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ON SOME NONLOCAL ISSUES: UNIQUE CONTINUATION FROM THE BOUNDARY AND CAPILLARITY PROBLEMS FOR ANISOTROPIC KERNELS

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Contents

Abstract

The aim of the present thesis is to discuss the results obtained during my PhD studies, mainly devoted to nonlocal issues. We first deal with strong unique continuation principles and local asymptotic expansions at certain boundary points for solutions of two different classes of elliptic equations. We start the investigation by the following class of fractional elliptic equations

$$(-\Delta)^s u = hu \tag{1}$$

in a bounded domain under some outer homogeneous Dirichlet boundary condition, with $s \in (0,1)$. More precisely, we are interested in proving the strong unique continuation property and local asymptotics of solutions at those boundary points where the domain is locally of class $C^{1,1}$. In order to do this, we exploit the Caffarelli-Silvestre extension procedure developed in [8], which allows us to get an equivalent formulation of the non-local problem as a local problem in one dimension more, consisting in a mixed Dirichlet-Neumann boundary value problem. Then, we use a classical idea by Garofalo and Lin [48] to obtain a doubling-type condition via a monotonicity formula for a suitable Almgren-type frequency function. To overcome the difficulties related to the lack of regularity at the Dirichlet-Neumann junction, we introduce a new technique based on an approximation argument, which leads us to derive a Pohozaev-type identity needed to estimate the derivative of the Almgren function. Thus we gain a strong unique continuation result in the local context, which is in turn combined with blow-up arguments to deduce local asymptotics and, consequently, a strong unique continuation result in the nonlocal setting as well.

We also provide a strong unique continuation result from the edge of a crack for the solutions to a specific class of second order elliptic equations in an open bounded domain with a fracture, on which a homogeneous Dirichlet boundary condition is prescribed, in the presence of potentials satisfying either a negligibility condition with respect to the inverse-square weight or some suitable integrability properties. This local problem is related to a particular case of the setting described above when $s = 1/2$, by virtue of a strong connection between this type of problems and the mixed Dirichlet-Neumann boundary value problems resulting from the Caffarelli-Silvestre extension associated to (1).

We also treat a capillarity theory of nonlocal type, inspired by the study performed in [60]. In our setting, we consider more general interaction kernels that are possibly anisotropic and not necessarily invariant under scaling. In particular, the lack of scale invariance is modeled via two different fractional exponents in order to take into account the possibility that the container and the environment present different features with respect to particle interactions. We determine a nonlocal Young's law for the contact angle between the droplet and the surface of the container and discuss the unique solvability of the corresponding equation in terms of the interaction kernels and of the relative adhesion coefficient.

Chapter 1

Introduction

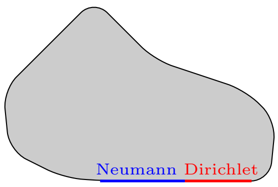
1.1 Motivations

The study of unique continuation from the boundary has been widely treated in literature. We refer to papers [3, 4, 38, 57, 77] for unique continuation at the boundary for solutions to elliptic equations under homogeneous Dirichlet conditions, to [76] for unique continuation at the boundary under zero Neumann conditions, and also to [26] for a strong unique continuation property from the vertex of a cone under non-homogeneous Neumann conditions. Once a strong unique continuation property is proved, infinite vanishing order for non-trivial solutions can be excluded. The problem of estimating and explicitely providing all possible vanishing orders is then naturally related to unique continuation; we quote e.g. [56] for quantitative uniqueness and bounds for the maximal order of vanishing and [38, 39, 44, 42, 43] for a precise description of the asymptotic behavior together with a classification of possible vanishing orders of solutions for several classes of problems, obtained by combining monotonicity methods with blow-up analysis for scaled solutions. Furthermore, we cite [36] for a unique continuation result and asymptotic expansions of solutions to fractional elliptic equations at interior points of the domain, achieved by Almgren type monotonicity formulas combined with blow-up arguments. We mention also [68] for quantitative unique continuation for fractional Schrödinger equations derived by Carleman estimates, [81] for fractional operators with variable coefficients, and [40, 41, 47, 70, 71, 72, 80] for higher order fractional problems.

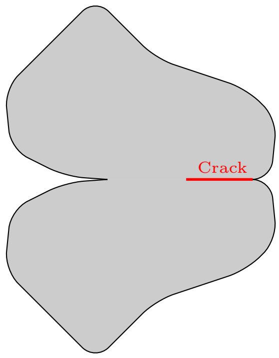
The issue of unique continuation from the boundary turns out to be particularly hard to study since a series of difficulties due to the geometry of the domain arise in the derivation of suitable monotonicity formulas and in the investigation of the asymptotic behavior of solutions. Indeed the regularity of the domain plays a crucial role in the behaviour of solutions at the boundary; for instance in [38] the asymptotic behavior at conical singularities of the boundary is proved to depend on the opening of the vertex of the cone.

In particular, concerning problem (1), our main goal consists in extending the results contained in [36] to boundary points of the domain, thus establishing sharp asymptotics and unique continuation from the boundary. A related problem is the regularity of solutions up to the boundary. Within this framework, we mention [64, 66] for regularity results at the boundary for solutions to fractional elliptic problems and also to [6], where quantitative upper and lower estimates at the boundary are exihibited for nonnegative solutions to semilinear nonlocal elliptic equations.

In connection with problem (1), we investigate also a class of second order elliptic problems in a domain with a crack, which are in fact related to mixed Dirichlet/Neumann boundary value problems. Indeed, if we consider an elliptic equation with mixed boundary conditions, in particular a homogeneous Dirichlet condition on a flat portion of the boundary and a homogeneous Neumann condition on the complement, applying an even reflection through the flat boundary we obtain an elliptic equation satisfied in the complement of the Dirichlet region as well. Then the Dirichlet portion becomes a crack (see Figure 1.1 below) and the edge of the crack corresponds to the Dirichlet-Neumann junction of the original mixed boundary value problem. We cite [37] for a unique continuation result and asymptotic expansions of solutions to planar mixed boundary value problems at Dirichlet-Neumann junctions. Our idea is to extend the monotonicity method developed in [37] to dimensions bigger than 2, with the aim of proving a strong unique continuation result. We refer to [54, 69] and references therein for some regularity results for second-order elliptic problems with mixed Dirichlet-Neumann type boundary conditions.



(a) Mixed Dirichlet/Neumann boundary conditions on a flat portion of the boundary



(b) After an even reflection the Dirichlet region becomes a crack

Figure 1.1: A relation between problems in domains with a crack and mixed Dirichlet/Neumann boundary value problems

Moreover, the study of second order elliptic problems in a domain with a crack is of particular interest itself since they occur in elasticity theory, see e.g. [18, 55, 58]. The non-smoothness of domains having slits produces strong singularities of solutions at edges of cracks; with regards to the structure of such singularities, we cite e.g. [12, 15, 32], and references therein. In particular, asymptotic expansions of solutions at edges of cracks play a crucial role in these problems, since the coefficients of such expansions are related to the so called *stress intensity factor*, see e.g. [18].

Concerning the nonlocal capillarity problem, it is well-known that in the classical capillarity theory (see e.g. [21, 22]) the contact angle is defined as the angle ϑ at which a liquid interface meets a solid surface. At the equilibrium, this angle is expressed by the Young's law equation in terms of the relative adhesion coefficient σ as the classical formula

$$\cos(\pi - \vartheta) = \sigma.$$

The contact angle plays also an important role in the fluid spreading on a solid surface, determining also the velocity of the moving contact lines (see e.g. [20] and the references therein). The contact angle is certainly the "macroscopic" outcome of several complex "microscopic" phenomena, involving physical chemistry, statistical physics and fluid dynamics, and ultimately relying on the effect of long-range and distance-dependent interactions between atoms or molecules (such as van der Waals forces). It is therefore of great interest to understand how the interplay between different microscopic effects generates an effective contact angle at a macroscopic scale, and to detect the proximal regions of the interfaces (likely, at a very small distance from the contact line) in which the effect of the singular long-range potentials may produce a significant effect, see e.g. [33, 53]. To further understand the role of long-range particle interactions in models related to capillarity theory, a modification of the classical Gauß free energy functional has been introduced in [60] that took into account surface tension energies of nonlocal type and modeled on the fractional perimeter presented in [9]. These new variational principles led to suitable equilibrium conditions that determine a specific contact angle depending on the relative adhesion coefficient and on the properties of the interaction kernel. The classical limit angle was then obtained from this long-range prescription via a limit procedure, and precise asymptotics have been provided in [27]. Local minimizers in the fractional capillarity model have been studied in [28], where their blow-up limits at boundary points have been considered, showing, by means of a new monotonicity formula, that these blow-up limits are cones, and giving a complete characterization of such cones in the planar case.

In our dissertation we present a capillarity theory of nonlocal type in which the long-range particle interactions are possibly anisotropic and not necessarily invariant under scaling. This setting is specifically motivated by the twofold objective to initiate and consolidate a nonlocal capillarity theory in an *anisotropic* scenario, and to model the case where the potential interactions of the droplet with the container and those with the environment are subject to different van der Waals forces.

In this setting, we determine a nonlocal Young's law for the contact angle, which extends the known one in the nonlocal isotropic setting and recovers the classical one as a limit case.

1.2 Organization of the thesis and main results

The first chapter of the present thesis is devoted to derive strong unique continuation results and local asymptotics at boundary points for solutions of two classes of elliptic equations. In particular, in Section 2.2 we recall some basic definitions related to the unique continuation property. In Section 2.3 we consider the following class of fractional elliptic equations

$$(-\Delta)^s u = hu \qquad \text{in } \Omega \tag{1.2.1}$$

where $s \in (0,1)$, $\Omega \subset \mathbb{R}^N$ is a bounded domain, $N \geq 2$ and

$$h \in W^{1,p}(\Omega) \quad \text{ with } p > N/2s. \tag{1.2.2}$$

More precisely, we are interested in a strong unique continuation property and local asymptotics of solutions at those boundary points where the domain is locally $C^{1,1}$ and some outer homogeneous Dirichlet boundary condition is prescribed. To this purpose, we assume there exists $x_0 \in \partial\Omega$ such that $\partial\Omega$ is of class $C^{1,1}$ in a neighbourhood of x_0, i.e. there exist a suitable radius $R > 0$ and a function $g \in C^{1,1}(\mathbb{R}^{N-1})$ such that, choosing a proper coordinate system $(x', x_N) \in \mathbb{R}^{N-1} \times \mathbb{R}$, it holds that

$$\begin{aligned} B'_R(x_0) \cap \Omega &= \{(x', x_N) \in B'_R(x_0) : x_N < g(x')\} \\ B'_R(x_0) \cap \partial\Omega &= \{(x', x_N) \in B'_R(x_0) : x_N = g(x')\} \end{aligned} \tag{1.2.3}$$

(see Section 2.1), and we prescribe for u the following local outer homogeneous Dirichlet boundary condition

$$u = 0 \quad \text{a.e. in } \Omega^c \cap B'_R(x_0). \tag{1.2.4}$$

In order to give a suitable weak formulation of (1.2.1), we introduce the functional space $\mathcal{D}^{s,2}(\mathbb{R}^N)$ defined as the completion of $C_c^\infty(\mathbb{R}^N)$ with respect to the scalar product

$$(u, v)_{\mathcal{D}^{s,2}(\mathbb{R}^N)} := \int_{\mathbb{R}^N} |\xi|^{2s}\, \widehat{u}(\xi)\overline{\widehat{v}(\xi)}\, d\xi \tag{1.2.5}$$

and the associated norm $\|u\|^2_{\mathcal{D}^{s,2}(\mathbb{R}^N)} = (u,u)_{\mathcal{D}^{s,2}(\mathbb{R}^N)}$, where $\hat{u}$ denotes the unitary Fourier transform of u in $\mathbb{R}^N$, i.e.

$$\widehat{u}(\xi) = \mathcal{F}u(\xi) := \frac{1}{(2\pi)^{N/2}} \int_{\mathbb{R}^N} e^{-ix\cdot\xi} u(x)\, dx.$$

The fractional Laplacian $(-\Delta)^s$ can be defined as the Riesz isomorphism of $\mathcal{D}^{s,2}(\mathbb{R}^N)$ with respect to the scalar product defined in (1.2.5), i.e.

$${}_{(\mathcal{D}^{s,2}(\mathbb{R}^N))^*}\langle (-\Delta)^s u, v\rangle_{\mathcal{D}^{s,2}(\mathbb{R}^N)} = (u, v)_{\mathcal{D}^{s,2}(\mathbb{R}^N)}$$

for all $u, v \in \mathcal{D}^{s,2}(\mathbb{R}^N)$. A weak solution to (1.2.1) is any function $u \in \mathcal{D}^{s,2}(\mathbb{R}^N)$ satisfying

$$(u, \varphi)_{\mathcal{D}^{s,2}(\mathbb{R}^N)} = \int_\Omega h(x)u(x)\varphi(x)\, dx \quad \text{ for all } \varphi \in C_c^\infty(\Omega). \tag{1.2.6}$$

We observe that the right hand side of (1.2.6) is well defined in view of assumption (1.2.2), by the Hölder's inequality and the following well-known Sobolev-type inequality

$$S_{N,s}\|u\|^2_{L^{2^*(s)}(\mathbb{R}^N)} \leq \|u\|^2_{\mathcal{D}^{s,2}(\mathbb{R}^N)}, \tag{1.2.7}$$

where $S_{N,s}$ is a positive constant depending only on N and s and

$$2^*(s) = \frac{2N}{N-2s}, \tag{1.2.8}$$

see [16]. By the extension technique introduced in [8], by adding an additional space variable $t \in [0, +\infty)$, we can reformulate the nonlocal problem (1.2.1) as a local degenerate or singular problem on the half space $\mathbb{R}^{N+1}_+$. For this, taking $z = (x,t) \in \mathbb{R}^{N+1}_+$, we define $\mathcal{D}^{1,2}(\mathbb{R}^{N+1}_+, t^{1-2s}\,dz)$ as the completion of $C_c^\infty(\overline{\mathbb{R}^{N+1}_+})$ with respect to the norm

$$\|U\|_{\mathcal{D}^{1,2}(\mathbb{R}^{N+1}_+, t^{1-2s}\,dz)} = \sqrt{\int_{\mathbb{R}^{N+1}_+} t^{1-2s}|\nabla U(x,t)|^2\,dx\,dt}.$$

It is well-known that there exists a continuous trace map

$$\mathrm{Tr} : \mathcal{D}^{1,2}(\mathbb{R}^{N+1}_+, t^{1-2s}\,dz) \to \mathcal{D}^{s,2}(\mathbb{R}^N)$$

(see e.g. [13]), which is onto, see [7]. By [8], for every $u \in \mathcal{D}^{s,2}(\mathbb{R}^N)$, the minimization problem

$$\min\left\{\|W\|^2_{\mathcal{D}^{1,2}(\mathbb{R}^{N+1}_+, t^{1-2s}\,dz)} : W \in \mathcal{D}^{1,2}(\mathbb{R}^{N+1}_+, t^{1-2s}\,dz),\ \mathrm{Tr}\,W = u\right\}$$

admits a unique minimizer $U = \mathcal{H}(u) \in \mathcal{D}^{1,2}(\mathbb{R}^{N+1}_+, t^{1-2s}\,dz)$, which can be obtained by convoluting u with the Poisson kernel of the half-space $\mathbb{R}^{N+1}_+$ and weakly solves

$$\begin{cases} -\mathrm{div}(t^{1-2s}\nabla U) = 0 & \text{in } \mathbb{R}^{N+1}_+, \\ -\lim\limits_{t\to 0^+} t^{1-2s}\partial_t U = \kappa_s(-\Delta)^s u & \text{in } \mathbb{R}^N \times \{0\}, \end{cases}$$

where

$$\kappa_s = \frac{\Gamma(1-s)}{2^{2s-1}\Gamma(s)} > 0,$$

that is, for all $W \in \mathcal{D}^{1,2}(\mathbb{R}^{N+1}_+, t^{1-2s}\,dz)$

$$\int_{\mathbb{R}^{N+1}_+} t^{1-2s}\nabla\mathcal{H}(u)(x,t)\cdot\nabla W(x,t)\,dx\,dt = \kappa_s(u, \mathrm{Tr}\,W)_{\mathcal{D}^{s,2}(\mathbb{R}^N)}.$$

As a relevant consequence, a function $u \in \mathcal{D}^{s,2}(\mathbb{R}^N)$ satisfies (1.2.6) if and only if its extension $U = \mathcal{H}(u)$ weakly solves

$$\begin{cases} -\mathrm{div}\left(t^{1-2s}\nabla U\right) = 0 & \text{in } \mathbb{R}^{N+1}_+, \\ \mathrm{Tr} U = u & \text{in } \mathbb{R}^N \times \{0\}, \\ -\lim\limits_{t\to 0^+} t^{1-2s}\partial_t U = \kappa_s h u & \text{in } \Omega \times \{0\}, \end{cases} \tag{1.2.9}$$

i.e

$$\int_{\mathbb{R}^{N+1}_+} t^{1-2s} \nabla U(x,t) \cdot \nabla \phi(x,t)\, dx\, dt = \kappa_s \int_\Omega hu \operatorname{Tr} \phi \, dx \tag{1.2.10}$$

for every $\phi \in C_c^\infty(\overline{\mathbb{R}^{N+1}_+})$ with $\operatorname{Tr}\phi \in C_c^\infty(\Omega)$.

After describing a more detailed functional setting for the extended problem (1.2.9) at the beginning of Section 2.3, in Subsection 2.3.1 we introduce the auxiliary problem (2.3.11) obtained by applying a diffeomorphism, inspired by [3], which straightens the boundary of the domain Ω. This deformation is thought to ensure that the extended equation is preserved by reflection through a straightened vertical boundary. In Subsection 2.3.2, first we provide some coercivity-type inequalities, and then we develop an approximation procedure in order to overcome the difficulties related to the lack of regularity at Dirichlet-Neumann junctions. Specifically, we approximate the potential h with potentials vanishing close to the boundary and the Dirichlet N-dimensional region with smooth $(N+1)$-dimensional sets having a straight vertical boundary. Then we construct a sequence of solutions to certain boundary value problems on the approximating domains which enjoy enough regularity to derive Pohozaev-type identities and, once we prove that such a sequence converges in the H^1-norm to the solution of (2.3.11), passing to the limit, we achieve a Pohozaev-type identity even for solutions to the straightened problem (2.3.11), see Subsection 2.3.3 for details. Subsection 2.3.4 is devoted to the proof of a monotonicity formula for the Almgren frequency function (2.3.89), which in turn is used to perform a blow-up analysis in Subsection 2.3.5. Here, the asymptotic behaviour at $x_0 \in \partial\Omega$ of solutions to (1.2.9), and consequently of solutions to (1.2.1), turn out to be related to the eigenvalues and the eigenfunctions of the following weighted spherical eigenvalue problem with mixed Dirichlet-Neumann boundary conditions on the unit half-sphere

$$\begin{cases} -\operatorname{div}_{\mathbb{S}^N}\left(\theta_{N+1}^{1-2s} \nabla_{\mathbb{S}^N} \psi\right) = \theta_{N+1}^{1-2s} \mu \psi & \text{in } \mathbb{S}^N_+, \\ \psi = 0 & \text{on } \mathbb{S}^{N-1} \cap \{\theta_N \geq 0\}, \\ \lim\limits_{\theta_{N+1} \to 0^+} \theta_{N+1}^{1-2s} \nabla_{\mathbb{S}^N} \psi \cdot \nu = 0 & \text{on } \mathbb{S}^{N-1} \cap \{\theta_N < 0\}, \end{cases} \tag{1.2.11}$$

where $\nu = (0, \dots, 0, -1)$ (see Section 2.1). In order to give the variational formulation of (1.2.11), we define $H^1(\mathbb{S}^N_+, \theta_{N+1}^{1-2s} dS)$ as the completion of $C^\infty(\overline{\mathbb{S}^N_+})$ with respect to the norm

$$\|\psi\|_{H^1(\mathbb{S}^N_+, \theta_{N+1}^{1-2s} dS)} = \left(\int_{\mathbb{S}^N_+} \theta_{N+1}^{1-2s} \left(|\nabla_{\mathbb{S}^N} \psi(\theta)|^2 + \psi^2(\theta) \right) dS \right)^{1/2}.$$

Let $\mathcal{H}_0$ be the closure of $C_c^\infty(\overline{\mathbb{S}^N_+} \setminus S_1^+)$ in $H^1(\mathbb{S}^N_+, \theta_{N+1}^{1-2s} dS)$. We say that $\mu \in \mathbb{R}$ is an *eigenvalue* of (1.2.11) if there exists $\psi \in \mathcal{H}_0 \setminus \{0\}$ such that

$$\int_{\mathbb{S}^N_+} \theta_{N+1}^{1-2s} \nabla_{\mathbb{S}^N} \psi \cdot \nabla_{\mathbb{S}^N} \phi \, dS = \mu \int_{\mathbb{S}^N_+} \theta_{N+1}^{1-2s} \psi \phi \, dS \quad \text{for any } \phi \in \mathcal{H}_0. \tag{1.2.12}$$

By classical spectral theory, problem (1.2.11) admits a diverging sequence of real eigenvalues with finite multiplicity $\{\mu_k\}_{k \geq 0}$. In Appendix A.2 we derive the following explicit formula for such eigenvalues

$$\mu_k = (k+s)(k+N-s), \quad k \in \mathbb{N}. \tag{1.2.13}$$

For all $k \in \mathbb{N}$, let $M_k \in \mathbb{N}\setminus\{0\}$ be the multiplicity of the eigenvalue μ_k and $\{Y_{k,m}\}_{m=1,2,...,M_k}$ be a $L^2(\mathbb{S}^N_+, \theta_{N+1}^{1-2s} dS)$-orthonormal basis of the eigenspace of problem (1.2.11) associated to μ_k. In particular,

$$\{Y_{k,m} : k \in \mathbb{N},\ m = 1, ..., M_k\} \tag{1.2.14}$$

is an orthonormal basis of $L^2(\mathbb{S}^N_+, \theta_{N+1}^{1-2s} dS)$.

A first result involving problem (1.2.11) is a sharp description of the asymptotic behaviour of solutions to (1.2.9) around $x_0 \in \partial\Omega$, contained in Theorem 2.3.30. More precisely, we prove that there exist $k_0 \in \mathbb{N}$ and an eigenfunction Y of problem (1.2.11) associated to the eigenvalue $\mu_{k_0} = (k_0+s)(k_0+N-s)$ such that, letting $z_0 = (x_0, 0)$,

$$\frac{U(z_0 + \lambda z)}{\lambda^{k_0+s}} \to |z|^{k_0+s} Y\left(\frac{z}{|z|}\right) \quad \text{in } H^1(B_1^+, t^{1-2s}dz) \text{ as } \lambda \to 0^+,$$

where $H^1(B_1^+, t^{1-2s}dz)$ is the weighted Sobolev space defined at the beginning of Section 2.3. Actually in the proof of Theorem 2.3.30 we give a more precise characterization of the angular limit profile Y as a linear combination of the orthonormalized eigenfunctions $\{Y_{k_0,m}\}_{m=1,2,...,M_{k_0}}$ of (1.2.11) associated to the eigenvalue μ_{k_0} with coefficients explicitely given by formula (2.3.176).

Then we are able to derive also a similar sharp description of the asymptotic behaviour of solutions to (1.2.1) at $x_0 \in \partial\Omega$ (we refer to Theorem 2.3.31), i.e. we infer that there exist $k_0 \in \mathbb{N}$ and an eigenfunction Y of problem (1.2.11) associated to the eigenvalue $\mu_{k_0} = (k_0+s)(k_0+N-s)$ such that

$$\frac{u(x_0 + \lambda x)}{\lambda^{k_0+s}} \to |x|^{k_0+s} Y\left(\tfrac{x}{|x|}, 0\right) \quad \text{in } H^s(B_1') \text{ as } \lambda \to 0^+,$$

where $H^s(B_1')$ is the usual fractional Sobolev space on the N-dimensional unit ball B_1', see e.g. [52].

As a consequence of the above asymptotic expansions, we deduce the following *strong unique continuation principles* for problems (1.2.9) and (1.2.1) (see Theorem 2.3.32), that is, respectively:

(i) if U is a weal solution to (1.2.9) such that $U(z) = O(|z-z_0|^k)$ as $z \to z_0$ for any $k \in \mathbb{N}$, then $U \equiv 0$ in $\mathbb{R}^{N+1}_+$;

(ii) if u is a weak solution to (1.2.1) such that $u(x) = O(|x-x_0|^k)$ as $x \to x_0$ for any $k \in \mathbb{N}$, then $u \equiv 0$ in $\mathbb{R}^N$.

Finally, in Appendix A.1 we present some boundary regularity results for singular/degenerate equations in cylinders, while in Appendix A.2 we prove (1.2.13), through a classification of possible homogeneity degrees of homogeneous solutions to (A.2.1).

In Section 2.4, we develop a monotonicity approach to the study of the asymptotic behavior and unique continuation from the edge of a crack for solutions to the following class of Dirichlet boundary value problems

$$\begin{cases} -\Delta u(z) = f(z)u(z) & \text{in } \Omega \setminus \Gamma, \\ u = 0 & \text{on } \Gamma, \end{cases} \tag{1.2.15}$$

where $\Omega \subset \mathbb{R}^{N+1}$ is a bounded open domain, $N \geq 2$, $\Gamma \subset \mathbb{R}^N$ is a closed set defined as

$$\Gamma = \{(x', x_N) = (x_1, \dots, x_{N-1}, x_N) \in \mathbb{R}^N : x_N \geq g(x')\}, \tag{1.2.16}$$

for some function $g \in C^2(\mathbb{R}^{N-1})$. In order to do this, we fix a point on the edge of Γ and, without loss of generality, we may select our coordinate system in such a way that the origin coincides with this point, and

$$g(0) = 0, \quad \nabla g(0) = 0, \tag{1.2.17}$$

namely the boundary of Γ is tangent to the hyperplane $x_N = 0$ at 0, thus having that

$$|g(x')| = O(|x'|^2) \quad \text{as } |x'| \to 0^+. \tag{1.2.18}$$

Moreover we assume that there exists a suitable radius $\hat{R} > 0$ such that

$$g(x') - x' \cdot \nabla g(x') \geq 0 \quad \text{for any } x' \in B'_{\hat{R}}. \tag{1.2.19}$$

We observe that this assumption can be removed arguing as in the fractional case, that is applying a suitable diffeomorphism to straighten the boundary before carrying out the approximation procedure.

In the setting described above, we are interested in studying local asymptotics and strong unique continuation property at the origin for solutions to the following boundary value problem

$$\begin{cases} -\Delta u = f\, u & \text{in } B_{\hat{R}} \setminus \Gamma, \\ u = 0 & \text{on } \Gamma, \end{cases} \tag{1.2.20}$$

where $f\colon B_{\hat{R}} \to \mathbb{R}$ is measurable and bounded in $B_{\hat{R}} \setminus B_\delta$ for every $\delta \in (0, \hat{R})$.

We contemplate two alternative sets of assumptions on f, namely we assume either that

$$\lim_{r \to 0^+} \xi_f(r) = 0, \tag{H1-1}$$

$$\frac{\xi_f(r)}{r} \in L^1(0, \hat{R}), \quad \frac{1}{r}\int_0^r \frac{\xi_f(s)}{s}\, ds \in L^1(0, \hat{R}), \tag{H1-2}$$

where the function ξ_f is defined as

$$\xi_f(r) := \sup_{z \in \overline{B_r}} |z|^2 |f(z)| \quad \text{for any } r \in (0, \hat{R}), \tag{H1-3}$$

or that

$$\lim_{r\to 0^+} \eta(r,f) = 0, \tag{H2-1}$$

$$\frac{\eta(r,f)}{r} \in L^1(0,\hat{R}), \quad \frac{1}{r}\int_0^r \frac{\eta(s,f)}{s}\,ds \in L^1(0,\hat{R}), \tag{H2-2}$$

and

$$\nabla f \in L^\infty_{\mathrm{loc}}(B_{\hat{R}} \setminus \{0\}), \tag{H2-3}$$

$$\frac{\eta(r,\nabla f\cdot z)}{r} \in L^1(0,\hat{R}), \quad \frac{1}{r}\int_0^r \frac{\eta(s,\nabla f\cdot z)}{s}\,ds \in L^1(0,\hat{R}), \tag{H2-4}$$

where

$$\eta(r,h) := \sup_{u\in H^1(B_r)\setminus\{0\}} \frac{\int_{B_r} |h|u^2\,dz}{\int_{B_r} |\nabla u|^2\,dz + \frac{N-1}{2r}\int_{\partial B_r} |u|^2\,dS}, \tag{H2-5}$$

for every $r \in (0,\hat{R})$ and $h \in L^\infty_{\mathrm{loc}}(B_{\hat{R}} \setminus \{0\})$. We refer to Section 2.4 for some examples of functions verifying the above assumptions.

In order to give a weak formulation of problem (1.2.20), we introduce for every $R > 0$ the space $H^1_\Gamma(B_R)$ defined as the closure in $H^1(B_R)$ of the subspace

$$C^\infty_{0,\Gamma}(\overline{B_R}) := \{u \in C^\infty(\overline{B_R}) : u = 0 \text{ in a neighborhood of } \Gamma\}. \tag{1.2.21}$$

It actually holds that

$$H^1_\Gamma(B_R) = \{u \in H^1(B_R) : \tau_\Gamma(u) = 0\},$$

where τ_Γ denotes the trace operator on Γ, see Lemma 2.4.1 in Section 2.4.

Hence we say that $u \in H^1(B_{\hat{R}})$ is a weak solution to (1.2.20) if

$$\begin{cases} u \in H^1_\Gamma(B_{\hat{R}}), \\ \displaystyle\int_{B_{\hat{R}}} \nabla u(z)\cdot\nabla\varphi(z)\,dz - \int_{B_{\hat{R}}} f(z)u(z)\varphi(z)\,dz = 0 \quad \text{for any } \varphi \in C^\infty_c(B_{\hat{R}} \setminus \Gamma). \end{cases} \tag{1.2.22}$$

Since our domain is highly non-smooth due to the presence of the crack, as in the above case we use an approximation argument to derive a monotonicity formula. Specifically, the proof of the monotonicity formula is based on the differentiation of the Almgren quotient defined in (2.4.47), which in turn requires a Pohozaev-type identity formally obtained by testing the equation with the function $\nabla u\cdot z$; however our domain with crack doesn't verify the exterior ball condition (which ensures L^2-integrability of second order derivatives, see [2]), thus $\nabla u \cdot z$ could be not sufficiently regular to be an admissible test function. Hence, in order to overcome this difficulty, in Subsection 2.4.1 we construct first a sequence of regular sets which approximate our cracked domain with the twofold features of satisfying the exterior ball condition and being star-shaped with respect to the origin, and then a sequence of solutions of some approximating problems on such domains, converging to the solution of the original problem with crack. Thus for each approximating problem we have enough regularity to derive a Pohozaev-type identity with some remainder terms, due to interference with the boundary, whose sign can nevertheless be established thanks to star-shapeness conditions (see Subsection 2.4.2). Then, passing to the limit in the resulting Pohozaev-type inequalities for the approximating problems, we obtain inequality (2.4.34), which allows us in Subsection 2.4.3 to estimate from below the derivative of the Almgren quotient and to prove that it has a finite limit at 0 (Lemma 2.4.23). Then, in Subsection 2.4.4, we perform a blow-up analysis for scaled solution: in particular, in the classification of possible vanishing orders and blow-up profiles of solutions, the following eigenvalue problem

$$\begin{cases} -\Delta_{\mathbb{S}^N}\psi = \mu\,\psi & \text{on } \mathbb{S}^N \setminus S_1^+, \\ \psi = 0 & \text{on } S_1^+, \end{cases} \tag{1.2.23}$$

on the unit N-dimensional sphere with an half-equator cut plays a crucial role.

We say that $\mu \in \mathbb{R}$ is an *eigenvalue* of (1.2.23) if there exists an eigenfunction $\psi \in H_0^1(\mathbb{S}^N \setminus S_1^+)$, $\psi \not\equiv 0$, such that

$$\int_{\mathbb{S}^N} \nabla_{\mathbb{S}^N}\psi \cdot \nabla_{\mathbb{S}^N}\phi \, dS = \mu \int_{\mathbb{S}^N} \psi\phi \, dS$$

for all $\phi \in H_0^1(\mathbb{S}^N \setminus S_1^+)$. By classical spectral theory, (1.2.23) admits a diverging sequence of real eigenvalues with finite multiplicity $\{\mu_k\}_{k\geq 1}$; these eigenvalues are explicitly given by the formula

$$\mu_k = \frac{k(k+2N-2)}{4}, \quad k \in \mathbb{N} \setminus \{0\}, \tag{1.2.24}$$

see Lemma 2.4.30. For all $k \in \mathbb{N}\setminus\{0\}$, let $M_k \in \mathbb{N}\setminus\{0\}$ be the multiplicity of the eigenvalue μ_k and

$$\begin{gathered} \{Y_{k,m}\}_{m=1,2,\dots,M_k} \text{ be a } L^2(\mathbb{S}^N)\text{-orthonormal basis} \\ \text{of the eigenspace of (1.2.23) associated to } \mu_k. \end{gathered} \tag{1.2.25}$$

In particular $\{Y_{k,m} : k \in \mathbb{N} \setminus \{0\}, m = 1, 2, \dots, M_k\}$ is an orthonormal basis of $L^2(\mathbb{S}^N)$.

In Subsection 2.4.5, by means of an auxiliary problem obtained by straightening the crack, a first result consists in proving that scaled solutions of problem (1.2.20) converge to a homogeneous limit profile, whose homogeneity order is related to the eigenvalues of problem (1.2.23). More precisely, we prove that

$$\frac{u(\lambda z)}{\lambda^{k_0/2}} \to |z|^{k_0/2}\psi\left(\frac{z}{|z|}\right) \quad \text{in } H^1(B_1) \text{ as } \lambda \to 0^+, \tag{1.2.26}$$

for some $k_0 \in \mathbb{N} \setminus \{0\}$ and some eigenfunction ψ of problem (1.2.23) associated with the eigenvalue μ_{k_0}. A strongest version of the above result can be found in Theorem 2.4.39, where we actually give a more precise description of the limit angular profile ψ: indeed, if $M_{k_0} \geq 1$ is the multiplicity of the eigenvalue μ_{k_0} and $\{Y_{k_0,i} : 1 \leq i \leq M_{k_0}\}$ is as in (1.2.25), then the eigenfunction ψ in (1.2.26) can be written as

$$\psi(\theta) = \sum_{i=1}^{m_{k_0}} \beta_i Y_{k_0,i}, \tag{1.2.27}$$

where the coefficients β_i are given by the *integral Cauchy-type formula* (2.4.151).

A relevant consequence of our asymptotic analysis is the following *strong unique continuation principle*, whose proof follows straightforwardly from (1.2.26) (see Theorem 2.4.40): if u is a weak solution to (1.2.20) such that $u(z) = O(|z|^k)$ as $|z| \to 0$ for any $k \in \mathbb{N}$, then $u \equiv 0$ in $B_{\hat{R}}$.

Finally, in Chapter 3 we present a capillarity theory of nonlocal type in which the long-range particle interactions are possibly anisotropic and not necessarily invariant under scaling. In particular, the lack of scale invariance will be modeled via two different fractional exponents $s_1, s_2 \in (0,1)$ which take into account the possibility that the container and the environment present different features with respect to particle interactions. In order to describe in more details our setting, we first discuss the type of particle interactions that we take into account and the variational structure of the corresponding anisotropic nonlocal capillarity theory. Owing to [9], the most widely studied interaction kernel of singular type in problems related to nonlocal surface tension is

$$K_s(\zeta) := \frac{1}{|\zeta|^{n+s}} \quad \text{for all } \zeta \in \mathbb{R}^n \setminus \{0\}, \tag{1.2.28}$$

with $s \in (0,1)$. We aim at considering more general kernels than the one in (1.2.28), with a twofold objective: on the one hand, we wish to initiate and consolidate a nonlocal capillarity theory in an *anisotropic* scenario; on the other hand, we want to also model the case in which the particle interaction of the container has a *different structure* with respect to the one of the external environment. The first of these two goals will be pursued by considering interaction kernels that are *not necessarily invariant under rotation*, the second by taking into account *interactions with different homogeneity* inside the container and in the external environment.

More specifically, given $n \geq 2$, $s \in (0,1)$, $\lambda \geq 1$ and $\varrho \in (0, \infty]$, we consider the family of interaction kernels, denoted by $\mathbf{K}(n, s, \lambda, \varrho)$, containing the even functions $K \colon \mathbb{R}^n \setminus \{0\} \to [0, +\infty)$ such that, for all $\zeta \in \mathbb{R}^n \setminus \{0\}$,

$$\frac{\chi_{B_\varrho(0)}(\zeta)}{\lambda |\zeta|^{n+s}} \leq K(\zeta) \leq \frac{\lambda}{|\zeta|^{n+s}}. \tag{1.2.29}$$

We use the notation $B_\varrho(0) = \mathbb{R}^n$ when $\varrho = \infty$. Also, for every $h \in \mathbb{N}$, we consider the class $\mathbf{K}^h(n, s, \lambda, \varrho)$ of all the kernels $K \in \mathbf{K}(n, s, \lambda, \varrho) \cap C^h(\mathbb{R}^n \setminus \{0\})$ such that, for all $\zeta \in \mathbb{R}^n \setminus \{0\}$,

$$|D^j K(\zeta)| \leq \frac{\lambda}{|\zeta|^{n+s+j}} \qquad \text{for all } 1 \leq j \leq h. \tag{1.2.30}$$

We also say that the kernel K admits a blow-up limit if for every $\zeta \in \mathbb{R}^n \setminus \{0\}$ the following limit exists:

$$K^*(\zeta) := \lim_{r \to 0^+} r^{n+s} K(r\zeta). \tag{1.2.31}$$

For each kernel K we consider the interaction induced by K between any two disjoint (measurable) subsets E, F of $\mathbb{R}^n$ defined by

$$I_K(E, F) := \int_E \int_F K(x - y)\, dx\, dy. \tag{1.2.32}$$

For instance, with this definition, the so-called K-nonlocal perimeter of a set E associated to K is given by the quantity $I_K(E, E^c)$, which is the interaction of the set E with its complement with respect to $\mathbb{R}^n$ (here, as usual, we use the notation $E^c := \mathbb{R}^n \setminus E$). See [14] for several results on the K-nonlocal perimeter. In particular, if K is the fractional kernel in (1.2.28), then the notion of K-perimeter boils down to the one introduced by Caffarelli, Roquejoffre and Savin in [9].

