EXPERIMENTAL OLIGOPOLIES MODELING: A DYNAMIC APPROACH BASED ON HETEROGENEOUS BEHAVIORS.

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ABSTRACT. In the rank of behavioral rules, imitation-based heuristics has received special attention in economics (see [Vega-Redondo, 1997] and [Schlag, 1998]). In particular, imitative behavior is considered in order to understand the evidences arising in experimental oligopolies which reveal that the Cournot-Nash equilibrium does not emerge as unique outcome and show that an important component of the production at the competitive level is observed (see e.g. [Apesteguia et al., 2007], [Oechssler et al., 2016], [Bigoni and Fort, 2013] or [Offerman et al., 2002], [Huck et al., 1999]). By considering the pioneering groundbreaking approach of [Apesteguia et al., 2010], we build a dynamical model of linear oligopolies where heterogeneous decision mechanisms of players are made explicit. In particular, we consider two different types of quantity setting players characterized by different decision mechanisms that coexist and operate simultaneously: agents that adaptively adjust their choices towards the direction that increases their profit are embedded with imitator agents. The latter ones use a particular form of *proportional imitation rule* that considers the awareness about the presence of strategic interactions. It is noteworthy that the Cournot-Nash outcome is a stationary state of our models. Our thesis is that the chaotic dynamics arousing from a dynamical model, where heterogeneous players are considered, are capable to qualitatively reproduce the outcomes of experimental oligopolies.

Keywords: Imitation, heterogeneity, dynamic instability, dynamical systems

1. INTRODUCTION

Several experimental oligopoly games of quantity setting players provide important evidence that the outcome of a competition among real players is not in line with the Cournot-Nash equilibrium of the corresponding one-shot game. In particular, such experiments show that none of the relevant notion of equilibrium, as well as in any other production level, emerges uniquely. In addition, outcomes where a unique average choice is reached with a Gaussian spread, due to random factors that affect decision processes, are also excluded. On the contrary, distributions of choices over a range of the quantity space that generally includes the Cournot-Nash and the competitive (Walrasian) equilibrium, are observed thus implying that no convergence towards a common decision is achieved. Therefore, it can be concluded that this circumstance is a strong indication of the fact that agents are not perfectly rational.

However, depending on the particular experimental setup, many peaks in the quantities' distributions, that correspond to certain production levels, can be distinguished. Specifically, the partial prevalence of choices that match with the Cournot-Nash production level is clearly recognizable in the histograms presented, for example, in [Offerman et al., 2002] or in [Bigoni and Fort, 2013]. The time series presented in [Oechssler et al., 2016] also reveals the occurrence of quantities at that level. This fact can be interpreted as an indirect indication of the presence of a relevant component of rationality within the decision mechanisms that determine the behavior of agents. Furthermore, from the same experiments as well as in many others, it emerges also that, besides the Cournot-Nash, the competitive production level is systematically observed (see a.g. [Huck et al., 1999], [Apesteguia et al., 2007], [Huck et al., 2004], [Bigoni and Fort, 2013], [Oechssler et al., 2016], [Friedman et al., 2015]). This fact can be interpreted as an indirect indication of the presence of the imitative behavior at the light of the result provided by [Vega-Redondo, 1997] that gives the theoretical underpinning for the emergence of the competitive equilibrium when imitative-based decision mechanism is assumed. More precisely, in [Vega-Redondo, 1997], the author considers a symmetric oligopoly of quantity setting agents that "*imitate the best*" by selecting the quantities that brought the highest profits in the previous period. Therefore, accounting for an arbitrarily small, but non-vanishing, mutation probability in agents' strategies, the competitive outcome is observed almost certainly in the long run, namely it is a *stochastically stable state*. "*The intuition of this result is straightforward: whenever price is higher than marginal cost, the agent with the highest quantity makes the largest profit (being thus imitated) and vice versa if profits are negative*" ([Huck et al., 1999]).

What can be guessed from such empirical evidences is that both the imitation and rationality behaviors are not sufficient to individually explain the experimental outcomes. However, those results suggest that the decision mechanisms of players may be represented as made up by heterogeneous components, specifically rationality and imitation. In this view, models for real decision mechanisms can be provided by embedding together different behaviors that concur together to decision processes. A first attempt towards this direction is provided in [Huck et al., 1999] and then taken up by [Apesteguia et al., 2010].

In those papers, the authors follow a direct approach by considering an explicit expression for individual behavior, modeled as a mixed combination of different adaptive decision mechanisms involved simultaneously in decision processes. Specifically, in [Apesteguia et al., 2010], imitation (in the sense of [Vega-Redondo, 1997]), best reply, fictitious play and relative profit maximization are considered. Together with those deterministic behavioral rules, the authors consider also a relevant contribution of random factors affecting decision processes in order to reproduce distributions of choices over the quantities' space observed in experiments. The stochastic model they get is characterized by not having any absorbing state but only a large absorbing set. We note that, besides this last generic result, the analysis of that model is limited only to simulations.

Here we echo the thought suggested by [Apesteguia et al., 2010] and develop a theoretical model of linear oligopolies of quantity setting players that considers explicit expressions of different decision mechanisms that coexist and operate simultaneously thus representing heterogeneous behaviors in the population. The model we consider is characterized by a deterministic structure and it is here studied analytically, together with its dynamical properties, going beyond a pure simulative approach. In particular we show that unpredictable dynamics related to experimental outcomes could be qualitative represented by the chaotic dynamics that the heterogeneity in decision mechanisms can give rise, with no need of any stochastic perturbation. In other words, experimental evidences, i.e. the distribution of choices and in the absence of convergence towards a common decision, can be understood through dynamical instabilities. This is a direct consequence of the approach suggested by [Apesteguia et al., 2010] since it is the presence of heterogeneities among decision mechanisms that determines complex dynamical scenarios. In this view we assert that chaotic dynamics are suitable modeling tools to represent the repeated decisions observed in experimental oligopolies. We then compare the outcomes of the model with those arising in experiments and reported in the cited literature.

Specifically, we introduce the heterogeneity among behaviors by considering different kinds of players. The first decision mechanism we consider is the gradient rule, first introduced in [Bischi and Naimzada, 2000], and we call the players that use such rule "*gradient*" agents. The second decision mechanism we consider is a kind of proportional imitation rule (similar to the one introduced in [Schlag, 1998]) that prescribes, at each period, the choice given by the weighted average of the previous period quantities, whose importance is proportional to the payoff they have respectively generated. We call the players that use such rule "*imitator*" agents. Such a particular form imitation accounts for the fact that *imitator* agents are aware about the presence of strategic interactions. Indeed, within the framework of experimental oligopolies, it is common knowledge that the price is a decreasing function of the total amount of goods in the marked, which in turn is determined by the choices of all players. The awareness about the presence of strategic interactions thus turns into the consciousness that an action that brought high profit in the previous period may not produce so good a result in the present time because of changes in environmental conditions. This indeterminacy about the profits that an action will produce is the reason that led us to consider such a *prudent imitative behavior*.

We note that the imitation rule here considered does not provide any profit-based selection. This marks the main difference from those rules considered by [Vega-Redondo, 1997] and by [Schipper, 2009] where players make a careless and incautious selection among the quantities to imitate. Indeed, according to the rule defined by [Vega-Redondo, 1997] every player refuses to imitate every strategy that did not produce the best result, while, according to the rule defined by [Schlag, 1998], every player refuses to imitate every strategy that has produced lower profits than his own.

Finally, it is worth noting that the model here derived accounts for heterogeneities only at a collective level, namely only different players characterized by their own decision mechanisms made up by a single behavior are considered. As deepened within the concluding part of the present work, a future stream of research may concern the study of a more structured model where heterogeneities also at an individual level, where mixed behaviors operate simultaneously in the decision mechanism of a single player, are taken into account.

The paper is organized as follows. In Section 2 we derive the model and we study it both with analytical and numerical tolls. Conditions for the stability of stationary states are provided, as well as comments on the role of the most relevant parameters in determining stability properties are given. In Section 3 we compare the outcomes of experimental oligopolies with the dynamical scenarios provided by the model. Section 4 concludes.

