathbf{1} - b\Sigma^{-1}\mathbf{r})}{\Delta} \qquad (2)
$$

where $a = \mathbf{r}^T\Sigma^{-1}\mathbf{r}$, $b = \mathbf{r}^T\Sigma^{-1}\mathbf{1}$, $c = \mathbf{1}^T\Sigma^{-1}\mathbf{1}$, and under the assumption $\Delta = ac - b^2 \neq 0$. Further, the set of these portfolio in the $(r_P; \sigma_P^2)$ plane[3] is the quadratic function

$$
\sigma^2(r_P) = \frac{cr_P^2 - 2br_P + a}{\Delta}.
$$

A portfolio is efficient (according to the mean-variance dominance principle) if there is no other portfolio with the same expected return (variance) and smaller variance (larger expected return). The set of efficient portfolios is named efficient frontier. The choice of the preferred efficient portfolio is done by means of a mean-variance utility function that encompasses the agent's risk aversion.
As recalled in the Introduction, optimal portfolios are in many cases highly sensitive of changes in the input parameters. To try to overcome this drawback, a first attempt is to consider an equally weighted (EW) portfolio where $\mathbf{x}_{EW}^*(i) = 1/n$, $i = 1, \dots, n$. Secondly, the minimum variance (MV) portfolio can be considered. Such portfolio results from a reduced version of (1) that reads

$$
\begin{array}{llll}
\min\limits_{\mathbf{x}} & \sigma_P^2 & = & \mathbf{x}^T\Sigma\mathbf{x} \\
\text{subject to} & & & \mathbf{1}^T\mathbf{x} = 1
\end{array}
$$

and whose explicit solution in vector form is

---

[2] Expression (2) contains the inverse matrix $\Sigma^{-1}$ of the variance/covariance matrix $\Sigma$. Determination of $\Sigma^{-1}$ requires the reciprocal of the determinant of matrix $\Sigma$. It is easy to verify that larger size variance/covariance matrices have determinants very close to 0. For instance, this determinant for the Chopra and Ziemba [5] dataset, that contains ten stocks, is $2.34 \cdot 10^{-25}$. Values of $\mathbf{x}^*$ in (2) can consequently swing greatly even with a tiny change in some input parameter.
[3] It is usual to plot portfolios in the $(\sigma_P; r_P)$ plane, where the standard deviation replaces variance. Under a geometrical point of view the set of such portfolios becomes a hyperbola rather than a parabola.

$$\mathbf{x}_{MV}^{*} = \frac{\Sigma^{-1}\mathbf{1}}{\mathbf{1}^{T}\Sigma^{-1}\mathbf{1}}$$

Another approach determines weights by means of the risk parity assumption (ERC): each stock in the portfolio contributes equally to its overall standard deviation $\sigma_P(x_1, \dots, x_n) = \sqrt{\mathbf{x}^T\Sigma\mathbf{x}}$. This result is obtained exploiting Euler's homogeneous functions theorem[4] that allows to decompose function $\sigma_P(\mathbf{x})$, homogeneous of degree 1, as follows

$$\sigma_P(\mathbf{x}) = \sum_{i=1}^{n} x_i \cdot \frac{\partial}{\partial x_i}\sigma_P(\mathbf{x}).$$

Marginal contributions are expressed as

$$\frac{\partial}{\partial x_i}\sigma_P(\mathbf{x}) = \frac{\sum_{k=1}^{n}\sigma_{i,k}\cdot x_k}{\mathbf{x}^T\Sigma\mathbf{x}}$$

while product $x_i \cdot \frac{\partial}{\partial x_i}\sigma_P(\mathbf{x})$ stands for the risk contribution of the $i$ −th stock. Imposing an equal contribution to the overall risk by each stock, that is,

$$x_i \cdot \frac{\partial}{\partial x_i}\sigma_P(\mathbf{x}) = x_j \cdot \frac{\partial}{\partial x_j}\sigma_P(\mathbf{x}), \ \ \forall i, j = 1, \dots, n, i \neq j$$

along with the usual budget constraint $\mathbf{1}^T\mathbf{x} = 1$ yields portfolio $\mathbf{x}_{ERP}^{*}$.
Unlike the EW and MV cases, there is no explicit expression for this portfolio when $n \geq 3$; its weights must be numerically determined by means of some constrained minimization algorithm[5]. An approach is by solving the optimization problem (see Maillard et al. [14])

$$\arg\min_{\mathbf{x}} \sum_{i=1}^{n}\left(\frac{\sqrt{\mathbf{x}^T\Sigma\mathbf{x}}}{n} - x_i \cdot \frac{\sum_{k=1}^{n}\sigma_{i,k}\cdot x_i}{\sqrt{\mathbf{x}^T\Sigma\mathbf{x}}}\right)$$