Given an open set $\Omega \subseteq \mathbb{R}^n$, s_1, $s_2 \in (0,1)$ and $\sigma \in \mathbb{R}$, for every $K_1 \in \mathbf{K}(n, s_1, \lambda, \varrho)$ and $K_2 \in \mathbf{K}(n, s_2, \lambda, \varrho)$ and every set $E \subseteq \Omega$ we define the functional

$$\mathcal{E}(E) := I_1(E, E^c \cap \Omega) + \sigma\, I_2(E, \Omega^c). \tag{1.2.33}$$

Throughout all Chapter 3, we will use the short notation $I_1 := I_{K_1}$ and $I_2 := I_{K_2}$.

We observe that when $\sigma > 0$, one could reabsorb it into the second interaction kernel up to redefining K_2 into σK_2. In general, one can think that σ "simply plays the role of a sign", say it suffices to understand the matter for $\sigma \in \{-1, +1\}$, up to changing K_2 into $|\sigma|\, K_2$. However, we thought it was convenient to consider σ as an "independent parameter", since this makes it easier to compare with the classical case.

Moreover, given a function $g \in L^\infty(\Omega)$, we let

$$\mathcal{C}(E) := \mathcal{E}(E) + \int_E g(x)\, dx. \tag{1.2.34}$$

The setting that we take into account is general enough to include anisotropic nonlocal perimeter functionals as in [59, 14], which, in turn, can be seen as nonlocal modifications of the classical anisotropic perimeter functional. In this spirit, the functional in (1.2.34) can be seen as a nonlocal generalization of classical anisotropic capillarity problems, such as the ones in [63]. As customary in the analysis of nonlocal problems arising from geometric functionals, the long-range interactions involved in (1.2.34) produce significant energy contributions which will give rise to structural differences with respect to the classical case.

In this context, our goal is to study the minimizers of the nonlocal capillarity functional $\mathcal{C}$ among all the sets E with a given volume. The case in which $K_1(\zeta) = K_2(\zeta) = K_s(\zeta)$ as in (1.2.28) has been studied in [60, 27, 28]. Here instead we are specifically interested in the nonlocal capillarity energy in (1.2.34) with two different types of interactions between E and $\Omega \cap E^c$ and between E and Ω^c, as modeled in (1.2.33).

Notice that the volume constrained minimization of the functional in (1.2.34) is well-posed, according to Proposition 3.1.2 in Section 3.1. In particular, we give a formulation of Proposition 3.1.2 which is new in literature, though its proof relies on an appropriate variation of standard techniques, see e.g. [9, 60].

In Section 3.2 we show that the volume constrained minimizers (and, more generally, the volume constrained critical points) obtained in Proposition 3.1.2 satisfy a suitable Euler-Lagrange equation (under reasonable regularity assumptions on the domain and on the interaction kernels), according to Proposition 3.2.1.

A crucial step of any capillarity theory is the determination of the contact angle between the droplet and the container (in jargon, the Young's law), which relies on a delicate cancellation of the singular kernel contributions, which requires the determination of an auxiliary angle which is "symmetric" (in a suitable sense of "measuring singular interactions") with respect to the contact angle itself. For this, in Section 3.3 we establish a cancellation property which has been thought in order to reproduce a cancellation of terms as in [60], highlighting that in this context a new construction is needed due to the fact that the function a_1 is anisotropic.

Then in Section 3.4 we present two versions of the nonlocal Young's law depending on whether $s_1 \neq s_2$ or $s_1 = s_2$, since in our setting the Young's law is very sensitive to the relative homogeneity of the interacting kernels. Loosely speaking, when $s_1 < s_2$, at small scales (which are the ones which we believe are more significant in the local determination of the contact angle), the interaction between the droplet and the exterior of the container prevails with respect to the one between the droplet and the interior of the container. Thus, in this situation, the sign of the relative adhesion coefficient σ becomes determinant: in the hydrophilic regime $\sigma < 0$ the droplet is "absorbed" by the boundary of the container, thus producing a zero contact angle; instead, in the hydrorepellent regime $\sigma > 0$ the droplet is "held off" the boundary of the container, thus producing a contact angle equal to π; finally, in the neutral case $\sigma = 0$ the behavior of the second interaction kernel becomes irrelevant. When $\sigma = 0$ and additionally the problem is isotropic, the contact angle becomes $\pi/2$. Conversely, when $s_1 > s_2$, the interaction between the droplet and the interior of the container is, at small scales, significantly stronger than that between the droplet and the exterior of the container. In this situation, the relative adhesion coefficient σ does not play any role and the contact angle is determined by an integral cancellation condition (that will be explicitly provided in (3.3.6)). When the first kernel is isotropic, this condition simplifies and the contact angle is proved to be $\pi/2$.

Section 3.5 deals with the possible complete stickiness or detachment of the nonlocal droplets. Indeed, we think that the detection of a contact angle in a nonlocal capillarity setting is an interesting feature in itself, especially when we compare this situation with the stickiness phenomenon for the nonlocal minimal surfaces, as discovered in [29]. More specifically, for nonlocal minimal surfaces, the long-range interactions make it possible for the surface to stick to a domain (even if the domain is smooth and convex), thus changing dramatically the boundary analysis (moreover, this phenomenon is essentially "generic", see [31]). The possible detection of the contact angle for the nonlocal capillarity theory instead highlights the fact that the boundary analysis of this theory is somewhat "sufficiently robust" with respect to the classical case. Roughly speaking, we believe that this important difference between nonlocal minimal surfaces and nonlocal capillarity theory is due to the fact that in the latter the mass is always placed in a bounded region, whence the energy contributions coming from infinity have a different nature than the ones occurring for nonlocal perimeter functionals.

We also stress that conditions (3.4.2) and (3.4.4) basically state that if the kernel K_2 is "too strong", then one cannot expect nontrivial minimizers. Roughly speaking, while Proposition 3.1.2 always guarantees the existence of a minimizer, when conditions (3.4.2) and (3.4.4) are violated the minimizer can "detach from the boundary" (or "completely stick to the boundary"), hence the notion of contact angle becomes degenerate or void.

That is, while for the existence of minimizers we do not need to require any bound on the relative adhesion coefficient σ in dependence of the interaction kernels, to speak about a contact angle some quantitative condition is in order (roughly speaking, otherwise the droplet does not meet the boundary of the container with a nontrivial angle, rather preferring to either detach from the container and float, or to completely stick at the boundary by surrounding it). The configuration in which the droplet tends to be squashed on the container, thus producing a contact angle ϑ close to zero, is sketched in Figure 1.2.

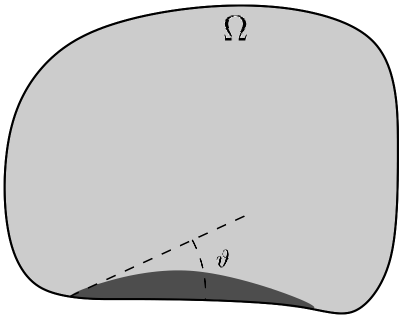


Figure 1.2: The configuration in which the droplet tends to stick to the container.

The opposite situation in which the droplet tends to detach from the container, thus producing a contact angle ϑ close to π, is depicted in Figure 1.3.

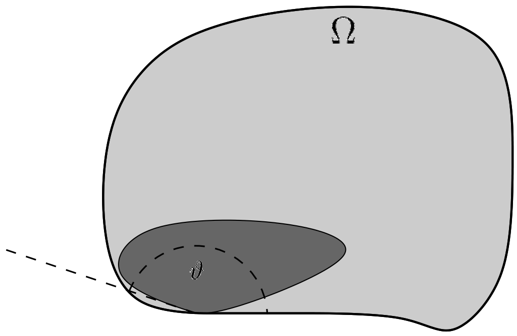


Figure 1.3: The configuration in which the droplet tends to detach from the container.

These concepts are made explicit in Theorems 3.5.1 and 3.5.2.

Section 3.6 is devoted to discuss the existence and uniqueness theory for the equation prescribing the nonlocal angle of contact between the droplet and the container. Additionally, in Remark 3.6.4 at the end of Section 3.6, we will point out that the uniqueness statement in Theorem 3.6.3 heavily depends on the strict positivity of the kernel and it fails for kernels that are merely nonnegative.

Chapter 2

Some strong unique continuation results from the boundary

2.1 Notations

We list below some symbols used throughout Chapter 2 and the relative description.

Symbol	**Description**
$\mathbb{R}^{N+1}_+$	The half-space $\mathbb{R}^N \times (0,+\infty)$
$\mathbb{S}^N$	The unit sphere $\{(\theta',\theta_N,\theta_{N+1}) \in \mathbb{R}^{N+1} : \lvert\theta'\rvert^2 + \theta_N^2 + \theta_{N+1}^2 = 1\}$
$\mathbb{S}^N_+$	The unit half-sphere $\{(\theta',\theta_N,\theta_{N+1}) \in \mathbb{S}^N : \theta_{N+1} > 0\}$
$\mathbb{S}^{N-1}$	The boundary of $\mathbb{S}^N_+$, i.e. $\mathbb{S}^{N-1} \times \{0\}$ identified with $\mathbb{S}^{N-1}$
$\mathbb{S}^{N-1}_-$	The set $\{(\theta_1,\dots,\theta_N) \in \mathbb{S}^{N-1} : \theta_N \le 0\}$
S_1^+	The set $\{(\theta',\theta_N,\theta_{N+1}) \in \mathbb{S}^N : \theta_{N+1} = 0 \text{ and } \theta_N \ge 0\}$
dS	The volume element on N-dimensional spheres
B_r	The ball in $\mathbb{R}^{N+1}$ centered at 0 with radius r, i.e. $\{z \in \mathbb{R}^{N+1} : \lvert z\rvert < r\}$
B_r^+	The half-ball in $\mathbb{R}^{N+1}$ given by $B_r \cap \mathbb{R}^{N+1}_+$
$\partial^+ B_r^+$	The spherical shell given by $\partial B_r \cap \mathbb{R}^{N+1}_+$
$B'_r(x_0)$	The ball in $\mathbb{R}^N$ centered at x_0 with radius r, i.e. $\{x \in \mathbb{R}^N : \lvert x - x_0\rvert < r\}$
B'_r	The ball in $\mathbb{R}^N$ centered at 0 with radius r, i.e. $B'_r(0)$

2.2 An introduction to the unique continuation property

In this section we exhibit a brief introduction to the unique continuation principle. With regard to this property, three different notions are available in the literature.

- The *strong unique continuation property* is said to hold for a family of functions, e.g. the set of solutions to a certain partial differential equation, if no solution in the family, except for the zero function, has a zero of infinite order. We notice that the sentence *has a zero of infinite order* acquires a different meaning depending on the context. In the case of a C^∞-function, we will say it has a zero of infinite order at some point x_0 if all its derivatives at x_0 are zero. For example, the set of analytic functions trivially satisfies the strong unique continuation principle. Instead, in the case of a non-smooth function, we will assert it has a zero of infinite order at some point x_0 if

$$u(x) = O(|x - x_0|^k) \text{ as } x \to x_0 \text{ for all } k \in \mathbb{N}.$$

- A weaker version of the strong unique continuation property is the following one: we say that the set of solutions to a certain partial differential equation satisties the so-called *weak unique continuation property* if no solution, except for the zero function, vanishes on some non-empty open set. We remark that if a family of solutions satisfies the strong unique continuation principle then it trivially verifies the weak unique continuation property.
- Finally, we assert that a family of functions enjoys the *unique continuation property from sets of positive measure* if no function, besides possibly the zero function, vanishes on a set of positive Lebesgue measure. In other words, if a function in the family is non-trivial then its nodal set has zero Lebesgue measure.

A way to obtain the strong unique continuation property for solutions to some linear second order elliptic equation on an open subset of $\mathbb{R}^N$ is to prove their analyticity, since in that case the strong unique continuation property would trivially follow. Hilbert's nineteenth problem asks whether the solutions to linear second order elliptic equations with analytic coefficients are themselves analytic. Several contributions occured over the years in order to give an answer to this problem. The first one was by Bernstein who proved in 1904 the analyticity of solutions of class C^3 in dimension 2; then Petrowsky improved this result in 1939 requiring less regularity for solutions.

When coefficients are not analytic, there is no hope for solutions to be analytic; thus in this case the strong unique continuation principle is not trivial to be proved, hence one should use different methods. For instance, if we consider a linear second order elliptic operator

$$\mathcal{L}u = -\sum_{i,j=1}^{N} \frac{\partial}{\partial x_i}\left(a_{ij}\frac{\partial u}{\partial x_j}\right) + \boldsymbol{W}\cdot\nabla u + Vu \tag{2.2.1}$$

for some vector field $\boldsymbol{W}$ and for some potential V, where a_{ij} are the components of some matrix-valued function $A(x)$, an approach to study the unique continuation property in the presence of non-analytic coefficients is the Carleman method, based on some weighted a priori estimates (see [11]); indeed, Carleman proved the strong unique continuation property in the case where $N = 2$, $a_{ij} \in C^2$, V and $\boldsymbol{W}$ belong to L^∞. Other contributions in this field were later given by Aronszajn, Jerison-Kenig and Sogge. In 1992 Wolff proved the weak unique continuation principle for $N \geq 3$, $a_{ij} \in C^{0,1}$, $V \in L^{N/2}$ and $\boldsymbol{W} \in L^N$. Under the same hypotheses, Koch and Tataru (2001) were able to prove the strong unique continuation principle using the Carleman estimates. We observe that in the context of Lebesgue spaces, the assumption $V \in L^{N/2}$ is sharp. In order to show this, we exhibit a counterexample provided by Jerison and Kenig in 1985.

Example 2.2.1. *We consider*

$$f(x) = \exp\left(-\left(\log\frac{1}{|x|}\right)^{1+\varepsilon}\right) \quad \textit{for every } |x| < 1, \textit{ with } \varepsilon > 0. \tag{2.2.2}$$

Then by direct calculations, $\Delta f = Vf$ *in* B_1*, with*

$$V(x) \sim (1+\varepsilon)^2 \left(\log\frac{1}{|x|}\right)^{2\varepsilon} \frac{1}{|x|^2} \quad \textit{as } |x| \to 0. \tag{2.2.3}$$

From this, we deduce that $V \in L^p(B_1)$ *if and only if*

$$\int_0^1 r^{N-1} (-\log r)^{2\varepsilon p} r^{-2p}\, dr < +\infty,$$

thus if and only if $2p - N + 1 < 1$*, that is* $p < N/2$*. We conclude by observing that* f *is non-trivial and for all* $k \in \mathbb{N}$ *it holds that*

$$\lim_{r\to 0} f(r) r^{-k} = 0,$$

namely f *has infinite order of vanishing at the origin. Thus the family of solutions to* $\Delta f = Vf$ *with* V *as in* (2.2.3) *does not satisfy the strong unique continuation principle.*

Another approach to get unique continuation results for solutions to elliptic equations has been developed by Garofalo and Lin, based on some local doubling properties obtained via the so-called monotonicity formula for the Almgren frequency function, which is defined as the local energy over the mass of non-trivial solutions near a fixed point x_0. To be more clear, we show the monotonicity formula in a simple case.

Example 2.2.2. *Using the same notation as in* (2.2.1)*, the frequency function associated to a non-trivial solution of* $\mathcal{L}u = 0$ *in* B_1*, with* $A = \mathrm{Id}_N$ *and* $\mathbf{W} \equiv 0$*, around* $x_0 = 0$ *is given by*

$$\mathcal{N}(r) = \frac{r\displaystyle\int_{B_r} \left[|\nabla u|^2 + Vu^2\right]\, dx}{\displaystyle\int_{\partial B_r} |u|^2\, dS}. \tag{2.2.4}$$

Once the boundedness of (2.2.4) *is proved, it is possible to derive a doubling type condition, that is*

$$\int_{B_{2r}} u^2\, dx \le C_{\mathrm{doub}} \int_{B_r} u^2\, dx,$$

for some positive constant C_{doub} *and then to prove the strong unique continuation property. To this goal, let us suppose that* $u(x) = O(|x|^k)$ *as* $x \to 0$ *for all* $k \in \mathbb{N}$ *and let* $k_0 \in \mathbb{N}$ *be such that* $\frac{C_{\mathrm{doub}}}{2^{2k_0+N}} < 1$*. In particular, we have that*

$$\begin{aligned}\int_{B_1} u^2\, dx \le C_{\mathrm{doub}} \int_{B_{1/2}} u^2\, dx \le \ldots \le C_{\mathrm{doub}}^k \int_{B_{1/2^k}} u^2\, dx &\underset{k \text{ large}}{\le} C_{\mathrm{doub}}^k C_0 \left(\frac{1}{2^k}\right)^{2k_0+N} \\ &\le C_0 \left(\frac{C_{\mathrm{doub}}}{2^{2k_0+N}}\right)^k \overset{k\to+\infty}{\longrightarrow} 0,\end{aligned}$$

which implies that $u \equiv 0$ in B_1, as desired.

In 1986 Garofalo and Lin proved the strong unique continuation principle for solutions to a perturbed problem in the presence of variable coefficients. In particular, they consider a class of potentials that are allowed to be very singular, namely $V(x) = c/|x|^m$ with $c \in \mathbb{R}$ and $0 \leq m \leq 2$. If $m > 2$ the strong unique continuation property fails. Furthermore, in 1990 Fabes, Garofal and Lin proved the weak unique continuation principle for V in some Kato class, see [34].

2.3 A fractional elliptic problem

In this part of the thesis we present the results contained in [25]. In particular, we investigate fractional elliptic equations of type (1.2.1) in a bounded domain $\Omega \subset \mathbb{R}^N$, where $N \geq 2$, $s \in (0,1)$, and the potential h satisfies (1.2.2), aiming to prove the strong unique continuation property and local asymptotics of solutions at any fixed point $x_0 \in \partial\Omega$ where the boundary of Ω is locally of class $C^{1,1}$ and some outer homogeneous Dirichlet boundary condition is assigned. These two assumptions are made explicit in (1.2.3) and (1.2.4) for some suitable $R > 0$ and function $g \in C^{1,1}(\mathbb{R}^{N-1})$. Without loss of generality, up to translation and rotation, we can assume that $x_0 = 0$ and

$$g(0) = 0 \quad \text{and} \quad \nabla g(0) = 0. \tag{2.3.1}$$

We recall that a weak solution to (1.2.1) is any function $u \in \mathcal{D}^{s,2}(\mathbb{R}^N)$ such that (1.2.6) holds true, where the space $\mathcal{D}^{s,2}(\mathbb{R}^N)$ is defined in Section 1.2.

In order to remedy the difficulty of defining a suitable Almgren's type frequency function in a non-local setting, we apply the Caffarelli-Silvestre extension technique to transform the non-local problem in the local problem performed in (1.2.9), see Section 1.2 for the contruction of the local problem. Thus, under assumptions (1.2.3) and (1.2.4), the extension $U = \mathcal{H}(u)$ solves

$$\begin{cases} -\mathrm{div}\left(t^{1-2s}\nabla U\right) = 0 & \text{in } B_R^+, \\ -\lim\limits_{t\to 0^+} t^{1-2s}\partial_t U = \kappa_s h u & \text{in } \Gamma^-_{g,R} := \{(x', x_N, 0) \in B'_R : x_N < g(x')\}, \\ U = 0 & \text{in } \Gamma^+_{g,R} := \{(x', x_N, 0) \in B'_R : x_N \geq g(x')\}. \end{cases} \tag{2.3.2}$$

In this new local context, we define for all $r > 0$ the weighted Sobolev space $H^1(B_r^+, t^{1-2s}\,dz)$ as the completion of $C^\infty(\overline{B_r^+})$ with respect to the norm

$$\|U\|_{H^1(B_r^+, t^{1-2s}\,dz)} = \sqrt{\int_{B_r^+} t^{1-2s}\left(|U|^2 + |\nabla U|^2\right) dz}.$$

It is well known, see e.g. [52, Proposition 2.1], that there exists a well-defined continuous trace operator

$$\mathrm{Tr} : H^1(B_r^+, t^{1-2s}\,dz) \to L^{2^*(s)}(B'_r);$$

in particular there exists a positive constant $C_{N,s}$ depending only on N and s such that, for all $r>0$ and $U \in H^1(B_r^+, t^{1-2s}\,dz)$,

$$\|\operatorname{Tr}(U)\|^2_{L^{2^*(s)}(B'_r)} \le C_{N,s}\int_{B_r^+} t^{1-2s}\left(r^{-2}|U(z)|^2 + |\nabla U(z)|^2\right)dz, \tag{2.3.3}$$

where $2^*(s)$ is given in (1.2.8).

The suitable weighted Sobolev space for energy solutions to (2.3.2) is $H^1_{\Gamma^+_{g,R}}(B_R^+, t^{1-2s}\,dz)$, defined as the closure of $C_c^\infty(\overline{B_R^+}\setminus\Gamma^+_{g,R})$ in $H^1(B_R^+, t^{1-2s}\,dz)$. By energy solution to (2.3.2) we mean a function $U \in H^1_{\Gamma^+_{g,R}}(B_R^+, t^{1-2s}\,dz)$ such that

$$\int_{B_R^+} t^{1-2s}\nabla U(x,t)\cdot\nabla\phi(x,t)\,dz - \kappa_s\int_{\Gamma^-_{g,R}} h\operatorname{Tr}U\operatorname{Tr}\phi\,dx = 0 \quad \text{for all } \phi \in C_c^\infty(B_R^+\cup\Gamma^-_{g,R}).$$

2.3.1 A diffeomorphism to straighten the boundary

In this section we exhibit a similar construction as in [3] in order to obtain the auxiliary problem (2.3.11) where the Dirichlet-Neumann junction coincides with the hyperplane $x_N = 0$. For this, we consider the set of variables $(y,t) \in \mathbb{R}^N \times [0,+\infty)$, with $y = (y', y_N) = (y_1, ..., y_{N-1}, y_N)$. Let $\rho \in C_c^\infty(\mathbb{R}^{N-1})$ be such that $\rho \ge 0$, $\operatorname{supp}(\rho) \subset B'_1$ and $\int_{\mathbb{R}^{N-1}} \rho(y')\,dy' = 1$. For every $\delta > 0$ we define

$$\rho_\delta(y') := \delta^{-N+1}\rho\left(\frac{y'}{\delta}\right).$$

Let us define also, for every $j = 1, ..., N-1$,

$$G_j(y', y_N) := \begin{cases} \left(\rho_{y_N} * \partial_{y_j} g\right)(y') & \text{if } y' \in \mathbb{R}^{N-1},\ y_N > 0,\\ \partial_{y_j} g(y') & \text{if } y' \in \mathbb{R}^{N-1},\ y_N = 0,\end{cases}$$

where $*$ denotes the convolution product.

It is easy to verify that, for all $j = 1, \dots, N-1$, $G_j \in C^\infty(\mathbb{R}^N_+)$, G_j is Lipschitz continous in $\overline{\mathbb{R}^N_+}$, and $\frac{\partial G_j}{\partial y_i} \in L^\infty(\mathbb{R}^N_+)$ for every $i \in \{1, \dots, N\}$. Moreover, for all $j = 1, \dots, N-1$ and $i = 1, \dots, N$,

$$y_N\frac{\partial G_j}{\partial y_i} \quad \text{is Lipschitz continuous in } \overline{\mathbb{R}^N_+}.$$

As a consequence, we have that, letting

$$\widetilde{G}_j : \mathbb{R}^N \to \mathbb{R}, \quad \widetilde{G}_j(y', y_N) := G_j(y', |y_N|)$$

and

$$\psi_j : \mathbb{R}^N \to \mathbb{R}, \quad \psi_j(y', y_N) = y_j - y_N\widetilde{G}_j(y', y_N),$$

$\widetilde{G}_j$ is Lipschitz continuous in $\mathbb{R}^N$ and $\psi_j \in C^{1,1}(\mathbb{R}^N)$ (i.e. ψ_j is continuously differentiable with Lipschitz gradient) for all $j = 1, \dots, N-1$. Let

$$\widetilde{G}(y', y_N) = (\widetilde{G}_1(y', y_N), \widetilde{G}_2(y', y_N), \dots, \widetilde{G}_{N-1}(y', y_N))$$

and denote as $J_{\widetilde{G}}(y', y_N)$ the Jacobian matrix of $\widetilde{G}$ at (y', y_N). Then $J_{\widetilde{G}} \in L^\infty(\mathbb{R}^N, \mathbb{R}^{N(N-1)})$ and

$$|\widetilde{G}(y', y_N) - \nabla g(y')| \le C\,|y_N| \quad \text{for all } (y', y_N) \in \mathbb{R}^N, \tag{2.3.4}$$

for some constant $C > 0$ independent of (y', y_N).

Let us consider the local diffeomorphism $F\colon \mathbb{R}^{N+1} \to \mathbb{R}^{N+1}$ defined as

$$F(y', y_N, t) = (\psi_1(y', y_N), ..., \psi_{N-1}(y', y_N), y_N + g(y'), t). \tag{2.3.5}$$

We observe that F is of class $C^{1,1}$ and $F(y', 0, 0) = (y', g(y'), 0)$, namely F^{-1} is straightening the boundary of the set $\{(x', x_N, 0) : x_N < g(x')\}$.

Direct computations and (2.3.4) yield that

$$\begin{aligned}
&\operatorname{Jac} F(y', y_N, t) = \operatorname{Jac} F(y', y_N) \qquad\qquad (2.3.6)\\
&= \begin{pmatrix}
1 - y_N \frac{\partial \widetilde{G}_1}{\partial y_1} & -y_N \frac{\partial \widetilde{G}_1}{\partial y_2} & \cdots & -y_N \frac{\partial \widetilde{G}_1}{\partial y_{N-1}} & -\widetilde{G}_1 - y_N \frac{\partial \widetilde{G}_1}{\partial y_N} & 0 \\
-y_N \frac{\partial \widetilde{G}_2}{\partial y_1} & 1 - y_N \frac{\partial \widetilde{G}_2}{\partial y_2} & \cdots & -y_N \frac{\partial \widetilde{G}_2}{\partial y_{N-1}} & -\widetilde{G}_2 - y_N \frac{\partial \widetilde{G}_2}{\partial y_N} & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
-y_N \frac{\partial \widetilde{G}_{N-1}}{\partial y_1} & -y_N \frac{\partial \widetilde{G}_{N-1}}{\partial y_2} & \cdots & 1 - y_N \frac{\partial \widetilde{G}_{N-1}}{\partial y_{N-1}} & -\widetilde{G}_{N-1} - y_N \frac{\partial \widetilde{G}_{N-1}}{\partial y_N} & 0 \\
\frac{\partial g}{\partial y_1}(y') & \frac{\partial g}{\partial y_2}(y') & \cdots & \frac{\partial g}{\partial y_{N-1}}(y') & 1 & 0 \\
0 & 0 & \dots & 0 & 0 & 1
\end{pmatrix} \\
&= \left(\begin{array}{c|c|c}
\mathrm{Id}_{N-1} - y_N J_{\widetilde{G}} & -\nabla g(y') + O(y_N) & \mathbf{0} \\
\hline
(\nabla g(y'))^T & 1 & 0 \\
\hline
\mathbf{0}^T & 0 & 1
\end{array}\right),
\end{aligned}$$

where $\nabla g(y')$ is meant as a column vector in $\mathbb{R}^{N-1}$, $\mathbf{0}$ is the null column vector in $\mathbb{R}^{N-1}$ and $(\nabla g(y'))^T, \mathbf{0}^T$ are their transpose; from now on, the notation $O(y_N)$ will be used to denote blocks of matrices with all entries being $O(y_N)$ as $y_N \to 0$ uniformly with respect to y' and t.

Setting $J(y', y_N) = \operatorname{Jac} F(y', y_N)$, from (2.3.1) and the fact that $g \in C^{1,1}(\mathbb{R}^{N-1})$ it follows that $\nabla g(y') = O(|y'|)$ as $|y'| \to 0$, then

$$\det J(y', y_N) = 1 + |\nabla g(y')|^2 + O(y_N) = 1 + O(|y'|^2) + O(y_N) \tag{2.3.7}$$

as $y_N \to 0$ and $|y'| \to 0$.

In particular we have that $\det J_F(0) = 1 \neq 0$; therefore, by the Inverse Function Theorem, F is invertible in a neighbourhood of the origin, i.e. there exists $R_1 > 0$ such that, from (2.3.7)

$$\alpha(y', y_N) := \det J(y', y_N) > 0 \quad \text{in } B'_{R_1} \tag{2.3.8}$$

and F is a diffeomorphism of class $C^{1,1}$ from B_{R_1} to $\mathcal{U} = F(B_{R_1})$ for some $\mathcal{U}$ open neighbourhood of 0 such that $\mathcal{U} \subset B_R$. Furthermore

$$F^{-1}(\mathcal{U} \cap \Gamma^-_{g,R}) = \Gamma^-_{R_1} \quad \text{and} \quad F^{-1}(\mathcal{U} \cap \Gamma^+_{g,R}) = \Gamma^+_{R_1},$$

where, for all $r > 0$, we denote

$$\Gamma^-_r := \{(y', y_N, 0) \in B'_r : y_N < 0\}, \quad \Gamma^+_r := \{(y', y_N, 0) \in B'_r : y_N \geq 0\}.$$

Since

$$F^{-1} \in C^{1,1}(\mathcal{U}, B_{R_1}), \quad F \in C^{1,1}(B_{R_1}, \mathcal{U}), \quad F(0) = F^{-1}(0) = 0, \quad J_F(0) = J_{F^{-1}}(0) = \mathrm{Id}_{N+1},$$

we have that

$$J_{F^{-1}}(x) = \mathrm{Id}_{N+1} + O(|x|) \quad \text{and} \quad F^{-1}(x) = x + O(|x|^2) \quad \text{as } |x| \to 0, \tag{2.3.9}$$

$$J_F(y) = \mathrm{Id}_{N+1} + O(|y|) \quad \text{and} \quad F(y) = y + O(|y|^2) \quad \text{as } |y| \to 0. \tag{2.3.10}$$

If U is a solution to (2.3.2), then $W = U \circ F$ solves

$$\begin{cases} -\mathrm{div}\left(t^{1-2s} A \nabla W\right) = 0 & \text{in } B^+_{R_1}, \\ \lim\limits_{t \to 0^+} \left(t^{1-2s} A \nabla W \cdot \nu\right) = \kappa_s \tilde{h} \ \mathrm{Tr}\, W & \text{in } \Gamma^-_{R_1}, \\ W = 0 & \text{in } \Gamma^+_{R_1}, \end{cases} \tag{2.3.11}$$

where $\nu = (0, 0, \ldots, 0, -1)$ is the vertical downward unit vector, A is the $(N+1) \times (N+1)$ variable coefficient matrix (not depending on t) given by

$$A(y) = (J(y))^{-1}((J(y))^{-1})^T |\det J(y)|, \tag{2.3.12}$$

and

$$\tilde{h}(y) = \alpha(y) h(F(y, 0)), \quad y \in \Gamma^-_{R_1}.$$

Equation (2.3.11) is meant in a weak sense, i.e. W belongs to $H^1_{\Gamma^+_{R_1}}(B^+_{R_1}, t^{1-2s}\, dz)$ (defined as the closure of $C^\infty_c(\overline{B^+_{R_1}} \setminus \Gamma^+_{R_1})$ in $H^1(B^+_{R_1}, t^{1-2s}\, dz)$) and satisfies

$$\int_{B^+_{R_1}} t^{1-2s} A(y) \nabla W(y, t) \cdot \nabla \phi(y, t)\, dz - \kappa_s \int_{\Gamma^-_{R_1}} \tilde{h}\, \mathrm{Tr}\, W\, \mathrm{Tr}\, \phi\, dy = 0 \tag{2.3.13}$$

for all $\phi \in C^\infty_c(B^+_{R_1} \cup \Gamma^-_{R_1})$.

We observe that A is symmetric and, in view of (2.3.9)–(2.3.10), uniformly elliptic if R_1 is chosen sufficiently small; furthermore A has $C^{0,1}$ coefficients. We also remark that, under assumption (1.2.2),

$$\tilde{h} \in W^{1,p}(\Gamma^-_{R_1}). \tag{2.3.14}$$

From (2.3.6) it follows that

$$J(y', y_N)^{-1} = \left(\begin{array}{c|c} (M(y', y_N))^{-1} & \mathbf{0} \\ \hline \mathbf{0}^T & 1 \end{array}\right),$$

where $\mathbf{0}$ is the null column vector in $\mathbb{R}^N$ and

$$M(y', y_N) = \left(\begin{array}{c|c} \mathrm{Id}_{N-1} - y_N J_{\widetilde{G}} & -\nabla g(y') + O(y_N) \\ \hline (\nabla g(y'))^T & 1 \end{array}\right). \tag{2.3.15}$$

From (2.3.6) and (2.3.8) one can deduce that

$$\det M(y', y_N) = \alpha(y', y_N) > 0 \quad \text{in } B'_{R_1}. \tag{2.3.16}$$

Let us define

$$B(y', y_N) := \det M(y', y_N)(M(y', y_N))^{-1}.$$

By (2.3.15) and a direct calculation we have that

$$B = \left(\begin{array}{cccc|c} 1 + \sum\limits_{j\neq 1} |\frac{\partial g}{\partial y_j}|^2 + O(y_N) & -\frac{\partial g}{\partial y_1}\frac{\partial g}{\partial y_2} + O(y_N) & \cdots & -\frac{\partial g}{\partial y_1}\frac{\partial g}{\partial y_{N-1}} + O(y_N) & \\ -\frac{\partial g}{\partial y_2}\frac{\partial g}{\partial y_1} + O(y_N) & 1 + \sum\limits_{j\neq 2} |\frac{\partial g}{\partial y_j}|^2 + O(y_N) & \cdots & -\frac{\partial g}{\partial y_2}\frac{\partial g}{\partial y_{N-1}} + O(y_N) & \nabla g + O(y_N) \\ \vdots & \vdots & \ddots & \vdots & \\ -\frac{\partial g}{\partial y_{N-1}}\frac{\partial g}{\partial y_1} + O(y_N) & -\frac{\partial g}{\partial y_{N-1}}\frac{\partial g}{\partial y_2} + O(y_N) & \cdots & 1 + \sum\limits_{j\neq N-1} |\frac{\partial g}{\partial y_j}|^2 + O(y_N) & \\ \hline \multicolumn{4}{c|}{-(\nabla g)^T + O(y_N)} & 1 + O(y_N) \end{array}\right). \tag{2.3.17}$$

Then $(J(y', y_N))^{-1}$ can be rewritten as follows

$$(J(y', y_N))^{-1} = \left(\begin{array}{c|c} \frac{1}{\alpha(y)} B(y) & \mathbf{0} \\ \hline \mathbf{0}^T & 1 \end{array}\right),$$

thus from (2.3.12) it turns out that

$$A(y) = \left(\begin{array}{c|c} D(y) & 0 \\ \hline 0 & \alpha(y) \end{array}\right), \tag{2.3.18}$$

where $D = \frac{1}{\alpha} BB^T$. From (2.3.17), (2.3.7), and (2.3.8) it follows that

$$D(y', y_N) = \left(\begin{array}{c|c} \mathrm{Id}_{N-1} + O(|y'|^2) + O(y_N) & O(y_N) \\ \hline O(y_N) & 1 + O(|y'|^2) + O(y_N) \end{array}\right), \tag{2.3.19}$$

where here $O(y_N)$, respectively $O(|y'|^2)$, denote blocks of matrices with all entries being $O(y_N)$ as $y_N \to 0$, respectively $O(|y'|^2)$ as $|y'| \to 0$. In particular we have that

$$A(y) = \mathrm{Id}_{N+1} + O(|y|) \quad \text{as } |y| \to 0. \tag{2.3.20}$$

We set

$$\mu(z) := \frac{A(y)z \cdot z}{|z|^2}, \tag{2.3.21}$$

observing that $\mu(z) > 0$ in B_{R_1}, possibly choosing R_1 smaller, thanks to (2.3.20). Thus we are allowed to define the vector

$$\beta(z) := \frac{A(y)z}{\mu(z)}, \tag{2.3.22}$$

having that

$$\beta(z) = (\beta'(z), \beta_{N+1}(z)) = \left(\frac{D(y)y}{\mu(z)}, \frac{\alpha(y)t}{\mu(z)}\right), \tag{2.3.23}$$

since the matrix A is of the form (2.3.18). Furthermore, up to choosing R_1 smaller, we have that

$$\|A(y)\|_{\mathcal{L}(\mathbb{R}^{N+1},\mathbb{R}^{N+1})} \le 2 \quad \text{for all } y \in B'_{R_1}. \tag{2.3.24}$$

Moreover, for every $\xi = (\xi_1, \dots, \xi_N, \xi_{N+1}) \in \mathbb{R}^{N+1}$ and $y \in B'_{R_1}$, we define a further vector in $\mathbb{R}^{N+1}$ denoted with the symbol $dA(y)\,\xi\,\xi$ such that for every $i = 1, ..., N+1$ the i-th component of this vector is given by

$$(dA(y)\,\xi\,\xi)_i = \sum_{j,k=1}^{N+1} \partial_{z_i} a_{jk}(y)\,\xi_j \xi_k. \tag{2.3.25}$$

Lemma 2.3.1. *Let μ be as in* (2.3.21) *and A as in* (2.3.12). *Then*

$$\mu(z) = 1 + O(|z|) \quad \text{as } |z| \to 0^+ \tag{2.3.26}$$

and

$$\nabla\mu(z) = O(1) \quad \text{as } |z| \to 0^+. \tag{2.3.27}$$

Proof. Estimate (2.3.26) follows directly from (2.3.21) and (2.3.20). In order to prove (2.3.27), we differentiate (2.3.21), obtaining that, for all $z = (y, t) \in B_{R_1}$,

$$\nabla\mu(z) = -2\frac{(A(y)z \cdot z)z}{|z|^4} + \frac{dA(y)zz}{|z|^2} + 2\frac{A(y)z}{|z|^2} = -2\frac{\mu(z)z}{|z|^2} + \frac{dA(y)zz}{|z|^2} + 2\frac{A(y)z}{|z|^2}.$$

Noting that $dA(y)zz = O(|z|^2)$ as $|z| \to 0^+$ since the matrix A has Lipschitz coefficients, and using (2.3.26) and (2.3.20), we deduce that

$$\nabla\mu(z) = -\frac{2z}{|z|^2}[1 + O(|z|)] + O(1) + \frac{2}{|z|^2}[z + O(|z|^2)] = O(1)$$

as $|z| \to 0^+$, thus proving (2.3.27). □

Lemma 2.3.2. *Let β be as in* (2.3.22) *and A as in* (2.3.12). *Then we have that, as $|z| \to 0^+$,*

$$\beta(z) = z + O(|z|^2) = O(|z|), \tag{2.3.28}$$

$$\operatorname{Jac}\beta(z) = A(y) + O(|z|) = \mathrm{Id}_{N+1} + O(|z|), \tag{2.3.29}$$

$$\operatorname{div}\beta(z) = N + 1 + O(|z|). \tag{2.3.30}$$

Proof. The result follows by combining (2.3.26), (2.3.27) and (2.3.20). □

2.3.2 Approximating domains

In this section we provide some important inequalities that will be pivotal throughout our discussion in Section 2.3, and then we construct some regular sets approximating the region on which an homogeneous Dirichlet boundary condition is prescribed. Then we build up a sequence of solutions to certain boundary value problems on such approximating domains converging in the $H^1(B^+_{R_0}, t^{1-2s}dz)$-norm to the solution of (2.3.11), for some suitable radius R_0.