2. THE MODEL

Let's consider a set of identical quantity setting agents $\mathcal{N} = \{1, 2, ..., N\}$ that compete in the same market for an homogeneous good, whose demand is summarized by a linear inverse-demand function $P(Q) = \max\{a - bQ, 0\}$. Let's denote by $q_{i,t}$ the quantity of goods that the generic i-th agent, with $i \in \mathcal{N}$, sells in the market at time-period t. Furthermore, all the agents bear the same constant marginal production cost c, so that the generic i-th agent earns the profit

$$\pi_i = P(Q)q_i - cq_i \tag{2.1}$$

We characterize the oligopoly by introducing heterogeneous decision mechanisms, used to decide how much quantity of goods to produce, by considering a population structured into two groups of agents of different kinds. The first group, denoted by $\mathcal{N}^{\mathsf{G}} \subseteq \mathcal{N}$ and with numerosity $N^{\mathsf{G}} := |\mathcal{N}^{\mathsf{G}}|$, includes boundedly rational players that use the *gradient* rule, first proposed in [Bischi and Naimzada, 2000], that will be called *gradient* players. The second group, denoted by $\mathcal{N}^{\mathsf{I}} \subseteq \mathcal{N}$ and with numerosity $N^{\mathsf{I}} := |\mathcal{N}^{\mathsf{I}}|$, includes agents that adopt an imitation-based decision mechanism, which will be discussed later, and that will be called *imitator* players. Clearly it is $\mathcal{N}^{\mathsf{G}} \cup \mathcal{N}^{\mathsf{I}} \equiv \mathcal{N}$ and $N^{\mathsf{G}} + N^{\mathsf{I}} = |\mathcal{N}| := N$. The population's splitting can be summarized by the fraction ω of *imitators*, that is $\omega = N^{\mathsf{I}}/N$. Consequently, the fraction of *gradient* players is given by $N^{\mathsf{G}}/N = (N - N^{\mathsf{I}})/N = 1 - \omega$.

Within the experimental oligopoly framework, the informational set provided to each agent usually includes his own previous period quantity and the related profit as well as those of his competitors (see [Friedman et al., 2015], [Apesteguia et al., 2010] and [Offerman et al., 2002] for example). It results that each agent does not have any knowledge of the demand function. Taking into account such a circumstance, we assume that a portion of the agents involved in the experiment use the informational set provided to them trying to infer how the environment will respond to their own production changes by an empirical estimate of the marginal profit. More precisely, we assume that the i-th player in a certain group $\mathcal{N}^{\mathsf{G}}$ mimics the simple heuristics according to which individual choice is adaptively adjusted based on past performances as follows:

$$q_{t+1} - q_t \propto G(t, t-1), \text{ where } G(t, t-1) = \frac{\pi(q_t) - \pi(q_{t-1})}{q_t - q_{t-1}} \tag{2.2}$$

According to (2.2), the next period quantity is increased or decreased provided that the ratio $G(t, t-1)$ is positive or negative respectively which, in turn, corresponds to having obtained higher or lower profits w.r.t. a variation in the production. As a result, the next quantity is increased or decreased along the trend that would lead to a rise in profits in the previous time-period, being implicitly assumed that the trend followed by the profit is maintained also in the next time. This implies that each player has static expectations, meaning by this that each of them believes that his competitors replicate the same choices performed at the previous time. It is here assumed that the decision mechanism that some agents in the subgroup $\mathcal{N}^{\mathsf{G}}$ adopt, comes from the heuristics (2.2) where the incremental relation $G(t, t-1)$ is replaced by its local approximation provided by the marginal profit $\partial \pi_i(\boldsymbol{q})/\partial q_i$. More precisely, the i-th player in the first group $\mathcal{N}^{\mathsf{G}}$, to which we will refer below as *gradient* player, adapts at each time-period $t+1$ his past decision $q_{i,t}$ taking into account only his individual choice and the related profit according to the gradient rule as introduced in [Bischi and Naimzada, 2000]:

$$
\begin{aligned}
q_{i,t+1} &:= q_{i,t} + \gamma q_{i,t} \left. \frac{\partial \pi_i(\boldsymbol{q})}{\partial q_i} \right|_{\boldsymbol{q}=\boldsymbol{q}_t} \\
&= q_{i,t} + \gamma q_{i,t} \left(a - c - bQ_{-i,t} - 2bq_{i,t} \right)
\end{aligned}
\tag{2.3}
$$

where $Q_{-i,t} = \sum_{j \neq i} q_{j,t}$ denotes the aggregate quantity that agent i observes in the market produced by the others at time t and $i \in \mathcal{N}^{\mathsf{G}}$. The parameter γ accounts for the reactivity of gradient players determining the amount of variation in quantity in the direction of increasing profits. The second equality in (2.3) is obtained within the linear oligopoly framework here considered.

On the contrary, we consider in the second group $\mathcal{N}^{\mathsf{I}}$, those players that adopt decisions taking into account not only their own past choices and related profits but also those of their competitors. The simplest heuristics that realizes such a decision mechanism is imitation. In particular, we consider *imitator* players characterized by a decision mechanism which is a kind of proportional imitation rule (similar to that introduced in [Schlag, 1998]) that prescribes, at each period, the choice given by the weighted average of the previous period quantities, whose importance is proportional to the payoff they have respectively generated. We call the players that use such rule "*imitator*" agents. Such a particular form of imitation accounts for the fact that *imitator* agents are aware about the presence of strategic interactions. Indeed, generally, within the framework of experimental oligopolies, it is common knowledge that the price is a decreasing function of the total amount of goods in the market, which in turn is determined by the choices of all players. The awareness about the presence of strategic interactions thus turns into the consciousness that an action that brought high profit in the previous period may not produce so good a result in the present time because of changes in environmental conditions. This indeterminacy about the performances that an action will produce is the reason that led us to consider such a *prudent imitative behavior*.

We note that the imitation rule here considered does not provide any performance-based selection. This marks the main difference from the rules considered by [Vega-Redondo, 1997] and by [Schipper, 2009] where players make a careless and incautious selection among the quantities to imitate. Indeed, according to the rule defined by [Vega-Redondo, 1997] every player refuses to imitate every strategy that did not produce the best result, while, according to the rule defined by [Schlag, 1998], every player refuses to imitate every strategy that has produced lower profits than his own.

According to the prudent imitation rule as described above, the generic j-th imitator chooses, at time-period t, the quantity

$$q^{\mathsf{I}}_{j,t+1} = \frac{\sum_{q_k \in \mathcal{A}_t} \pi(q_k) q_k}{\sum_{q_k \in \mathcal{A}_t} \pi(q_k)} \tag{2.4}$$

where $\mathcal{A}_t$ denotes the set of quantities chosen at time-period t. From the recurrence (2.4) it follows that the strategies of imitators are the same after the first time period. This means that $q^{\mathsf{I}}_{j,t+1} = q^{\mathsf{I}}_{k,t+1} = q^{\mathsf{I}}$ for every $j, k \in \mathcal{N}^{\mathsf{I}}$ and for $t \geqslant 1$.

In what follows we assume that all the players are characterized by the same initial conditions, namely $q_{i,0} = q_{j,0}$ for all $i, j \in \mathcal{N}$. In this event, players of each kind will make the same choices in each time step. It results that *gradient* agents will make the same choices in successive periods, that is $q_{i,t} = q_{j,t}$ for every $t \geqslant 0$ and for every $i, j \in \mathcal{N}^{\mathsf{G}}$, so their collective behavior can be described in terms of a single *gradient* representative agent whose production at time t is provided by the new variable q^{G}_t. Similarly, in each period t, the *imitator* agents' productions is at the same level for all of them, that is $q_{i,t} = q_{j,t}$ for every $t \geqslant 0$ and for every $i, j \in \mathcal{N}^{\mathsf{I}}$. Again, the collective behavior of such players can be described in terms of a single *imitator* representative agent whose production at time t is provided by the new variable q^{I}_t.