In the numerical part of this article, portfolio weights have been obtained exploiting Matlab's website on-line resources[6].
An important departure from the above models is due to Black and Litterman (BL) [4]. Their approach considers a 'prior', that is a set of information merely subsumed from historical market data. A key point here is the reference portfolio, that, in the original BL setting, is the CAPM's (Sharpe [22]) market one. Its weights are obtained by means of "reverse optimization"[7] that considers also a risk aversion parameter $\delta$. Investors' sentiments and knowledge (the 'posterior') are introduced by means of $k$ 'views' that can be either absolute or relative. An absolute view is a claim made on the future behaviour of a specific stock. A relative view allows, instead, to include an opinion on the relative performance on two or more stocks. The matrix that identifies such views is denoted by $P \in \mathcal{M}(k, n)$ while $Q \in \mathbb{R}^k$ is the view vector. Each row in $P$ introduces a view whose numerical claim is an element in $Q$. Vector $\Pi \in \mathbb{R}^n$ is the reference portfolio. Along with Bayesian decision theory, errors in judgement should be attached to predictions. The numerical measurements of these quantities are expressed in terms of variances of the views and contained into a diagonal matrix $\Omega = [\omega_{u,v}] \in \mathcal{M}(k, k),\ u, v = 1, \dots, k$ where $\omega_{u,v} = 0$ whenever $u \neq v$. A way to determine $\Omega$ is proposed by Meucci [18]:

$$\Omega = \left(\frac{1}{c} - 1\right) P\Sigma P' \qquad (3)$$

where $c \in (0; 1)$ so that if $c \to 0$ views are not deemed informative while when $c \to 1$ views are entirely trusted. A final parameter needed is $\tau$, whose aim is to shift the model's focus to either market portfolios or views. All these pieces of information lead to a view-corrected vector of expected returns (Meucci [18])

$$\mathbf{r}_{\mathrm{BL}} = \mathbf{r} - \Sigma P'(P\Sigma P' + \Omega)^{-1}(Q - P\mathbf{r}) \qquad (4)$$

and a view corrected variance-covariance matrix

$$\Sigma_{BL} = \Sigma - \Sigma P'(P\Sigma P' + \Omega)^{-1}P\Sigma. \qquad (5)$$

---

[4] Let $f(\mathbf{x})\colon \mathbb{R}_+^n \to \mathbb{R}$ be a $C^1$ homogeneous function of degree $\gamma$. Then, for all $\mathbf{x}$

$$x_1 \cdot \frac{\partial}{\partial x_1}f(\mathbf{x}) + \dots x_n \cdot \frac{\partial}{\partial x_n}f(\mathbf{x}) = \gamma \cdot f(\mathbf{x})$$

[5] In Appendix, a brief discussion of the $n = 2$ case, that has an analytic solution, is presented.
[6] Refer to https://it.mathworks.com/matlabcentral/answers/278745-risk-parity-equal-risk-contribution-optimization, where MATLAB's function fmincon is exploited.
[7] For the full mathematical description of the BL model, refer to (Idzorek [12]).

Having concluded the theoretical part and armed with all required formulas, the rest of this article tackles a real-market application and analyses its results.

## 3) Data and numerical results

In order to test the claim this article evaluating, historical weekly prices of five stocks[8] (ENI, E.ON, Generali, SAP, and Volkswagen) have been considered. This choice has no specific reason but to encompass companies whose random returns can be considered sufficiently diversified. Correlations between random returns in Table 3a confirms this and hints that portfolios containing these stocks should allow for some degree of diversification.

Overall data ranging from the beginning of 2016 to the end of 2020, for a total of 260 weekly log-returns, have been divided in two subgroups: the first 208 returns (years 2016-2019) are used to determine the historical vector of expected returns $\mathbf{r}$ (Table 1, second column), the variance/covariance matrix $\Sigma$ (Table 2a), and the correlation matrix (Table 3a). These are the input required to determine portfolio weights in the cases described before. Variance-covariance (Table 2b) and correlation (Table 3b) matrices come from the remaining 52 observations (year 2020). Table 1 also contains additional descriptive statistics; as a remark, excess kurtosis for all stocks is positive, suggesting that, as usual with stocks, historical returns show a leptokurtic (i.e. fat-tail) behaviour for which 'rare' events occur with a frequency unattainable to phenomena described with the normal distribution.