We start by recalling from [36, Lemma 2.4] the following Hardy type inequality with boundary terms, which will be used throughout the paper.

Lemma 2.3.3. *For all $r > 0$ and $w \in H^1(B_r^+, t^{1-2s}\, dz)$*

$$\left(\frac{N-2s}{2}\right)^2 \int_{B_r^+} t^{1-2s} \frac{w^2(z)}{|z|^2}\, dz$$
$$\leq \int_{B_r^+} t^{1-2s} \left(\nabla w(z) \cdot \frac{z}{|z|}\right)^2 dz + \left(\frac{N-2s}{2r}\right) \int_{\partial^+ B_r^+} t^{1-2s} w^2 dS.$$

We refer to Section 2.1 for the definition of $\partial^+ B_r^+$. In order to prove the coercivity-type inequality (2.3.32), we provide the following Sobolev-type inequality with boundary terms (see Lemma 2.6 in [36]).

Lemma 2.3.4. *There exists a positive constant $\tilde{S}_{N,s} > 0$ depending on N and s such that, for all $r > 0$ and $V \in H^1(B_r^+, t^{1-2s}dz)$,*

$$\left(\int_{B_r'} |\operatorname{Tr} V|^{2^*(s)} dy\right)^{\frac{2}{2^*(s)}} \leq \tilde{S}_{N,s}\left[\frac{N-2s}{2r}\int_{\partial^+ B_r^+} t^{1-2s} V^2 dS + \int_{B_r^+} t^{1-2s}|\nabla V|^2 dz\right]. \tag{2.3.31}$$

Lemma 2.3.5. *For every $\bar{\alpha} > 0$, there exists $r(\bar{\alpha}) \in (0, R_1)$ such that, for any $0 < r \leq r(\bar{\alpha})$, $\zeta \in L^p(B'_{R_1})$ such that $\|\zeta\|_{L^p(B'_{R_1})} \leq \bar{\alpha}$ and $V \in H^1(B_r^+, t^{1-2s}dz)$,*

$$\int_{B_r^+} t^{1-2s} A\nabla V \cdot \nabla V\, dz - \kappa_s \int_{B_r'} \zeta |\operatorname{Tr} V|^2 dy + \frac{N-2s}{2r}\int_{\partial^+ B_r^+} t^{1-2s}\mu V^2\, dS \tag{2.3.32}$$
$$\geq \tilde{C}_{N,s}\left(\int_{B_r^+} t^{1-2s}|\nabla V|^2\, dz + \left(\int_{B_r'} |\operatorname{Tr} V|^{2^*(s)} dy\right)^{\frac{2}{2^*(s)}}\right),$$

for some positive constant $\tilde{C}_{N,s} > 0$ depending only on N and s.

Proof. Let us estimate from below each term on the left hand side of (2.3.32). To this aim, exploiting (2.3.20), we can choose $r_1 \in (0, R_1)$ such that, for all $0 < r \le r_1$ and $V \in H^1(B_r^+, t^{1-2s}\,dz)$,

$$\int_{B_r^+} t^{1-2s} A\nabla V \cdot \nabla V \; dz \ge \frac{1}{2} \int_{B_r^+} t^{1-2s} |\nabla V|^2 \; dz. \tag{2.3.33}$$

Furthermore, thanks to (2.3.26), we can assert that $\mu \ge 1/4$ in B_r if $0 < r \le r_2$, for some $r_2 \in (0, R_1)$. Hence, exploiting (2.3.31), we deduce that, for all $0 < r \le r_2$ and $V \in H^1(B_r^+, t^{1-2s}\,dz)$,

$$\frac{N-2s}{2r} \int_{\partial^+ B_r^+} t^{1-2s} \mu V^2 \, dS \ge \frac{1}{4\tilde{S}_{N,s}} \left(\int_{B_r'} |\operatorname{Tr} V|^{2^*(s)} dy \right)^{\frac{2}{2^*(s)}} - \frac{1}{4} \int_{B_r^+} t^{1-2s} |\nabla V|^2 \, dz. \tag{2.3.34}$$

Let $\bar{\alpha} > 0$ and let us observe that by (2.3.31) $\operatorname{Tr} V \in L^{2^*(s)}(B_r')$. Hence applying the Hölder's inequality, we infer that for all $r \in (0, R_1)$, $V \in H^1(B_r^+, t^{1-2s}dz)$, and $\zeta \in L^p(B'_{R_1})$ such that $\|\zeta\|_{L^p(B'_{R_1})} \le \bar{\alpha}$,

$$\begin{aligned} \int_{B_r'} \zeta \, |\operatorname{Tr} V|^2 \, dy &\le \tilde{c}_{N,s,p} \, r^{\bar{\varepsilon}} \|\zeta\,\|_{L^p(B'_{R_1})} \left(\int_{B_r'} |\operatorname{Tr} V|^{2^*(s)} dy \right)^{\frac{2}{2^*(s)}} \\ &\le \tilde{c}_{N,s,p} \bar{\alpha} \, r^{\bar{\varepsilon}} \left(\int_{B_r'} |\operatorname{Tr} V|^{2^*(s)} dy \right)^{\frac{2}{2^*(s)}} \end{aligned} \tag{2.3.35}$$

for some positive constant $\tilde{c}_{N,s,p} > 0$ (depending only on p, N, s), where

$$\bar{\varepsilon} = \frac{2sp - N}{p} > 0. \tag{2.3.36}$$

Selecting $r_3 = r_3(\bar{\alpha}) \in (0, R_1)$ such that

$$\kappa_s \tilde{c}_{N,s,p} \bar{\alpha} r^{\bar{\varepsilon}} \le \frac{1}{8\tilde{S}_{N,s}} \quad \text{for all } 0 < r \le r_3 \tag{2.3.37}$$

and combining (2.3.33), (2.3.34), and (2.3.35), we obtain that, for all $0 < r \le r(\bar{\alpha}) := \min\{r_1, r_2, r_3\}$ and $V \in H^1(B_r^+, t^{1-2s}\,dz)$,

$$\begin{aligned} &\int_{B_r^+} t^{1-2s} A\nabla V \cdot \nabla V \, dz - \kappa_s \int_{B_r'} \zeta |\operatorname{Tr} V|^2 dy + \frac{N-2s}{2r} \int_{\partial^+ B_r^+} t^{1-2s} \mu V^2 \, dS \\ &\qquad \ge \frac{1}{4} \int_{B_r^+} t^{1-2s} |\nabla V|^2 dz + \left(\frac{1}{4\tilde{S}_{N,s}} - \kappa_s \tilde{c}_{N,s,p} \bar{\alpha} \, r^{\bar{\varepsilon}} \right) \left(\int_{B_r'} |\operatorname{Tr} V|^{2^*(s)} dy \right)^{\frac{2}{2^*(s)}} \\ &\qquad \ge \tilde{C}_{N,s} \left(\int_{B_r^+} t^{1-2s} |\nabla V|^2 dz + \left(\int_{B_r'} |\operatorname{Tr} V|^{2^*(s)} dy \right)^{\frac{2}{2^*(s)}} \right), \end{aligned}$$

where $\tilde{C}_{N,s} := \min\left\{\frac{1}{4}, \frac{1}{8\tilde{S}_{N,s}}\right\}$, thus proving (2.3.32). $\square$

For purposes that will be clear in the sequel we provide the following remark.

Remark 2.3.6. *For $\bar\alpha > 0$, let $r(\bar\alpha)$ and $\tilde{c}_{N,s,p}$ be as in Lemma 2.3.5 and let $\zeta \in L^p(B'_{R_1})$ be such that $\|\zeta\|_{L^p(B'_{R_1})} \le \bar\alpha$. Then, for every $r \in (0, r(\bar\alpha)]$ and $V \in H^1(B_r^+, t^{1-2s}dz)$, we have that*

$$\int_{B'_r} \zeta |\operatorname{Tr} V|^2 dy \le \tilde{S}_{N,s}\tilde{c}_{N,s,p} r^{\bar\varepsilon}\bar\alpha \frac{2(N-2s)}{r}\int_{\partial^+ B_r^+} t^{1-2s}\mu V^2 dS + \frac{1}{8\kappa_s}\int_{B_r^+} t^{1-2s}|\nabla V|^2\, dz. \tag{2.3.38}$$

Proof. Applying (2.3.35) and (2.3.31), we obtain (2.3.38), taking into account that, for all $0 < r \le r(\bar\alpha)$, (2.3.37) holds and $\mu \ge 1/4$. $\square$

The main difficulty in the proof of a Pohozaev type identity for problem (2.3.11), which is needed to differentiate the Almgren quotient, relies in a substancial lack of regularity at Dirichlet-Neumann junctions. We face this difficulty by a double approximation procedure, involving both the potential h and the N-dimensional region $\Gamma^+_{R_1}$ where the solution to (2.3.11) is forced to vanish. In order to construct our approximation procedure, let $\eta \in C^\infty([0,+\infty))$ be such that

$$\eta \equiv 1 \text{ in } [0,1/2], \quad \eta \equiv 0 \text{ in } [1,+\infty), \quad 0 \le \eta \le 1 \quad \text{and} \quad \eta' \le 0. \tag{2.3.39}$$

Let

$$f : [0,+\infty) \to \mathbb{R}, \quad f(t) = \eta(t) + (1-\eta(t))t^{1/4}.$$

We observe that

$$f \in C^\infty([0,+\infty)), \quad f(t) = 1 \text{ for all } t \in [0,1/2], \quad \text{and} \quad f(t) - 4t\, f'(t) \ge 0 \text{ for all } t \ge 0. \tag{2.3.40}$$

Furthermore

$$f(t) \ge \frac{1}{2} \quad \text{and} \quad f(t) \ge t^{1/4} \quad \text{for all } t \ge 0. \tag{2.3.41}$$

For every $n \in \mathbb{N}\setminus\{0\}$, we introduce the sequence of functions

$$f_n(t) = \frac{f(nt)}{n^{1/8}}.$$

Then, (2.3.40) implies that

$$f_n \in C^\infty([0,+\infty)), \quad f_n(t) = n^{-1/8} \text{ for all } t \in [0,1/2n], \quad f_n(t) - 4t\, f_n'(t) \ge 0 \text{ for all } t \ge 0, \tag{2.3.42}$$

whereas (2.3.41) yields

$$f_n(t) \ge \frac{1}{2}n^{-1/8} \quad \text{and} \quad f_n(t) \ge n^{1/8}t^{1/4} \quad \text{for all } t \ge 0. \tag{2.3.43}$$

By (2.3.14) and density of smooth functions in Sobolev spaces, there exists a sequence of potential terms $h_n \in C^\infty\big(\overline{\Gamma^-_{R_1}}\big)$ such that

$$h_n \to \tilde{h} \quad \text{in } W^{1,p}(\Gamma^-_{R_1}). \tag{2.3.44}$$

Let

$$\bar{\alpha}_0 = \sup_n \|h_n\|_{L^p(\Gamma^-_{R_1})} \tag{2.3.45}$$

and set

$$R_0 = r(\bar{\alpha}_0) \tag{2.3.46}$$

according to the notation introduced in Lemma 2.3.5.

Remark 2.3.7. *Because of the above choice of R_0, we have that* (2.3.32) *holds with any $\zeta \in L^p(B'_{R_1})$ such that $|\zeta| \le |h_n|$ a.e. (being h_n trivially extended in $B'_{R_1} \setminus \Gamma^-_{R_1}$), for any $n \in \mathbb{N} \setminus \{0\}$, $r \le R_0$, and for all $V \in H^1(B_r^+, t^{1-2s}\,dz)$.*

Let us define, for all $n \in \mathbb{N} \setminus \{0\}$,

$$\gamma_n = \{(y', y_N, t) \in \overline{B^+_{R_0}} :\ y_N = f_n(t)\}, \tag{2.3.47}$$

with R_0 as in (2.3.46). If $(y', y_N, t) \in \gamma_n$, then from (2.3.43) it follows that

$$n^{1/8} t^{1/4} \le f_n(t) = y_N \le R_0,$$

thus obtaining that

$$t \le \frac{R_0^4}{\sqrt{n}} \quad \text{if } (y', y_N, t) \in \gamma_n. \tag{2.3.48}$$

The approximating domains are defined as

$$\mathcal{U}_n := \{(y', y_N, t) \in B^+_{R_0} : y_N < f_n(t)\} \tag{2.3.49}$$

with topological boundary

$$\partial\mathcal{U}_n = \sigma_n \cup \gamma_n \cup \tau_n,$$

where γ_n has been defined in (2.3.47) and

$$\sigma_n = \left\{(y', y_N) \in B'_{R_0} : y_N < \frac{1}{n^{1/8}}\right\}, \quad \tau_n = \left\{(y', y_N, t) \in \partial B_{R_0} : t \ge 0,\ y_N < f_n(t)\right\},$$

see Figure 2.1.

Functions f_n have been constructed with the precise aim to have that $\mathcal{U}_n$ satisfy the following geometric property, which will be used to estimate some boundary terms in the Pohozaev-type identity.

Lemma 2.3.8. *There exists $\bar{n} \in \mathbb{N} \setminus \{0\}$ such that, for all $n \ge \bar{n}$ and $z = (y, t) \in \gamma_n \cap B^+_{R_0}$,*

$$A(y)z \cdot \nu \ge 0 \quad \text{on } \gamma_n, \tag{2.3.50}$$

where γ_n has been defined in (2.3.47) *and $\nu = \nu(z)$ is the outward unit normal vector at $z \in \partial\mathcal{U}_n$.*

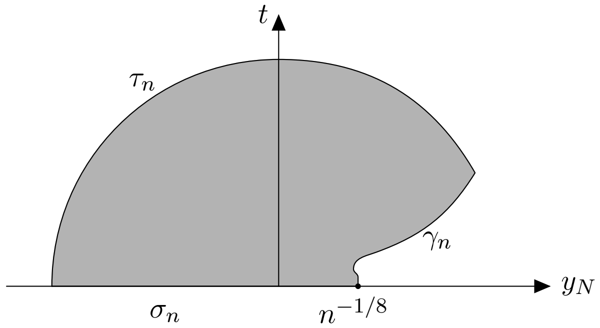


Figure 2.1: Section of the approximating domain $\mathcal{U}_n$.

Proof. For all $z=(y,t)\in\gamma_n\cap B^+_{R_0}$ we have that $\nu=\nu(z)=\frac{\mathfrak{n}}{|\mathfrak{n}|}$, where $\mathfrak{n}=(\mathbf{0},1,-f_n'(t))$. Hence, from (2.3.18) and (2.3.19) it follows that

$$A(y)(y,t)\cdot\mathfrak{n}=(D(y)y,\alpha(y)t)\cdot((\mathbf{0},1),-f_n'(t))=y_N(1+O(|y'|)+O(y_N))-\alpha(y)tf_n'(t).$$

Therefore, possibly choosing R_1 (and consequently R_0) smaller from the beginning and recalling (2.3.7)–(2.3.8), we obtain that

$$A(y)(y,t)\cdot\mathfrak{n}\geq\begin{cases}\frac{y_N}{2}-2tf_n'(t)=\frac{1}{2}(f_n(t)-4tf_n'(t)) & \text{if } f_n'(t)\geq 0\\ \frac{y_N}{2} & \text{if } f_n'(t)\leq 0\end{cases}$$

thus concluding that $A(y)(y,t)\cdot\mathfrak{n}\geq 0$ in view of (2.3.42). □

Now we construct a sequence U_n of solutions to some suitable approximating problems on $\mathcal{U}_n$ that converges strongly to W in the weighted Sobolev space $H^1(B^+_{R_0},t^{1-2s}\,dz)$. Functions U_n will be sufficiently regular to satisfy a Rellich-Nečas identity and make it integrable on $\mathcal{U}_n$, thus allowing us to obtain a Pohozaev type identity for U_n with some remainder terms produced by the transition of the boundary conditions, whose sign can anyway be understood thanks to the geometric property (2.3.50); therefore, passing to the limit in the Pohozaev identity satisfied by U_n, we end up with inequality (2.3.67) for W, which will be used to estimate from below the derivative of the Almgren frequency function (2.3.89) and then to prove that such frequency has a finite limit at 0 (Proposition 2.3.19).

Let $W\in H^1(B^+_{R_1},t^{1-2s}\,dz)$ be a non-trivial energy solution to (2.3.11), in the sense clarified in (2.3.13). By density, there exists a sequence of functions $G_n\in C^\infty_c(\overline{B^+_{R_1}}\setminus\Gamma^+_{R_1})$ such that $G_n\to W$ strongly in $H^1(B^+_{R_1},t^{1-2s}\,dz)$. Thanks to (2.3.48), without loss of generality we can assume that $G_n=0$ on γ_n.

We construct a sequence of cut-off functions letting $\eta\in C^\infty([0,+\infty))$ as in (2.3.39) and defining

$$\eta_n:\mathbb{R}^N\to\mathbb{R},\quad \eta_n(y',y_N)=\begin{cases}1-\eta\left(-\frac{ny_N}{2}\right) & \text{if } y_N\leq 0,\\ 0 & \text{if } y_N>0.\end{cases}\tag{2.3.51}$$

For any fixed $n \in \mathbb{N}$, we consider the following boundary value problem

$$\begin{cases} -\mathrm{div}\left(t^{1-2s} A\nabla U_n\right) = 0 & \text{in } \mathcal{U}_n, \\ \lim_{t\to 0^+} \left(t^{1-2s} A\nabla U_n \cdot \nu\right) = \kappa_s \eta_n h_n \operatorname{Tr} U_n & \text{in } \sigma_n, \\ U_n = G_n & \text{in } \tau_n \cup \gamma_n, \end{cases} \tag{2.3.52}$$

in a weak sense, i.e.

$$\begin{cases} U_n - G_n \in \mathcal{H}_n, \\ \displaystyle\int_{\mathcal{U}_n} t^{1-2s} A\nabla U_n \cdot \nabla\Phi \, dz - \kappa_s \int_{\sigma_n} \eta_n h_n \operatorname{Tr} U_n \operatorname{Tr}\Phi \, dy = 0 \quad \text{for all } \Phi \in \mathcal{H}_n, \end{cases} \tag{2.3.53}$$

where $\mathcal{H}_n$ is defined as the closure of $C_c^\infty(\mathcal{U}_n \cup \sigma_n)$ in $H^1(\mathcal{U}_n, t^{1-2s}\, dz)$.

We aim to prove that for any fixed $n \in \mathbb{N}$ problem (2.3.53) admits a unique weak solution. To this purpose, we premise the Urysohn's subsequence principle.

Lemma 2.3.9. *Let X be a topological space and let us consider a sequence $\{x_n\}_{n\in\mathbb{N}}$ of elements of X. We suppose that*

(i) *every subsequence of $\{x_n\}_{n\in\mathbb{N}}$ admits a convergent subsequence in X,*

(ii) *all convergent subsequences of $\{x_n\}_{n\in\mathbb{N}}$ have the same limit $\bar{x} \in X$.*

Then it holds that $\{x_n\}_{n\in\mathbb{N}}$ converges to $\bar{x}$ in X.

Proof. We assume by contradiction that $\{x_n\}_{n\in\mathbb{N}}$ does not converge to $\bar{x}$. Hence there exists a neighbourhood W of $\bar{x}$ such that for all $n \in \mathbb{N}$ there exists $\bar{n} > n$ such that $x_{\bar{n}} \notin W$. In this way we are able to costruct a subsequence $\{x_{n_k}\}_{k\in\mathbb{N}}$ such that $x_{n_k} \notin W$. In virtue of (i), there exists a subsequence $\{x_{n_{k_h}}\}_{h\in\mathbb{N}}$ and $y \in X$ such that $x_{n_{k_h}} \to y$ in X. Therefore, by (ii), it follows that $y = \bar{x}$ and this implies that for every neighbourhood V of $\bar{x}$ there exists $\nu \in \mathbb{N}$ such that $x_{n_{k_h}} \in V$ for all $h \geq \nu$. Thus if we choose $V = W$ we obtain that $x_{n_{k_h}} \in W$ definitely and this is a contradiction since $x_{n_{k_h}}$ is a subsequence of x_{n_k}, thus completing the proof. ☐

In the following proposition we establish the existence of a unique solution U_n of (2.3.53) for every $n \in \mathbb{N}$ and we also show the convergence of such sequence to W.

Proposition 2.3.10. *For any fixed $n \in \mathbb{N}$, there exists a unique solution U_n to (2.3.53). Moreover $U_n \to W$ strongly in $H^1(B^+_{R_0}, t^{1-2s}dz)$ (where U_n is extended trivially to zero in $B^+_{R_0} \setminus \mathcal{U}_n$) and R_0 is taken as in (2.3.46).*

Proof. For any fixed $n \geq 1$, U_n solves (2.3.53) if and only if $V_n = U_n - G_n$ satisfies

$$V_n \in \mathcal{H}_n \quad \text{and} \quad b_n(V_n, \Phi) = \langle F_n, \Phi\rangle \quad \text{for all } \Phi \in \mathcal{H}_n, \tag{2.3.54}$$

where

$$b_n : \mathcal{H}_n \times \mathcal{H}_n \to \mathbb{R}, \quad b_n(V, \Phi) = \int_{\mathcal{U}_n} t^{1-2s} A\nabla V \cdot \nabla\Phi \, dz - \kappa_s \int_{\sigma_n} \eta_n h_n \operatorname{Tr} V \operatorname{Tr}\Phi \, dy \tag{2.3.55}$$

and

$$F_n : \mathcal{H}_n \to \mathbb{R}, \quad \langle F_n, \Phi\rangle = -\int_{\mathcal{U}_n} t^{1-2s} A\nabla G_n \cdot \nabla\Phi \, dz + \kappa_s \int_{\sigma_n} \eta_n h_n \operatorname{Tr} G_n \operatorname{Tr}\Phi \, dy. \quad (2.3.56)$$

From Hölder's inequality, (2.3.31), and the boundedness of $\{h_n\}_{n\geq 1}$ and $\{G_n\}_{n\geq 1}$ respectively in $L^p(\Gamma^-_{R_1})$ and in $H^1(B^+_{R_1}, t^{1-2s}\,dz)$, it follows that

$$|\langle F_n, \Phi\rangle| \leq c\,\|\Phi\|_{\mathcal{H}_n} \quad \text{for all } \Phi \in \mathcal{H}_n \quad (2.3.57)$$

for some constant $c > 0$ which does not depend on n. In particular $F_n \in \mathcal{H}_n^\star$, being $\mathcal{H}_n^\star$ the dual space of $\mathcal{H}_n$, and $\|F_n\|_{\mathcal{H}_n^\star} \leq c$ uniformly in n.

The idea is to apply the Lax-Milgram Theorem. In order to do this, we remark that, using the Hardy inequality in Lemma 2.3.3, after extending functions V_n trivially to zero in $B^+_{R_0} \setminus \mathcal{U}_n$, the weighted L^2-norm of the gradient

$$\left(\int_{\mathcal{U}_n} t^{1-2s}|\nabla V_n|^2 dz\right)^{1/2}$$

turns out to be an equivalent norm in the space $\mathcal{H}_n$ that will be still denoted as $\|\cdot\|_{\mathcal{H}_n}$. It follows that b_n is coercive: indeed, for every $V \in \mathcal{H}_n$, we have that

$$\begin{aligned} b_n(V,V) &= \int_{\mathcal{U}_n} t^{1-2s} A\nabla V \cdot \nabla V\, dz - \kappa_s \int_{\sigma_n} \eta_n h_n |\operatorname{Tr} V|^2 dy \qquad (2.3.58)\\ &= \int_{B^+_{R_0}} t^{1-2s} A\nabla V \cdot \nabla V\; dz - \kappa_s \int_{B'_{R_0}} \eta_n h_n |\operatorname{Tr} V|^2 dy \\ &\geq \tilde{C}_{N,s} \int_{B^+_{R_0}} t^{1-2s}|\nabla V|^2 dz = \tilde{C}_{N,s} \int_{\mathcal{U}_n} t^{1-2s}|\nabla V|^2 dz = \tilde{C}_{N,s}\|V\|^2_{\mathcal{H}_n}, \end{aligned}$$

as a consequence of Lemma 2.3.5, with $\zeta = \eta_n h_n$, see Remark 2.3.7. Furthermore, from (2.3.24) and (2.3.38) it follows that

$$|b_n(V,W)| \leq \left(2 + \frac{1}{8}\right)\|V\|_{\mathcal{H}_n}\|W\|_{\mathcal{H}_n} \leq 3\|V\|_{\mathcal{H}_n}\|W\|_{\mathcal{H}_n} \quad (2.3.59)$$

for all $V, W \in \mathcal{H}_n$. In particular b_n is continuous.

Hence, from (2.3.58), (2.3.59) and the Lax-Milgram Theorem we can conclude that there exists a unique $V_n \in \mathcal{H}_n$ solving (2.3.54), which implies also the existence and uniqueness of a solution U_n to (2.3.53). Moreover, combining (2.3.58) and (2.3.57) we also obtain that, extending V_n trivially to zero in $B^+_{R_0} \setminus \mathcal{U}_n$,

$$\|V_n\|_{H^1(B^+_{R_0}, t^{1-2s}\,dz)} \leq \frac{c}{\tilde{C}_{N,s}} \quad \text{for all } n,$$

thus V_n is bounded in $H^1(B^+_{R_0}, t^{1-2s}\,dz)$. From this, it follows that there exist $V \in H^1(B^+_{R_0}, t^{1-2s}\,dz)$ and a subsequence $\{V_{n_k}\}$ of $\{V_n\}$ such that

$$V_{n_k} \rightharpoonup V \quad \text{weakly in } H^1(B^+_{R_0}, t^{1-2s}\,dz). \quad (2.3.60)$$

From the fact that $V_n \in \mathcal{H}_n$, we easily deduce that V has null trace on $\partial^+ B^+_{R_0}$ and on $\Gamma^+_{R_0}$. Hence it can be taken as a test function in (2.3.13) yielding

$$\int_{B^+_{R_0}} t^{1-2s} A\nabla W \cdot \nabla V \, dz - \kappa_s \int_{\Gamma^-_{R_0}} \tilde{h} \operatorname{Tr} W \operatorname{Tr} V \, dy = 0. \tag{2.3.61}$$

Since $G_n \to W$ strongly in $H^1(B^+_{R_1}, t^{1-2s}\, dz)$, from (2.3.3) we deduce that $\operatorname{Tr} G_n \to \operatorname{Tr} W$ in $L^{2^*(s)}(B'_{R_1})$. By (2.3.3) and (2.3.60) we have that $\operatorname{Tr} V_{n_k} \rightharpoonup \operatorname{Tr} V$ weakly in $L^{2^*(s)}(B'_{R_1})$. Furthermore $\eta_n h_n \to \tilde{h}$ in $L^{\frac{N}{2s}}(\Gamma^-_{R_1})$. Hence from (2.3.61) it follows that

$$\begin{aligned} 0 &= \int_{B^+_{R_0}} t^{1-2s} A\nabla W \cdot \nabla V \, dz - \kappa_s \int_{\Gamma^-_{R_0}} \tilde{h} \operatorname{Tr} W \operatorname{Tr} V \, dy \\ &= \lim_{k\to+\infty} \int_{B^+_{R_0}} t^{1-2s} A\nabla G_{n_k} \cdot \nabla V_{n_k} \, dz - \kappa_s \int_{\Gamma^-_{R_0}} \eta_{n_k} h_{n_k} \operatorname{Tr} G_{n_k} \operatorname{Tr} V_{n_k} \, dy \\ &= - \lim_{k\to+\infty} \langle F_{n_k}, V_{n_k} \rangle = - \lim_{k\to\infty} b_{n_k}(V_{n_k}, V_{n_k}) \end{aligned}$$

thus obtaining that $\|V_{n_k}\|_{H^1(B^+_{R_0}, t^{1-2s}\, dz)} \to 0$ as $k \to +\infty$ in view of (2.3.58). Hence $V_{n_k} \to 0$ strongly in $H^1(B^+_{R_0}, t^{1-2s}\, dz)$. By Lemma 2.3.9, we can deduce that actually $V_n \to 0$ strongly in $H^1(B^+_{R_0}, t^{1-2s}\, dz)$. Indeed assumption (i) of Lemma 2.3.9 is a trivial consequence of the boundedness of V_n in $H^1(B^+_{R_0}, t^{1-2s}\, dz)$, and, as far as assumption (ii) is concerned, if V_{n_h} is any other convergent subsequence of V_n, namely such that $V_{n_h} \rightharpoonup \overline{V}$ for some $\overline{V} \in H^1(B^+_{R_0}, t^{1-2s}\, dz)$, repeating the same argument as above, we are able to prove that $V_{n_h} \to 0$ in $H^1(B^+_{R_0}, t^{1-2s}\, dz)$ as well. Then also assumption (ii) is proved. Therefore, by Lemma 2.3.9, we can conclude that $V_n \to 0$ strongly in $H^1(B^+_{R_0}, t^{1-2s}\, dz)$ and, consequently, it holds that $U_n = V_n + G_n \to W$ in $H^1(B^+_{R_0}, t^{1-2s}\, dz)$ as $n \to +\infty$. □

2.3.3 Pohozaev-type inequality for the extended problem

The aim of this section is to prove a Pohozaev-type inequality for the energy solution $W \in H^1(B^+_{R_1}, t^{1-2s}\, dz)$ to (2.3.11); in this situation we have to settle for an inequality instead of a classical Pohozev-type identity because of the mixed boundary conditions, which produce some extra singular terms with a recognizable sign when integrating the Rellich-Nečas identity.

The idea is to obtain the inequality as limit of ones for the approximating sequence U_n. For every $r \in (0, R_0)$, $n \in \mathbb{N}$ such that $n > r^{-8}$, and $\delta \in \left(0, \frac{1}{4n}\right)$, we consider the following domain

$$O_{r,n,\delta} := \mathcal{U}_n \cap \{(y,t) \in B_r : t > \delta\}.$$

We note that, if $\delta \in \left(0, \frac{1}{4n}\right)$, then $f_n(t) = n^{-1/8}$ for $0 \le t \le 2\delta$, see (2.3.42). We can split its topological boundary as follows

$$\partial O_{r,n,\delta} = \sigma_{r,n,\delta} \cup \gamma_{r,n,\delta} \cup \tau_{r,n,\delta}, \tag{2.3.62}$$

with

$$\sigma_{r,n,\delta} := \left\{(y', y_N, t) \in B_r : y_N < \frac{1}{n^{1/8}},\ t = \delta\right\}, \tag{2.3.63}$$

$$\gamma_{r,n,\delta} := \left\{(y', y_N, t) \in \overline{B_r^+} : y_N = f_n(t),\ t \geq \delta\right\}, \tag{2.3.64}$$

$$\tau_{r,n,\delta} := \left\{(y', y_N, t) \in \partial^+ B_r^+ : y_N < f_n(t),\ t \geq \delta\right\}, \tag{2.3.65}$$

see Figure 2.2.

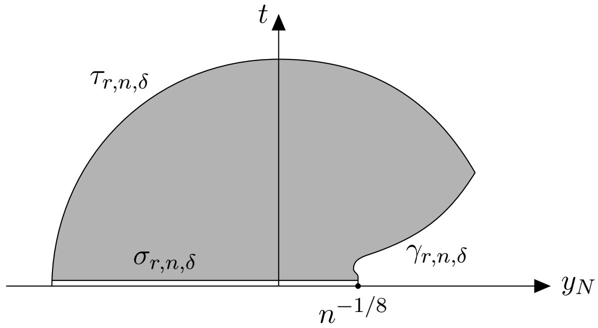


Figure 2.2: Section of $O_{r,n,\delta}$.

We define also the set

$$S_r^- := \{(y', y_N, t) \in \partial B_r : t = 0 \text{ and } y_N < 0\}. \tag{2.3.66}$$

Proposition 2.3.11 (Pohozaev-type inequality)**.** *Let $W \in H^1(B_{R_1}^+, t^{1-2s}\,dz)$ weakly solve (2.3.11). Then, for almost every $r \in (0, R_0)$,*

$$\begin{aligned} &\frac{r}{2}\int_{\partial^+ B_r^+} t^{1-2s} A\nabla W \cdot \nabla W \, dS - r\int_{\partial^+ B_r^+} t^{1-2s}\frac{|A\nabla W\cdot\nu|^2}{\mu}\, dS \\ &\quad + \frac{\kappa_s}{2}\int_{\Gamma_r^-}\left(\nabla\tilde{h}\cdot\beta' + \tilde{h}\,\mathrm{div}\beta'\right)|\mathrm{Tr}W|^2 dy - \frac{\kappa_s r}{2}\int_{S_r^-}\tilde{h}|\mathrm{Tr}W|^2 dS' \\ &\geq \frac{1}{2}\int_{B_r^+} t^{1-2s} A\nabla W\cdot\nabla W\ \mathrm{div}\beta\, dz - \int_{B_r^+} t^{1-2s}\mathrm{Jac}\beta(A\nabla W)\cdot\nabla W\, dz \\ &\quad + \frac{1}{2}\int_{B_r^+} t^{1-2s}(dA\nabla W\nabla W)\cdot\beta\, dz + \frac{1-2s}{2}\int_{B_r^+} t^{1-2s}\frac{\alpha}{\mu}A\nabla W\cdot\nabla W\, dz \end{aligned} \tag{2.3.67}$$

and

$$\int_{B_r^+} t^{1-2s} A\nabla W\cdot\nabla W\, dz = \int_{\partial^+ B_r^+} t^{1-2s}(A\nabla W\cdot\nu)\, W\, dS + \kappa_s\int_{\Gamma_r^-}\tilde{h}|\mathrm{Tr}W|^2 dy. \tag{2.3.68}$$

Remark 2.3.12. *The term $\int_{S_r^-}\tilde{h}|\mathrm{Tr}W|^2 dS'$ is understood for a.e. $r \in (0, R_0)$ as the L^1-function given by the weak derivative of the $W^{1,1}(0, R_0)$-function $r \mapsto \int_{\Gamma_r^-}\tilde{h}|\mathrm{Tr}W|^2 dy$.*

Likewise, the two terms

$$\int_{\partial^+ B_r^+} t^{1-2s} A\nabla W \cdot \nabla W \, dS \quad \text{and} \quad \int_{\partial^+ B_r^+} t^{1-2s} \frac{|A\nabla W \cdot \nu|^2}{\mu} \, dS$$

are understood for a.e. $r \in (0, R_0)$ as the L^1-functions given by the weak derivative of the $W^{1,1}(0, R_0)$-functions $r \mapsto \int_{B_r^+} A\nabla W \cdot \nabla W \, dz$ and $r \mapsto \int_{B_r^+} t^{1-2s} \frac{|A\nabla W \cdot \nu|^2}{\mu} \, dz$ respectively.