Based on the above assumptions, the collective behavior of the whole heterogeneous population of N players is described by the following 2-dimensional discrete-time dynamical system:

$$T : \begin{cases} q^{\mathsf{G}}_{t+1} &= q^{\mathsf{G}}_t + \gamma q^{\mathsf{G}}_t \left(a - b\left((N(1-\omega)+1)q^{\mathsf{G}}_t + \omega N q^{\mathsf{I}}_t\right) - c\right) \\ q^{\mathsf{I}}_{t+1} &= \dfrac{\pi^{\mathsf{I}}_t}{\pi^{\mathsf{I}}_t + \pi^{\mathsf{G}}_t} q^{\mathsf{I}}_t + \dfrac{\pi^{\mathsf{G}}_t}{\pi^{\mathsf{I}}_t + \pi^{\mathsf{G}}_t} q^{\mathsf{G}}_t \end{cases} \tag{2.5}$$

where

$$\pi^{\mathsf{G}}_t = \left(a - c - bN\left((1-\omega)q^{\mathsf{G}}_t + \omega q^{\mathsf{I}}_t\right)\right) q^{\mathsf{G}}_t \tag{2.6}$$

$$\pi^{\mathsf{I}}_t = \left(a - c - bN\left((1-\omega)q^{\mathsf{G}}_t + \omega q^{\mathsf{I}}_t\right)\right) q^{\mathsf{I}}_t \tag{2.7}$$

Proposition 1. The dynamical system (2.5) has the stationary state E_{N} given by

$$E_{\mathsf{N}} = (q_{\mathsf{N}}, q_{\mathsf{N}}) \quad \text{where} \quad q^{\mathsf{G}}_{\mathsf{N}} = q^{\mathsf{I}}_{\mathsf{N}} = q^{\mathsf{N}} := \frac{a-c}{b(N+1)} \tag{2.8a}$$

The model (2.5) has further stationary states $E_{\mathcal{L}}$ located along the half line $\mathcal{L} := \{(x,y) \mid x = 0, y \geqslant 0\}$ (where $q^{\mathsf{G}} = 0$ and $q^{\mathsf{I}} = y$).

Proof. See Appendix 5.1 □

It is noteworthy that the stationary state E_{N} of the dynamical system (2.5) corresponds to the Cournot-Nash notion of equilibrium in the linear oligopoly outlined above. Finally, the further stationary states that are located along the half line $\mathcal{L}$ represent states of the system where the decision mechanism of *gradient* players results in vanishing productions, meaning by this that they are excluded from the competition. At the stationary states belonging to $\mathcal{L}$, the only active agents in the market are the *imitator* ones.

2.1. **Local stability analysis: extreme homogeneous cases.** The model (2.5) derived above provides two different types of agents characterized by heterogeneous decision mechanisms. Under the usual assumption of identical agents starting from identical initial conditions, that model reduces to a one dimensional dynamical system if only a type is present. In the case where only identical imitator agents are present, that corresponds to $\omega = 1$, the dynamics is simply given by the replication of initial conditions, that is $q^{\mathsf{I}}_{t+1} = q^{\mathsf{I}}_t = q^{\mathsf{I}}_0$. Otherwise, if only gradient players are present, that corresponds to $\omega = 0$, the model (2.5) reduces to

$$q^{\mathsf{G}}_{t+1} = q^{\mathsf{G}}_t + \gamma q^{\mathsf{G}}_t \left(a - b(N+1)q^{\mathsf{G}}_t - c\right) \tag{2.9}$$

The nontrivial fixed point of that recurrence is again the Cournot-Nash production level $E^{1D}_{\mathsf{N}} = q_{\mathsf{N}}$, the stability condition of which is obtained by imposing that the slope of the map (2.9) lies in the unitary interval:

$$\left|\frac{\partial q^{\mathsf{G}}_{t+1}}{\partial q^{\mathsf{G}}_{t}}\right| < 1 \;\Rightarrow\; \left(\frac{1}{2} - \frac{1}{\gamma(a-c)}\right) < 0 \tag{2.10}$$

From this last result it can be noted that, in the framework of linear oligopoly where only gradient players compete among them, the stability of the symmetric Cournot-Nash equilibrium E^{1D}_{N} does not depend on the parameter N. This result is noteworthy since gradient players are similar to best replier players for which the Theocharis' rule apply, that is the Cournot-Nash equilibrium is stable up to $N < 3$ (see [Theocharis, 1960]).

2.2. **Local stability analysis: general case.**

Proposition 2. The stationery state E_{N} is locally asymptotically stable provided that

$$\omega > \omega_f := \frac{3}{2}\frac{N+1}{N}\left(\frac{1}{2} - \frac{1}{\gamma(a-c)}\right) \tag{2.11}$$

$$\omega < \omega_{ns} := \frac{1}{2}\frac{N+1}{N}\left(\frac{1}{\gamma(a-c)} + 1\right) \tag{2.12}$$

The fixed point E_{N} undergoes to a flip bifurcation at $\omega = \omega_f$ and to a Neimark-Saker bifurcation at $\omega = \omega_{ns}$.

The stationary states on $\mathcal{L} = (0, y)$, with $y \geqslant 0$, are (hyperbolic) stable fixed points whenever

$$y > Y_{ns} := \left(a - c - \frac{1}{\gamma}\right)\frac{1}{b\omega N} \tag{2.13}$$

At $y = Y_{ns}$ the point $(0, y)$ undergoes to a Neimark-Saker bifurcation.

Proof. See Appendix 5.2 □

The destabilizing role of the parameters γ and N is now considered. For increasing values of γ the fixed point E_{N} loses its stability. Indeed, for sufficiently small values of γ and for whatever values of the other parameters, E_{N} is stable:

$$\lim_{\gamma\to 0}\omega_f \to -\infty \tag{2.14}$$

$$\lim_{\gamma\to 0}\omega_{ns} \to +\infty \tag{2.15}$$

and both conditions (2.11) are fulfilled. The stability region in the plane $\gamma - \omega$ is shown in figure 2.1. Furthermore, there are values of γ for which, given any set of the other parameters' values, the stationary state E_{N} can never be stable. This happens if $\omega_f \geqslant \omega_{ns}$ whereby both conditions (2.11) can never be satisfied

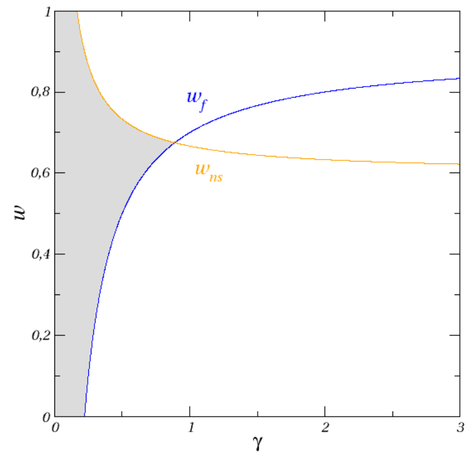

FIGURE 2.1. The stability regions of E_{N} in the $\gamma - \omega$ plane. The blue line marks the occurrence of the flip bifurcation, while the orange line marks the occurrence of the Neimark-Saker bifurcation. The grey region represents points where both conditions (2.11) are satisfied. The other parameters are: $a = 10$, $b = c = 1$ and $N = 5$.

and this occurs given that $\gamma(a - c) \geqslant 8$. Those stability and instability results can be interpreted at the light of the meaning of γ as a measure of the reactiveness of gradient players. Indeed, small values of γ induces small production changes along the direction of increasing profits provided by the marginal profit $\partial\pi(\boldsymbol{q})/\partial q$ and the optimum quantity is bit by bit approached even if changes in the profit function, due to collective action of players that determine the variation of the total output $Q_t = \sum_{i \in \mathcal{N}} q_{i,t}$, occur. On the contrary, at values of γ beyond the threshold value $8/(a-c)$, no convergence is achieved due to strong changes of outputs beyond the optimum values of profits, even when their changes along the competition take place.

Those analytical considerations are confirmed by numerical simulations, see figures 2.2 and 2.3, where the bifurcation diagrams of the dynamical variables q^{I}, q^{G} are presented as the parameter γ increases.