*Table 1a – Descriptive Statistics for 2016-2019 weekly log-returns of companies E.On, ENI, Generali, SAP, and Volkswagen. (Total number of observations: 208)*

| stock | Mean ($\mathbf{r}_{2015-2019}$) | median | min | max | st. dev. | skewness | exc. Kurtosis |
|---|---|---|---|---|---|---|---|
| E.ON | 0.0022 | -0.0007 | -0.1517 | 0.1161 | 0.0366 | -0.1862 | 1.6377 |
| ENI | 0.0016 | 0.0032 | -0.0731 | 0.1127 | 0.0291 | -0.0097 | 0.6887 |
| GEN | 0.002 | 0.004 | -0.1192 | 0.1301 | 0.0329 | -0.1077 | 2.3352 |
| SAP | 0.0028 | 0.0049 | -0.0778 | 0.1292 | 0.028 | 0.3308 | 2.1864 |
| VW | 0.0019 | 0.0005 | -0.0779 | 0.1248 | 0.0343 | 0.391 | 0.5606 |

*Table 1b – Mean of weekly log-returns for companies under scrutiny for year 2020 - (a): Jan 1st-Jun 30th; (b): Jan 1st-Dec 31st (Total number of observations: 52)*

| stock | 6-month return (a) | 12-month return (b) |
|---|---|---|
| E.ON | 0.2323 | -0.00613 |
| ENI | -0.82225 | -0.41717 |
| GEN | -0.50433 | -0.22083 |
| SAP | 0.15215 | -0.09307 |
| VW | -0.35566 | 0.00618 |

*Table 2a – Variance/Covariance matrix ($\Sigma_{2015-2019}$) for observed weekly log-returns (2016-1019)*

| | E.ON | ENI | GEN | SAP | VW |
|---|---|---|---|---|---|
| E.ON | 0.00134 | | | | |
| ENI | 0.00045 | 0.00085 | | | |
| GEN | 0.00029 | 0.00044 | 0.00108 | | |
| SAP | 0.00026 | 0.00027 | 0.00032 | 0.00078 | |
| VW | 0.00041 | 0.00038 | 0.00044 | 0.00038 | 0.00118 |

*Table 2b – Variance/Covariance matrix for observed weekly log-returns (2020)*

| | E.ON | ENI | GEN | SAP | VW |
|---|---|---|---|---|---|
| E.ON | 0.00221 | | | | |
| ENI | 0.00274 | 0.00677 | | | |
| GEN | 0.00225 | 0.00414 | 0.00304 | | |
| SAP | 0.00202 | 0.00289 | 0.00246 | 0.00421 | |
| VW | 0.00241 | 0.00389 | 0.00336 | 0.00297 | 0.00436 |

*Table 3a – Correlation matrix for observed weekly log-returns (2016-2019)*

| | E.ON | ENI | GEN | SAP | VW |
|---|---|---|---|---|---|
| E.ON | 1 | | | | |
| ENI | 0.42174 | 1 | | | |
| GEN | 0.23997 | 0.46369 | 1 | | |
| SAP | 0.2505 | 0.33118 | 0.34307 | 1 | |
| VW | 0.32716 | 0.37912 | 0.39445 | 0.392998 | 1 |

A comparison between values in Tables 2a, 2b, 3a, and 3b clearly displays that, due to the COVID-19 induced financial crisis, all variances, covariances, and correlations in year 2020 substantially increase. This means that portfolios whose composition has been obtained using historical data from 2016 to 2019 (Table 4) might suffer of severe misspecification if applied to 2020 data, resulting in large, unexpected risk levels.