Proof. Since the matrix A has Lipschitz coefficients and being the equation satisfied in a smooth domain containing $O_{r,n,\delta}$, by classical elliptic regularity theory (see e.g. [51, Theorem 2.2.2.3]) we have that

$$U_n \in H^2(O_{r,n,\delta}). \tag{2.3.69}$$

Hence from (2.3.69) the following Rellich-Nečas identity holds in a distributional sense in $O_{r,n,\delta}$:

$$\begin{aligned} &\operatorname{div}\Big((\tilde{A}\nabla U_n \cdot \nabla U_n)\beta - 2(\beta \cdot \nabla U_n)\tilde{A}\nabla U_n\Big) \\ &= (\tilde{A}\nabla U_n \cdot \nabla U_n)\operatorname{div}\beta - 2(\beta \cdot \nabla U_n)\operatorname{div}(\tilde{A}\nabla U_n) + (d\tilde{A}\nabla U_n \nabla U_n) \cdot \beta \\ &\qquad - 2\operatorname{Jac}\beta(\tilde{A}\nabla U_n) \cdot \nabla U_n, \end{aligned} \tag{2.3.70}$$

where $\tilde{A}(z) = t^{1-2s}A(y)$ and β has been defined in (2.3.22). Moreover we have that

$$(\tilde{A}\nabla U_n \cdot \nabla U_n)\beta - 2(\beta \cdot \nabla U_n)\tilde{A}\nabla U_n \in W^{1,1}(O_{r,n,\delta}),$$

as a consequence of (2.3.69) and that $\tilde{A}$ and β have Lipschitz components.

Thus we can use the integration by parts formula for Sobolev functions on the Lipschitz domain $O_{r,n,\delta}$, obtaining that

$$\begin{aligned} &\int_{\partial O_{r,n,\delta}} \Big((\tilde{A}\nabla U_n \cdot \nabla U_n)\beta - 2(\beta \cdot \nabla U_n)\tilde{A}\nabla U_n\Big) \cdot \nu \, dS \\ &= \int_{O_{r,n,\delta}} t^{1-2s} A\nabla U_n \cdot \nabla U_n \operatorname{div}\beta \, dz - 2\int_{O_{r,n,\delta}} t^{1-2s} \operatorname{Jac}\beta(A\nabla U_n) \cdot \nabla U_n \, dz \\ &\quad + \int_{O_{r,n,\delta}} t^{1-2s} (dA\nabla U_n \nabla U_n) \cdot \beta \, dz + (1-2s)\int_{O_{r,n,\delta}} t^{1-2s} \frac{\alpha}{\mu} A\nabla U_n \cdot \nabla U_n \, dz, \end{aligned} \tag{2.3.71}$$

by (2.3.25) and (2.3.52). Taking into account (2.3.62), (2.3.63), (2.3.64), (2.3.65), we estimate each term on the left hand side of (2.3.71). For this, by (2.3.22), (2.3.21), using that A is symmetric and observing that on $\tau_{r,n,\delta}$ the outward unit normal vector ν can be written as z/r, we have that

$$\begin{aligned} \int_{\tau_{r,n,\delta}} (t^{1-2s} A\nabla U_n \cdot \nabla U_n)\beta \cdot \nu \, dz &= \int_{\tau_{r,n,\delta}} (t^{1-2s} A\nabla U_n \cdot \nabla U_n) \frac{A(y)z}{\mu} \cdot \frac{z}{r} \, dz \\ &= \int_{\tau_{r,n,\delta}} (t^{1-2s} A\nabla U_n \cdot \nabla U_n)\mu r^2 \frac{1}{\mu r} \, dz \\ &= r\int_{\tau_{r,n,\delta}} t^{1-2s} A\nabla U_n \cdot \nabla U_n \, dz, \end{aligned}$$

and

$$
\begin{aligned}
-2\int_{\tau_{r,n,\delta}} t^{1-2s}(\beta\cdot\nabla U_n)A\nabla U_n\cdot\nu\,dS &= -2\int_{\tau_{r,n,\delta}} t^{1-2s}\left(\frac{A(y)z}{\mu}\cdot\nabla U_n\right)A\nabla U_n\cdot\nu\,dS\\
&= -2\int_{\tau_{r,n,\delta}} t^{1-2s}\left(\frac{A\nabla U_n}{\mu}\cdot r\nu\right)A\nabla U_n\cdot\nu\,dS\\
&= -2r\int_{\tau_{r,n,\delta}} t^{1-2s}\frac{|A\nabla U_n\cdot\nu|^2}{\mu}\,dS.
\end{aligned}
$$

As far as the integral on $\gamma_{r,n,\delta}$ is concerned, since ∇U_n boils down to $\dfrac{\partial U_n}{\partial\nu}$ on $\gamma_{r,n,\delta}$, it holds that

$$
\begin{aligned}
\int_{\gamma_{r,n,\delta}} (t^{1-2s}A\nabla U_n\cdot\nabla U_n)(\beta\cdot\nu)\,dz - 2\int_{\gamma_{r,n,\delta}} t^{1-2s}(\beta\cdot\nabla U_n)(A\nabla U_n\cdot\nu)\,dS&\\
= -\int_{\gamma_{r,n,\delta}} \frac{t^{1-2s}}{\mu}\left|\frac{\partial U_n}{\partial\nu}\right|^2(A\nu\cdot\nu)(Az\cdot\nu)\,dS.&
\end{aligned}
$$

Finally, we notice that on $\sigma_{r,n,\delta}$, $\nu=(0,\dots,0,-1)$, hence

$$
\begin{aligned}
\int_{\sigma_{r,n,\delta}} t^{1-2s}(A\nabla U_n\cdot\nabla U_n)(\beta\cdot\nu)\,dy &= \int_{\sigma_{r,n,\delta}} t^{1-2s}(A\nabla U_n\cdot\nabla U_n)\left(\frac{A(y)z}{\mu}\cdot\nu\right)\,dy\\
&= -\int_{\sigma_{r,n,\delta}} \delta^{2-2s}\frac{\alpha}{\mu}A\nabla U_n\cdot\nabla U_n\,dy,
\end{aligned}
$$

and

$$
\begin{aligned}
-2\int_{\sigma_{r,n,\delta}} t^{1-2s}(\beta\cdot\nabla U_n)(A\nabla U_n\cdot\nu)\,dy &= 2\int_{\sigma_{r,n,\delta}} \delta^{1-2s}\left(\frac{A(y)z}{\mu}\cdot\nabla U_n\right)(\alpha\partial_t U_n)\,dy\\
&= 2\int_{\sigma_{r,n,\delta}} \delta^{1-2s}\frac{D\nabla_y U_n\cdot y+\alpha\delta\partial_t U_n}{\mu}(\alpha\partial_t U_n)\,dy\\
&= 2\int_{\sigma_{r,n,\delta}} \delta^{1-2s}\frac{\alpha}{\mu}\partial_t U_n(D\nabla_y U_n\cdot y)\,dy\\
&+ 2\int_{\sigma_{r,n,\delta}} \delta^{2-2s}\frac{\alpha^2}{\mu}|\partial_t U_n|^2\,dy.
\end{aligned}
$$

Putting together all the above computations, (2.3.71) can be rewritten as follows

$$\begin{aligned}
& r\int_{\tau_{r,n,\delta}} t^{1-2s} A\nabla U_n\cdot\nabla U_n\,dS - 2r\int_{\tau_{r,n,\delta}} t^{1-2s}\frac{|A\nabla U_n\cdot\nu|^2}{\mu}\,dS \\
&\qquad - \int_{\gamma_{r,n,\delta}} \frac{t^{1-2s}}{\mu}|\partial_\nu U_n|^2 (A\nu\cdot\nu)(Az\cdot\nu)\,dS - \int_{\sigma_{r,n,\delta}} \delta^{2-2s}\frac{\alpha}{\mu} A\nabla U_n\cdot\nabla U_n\,dy \\
&\qquad + 2\int_{\sigma_{r,n,\delta}} \delta^{1-2s}\frac{\alpha}{\mu}\partial_t U_n (D\nabla_y U_n\cdot y)\,dy + 2\int_{\sigma_{r,n,\delta}} \delta^{2-2s}\frac{\alpha^2}{\mu}|\partial_t U_n|^2\,dy \\
&= \int_{O_{r,n,\delta}} t^{1-2s} A\nabla U_n\cdot\nabla U_n\ \mathrm{div}\beta\,dz - 2\int_{O_{r,n,\delta}} t^{1-2s}\mathrm{Jac}\beta(A\nabla U_n)\cdot\nabla U_n\,dz \\
&\qquad + \int_{O_{r,n,\delta}} t^{1-2s}(dA\nabla U_n\nabla U_n)\cdot\beta\,dz + (1-2s)\int_{O_{r,n,\delta}} t^{1-2s}\frac{\alpha}{\mu} A\nabla U_n\cdot\nabla U_n\,dz.
\end{aligned}$$

From Lemma 2.3.8 and uniform ellipticity of A it follows that

$$\int_{\gamma_{r,n,\delta}} \frac{t^{1-2s}}{\mu}|\partial_\nu U_n|^2 (A\nu\cdot\nu)(Az\cdot\nu)\,dS \ge 0.$$

Hence, from this we get the following inequality

$$\begin{aligned}
& r\int_{\tau_{r,n,\delta}} t^{1-2s} A\nabla U_n\cdot\nabla U_n\,dS - 2r\int_{\tau_{r,n,\delta}} t^{1-2s}\frac{|A\nabla U_n\cdot\nu|^2}{\mu}\,dS \\
&\qquad - \int_{\sigma_{r,n,\delta}} \delta^{2-2s}\frac{\alpha}{\mu} A\nabla U_n\cdot\nabla U_n\,dy + 2\int_{\sigma_{r,n,\delta}} \delta^{1-2s}\frac{\alpha}{\mu}\partial_t U_n (D\nabla_y U_n\cdot y)\,dy \\
&\qquad + 2\int_{\sigma_{r,n,\delta}} \delta^{2-2s}\frac{\alpha^2}{\mu}|\partial_t U_n|^2\,dy \\
&\ge \int_{O_{r,n,\delta}} t^{1-2s} A\nabla U_n\cdot\nabla U_n\ \mathrm{div}\beta\,dz - 2\int_{O_{r,n,\delta}} t^{1-2s} J_\beta(A\nabla U_n)\cdot\nabla U_n\,dz \\
&\qquad + \int_{O_{r,n,\delta}} t^{1-2s}(dA\nabla U_n\nabla U_n)\cdot\beta\,dz + (1-2s)\int_{O_{r,n,\delta}} t^{1-2s}\frac{\alpha}{\mu} A\nabla U_n\cdot\nabla U_n\,dz.
\end{aligned} \tag{2.3.72}$$

At this point, we want to pass to the limit as $\delta\to 0$. We denote as $O_{r,n}$ the limit domain whose boundary is given by $\partial O_{r,n} = \sigma_{r,n}\cup\gamma_{r,n}\cup\tau_{r,n}$, i.e.

$$\begin{aligned}
& O_{r,n} = \mathcal{U}_n\cap B_r, \quad \tau_{r,n} = \left\{(y', y_N, t)\in\partial B_r : y_N < f_n(t),\ t\ge 0\right\}, \\
& \gamma_{r,n} = \left\{(y', y_N, t)\in\overline{B_r^+} : y_N = f_n(t)\right\}, \quad \sigma_{r,n} = \left\{(y', y_N)\in B_r' : y_N < n^{-1/8}\right\}.
\end{aligned}$$

We claim that, for every fixed $r\in(0,R_0)$ and $n > r^{-8}$, there exists a sequence $\delta_k\to 0^+$ such that

$$-\int_{\sigma_{r,n,\delta_k}} \delta_k^{2-2s}\frac{\alpha}{\mu} A\nabla U_n\cdot\nabla U_n\,dy + 2\int_{\sigma_{r,n,\delta_k}} \delta_k^{2-2s}\frac{\alpha^2}{\mu}|\partial_t U_n|^2\,dy \to 0 \quad \text{as } k\to\infty.$$

Using that α defined in (2.3.8) is bounded by (2.3.7), $\mu \geq 1/4$ in B_{R_0}, and A has bounded coefficients, it is enough to prove that there exists a sequence $\delta_k \to 0^+$ such that $\lim_{k\to\infty} \int_{\sigma_{r,n,\delta_k}} \delta_k^{2-2s} |\nabla U_n|^2 \, dy = 0$. To prove this, we argue by contradiction and assume that there exist a positive constant $c > 0$ and $\delta_0 > 0$ such that, for any $\delta \in (0, \delta_0)$,

$$\frac{c}{\delta} \leq \int_{\sigma_{r,n,\delta}} \delta^{1-2s} |\nabla U_n(y, \delta)|^2 \, dy,$$

which, after integration over $(0, \delta_0)$, gives a contradiction, since it holds

$$\int_0^{\delta_0} \frac{c}{\delta} \, d\delta \leq \int_0^{\delta_0} \delta^{1-2s} \left(\int_{\sigma_{r,n,\delta}} |\nabla U_n(y, \delta)|^2 \, dy \right) d\delta \leq \|U_n\|^2_{H^1(B^+_{R_0}, t^{1-2s}\, dz)},$$

where the first integral diverges.

In order to prove the convergence

$$2 \int_{\sigma_{r,n,\delta}} \delta^{1-2s} \frac{\alpha}{\mu} \partial_t U_n (D \nabla_y U_n \cdot y) \, dy \underset{\delta \to 0}{\longrightarrow} -2\kappa_s \int_{\sigma_{r,n}} \frac{1}{\mu} \eta_n h_n \mathrm{Tr} U_n (D \nabla_y \mathrm{Tr} U_n \cdot y) \, dy,$$

we exploit a continuity result for $t^{1-2s}\partial_t U_n$ and $\nabla_y U_n$ over $\overline{\mathcal{U}_n \cap B_r}$, which allows us to pass to the limit by the Dominated Convergence Theorem. More precisely we claim that, for all $r \in (0, R_0)$ and $n > r^{-8}$,

$$t^{1-2s}\partial_t U_n \in C^0(\overline{\mathcal{U}_n \cap B_r}), \quad \nabla_y U_n \in C^0(\overline{\mathcal{U}_n \cap B_r}). \tag{2.3.73}$$

The continuity of $t^{1-2s}\partial_t U_n$ and $\nabla_y U_n$ away from $\{t = 0\}$ easily follows from classical elliptic regularity theory, since U_n is solution of an uniformly elliptic equation (we refer to [50, Corollary 8.36]). Nevertheless, Lemma 3.3 in [36] allows us to prove the continuity of $t^{1-2s}\partial_t U_n$ and $\nabla_y U_n$ up to $\{t = 0\}$ when we stay away from the corner between $\sigma_{r,n}$ and $\gamma_{r,n}$, i.e. away from the edge $\{(y', y_N, t) \in \overline{B_r} : t = 0 \text{ and } y_N = n^{-1/8}\}$: to this aim it is enough to apply [36, Lemma 3.3] to the function $U_n \circ F^{-1}$. Eventually, we can deduce the continuity of $t^{1-2s}\partial_t U_n$ and $\nabla_y U_n$ also in the set

$$\{(y', y_N, t) \in \overline{B_r} : t \in [0, 1/2n] \text{ and } y_N \in [0, n^{-1/8}]\}$$

as a consequence of the regularity result given in Lemma A.1.1 applied to the function $U_n \circ F^{-1}$.

We remark that for all $r \in (0, R_0)$ and $n > r^{-8}$, the terms integrated over $\tau_{r,n,\delta}$ belong to $L^1(\tau_{r,n})$ in view of (2.3.73) and the terms integrated over $O_{r,n,\delta}$ belong to $L^1(\mathcal{U}_n \cap B_r)$ since $U_n \in H^1(\mathcal{U}_n, t^{1-2s}\, dz)$. These facts allow us passing to the limit in (2.3.72) along $\delta = \delta_k$ as $k \to +\infty$ by absolute continuity of the Lebesgue integral, thus ending up with the following inequality

$$
\begin{aligned}
&r \int_{\tau_{r,n}} t^{1-2s} A\nabla U_n \cdot \nabla U_n \, dS - 2r \int_{\tau_{r,n}} t^{1-2s} \frac{|A\nabla U_n \cdot \nu|^2}{\mu} \, dS \\
&\qquad - 2\kappa_s \int_{\sigma_{r,n}} \frac{1}{\mu} \eta_n h_n \mathrm{Tr} U_n (D\nabla_y \mathrm{Tr} U_n \cdot y) \, dy \\
&\geq \int_{O_{r,n}} t^{1-2s} A\nabla U_n \cdot \nabla U_n \mathrm{div}\beta \, dz - 2 \int_{O_{r,n}} t^{1-2s} \mathrm{Jac}\beta(A\nabla U_n) \cdot \nabla U_n \, dz \\
&\qquad + \int_{O_{r,n}} t^{1-2s} (dA\nabla U_n \nabla U_n) \cdot \beta \, dz \\
&\qquad + (1-2s) \int_{O_{r,n}} t^{1-2s} \frac{\alpha}{\mu} A\nabla U_n \cdot \nabla U_n \, dz,
\end{aligned}
\tag{2.3.74}
$$

for all $r \in (0, R_0)$ and $n > r^{-8}$.

Now for $r \in (0, R_0)$ fixed, we aim to pass to the limit in (2.3.74) as $n \to +\infty$. Therefore, we extend the functions U_n to be zero in $B_r^+ \setminus \mathcal{U}_n$. By the strong convergence $U_n \to W$ in $H^1(B_{R_0}^+, t^{1-2s} dz)$ (see Proposition 2.3.10), it follows that

$$
\int_0^{R_0} \left(\int_{\partial^+ B_r^+} t^{1-2s} \left(|\nabla (U_n - W)|^2 + |U_n - W|^2 \right) \, dS \right) dr \to 0,
$$

i.e. the sequence of functions

$$
u_n(r) := \int_{\partial^+ B_r^+} t^{1-2s} \left(|\nabla (U_n - W)|^2 + |U_n - W|^2 \right) \, dS
$$

converges to 0 in $L^1(0, R_0)$ and hence a.e. along a subsequence u_{n_k}. In particular we have that

$$
U_{n_k} \to W \text{ as } k \to \infty \text{ in } H^1(\partial^+ B_r^+, t^{1-2s} \, dS) \text{ for a.e. } r \in (0, R_0), \tag{2.3.75}
$$

where $H^1(\partial^+ B_r^+, t^{1-2s} \, dS)$ is the completion of $C^\infty(\overline{\partial^+ B_r^+})$ with respect to the norm

$$
\|\psi\|_{H^1(\partial^+ B_r^+, t^{1-2s} \, dS)} = \left(\int_{\partial^+ B_r^+} t^{1-2s} \left(|\nabla \psi|^2 + \psi^2 \right) \, dS \right)^{1/2}.
$$

Let us now discuss the behavior of the term $\int_{\sigma_{r,n}} \frac{1}{\mu} \eta_n h_n \mathrm{Tr} U_n (D\nabla_y \mathrm{Tr} U_n \cdot y) \, dy$ as $n \to \infty$.

Since $\eta_n(y', y_N) = 0$ if $y_N > -\frac{1}{n}$, by the Divergence Theorem we have that

$$\int_{\sigma_{r,n}} \frac{1}{\mu}\eta_n h_n \mathrm{Tr}U_n (D\nabla_y \mathrm{Tr}U_n \cdot y)\, dy = \int_{\Gamma_r^-} \frac{1}{\mu}\eta_n h_n \mathrm{Tr}U_n (D\nabla_y \mathrm{Tr}U_n \cdot y)\, dy \tag{2.3.76}$$
$$= \frac{1}{2}\int_{\Gamma_r^-} \mathrm{div}_y \left(\frac{1}{\mu}\eta_n h_n |\mathrm{Tr}U_n|^2 Dy\right) dy - \frac{1}{2}\int_{\Gamma_r^-} |\mathrm{Tr}U_n|^2 \mathrm{div}_y \left(\frac{1}{\mu}\eta_n h_n Dy\right) dy$$
$$= \frac{1}{2}\int_{S_r^-} \frac{1}{\mu}\eta_n h_n |\mathrm{Tr}U_n|^2 Dy \cdot \nu\, dS' - \frac{1}{2}\int_{\Gamma_r^-} |\mathrm{Tr}U_n|^2 \mathrm{div}_y \left(\eta_n h_n \beta'\right) dy$$
$$= \frac{r}{2}\int_{S_r^-} \eta_n h_n |\mathrm{Tr}U_n|^2\, dS' - \frac{1}{2}\int_{\Gamma_r^-} |\mathrm{Tr}U_n|^2 \left(\eta_n \nabla_y h_n \cdot \beta' + \eta_n h_n \mathrm{div}_y \beta'\right) dy$$
$$- \frac{1}{2}\int_{\Gamma_r^-} |\mathrm{Tr}U_n|^2 h_n \nabla_y \eta_n \cdot \beta'\, dy,$$

where S_r^- has been introduced in (2.3.66) and β' has been defined in (2.3.23). From the strong convergence of U_n to W in $H^1(B_{R_0}^+, t^{1-2s}dz)$ proved in Proposition 2.3.10, (2.3.44) and (2.3.3), it follows that

$$\int_0^{R_0} \left(\int_{S_r^-} \left(\eta_n h_n |\mathrm{Tr}U_n|^2 - \tilde{h}|\mathrm{Tr}W|^2\right) dS'\right) dr \to 0,$$

i.e. the sequence of functions $r \mapsto \int_{S_r^-} \left(\eta_n h_n |\mathrm{Tr}U_n|^2 - \tilde{h}|\mathrm{Tr}W|^2\right) dS'$ converges to 0 in $L^1(0, R_0)$ and hence a.e. along a further subsequence, which we still index by n_k. In particular we deduce that

$$\int_{S_r^-} \eta_{n_k} h_{n_k} |\mathrm{Tr}U_{n_k}|^2\, dS' \to \int_{S_r^-} \tilde{h}|\mathrm{Tr}W|^2\, dS' \text{ as } k \to \infty \text{ for a.e. } r \in (0, R_0). \tag{2.3.77}$$

The strong convergence of U_n to W in $H^1(B_{R_0}^+, t^{1-2s}dz)$ implies that $\mathrm{Tr}U_n \to \mathrm{Tr}W$ in $L^{2^*(s)}(B'_{R_0})$ by (2.3.3). Combining this fact with (2.3.44) and that $\eta_n \to 1$ a.e. in $\Gamma_{R_0}^-$, we obtain that

$$\int_{\Gamma_r^-} |\mathrm{Tr}U_n|^2 \left(\eta_n \nabla_y h_n \cdot \beta' + \eta_n h_n \mathrm{div}_y \beta'\right) dy \to \int_{\Gamma_r^-} |\mathrm{Tr}W|^2 \left(\nabla_y \tilde{h} \cdot \beta' + \tilde{h}\, \mathrm{div}_y \beta'\right) dy \tag{2.3.78}$$

as $n \to \infty$ for all $r \in (0, R_0)$. Finally, we observe that, by (2.3.51) and (2.3.23),

$$\nabla_y \eta_n \cdot \beta' = \frac{1}{\mu}\frac{n}{2}\eta'\left(-\tfrac{n y_N}{2}\right)(D(y)y)_N.$$

By (2.3.19) we have that $(D(y)y)_N = O(y_N)$ as $y_N \to 0$ and (2.3.39) allows us to assert that $\eta'\left(-\frac{n y_N}{2}\right) \neq 0$ only for $y_N \in \left(-\frac{2}{n}, -\frac{1}{n}\right)$. Hence we can conclude that

$$\nabla_y \eta_n \cdot \beta' \quad \text{is bounded in } \Gamma_r^- \text{ uniformly with respect to } n.$$

Therefore, by the Hölder's inequality,

$$\left|\int_{\Gamma_r^-} |\mathrm{Tr}U_n|^2 h_n \nabla_y \eta_n \cdot \beta' \, dy\right|$$
$$\leq \text{const}\, \|\mathrm{Tr}U_n\|^2_{L^{2^*(s)}(\Gamma_r^-)} \|h_n\|_{L^p(\Gamma_r^-)} \left|\{(y', y_N) \in \Gamma_r^- : -\tfrac{2}{n} < y_N < -\tfrac{1}{n}\}\right|_N^{\frac{2s}{pN}(p-\frac{N}{2s})}$$

where $|\cdot|_N$ stands for the N-dimensional Lebesgue measure; hence, since $\{\mathrm{Tr}U_n\}$ is bounded in $L^{2^*(s)}(\Gamma_r^-)$ and the same hols true for $\{h_n\}$ in $L^p(\Gamma_r^-)$, we infer that

$$\lim_{n\to\infty} \int_{\Gamma_r^-} |\mathrm{Tr}U_n|^2 h_n \nabla_y \eta_n \cdot \beta' \, dy = 0. \tag{2.3.79}$$

Combining (2.3.77), (2.3.78) and (2.3.79), passing to the limit in (2.3.76) along the subsequence, we obtain that

$$\lim_{k\to\infty} \int_{\sigma_{r,n_k}} \frac{1}{\mu} \eta_{n_k} h_{n_k} \mathrm{Tr}U_{n_k} (D\nabla_y \mathrm{Tr}U_{n_k} \cdot y) \, dy$$
$$= \frac{r}{2} \int_{S_r^-} \tilde{h} |\mathrm{Tr}W|^2 \, dS' - \frac{1}{2} \int_{\Gamma_r^-} |\mathrm{Tr}W|^2 \left(\nabla_y \tilde{h} \cdot \beta' + \tilde{h}\,\mathrm{div}_y \beta'\right) dy. \tag{2.3.80}$$

In virtue of (2.3.75), (2.3.80) and the strong convergence of U_n to W in $H^1(B^+_{R_0}, t^{1-2s}dz)$, we can pass to the limit as $n = n_k \to \infty$ in (2.3.74) obtaining the desired Pohozaev-type inequality (2.3.67) for the solution W.

Finally, to prove (2.3.68), we first multiply equation (2.3.52) by U_n itself and integrate by parts over $O_{r,n,\delta}$; then we pass to the limit as $\delta \to 0^+$ using (2.3.73) and as $n = n_k \to \infty$, taking into account (2.3.75). □

2.3.4 The Almgren frequency function for the extended problem

In this section we analyze the properties of the Almgren frequency function $\mathcal{N}(r)$ associated to (2.3.11), defined in (2.3.89): in particular we will prove the boundedness of the frequency and that $\mathcal{N}$ possesses a nonnegative finite limit as $r \to 0^+$.

To this aim, let $W \in H^1_{\Gamma^+_{R_1}}(B^+_{R_1}, t^{1-2s}\, dz)$ be a nontrivial weak solution to (2.3.11). For all $r \in (0, R_1)$, we define

$$E(r) = \frac{1}{r^{N-2s}} \left(\int_{B_r^+} t^{1-2s} A\nabla W \cdot \nabla W \, dz - \kappa_s \int_{\Gamma_r^-} \tilde{h} |\operatorname{Tr} W|^2 \, dy \right) \tag{2.3.81}$$

and

$$H(r) = \frac{1}{r^{N+1-2s}} \int_{\partial^+ B_r^+} t^{1-2s} \mu(z) W^2(z) \, dS. \tag{2.3.82}$$

Let us first estimate the derivative of H.

Lemma 2.3.13. *Let E and H be the functions defined as in (2.3.81) and (2.3.82). Then $H \in W^{1,1}_{\mathrm{loc}}(0, R_1)$ and*

$$H'(r) = \frac{2}{r^{N+1-2s}} \int_{\partial^+ B_r^+} t^{1-2s} \mu W \frac{\partial W}{\partial \nu} \, dS + H(r) O(1) \quad \text{as } r \to 0^+ \tag{2.3.83}$$

in a distributional sense and for a.e. $r \in (0, R_1)$, where $\nu = \nu(z) = \frac{z}{|z|}$ denotes the unit outer normal vector to $\partial^+ B_r^+$. Moreover

$$H'(r) = \frac{2}{r^{N+1-2s}} \int_{\partial^+ B_r^+} t^{1-2s} (A \nabla W \cdot \nu) W \, dS + H(r) O(1) \tag{2.3.84}$$

and

$$H'(r) = \frac{2}{r} E(r) + H(r) O(1) \tag{2.3.85}$$

as $r \to 0^+$.

Proof. We observe that $H \in L^1_{\mathrm{loc}}(0, R_1)$ by definition and it can be rewritten as

$$H(r) = \int_{\mathbb{S}^N_+} \theta_{N+1}^{1-2s} \mu(r\theta) W^2(r\theta) \, dS.$$

Thus, for all test functions $\varphi \in C^\infty_c(0, R_1)$, we have that

$$\begin{aligned}
-\int_0^{R_1} H(r) \varphi'(r) dr &= -\int_0^{R_1} \left(\int_{\mathbb{S}^N_+} \theta_{N+1}^{1-2s} \mu(r\theta) W^2(r\theta) \, dS \right) \varphi'(r) \, dr \\
&= -\int_0^{R_1} \int_{\mathbb{S}^N_+} \theta_{N+1}^{1-2s} \mu(r\theta) W^2(r\theta) \nabla \tilde{\varphi}(r\theta) \cdot \theta \, dS \, dr \\
&= -\int_{B^+_{R_1}} t^{1-2s} \frac{\mu(z) W^2(z)}{|z|^{N+2-2s}} \nabla \tilde{\varphi}(z) \cdot z \, dz \\
&= \int_{B^+_{R_1}} \mathrm{div} \left(\frac{t^{1-2s} \mu(z) W^2(z) z}{|z|^{N+2-2s}} \right) \tilde{\varphi}(z) dz \\
&= \int_{B^+_{R_1}} t^{1-2s} \left(\frac{2\mu(z) W(z) \nabla W(z) + W^2(z) \nabla \mu(z)}{|z|^{N+2-2s}} \cdot z \right) \tilde{\varphi}(z) dz \\
&= \int_0^{R_1} \left(\int_{\mathbb{S}^N_+} \theta_{N+1}^{1-2s} \big[2\mu(r\theta) W(r\theta) \nabla W(r\theta) \cdot \theta + W^2(r\theta) \nabla \mu(r\theta) \cdot \theta \big] dS \right) \varphi(r) \, dr,
\end{aligned}$$

where we set $\tilde{\varphi}(r\theta) := \varphi(r)$ for every $r \in (0, R_1)$ and $\theta \in \mathbb{S}^N_+$, having that $\varphi'(r) = \nabla \tilde{\varphi}(r\theta) \cdot \theta$ and that $\tilde{\varphi}(R_1 \theta) = 0$. Hence the distributional derivative of H in $(0, R_1)$ is given by

$$H'(r) = \frac{2}{r^{N+1-2s}} \int_{\partial^+ B_r^+} t^{1-2s} \mu W \frac{\partial W}{\partial \nu} \, dS + \frac{1}{r^{N+1-2s}} \int_{\partial^+ B_r^+} t^{1-2s} W^2 \nabla \mu \cdot \nu \, dS. \tag{2.3.86}$$

Since $W, \nabla W \in L^2(B^+_{R_1}, t^{1-2s}dz)$, from (2.3.26) and (2.3.27) we easily infer that $H \in W^{1,1}_{\rm loc}(0, R_1)$ and (2.3.86) also holds for a.e. $r \in (0, R_1)$. Moreover, combining (2.3.26), (2.3.27), (2.3.82) and (2.3.86), we obtain (2.3.83).

In order to prove (2.3.84), we introduce $\gamma(z) := \mu(z)(\beta(z) - z)/|z|$, observing that

$$\begin{aligned} \gamma(z) \cdot z &= 0, \\ \mathrm{div}(t^{1-2s}\gamma) &= t^{1-2s}\mathrm{div}\gamma + (1-2s)\gamma_{N+1}t^{-2s}, \\ \gamma_{N+1}(z) &= t\,O(1) \quad \text{ as } |z| \to 0^+, \end{aligned}$$

and

$$\mathrm{div}\gamma = \left(\frac{\nabla\mu(z)}{|z|} - \frac{\mu(z)z}{|z|^3}\right)(\beta(z) - z) + \frac{\mu(z)}{|z|}(\mathrm{div}\beta - (N+1)) = O(1) \quad \text{as } |z| \to 0^+,$$

as a consequence of (2.3.26), (2.3.27), (2.3.28), (2.3.30). From all these facts, we deduce that for a.e. $r \in (0, R_1)$,

$$\begin{aligned} \int_{\partial^+ B_r^+} t^{1-2s}(A\nabla W \cdot \nu)W\,dS &= \int_{\partial^+ B_r^+} t^{1-2s}\mu W\frac{\partial W}{\partial\nu}dS + \frac{1}{2}\int_{\partial^+ B_r^+} t^{1-2s}\gamma \cdot \nabla(W^2)\,dS \\ &= \int_{\partial^+ B_r^+} t^{1-2s}\mu W\frac{\partial W}{\partial\nu}dS - \frac{1}{2}\int_{\partial^+ B_r^+} \mathrm{div}(t^{1-2s}\gamma)W^2\,dS \\ &= \int_{\partial^+ B_r^+} t^{1-2s}\mu W\frac{\partial W}{\partial\nu}dS + H(r)O(r^{N+1-2s}), \end{aligned} \tag{2.3.87}$$

using (2.3.82) and (2.3.26). Hence, from (2.3.83) and (2.3.87), it follows (2.3.84). From (2.3.81), (2.3.68) and (2.3.84) we infer that

$$r^{N-2s}E(r) = \int_{\partial^+ B_r^+} t^{1-2s}(A\nabla W \cdot \nu)W\,dS = \frac{r^{N+1-2s}}{2}H'(r) + H(r)O(r^{N+1-2s}),$$

as $r \to 0^+$, which gives (2.3.85), thus proving the lemma. □

Lemma 2.3.14. *The function H defined as in* (2.3.82) *is strictly positive for every* $0 < r \le R_0$*, with* R_0 *being defined in* (2.3.46).

Proof. We prove the statement arguing by contradiction. To this aim, we suppose that there exists $\overline{R} \le R_0$ such that $H(\overline{R}) = 0$. Then, using that $\mu \ge 1/4$ in B_r for every $r \le R_0$, we obtain that $\int_{\partial^+ B^+_{\overline{R}}} t^{1-2s}W^2\,dS = 0$, hence $W \equiv 0$ on $\partial^+ B^+_{\overline{R}}$. From (2.3.85) it follows that H is differentiable in a classical sense in $\overline{R}$ and $H'(\overline{R}) = 2\overline{R}^{-1}E(\overline{R})$; on the other hand, $H(r) \ge 0 = H(\overline{R})$ implies that $0 = H'(\overline{R}) = 2\overline{R}^{-1}E(\overline{R})$ and hence $E(\overline{R}) = 0$. Then from (2.3.32) it follows that

$$0 = \int_{B^+_{\overline{R}}} t^{1-2s}A\nabla W \cdot \nabla W\,dz - \kappa_s\int_{\Gamma^-_{\overline{R}}} \tilde{h}|\,\mathrm{Tr}\,W|^2\,dy \ge \tilde{C}_{N,s}\int_{B^+_{\overline{R}}} t^{1-2s}|\nabla W|^2\,dz. \tag{2.3.88}$$

By (2.3.88) and Lemma 2.3.3, we can deduce that $W \equiv 0$ in $B^+_{\bar{R}}$, which in turn leads to $W \equiv 0$ in $B^+_{R_1} \cap \{t > \delta\}$ from classical unique continuation principles for second order elliptic equations with Lipschitz coefficients (see [48]). Since $\delta > 0$ can be taken arbitrarily small, we end up with $W \equiv 0$ in $B^+_{R_1}$, which is a contradiction. □

As a consequence of Lemma 2.3.14, the *Almgren type frequency* function

$$\mathcal{N}(r) = \frac{E(r)}{H(r)} \tag{2.3.89}$$

is well defined in $(0, R_0]$, with R_0 as in (2.3.46).

In the following lemma we provide an estimate for the derivative of the function E.