The role of the parameter N in determining the stability features of the stationary state E_{N} is conditioned by the values of γ. Indeed, provided that $\gamma(a - c) \geqslant 8$, otherwise E_{N} is always unstable, different bifurcation successions are observed. In particular, if $\gamma(a - c) \leqslant 2$, the values of ω_f are non positive and the fixed point E_{N} can lose its stability only through a Neimark-Saker bifurcation for increasing values of N (see figure 2.4 left panel). Indeed

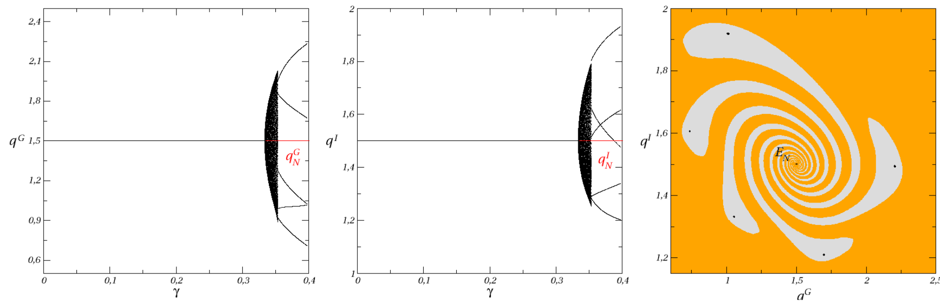

FIGURE 2.2. Bifurcation diagrams varying γ for $N = 5$, $a = 10$, $b = c = 1$ and $\omega = 0.8$. Left. Bifurcation diagram of q^{G} (black) and the value of $q^{\mathsf{G}}_{\mathsf{N}}$ (red). Center. Bifurcation diagram of q^{I} and the value of $q^{\mathsf{I}}_{\mathsf{N}}$ (red). Right. Basin of attraction at $\gamma = 0.39$ where grey points generate convergent trajectories towards the 5-period cycle whereas orange points originate unfeasible trajectories.

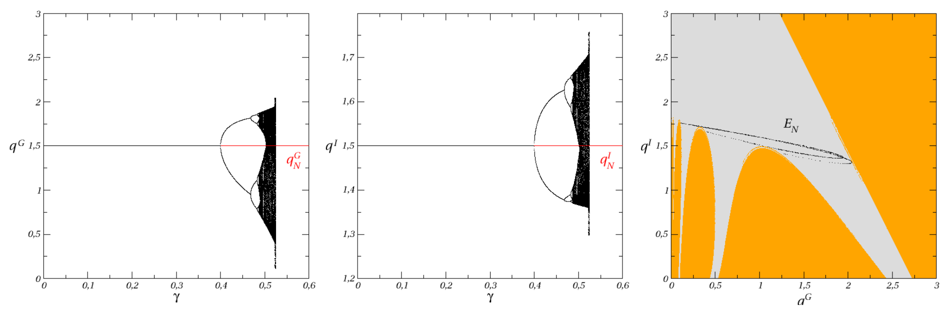

FIGURE 2.3. Bifurcation diagrams varying γ for $N = 5$, $a = 10$, $b = c = 1$ and $\omega = 0.4$. Left. Bifurcation diagram of q^{G} (black) and the value of $q^{\mathsf{G}}_{\mathsf{N}}$ (red). Center. Bifurcation diagram of q^{I} and the value of $q^{\mathsf{I}}_{\mathsf{N}}$ (red). Right. Basin of attraction at $\gamma = 0.5244$ where grey points generate convergent trajectories towards the chaotic attractor whereas orange points originate unfeasible trajectories.

it is:

(2.16) $$\lim_{N \to 0} \omega_f \to -\infty \text{ provided that } \gamma(a-c) < 2$$

Furthermore, for sufficiently small values of N, E_{N} is stable

(2.17) $$\lim_{N \to 0} \omega_{ns} \to +\infty$$

and both conditions (2.11) are met. On the contrary, if $\gamma(a-c) > 2$, a double threshold is observed as shown in figure 2.4, right panel. In this latter case the fixed point E_{N} is unstable for sufficiently small values of N:

$$\lim_{N\to 0} \omega_f \to +\infty \text{ provided that } \gamma(a-c) > 2 \tag{2.18}$$

$$\lim_{N\to 0} \omega_{ns} \to +\infty \tag{2.19}$$

and the first condition (2.11) can never be satisfied. It is worth noticing that the

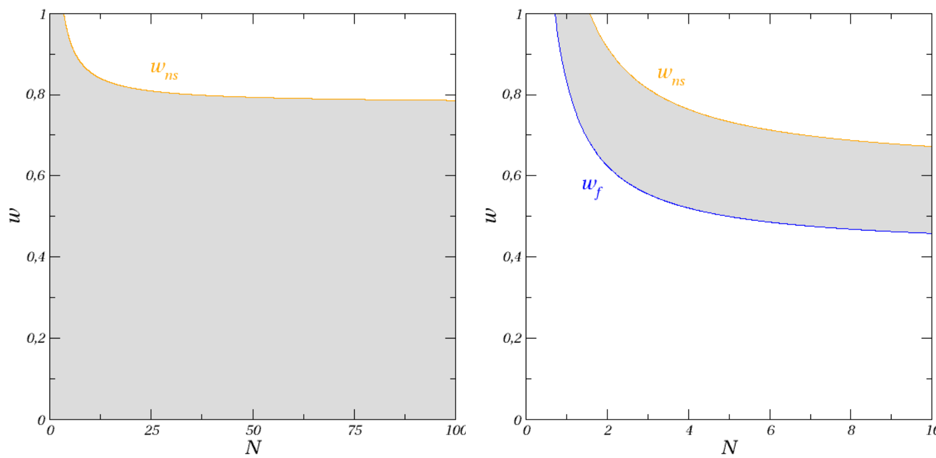

FIGURE 2.4. The stability regions of E_{N} in the $N-\omega$ plane. The blue line denotes the occurrence of the flip bifurcation, while the orange line denotes the occurrence of the Neimark-Saker bifurcation. The grey region denotes the points where both the conditions (2.11) are satisfied. Left. $\gamma = 0.2$. Right. $\gamma = 1$. The other parameters are: $a = 10$, $b = c = 1$.

double threshold is present when the amount of imitator agents is greater than the gradient agents. This fact can be understood by observing that the second condition (2.11) can be violated provided that $\omega > 1/2$:

$$\lim_{N\to\infty} \omega_{ns} = \frac{1}{2}\left(\frac{1}{\gamma(a-c)+1}\right) > \frac{1}{2} \tag{2.20}$$

Those analytical considerations are confirmed by numerical simulations, see figure 2.5 and 2.6, where the bifurcation diagrams of the dynamical variables q^{I}, q^{G} are presented as the parameter N increases in both cases where $\gamma(a-c) < 2$ and $\gamma(a-c) > 2$.

A last comment is devoted to outline the destabilizing role of the fraction of imitator agents ω. From figure 2.1 it can be guessed that at small values of γ the

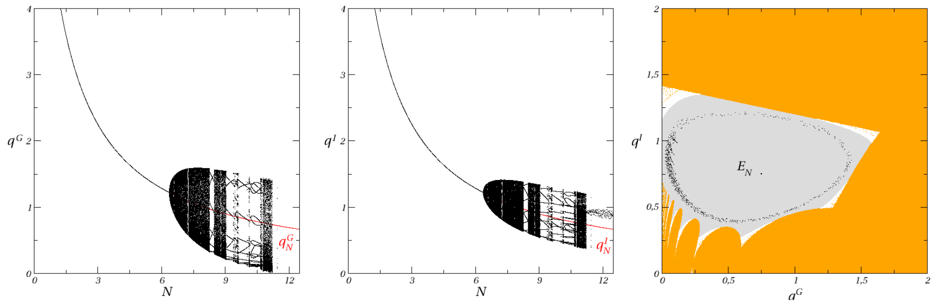

FIGURE 2.5. Bifurcation diagrams varying N for $\gamma(a-c) < 2$. Parameter are $a = 10$, $b = c = 1$, $\omega = 0.9$ and $\gamma = 0.2$. Left. Bifurcation diagram of q^{G} (black) and the value of $q_{\mathsf{N}}^{\mathsf{G}}$ (red). Center. Bifurcation diagram of q^{I} and the value of $q_{\mathsf{N}}^{\mathsf{I}}$ (red). Right. Basin of attraction at $N = 11$ where grey points generate convergent trajectories towards the chaotic attractor whereas orange points originate unfeasible trajectories. White points generate trajectories convergent to $\mathcal{L}$.