[8] Data (adjusted closing prices) retrieved from https://it.finance.yahoo.com/

*Table 3b – Correlation matrix for observed weekly log-returns (2020)*

| | E.ON | ENI | GEN | SAP | VW |
|---|---|---|---|---|---|
| E.ON | 1 | | | | |
| ENI | 0.70728 | 1 | | | |
| GEN | 0.81096 | 0.85554 | 1 | | |
| SAP | 0.66282 | 0.54573 | 0.64565 | 1 | |
| VW | 0.77441 | 0.72173 | 0.86674 | 0.69235 | 1 |

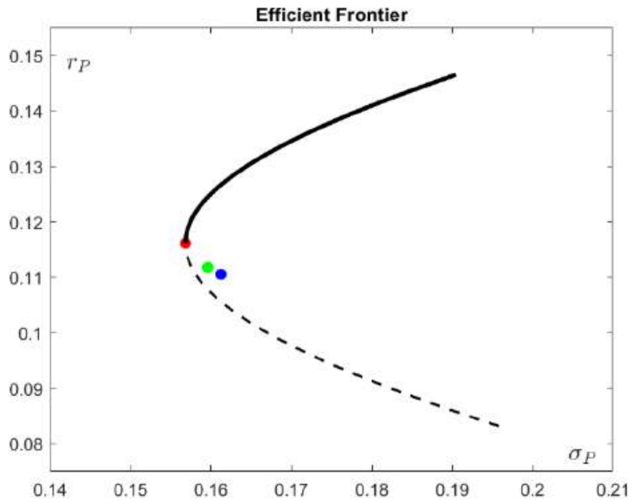
*Table 4 – Portfolio weights for the equally weighted (second column), minimum variance (third column) and equally risk contribution (fourth column) portfolios obtained using the 2016-2019 period data*

| | EW | MV | ERC |
|---|---|---|---|
| E.ON | 0.2 | 0.1376 | 0.1847 |
| ENI | 0.2 | 0.2433 | 0.206 |
| GEN | 0.2 | 0.1534 | 0.193 |
| SAP | 0.2 | 0.3656 | 0.2354 |
| VW | 0.2 | 0.1003 | 0.1809 |

A first remark that can be drawn looking at weights in Table 4 is that, according to the peculiar choice of stocks made here, the EW composition acts as a 'continental divide' between stocks whose relative quantities are smaller (larger) than 0.2. If the weight of a stock is less than the EW one in the MV portfolio it does not trespass this threshold in the ERC portfolio as well. A possible explanation to this fact is that the minimum variance and equal risk contribution approaches treat risky and less risky stocks alike, assigning them smaller or larger weights.

Figure 1 represents the efficient frontier (continuous plot) in the usual standard deviation/expected return two-dimensional plane; the red point pinpoints the MV portfolio, the blue and green ones, respectively, the EW and ERC portfolios. For ease of display, returns and standard deviations have been transformed on a yearly basis.

*Figure 1 – efficient frontier (black continuous curve) in the $(\sigma_P, r_P)$ plane with dots depicting portfolios: EW (blue dot), MV (red dot) and ERC (green dot). Data have been annualized.*




It is interesting to notice (Table 5) that MV portfolio dominates, according to the mean-variance criterion, the ERC one and that EW portfolio is dominated by the other two. This result seems acceptable as the latter portfolio is determined without trying to actively manage its risk.

*Table 5 – realized yearly expected returns and standard deviations for portfolios under scrutiny (2016-2019)*

| | expected return | standard deviation |
|---|---|---|
| MV | 0.1161 | 0.1568 |
| EW | 0.1105 | 0.1612 |
| ERC | 0.1118 | 0.1596 |

Efficient portfolios and their expected returns with the same standard deviation as the EW and ERC ones are reported in Table 6. Even if only two efficient portfolios are displayed, it seems evident that both assign a large weight to SAP stock. This result might derive to the fact that SAP Sharpe's ratio is larger than the other ones. Even if this is a theoretically correct choice, such feature could deliver portfolios that are not well-balanced as they are exposed to any negative changes in the performance of SAP's stock.

*Table 6 – weights and expected returns of efficient portfolios with the same standard deviation of the EW and ERC ones (annualized data).*

| | $\sigma_P = 0.1612$ (EW) | $\sigma_P = 0.1596$ (ERC) |
|---|---|---|
| E.ON | 0.17024 | 0.16351 |
| ENI | 0.09708 | 0.12719 |
| GEN | 0.15376 | 0.15368 |
| SAP | 0.5278 | 0.49438 |
| VW | 0.05112 | 0.06125 |
| $r_P$ | 0.1266 | 0.12445 |

The first result in the empirical analysis deals with the performance of 'historically' determined (i.e., based on 2016-2019 data) EW, MV, and ERC portfolios when applied to year 2020 values. This is achieved by computing the realized yearly return[9] using data for time interval Jan 1st-June 30th, 2020 (interval a) and the realized yearly return and standard deviation for time interval Jan 1st-Dec 31st, 2020 (interval b). Period Jan 1st – Mar 31st, 2020 cannot be unfortunately considered as its yearly log-returns show an abnormal behaviour as some returns are so negative that their transformation on an yearly basis makes them with no sensible financial meaning. Table 7 displays these values.