Lemma 2.3.15. *Let E be the function defined as in* (2.3.81)*. Then $E \in W^{1,1}_{\mathrm{loc}}(0, R_1)$ and*

$$E'(r) \geq \frac{2}{r^{N-2s}} \int_{\partial^+ B_r^+} t^{1-2s} \frac{|A\nabla W \cdot \nu|^2}{\mu} + O(r^{-1+\bar{\delta}}) \left[E(r) + \frac{N-2s}{2} H(r)\right] \quad \textit{as } r \to 0^+ \tag{2.3.90}$$

for a.e. $r \in (0, R_0)$, where

$$\bar{\delta} = \min\{\bar{\varepsilon}, 1\} \in (0, 1] \tag{2.3.91}$$

and $\bar{\varepsilon}$ is defined as in (2.3.36)*.*

Proof. From (2.3.81) we deduce that $E \in L^1_{\mathrm{loc}}(0, R_1)$. Using the coarea formula we obtain that

$$\begin{aligned} E'(r) =& \frac{2s-N}{r^{N+1-2s}} \left(\int_{B_r^+} t^{1-2s} A\nabla W \cdot \nabla W \, dz - \kappa_s \int_{\Gamma_r^-} \tilde{h} |\operatorname{Tr} W|^2 \, dy \right) \\ &+ \frac{1}{r^{N-2s}} \left(\int_{\partial^+ B_r^+} t^{1-2s} A\nabla W \cdot \nabla W \, dS - \kappa_s \int_{S_r^-} \tilde{h} |\operatorname{Tr} W|^2 \, dS' \right) \end{aligned} \tag{2.3.92}$$

in a distributional sense and a.e. in $(0, R_1)$, thus having that $E \in W^{1,1}_{\mathrm{loc}}(0, R_1)$. Using (2.3.67), Lemma 2.3.1 and Lemma 2.3.2, we infer that

$$\begin{aligned} E'(r) \geq& \frac{2}{r^{N-2s}} \int_{\partial^+ B_r^+} t^{1-2s} \frac{|A\nabla W \cdot \nu|^2}{\mu} \, dS + \frac{O(r)}{r^{N+1-2s}} \int_{B_r^+} t^{1-2s} A\nabla W \cdot \nabla W \, dz \\ &+ \frac{O(1)}{r^{N+1-2s}} \int_{\Gamma_r^-} (|\tilde{h}| + |\nabla_y \tilde{h}|) |\operatorname{Tr} W|^2 \, dy \end{aligned} \tag{2.3.93}$$

as $r \to 0^+$, for a.e. $r \in (0, R_0)$. We can estimate the last two terms on the right hand side in (2.3.93) exploiting (2.3.32). Indeed, observing that

$$\frac{O(r)}{r^{N+1-2s}} \int_{B_r^+} t^{1-2s} A\nabla W \cdot \nabla W \, dz = \frac{O(r)}{r^{N+1-2s}} \int_{B_r^+} t^{1-2s} |\nabla W|^2 \, dz,$$

as a consequence of (2.3.20), we obtain that

$$\frac{O(r)}{r^{N+1-2s}} \int_{B_r^+} t^{1-2s} A\nabla W \cdot \nabla W \, dz = O(1) \left[E(r) + \frac{N-2s}{2} H(r)\right] \tag{2.3.94}$$

and, taking into account (2.3.35), we also derive that

$$\frac{O(1)}{r^{N+1-2s}}\int_{\Gamma_r^-}(|\tilde{h}|+|\nabla_y\tilde{h}|)|\operatorname{Tr}W|^2\,dy=\frac{O(r^{\bar{\varepsilon}})}{r^{N+1-2s}}\left(\int_{\Gamma_r^-}|\operatorname{Tr}W|^{2^*(s)}\,dy\right)^{2/2^*(s)}$$
$$=O(r^{-1+\bar{\varepsilon}})\left[E(r)+\frac{N-2s}{2}H(r)\right]. \tag{2.3.95}$$

Estimate (2.3.90) follows from (2.3.93), (2.3.94) and (2.3.95). □

Lemma 2.3.16. *Let $\mathcal{N}$ be the function defined in* (2.3.89)*. Then, for every* $0<r\le R_0$*,*

$$\mathcal{N}(r)>-\frac{N-2s}{2} \tag{2.3.96}$$

and

$$\liminf_{r\to 0^+}\mathcal{N}(r)\ge 0. \tag{2.3.97}$$

Proof. We deduce (2.3.96) from (2.3.32). By (2.3.81), (2.3.82), (2.3.38) and (2.3.33), we obtain that for all $0<r\le R_0$

$$\begin{aligned}r^{N-2s}E(r)&=\int_{B_r^+}t^{1-2s}A\nabla W\cdot\nabla W\,dz-\kappa_s\int_{\Gamma_r^-}\tilde{h}|\operatorname{Tr}W|^2\,dy\\&\ge\frac{3}{8}\int_{B_r^+}t^{1-2s}|\nabla W|^2\,dz-\kappa_s\tilde{S}_{N,s}\tilde{c}_{N,s,p}r^{\bar{\varepsilon}}\bar{\alpha}_0\frac{2(N-2s)}{r}\int_{\partial^+B_r^+}t^{1-2s}\mu W^2\,dS\\&\ge-\tilde{C}r^{\bar{\varepsilon}+N-2s}H(r),\end{aligned}$$

with $\bar{\alpha}_0$ as in (2.3.45) and $\tilde{C}:=2(N-2s)\kappa_s\tilde{S}_{N,s}\tilde{c}_{N,s,p}\bar{\alpha}_0>0$. From this and (2.3.89) it follows that, for every $0<r\le R_0$,

$$\mathcal{N}(r)\ge-\tilde{C}r^{\bar{\varepsilon}},$$

which in turn leads to (2.3.97). □

Lemma 2.3.17. *Let $\mathcal{N}$ be the function defined in* (2.3.89)*. Then* $\mathcal{N}\in W^{1,1}_{\text{loc}}((0,R_0])$*.*

Proof. Let us consider any interval $[a,b]$ with $0<a<b\le R_0$ and notice that $\mathcal{N}\in L^1([a,b])$ trivially since E and H are continuous functions by Lemma 2.3.15 and Lemma 2.3.13. A bit more difficult is to show that also $\mathcal{N}'\in L^1([a,b])$. In order to do this, we prove two statements:

1) if $f\in W^{1,1}([a,b])$ and $f>0$, then $1/f\in W^{1,1}([a,b])$;

2) if $f,g\in W^{1,1}([a,b])$, then $fg\in W^{1,1}([a,b])$.

Then it will be sufficient to apply the above results with $f = E$ and $g = 1/H$ to prove the thesis, taking into account Lemmas 2.3.13, 2.3.14, and 2.3.15.

As far as 1) is concerned, we first observe that $f \geq C > 0$ for some positive constant $C > 0$ by assumption and then we introduce a real-valued function $G \in C^1(\mathbb{R})$ as in Figure 2.3 such that $G(t) = 1/t$ for every $t \geq C$.

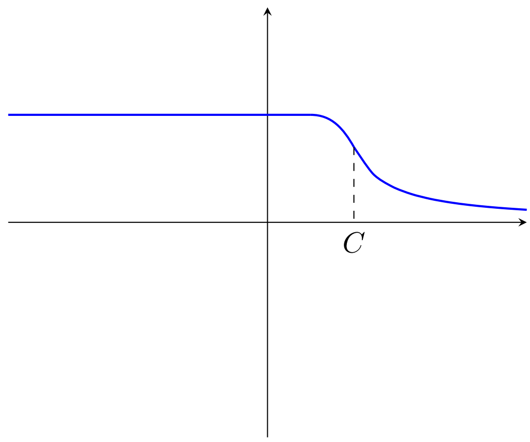


Figure 2.3: The graph of the function G

Thus, using that $f \geq C$ and f is continuous, it follows that $G \circ f = 1/f \in L^1([a, b])$. Moreover it holds that $|G'(t)| \leq \text{const}$. Hence,

$$\left(\frac{1}{f}\right)' = (G \circ f)' = (G' \circ f)f' \in L^1([a, b])$$

since $f' \in L^1([a, b])$ by assumption, thus obtaining that $1/f \in W^{1,1}([a, b])$.

Now let us move on to prove 2). For this, we observe that there exist $\{f_n\}_n$, $\{g_n\}_n \subset C^\infty([a, b])$ such that

$$f_n \to f \quad \text{and} \quad g_n \to g \quad \text{in } W^{1,1}((a, b)). \tag{2.3.98}$$

From this, it also follows that

$$f_n \to f \quad \text{and} \quad g_n \to g \quad \text{in } L^\infty((a, b)), \tag{2.3.99}$$

in virtue of the Sobolev embedding $W^{1,1}((a, b)) \hookrightarrow L^\infty((a, b))$, which allows us also to conclude that $fg \in L^1((a, b))$. Thus, expoliting (2.3.98) and (2.3.99), we deduce that

$$(f_n g_n)' = f_n' g_n + f_n g_n' \to f'g + fg' \quad \text{in } L^1((a, b)). \tag{2.3.100}$$

Neverthless, since $f_n g_n \to fg$ in $L^1((a, b))$ as a consequence of (2.3.98) and (2.3.99), then it follows that $f_n g_n \to fg$ in $\mathcal{D}'((a, b))$ as well. This leads to

$$(f_n g_n)' \to (fg)' \quad \text{in } \mathcal{D}'((a, b)). \tag{2.3.101}$$

Therefore, combining (2.3.100) and (2.3.101) and by the uniqueness of limits in the distributional sense, we obtain that $(fg)' = f'g + fg' \in L^1((a, b))$, thus proving that $fg \in W^{1,1}((a, b))$. □

Lemma 2.3.18. *Let $\mathcal{N}$ be the function defined in* (2.3.89). *Then*

$$\mathcal{N}'(r) \geq V_1(r) + V_2(r) \tag{2.3.102}$$

for almost every $r \in (0, R_0)$, where

$$V_1(r) = \frac{2r\left[\left(\int_{\partial^+ B_r^+} t^{1-2s}\frac{|A\nabla W\cdot\nu|^2}{\mu}\,dS\right)\left(\int_{\partial^+ B_r^+} t^{1-2s}\mu W^2\,dS\right) - \left(\int_{\partial^+ B_r^+} t^{1-2s}(A\nabla W\cdot\nu)W\,dS\right)^2\right]}{\left(\int_{\partial^+ B_r^+} t^{1-2s}\mu W^2\,dS\right)^2}$$

and

$$V_2(r) = O(r^{-1+\bar{\delta}})\left(\mathcal{N}(r) + \frac{N-2s}{2}\right) \quad \text{as } r \to 0^+,$$

with $\bar{\delta}$ as in (2.3.91).

Proof. Exploiting (2.3.84), (2.3.85) and (2.3.90), we obtain that

$$\begin{aligned}\mathcal{N}'(r) &= \frac{E'(r)H(r) - H'(r)E(r)}{H^2(r)} = \frac{E'(r)H(r)}{H^2(r)} - \frac{H'(r)}{H^2(r)}\left(\frac{r}{2}H'(r) + H(r)O(r)\right)\\ &\geq \frac{2r\left[\left(\int_{\partial^+ B_r^+} t^{1-2s}\frac{|A\nabla W\cdot\nu|^2}{\mu}dS\right)\left(\int_{\partial^+ B_r^+} t^{1-2s}\mu W^2 dS\right) - \left(\int_{\partial^+ B_r^+} t^{1-2s}(A\nabla W\cdot\nu)W dS\right)^2\right]}{\left(\int_{\partial^+ B_r^+} t^{1-2s}\mu W^2 dS\right)^2}\\ &\quad + O(r^{-1+\bar{\delta}})\left(\mathcal{N}(r) + \frac{N-2s}{2}\right) + O(r) + O(1)\frac{1}{H(r)}\frac{1}{r^{N-2s}}\int_{\partial^+ B_r^+} t^{1-2s}(A\nabla W\cdot\nu)W\,dS\end{aligned} \tag{2.3.103}$$

as $r \to 0^+$, for a.e. $r \in (0, R_0)$. In order to estimate the last term in (2.3.103), we observe that

$$\begin{aligned}O(1)\frac{1}{H(r)}\frac{1}{r^{N-2s}}\int_{\partial^+ B_r^+} t^{1-2s}(A\nabla W\cdot\nu)W\,dS &= \frac{H'(r)}{H(r)}O(r) + O(r)\\ &= \mathcal{N}(r)O(1) + O(r),\end{aligned} \tag{2.3.104}$$

as $r \to 0^+$, by (2.3.84) and (2.3.85). Inserting (2.3.104) into (2.3.103), we obtain that

$$\begin{aligned}\mathcal{N}'(r) &\geq \frac{2r\left[\left(\int_{\partial^+ B_r^+} t^{1-2s}\frac{|A\nabla W\cdot\nu|^2}{\mu}dS\right)\left(\int_{\partial^+ B_r^+} t^{1-2s}\mu W^2 dS\right) - \left(\int_{\partial^+ B_r^+} t^{1-2s}(A\nabla W\cdot\nu)W dS\right)^2\right]}{\left(\int_{\partial^+ B_r^+} t^{1-2s}\mu W^2 dS\right)^2}\\ &\quad + \mathcal{N}(r)O(1) + O(r) + O(r^{-1+\bar{\delta}})\left(\mathcal{N}(r) + \frac{N-2s}{2}\right) \quad \text{as } r \to 0^+,\end{aligned}$$

which yields (2.3.102) in view of (2.3.97). □

Proposition 2.3.19. *Let $\mathcal{N}$ be the function defined in* (2.3.89). *Then there exists $C_1 > 0$ such that, for every $r \in (0, R_0]$,*

$$\mathcal{N}(r) \le C_1. \tag{2.3.105}$$

Moreover the limit

$$\gamma := \lim_{r\to 0^+} \mathcal{N}(r) \tag{2.3.106}$$

exists, is finite and nonnegative.

Proof. From Lemma 2.3.18, we deduce that $\mathcal{N}'(r) \ge V_2(r)$ a.e. in $(0, R_0)$, since $V_1(r) \ge 0$ as a consequence of Schwarz's inequality. Hence there exist $0 < \hat{R} < R_0$ and $C_2 > 0$ such that

$$\mathcal{N}'(r) \ge -C_2 r^{-1+\bar{\delta}} \left(\mathcal{N}(r) + \frac{N-2s}{2}\right), \tag{2.3.107}$$

for a.e. $r \in (0, \hat{R})$. Then

$$\frac{d}{dr}\left[\log\left(\mathcal{N}(r) + \frac{N-2s}{2}\right)\right] \ge -C_2 r^{-1+\bar{\delta}} \quad \text{a.e. in } (0, \hat{R}),$$

and, integrating the above inequality between $(r, \hat{R})$ with $r < \hat{R}$, we obtain the upper bound

$$\mathcal{N}(r) \le \left(\mathcal{N}(\hat{R}) + \frac{N-2s}{2}\right) e^{C_2 \frac{\hat{R}^{\bar{\delta}}}{\bar{\delta}}} - \frac{N-2s}{2} \quad \text{for all } r \in (0, \hat{R}),$$

which yields (2.3.105), in view of the continuity of $\mathcal{N}$ on $(0, R_0]$. From (2.3.107), we deduce that

$$\frac{d}{dr}\left[e^{C_2 \frac{r^{\bar{\delta}}}{\bar{\delta}}} \left(\mathcal{N}(r) + \frac{N-2s}{2}\right)\right] \ge 0 \quad \text{a.e. in } (0, \hat{R}),$$

hence

$$r \mapsto e^{C_2 \frac{r^{\bar{\delta}}}{\bar{\delta}}} \left(\mathcal{N}(r) + \frac{N-2s}{2}\right)$$

is a monotonically increasing function on the interval $(0, \hat{R})$, thus its limit as $r \to 0^+$ does exist, and the same holds true for the limit of the function $\mathcal{N}$. From (2.3.105) we can conclude that the limit $\gamma := \lim_{r\to 0^+} \mathcal{N}(r)$ is finite and it is nonnegative by Lemma 2.3.16. □

Lemma 2.3.20. *Let $\gamma = \lim_{r\to 0^+} \mathcal{N}(r)$. Then:*

(i) *there exists $k_1 > 0$ such that, for all $r \in (0, R_0]$,*

$$H(r) \le k_1 r^{2\gamma}; \tag{2.3.108}$$

(ii) *for any $\sigma > 0$, there exists $k_2(\sigma) > 0$ such that, for all $r \in (0, R_0)$,*

$$H(r) \ge k_2(\sigma) r^{2\gamma+\sigma}.$$

Proof. To prove (i), we write

$$\mathcal{N}'(r) = \beta_1(r) + \beta_2(r), \tag{2.3.109}$$

where

$$\beta_1(r) := \mathcal{N}'(r) + C_2 r^{-1+\bar{\delta}} \left(C_1 + \tfrac{N-2s}{2}\right) \geq 0 \quad \text{for a.e. } r \in (0, \hat{R}), \tag{2.3.110}$$

as a consequence of (2.3.107) and (2.3.105), and

$$\beta_2(r) := -C_2 r^{-1+\bar{\delta}} \left(C_1 + \tfrac{N-2s}{2}\right) \in L^1(0, \hat{R}). \tag{2.3.111}$$

Since $\mathcal{N} \in W^{1,1}_{\mathrm{loc}}((0, R_0])$ and by (2.3.109), it holds that

$$\begin{aligned} \mathcal{N}(r) - \mathcal{N}(\varepsilon) &= \int_\varepsilon^r \mathcal{N}'(s)\, ds = \int_\varepsilon^r \beta_1(s)\, ds + \int_\varepsilon^r \beta_2(s)\, ds \\ &= \int_0^r \chi_{(\varepsilon,r)} \beta_1(s)\, ds + \int_0^r \chi_{(\varepsilon,r)} \beta_2(s)\, ds \end{aligned} \tag{2.3.112}$$

for every $r \in (0, \hat{R})$. Passing to the limit as $\varepsilon \to 0^+$ into (2.3.112), taking into account (2.3.106), (2.3.110) and (2.3.111), we obtain that

$$\mathcal{N}(r) - \gamma = \int_0^r \mathcal{N}'(s)\, ds \quad \text{for all } r \in (0, \hat{R}). \tag{2.3.113}$$

From this, by (2.3.110), we easily deduce that

$$\mathcal{N}(r) - \gamma \geq -C_2 \left(C_1 + \tfrac{N-2s}{2}\right) \frac{r^{\bar{\delta}}}{\bar{\delta}} \quad \text{for all } r \in (0, \hat{R}). \tag{2.3.114}$$

By (2.3.85) we have that there exist a positive constant $C > 0$ and a suitable radius $\tilde{R}_0 > 0$ such that

$$\frac{H'(r)}{H(r)} \geq \frac{2\mathcal{N}(r)}{r} - C \geq \frac{2\gamma}{r} - 2C_2 \left(C_1 + \tfrac{N-2s}{2}\right) \frac{r^{-1+\bar{\delta}}}{\bar{\delta}} - C$$

for all $r \in (0, \min\{\tilde{R}_0, \hat{R}\})$. Integrating the above estimate we gain (2.3.108) for all $r \in (0, \min\{\tilde{R}_0, \hat{R}\})$. Taking into account that H is continuous and positive on $(0, R_0]$, we obtain (2.3.108) for all $r \in (0, R_0]$, since

$$\frac{H(r)}{r^{2\gamma}} \leq \max_{r \in [\min\{\tilde{R}_0, \hat{R}\}, R_0]} \frac{H(r)}{r^{2\gamma}}.$$

Now we move on to prove (ii). Since $\gamma = \lim_{r \to 0^+} \mathcal{N}(r)$, for any $\sigma > 0$ there exists $r_\sigma > 0$ such that, for any $r \in (0, r_\sigma)$,

$$\mathcal{N}(r) < \gamma + \frac{\sigma}{2},$$

and hence by (2.3.85)

$$\frac{H'(r)}{H(r)} = \frac{2\mathcal{N}(r)}{r} + O(1) < \frac{2\gamma + \sigma}{r} + \text{const},$$

up to taking r_σ smaller arguing as above. Integrating over the interval (r, r_σ) and taking into account that H is continuous and positive in $(0, R_0]$, we also complete the proof of the second statement. □

2.3.5 Blow-up analysis and local asymptotics

Let $W \in H^1_{\Gamma^+_{R_1}}(B^+_{R_1}, t^{1-2s}\,dz)$ be a nontrivial weak solution to (2.3.11). For every $\lambda \in (0, R_0)$, with R_0 being as in (2.3.46), let us define

$$w^\lambda(z) = \frac{W(\lambda z)}{\sqrt{H(\lambda)}}. \tag{2.3.115}$$

We have that w^λ is a weak solution to

$$\begin{cases} -\mathrm{div}\left(t^{1-2s}A(\lambda\,\cdot)\nabla w^\lambda\right) = 0 & \text{in } B^+_{R_1/\lambda}, \\ \lim\limits_{t\to 0^+}\left(t^{1-2s}A(\lambda\,\cdot)\nabla w^\lambda\cdot\nu\right) = \kappa_s\lambda^{2s}\tilde{h}(\lambda\cdot)\operatorname{Tr} w^\lambda & \text{on } \Gamma^-_{R_1/\lambda}, \\ w^\lambda = 0 & \text{on } \Gamma^+_{R_1/\lambda}. \end{cases} \tag{2.3.116}$$

Moreover we have that

$$\int_{\mathbb{S}^N_+} \theta_{N+1}^{1-2s}\mu(\lambda\theta)|w^\lambda(\theta)|^2\,dS = 1. \tag{2.3.117}$$

Lemma 2.3.21. *The family of functions $\{w^\lambda\}_{\lambda\in(0,R_0)}$ is bounded in $H^1(B_1^+, t^{1-2s}dz)$.*

Proof. By (2.3.89) and using (2.3.33), (2.3.37) and (2.3.38), we obtain that, for every $\lambda \in (0, R_0)$,

$$\begin{aligned} \mathcal{N}(\lambda) &= \frac{\lambda^{2s-N}}{H(\lambda)}\left(\int_{B^+_\lambda} t^{1-2s}A\nabla W\cdot\nabla W\;dz - \kappa_s\int_{\Gamma^-_\lambda}\tilde{h}|\operatorname{Tr} W|^2\;dy\right) \\ &\geq \frac{\lambda^{2s-N}}{H(\lambda)}\left[\frac{3}{8}\int_{B^+_\lambda} t^{1-2s}|\nabla W|^2\;dz - \kappa_s\tilde{S}_{N,s}\tilde{c}_{N,s,p}\lambda^{\bar{\varepsilon}}\bar{\alpha}_0\frac{2(N-2s)}{\lambda}\int_{\partial^+B^+_\lambda} t^{1-2s}\mu W^2 dS\right] \\ &\geq \frac{3}{8}\int_{B_1^+} t^{1-2s}|\nabla w^\lambda|^2\;dz - 2(N-2s)\kappa_s\tilde{S}_{N,s}\tilde{c}_{N,s,p}\lambda^{\bar{\varepsilon}}\bar{\alpha}_0 \\ &\geq \frac{3}{8}\int_{B_1^+} t^{1-2s}|\nabla w^\lambda|^2\;dz - \frac{N-2s}{4}, \end{aligned}$$

which together with (2.3.105) implies that $\left\{\|\nabla w^\lambda\|_{L^2(B_1^+,t^{1-2s}dz)}\right\}_{\lambda\in(0,R_0)}$ is bounded. From this and (2.3.117), the boundedness of $\{w^\lambda\}_{\lambda\in(0,R_0)}$ in $H^1(B_1^+, t^{1-2s}dz)$ follows by Lemma 2.3.3. □

We aim to prove strong convergence in $H^1(B_1^+, t^{1-2s}dz)$ of $\{w^\lambda\}$ along a proper vanishing sequence of λ's; to this purpose, we first need to establish the following doubling properties.

Lemma 2.3.22. *There exists $C_3 > 0$ such that*

$$\frac{1}{C_3}H(\lambda) \leq H(R\lambda) \leq C_3 H(\lambda), \tag{2.3.118}$$

$$\int_{B_R^+} t^{1-2s}|\nabla w^\lambda|^2 dz \le C_3 2^{N-2s} \int_{B_1^+} t^{1-2s}|\nabla w^{R\lambda}|^2 dz, \tag{2.3.119}$$

and

$$\int_{B_R^+} t^{1-2s}|w^\lambda|^2 dz \le C_3 2^{N+2-2s} \int_{B_1^+} t^{1-2s}|w^{R\lambda}|^2 dz. \tag{2.3.120}$$

for any $\lambda < R_0/2$ *and* $R \in [1,2]$.

Proof. From (2.3.85) we deduce that, for a.e. $r \in (0, R_0)$,

$$\frac{H'(r)}{H(r)} = \frac{2\mathcal{N}(r)}{r} + O(1) \quad \text{as } r \to 0^+.$$

Hence for all $r \in (0, \tilde{R}_0)$,

$$-C - \frac{N-2s}{r} \le \frac{H'(r)}{H(r)} \le C + \frac{2C_1}{r},$$

with $C > 0$ and $\tilde{R}_0$ as in the proof of Lemma 2.3.20, in virtue of (2.3.96) and (2.3.105). Integrating the above inequalities over the interval $(\lambda, R\lambda)$, with $R \in (1,2]$ and $\lambda < \tilde{R}_0/R$, we obtain that

$$2^{-(N-2s)} e^{-C\frac{\tilde{R}_0}{R}(R-1)} \le \frac{H(R\lambda)}{H(\lambda)} \le 4^{C_1} e^{C\frac{\tilde{R}_0}{R}(R-1)}. \tag{2.3.121}$$

The above chain of inequalities trivially extends to the case $R = 1$. Estimate (2.3.118) follows from (2.3.121) and the fact that H is continuous and strictly positive on $(0, R_0]$ (Lemmas 2.3.13 and 2.3.14). By scaling and (2.3.118), we easily deduce (2.3.119) as follows

$$\begin{aligned}
\int_{B_R^+} t^{1-2s}|\nabla w^\lambda|^2 \, dz &= \frac{\lambda^{2s-N}}{H(\lambda)} \int_{B_{R\lambda}^+} t^{1-2s}|\nabla W(z)|^2 \, dz \\
&= \frac{\lambda^2 R^{2-2s+N}}{H(\lambda)} \int_{B_1^+} t^{1-2s}|\nabla W(R\lambda z)|^2 \, dz = \frac{R^{N-2s} H(R\lambda)}{H(\lambda)} \int_{B_1^+} t^{1-2s}|\nabla w^{R\lambda}(z)|^2 \, dz \\
&\le R^{N-2s} C_3 \int_{B_1^+} t^{1-2s}|\nabla w^{R\lambda}(z)|^2 \, dz.
\end{aligned}$$

With a similar argument we obtain also (2.3.120). □

Lemma 2.3.23. *Let* w^λ *be as in* (2.3.115)*, with* $\lambda \in (0, R_0)$*. Then there exist* $M > 0$ *and* $\lambda_0 > 0$ *such that, for any* $\lambda \in (0, \lambda_0)$*, there exists* $R_\lambda \in [1,2]$ *such that*

$$\int_{\partial^+ B_{R_\lambda}^+} t^{1-2s}|\nabla w^\lambda|^2 \, dS \le M \int_{B_{R_\lambda}^+} t^{1-2s}|\nabla w^\lambda|^2 \, dz.$$

Proof. We recall that, by Lemma 2.3.21, the family $\{w^\lambda\}_{\lambda \in (0,R_0)}$ is bounded in the space $H^1(B_1^+, t^{1-2s}dz)$ and trivially

$$w^\lambda \in H^1_{\Gamma_1^+}(B_1^+, t^{1-2s}\, dz) \quad \text{for all } \lambda \in (0, R_0). \tag{2.3.122}$$

Moreover, by Lemma 2.3.22, we have that $\{w^\lambda\}_{\lambda\in(0,R_0/2)}$ is bounded in $H^1(B_2^+, t^{1-2s}dz)$, hence

$$\limsup_{\lambda\to 0^+} \int_{B_2^+} t^{1-2s}|\nabla w^\lambda|^2\,dz < +\infty. \tag{2.3.123}$$

For every $\lambda \in (0, R_0/2)$, let

$$f_\lambda(r) := \int_{B_r^+} t^{1-2s}|\nabla w^\lambda|^2\,dz.$$

Then f_λ is absolutely continuous in $[0, 2]$ with distributional derivative given by

$$f'_\lambda(r) = \int_{\partial^+ B_r^+} t^{1-2s}|\nabla w^\lambda|^2\,dS \quad \text{for almost every } r \in (0,2).$$

Let us suppose by contradiction that for any $M > 0$ there exists a sequence $\lambda_n \to 0^+$ such that

$$\int_{\partial^+ B_r^+} t^{1-2s}|\nabla w^{\lambda_n}|^2\,dS > M \int_{B_r^+} t^{1-2s}|\nabla w^{\lambda_n}|^2\,dz$$

for all $r \in [1, 2]$ and $n \in \mathbb{N}$, i.e.

$$f'_{\lambda_n}(r) > M f_{\lambda_n}(r) \tag{2.3.124}$$

for a.e. $r \in (1, 2)$ and any $n \in \mathbb{N}$. Integrating (2.3.124) over $[1, 2]$, we obtain that, for any $n \in \mathbb{N}$, $f_{\lambda_n}(2) > e^M f_{\lambda_n}(1)$, and hence

$$\liminf_{n\to+\infty} f_{\lambda_n}(1) \le \limsup_{n\to+\infty} f_{\lambda_n}(1) \le e^{-M} \limsup_{n\to+\infty} f_{\lambda_n}(2),$$

which implies that

$$\liminf_{\lambda\to 0^+} f_\lambda(1) \le e^{-M} \limsup_{\lambda\to 0^+} f_\lambda(2), \tag{2.3.125}$$

for all $M > 0$. From (2.3.125) and (2.3.123), letting $M \to +\infty$ we deduce that

$$\liminf_{\lambda\to 0^+} f_\lambda(1) = 0.$$

Then there exist a sequence $\tilde\lambda_n \to 0^+$ and some $w \in H^1(B_1^+, t^{1-2s}dz)$ such that $w^{\tilde\lambda_n} \rightharpoonup w$ in $H^1(B_1^+, t^{1-2s}dz)$ with

$$\lim_{n\to+\infty} \int_{B_1^+} t^{1-2s}|\nabla w^{\tilde\lambda_n}|^2 dz = 0.$$

However, by compactness of trace map $H^1(B_1^+, t^{1-2s}dz) \hookrightarrow\hookrightarrow L^2(\partial^+ B_1^+, t^{1-2s}dS)$, (2.3.117), (2.3.26), and weak lower semicontinuity of norms, we necessarily have that

$$\int_{B_1^+} t^{1-2s}|\nabla w|^2 dz = 0 \qquad \text{and} \qquad \int_{\partial^+ B_1^+} t^{1-2s} w^2 dS = 1.$$

Hence there exists $c \in \mathbb{R}$ such that $w \equiv c$ in B_1^+ and $c \neq 0$. Since $H^1_{\Gamma_1^+}(B_1^+, t^{1-2s}\,dz)$ is weakly closed in $H^1(B_1^+, t^{1-2s}\,dz)$, from (2.3.122) we deduce that

$$w \equiv c \in H^1_{\Gamma_1^+}(B_1^+, t^{1-2s}\,dz),$$

so that $0 = \operatorname{Tr} w\big|_{\Gamma_1^+} = c$, a contradiction. □

Lemma 2.3.24. *Let w^λ and R_λ be as in the statement of Lemma 2.3.23. Then there exists $\overline{M} > 0$ such that*

$$\int_{\partial^+ B_1^+} t^{1-2s} |\nabla w^{\lambda R_\lambda}|^2 \, dS \le \overline{M}$$

for any $\lambda \in (0, \min\{\lambda_0, R_0/2\})$.

Proof. We observe that, by scaling and (2.3.115),

$$\int_{\partial^+ B_1^+} t^{1-2s} |\nabla w^{\lambda R_\lambda}|^2 \, dS = \frac{R_\lambda^{1-N+2s} H(\lambda)}{H(\lambda R_\lambda)} \int_{\partial^+ B_{R_\lambda}^+} t^{1-2s} |\nabla w^\lambda|^2 \, dS,$$

so that, in view of Lemmas 2.3.22, 2.3.23 and 2.3.21, we have that

$$\begin{aligned}\int_{\partial^+ B_1^+} t^{1-2s} |\nabla w^{\lambda R_\lambda}|^2 \, dS &\le 2C_3 M \int_{B_{R_\lambda}^+} t^{1-2s} |\nabla w^\lambda|^2 \, dz \\ &\le 2^{1+N-2s} M C_3^2 \int_{B_1^+} t^{1-2s} |\nabla w^{\lambda R_\lambda}|^2 \, dz \le \overline{M} < +\infty,\end{aligned}$$

for any $\lambda \in (0, \min\{\lambda_0, R_0/2\})$. The proof is thereby complete. □

Proposition 2.3.25. *Let $W \in H^1_{\Gamma^+_{R_1}}(B^+_{R_1}, t^{1-2s}\, dz)$, $W \not\equiv 0$, be a nontrivial weak solution to (2.3.11). Let γ be as in Proposition 2.3.19. Then*

(i) *there exists $k_0 \in \mathbb{N}$ such that $\gamma = s + k_0$;*

(ii) *for any sequence $\lambda_n \to 0^+$, there exist a subsequence $\{\lambda_{n_k}\}$ and an eigenfunction ψ of problem (1.2.11) associated to the eigenvalue $\mu_{k_0} = (k_0 + s)(k_0 + N - s)$ such that $\|\psi\|_{L^2(\mathbb{S}^N_+, \theta_{N+1}^{1-2s} dS)} = 1$ and*

$$w^{\lambda_{n_k}}(z) = \frac{W(\lambda_{n_k} z)}{\sqrt{H(\lambda_{n_k})}} \to |z|^\gamma \psi\left(\frac{z}{|z|}\right)$$

strongly in $H^1(B_1^+, t^{1-2s} dz)$.

Proof. Let $w^\lambda \in H^1_{\Gamma_1^+}(B_1^+, t^{1-2s}\, dz)$ be as in (2.3.115) and R_λ as in Lemma 2.3.23. From Lemma 2.3.21 we deduce that the set $\{w^{\lambda R_\lambda}\}_{\lambda \in (0, \min\{\lambda_0, R_0/2\})}$ is bounded in the space $H^1(B_1^+, t^{1-2s} dz)$. Let us consider a sequence $\lambda_n \to 0^+$. Then there exist a subsequence $\{\lambda_{n_k}\}_k$ and $w \in H^1_{\Gamma_1^+}(B_1^+, t^{1-2s}\, dz)$ such that $w^{\lambda_{n_k} R_{\lambda_{n_k}}} \rightharpoonup w$ weakly in $H^1(B_1^+, t^{1-2s} dz)$. Moreover we have that

$$\int_{\partial^+ B_1^+} t^{1-2s} w^2 dS = 1 \tag{2.3.126}$$

by compactness of trace map $H^1(B_1^+, t^{1-2s} dz) \hookrightarrow\hookrightarrow L^2(\partial^+ B_1^+, t^{1-2s} dS)$, (2.3.117), and (2.3.26). This allows us to conclude that w is non-trivial.