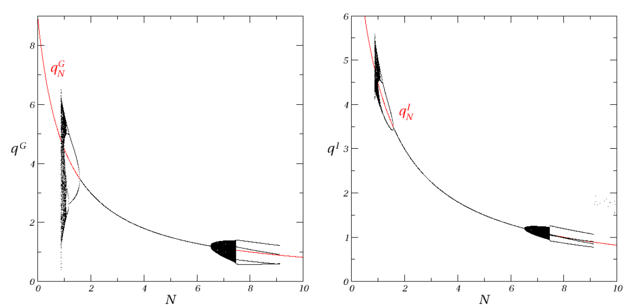

FIGURE 2.6. Bifurcation diagrams varying N for $\gamma(a-c) > 2$. Parameter are $a = 10$, $b = c = 1$, $\omega = 0.9$ and $\gamma = 0.5$. Left. Bifurcation diagram of q^{G} (black) and the value of $q_{\mathsf{N}}^{\mathsf{G}}$ (red). Right. Bifurcation diagram of q^{I} and the value of $q_{\mathsf{N}}^{\mathsf{I}}$ (red).

stationary state E_{N} is stable for whatever value of ω. Such a result is consistent with the interpretation of γ as the reactiveness of gradient players that leads the oligopoly to the achievement of the Cournot-Nash equilibrium provided that quantity changes are fairly gradual. Moreover, in the range $\gamma \geqslant 2/(a-c)$, a flip bifurcation occurs at $\omega = \omega_f$ and an increase of ω retrieves the stability of E_{N}. If, also, $\gamma \geqslant (N+1)/((N-1)(a-c))$ a double instability threshold is present and the Neimark-Saker bifurcation occurs at $\omega = \omega_{ns}$, thus implying that a further increase of ω causes the loss of stability of E_{N}. The first stability result is somehow expected and can be interpreted by saying the imitators give inertia to the system, meaning by this that they do not generate new choices by choosing some among those that are already present, with the effect of slowing down the dynamics. In this event the trend provided by gradient players, and followed by imitators, turns to be convergent. However, the second instability result, which is due to a prevalence of imitators, may appear as a surprising result as their action tends to replicate past choices. However, if their inertia is very important, final decisions are not achieved causing the emergence of cyclic trajectories after a Neimark-Saker bifurcation. Finally, if $\gamma \geqslant 8/(a-c)$, the stationary state E_{N} is always unstable. At the light of previous interpretation, such a fact is due to an overcrowding of too reactive gradient players or to an excessive inertia introduced by imitators, or to both such factors.

3. Experimental oligopolies modeling

In this section we discuss whether to represent experimental oligopoly dynamics by using the model (2.5). The starting point is the main result arising from experimental oligopolies where no single production level, and then even no notions of equilibria such as the Cournot-Nash or the competitive (Walrasian) production levels, emerges as the unique outcome of the competition. Instead, the choices of agents are confined and distributed over a wide range in the quantity space in which, generally, the Cournot-Nash and the competitive notions of equilibria are included. This circumstance, found in most of the experiments, is the main coordination phenomenon that spontaneously emerges (see for example [Oechssler et al., 2016], [Bigoni and Fort, 2013] [Bosch-Domènech and Vriend, 2003] or [Offerman et al., 2002]). As those evidences do not reveal the prevalence of a unique common choice, one concludes that no convergence towards a common decision is achieved. Repeated decisions seems to be mainly uncoordinated and, sometimes, erratic. However, weak phenomena of coordination and synchronization around the Cournot-Nash and the competitive equilibria are found in most of the experiments (see for example [Oechssler et al., 2016], [Friedman et al., 2015], [Apesteguia et al., 2010], [Bigoni and Fort, 2013]) even though they were characterized by different setups designed to reveal specific features of the behaviors of real players. As an example, great attention is aimed either to show how the informational set the agents were provided with induces a particular decision mechanism rather than another (see [Offerman et al., 2002] or [Apesteguia et al., 2007]) or to reveal the effect of the decision time (see [Friedman et al., 2015]).

In particular, the systematic emergence of the competitive notion of equilibrium is detected, thus suggesting, at the light of the result provided in [Vega-Redondo, 1997], that the imitative behavior constitutes a partial component of the decision mechanism of the player. Analogously, the systematic occurrence of decisions that match with the Cournot-Nash production level suggests that the rational behavior can be thought as a further component of the same decision mechanism. However, both the imitative and the rational behaviors are not sufficient to individually explain the experimental outcomes. If so, decisions would match with a singular notion of equilibrium and the corresponding production level would be observed with an eventual Gaussian spread due to random factors that affect decision processes. On the contrary, the presence of such weak coordinations suggests that the decision mechanisms of agents may include both components of rational behavior and imitative behavior.

An attempt to describe the experimental evidences by considering an explicit model for the agents' decision mechanisms is provided first in [Huck et al., 1999] and, later, in [Apesteguia et al., 2010]. In those papers, the decision mechanism of agents is considered as if it were made up of different behavioral components, substantially related to rational behavior and imitative behavior, simultaneously involved in decision processes. Together with those deterministic behavioral rules, the authors also consider a relevant contribution of random factors affecting the decision processes to reproduce the distributions of choices over the quantities' space observed in experiments.

Here we echo the thought suggested by [Huck et al., 1999] and [Apesteguia et al., 2010] in order to derive the model (2.5) which describes the competition among players in a linear oligopoly framework by providing explicit expressions for their decision mechanisms made up by heterogeneous components (gradient rule and imitation) that coexists and operate simultaneously. Players' decisions are thus modeled as mixed processes. At the same time, differently from the above cited literature, our model is characterized by having a deterministic structure and we show that the unpredictable empirical dynamics, taking place within a finite range of the quantity space that includes both the Cournot-Nash and the competitive production levels, can be qualitative represented by the chaotic dynamics that are originated by heterogeneous decision mechanisms, with no need of any stochastic perturbation. Indeed, the distribution of choices over the quantity space and the absence of convergence towards a common decision can be understood through dynamical instabilities. This is a direct consequence of the approach suggested by [Apesteguia et al., 2010] since it is exactly the presence of heterogeneities among decision mechanisms that determines the complexity of dynamical scenarios. In this view we assert that chaotic dynamics can represent experimental outcomes.

It is worth noticing that our modeling purpose does not claim to qualitative reproduce each specific finding from the various experiments provided in the literature. The model only grasps the main qualitative evidences outlined above. In this view, the parameters' selection is performed in order to make the model capable to give rise to a wide range of complex dynamics in order to represent the rich outcomes from experiments. This is facilitated by the fact that relations among parameters placing the dynamical scenarios within chaotic regimes allow to perform a not too much accurate selection. In particular, the stability relations (2.11)), as well as the bifurcation diagrams provided in figures 2.2, 2.3, 2.5 and 2.6, reveal that chaotic regimes persist in a wide range (with positive measure) of parameters' spaces (look at white regions in figures 2.1 and 2.4). We finally note that the usage of a linear demand function and constant marginal costs is in line with the underlying oligopoly framework usually adopted in experiments and the values of fundamentals, such as the maximum price a and the marginal cost c, are selected to maintain the numerical values of the Cournot-Nash and the competitive production levels distinct between them.

Furthermore, in order to make our description closer to the real world experimental conditions, we consider the possibility for the *imitator* agents to make mistakes in choosing their own strategies. The sources of error may rely both on the information retrieval, namely how the knowledge about the latests quantities and related profits in the oligopoly is obtained, and on the evaluation of the weighted sums (2.4). However, we do not wish to describe in full details such noise effects and we assume that such perturbations can be modeled by an additive white and Gaussian noise. So the model (2.5) can thus be rewritten as follows:

$$(3.1) \qquad T : \begin{cases} q^{\mathsf{G}}_{t+1} & = q^{\mathsf{G}}_t + \gamma q^{\mathsf{G}}_t \left(a - b\left((N(1-\omega)+1)q^{\mathsf{G}}_t + \omega N q^{\mathsf{I}}_t\right) - c\right) \\ q^{\mathsf{I}}_{t+1} & = \dfrac{\pi^{\mathsf{I}}_t}{\pi^{\mathsf{I}}_t + \pi^{\mathsf{G}}_t} q^{\mathsf{I}}_t + \dfrac{\pi^{\mathsf{G}}_t}{\pi^{\mathsf{I}}_t + \pi^{\mathsf{G}}_t} q^{\mathsf{G}}_t + \xi_t \end{cases}$$

where ξ_t is the noise that, at each time period, is assumed to be a white Gaussian random variable with 0 mean and variance σ^2.