*Table 7: realized yearly returns and realized 12-month standard deviations for EW, MV, and ERC portfolios in 2020. Due to the negative dynamics of stock prices in 2020, losses occur in all cases.*

| portfolios | 6-month return (interval a) | 12-month return (interval b) | 12-month st dev (interval b) |
|---|---|---|---|
| EW | -0.2596 | -0.1462 | 0.40605 |
| MV | -0.2255 | -0.1696 | 0.41346 |
| ERC | -0.2524 | -0.1505 | 0.40687 |

A comparison between Tables 5 and 7 further shows how riskier and poorly performing portfolios made of the chosen stocks ended up being in 2020.
Another benchmark is displayed in Table 8 where portfolio weights for the MV and ERC cases obtained using Table 2b data (year 2020 variance/covariance matrix) are:

*Table 8 – Portfolio weights for the minimum variance (second column) and equally risk contribution (third column) portfolios obtained using variance/covariance matrix in Table 8.*

| | MV | ERC |
|---|---|---|
| E.ON | 0.9532 | 0.6589 |
| ENI | -0.2569 | -1.2909 |
| GEN | 0.4165 | 0.7799 |
| SAP | 0.1237 | 0.4065 |
| VW | -0.2364 | 0.4455 |

These weights produce abnormal portfolios with both short selling and large fractions invested in some stocks, features that can hardly be associated to proper risk reduction.
The second analysis performed relies on the Black-Litterman version of EW, MV, and ERC portfolios. In the BL framework the starting portfolio is the implied equilibrium one, out of which excess equilibrium returns are obtained. Here, partially departing from the underlying theory, such returns are the historical ones (Table 1a, second column). Further, as risk-less rates during period 2015-2020 have been very close to zero (or even negative), excess returns can be set equal to realized ones. Views are assumed to be pessimistic. In a first instance, all expected returns are obtained subtracting to each historical expected return a portion of the respective standard deviation. Matrix $P$ is

$$P = \begin{bmatrix} -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 \end{bmatrix}.$$

The single, negative value in each of its rows translates the absolute view made on each stock: to the expected return of the $i$ −th stock ($i$ −th element of the $i$ −th row) is assigned the $i$ −th element of view vector

$$Q = \begin{bmatrix} 0.00223 - 0.5 \cdot 0.03661 \\ 0.0016 - 0.5 \cdot 0.02913 \\ 0.00204 - 0.5 \cdot 0.03285 \\ 0.00282 - 0.5 \cdot 0.02797 \\ 0.00194 - 0.5 \cdot 0.03432 \end{bmatrix} = \begin{bmatrix} -0.01597 \\ -0.012965 \\ -0.014386 \\ -0.011163 \\ -0.015221 \end{bmatrix} \qquad (6)$$

[9] For the six-month interval, the number of available observations is deemed to be not sufficient to be safely trusted as standard deviations might end up with misleading results.

where 50% of the 2015-2019 standard deviation is assumed to be large enough to penalize historical expected returns. Diagonal matrix $\Omega$ is obtained according to (3), letting $c = 0.25$ (views are mildly informative) and $c = 0.75$ (views display a large degree of trustworthiness). Posterior estimates for the expected returns are, according to (4)

$$\bar{\mathbf{r}}_{BL} = \begin{bmatrix} -0.00108 \\ -0.00124 \\ -0.00105 \\ 0.00073 \\ -0.00138 \end{bmatrix} \text{ when } c = 0.25 \text{ and } \bar{\mathbf{r}}_{BL} = \begin{bmatrix} -0.0079 \\ -0.00692 \\ -0.00722 \\ -0.00344 \\ -0.00802 \end{bmatrix} \text{ when } c = 0.75.$$