We now claim strong convergence

$$w^{\lambda_{n_k} R_{\lambda_{n_k}}} \to w \quad \text{in } H^1(B_1^+, t^{1-2s}dz). \tag{2.3.127}$$

We note that $w^{\lambda_{n_k} R_{\lambda_{n_k}}}$ weakly solves (2.3.116) with $\lambda = \lambda_{n_k} R_{\lambda_{n_k}}$. Since

$$B_1^+ \subset B^+_{R_1/(\lambda_{n_k} R_{\lambda_{n_k}})}$$

for sufficiently large k, we then have that

$$\begin{aligned}\int_{B_1^+} & t^{1-2s} A(\lambda_{n_k} R_{\lambda_{n_k}} y) \nabla w^{\lambda_{n_k} R_{\lambda_{n_k}}}(z) \cdot \nabla \Phi(z)\, dz \\ &= \kappa_s (\lambda_{n_k} R_{\lambda_{n_k}})^{2s} \int_{\Gamma_1^-} \tilde{h}(\lambda_{n_k} R_{\lambda_{n_k}} y) \operatorname{Tr} w^{\lambda_{n_k} R_{\lambda_{n_k}}}(y) \operatorname{Tr} \Phi(y)\, dy \\ &\quad + \int_{\partial^+ B_1^+} \big(t^{1-2s} A(\lambda_{n_k} R_{\lambda_{n_k}} y) \nabla w^{\lambda_{n_k} R_{\lambda_{n_k}}}(z) \cdot \nu\big) \Phi(z)\, dS\end{aligned} \tag{2.3.128}$$

for sufficiently large k and for every $\Phi \in C_c^\infty(\overline{B_1^+} \setminus \Gamma_1^+)$, hence by density for every $\Phi \in H^1_{\Gamma_1^+}(B_1^+, t^{1-2s}dz)$. We want to pass to the limit in (2.3.128). To this aim, we observe that (2.3.20) implies that

$$\begin{aligned}&\left| \int_{B_1^+} t^{1-2s} (A(\lambda y) \nabla w^\lambda(z) - \nabla w(z)) \cdot \nabla \Phi(z)\, dz \right| \\ &\le \left| \int_{B_1^+} t^{1-2s} \nabla (w^\lambda - w) \cdot \nabla \Phi\, dz \right| + C\lambda \int_{B_1^+} t^{1-2s} |\nabla w^\lambda| |\nabla \Phi|\, dz \\ &\le \left| \int_{B_1^+} t^{1-2s} \nabla (w^\lambda - w) \cdot \nabla \Phi\, dz \right| + C\lambda \left(\int_{B_1^+} t^{1-2s} |\nabla w^\lambda|^2\, dz \right)^{1/2} \left(\int_{B_1^+} t^{1-2s} |\nabla \Phi|^2\, dz \right)^{1/2}\end{aligned} \tag{2.3.129}$$

for some $C > 0$ and for sufficiently small λ, and

$$\begin{aligned}&\lambda^{2s} \left| \int_{\Gamma_1^-} \tilde{h}(\lambda y) \operatorname{Tr} w^\lambda(y) \operatorname{Tr} \Phi(y)\, dy \right| \\ &\quad \le \lambda^{2s} \left(\int_{B_1'} |\operatorname{Tr} w^\lambda(y)|^{2^*(s)}\, dy \right)^{\frac{1}{2^*(s)}} \left(\int_{B_1'} |\operatorname{Tr} \Phi(y)|^{2^*(s)}\, dy \right)^{\frac{1}{2^*(s)}} \left(\int_{\Gamma_1^-} |\tilde{h}(\lambda y)|^{\frac{N}{2s}}\, dy \right)^{\frac{2s}{N}} \\ &\quad = O(1) \left(\int_{\Gamma_\lambda^-} |\tilde{h}(y)|^{\frac{N}{2s}}\, dy \right)^{\frac{2s}{N}} = o(1) \quad \text{as } \lambda \to 0^+,\end{aligned} \tag{2.3.130}$$

from Hölder's inequality, Lemma 2.3.4, Lemma 2.3.21 and (2.3.117), using that $\mu(\lambda y) \ge 1/4$ for all $\lambda \le R_0$. Taking $\lambda = \lambda_{n_k} R_{\lambda_{n_k}}$ in (2.3.129) and (2.3.130), and recalling that $w^{\lambda_{n_k} R_{\lambda_{n_k}}} \rightharpoonup w$ weakly in $H^1(B_1^+, t^{1-2s}dz)$ as $k \to +\infty$, we obtain that

$$\lim_{k \to +\infty} \int_{B_1^+} t^{1-2s} A(\lambda_{n_k} R_{\lambda_{n_k}} y) \nabla w^{\lambda_{n_k} R_{\lambda_{n_k}}}(z) \cdot \nabla \Phi(z)\, dz = \int_{B_1^+} t^{1-2s} \nabla w \cdot \nabla \Phi\, dz \tag{2.3.131}$$

and

$$\lim_{k\to+\infty}(\lambda_{n_k}R_{\lambda_{n_k}})^{2s}\int_{\Gamma_1^-}\tilde h(\lambda_{n_k}R_{\lambda_{n_k}}y)\,\mathrm{Tr}\,w^{\lambda_{n_k}R_{\lambda_{n_k}}}(y)\,\mathrm{Tr}\,\Phi(y)\,dy=0. \tag{2.3.132}$$

Thanks to (2.3.20), we also have that

$$\begin{aligned}
&\int_{\partial^+B_1^+}t^{1-2s}\Big(A(\lambda y)\nabla w^\lambda(z)\cdot\nu\Big)\Phi(z)\,dS \qquad (2.3.133)\\
&=\int_{\partial^+B_1^+}t^{1-2s}\frac{\partial w^\lambda}{\partial\nu}\Phi\,dS+\int_{\partial^+B_1^+}t^{1-2s}\Big((A(\lambda y)-\mathrm{Id}_N)\nabla w^\lambda(z)\cdot\nu\Big)\,\Phi(z)\,dS\\
&=\int_{\partial^+B_1^+}t^{1-2s}\frac{\partial w^\lambda}{\partial\nu}\Phi\,dS+O(\lambda)\left(\int_{\partial^+B_1^+}t^{1-2s}|\nabla w^\lambda|^2\,dS\right)^{1/2}\left(\int_{\partial^+B_1^+}t^{1-2s}\Phi^2\,dS\right)^{1/2}.
\end{aligned}$$

Moreover, from Lemma 2.3.24, up to a further subsequence, we have that

$$\frac{\partial w^{\lambda_{n_k}R_{\lambda_{n_k}}}}{\partial\nu}\rightharpoonup f\quad\text{weakly in } L^2(\partial^+B_1^+,t^{1-2s}dS) \tag{2.3.134}$$

for some $f\in L^2(\partial^+B_1^+,t^{1-2s}dS)$. Then, taking $\lambda=\lambda_{n_k}R_{\lambda_{n_k}}$ in (2.3.133) and passing to the limit as $k\to+\infty$, we obtain that

$$\lim_{k\to+\infty}\int_{\partial^+B_1^+}\big(t^{1-2s}A(\lambda_{n_k}R_{\lambda_{n_k}}y)\nabla w^{\lambda_{n_k}R_{\lambda_{n_k}}}(z)\cdot\nu\big)\Phi(z)\,dS=\int_{\partial^+B_1^+}t^{1-2s}f\Phi\,dS, \tag{2.3.135}$$

as a consequence of Lemma 2.3.24 and (2.3.134). Hence, passing to the limit as $k\to+\infty$ in (2.3.128) and combining (2.3.131), (2.3.132) and (2.3.135), we find that

$$\int_{B_1^+}t^{1-2s}\nabla w\cdot\nabla\Phi\,dz=\int_{\partial^+B_1^+}t^{1-2s}f\Phi\,dS\quad\text{for any }\Phi\in H^1_{\Gamma_1^+}(B_1^+,t^{1-2s}dz). \tag{2.3.136}$$

On the other hand, if we take $\Phi=w^{\lambda_{n_k}R_{\lambda_{n_k}}}$ in (2.3.128), we have that

$$\begin{aligned}
&\int_{B_1^+}t^{1-2s}A(\lambda_{n_k}R_{\lambda_{n_k}}y)\nabla w^{\lambda_{n_k}R_{\lambda_{n_k}}}(z)\cdot\nabla w^{\lambda_{n_k}R_{\lambda_{n_k}}}(z)\,dz\\
&\qquad=\kappa_s(\lambda_{n_k}R_{\lambda_{n_k}})^{2s}\int_{\Gamma_1^-}\tilde h(\lambda_{n_k}R_{\lambda_{n_k}}y)|\,\mathrm{Tr}\,w^{\lambda_{n_k}R_{\lambda_{n_k}}}(y)|^2\,dy\\
&\qquad\quad+\int_{\partial^+B_1^+}\big(t^{1-2s}A(\lambda_{n_k}R_{\lambda_{n_k}}z)\nabla w^{\lambda_{n_k}R_{\lambda_{n_k}}}(z)\cdot\nu\big)w^{\lambda_{n_k}R_{\lambda_{n_k}}}(z)\,dS.
\end{aligned}$$

From this, by (2.3.20), using (2.3.132) with $\Phi=w^{\lambda_{n_k}R_{\lambda_{n_k}}}$, (2.3.133) with $\lambda=\lambda_{n_k}R_{\lambda_{n_k}}$, we obtain that

$$\lim_{k\to+\infty}\int_{B_1^+}|\nabla w^{\lambda_{n_k}R_{\lambda_{n_k}}}|^2=\lim_{k\to+\infty}\int_{B_1^+}t^{1-2s}A(\lambda_{n_k}R_{\lambda_{n_k}}y)\nabla w^{\lambda_{n_k}R_{\lambda_{n_k}}}(z)\cdot\nabla w^{\lambda_{n_k}R_{\lambda_{n_k}}}(z)\,dz \tag{2.3.137}$$

$$=\lim_{k\to+\infty}\int_{\partial^+B_1^+}t^{1-2s}\frac{\partial w^{\lambda_{n_k}R_{\lambda_{n_k}}}}{\partial\nu}w^{\lambda_{n_k}R_{\lambda_{n_k}}}\,dS$$

$$=\int_{\partial^+B_1^+}t^{1-2s}fw\,dS=\int_{B_1^+}t^{1-2s}|\nabla w|^2,$$

where we used also that the trace operator from $H^1(B_1^+,t^{1-2s}dz)$ to $L^2(\partial^+B_1^+,t^{1-2s}dS)$ is compact, (2.3.134) and (2.3.136) with $\Phi=w$. The weak convergence $w^{\lambda_{n_k}R_{\lambda_{n_k}}}\rightharpoonup w$ in $H^1(B_1^+,t^{1-2s}dz)$ together with (2.3.137) imply (2.3.127).

For every $k\in\mathbb{N}$ and $r\in(0,1]$, let us define

$$E_k(r)=\frac{1}{r^{N-2s}}\Bigg[\int_{B_r^+}t^{1-2s}A(\lambda_{n_k}R_{\lambda_{n_k}}y)\nabla w^{\lambda_{n_k}R_{\lambda_{n_k}}}\cdot\nabla w^{\lambda_{n_k}R_{\lambda_{n_k}}}\,dz - \kappa_s\lambda_{n_k}^{2s}R_{\lambda_{n_k}}^{2s}\int_{\Gamma_r^-}\tilde h(\lambda_{n_k}R_{\lambda_{n_k}}y)|\operatorname{Tr}w^{\lambda_{n_k}R_{\lambda_{n_k}}}|^2dy\Bigg]$$

and

$$H_k(r)=\frac{1}{r^{N+1-2s}}\int_{\partial^+B_r^+}t^{1-2s}\mu(\lambda_{n_k}R_{\lambda_{n_k}}z)|w^{\lambda_{n_k}R_{\lambda_{n_k}}}(z)|^2dS.$$

We also define, for any $r\in(0,1]$,

$$E_w(r)=\frac{1}{r^{N-2s}}\int_{B_r^+}t^{1-2s}|\nabla w(z)|^2dz \tag{2.3.138}$$

and

$$H_w(r)=\frac{1}{r^{N+1-2s}}\int_{\partial^+B_r^+}t^{1-2s}w^2(z)\,dS. \tag{2.3.139}$$

By scaling, one can easily verify that

$$\mathcal{N}_k(r):=\frac{E_k(r)}{H_k(r)}=\frac{E(\lambda_{n_k}R_{\lambda_{n_k}}r)}{H(\lambda_{n_k}R_{\lambda_{n_k}}r)}=\mathcal{N}(\lambda_{n_k}R_{\lambda_{n_k}}r)\quad\text{for all } r\in(0,1]. \tag{2.3.140}$$

From (2.3.127), (2.3.20), and (2.3.130), it follows that, for any fixed $r\in(0,1]$,

$$E_k(r)\to E_w(r). \tag{2.3.141}$$

On the other hand, by compactness of the trace operator and (2.3.26), we also have, for any fixed $r\in(0,1]$,

$$H_k(r)\to H_w(r). \tag{2.3.142}$$

In order to prove that H_w is strictly positive, we argue by contradiction and assume that there exists $r \in (0,1]$ such that $H_w(r) = 0$; then r is a minimum point for H_w and hence, arguing as in Lemma 2.3.13, we obtain that necessarily

$$0 = H_w'(r) = 2r^{2s-N-1} \int_{B_r^+} t^{1-2s} |\nabla w(z)|^2 dz$$

and hence w is constant in B_r^+. From Lemma 2.3.3 we conclude that $w \equiv 0$ in B_r^+, which implies that $w \equiv 0$ in B_1^+ from classical unique continuation principles for second order elliptic equations, thus contradicting (2.3.126).

Hence $H_w(r) > 0$ for all $r \in (0,1]$, thus the function

$$\mathcal{N}_w : (0,1] \to \mathbb{R}, \quad \mathcal{N}_w(r) := \frac{E_w(r)}{H_w(r)}$$

is well defined and, arguing as in Lemma 2.3.17, one can easily prove that $\mathcal{N}_w$ belongs to $W^{1,1}_{\rm loc}((0,1])$, since E_w and H_w belong to $W^{1,1}_{\rm loc}((0,1])$. From (2.3.140), (2.3.141), (2.3.142) and Proposition 2.3.19, we deduce that

$$\mathcal{N}_w(r) = \lim_{k \to +\infty} \mathcal{N}(\lambda_{n_k} R_{\lambda_{n_k}} r) = \gamma \tag{2.3.143}$$

for all $r \in (0,1]$. Therefore $\mathcal{N}_w$ is constant in $(0,1]$, hence

$$\mathcal{N}_w'(r) = 0 \quad \text{for any } r \in (0,1). \tag{2.3.144}$$

Recalling the equation satisfied by w, i.e. (2.3.136), and arguing as in Lemma 2.3.18 with $A = \mathrm{Id}_N$ and $\tilde{h} \equiv 0$, we can prove that, for a.e. $r \in (0,1)$,

$$\mathcal{N}_w'(r) \geq \frac{2r \left[\left(\int_{\partial^+ B_r^+} t^{1-2s} \left| \partial_\nu w \right|^2 dS \right) \left(\int_{\partial^+ B_r^+} t^{1-2s} w^2 dS \right) - \left(\int_{\partial^+ B_r^+} t^{1-2s} \partial_\nu w \, w \, dS \right)^2 \right]}{\left(\int_{\partial^+ B_r^+} t^{1-2s} w^2 dS \right)^2}. \tag{2.3.145}$$

Combining (2.3.144) and (2.3.145) with Schwarz's inequality, we obtain that, for a.e. $r \in (0,1)$,

$$\left(\int_{\partial^+ B_r^+} t^{1-2s} \left| \partial_\nu w \right|^2 dS \right) \left(\int_{\partial^+ B_r^+} t^{1-2s} w^2 dS \right) - \left(\int_{\partial^+ B_r^+} t^{1-2s} \partial_\nu w \, w \, dS \right)^2 = 0.$$

Therefore, for a.e. $r \in (0,1)$, w and $\partial_\nu w$ have the same direction as vectors in the space $L^2(\partial^+ B_r^+, t^{1-2s} dS)$, so that there exists a function $\eta = \eta(r)$, defined a.e. in $(0,1)$, such that $\partial_\nu w(r\theta) = \eta(r) w(r\theta)$ for a.e. $r \in (0,1)$ and for all $\theta \in \mathbb{S}^N_+$. It is easy to verify that $\eta(r) = \frac{H_w'(r)}{2H_w(r)}$ for a.e $r \in (0,1)$, so that $\eta \in L^1_{\rm loc}((0,1])$, by Lemma 2.3.17. After integration we obtain that

$$w(r\theta) = e^{\int_1^r \eta(s) ds} w(\theta) = g(r) \psi(\theta), \quad r \in (0,1), \ \theta \in \mathbb{S}^N_+, \tag{2.3.146}$$

where $g(r) = e^{\int_1^r \eta(s)ds}$ and $\psi = w\big|_{\mathbb{S}^N_+}$. We observe that (2.3.126) implies that

$$\|\psi\|_{L^2(\mathbb{S}^N_+,\theta_{N+1}^{1-2s}dS)} = 1. \tag{2.3.147}$$

From the fact that $w \in H^1_{\Gamma_1^+}(B_1^+, t^{1-2s}\,dz)$ it follows that $\psi \in \mathcal{H}_0$, where $\mathcal{H}_0$ is defined in Section 1.2; moreover, plugging (2.3.146) into (2.3.136) we obtain that ψ satisfies (1.2.12) for some $\mu \in \mathbb{R}$, so that ψ is an eigenfunction of (1.2.11). Recalling (1.2.13) and letting $k_0 \in \mathbb{N}$ be such that $\mu = \mu_{k_0} = (k_0 + s)(k_0 + N - s)$, we can rewrite the equation $-\text{div}\left(t^{1-2s}\nabla w\right) = 0$ in polar coordinates exploiting [36, Lemma 2.1], thus obtaining, for all $r \in (0,1)$ and $\theta \in \mathbb{S}^N_+$,

$$\begin{aligned} 0 &= \frac{1}{r^N}(r^{N+1-2s}g')'\theta_{N+1}^{1-2s}\psi(\theta) + r^{-1-2s}g(r)\text{div}_{\mathbb{S}^N}(\theta_{N+1}^{1-2s}\nabla_{\mathbb{S}^N}\psi(\theta)) \\ &= \frac{1}{r^N}(r^{N+1-2s}g')'\theta_{N+1}^{1-2s}\psi(\theta) - r^{-1-2s}g(r)\theta_{N+1}^{1-2s}\mu_{k_0}\psi(\theta). \end{aligned}$$

Then $g(r)$ solves the equation

$$-\frac{1}{r^N}(r^{N+1-2s}g')' + \mu_{k_0}r^{-1-2s}g(r) = 0 \quad \text{in } (0,1)$$

i.e.

$$-g''(r) - \frac{N+1-2s}{r}g'(r) + \frac{\mu_{k_0}}{r^2}g(r) = 0 \quad \text{in } (0,1).$$

Hence $g(r)$ is of the form

$$g(r) = c_1 r^{k_0+s} + c_2 r^{s-N-k_0}$$

for some $c_1, c_2 \in \mathbb{R}$. Since $w \in H^1(B_1^+, t^{1-2s}\,dz)$ and the function $|z|^{-1}|z|^{s-N-k_0}\psi\big(\frac{z}{|z|}\big) \notin L^2(B_1^+, t^{1-2s}\,dz)$, from Lemma 2.3.3 we deduce that necessarily $c_2 = 0$ and $g(r) = c_1 r^{k_0+s}$. Moreover, from $g(1) = 1$, we obtain that $c_1 = 1$ and then

$$w(r\theta) = r^{k_0+s}\psi(\theta), \quad \text{for all } r \in (0,1) \text{ and } \theta \in \mathbb{S}^N_+. \tag{2.3.148}$$

Let us now consider the sequence $\{w^{\lambda_{n_k}}\}$. Up to a further subsequence still denoted by $\{w^{\lambda_{n_k}}\}$, we may suppose that $w^{\lambda_{n_k}} \rightharpoonup \overline{w}$ weakly in $H^1(B_1^+, t^{1-2s}\,dz)$ for some $\overline{w} \in H^1(B_1^+, t^{1-2s}\,dz)$ and that $R_{\lambda_{n_k}} \to \overline{R}$ for some $\overline{R} \in [1,2]$.

Strong convergence of $w^{\lambda_{n_k}R_{\lambda_{n_k}}}$ in $H^1(B_1^+, t^{1-2s}dz)$ implies that, up to a subsequence, both $w^{\lambda_{n_k}R_{\lambda_{n_k}}}$ and $|\nabla w^{\lambda_{n_k}R_{\lambda_{n_k}}}|$ are a.e. dominated by a $L^2(B_1^+, t^{1-2s}dz)$-function uniformly with respect to k. Moreover, by (2.3.118), up to a further subsequence, we may assume that the limit

$$\ell := \lim_{k\to+\infty} \frac{H(\lambda_{n_k}R_{\lambda_{n_k}})}{H(\lambda_{n_k})}$$

exists and is finite, with $\ell > 0$. Then, by the Dominated Convergence Theorem, we have that

$$\lim_{k\to+\infty}\int_{B_1^+} t^{1-2s} w^{\lambda_{n_k}}(z)v(z)dz = \lim_{k\to+\infty} R_{\lambda_{n_k}}^{N+2-2s}\int_{B^+_{1/R_{\lambda_{n_k}}}} t^{1-2s} w^{\lambda_{n_k}}(R_{\lambda_{n_k}}z)v(R_{\lambda_{n_k}}z)dz$$

$$= \lim_{k\to+\infty} R_{\lambda_{n_k}}^{N+2-2s}\sqrt{\frac{H(\lambda_{n_k}R_{\lambda_{n_k}})}{H(\lambda_{n_k})}}\int_{B_1^+} t^{1-2s}\chi_{B^+_{1/R_{\lambda_{n_k}}}}(z)\, w^{\lambda_{n_k}R_{\lambda_{n_k}}}(z)v(R_{\lambda_{n_k}}z)dz$$

$$= \overline{R}^{N+2-2s}\sqrt{\ell}\int_{B_1^+} t^{1-2s}\chi_{B^+_{1/\overline{R}}}(z)w(z)v(\overline{R}z)dz$$

$$= \overline{R}^{N+2-2s}\sqrt{\ell}\int_{B^+_{1/\overline{R}}} t^{1-2s}w(z)v(\overline{R}z)dz = \sqrt{\ell}\int_{B_1^+} t^{1-2s}w(z/\overline{R})v(z)dz$$

for any $v \in C^\infty(\overline{B_1^+})$. By density, the above convergence actually holds for all $v \in L^2(B_1^+, t^{1-2s}dz)$. This proves that $w^{\lambda_{n_k}} \rightharpoonup \sqrt{\ell}w(\cdot/\overline{R})$ weakly in $L^2(B_1^+, t^{1-2s}dz)$. Since we know that $w^{\lambda_{n_k}} \rightharpoonup \overline{w}$ weakly in $H^1(B_1^+, t^{1-2s}\, dz)$, we conclude that $\overline{w} = \sqrt{\ell}w(\cdot/\overline{R})$ and then $w^{\lambda_{n_k}} \rightharpoonup \sqrt{\ell}w(\cdot/\overline{R})$ in $H^1(B_1^+, t^{1-2s}\, dz)$. Moreover

$$\lim_{k\to+\infty}\int_{B_1^+} t^{1-2s}|\nabla w^{\lambda_{n_k}}(z)|^2dz = \lim_{k\to+\infty} R_{\lambda_{n_k}}^{N+2-2s}\int_{B^+_{1/R_{\lambda_{n_k}}}} t^{1-2s}|\nabla w^{\lambda_{n_k}}(R_{\lambda_{n_k}}z)|^2dz$$

$$= \lim_{k\to+\infty} R_{\lambda_{n_k}}^{N-2s}\frac{H(\lambda_{n_k}R_{\lambda_{n_k}})}{H(\lambda_{n_k})}\int_{B_1^+} t^{1-2s}\chi_{B^+_{1/R_{\lambda_{n_k}}}}|\nabla w^{\lambda_{n_k}R_{\lambda_{n_k}}}(z)|^2dz$$

$$= \overline{R}^{N-2s}\ell\int_{B_1^+} t^{1-2s}\chi_{B^+_{1/\overline{R}}}(z)|\nabla w(z)|^2dz = \overline{R}^{N-2s}\ell\int_{B^+_{1/\overline{R}}} t^{1-2s}|\nabla w(z)|^2dz$$

$$= \int_{B_1^+} t^{1-2s}\Big|\sqrt{\ell}\nabla\Big(w\Big(\frac{z}{\overline{R}}\Big)\Big)\Big|^2dz.$$

This shows that $w^{\lambda_{n_k}} \to \overline{w} = \sqrt{\ell}w(\cdot/\overline{R})$ strongly in $H^1(B_1^+, t^{1-2s}dz)$.

By (2.3.148) w is homogeneous of degree k_0+s, hence $\overline{w} = \sqrt{\ell}\,\overline{R}^{-k_0-s}w$. Furthermore (2.3.117), (2.3.26) and the strong convergence $w^{\lambda_{n_k}} \to \overline{w}$ in $L^2(\partial^+B_1^+, t^{1-2s}dS)$ imply that

$$1 = \int_{\partial^+B_1^+} t^{1-2s}\overline{w}^2\, dS = \ell\,\overline{R}^{-2k_0-2s}\int_{\partial^+B_1^+} t^{1-2s}w^2\, dS = \ell\,\overline{R}^{-2k_0-2s}$$

in view of (2.3.126), thus implying that $\overline{w} = w$.

It remains to prove part (i). By (2.3.148), (2.3.147) and the fact that ψ is an eigenfunction of (1.2.11) with associated eigenvalue $\mu_{k_0} = (k_0+s)(k_0+N-s)$, we have that

$$\int_{B_r^+} t^{1-2s}|\nabla w(z)|^2dz = \frac{r^{N+2k_0}}{N+2k_0}\big((k_0+s)^2+\mu_{k_0}\big) = (k_0+s)r^{N+2k_0}$$

and

$$\int_{\partial^+B_r^+} t^{1-2s}w^2dS = r^{N+1-2s}\int_{\mathbb{S}^N_+}\theta_{N+1}^{1-2s}w^2(r\theta)\, dS = r^{N+2k_0+1}.$$

Therefore, by (2.3.138), (2.3.139) and (2.3.143), it follows that

$$\gamma = \mathcal{N}_w(r) = \frac{E_w(r)}{H_w(r)} = \frac{r \displaystyle\int_{B_r^+} t^{1-2s} |\nabla w(z)|^2 dz}{\displaystyle\int_{\partial^+ B_r^+} t^{1-2s} w^2 dS} = k_0 + s.$$

This completes the proof. □

To complete the blow-up analysis and detect the sharp asymptotic behaviour of W at 0, it remains to describe the behavior of $H(\lambda)$ as $\lambda \to 0^+$.

Lemma 2.3.26. *Let $\gamma = \lim_{r\to 0} \mathcal{N}(r)$ be as in Proposition 2.3.19. Then the limit*

$$\lim_{r\to 0^+} r^{-2\gamma} H(r)$$

exists and is finite.

Proof. Thanks to (2.3.108), it is enough to show that the limit exists. From (2.3.85) and (2.3.113), we deduce that, a.e. in $(0, \hat{R})$,

$$\begin{aligned} \frac{d}{dr}\frac{H(r)}{r^{2\gamma}} &= \frac{H'(r)}{r^{2\gamma}} - 2\gamma\frac{H(r)}{r^{2\gamma+1}} = \frac{2}{r^{2\gamma+1}}\left[E(r) + H(r)O(r) - \gamma H(r)\right] \\ &= \frac{2H(r)}{r^{2\gamma+1}}\left[\mathcal{N}(r) - \gamma + O(r)\right] = \frac{2H(r)}{r^{2\gamma+1}}\left(\int_0^r \mathcal{N}'(s)\, ds + O(r)\right) \end{aligned} \tag{2.3.149}$$

as $r \to 0^+$. Using the same notation as in the proof of Lemma 2.3.20, we write $\mathcal{N}' = \beta_1 + \beta_2$ in $(0, \hat{R})$, with β_1 and β_2 defined as in (2.3.110) and (2.3.111). Integrating (2.3.149) between $(r, \hat{R})$, we obtain that

$$\begin{aligned} \frac{H(\hat{R})}{\hat{R}^{2\gamma}} - \frac{H(r)}{r^{2\gamma}} &= \int_r^{\hat{R}} \tfrac{2H(\rho)}{\rho^{2\gamma+1}}\left(\int_0^\rho \beta_1(\tau)\, d\tau\right) d\rho + \int_r^{\hat{R}} \tfrac{2H(\rho)}{\rho^{2\gamma+1}}\left(\int_0^\rho \beta_2(\tau)\, d\tau\right) d\rho \\ &\quad + \int_r^{\hat{R}} \tfrac{H(\rho)}{\rho^{2\gamma}} O(1)\, d\rho \\ &= \int_r^{\hat{R}} \tfrac{2H(\rho)}{\rho^{2\gamma+1}}\left(\int_0^\rho \beta_1(\tau)\, d\tau\right) d\rho - \int_r^{\hat{R}} \frac{H(\rho)}{\rho^{2\gamma}}\left(\frac{2C_4}{\bar{\delta}}\rho^{-1+\bar{\delta}} + O(1)\right) d\rho, \end{aligned}$$

where $C_4 := C_2\left(C_1 + \frac{N-2s}{2}\right)$. By (2.3.110) we have that

$$\lim_{r\to 0^+} \int_r^{\hat{R}} \frac{2H(\rho)}{\rho^{2\gamma+1}}\left(\int_0^\rho \alpha_1(\tau)\, d\tau\right) d\rho \quad \text{exists.}$$

On the other hand, estimate (2.3.108) ensures that

$$\rho \mapsto \frac{H(\rho)}{\rho^{2\gamma}}\left(-\frac{2C_4}{\bar{\delta}}\rho^{-1+\bar{\delta}} + O(1)\right) \in L^1(0, \hat{R}),$$

so that the limit $\lim_{r\to 0^+}\int_r^{\hat R}\frac{H(\rho)}{\rho^{2\gamma}}\left(-\frac{2C_4}{\bar\delta}\rho^{-1+\bar\delta}+O(1)\right)d\rho$ exists and is finite. The lemma is thereby proved. □

The next step is to prove that the limit $\lim_{r\to 0^+} r^{-2\gamma}H(r)$ is actually strictly positive. To this aim, we first define the Fourier coefficients associated with W, with respect to the orthonormal basis (1.2.14) of $L^2(\mathbb{S}^N_+,\theta_{N+1}^{1-2s}dS)$, as

$$\varphi_{k,m}(\lambda)=\int_{\mathbb{S}^N_+}\theta_{N+1}^{1-2s}W(\lambda\theta)Y_{k,m}(\theta)dS,\quad \lambda\in(0,R_1],\ k\in\mathbb{N},\ m=1,\dots,M_k. \tag{2.3.150}$$

We also define

$$\Upsilon_{k,m}(\lambda)=-\int_{B^+_\lambda}t^{1-2s}(A-\mathrm{Id}_{N+1})\nabla W(z)\cdot\frac{\nabla_{\mathbb{S}^N}Y_{k,m}(z/|z|)}{|z|}\,dz \tag{2.3.151}$$
$$+\kappa_s\int_{\Gamma^-_\lambda}\tilde h(y)\mathrm{Tr}W(y)\mathrm{Tr}Y_{k,m}\left(\frac{y}{|y|}\right)dy+\int_{\partial^+B^+_\lambda}t^{1-2s}(A-\mathrm{Id}_{N+1})\nabla W\cdot\frac{z}{|z|}Y_{k,m}\left(\frac{z}{|z|}\right)dS,$$

for a.e. $\lambda\in(0,R_1]$, $k\in\mathbb{N}$ and $m\in\{1,2,...,M_k\}$.

Lemma 2.3.27. *Let k_0 be as in Proposition 2.3.25. Then, for all $m\in\{1,2,\dots,M_{k_0}\}$ and $R\in(0,R_0]$,*

$$\varphi_{k_0,m}(\lambda)=\lambda^{k_0+s}\Bigg(R^{-k_0-s}\varphi_{k_0,m}(R)+\frac{(k_0+s)R^{-N-2k_0}}{N+2k_0}\int_0^R\rho^{k_0+s-1}\Upsilon_{k_0,m}(\rho)\,d\rho$$
$$+\frac{N-s+k_0}{N+2k_0}\int_\lambda^R\rho^{-N-1+s-k_0}\Upsilon_{k_0,m}(\rho)\,d\rho\Bigg)+O(\lambda^{k_0+s+\bar\delta})\quad\text{as }\lambda\to0^+, \tag{2.3.152}$$

where $\bar\delta$ is defined in (2.3.91).

Proof. Let $k\in\mathbb{N}$ and $m\in\{1,2,...,M_k\}$. Testing (2.3.11) with $\phi=\frac{\omega(|z|)}{|z|^{N+1-2s}}Y_{k,m}(z/|z|)$ for any test function $\omega\in C^\infty_c(0,R_1)$ and using (1.2.12), we can easily verify that $\varphi_{k,m}$ solves the following second order differential equation

$$-\varphi''_{k,m}(\lambda)-\frac{N+1-2s}{\lambda}\varphi'_{k,m}(\lambda)+\frac{\mu_k}{\lambda^2}\varphi_{k,m}(\lambda)=\zeta_{k,m}(\lambda)\quad\text{in }(0,R_1) \tag{2.3.153}$$

in a distributional sense, with μ_k as in (1.2.13), where the distribution $\zeta_{k,m}\in\mathcal{D}'(0,R_1)$ is defined by

$$_{\mathcal{D}'(0,R_1)}\langle\zeta_{k,m},\omega\rangle_{\mathcal{D}(0,R_1)}=\kappa_s\int_0^{R_1}\frac{\omega(\lambda)}{\lambda^{2-2s}}\left(\int_{\mathbb{S}^{N-1}_-}\tilde h(\lambda\theta')\,\mathrm{Tr}\,W(\lambda\theta')Y_{k,m}(\theta',0)dS'\right)d\lambda$$
$$-\int_{B^+_{R_1}}t^{1-2s}(A-\mathrm{Id}_{N+1})\nabla W\cdot\nabla\left(\omega(|z|)|z|^{-N-1+2s}Y_{k,m}(z/|z|)\right)dz$$

for any $\omega \in C_c^\infty(0, R_1)$ (we refer to Section 2.1 for the definition of $\mathbb{S}^{N-1}_-$).

Letting $\Upsilon_{k,m}$ be as in (2.3.151), by direct calculations we have that $\Upsilon_{k,m} \in L^1(0, R_1)$ and

$$\Upsilon'_{k,m}(\lambda) = \lambda^{N+1-2s}\zeta_{k,m}(\lambda) \qquad \text{in } \mathcal{D}'(0, R_1). \tag{2.3.154}$$

In view of (2.3.154) and (1.2.13), we have that (2.3.153) is equivalent to

$$-\left(\lambda^{N+1+2k}\left(\lambda^{-k-s}\varphi_{k,m}\right)'\right)' = \lambda^{k+s}\Upsilon'_{k,m} \quad \text{in } \mathcal{D}'(0, R_1).$$

Integrating the above equation, we obtain that, for every $R \in (0, R_1]$, $k \in \mathbb{N}$ and $m \in \{1, 2, \dots, M_k\}$, there exists a real number $c_{k,m}(R)$ (depending also on R) such that

$$\begin{aligned}\left(\lambda^{-k-s}\varphi_{k,m}(\lambda)\right)' = & -\lambda^{-N-1+s-k}\Upsilon_{k,m}(\lambda) \\ & - (k+s)\lambda^{-N-1-2k}\left(c_{k,m}(R) + \int_\lambda^R \rho^{k+s-1}\Upsilon_{k,m}(\rho)\,d\rho\right),\end{aligned} \tag{2.3.155}$$

in the sense of distributions in $(0, R_1)$. From (2.3.155) we infer that $\varphi_{k,m} \in W^{1,1}_{\rm loc}((0, R_1])$, thus a new integration leads to

$$\begin{aligned}\varphi_{k,m}(\lambda) = \lambda^{k+s}&\left(\frac{\varphi_{k,m}(R)}{R^{k+s}} - \frac{(k+s)c_{k,m}(R)}{(N+2k)R^{N+2k}} + \frac{N+k-s}{N+2k}\int_\lambda^R \rho^{-N-k+s-1}\Upsilon_{k,m}(\rho)\,d\rho\right) \\ & + \frac{(k+s)\lambda^{-N-k+s}}{N+2k}\left(c_{k,m}(R) + \int_\lambda^R \rho^{k+s-1}\Upsilon_{k,m}(\rho)\,d\rho\right)\end{aligned} \tag{2.3.156}$$

for all $\lambda \in (0, R_1]$. From now on, we fix k_0 as in Proposition 2.3.25, R_0 as in (2.3.46), and $m \in \{1, 2, \dots, M_{k_0}\}$. We prove that

$$\int_0^{R_0} \rho^{-N-k_0+s-1}|\Upsilon_{k_0,m}(\rho)|\,d\rho < +\infty. \tag{2.3.157}$$

To this purpose, exploiting (2.3.20) and using Hölder's inequality, one can estimate the first term in (2.3.151) for all $\rho \in (0, R_0)$ as follows

$$\begin{aligned}&\left|\int_{B_\rho^+} t^{1-2s}(A - \mathrm{I}_{N+1})\nabla W(z) \cdot \frac{\nabla_{\mathbb{S}^N} Y_{k_0,m}(z/|z|)}{|z|}dz\right| \\ &\qquad \le \text{const}\sqrt{\int_{B_\rho^+} t^{1-2s}|\nabla W|^2 dz} \cdot \sqrt{\int_{B_\rho^+} t^{1-2s}|\nabla_{\mathbb{S}^N} Y_{k_0,m}(z/|z|)|^2 dz} \\ &\qquad =: \text{const}\, I_1(\rho)\cdot I_2(\rho),\end{aligned} \tag{2.3.158}$$

where

$$\begin{aligned}I_1(\rho) = \sqrt{\rho^{N+2-2s}\int_{B_1^+} t^{1-2s}|\nabla W(\rho z)|^2 dz} = \rho^{\frac{N-2s}{2}}\sqrt{H(\rho)}\sqrt{\int_{B_1^+} t^{1-2s}|\nabla w^\rho(z)|^2 dz} \\ \le \text{const}\,\rho^{\frac{N-2s}{2}}\sqrt{H(\rho)},\end{aligned} \tag{2.3.159}$$

as a consequence of Lemma 2.3.21, and

$$I_2(\rho) = \sqrt{\int_0^\rho \tau^{N+1-2s}\left(\int_{\mathbb{S}^N_+} \theta_{N+1}^{1-2s}|\nabla_{\mathbb{S}^N} Y_{k_0,m}(\theta)|^2 dS\right) d\tau} = \frac{\sqrt{\mu_{k_0}}}{\sqrt{N+2-2s}}\rho^{\frac{N+2-2s}{2}}, \tag{2.3.160}$$

due to (1.2.12) and taking into account that elements of (1.2.14) have $L^2(\mathbb{S}^N_+, \theta_{N+1}^{1-2s}dS)$-norm equals 1. Combining (2.3.158), (2.3.159), (2.3.160), and (2.3.108) we obtain that, for every $R \in (0, R_0]$,

$$\int_0^R \rho^{-N-1+s-k_0}\left|\int_{B_\rho^+} t^{1-2s}(A - \mathrm{I}_{N+1})\nabla W(z)\cdot \frac{\nabla_{\mathbb{S}^N} Y_{k_0,m}(z/|z|)}{|z|}dz\right| d\rho$$
$$\leq \text{const} \int_0^R \rho^{-s-k_0}\sqrt{H(\rho)}\, ds \leq \text{const}\, R. \tag{2.3.161}$$

Moreover, as regards the second term in (2.3.151), Hölder's inequality implies that

$$\left|\int_{\Gamma_\lambda^-} \tilde h(y)\,\mathrm{Tr}\, W(y)\,\mathrm{Tr}\, Y_{k_0,m}\big(\tfrac{y}{|y|}\big)\, dy\right| \leq \sqrt{\int_{\Gamma_\lambda^-} |\tilde h||\,\mathrm{Tr}\, W|^2 dy}\cdot\sqrt{\int_{\Gamma_\lambda^-} |\tilde h(y)||\,\mathrm{Tr}\, Y_{k_0,m}\big(\tfrac{y}{|y|}\big)|^2 dy}. \tag{2.3.162}$$

Arguing as in (2.3.35) and using homogeneity of the function $Y_{k_0,m}(y/|y|)$, we have that, for all $\rho \in (0, R_0)$,

$$\sqrt{\int_{\Gamma_\rho^-} |\tilde h(y)||\,\mathrm{Tr}\, Y_{k_0,m}\big(\tfrac{y}{|y|}\big)|^2 dy} \leq \sqrt{\tilde c_{N,s,p}}\, \|\tilde h\|^{1/2}_{L^p(\Gamma_{R_1}^-)}\rho^{\bar\varepsilon/2}\left(\int_{\Gamma_\rho^-} |\,\mathrm{Tr}\, Y_{k_0,m}\big(\tfrac{y}{|y|}\big)|^{2^*(s)}\, dy\right)^{\frac{1}{2^*(s)}}$$
$$= \sqrt{\tilde c_{N,s,p}}\, \|\tilde h\|^{1/2}_{L^p(\Gamma_{R_1}^-)}\rho^{\frac{\bar\varepsilon+N-2s}{2}}\left(\int_{\Gamma_1^-} |\,\mathrm{Tr}\, Y_{k_0,m}\big(\tfrac{y}{|y|}\big)|^{2^*(s)}\, dy\right)^{\frac{1}{2^*(s)}}.$$

Furthermore, using (2.3.32), and (2.3.105), we deduce that, for all $\rho \in (0, R_0)$,

$$\sqrt{\int_{\Gamma_\rho^-} |\tilde h||\,\mathrm{Tr}\, W|^2 dy} \leq \sqrt{\tilde c_{N,s,p}\|\tilde h\|_{L^p(\Gamma_{R_1}^-)}\rho^{\bar\varepsilon}\left(\int_{\Gamma_\rho^-} |\mathrm{Tr} W|^{2^*(s)}\, dy\right)^{2/2^*(s)}}$$
$$\leq \sqrt{\frac{\tilde c_{N,s,p}}{\tilde C_{N,s}}\|\tilde h\|_{L^p(\Gamma_{R_1}^-)}\rho^{\bar\varepsilon+N-2s}H(\rho)\left(\mathcal{N}(\rho) + \frac{N-2s}{2}\right)}$$
$$\leq \text{const}\,\rho^{\frac{N-2s+\bar\varepsilon}{2}}\sqrt{H(\rho)}.$$

Putting the above estimates together and recalling (2.3.108), we conclude that, for every $R \in (0, R_0]$,

$$\int_0^R \rho^{-N-k_0+s-1}\left|\int_{\Gamma_\rho^-} \tilde h(y)\mathrm{Tr}W(y)\mathrm{Tr}Y_{k_0,m}\big(\tfrac{y}{|y|}\big)\, dy\right| d\rho$$
$$\leq \text{const}\int_0^R \rho^{-1+\bar\varepsilon-k_0-s}\sqrt{H(\rho)}\, d\rho \leq \text{const}\, R^{\bar\varepsilon}. \tag{2.3.163}$$

In order to estimate the last term in (2.3.151), we observe that

$$\int_{B_\lambda^+} t^{1-2s} \big|Y_{k_0,m}\big(\tfrac{z}{|z|}\big)\big|^2 dz = \int_0^\lambda \tau^{N+1-2s}\bigg(\int_{\mathbb{S}^N_+} \theta_{N+1}^{1-2s}|Y_{k_0,m}(\theta)|^2 dS\bigg)\,d\tau$$
$$= \frac{\lambda^{N+2-2s}}{N+2-2s}. \tag{2.3.164}$$

Hence, thanks to (2.3.20), Hölder inequality, (2.3.159), (2.3.164) and (2.3.108), integrating by parts, we have that, for every $R \in (0, R_0]$,

$$\int_0^R \rho^{-N+s-1-k_0}\bigg|\int_{\partial^+ B_\rho^+} t^{1-2s}(A - \mathrm{Id}_{N+1})\nabla W \cdot \frac{z}{|z|} Y_{k_0,m}\big(\tfrac{z}{|z|}\big)\,dS\bigg|\,d\rho \tag{2.3.165}$$
$$\leq \mathrm{const} \int_0^R \rho^{-N+s-k_0}\left(\int_{\partial^+ B_\rho^+} t^{1-2s}|\nabla W|\big|Y_{k_0,m}\big(\tfrac{z}{|z|}\big)\big|\,dS\right)d\rho$$
$$= \mathrm{const}\bigg(R^{-N+s-k_0}\int_{B_R^+} t^{1-2s}|\nabla W|\big|Y_{k_0,m}\big(\tfrac{z}{|z|}\big)\big|\,dz$$
$$+ (N + k_0 - s)\int_0^R \rho^{-N+s-1-k_0}\bigg(\int_{B_\rho^+} t^{1-2s}|\nabla W|\big|Y_{k_0,m}\big(\tfrac{z}{|z|}\big)\big|\,dz\bigg)d\rho\bigg)$$
$$\leq \mathrm{const}\bigg(R^{1-s-k_0}\sqrt{H(R)} + \int_0^R \rho^{-s-k_0}\sqrt{H(\rho)}\,d\rho\bigg) \leq \mathrm{const}\,R.$$

Thus from (2.3.151), (2.3.161), (2.3.163) and (2.3.165) it follows that, for every $R \in (0, R_0]$,

$$\int_0^R \rho^{-N-k_0+s-1}|\Upsilon_{k_0,m}(\rho)|\,d\rho \leq \mathrm{const}\,R^{\bar\delta} \tag{2.3.166}$$

where $\bar\delta$ is defined in (2.3.91). From (2.3.166), it immediately follows (2.3.157).