In figures 3.1 and 3.2 some simulations of the model (3.1) for different sets of parameters are shown. Time series related to the same simulations are presented in figure 3.3. In those figures the main coordination phenomenon that confines choices to a finite range of the quantity space, that includes the

Cournot-Nash and the competitive equilibria, is well represented. Furthermore, the weaker coordination that makes those notions of equilibrium especially visited, highlighted by the high frequencies around that production levels, is also reproduced. More precisely, in the simulations related to the histograms in figure 3.1 and to the time series on the top of figure 3.3, the dynamical instabilities are due to a high reactiveness of gradient players. Choices jump on both sides of the Cournot-Nash equilibria as a result of the loss of stability of E_{N} through the flip bifurcation and spread away from it to cover a portion of the feasible region due to the overshooting actions of gradient players. Differently, in the simulations related to the histograms in figures 3.2 and to the time series on the bottom of figure 3.3, the dynamical instabilities arise from the preponderance of imitators and the choices oscillate around the Cournot-Nash equilibria as a result of the loss of stability of E_{N} through the Neimark-Saker bifurcation. Both the simulations highlight that the presence of imitators brings the oligopoly towards high competitive levels and quantities between the Cournot-Nash and the competitive equilibria are visited with significantly high frequency w.r.t. other choices. However, it can be further noted that the overshooting action of gradient players makes the outcome particularly undetermined whether such a rational-like behavior prevails. On the contrary, the preponderance of imitation behavior brings to a less erratic dynamics as a result of the regularity of the oscillations around E_{N}. The distribution of quantities is characterized by a greater smoothness than in the previous case, meaning by this that differences among the frequencies of choices close to each other are, on average, less pronounced. Such occurrences are almost preserved even in the presence of small noise perturbations which, despite giving further instability to chaotic dynamics, do not affect the qualitative behavior of trajectories. At the light of experimental outcomes, in such a modeling framework imitation behavior seems to be not so preponderant in favor of the prevalence of the gradient decision mechanism.

Finally we focus on the possibility to observe such complex scenarios depending on initial conditions. In this section we provide some simulations that reveal the complexity not only in terms of chaotic trajectories but also with respect to basins of attraction of the attracting sets of the model (2.5). Indeed, such simulations reveal the circumstance that the Cournot-Nash equilibrium E_{N}, or a chaotic attractor that is originated once it has lost its stability, is not globally attracting. This path dependence provides a further modeling tool to describe the experimental outcomes and the possibilities to observe chaotic

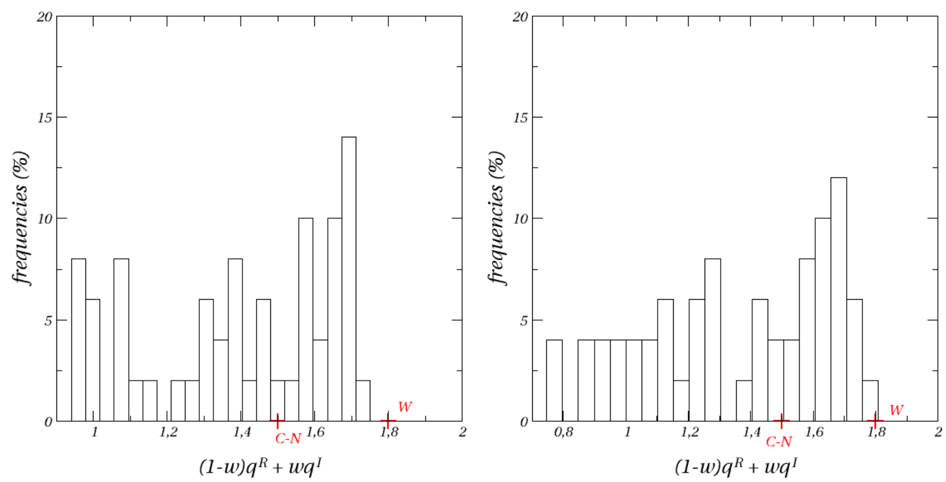

FIGURE 3.1. Frequencies obtained for the first 50 iterations of the model (2.5). Parameters are $N = 5$, $a = 10$, $b = c = 1$, $\gamma = 0.52$, $\omega = 0.4$. Left. Unperturbed simulation. Right. Noisy model with $\sigma^2 = 0.04$.

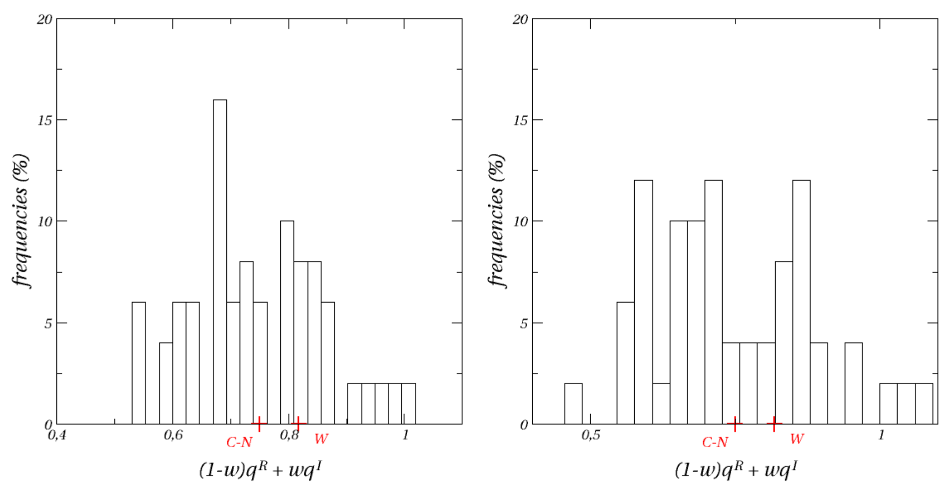

FIGURE 3.2. Frequencies obtained for the first 50 iterations of the model (2.5). Parameters are $N = 11$, $a = 10$, $b = c = 1$, $\gamma = 0.2$, $\omega = 0.9$. Left. Unperturbed simulation. Right. Noisy model with $\sigma^2 = 0.1$.

dynamics are conditioned upon the choice of initial conditions. From an interpretative point of view, this gives value to the selection of the initial choices in the experiments that are represented, in such a modeling framework, precisely by initial conditions. An example is represented in figure 3.4 where a chaotic

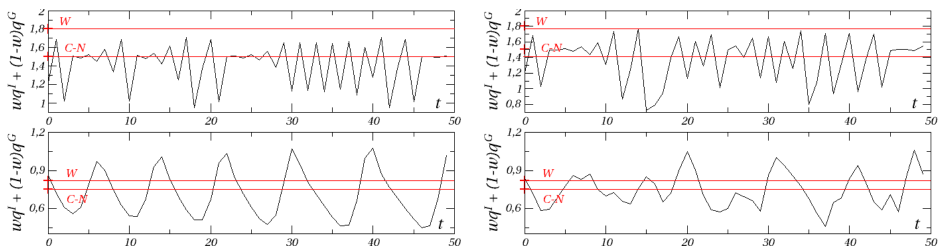

FIGURE 3.3. Time series of the first 50 iterations of the model (2.5). From the top down to the bottom parameters are as in figures 3.1 and 3.2 respectively. In particular on the left the unperturbed cases ($\sigma^2 = 0$) are shown and, on the right, the perturbed ones ($\sigma^2 = 0.04; 0.1$ from the top down to the bottom).

cycle, originated from a Neimark-Saker bifurcation of E_{N}, together with its basin of attraction are shown.