Needless to say, all estimates but the one for SAP in the first case, are smaller than historical returns. Posterior estimates of the variance-covariance matrices (see formula (5)) are

$$\Sigma_{\mathrm{BL}} = \begin{bmatrix} 0.00168 & & & & \\ 0.00056 & 0.00106 & & & \\ 0.00036 & 0.00055 & 0.00135 & & \\ 0.00032 & 0.00034 & 0.00039 & 0.00098 & \\ 0.00051 & 0.00047 & 0.00056 & 0.00047 & 0.0015 \end{bmatrix} (c_1 = 0.25),$$

$$\text{and } \Sigma_{BL} = \begin{bmatrix} 0.00235 & & & & \\ 0.00079 & 0.00149 & & & \\ 0.00051 & 0.00078 & 0.00189 & & \\ 0.00045 & 0.00047 & 0.00055 & 0.00137 & \\ 0.00072 & 0.00066 & 0.00078 & 0.00066 & 0.00206 \end{bmatrix} (c_1 = 0.75).$$

Using these latter inputs, it results that MV and ERC portfolios are the same obtained above. This result is due to the fact that both matrices embed larger variances and covariances but correlations are the same. This particular instance shows that when all stocks share the same view, the Black-Litterman model correctly changes expected return vectors and variance-covariance matrices but is unable to adjust correlations.
Secondly, absolute views are provided only for a stock at a time. Matrix $P$ shrinks to a row vector with all null elements but one, equal to $-1$, its position in $P$ identifying which stock the views relates to. The view value is corresponding value in $Q$. If, for instance, the absolute negative view is about E.ON, the first stock in matrix $Q$ (see (6)), posterior estimates for the expected returns are now

$$\bar{\mathbf{r}}_{BL} = \begin{bmatrix} -0.00108 \\ 0.000455 \\ 0.001305 \\ 0.002167 \\ 0.000894 \end{bmatrix} \text{ when } c_1 = 0.25 \text{ and } \bar{\mathbf{r}}_{BL} = \begin{bmatrix} -0.0079 \\ -0.00183 \\ -0.00016 \\ 0.00086 \\ -0.0012 \end{bmatrix} \text{ when } c_1 = 0.75$$

while the variance-covariance matrices here read

$$\Sigma_{\mathrm{BL}} = \begin{bmatrix} 0.00168 & & & & \\ 0.00056 & 0.00089 & & & \\ 0.00036 & 0.00047 & 0.00109 & & \\ 0.00032 & 0.00029 & 0.00033 & 0.00079 & \\ 0.00051 & 0.00041 & 0.00047 & 0.0004 & 0.00121 \end{bmatrix} (c = 0.25),$$

$$\text{and } \Sigma_{BL} = \begin{bmatrix} 0.00235 & & & & \\ 0.00079 & 0.00096 & & & \\ 0.00051 & 0.00052 & 0.00112 & & \\ 0.00045 & 0.00033 & 0.00036 & 0.00082 & \\ 0.00072 & 0.00048 & 0.00051 & 0.00044 & 0.00127 \end{bmatrix} (c = 0.75).$$

These results deserve some comments. Expected returns, variances and covariances of stocks unaffected by the view improve, in the sense that the BL adjustment leads to larger expected returns and smaller variances and covariances while data for E.ON remain unchanged. Remarkably, an absolute view does not directly affect the stock its posterior estimates but estimates of the other stock involved in the analysis. MV and ERC portfolios are

*Table 9 – Portfolio weights for minimum variance (second and third columns) and equally risk contribution (fourth and fifth columns) portfolios obtained using variance-covariance matrices encompassing a negative, absolute view on E.ON*

| | MV ($c = 0.25$) | MV ($c = 0.75$) | ERC ($c = 0.25$) | ERC ($c = 0.75$) |
|---|---|---|---|---|
| E.ON | 0.0721 | -0.0161 | -0.6623 | -0.5802 |
| ENI | 0.2617 | 0.2866 | 0.485 | 0.4682 |
| GEN | 0.165 | 0.1807 | 0.3536 | 0.3351 |
| SAP | 0.3933 | 0.4307 | 0.4483 | 0.4253 |
| VW | 0.1079 | 0.1181 | 0.3754 | 0.357 |

Weights in Table 9 reflect the absolute and pessimistic view attached to E.ON stock. Its contribution to the optimal portfolio decreases in all cases. On top of this, in three instances the optimal portfolios carry negative weights of E.ON stock. The level of its risk is deemed so large when compared to the ones of the other stocks that an advantage in short selling the stock appears.