From (2.3.157) we infer that, for every $R \in (0, R_0]$,

$$\lambda^{k_0+s}\bigg(\frac{\varphi_{k_0,m}(R)}{R^{k_0+s}} - \frac{(k_0+s)c_{k_0,m}(R)}{(N+2k_0)R^{N+2k_0}} + \frac{N+k_0-s}{N+2k_0}\int_\lambda^R \rho^{-N-k_0+s-1}\Upsilon_{k_0,m}(\rho)\,d\rho\bigg)$$
$$= O(\lambda^{k_0+s}) = o(\lambda^{-N-k_0+s}) \quad \text{as } \lambda \to 0^+. \tag{2.3.167}$$

Now we prove that, for every $R \in (0, R_0]$,

$$c_{k_0,m}(R) + \int_0^R \rho^{k_0+s-1}\Upsilon_{k_0,m}(\rho)\,d\rho = 0. \tag{2.3.168}$$

To this aim, first we observe that

$$\int_0^{R_0} \rho^{k_0+s-1}|\Upsilon_{k_0,m}(\rho)|d\rho < +\infty, \tag{2.3.169}$$

as a direct consequence of (2.3.157), since $k_0 + s - 1 > -N - k_0 + s - 1$. Suppose by contradiction that (2.3.168) does not hold true for some $R \in (0, R_0]$; then from (2.3.156), (2.3.167) and (2.3.169), we would have that

$$\varphi_{k_0,m}(\lambda) \sim \frac{(k_0+s)\lambda^{-N-k_0+s}}{N+2k_0}\left(c_{k_0,m}(R) + \int_0^R \rho^{k_0+s-1}\Upsilon_{k_0,m}(\rho)\,d\rho\right) \quad \text{as } \lambda \to 0^+,$$

and hence

$$\int_0^{R_0} \lambda^{N-1-2s}|\varphi_{k_0,m}(\lambda)|^2 d\lambda = +\infty.$$

On the other hand, by (2.3.150), we have that

$$\begin{aligned}\int_0^{R_0} \lambda^{N-1-2s}|\varphi_{k_0,m}(\lambda)|^2 d\lambda &\le \int_0^{R_0} \lambda^{N-1-2s}\left(\int_{\mathbb{S}^N_+} \theta_{N+1}^{1-2s}|W(\lambda\theta)|^2 dS\right) d\lambda \\ &= \int_{B^+_{R_0}} t^{1-2s}\frac{W^2(z)}{|z|^2}dz < \infty,\end{aligned}$$

as a consequence of Lemma 2.3.3, giving rise to a contradiction. Hence (2.3.168) is proved. From (2.3.168) and (2.3.166) we deduce that, for every $R \in (0, R_0]$,

$$\begin{aligned}\left|\lambda^{-N-k_0+s}\left(c_{k_0,m}(R) + \int_\lambda^R \rho^{k_0+s-1}\Upsilon_{k_0,m}(\rho)\,d\rho\right)\right| &= \lambda^{-N+s-k_0}\left|\int_0^\lambda \rho^{k_0+s-1}\Upsilon_{k_0,m}(\rho)\,d\rho\right| \\ \le \lambda^{-N+s-k_0}\int_0^\lambda \rho^{N+2k_0}|\rho^{-N-1+s-k_0}\Upsilon_{k_0,m}(\rho)|\,d\rho& \\ \le \lambda^{k_0+s}\int_0^\lambda \rho^{-N-1+s-k_0}|\Upsilon_{k_0,m}(\rho)|\,d\rho = O(\lambda^{k_0+s+\bar\delta}) \quad \text{as } \lambda \to 0^+.&\end{aligned}$$

Combining this last information with (2.3.168) and (2.3.156), we finally obtain (2.3.152). □

Using Lemma 2.3.27, we now prove that $\lim_{r\to 0^+} r^{-2\gamma}H(r) = \lim_{r\to 0^+} r^{-2(k_0+s)}H(r) > 0$.

Lemma 2.3.28. *Let* $\gamma = \lim_{r\to 0}\mathcal{N}(r)$ *be as in Proposition 2.3.19. Then*

$$\lim_{r\to 0^+} r^{-2\gamma}H(r) > 0.$$

Proof. By (2.3.26) and using the Parseval identity we have that

$$\begin{aligned}H(\lambda) &= \int_{\mathbb{S}^N_+} \theta_{N+1}^{1-2s}\mu(\lambda\theta)|W(\lambda\theta)|^2 dS \qquad (2.3.170)\\ &= (1+O(\lambda))\int_{\mathbb{S}^N_+} \theta_{N+1}^{1-2s}|W(\lambda\theta)|^2 dS = (1+O(\lambda))\sum_{k=0}^{\infty}\sum_{m=1}^{M_k}|\varphi_{k,m}(\lambda)|^2.\end{aligned}$$

Let $k_0 \in \mathbb{N}$ be as in Proposition 2.3.25, thus $\gamma = k_0 + s$. We argue by contradiction, assuming that

$$\lim_{\lambda\to 0^+} \lambda^{-2\gamma} H(\lambda) = 0. \tag{2.3.171}$$

Hence from (2.3.170) it follows that $\lim_{\lambda\to 0^+} \lambda^{-(k_0+s)}\varphi_{k_0,m}(\lambda) = 0$ for any $m \in \{1, 2, \dots, M_{k_0}\}$. This, together with (2.3.157) and Lemma 2.3.27, leads to

$$R^{-k_0-s}\varphi_{k_0,m}(R) + \frac{(k_0+s)R^{-N-2k_0}}{N+2k_0}\int_0^R \rho^{k_0+s-1}\Upsilon_{k_0,m}(\rho)\,d\rho \\ + \frac{N-s+k_0}{N+2k_0}\int_0^R \rho^{-N-1+s-k_0}\Upsilon_{k_0,m}(\rho)\,d\rho = 0, \tag{2.3.172}$$

for all $m \in \{1, 2, \dots, M_{k_0}\}$ and for every $R \in (0, R_0]$. From (2.3.172), (2.3.152) and (2.3.166) it follows that

$$\varphi_{k_0,m}(\lambda) = -\lambda^{k_0+s}\frac{N-s+k_0}{N+2k_0}\int_0^\lambda \rho^{-N-1+s-k_0}\Upsilon_{k_0,m}(\rho)\,d\rho + O(\lambda^{k_0+s+\bar\delta}) = O(\lambda^{k_0+s+\bar\delta})$$

as $\lambda \to 0^+$ for all $m \in \{1, 2, \dots, M_{k_0}\}$. Hence

$$\sqrt{H(\lambda)}\,(w^\lambda, \psi)_{L^2(\mathbb{S}^N_+,\theta_{N+1}^{1-2s}dS)} = O(\lambda^{k_0+s+\bar\delta}) \quad \text{as } \lambda \to 0^+ \tag{2.3.173}$$

for every $\psi \in \operatorname{span}\{Y_{k_0,m} : m = 1, \dots, M_{k_0}\}$. From Lemma 2.3.20-(ii), $\sqrt{H(\lambda)} \geq \sqrt{k_2(\bar\delta)}\lambda^{k_0+s+\frac{\bar\delta}{2}}$ for λ small, so that (2.3.173) yields

$$(w^\lambda, \psi)_{L^2(\mathbb{S}^N_+,\theta_{N+1}^{1-2s}dS)} = O(\lambda^{\bar\delta/2}) \quad \text{as } \lambda \to 0^+ \tag{2.3.174}$$

for every $\psi \in \operatorname{span}\{Y_{k_0,m} : m = 1, \dots, M_{k_0}\}$. On the other hand, by Proposition 2.3.25 and continuity of the trace map from $H^1(B_1^+, t^{1-2s}dz)$ into $L^2(\partial^+B_1^+, t^{1-2s}dS)$, for any sequence $\lambda_n \to 0^+$, there exist a subsequence $\{\lambda_{n_k}\}$ and $\psi_0 \in \operatorname{span}\{Y_{k_0,m} : m = 1, \dots, M_{k_0}\}$ such that

$$\|\psi_0\|_{L^2(\mathbb{S}^N_+,\theta_{N+1}^{1-2s}dS)} = 1 \quad \text{and} \quad w^{\lambda_{n_k}} \to \psi_0 \quad \text{in } L^2(\mathbb{S}^N_+, \theta_{N+1}^{1-2s}dS). \tag{2.3.175}$$

From (2.3.174) and (2.3.175) we deduce that

$$0 = \lim_{k\to+\infty}(w^{\lambda_{n_k}}, \psi_0)_{L^2(\mathbb{S}^N_+,\theta_{N+1}^{1-2s}dS)} = \|\psi_0\|^2_{L^2(\mathbb{S}^N_+,\theta_{N+1}^{1-2s}dS)} = 1,$$

thus reaching a contradiction. □

Theorem 2.3.29. *Let $k_0 \in \mathbb{N}$ be as in Proposition 2.3.25. Let $M_{k_0} \in \mathbb{N} \setminus \{0\}$ be the multiplicity of the eigenvalue $\mu_{k_0} = (k_0+s)(k_0+N-s)$ and let $\{Y_{k_0,m}\}_{m=1,\dots,M_{k_0}}$ be a $L^2(\mathbb{S}^N_+, \theta_{N+1}^{1-2s}dS)$-orthonormal basis of the eigenspace of* (1.2.11) *associated to μ_{k_0}.*

Then, for every $m \in \{1, 2, \dots, M_{k_0}\}$*, there exists* $\beta_m \in \mathbb{R}$ *such that* $(\beta_1, \beta_2, \dots, \beta_{M_{k_0}}) \neq (0, 0, \dots, 0)$*,*

$$\frac{W(\lambda z)}{\lambda^{k_0+s}} \to |z|^{k_0+s} \sum_{m=1}^{M_{k_0}} \beta_m Y_{k_0,m}(z/|z|) \quad \text{in } H^1(B_1^+, t^{1-2s}dz) \text{ as } \lambda \to 0^+,$$

and

$$\beta_m = R^{-(k_0+s)} \varphi_{k_0,m}(R) + \frac{(k_0+s)R^{-N-2k_0}}{N+2k_0} \int_0^R \rho^{k_0+s-1} \Upsilon_{k_0,m}(\rho)\, d\rho + \frac{N-s+k_0}{N+2k_0} \int_0^R \rho^{-N-1+s-k_0} \Upsilon_{k_0,m}(\rho)\, d\rho \quad \text{for all } R \in (0, R_0], \tag{2.3.176}$$

with $\varphi_{k_0,m}$ *and* $\Upsilon_{k_0,m}$ *given by* (2.3.150) *and* (2.3.151) *respectively.*

Proof. If we consider any sequence of strictly positive real numbers $\lambda_n \to 0^+$, then from Proposition 2.3.25 and Lemmas 2.3.26 and 2.3.28, we deduce that there exist a subsequence $\{\lambda_{n_k}\}_{k\in\mathbb{N}}$ and real numbers $\beta_1, \beta_2, \dots \beta_{M_{k_0}}$ not all equal to 0 such that

$$\frac{W(\lambda_{n_k} z)}{\lambda_{n_k}^{k_0+s}} \to |z|^{k_0+s} \sum_{m=1}^{M_{k_0}} \beta_m Y_{k_0,m}(z/|z|) \quad \text{in } H^1(B_1^+, t^{1-2s}dz) \text{ as } k \to \infty. \tag{2.3.177}$$

We claim that the coefficients β_m depend neither on the sequence $\{\lambda_n\}_{n\in\mathbb{N}}$, nor on its subsequence $\{\lambda_{n_k}\}_{k\in\mathbb{N}}$. To this aim, we observe that (2.3.150), (2.3.177), and the continuity of the trace map from $H^1(B_1^+, t^{1-2s}dz)$ into $L^2(\partial^+ B_1^+, t^{1-2s}dS)$ imply that

$$\begin{aligned}\lim_{k\to+\infty} \lambda_{n_k}^{-(k_0+s)} \varphi_{k_0,m}(\lambda_{n_k}) &= \lim_{k\to+\infty} \int_{\mathbb{S}^N_+} \theta_{N+1}^{1-2s} \lambda_{n_k}^{-(k_0+s)} W(\lambda_{n_k}\theta) Y_{k_0,m}(\theta) dS \\ &= \sum_{i=1}^{M_{k_0}} \beta_i \int_{\mathbb{S}^N_+} \theta_{N+1}^{1-2s} Y_{k_0,i}(\theta) Y_{k_0,m}(\theta) dS = \beta_m,\end{aligned}$$

for all $m \in \{1, 2, \dots, M_{k_0}\}$. At the same time, after fixing $R \le R_0$, by (2.3.152) we have that

$$\begin{aligned}\lim_{k\to\infty} \lambda_{n_k}^{-(k_0+s)} \varphi_{k_0,m}(\lambda_{n_k}) =& R^{-(k_0+s)} \varphi_{k_0,m}(R) + \frac{(k_0+s)R^{-N-2k_0}}{N+2k_0} \int_0^R \rho^{k_0+s-1} \Upsilon_{k_0,m}(\rho)\, d\rho \\ &+ \frac{N-s+k_0}{N+2k_0} \int_0^R \rho^{-N-1+s-k_0} \Upsilon_{k_0,m}(\rho)\, d\rho,\end{aligned}$$

hence, by uniqueness of the limit, we can deduce that, for all $m \in \{1, 2, \dots, M_{k_0}\}$,

$$\begin{aligned}\beta_m =& R^{-(k_0+s)} \varphi_{k_0,m}(R) + \frac{(k_0+s)R^{-N-2k_0}}{N+2k_0} \int_0^R \rho^{k_0+s-1} \Upsilon_{k_0,m}(\rho)\, d\rho \\ &+ \frac{N-s+k_0}{N+2k_0} \int_0^R \rho^{-N-1+s-k_0} \Upsilon_{k_0,m}(\rho)\, d\rho.\end{aligned}$$

This is enough to conclude that the coefficients β_m depend neither on the sequence $\{\lambda_n\}_{n\in\mathbb{N}}$, nor on its subsequence $\{\lambda_{n_k}\}_{k\in\mathbb{N}}$. Lemma 2.3.9 allows us to conclude that the convergence in (2.3.177) holds as $\lambda \to 0^+$, thus completing the proof. $\square$

We are now in position to prove the following convergence result for scaled solutions to (1.2.9).

Theorem 2.3.30. *Let Ω be a bounded domain in $\mathbb{R}^N$ such that there exist $g \in C^{1,1}(\mathbb{R}^{N-1})$, $x_0 \in \partial\Omega$ and $R > 0$ satisfying* (1.2.3)*. Let h satisfy* (1.2.2) *and $U \in \mathcal{D}^{1,2}(\mathbb{R}^{N+1}_+, t^{1-2s}\,dz)$ be a weak solution to* (1.2.9) *in the sense of* (1.2.10)*, with $U \not\equiv 0$ and $\operatorname{Tr} U = u$ satisfying* (1.2.4)*. Then there exist $k_0 \in \mathbb{N}$ and an eigenfunction Y of problem* (1.2.11) *associated to the eigenvalue $\mu_{k_0} = (k_0 + s)(k_0 + N - s)$ such that, letting $z_0 = (x_0, 0)$,*

$$\frac{U(z_0 + \lambda z)}{\lambda^{k_0+s}} \to |z|^{k_0+s} Y\left(\frac{z}{|z|}\right) \quad \text{in } H^1(B_1^+, t^{1-2s}dz) \text{ as } \lambda \to 0^+. \tag{2.3.178}$$

Proof. Up to a translation, we can assume that $x_0 = 0$. If U is as in the assumptions of Theorem 2.3.30, then, letting F as in Subsection 2.3.1, $W = U \circ F \in H^1_{\Gamma^+_{R_1}}(B^+_{R_1}, t^{1-2s}\,dz)$ is a nontrivial weak solution to (2.3.11). We notice that the nontriviality of U in any neighbourhood of 0, and consequently of W in $B^+_{R_1}$, can be easily deduced from nontriviality of U in $\mathbb{R}^{N+1}_+$ and classical unique continuation principles for second order elliptic equations with Lipschitz coefficients [48].

Then, by Proposition 2.3.25 and Theorem 2.3.29, there exist $k_0 \in \mathbb{N}$ and an eigenfunction Y of problem (1.2.11) associated to the eigenvalue $\mu_{k_0} = (k_0 + s)(k_0 + N - s)$ such that

$$\frac{W(\lambda z)}{\lambda^{k_0+s}} \to |z|^{k_0+s} Y(z/|z|) \quad \text{in } H^1(B_1^+, t^{1-2s}dz) \text{ as } \lambda \to 0^+. \tag{2.3.179}$$

We observe that

$$\frac{U(\lambda z)}{\lambda^{k_0+s}} = \frac{W(\lambda G_\lambda(z))}{\lambda^{k_0+s}}, \quad \nabla\left(\frac{U(\lambda\cdot)}{\lambda^{k_0+s}}\right)(z) = \nabla\left(\frac{W(\lambda\cdot)}{\lambda^{k_0+s}}\right)(G_\lambda(z)) \operatorname{Jac} G_\lambda(z), \tag{2.3.180}$$

where

$$G_\lambda(z) = \frac{1}{\lambda} F^{-1}(\lambda z).$$

From (2.3.9) we have that

$$G_\lambda(z) = z + O(\lambda) \quad \text{and} \quad \operatorname{Jac} G_\lambda(z) = \mathrm{Id}_{N+1} + O(\lambda) \tag{2.3.181}$$

as $\lambda \to 0^+$ uniformly with respect to $z \in B_1^+$. From (2.3.181) one can easily deduce that, if $f_\lambda \to f$ in $L^2(B_1^+, t^{1-2s}dz)$, then $f_\lambda \circ G_\lambda \to f$ in $L^2(B_1^+, t^{1-2s}dz)$. In view of (2.3.179) and (2.3.180), this yields the conclusion. $\square$

As a direct consequence of Theorem 2.3.30 and of the equivalent formulation of problem (1.2.1) given in (1.2.9), we obtain also a convergence result for scaled solutions to (1.2.1).

Theorem 2.3.31. *Let Ω be a bounded domain in $\mathbb{R}^N$ such that there exist $g \in C^{1,1}(\mathbb{R}^{N-1})$, $x_0 \in \partial\Omega$ and $R > 0$ satisfying* (1.2.3)*. Let h satisfy* (1.2.2) *and $u \in \mathcal{D}^{s,2}(\mathbb{R}^N)$, $u \not\equiv 0$, be a weak solution to* (1.2.1) *in the sense of* (1.2.6)*, satisfying* (1.2.4)*. Then there exist $k_0 \in \mathbb{N}$ and an eigenfunction Y of problem* (1.2.11) *associated to the eigenvalue $\mu_{k_0} = (k_0 + s)(k_0 + N - s)$ such that*

$$\frac{u(x_0 + \lambda x)}{\lambda^{k_0+s}} \to |x|^{k_0+s} Y\left(\tfrac{x}{|x|}, 0\right) \quad \text{in } H^s(B_1') \text{ as } \lambda \to 0^+, \tag{2.3.182}$$

where $H^s(B_1')$ is the usual fractional Sobolev space on the N-dimensional unit ball B_1'.

Proof. If $u \in \mathcal{D}^{s,2}(\mathbb{R}^N)$, $u \not\equiv 0$, is a nontrivial weak solution to (1.2.1), then its extension $U = \mathcal{H}(u) \in \mathcal{D}^{1,2}(\mathbb{R}^{N+1}_+, t^{1-2s}\,dz)$ weakly solves (1.2.9) in the weak sense specified in (1.2.10), see [8] and Section 1.2. Then the conclusion follows from Theorem 2.3.30 applied to U and the continuity of the trace map from $H^1(B_1^+, t^{1-2s}dz)$ into $H^s(B_1')$, see e.g. [52, Proposition 2.1]. □

The salient consequence of the precise asymptotic expansions given in Theorem 2.3.30 and Theorem 2.3.31 is the following strong unique continuation principle for problems (1.2.1) and (1.2.9).

Theorem 2.3.32.

(i) *Under the same assumptions as in Theorems 2.3.30, let $U \in \mathcal{D}^{1,2}(\mathbb{R}^{N+1}_+, t^{1-2s}\,dz)$ be a weak solution to* (1.2.9) *(in the sense of* (1.2.10)*) with $\operatorname{Tr} U = u$ satisfying* (1.2.4) *and such that $U(z) = O(|z - z_0|^k)$ as $z \to z_0$, for any $k \in \mathbb{N}$. Then $U \equiv 0$ in $\mathbb{R}^{N+1}_+$.*

(ii) *Under the same assumptions as in Theorems 2.3.31, let $u \in \mathcal{D}^{s,2}(\mathbb{R}^N)$ be a weak solution to* (1.2.1) *(in the sense of* (1.2.6)*) satisfying* (1.2.4) *and such that $u(x) = O(|x - x_0|^k)$ as $x \to x_0$, for any $k \in \mathbb{N}$. Then $u \equiv 0$ in $\mathbb{R}^N$.*

In order to prove it we premise the following remark.

Remark 2.3.33. *It is worth highlighting the fact that eigenfunctions of problem* (1.2.11) *cannot vanish identically on $\mathbb{S}^{N-1} \cap \{\theta_N < 0\}$, i.e. on the boundary portion where a Neumann homogeneous condition is assigned. Indeed, if an eigenfunction ψ associated to the eigenvalue $\mu_k = (k + s)(k + N - s)$ vanishes on $\mathbb{S}^{N-1} \cap \{\theta_N < 0\}$, then the function $\Psi(\rho\theta) = \rho^{k+s}\psi(\theta)$ would be a weak solution to the equation $\operatorname{div}(t^{1-2s}\nabla\Psi) = 0$ in $\mathbb{R}^{N-1} \times (-\infty, 0) \times (0, +\infty)$ satisfying both Dirichlet and weighted Neumann homogeneous boundary conditions on $\mathbb{R}^{N-1} \times (-\infty, 0) \times \{0\}$; then its trivial extension to $\mathbb{R}^{N-1} \times (-\infty, 0) \times \mathbb{R}$ would violate the unique continuation principle for elliptic equations with Muckenhoupt weights proved in [77] (see also [48], [73, Corollary 3.3], and [67, Proposition 2.2]).*

Proof of Theorem 2.3.32. In order to prove (i), let $U \in \mathcal{D}^{1,2}(\mathbb{R}^{N+1}_+, t^{1-2s}\,dz)$ be a nontrivial weak solution to (1.2.9). Exploiting that by assumption $U(z) = O(|z - z_0|^k)$ as $z \to z_0$ for any $k \in \mathbb{N}$, we have that for any fixed $k \in \mathbb{N}$

$$\left|\frac{U(z_0 + \lambda z)}{\lambda^{k_0+s}}\right| \le \text{const}\, \lambda^{k-k_0-s} \tag{2.3.183}$$

for λ sufficiently small. Taking $k > k_0 + s$, from (2.3.183) il follows that $\frac{U(z_0+\lambda z)}{\lambda^{k_0+s}}$ tends to 0 in $L^2(B_1^+, t^{1-2s}dz)$ as $\lambda \to 0$, thus contradicting the assumption that U is non trivial and (2.3.178). As far as the proof of (ii) is concerned, we argue by combining a similar argument to the one used for the proof of (i) with Remark 2.3.33, which ensures that the right hand side on (2.3.182) is non trivial. □

2.4 Second order elliptic equations in a domain with a crack

In this section we present the results contained in [24]. Specifically, we carry out the study of local asymptotics and the strong unique continuation property from the edge of a crack for solutions to the class of boundary value problems of type (1.2.15), where $N \geq 2$, $\Omega \subset \mathbb{R}^{N+1}$ is a bounded open domain, $\Gamma \subset \mathbb{R}^N$ is a closed set defined as in (1.2.16). The function g that parametrizes the edge of Γ is assumed to be of class C^2 and, without loss of generality, we suppose (1.2.17) holds true after fixing at the origin of our coordinate system a point of the edge of the crack. Then in particular we focus on the study of the strong unique continuation principle at the origin for solutions to problem (1.2.20), where the radius $\hat{R}$ is choosen in assumption (1.2.19) and the potential f satisfies either (H1-1)-(H1-3) or (H2-1)-(H2-5). We recall that a weak solution to (1.2.20) is a function $u \in H^1(B_{\hat{R}})$ satisfying (1.2.22), where the space $H^1_\Gamma(B_{\hat{R}})$ is defined as the closure with respect to the H^1-norm of the subspace defined in (1.2.21). The above space can be explicitly characterized as follows.

Lemma 2.4.1. *The space $H^1_\Gamma(B_{\hat{R}})$ coincides with the subset of $H^1(B_{\hat{R}})$ of those functions with null trace on Γ.*

The proof is based on the following Hardy-type inequality with boundary terms, due to Wang and Zhu [78].

Lemma 2.4.2 ([78], Theorem 1.1)**.** *For every $r > 0$ and $u \in H^1(B_r)$,*

$$\int_{B_r} |\nabla u(z)|^2 \, dz + \frac{N-1}{2r} \int_{\partial B_r} |u(z)|^2 \, dS \geq \left(\frac{N-1}{2}\right)^2 \int_{B_r} \frac{|u(z)|^2}{|z|^2} \, dz. \tag{2.4.1}$$

It is also useful to give an adapted version of [5, Theorem 3.1] to our setting in order to prove Lemma 2.4.1.

Theorem 2.4.3. *Let $\mathring{\Gamma}$ be the interior of the crack Γ. Then the space of all smooth functions defined in the closure of the ball $B_{\hat{R}}$ vanishing in a neighbourhood of $\mathring{\Gamma}$ is dense in the set of functions in $H^1(B_{\hat{R}})$ having null trace on $\mathring{\Gamma}$.*

Now we can move on to prove Lemma 2.4.1.

Proof of Lemma 2.4.1. It is sufficient to prove that any function in $H^1(B_{\hat{R}})$ having null trace on Γ can be approximated by smooth functions defined in the closure of the ball $B_{\hat{R}}$ vanishing in a neighbourhood of Γ. In order to do this, we exploit Theorem 2.4.3, taking into account that $\partial\Gamma$ has zero capacity in $B_{\hat{R}}$, being contained in a 2-codimensional manifold (see [49]). For this, we recall that the capacity of a compact set K contained in an open set $\Omega \subset \mathbb{R}^{N+1}$ is defined as

$$\mathrm{cap}_\Omega K := \inf\left\{\int_{\mathbb{R}^{N+1}} |\nabla u|^2\,dz : u \in \mathcal{D}(K,\Omega)\right\},$$

where $\mathcal{D}(K,\Omega) := \{u \in C_c^\infty(\Omega) : 0 \le u \le 1,\ u = 1 \text{ in a neighbourhood of } K\}$.

Let u be any function in $H^1(B_{\hat{R}})$ with null trace on Γ and let $\varepsilon > 0$. By Theorem 2.4.3 we deduce that there exists a function $g_\varepsilon \in C^\infty(\overline{B_{\hat{R}}})$ such that

$$g_\varepsilon = 0 \text{ in a neighbourhood of } \mathring{\Gamma} \text{ and } \|u - g_\varepsilon\|_{H^1(B_{\hat{R}})} < \varepsilon/2.$$

Furthermore, since $\partial\Gamma$ has zero capacity in $B_{\hat{R}}$, there exists a sequence of functions $\{\eta_n\}_{n\in\mathbb{N}} \subset \mathcal{D}(\partial\Gamma, B_{\hat{R}})$ such that

$$\int_{B_{\hat{R}}} |\nabla\eta_n|^2\,dz \to 0 \quad \text{as } n \to \infty. \tag{2.4.2}$$

We claim that $g_\varepsilon(1-\eta_n) \to g_\varepsilon$ in $H^1(B_{\hat{R}})$ as $n \to \infty$. In order to show it, we first prove that

$$\int_{B_{\hat{R}}} |g_\varepsilon - g_\varepsilon(1-\eta_n)|^2\,dz \to 0 \quad \text{as } n \to \infty.$$

Indeed, since $g_\varepsilon - g_\varepsilon(1-\eta_n) = g_\varepsilon\eta_n$, it is sufficient to observe that g_ε is bounded and

$$\int_{B_{\hat{R}}} |\eta_n|^2\,dz \le \hat{R}^2 \int_{B_{\hat{R}}} \frac{\eta_n^2}{|z|^2}\,dz \le \mathrm{const}\,\hat{R}^2 \int_{B_{\hat{R}}} |\nabla\eta_n|^2\,dz \to 0 \quad \text{as } n \to \infty, \tag{2.4.3}$$

where we used (2.4.1) and (2.4.2). Moreover, we have that $\nabla g_\varepsilon - \nabla\left(g_\varepsilon(1-\eta_n)\right) = \eta_n\nabla g_\varepsilon + g_\varepsilon\nabla\eta_n$, and

$$\begin{aligned}\int_{B_{\hat{R}}} |\eta_n\nabla g_\varepsilon + g_\varepsilon\nabla\eta_n|^2\,dz &\le 2\left(\int_{B_{\hat{R}}} |\eta_n\nabla g_\varepsilon|^2\,dz + \int_{B_{\hat{R}}} |g_\varepsilon\nabla\eta_n|^2\,dz\right)\\ &\le 2\,\mathrm{const}\left(\int_{B_{\hat{R}}} |\eta_n|^2\,dz + \int_{B_{\hat{R}}} |\nabla\eta_n|^2\,dz\right) \to 0\end{aligned}$$

as $n \to \infty$, exploiting the boundedness of ∇g_ε, (2.4.3) and (2.4.2). Hence there exists $\nu = \nu(\varepsilon) \in \mathbb{N}$ such that

$$\|g_\varepsilon(1-\eta_{\nu(\varepsilon)}) - g_\varepsilon\|_{H^1(B_{\hat{R}})} < \varepsilon/2.$$

Putting together all the above information we achieve the desired convergence because $g_\varepsilon(1-\eta_{\nu(\varepsilon)})$ vanishes in a neighbourhood of Γ and

$$\|g_\varepsilon(1-\eta_{\nu(\varepsilon)}) - u\|_{H^1(B_{\hat{R}})} \le \|g_\varepsilon(1-\eta_{\nu(\varepsilon)}) - g_\varepsilon\|_{H^1(B_{\hat{R}})} + \|g_\varepsilon - u\|_{H^1(B_{\hat{R}})} < \varepsilon/2 + \varepsilon/2 = \varepsilon,$$

thus completing the proof. □

We provide some examples of functions satisfying our assumptions on potential f.

Remark 2.4.4. *Conditions* (H1-1)-(H1-3) *are satisfied e.g. if* $|f(z)| = O(|z|^{-2+\delta})$ *as* $|z| \to 0^+$ *for some* $\delta > 0$*, whereas assumptions* (H2-1)-(H2-5) *hold e.g. if* $f \in W^{1,\infty}_{\rm loc}(B_{\hat{R}} \setminus \{0\})$ *and* $f, \nabla f \in L^p(B_{\hat{R}})$ *for some* $p > \frac{N+1}{2}$*. We also observe that condition* (H2-1) *is satisfied if* f *belongs to the Kato class* K_{n+1}*, see [34].*

We make also some observations on assumption (1.2.19).

Remark 2.4.5. *Assumption* (1.2.19) *says that the complement of* Γ *is star-shaped with respect to the origin in a neighbourood of 0. This fact can be easily seen taking into account that if* $x = (x', x_N) \in \partial\Gamma^c$*, then* $x_N = g(x')$ *and the outward unit normal vector at* x *denoted with* $\nu(x)$ *is given by*

$$\frac{(-\nabla g(x'), 1)}{\sqrt{1 + |\nabla g(x')|^2}}.$$

In particular, we observe that (1.2.19) *is satisfied for instance if the function* g *is concave in a neighbourood of the origin, see Figure 2.4. Indeed, under this assumption the Hessian matrix is negative semi-definite for any point in a neighbourood of the origin; in particular, using condition* (1.2.17) *and considering the asymptotic expansions of* g *and* ∇g *around 0 we deduce that*

$$g(x') = \frac{1}{2} \sum_{i,j=1}^{N-1} \frac{\partial^2 g(0)}{\partial x_i \partial x_j} x_i x_j + o(|x'|^2) \quad \text{as } |x'| \to 0^+ \tag{2.4.4}$$

$$\nabla g(x') \cdot x' = \sum_{i,j=1}^{N-1} \frac{\partial^2 g(0)}{\partial x_i \partial x_j} x_i x_j + o(|x'|^2) \quad \text{as } |x'| \to 0^+,$$

hence

$$g(x') - \nabla g(x') \cdot x' = -\frac{1}{2} \sum_{i,j=1}^{N-1} \frac{\partial^2 g(0)}{\partial x_i \partial x_j} x_i x_j + o(|x'|^2) \quad \text{as } |x'| \to 0^+, \tag{2.4.5}$$

thus implying (1.2.19).

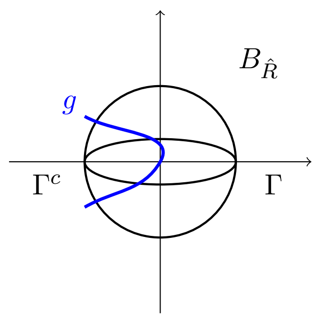


Figure 2.4: An example of g satistying (1.2.19)

2.4.1 Approximation argument

In this section we carry out an approximation argument based on the construction of a sequence of domains approximating our cracked domain with the twofold features of satisfying the exterior ball condition and being star-shaped with respect to the origin. In order to have the latter property, condition (1.2.19) turns out to be crucial (see the proof of Lemma 2.4.8).

Consequently, we consider a sequence of solutions of some boundary value problems on such domains converging to the solution of the original problem with crack.

We start by providing a coercivity type result for the quadratic form associated to problem (1.2.20) in small neighbourhoods of the origin.