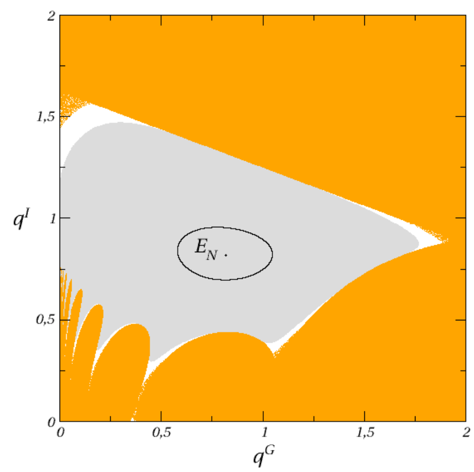

FIGURE 3.4. A dynamical scenario from the model (2.5) where a chaotic cycle (black) around E_{N} (black dot) and its basin of attraction (grey points) are depicted. From the orange points unfeasible trajectories are originated. Finally, white points generate trajectories towards the half line $\mathcal{L}$. Parameters are $N = 10$, $a = 10$, $b = c = 1$, $\gamma = 0.25$, $\omega = 0.8$.

Furthermore, the model (2.5) may give rise not only to a unique attractor, but also to the presence of multi-stabilities, that is the presence of two different attracting sets that coexist together. A first evidence of their presence is provided by the bifurcation diagrams presented in figures (3.5). This aspect is better

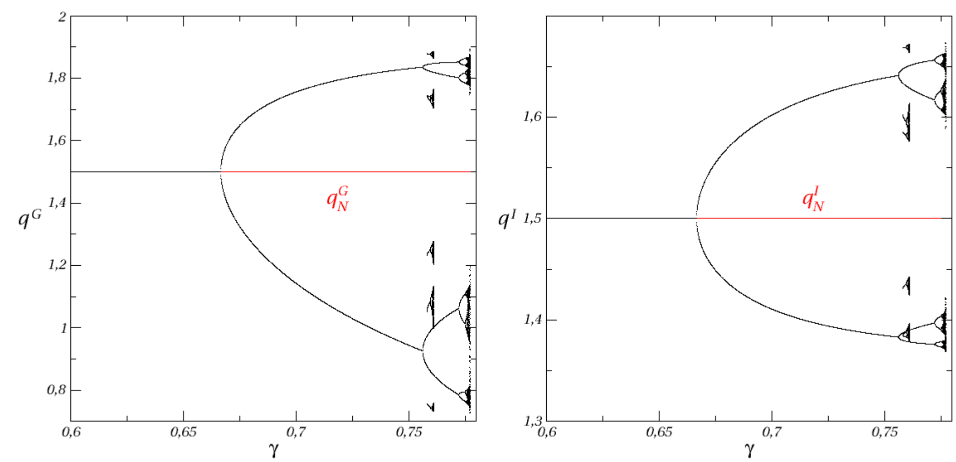

FIGURE 3.5. Bifurcation diagrams of q^{G} and q^{I}, respectively on the left and on the right, as the parameter γ varies. The coexistence of two attractors is observed. Parameters are $N = 5$, $a = 10$, $b = c = 1$, and $\omega = 0.6$.

understood by looking at the basins of attraction of these two attractors in the phase space of the model (2.5) shown in figure 3.6. The coexistence of multiple attractors introduces a further degree of complexity that the model is able to represent. Again, in order to observe chaotic trajectories, not only the stability properties of the equilibrium point E_{N} matter, but initial conditions matter too. From an interpretative point of view this confirms the relevance of initial choices in experiments and adds a further contribution in terms of indeterminacy within the experimental outcomes modeling framework.

4. CONCLUSION

Within the modeling framework outlined first by [Huck et al., 1999] and later by [Apesteguia et al., 2010], we derived an explicit expression of decision mechanisms of quantity setting players that compete in a linear oligopoly framework in order to model their behaviors on the basis of empirical observations. Experimental results, indeed, show that no convergence towards a common decision is achieved and several distributions of choices over a range of the quantity space, that includes generally the Cournot-Nash and the competitive (Walrasian) equilibrium, are observed. The same experiments further suggest that real agents decide how much to produce by the simultaneous operating of different and coexisting behavioral components, the two main of which may be

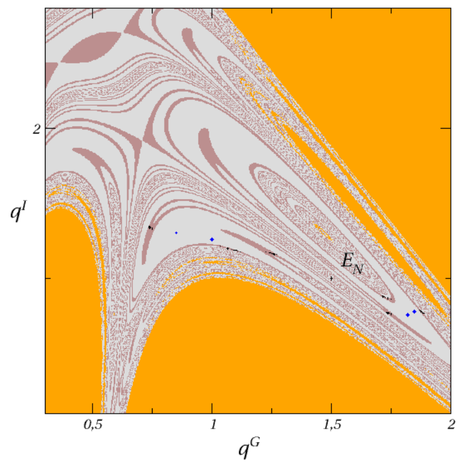

FIGURE 3.6. A dynamical scenario from the model (2.5) where a 4-period cycle (blue dots), originated from the fixed point E_{N} (black dot) through a flip bifurcation, and its basin of attraction (grey points) are present together with a further chaotic attractor, the basin of which is represented by the set of brown points. The other orange points originate unfeasible trajectories. Parameters are $N = 5$, $a = 10$, $b = c = 1$, $\gamma = 0.76062$, $\omega = 0.6$.

represented by rationality and imitation. We found that the dynamical scenarios arising from the model (2.5), derived by considering the presence of heterogeneous behaviors, are capable to qualitatively reproduce the experimental outcomes by means of dynamical instabilities and chaotic trajectories even in the absence of stochastic perturbations. Simulations of (2.5) reveal that the main coordination phenomenon that limits choices to a finite range of the quantity's space is explained by the coexistence of rational and imitative behaviors. Furthermore, the same heterogeneity is capable to represent the weaker coordination that makes the Cournot-Nash and the competitive equilibria especially visited in simulations.

We finally note that a future stream of research can be devoted to deepen the dynamical consequences by considering also heterogeneities at an individual level, namely where mixed behaviors operate simultaneously in the decision mechanism of a single player. Indeed, the model here discussed is derived by assuming heterogeneity only on a collective basis, where different players, characterized by their own decision mechanisms made up by a single behavior, are considered. A first step in this direction can be taken for an $\mathcal{N}$-player dynamic oligopoly assuming that each agent's decision mechanism consists of a mixed combination of the gradient rule and the imitative rule, as derived above. Further on, introducing positive parameters $\sigma_i \leqslant 1$, with $i = 1, ..., N$, that weight the importance of the heterogeneous components within each decision mechanism, it results the following N-dimensional recurrence:

(4.1)

$$q_{i,t+1} = \sigma_i \left(q_{1,t} + \gamma q_{i,t} \frac{\partial \pi(\boldsymbol{q}(t)}{\partial q_{i,t})} \right) + (1 - \sigma_i) \left(\frac{\sum_{q_k \in \mathcal{A}(t)} \pi(q_k) q_k}{\sum_{q_k \in \mathcal{A}(t)} \pi_k} \right), \; i = 1, ..., N$$

where $i \in N$ marks the i-th player. By assuming identical initial conditions, the model (4.1) reduces to the one considered in this paper (2.3) only if $\sigma_i = 1$ for $i \in \mathcal{N}^{\mathsf{G}} \subseteq \mathcal{N}$ and $\sigma_i = 0$ otherwise. However, even if the consequences by considering heterogeneities both at a collective and at an individual level have to be deepened, complexities arising from these two different situations give rise to significantly different outcomes. Indeed, letting $\sigma_i = \sigma_j$ for all $i, j = 1, ..., N$, each player has an identical mixed decision mechanism to its competitors and the whole $\mathcal{N}$-player dynamic game can be described in terms of a one dimensional recurrence at an aggregate level, thus having qualitatively different features rather than the model (2.3) which is a two dimensional one.