Finally, Tables 10a and 10b display the view-adjusted correlation matrices.

*Table 10a – Correlation matrix for view-adjusted variance-covariance matrix* ($c = 0.25$)

| | E.ON | ENI | GEN | SAP | VW |
|---|---|---|---|---|---|
| E.ON | 1 | | | | |
| ENI | 0.5793 | 1 | | | |
| GEN | 0.46406 | 0.72662 | 1 | | |
| SAP | 0.43745 | 0.57332 | 0.54819 | 1 | |
| VW | 0.55384 | 0.68308 | 0.65305 | 0.70684 | 1 |

*Table 10b – Correlation matrix for view-adjusted variance-covariance matrix* ($c = 0.75$)

| | E.ON | ENI | GEN | SAP | VW |
|---|---|---|---|---|---|
| E.ON | 1 | | | | |
| ENI | 0.5793 | 1 | | | |
| GEN | 0.46406 | 0.73273 | 1 | | |
| SAP | 0.43745 | 0.58961 | 0.5628 | 1 | |
| VW | 0.55384 | 0.70831 | 0.67381 | 0.72985 | 1 |

It is worth stressing that even if the view regards only one stock, correlations between the other stocks end up being affected by the view itself.
Similar findings occur when applying an absolute view to one of the remaining stocks. For sake of paucity numerical results have been omitted.

## 4) Conclusions

Since finance started being, in the second half of the twentieth century, a topic of its own, risk management has strived to tackle real market problems under both a theoretical and a practical point of view. Among the tools that can be used in this context, mean-variance portfolio analysis has been, by far, the most studied and investigated. Unfortunately, this methodology (at least in the range this article has dealt with) proves to be unsuccessful when markets are hit by crashes and severe turbulences. This article has also shown that the Black-Litterman approach, is uncapable of modifying correlations between random returns when views are absolute and share the same structure. This might mean that the application of the BL methodology, that confirms its importance when applied to the forecasted behaviour of single stocks, needs to be finely tuned when an overall drop in stock prices is expected. Even if Markowitz's model provides some useful insights and the basis for more sophisticated approaches, up to the extent of stocks considered in this contribution its solid market application appears to be debatable.
It might be interesting, and left for subsequent research, to perform a similar analysis where more recent risk measures, such as Value-At-Risk and Expected Shortfall (Artzner et al. [1], Bagnato et al. [2]) replace variance. With tools capable of detecting tail and non-normal shapes of risk an analysis similar to the one performed here might result with a more positive ending.

## Appendix

In order to provide an intuition of the structure of these weights, consider case $n = 2$ where an explicit expression for $\mathbf{x}^*_{ERP}$ exists. Starting from the risk contributions of stocks 1 and 2

$$\frac{x_1^2\sigma_1^2 + x_1x_2\sigma_{1,2}}{x_1^2\sigma_1^2 + 2x_1x_2\sigma_{1,2} + x_2^2\sigma_2^2}, \qquad \frac{x_1x_2\sigma_{1,2} + x_2^2\sigma_2^2}{x_1^2\sigma_1^2 + 2x_1x_2\sigma_{1,2} + x_2^2\sigma_2^2}$$

and recalling that $x_1 + x_2 = 1$, equating the two above quantities yields

$$\mathbf{x}^*_{ERC} = \left[\frac{\sigma_2}{\sigma_1+\sigma_2}, \qquad \frac{\sigma_1}{\sigma_1+\sigma_2}\right]$$

It is easy to verify that in this case portfolio weight of the first (second) stock is linearly dependant on standard deviation of the second (first) stock and that these weights do not depend on the correlation between random returns of the two stocks (Maillard et al. [15]).

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## Bibliography


[1] Artzner, P., Delbaen, F., Eber, J. M., Heath, D. (1999). “Coherent measures of risk”. *Mathematical Finance*, 9(3), 203-228. (DOI: https://doi.org/10.1111/1467-9965.00068)

[2] Bagnato, M., Bottasso, A., Giribone, P.G. (2021). “Design of an algorithm for an adaptive Value at Risk measurement through the implementation of robust methods in relation to asset cross-correlation”. Vol. 16, n. 1. (https://www.aifirm.it/wp-content/uploads/2021/05/RMM-2021-01-Excerpt-5.pdf)

[3] Behr, P., Guettler, A., Miebs, F. (2013). “On portfolio optimization: Imposing the right constraints”. *Journal of Banking and Finance*, 37(4), 1232-1242. (DOI: https://doi.org/10.1016/j.jbankfin.2012.11.020)

[4] Black, F., Litterman, R., (1992). “Global portfolio optimization”. Financial Analysts Journal, 48(5), 28-43. (DOI: https://doi.org/10.2469/faj.v48.n5.28)

[5] Chopra V.K., Ziemba, W.T., (1993), “The Effect of Errors in Means, Variances, and Covariances on Optimal Portfolio Choice”, *Journal of Portfolio Management*, 6. (DOI: https://doi.org/10.3905/jpm.1993.409440)

[6] Coqueret, G. (2015). “Diversified minimum-variance portfolios”. *Annals of Finance*, 11(2), 221-241. (DOI: https://doi.org/10.1007/s10436-014-0253-x)

[7] DeMiguel, V., Garlappi, L., Uppal, R., (2007). “Optimal versus naive diversification: How inefficient is the 1/N portfolio strategy?”. *The Review of Financial Studies*, 22(5), 1915-1953. (DOI: https://doi.org/10.1093/rfs/hhm075)

[8] DeMiguel, V., Garlappi, L., Nogales, F. J., Uppal, R., (2009). “A generalized approach to portfolio optimization: Improving performance by constraining portfolio norms”. *Management science*, 55(5), 798-812. (DOI: https://doi.org/10.1287/mnsc.1080.0986)

[9] Fliege, J., Werner, R., (2014). “Robust multiobjective optimization & applications in portfolio optimization”. *European Journal of Operational Research*, 234(2), 422-433.

[10] Goldfarb, D., Iyengar, G., (2003). “Robust portfolio selection problems”. *Mathematics of operations research*, 28(1), 1-38. (DOI: https://doi.org/10.1287/moor.28.1.1.14260)

[11] Goldfarb, D., Iyengar, G., (2003). “Robust convex quadratically constrained programs”. *Mathematical Programming*, 97(3), 495-515. (DOI: https://doi.org/10.1007/s10107-003-0425-3)

[12] Idzorek T.M. (2007). “A step-by-step guide to the Black-Litterman model: Incorporating user-specified confidence levels”. In Forecasting expected returns in the financial markets, 17-38. Academic Press.

[13] Jagannathan, R., Ma, T., (2003. “Risk reduction in large portfolios: why imposing the wrong constraints helps”. *Journal of Finance* 58, 1651–1684. (DOI: https://doi.org/10.1111/1540-6261.00580)

[14] Maillard, S., Roncalli, T., Teiletche, J. (2009). “On the properties of equally-weighted risk contributions portfolios”. Working paper retrieved from http://www.thierry-roncalli.com/download/erc.pdf on June 22nd, 2021.

[15] Maillard, S., Roncalli, T., Teïletche, J. (2010). “The properties of equally weighted risk contribution portfolios”. *The Journal of Portfolio Management*, 36(4), 60-70. (DOI: https://doi.org/10.3905/jpm.2010.36.4.060)

[16] Markowitz, H., (1952), “Portfolio selection”, *Journal of Finance* 7, 77-91. (DOI: https://doi.org/10.1111/j.1540-6261.1952.tb01525.x)

[17] Markowitz, H., (1959), Portfolio Selection: Efficient Diversification of Investments, Cowles Foundation Monograph #16 (Wiley, New York)

[18] Meucci, A. (2009). Risk and asset allocation. Springer Science & Business Media

[19] Roncalli, T. (2016), Introduction to risk parity and budgeting, CRC Press (Roca Raton, Florida, USA)

[20] Schöttle, K., Werner, R. (2006). “Towards reliable efficient frontiers”. *Journal of Asset Management*, 7(2), 128-141. (DOI: https://doi.org/10.1057/palgrave.jam.2240208)

[21] Schöttle, K., Werner, R. (2009). “Robustness properties of mean-variance portfolios”. *Optimization*, 58(6), 641-663. (DOI: https://doi.org/10.1080/02331930902819220)

[22] Sharpe, W. (1964). Capital Asset Prices: A Theory of Market Equilibrium Under Conditions of Risk. *Journal of Finance*, 19, 425–42.

[23] Szegö, G. P. (1980). Portfolio theory: with application to bank asset management. Academic Press. (New York, USA)