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5. Appendix

5.1. **Proof of Proposition 1.** The fixed points of (2.5) are the solutions of the following equations:

$$T:\begin{cases} q^{\mathsf{G}} &= q^{\mathsf{G}} + \gamma q^{\mathsf{G}}\left(a - b\left((N(1-\omega)+1)q^{\mathsf{G}} + \omega N q^{\mathsf{I}}\right) - c\right) \\ q^{\mathsf{I}} &= \dfrac{\pi^{\mathsf{I}}}{\pi^{\mathsf{I}}+\pi^{\mathsf{G}}} q^{\mathsf{I}} + \dfrac{\pi^{\mathsf{G}}}{\pi^{\mathsf{I}}+\pi^{\mathsf{G}}} q^{\mathsf{G}} \end{cases} \tag{5.1}$$

which reduces to:

$$\begin{cases} q^{\mathsf{G}}\left(a - b\left((N(1-\omega)+1)q^{\mathsf{G}} + \omega N q^{\mathsf{I}}\right) - c\right) = 0 \\ \\ q^{\mathsf{I}}\pi^{\mathsf{G}} = q^{\mathsf{G}}\pi^{\mathsf{G}} \end{cases} \tag{5.2}$$

The second equation in (5.1) holds true both if $q^{\mathsf{I}} = q^{\mathsf{G}}$ or if $\pi^{\mathsf{G}} = 0$, thus fixing, at least, two stationary states for the dynamical recurrence (2.5).

5.1.1. *The first stationary state of (2.5) at which $q^{I} = q^{G}$.* In this occurrence, the stationary production level of the representative best replier matches with the Cournot-Nash production level in a linear oligopoly and it is obtained by using the former equality together with the first equation in (5.1), thus giving $q^{\mathsf{G}} = q_{\mathsf{N}} := (a-c)/b(N+1)$. This, together with the equality $q^{\mathsf{I}} = q^{\mathsf{G}} = q_{\mathsf{N}}$, fixes the same Cournot-Nash production level for the representative imitator agent. The point $E_{\mathsf{N}} = (q_{\mathsf{N}}, q_{\mathsf{N}})$ is shown to be a stationary state of (2.5).

5.1.2. *Other stationary states of (2.5) at which $\pi^{G} = 0$.* Since $\pi^{\mathsf{G}} = (a - c - bN((1-\omega)q^{\mathsf{G}} + \omega q^{\mathsf{I}})q^{\mathsf{G}}$ the condition $\pi^{\mathsf{G}} = 0$ is matched if, alternatively, $q^{\mathsf{G}} = 0$ or $(a - c - bN((1-\omega)q^{\mathsf{G}} + \omega q^{\mathsf{I}}) = 0$. In the first occurrence every production level of imitators is a stationary state, that is from the second recurrence in (2.5) one gets $q^{\mathsf{I}}_{t+1} = q^{\mathsf{I}}_{t} = q^{\mathsf{I}}_{0}$. So, every point that belongs to the line $\mathcal{L} := \{(0,x) \mid x \geqslant 0\}$ is a stationary state. In the second event one gets, from the first equation in (5.2), the following correspondence

$$q^{\mathsf{G}} = \frac{a - c - b\omega N q^{\mathsf{I}}}{bN(1-\omega)} \tag{5.3}$$

By using the relation (5.3) into the equation $\pi^{\mathsf{G}} = 0$ one gets $q^{\mathsf{I}} = \dfrac{a-c}{b\omega N}$ which, in turn, implies that $q^{\mathsf{G}} = 0$. It follows that such stationary state, given by $(0, \dfrac{a-c}{b\omega N})$, is included in the set $\mathcal{L}$.

5.2. **Proof of Proposition 2.** The stability properties of the stationary states E_{N} and those included in the set $\mathcal{L}$ are studied by using the characteristic polynomial of the Jacobian matrix $\boldsymbol{J}(q^{\mathsf{G}}, q^{\mathsf{I}})$ of the system (2.5) evaluated at those fixed points. For the sake of completeness the Jacobian matrix $\boldsymbol{J}(q^{\mathsf{G}}, q^{\mathsf{I}})$ is here presented.

$$\boldsymbol{J}(q^{\mathsf{G}}, q^{\mathsf{I}}) = \begin{pmatrix} \dfrac{q^{\mathsf{I}}q^{\mathsf{I}} + 2q^{\mathsf{I}}q^{\mathsf{G}} - q^{\mathsf{G}}q^{\mathsf{G}}}{(q^{\mathsf{I}} + q^{\mathsf{G}})^2} & \dfrac{q^{\mathsf{G}}q^{\mathsf{G}} + 2q^{\mathsf{I}}q^{\mathsf{G}} - q^{\mathsf{I}}q^{\mathsf{I}}}{(q^{\mathsf{I}} + q^{\mathsf{G}})^2} \\ & \\ -\gamma b\omega N q^{\mathsf{G}} & 1 + \gamma\left(a - c - b(2(N(1-\omega)+1)q^{\mathsf{G}} + \omega N q^{\mathsf{I}})\right) \end{pmatrix}$$

5.2.1. *Stability of E_{N}.* The characteristic polynomial of $\boldsymbol{J}_{\mathsf{N}} = \boldsymbol{J}(q_{\mathsf{N}}, q_{\mathsf{N}})$ is $p_{\mathsf{N}}(\lambda) = \lambda^2 - \mathrm{tr}\boldsymbol{J}_{\mathsf{N}}\lambda + \det\boldsymbol{J}_{\mathsf{N}}$ where trace $\mathrm{tr}\boldsymbol{J}_{\mathsf{N}}$ and the determinant $\det\boldsymbol{J}_{\mathsf{N}}$ of $\boldsymbol{J}_{\mathsf{N}}$ are

$$\mathrm{tr}\boldsymbol{J}_{\mathsf{N}} = \frac{3}{2} - \gamma(a-c) + \gamma\omega N\frac{a-c}{N+1}$$
$$\det\boldsymbol{J}_{\mathsf{N}} = \frac{1}{2}\left(1 - \gamma(a-c)\right) + \gamma\omega N\frac{a-c}{N+1}$$

From the Jury's stability condition, the Cournot-Nash equilibrium E_{N} is stable provided that

$$p(1) = \frac{1}{2}\gamma(a-c) > 0 \ \text{ always}$$
$$p(-1) = 3 - \frac{3}{2}\gamma(a-c) + 2\gamma\omega N\frac{a-c}{N+1} > 0 \quad\Leftrightarrow\quad \gamma(a-c)\left(\frac{3}{2} - \frac{2\omega N}{N+1}\right) < 3$$
$$\det\boldsymbol{J}(E_{\mathsf{N}}) < 1 \quad\Leftrightarrow\quad \gamma(a-c)\left(\frac{1}{2} - \frac{\omega N}{N+1}\right) > -\frac{1}{2}$$

from which the conditions in Proposition 2 follow.

5.2.2. *Stability of $\mathcal{L}$.* The characteristic polynomial of the Jacobian matrix $\boldsymbol{J}_{\mathcal{L}} = \boldsymbol{J}(0, y)$ computed along the set $\mathcal{L}$ is $p_{\mathcal{L}}(\lambda) = \lambda^2 - \mathrm{tr}\boldsymbol{J}_{\mathcal{L}}\lambda + \det\boldsymbol{J}_{\mathcal{L}}$ where the trace $\mathrm{tr}\boldsymbol{J}_{\mathcal{L}}$ and the determinant $\det\boldsymbol{J}_{\mathcal{L}}$ are

$$\mathrm{tr}\boldsymbol{J}_{\mathcal{L}} = 2 + \gamma(a - c - b\omega N y)$$
$$\det\boldsymbol{J}_{\mathcal{L}} = 1 + \gamma(a - c - b\omega N y)$$

The Jury's stability condition for the stability of the points in $\mathcal{L}$ are as follows:

$$p(1) = 0 \tag{5.4}$$

$$p(-1) = 3 + \gamma(a - c - b\omega N y) > 0 \ \text{always} \tag{5.5}$$

$$\det\boldsymbol{J}_{\mathcal{L}} = 1 + \gamma(a - c - b\omega N y) < 1 \quad\Leftrightarrow\quad y > \left(a - c - \frac{1}{\gamma}\right)\frac{1}{b\omega N} \tag{5.6}$$

In particular, from the first condition $P(1) = 0$ one gets that points in $\mathcal{L}$ are hyperbolic fixed points. Indeed, they are characterized by the 1-valued eigenvalue related to the horizontal direction. So their stability derives only from their transversal attractiveness determined by the third condition in (5.6) (that corresponds to (2.13) in Proposition 2), thus implying that the the point $(0, y)$ undergoes a Neimark-Saker bifurcation at $y = \left(a - c - \frac{1}{\gamma}\right)\frac{1}{b\omega N}$.

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