Lemma 2.4.6. *Let f satisfy either* (H1-1) *or* (H2-1)*. Then there exists $r_0 \in (0, \hat{R})$ such that for every $r \in (0, r_0]$ and $u \in H^1(B_r)$*

$$\int_{B_r} (|\nabla u|^2 - |f|u^2)\, dz \geq \frac{1}{2} \int_{B_r} |\nabla u|^2\, dz - \omega(r) \int_{\partial B_r} u^2\, dS \tag{2.4.6}$$

where

$$\omega(r) = \begin{cases} \dfrac{2}{N-1} \dfrac{\xi_f(r)}{r}, & \textit{under assumption } \text{(H1-1)}, \\ \dfrac{N-1}{2} \dfrac{\eta(r,f)}{r}, & \textit{under assumption } \text{(H2-1)}, \end{cases} \tag{2.4.7}$$

and

$$r\omega(r) < \frac{N-1}{4}. \tag{2.4.8}$$

Remark 2.4.7. *For future use, we notice that* (2.4.6) *can be rewritten as follows*

$$\int_{B_r} |f|u^2\, dz \leq \frac{1}{2} \int_{B_r} |\nabla u|^2\, dz + \omega(r) \int_{\partial B_r} u^2\, dS \tag{2.4.9}$$

for all $u \in H^1(B_r)$ and $r \in (0, r_0]$.

Proof of Lemma 2.4.6. We first prove the lemma under assumption (H1-1). Using (H1-3) and (2.4.1), we infer that for any $r \in (0, \hat{R})$ and $u \in H^1(B_r)$

$$\begin{aligned} \int_{B_r} |f|u^2\, dz &\leq \xi_f(r) \int_{B_r} \frac{|u(z)|^2}{|z|^2}\, dz \\ &\leq \frac{4\xi_f(r)}{(N-1)^2} \left[\int_{B_r} |\nabla u|^2\, dz + \frac{N-1}{2r} \int_{\partial B_r} u^2\, dS \right]. \end{aligned} \tag{2.4.10}$$

From (H1-1) we can deduce that there exists $r_0 \in (0, \hat{R})$ such that

$$\frac{4\xi_f(r)}{(N-1)^2} < \frac{1}{2} \quad \text{for all } r \in (0, r_0]. \tag{2.4.11}$$

Thus, for every $r \in (0, r_0]$, combining (2.4.11) and (2.4.10), we obtain that

$$\begin{aligned}\int_{B_r} \left(|\nabla u|^2 - |f|u^2\right) dz &\geq \left(1 - \frac{4\xi_f(r)}{(N-1)^2}\right) \int_{B_r} |\nabla u|^2 \, dz - \frac{2}{N-1} \frac{\xi_f(r)}{r} \int_{\partial B_r} u^2 \, dS \\ &\geq \frac{1}{2} \int_{B_r} |\nabla u|^2 \, dz - \frac{2}{N-1} \frac{\xi_f(r)}{r} \int_{\partial B_r} u^2 \, dS\end{aligned}$$

and this completes the proof of (2.4.6) under assumption (H1-1).

Now we move on to prove the lemma under assumption (H2-1). Then by (H2-5), it follows that for every $r \in (0, \hat{R})$ and $u \in H^1(B_r)$

$$\int_{B_r} |f|u^2 \, dz \leq \eta(r, f) \left[\int_{B_r} |\nabla u|^2 \, dz + \frac{N-1}{2r} \int_{\partial B_r} u^2 \, dS\right]. \tag{2.4.12}$$

From (H2-1) we can deduct that there exists $r_0 \in (0, \hat{R})$ be such that

$$\eta(r, f) < \frac{1}{2} \quad \text{for all } r \in (0, r_0]. \tag{2.4.13}$$

Hence, for every $r \in (0, r_0]$, putting together (2.4.13) and (2.4.12) we deduce that

$$\begin{aligned}\int_{B_r} \left(|\nabla u|^2 - |f|u^2\right) dz &\geq (1 - \eta(r, f)) \int_{B_r} |\nabla u|^2 \, dz - \frac{N-1}{2} \frac{\eta(r, f)}{r} \int_{\partial B_r} u^2 \, dS \\ &\geq \frac{1}{2} \int_{B_r} |\nabla u|^2 \, dz - \frac{N-1}{2} \frac{\eta(r, f)}{r} \int_{\partial B_r} u^2 \, dS,\end{aligned}$$

hence concluding the proof of (2.4.6) under assumption (H2-1). Estimate (2.4.8) follows from the definition of ω in (2.4.7), (2.4.11), and (2.4.13). □

Now we construct suitable regular sets approximating our cracked domain which are star-shaped with respect to the origin and satisfy the exterior ball condition. In order to do this, for any $n \in \mathbb{N} \setminus \{0\}$ let $f_n \colon \mathbb{R} \to \mathbb{R}$ be defined as

$$f_n(t) = \begin{cases} n|t| + \frac{1}{n} e^{\frac{2n^2|t|}{n^2|t|-2}}, & \text{if } |t| < 2/n^2, \\ n|t|, & \text{if } |t| \geq 2/n^2, \end{cases}$$

so that $f_n \in C^2(\mathbb{R})$, $f_n(t) \geq n|t|$ for all $t \in \mathbb{R}$, and $f_n'(t) \leq n$ for every $t > 0$ and $f_n' \geq -n$ for every $t < 0$; from these information, we can easily deduce that

$$f_n(t) - t\, f_n'(t) \geq 0 \quad \text{for every } t \in \mathbb{R}. \tag{2.4.14}$$

This condition reveals to be fundamental to obtain domains that are star-shaped with respect to the origin (see Lemma 2.4.8 below). Then for all $r > 0$, we define

$$\tilde{B}_{r,n} := \{z = (x', x_N, t) \in B_r : x_N < g(x') + f_n(t)\}, \tag{2.4.15}$$

see Figure 2.5.

(a) The set $\tilde{B}_{r,n}$ **(b)** Section of $\tilde{B}_{r,n}$

Figure 2.5: Approximating domains

Let $\tilde{\gamma}_{r,n}$ that part of the boundary of $\tilde{B}_{r,n}$ contained in B_r given by the set

$$\{z = (x', x_N, t) \in B_r : x_N = g(x') + f_n(t)\}$$

and $\tilde{S}_{r,n}$ denote its complement with respect to $\partial \tilde{B}_{r,n}$. For any fixed $r > 0$, the set $\tilde{\gamma}_{r,n}$ is not empty and, consequently, $\tilde{B}_{r,n} \neq B_r$, provided that n is sufficiently large.

Lemma 2.4.8. *Let $0 < r \leq \hat{R}$. Then, for all $n \in \mathbb{N} \setminus \{0\}$, the set $\tilde{B}_{r,n}$ is star-shaped with respect to the origin, i.e. $z \cdot \nu(z) \geq 0$ for a.e. $z \in \partial \tilde{B}_{r,n}$, where ν is the outward unit normal vector.*

Proof. If $\tilde{\gamma}_{r,n}$ is empty, then $\tilde{B}_{r,n} = B_r$ and the conclusion is obvious. Let $\tilde{\gamma}_{r,n}$ be not empty.

The thesis is trivial if one considers a point $z \in \partial \tilde{B}_{r,n} \setminus \tilde{\gamma}_{r,n}$.

If $z \in \tilde{\gamma}_{r,n}$, then $z = (x', g(x') + f_n(t), t)$ and the outward unit normal vector at this point is given by

$$\nu(z) = \frac{(-\nabla g(x'), 1, -f_n'(t))}{\sqrt{1 + |f_n'(t)|^2 + |\nabla g(x')|^2}},$$

hence

$$z \cdot \nu(z) = \frac{g(x') - \nabla g(x') \cdot x' + f_n(t) - t\ f_n'(t)}{\sqrt{1 + |f_n'(t)|^2 + |\nabla g(x')|^2}} \geq 0$$

as a consequence of assumption (1.2.19) and by (2.4.14). □

We fix once and for all $u \in H^1(B_{\hat{R}})$ a non-trivial weak solution to problem (1.2.20), as clarified in (1.2.22). Hence there exists a sequence of functions $G_n \in C^\infty_{0,\Gamma}(\overline{B_{\hat{R}}})$ such that $G_n \to u$ in $H^1(B_{\hat{R}})$. Starting from the functions G_n, we can easily construct a sequence of functions g_n such that

$$g_n(x', x_N, t) = 0 \quad \text{if } (x', x_N) \in \Gamma \text{ and } |t| \leq \frac{\tilde{C}}{n}, \tag{2.4.16}$$

with

$$\tilde{C} > \sqrt{2(r_0^2 + M^2)}, \quad \text{where } M := \max\{|g(x')| : |x'| \leq r_0\}. \tag{2.4.17}$$

Indeed, since G_1 vanishes in a neighbourhood of Γ, then G_1 vanishes on a set of type $\Gamma \times \left(-\frac{\tilde{C}}{n_1}, \frac{\tilde{C}}{n_1}\right)$ for sufficiently large $n_1 \in \mathbb{N}^*$. Thus, choosing $g_i = G_1$ for every $1 \le i \le n_1$, we will have

$$g_i(x', x_N, t) = 0 \quad \text{if } (x', x_N) \in \Gamma \text{ and } |t| \le \frac{\tilde{C}}{n_1} \le \frac{\tilde{C}}{i}$$

for every $1 \le i \le n_1$, which implies property (2.4.16), as desired. We proceed by selecting $n_2 > n_1$ such that $G_2 = 0$ on $\Gamma \times \left(-\frac{\tilde{C}}{n_2}, \frac{\tilde{C}}{n_2}\right)$ and letting $g_j = G_2$ for every $n_1+1 \le j \le n_2$. In this way, (2.4.16) holds true also for g_j, with $n_1+1 \le j \le n_2$. Via an iteration argument, as shown below

g_1	g_2	$\cdots$	g_{n_1}	g_{n_1+1}	$\cdots$	g_{n_2}	$\cdots$
G_1	G_1	$\cdots$	G_1	G_2	$\cdots$	G_2	$\cdots$

we obtain a sequence of functions g_n satisfying (2.4.16).

Remark 2.4.9. *It holds that $g_n \equiv 0$ in $B_{r_0} \setminus \tilde{B}_{r_0,n}$. Indeed, if $z = (x', x_N, t) \in B_{r_0} \setminus \tilde{B}_{r_0,n}$, then*

$$x_N \ge g(x') + f_n(t) > g(x'),$$

and hence $(x', x_N) \in \Gamma$. Moreover

$$x_N \ge f_n(t) + g(x') \ge n|t| - M,$$

with M defined as in (2.4.17). Thus either $|t| \le \frac{M}{n}$ or $r_0^2 \ge x_N^2 \ge (n|t| - M)^2 \ge \frac{n^2}{2}|t|^2 - M^2$, implying that $|t| \le \frac{\sqrt{2(r_0^2+M^2)}}{n} < \frac{\tilde{C}}{n}$, if we take $\tilde{C}$ as in (2.4.17). Then $g_n(z) = 0$ in view of (2.4.16).

We go ahead with our construction by considering a sequence of solutions $\{u_n\}_{n\in\mathbb{N}}$ to some boundary value problems on the approximating domains $\tilde{B}_{r_0,n}$. Therefore, for every $n \in \mathbb{N}$, we claim that there exists a unique weak solution u_n to the following boundary value problem

$$\begin{cases} -\Delta u_n = f u_n & \text{in } \tilde{B}_{r_0,n}, \\ \quad u_n = g_n & \text{on } \partial \tilde{B}_{r_0,n}. \end{cases} \tag{2.4.18}$$

Letting $v_n := u_n - g_n$, we have that u_n weakly solves (2.4.18) if and only if $v_n \in H^1(\tilde{B}_{r_0,n})$ is a weak solution to the homogeneous boundary value problem

$$\begin{cases} -\Delta v_n - f v_n = f g_n + \Delta g_n & \text{in } \tilde{B}_{r_0,n}, \\ \quad v_n = 0 & \text{on } \partial \tilde{B}_{r_0,n}, \end{cases} \tag{2.4.19}$$

that is equivalent to assert that

$$\begin{cases} v_n \in H_0^1(\tilde{B}_{r_0,n}), \\ \displaystyle\int_{\tilde{B}_{r_0,n}} (\nabla v_n \cdot \nabla \phi - f v_n \phi)\, dz = \int_{\tilde{B}_{r_0,n}} (f g_n + \Delta g_n)\, \phi \, dz \quad \text{for any } \phi \in H_0^1(\tilde{B}_{r_0,n}). \end{cases}$$

Lemma 2.4.10. *Let r_0 be as in Lemma 2.4.6. Then, for all $n \in \mathbb{N}$, problem* (2.4.19) *has one and only one weak solution $v_n \in H_0^1(\tilde{B}_{r_0,n})$, where $\tilde{B}_{r_0,n}$ is defined as in* (2.4.15).

Proof. For every $v, w \in H_0^1(\tilde{B}_{r_0,n})$ we introduce the bilinear form

$$a(v,w) = \int_{\tilde{B}_{r_0,n}} (\nabla v \cdot \nabla w - f v w)\, dz,$$

and by Lemma 2.4.6 we deduce that a is coercive on $H_0^1(\tilde{B}_{r_0,n})$, namely there exists a positive constant $\beta > 0$ such that for every $v \in H_0^1(\tilde{B}_{r_0,n})$

$$a(v,v) \geq \beta \|v\|^2_{H_0^1(\tilde{B}_{r_0,n})}.$$

Indeed, observing that the boundary term in (2.4.6) vanishes since $v \in H_0^1(\tilde{B}_{r_0,n})$, we have that

$$a(v,v) = \int_{\tilde{B}_{r_0,n}} \left[|\nabla v|^2 - f v^2 \right]\, dz \geq \frac{1}{2} \int_{\tilde{B}_{r_0,n}} |\nabla v|^2 \, dz = \frac{1}{2} \|v\|^2_{H_0^1(\tilde{B}_{r_0,n})}. \tag{2.4.20}$$

Furthermore, from estimate (2.4.9) we easily infer that a is continuous, i.e. there exists a positive constant $C > 0$ such that for every $v, w \in H_0^1(\tilde{B}_{r_0,n})$

$$|a(v,w)| \leq C \|v\|_{H_0^1(\tilde{B}_{r_0,n})} \|w\|_{H_0^1(\tilde{B}_{r_0,n})}.$$

In order to show this we introduce

$$\tilde{\omega}(r) := \begin{cases} \dfrac{4\xi_f(r_0)}{(N-1)^2}, & \text{under assumption (H1-1),} \\ \eta(r,f), & \text{under assumption (H2-1),} \end{cases} \tag{2.4.21}$$

obtaining that by (2.4.11) and (2.4.13)

$$\tilde{\omega}(r_0) < \frac{1}{2}.$$

Then applying the Hölder inequality and proceeding as in the proof of Lemma 2.4.6

$$\begin{aligned} |a(v,w)| &\leq \int_{\tilde{B}_{r_0,n}} |\nabla v \cdot \nabla w| \, dz + \int_{\tilde{B}_{r_0,n}} |f v w| \, dz \\ &\leq \left(\int_{\tilde{B}_{r_0,n}} |\nabla v|^2 \, dz \right)^{1/2} \left(\int_{\tilde{B}_{r_0,n}} |\nabla w|^2 \, dz \right)^{1/2} + \left(\int_{\tilde{B}_{r_0,n}} |f| v^2 \, dz \right)^{1/2} \left(\int_{\tilde{B}_{r_0,n}} |f| w^2 \, dz \right)^{1/2} \\ &\leq (1 + \tilde{\omega}(r_0)) \|v\|_{H_0^1(\tilde{B}_{r_0,n})} \|w\|_{H_0^1(\tilde{B}_{r_0,n})} \\ &\leq \frac{3}{2} \|v\|_{H_0^1(\tilde{B}_{r_0,n})} \|w\|_{H_0^1(\tilde{B}_{r_0,n})}. \end{aligned}$$

The thesis follows from the Lax-Milgram Theorem. □

Proposition 2.4.11. *Under the same assumptions of Lemma 2.4.10, there exists a positive constant $C > 0$ such that $\|v_n\|_{H^1_0(B_{r_0})} \le C$ for every $n \in \mathbb{N}$, after extending v_n trivially to zero in $B_{r_0} \setminus \tilde{B}_{r_0,n}$.*

Proof. First we observe that fg_n and $-\Delta g_n$ interpreted as linear and continuous operators on $H^1_0(B_{r_0})$ are bounded in $H^{-1}(B_{r_0})$: indeed, by the Hölder inequality and (2.4.9), for any $\phi \in H^1_0(B_{r_0})$,

$$\begin{aligned} \left| \int_{B_{r_0}} f g_n \phi \, dz \right| &\le \left(\int_{B_{r_0}} |f| g_n^2 \, dz \right)^{1/2} \left(\int_{B_{r_0}} |f| \phi^2 \, dz \right)^{1/2} \\ &\le \frac{1}{2} \left(\frac{1}{2} \int_{B_{r_0}} |\nabla g_n|^2 \, dz + \omega(r_0) \int_{\partial B_{r_0}} g_n^2 \, dS \right)^{1/2} \left(\int_{B_{r_0}} |\nabla \phi|^2 \, dz \right)^{1/2} \\ &\le \text{const} \|g_n\|_{H^1(B_{r_0})} \|\phi\|_{H^1_0(B_{r_0})} \le \text{const} \|\phi\|_{H^1_0(B_{r_0})}, \end{aligned}$$

where we used also the continuity of the trace map from $H^1(B_{r_0})$ to $L^2(\partial B_{r_0})$ and the boundedness of functions g_n in $H^1(B_{r_0})$; moreover we have also

$$\begin{aligned} \left| - \int_{B_{r_0}} \Delta g_n \phi \, dz \right| = \left| \int_{B_{r_0}} \nabla g_n \cdot \nabla \phi \, dz \right| &\le \text{const} \|g_n\|_{H^1(B_{r_0})} \|\phi\|_{H^1_0(B_{r_0})} \\ &\le \text{const} \|\phi\|_{H^1_0(B_{r_0})}. \end{aligned}$$

Thus exploiting the equation (2.4.19) and Lemma 2.4.6, it follows that

$$\begin{aligned} \|v_n\|^2_{H^1_0(B_{r_0})} = \int_{B_{r_0}} |\nabla v_n|^2 \, dz \le 2 \int_{B_{r_0}} (|\nabla v_n|^2 - f v_n^2) \, dz = 2 \int_{B_{r_0}} (f g_n + \Delta g_n) v_n \, dz \\ \le \text{const} \|v_n\|_{H^1_0(B_{r_0})}, \end{aligned}$$

thus completing the proof. □

Proposition 2.4.12. *Under the same assumptions of Lemma 2.4.10, it holds that $u_n \rightharpoonup u$ weakly in $H^1(B_{r_0})$, after extending u_n trivially to zero in $B_{r_0} \setminus \tilde{B}_{r_0,n}$.*

Proof. We observe that the trivial extension to zero of u_n in $B_{r_0} \setminus \tilde{B}_{r_0,n}$ belongs to $H^1(B_{r_0})$ since the trace of u_n on $\tilde{\gamma}_{r_0,n}$ is null in view of Remark 2.4.9.

From Proposition 2.4.11 it follows that there exist $\tilde{v} \in H^1_0(B_{r_0})$ and a subsequence $\{v_{n_k}\}$ of $\{v_n\}$ such that $v_{n_k} \rightharpoonup \tilde{v}$ weakly in $H^1_0(B_{r_0})$. Then $u_{n_k} = v_{n_k} + g_{n_k} \rightharpoonup \tilde{u}$ weakly in $H^1(B_{r_0})$, where $\tilde{u} := \tilde{v} + u$. Let $\phi \in C^\infty_c(B_{r_0} \setminus \Gamma)$. Arguing as in Remark 2.4.9, we can prove that $\phi \in H^1_0(\tilde{B}_{r_0,n_k})$ for all sufficiently large k. Hence, from (2.4.18) it follows that, for all sufficiently large k,

$$\int_{B_{r_0}} \nabla u_{n_k} \cdot \nabla \phi \, dz = \int_{B_{r_0}} f u_{n_k} \phi \, dz, \tag{2.4.22}$$

where u_{n_k} is extended trivially to zero in $B_{r_0} \setminus \tilde{B}_{r_0,n_k}$. Passing to the limit into (2.4.22), we obtain that

$$\int_{B_{r_0}} \nabla \tilde{u} \cdot \nabla \phi \, dz = \int_{B_{r_0}} f \tilde{u} \phi \, dz$$

for every $\phi \in C_c^\infty(B_{r_0} \setminus \Gamma)$. Furthermore $\tilde{u} = u$ on ∂B_{r_0} in the trace sense: indeed, due to compactness of the trace map $\gamma : H^1(B_{r_0}) \to L^2(\partial B_{r_0})$, we have that $\gamma(u_{n_k}) \to \gamma(\tilde{u})$ in $L^2(\partial B_{r_0})$ and $\gamma(u_{n_k}) = \gamma(g_{n_k}) \to \gamma(u)$ in $L^2(\partial B_{r_0})$, since $g_n \to u$ in $H^1(B_{r_0})$.

Finally, we prove that $\tilde{u} \in H^1_\Gamma(B_{r_0})$. To this aim, for every $\delta > 0$ let $\Gamma_\delta := \{(x', x_N) \in \mathbb{R}^N : x_N \geq g(x') + \delta\}$. For every $\delta > 0$ we have that $\Gamma_\delta \cap B_{r_0} \subset B_{r_0} \setminus \tilde{B}_{r_0,n}$ provided n is sufficiently large. Hence, since u_n is extended trivially to zero in $B_{r_0} \setminus \tilde{B}_{r_0,n}$, we have that, for every $\delta > 0$, $u_n \in H^1_{\Gamma_\delta}(B_{r_0})$ provided n is sufficiently large. Since $H^1_{\Gamma_\delta}(B_{r_0})$ is weakly closed in $H^1(B_{r_0})$, it follows that $\tilde{u} \in H^1_{\Gamma_\delta}(B_{r_0})$ for every $\delta > 0$, and hence $\tilde{u} \in H^1_\Gamma(B_{r_0})$.

Thus $\tilde{u}$ weakly solves

$$\begin{cases} -\Delta \tilde{u} = f\tilde{u} & \text{in } B_{r_0} \setminus \Gamma, \\ \tilde{u} = u & \text{on } \partial B_{r_0}, \\ \tilde{u} = 0 & \text{on } \Gamma. \end{cases}$$

Now we consider the function $U := \tilde{u} - u$: it weakly solves the following problem

$$\begin{cases} -\Delta U = fU & \text{in } B_{r_0} \setminus \Gamma, \\ U = 0 & \text{on } \partial B_{r_0}, \\ U = 0 & \text{on } \Gamma. \end{cases} \tag{2.4.23}$$

Testing equation (2.4.23) with U itself and using Lemma 2.4.6, we obtain that

$$\frac{1}{2} \int_{B_{r_0}} |\nabla U|^2 \, dz \leq \int_{B_{r_0}} (|\nabla U|^2 - fU^2) \, dz = 0,$$

so that $U = 0$, hence $u = \tilde{u}$. We observe that, since v_n is bounded, then assumption (i) of Lemma 2.3.9 is trivially satisfied and if v_{n_h} is any subsequence of v_n such that $v_{n_h} \rightharpoonup \bar{v}$ for some $\bar{v} \in H^1_0(B_{r_0})$, then $u_{n_h} = v_{n_h} + g_{n_h} = \bar{v} + u =: \bar{u}$. Arguing as above we are able to prove that $\bar{u} = u$, thus having that $\bar{v} = 0$ hence the limit does not depend on the specific subsequence and also assumption (ii) holds true. Therefore, by Lemma 2.3.9, we can conclude that $v_n \rightharpoonup 0$ weakly in $H^1(B_{r_0})$ and, consequently, $u_n \rightharpoonup u$ weakly in $H^1(B_{r_0})$. □

We are now able to prove that actually there is strong convergence of the sequence $\{u_n\}_{n\in\mathbb{N}}$ to u in $H^1(B_{r_0})$.

Proposition 2.4.13. *Under the same assumptions of Lemma 2.4.10, it holds that $u_n \to u$ strongly in $H^1(B_{r_0})$.*

Proof. From Proposition 2.4.12 it follows that $v_n \rightharpoonup 0$ in $H^1(B_{r_0})$, hence testing (2.4.19) with v_n itself, we have that

$$\begin{aligned}\int_{B_{r_0}} (|\nabla v_n|^2 - f v_n^2)\, dz &= \int_{\tilde{B}_{r_0,n}} (|\nabla v_n|^2 - f v_n^2)\, dz \\ &= \int_{\tilde{B}_{r_0,n}} (f g_n v_n - \nabla g_n \cdot \nabla v_n)\, dz = \int_{B_{r_0}} (f g_n v_n - \nabla g_n \cdot \nabla v_n)\, dz \to 0\end{aligned}$$

as $n \to \infty$. Thus, by Lemma 2.4.6, we deduce that $\|v_n\|_{H^1_0(B_{r_0})} \to 0$ as $n \to \infty$, therefore $v_n \to 0$ in $H^1(B_{r_0})$. This yields that $u_n = g_n + v_n \to u$ in $H^1(B_{r_0})$. □

2.4.2 Pohozaev-type inequality

In the present section we provide a Pohozaev-type inequality for problem (1.2.20) in order to estimate the derivative of the Almgren function (2.4.47) in Section 2.4.3. In particular, in this case due to the high non-smoothness of the domain, it is not possible directly to infer a Pohozaev-type identity for problem (1.2.20). Then the idea is to derive Pohozaev-type identities for problems (2.4.18) exploiting the higher regularity of the approximating domains. Thus, using the star-shapeness of such domains exhibited in Lemma 2.4.8, we are able to estimate some boundary terms appearing in the above identities. Then, passing to the limit in the resulting inequalities, thanks to the convergence shown in Proposition 2.4.13, we obtain inequality (2.4.34).

To this aim, for every $r \in (0, r_0)$ and $v \in H^1(B_r)$, we define

$$\mathcal{R}(r, v) = \begin{cases} \displaystyle\int_{B_r} f v (z \cdot \nabla v)\, dz, & \text{if } f \text{ satisfies (H1-1)-(H1-3)}, \\ \displaystyle\frac{r}{2}\int_{\partial B_r} f\, v^2\, dS - \frac{1}{2}\int_{B_r} \big(\nabla f \cdot z + (N+1) f\big) v^2\, dz, & \text{if } f \text{ satisfies (H2-1)-(H2-5)}. \end{cases}$$

Lemma 2.4.14. *Let $r \in (0, r_0)$. Then there exists $n_0 = n_0(r) \in \mathbb{N} \setminus \{0\}$ such that, for all $n \geq n_0$,*

$$\begin{aligned}-\frac{N-1}{2} \int_{\tilde{B}_{r,n}} |\nabla u_n|^2\, dz + \frac{r}{2} \int_{\tilde{S}_{r,n}} |\nabla u_n|^2\, dS \\ - \frac{1}{2} \int_{\tilde{\gamma}_{r,n}} \left|\frac{\partial u_n}{\partial \nu}\right|^2 z \cdot \nu\, dS - r \int_{\tilde{S}_{r,n}} \left|\frac{\partial u_n}{\partial \nu}\right|^2 dS - \mathcal{R}(r, u_n) = 0, \quad (2.4.24)\end{aligned}$$

where u_n is a weak solution to problem (2.4.18) for each fixed $n \in \mathbb{N} \setminus \{0\}$.

Proof. Since u_n solves (2.4.18) in $\tilde{B}_{r_0,n}$ that satisfies the exterior ball condition, and $f u_n \in L^2_{\text{loc}}(\tilde{B}_{r_0,n} \setminus \{0\})$, by elliptic regularity theory (see [2]), we can conclude that $u_n \in H^2(\tilde{B}_{r,n} \setminus B_\delta)$ for all $r \in (0, r_0)$, n sufficiently large and all $\delta < r_n$, where r_n is such that $B_{r_n} \subset \tilde{B}_{r,n}$. Furthermore from the fact that

$$\int_0^{r_n} \left[\int_{\partial B_t} \big(|\nabla u_n|^2 + |f| u_n^2\big)\, dS\right] dt = \int_{B_{r_n}} \big(|\nabla u_n|^2 + |f| u_n^2\big)\, dx < +\infty,$$

we deduce that there exists a sequence $\{\delta_k\}_{k\in\mathbb{N}} \subset (0, r_n)$ such that $\lim\limits_{k\to\infty} \delta_k = 0$ and

$$\delta_k \int_{\partial B_{\delta_k}} |\nabla u_n|^2 \, dS \to 0, \quad \delta_k \int_{\partial B_{\delta_k}} |f| u_n^2 \, dS \to 0 \quad \text{as } k \to \infty. \tag{2.4.25}$$

Thus testing (2.4.18) with $z \cdot \nabla u_n$ and integrating over $\tilde{B}_{r,n} \setminus B_{\delta_k}$, we obtain that

$$-\int_{\tilde{B}_{r,n}\setminus B_{\delta_k}} \Delta u_n (z \cdot \nabla u_n) \, dz = \int_{\tilde{B}_{r,n}\setminus B_{\delta_k}} f u_n (z \cdot \nabla u_n) \, dz. \tag{2.4.26}$$

Integration by parts allows us to rewrite the first term in (2.4.26) as follows

$$\begin{aligned} -\int_{\tilde{B}_{r,n}\setminus B_{\delta_k}} \Delta u_n (z \cdot \nabla u_n) \, dz = &\int_{\tilde{B}_{r,n}\setminus B_{\delta_k}} \nabla u_n \cdot \nabla (z \cdot \nabla u_n) \, dz - r \int_{\tilde{S}_{r,n}} \left| \frac{\partial u_n}{\partial \nu} \right|^2 dS \\ &- \int_{\tilde{\gamma}_{r,n}} \left| \frac{\partial u_n}{\partial \nu} \right|^2 z \cdot \nu \, dS + \delta_k \int_{\partial B_{\delta_k}} \left| \frac{\partial u_n}{\partial \nu} \right|^2 dS, \end{aligned} \tag{2.4.27}$$

where we used that $z = r\nu$ on $\tilde{S}_{r,n}$, $z = -\delta_k \nu$ on ∂B_{δ_k} and the gradient ∇u_n is orthogonal to $\tilde{\gamma}_{r,n}$, i.e. $\nabla u_n = \frac{\partial u_n}{\partial \nu} \nu$ on $\tilde{\gamma}_{r,n}$. Furthermore, by direct calculations, the first term on the right hand side in (2.4.27) can be rewritten as

$$\begin{aligned} &\int_{\tilde{B}_{r,n}\setminus B_{\delta_k}} \nabla u_n \cdot \nabla (z \cdot \nabla u_n) \, dz \\ &\quad = \sum_{i,j=1}^{N+1} \int_{\tilde{B}_{r,n}\setminus B_{\delta_k}} \frac{\partial u_n}{\partial z_i} \frac{\partial}{\partial z_i} \left(\frac{\partial u_n}{\partial z_j} z_j \right) dz \sum_{i,j=1}^{N+1} \int_{\tilde{B}_{r,n}\setminus B_{\delta_k}} \frac{\partial u_n}{\partial z_i} \left[\frac{\partial^2 u_n}{\partial z_i \partial z_j} z_j + \delta_{ij} \frac{\partial u_n}{\partial z_j} \right] dz \\ &\quad = \int_{\tilde{B}_{r,n}\setminus B_{\delta_k}} |\nabla u_n|^2 \, dz + \frac{1}{2} \sum_{i,j=1}^{N+1} \int_{\tilde{B}_{r,n}\setminus B_{\delta_k}} z_j \frac{\partial}{\partial z_j} \left[\left(\frac{\partial u_n}{\partial z_i} \right)^2 \right] dz \\ &\quad = \int_{\tilde{B}_{r,n}\setminus B_{\delta_k}} |\nabla u_n|^2 \, dz - \frac{N+1}{2} \int_{\tilde{B}_{r,n}\setminus B_{\delta_k}} |\nabla u_n|^2 \, dz \\ &\qquad + \frac{1}{2} \int_{\partial(\tilde{B}_{r,n}\setminus B_{\delta_k})} \sum_{i,j=1}^{N+1} \left| \frac{\partial u_n}{\partial z_i} \right|^2 z_j \nu_j \, dS \\ &\quad = -\frac{N-1}{2} \int_{\tilde{B}_{r,n}\setminus B_{\delta_k}} |\nabla u_n|^2 \, dz + \frac{r}{2} \int_{\tilde{S}_{r,n}} |\nabla u_n|^2 \, dS + \frac{1}{2} \int_{\tilde{\gamma}_{r,n}} \left| \frac{\partial u_n}{\partial \nu} \right|^2 z \cdot \nu \, dS \\ &\qquad - \frac{\delta_k}{2} \int_{\partial B_{\delta_k}} |\nabla u_n|^2 \, dS. \end{aligned} \tag{2.4.28}$$

Putting together (2.4.26), (2.4.27) and (2.4.28), we obtain that

$$
\begin{aligned}
&-\frac{N-1}{2}\int_{\tilde{B}_{r,n}\setminus B_{\delta_k}}|\nabla u_n|^2\,dz+\frac{r}{2}\int_{\tilde{S}_{r,n}}|\nabla u_n|^2\,dS-\frac{1}{2}\int_{\tilde{\gamma}_{r,n}}\left|\frac{\partial u_n}{\partial\nu}\right|^2 z\cdot\nu\,dS-r\int_{\tilde{S}_{r,n}}\left|\frac{\partial u_n}{\partial\nu}\right|^2 dS\\
&-\frac{\delta_k}{2}\int_{\partial B_{\delta_k}}|\nabla u_n|^2\,dS+\delta_k\int_{\partial B_{\delta_k}}\left|\frac{\partial u_n}{\partial\nu}\right|^2 dS-\int_{\tilde{B}_{r,n}\setminus B_{\delta_k}}fu_n(z\cdot\nabla u_n)\,dz=0. \quad (2.4.29)
\end{aligned}
$$

Under assumptions (H1-1)-(H1-3), we have that $f\,u_n(z\cdot\nabla u_n)\in L^1(B_r)$, indeed by the Hardy inequality (2.4.1)

$$
\begin{aligned}
\int_{B_r}|f\,u_n(z\cdot\nabla u_n)|\,dz&\le\xi_f(r)\int_{B_r}\frac{|u_n(z)|}{|z|}|\nabla u_n|\,dz\\
&\le\xi_f(r)\left(\int_{B_r}\frac{|u_n(z)|^2}{|z|^2}\,dz\right)^{1/2}\left(\int_{B_r}|\nabla u_n|^2\,dz\right)^{1/2}\\
&\le\mathrm{const}\,\xi_f(r)\left(\int_{B_r}|\nabla u_n|^2\,dz+\frac{N-1}{2r}\int_{\partial B_r}|u_n|^2dS\right)^{1/2}\left(\int_{B_r}|\nabla u_n|^2\,dz\right)^{1/2}<\infty,
\end{aligned}
$$

since $\xi_f(r)$ is bounded thus finite for sufficiently small r, as a consequence of assumption (H1-1). Hence we use the Lebesgue's dominated convergence theorem to conclude that

$$
\begin{aligned}
\lim_{k\to\infty}\int_{\tilde{B}_{r,n}\setminus B_{\delta_k}}fu_n(z\cdot\nabla u_n)\,dz&=\lim_{k\to\infty}\int_{B_r\setminus B_{\delta_k}}fu_n(z\cdot\nabla u_n)\,dz\\
&=\int_{B_r}fu_n(z\cdot\nabla u_n)\,dz.
\end{aligned}
\quad (2.4.30)
$$

On the other hand, if (H2-1)-(H2-5) hold, we can use the Divergence Theorem to obtain that

$$
\begin{aligned}
&\int_{\tilde{B}_{r,n}\setminus B_{\delta_k}}fu_n(z\cdot\nabla u_n)\,dz=\frac{1}{2}\int_{\tilde{B}_{r,n}\setminus B_{\delta_k}}fz\cdot\nabla(u_n^2)\,dz\\
&\quad=\frac{r}{2}\int_{\tilde{S}_{r,n}}fu_n^2\,dS-\frac{1}{2}\int_{\tilde{B}_{r,n}\setminus B_{\delta_k}}\big(\nabla f\cdot z+(N+1)f\big)u_n^2\,dz-\frac{\delta_k}{2}\int_{\partial B_{\delta_k}}fu_n^2\,dS\\
&\quad=\frac{r}{2}\int_{\partial B_r}fu_n^2\,dS-\frac{1}{2}\int_{B_r\setminus B_{\delta_k}}\big(\nabla f\cdot z+(N+1)f\big)u_n^2\,dz-\frac{\delta_k}{2}\int_{\partial B_{\delta_k}}fu_n^2\,dS. \quad (2.4.31)
\end{aligned}
$$

Under assumptions (H2-1)-(H2-5), it holds that $\big(\nabla f\cdot z+(N+1)f\big)u_n^2\in L^1(B_r)$, indeed

$$
\begin{aligned}
\int_{B_r}|\nabla f\cdot z+(N+1)f|u_n^2\,dz&\le\int_{B_r}|\nabla f\cdot z|u_n^2\,dz+(N+1)\int_{B_r}|f|u_n^2\,dz\\
&\le(\eta(r,\nabla f\cdot z)+(N+1)\eta(r,f))\left(\int_{B_r}|\nabla u_n|^2\,dz+\frac{N-1}{2r}\int_{\partial B_r}|u_n|^2\,dS\right)<\infty,
\end{aligned}
$$

since $\eta(r, \nabla f \cdot z)$ is finite a.e. by assumption (H2-4) and $\eta(r, f)$ is finite for sufficiently small r in virtue of (H2-2). Then passing to the limit as $k \to \infty$ in (2.4.31), taking into account also (2.4.25), we deduce that

$$\lim_{k\to\infty} \int_{\tilde{B}_{r,n}\setminus B_{\delta_k}} f u_n (z\cdot\nabla u_n)\, dz = \frac{r}{2}\int_{\partial B_r} f u_n^2\, dS - \frac{1}{2}\int_{B_r} \big(\nabla f\cdot z + (N+1)f\big) u_n^2\, dz. \quad (2.4.32)$$

Letting $k \to +\infty$ in (2.4.29), by (2.4.25), (2.4.30), and (2.4.32), we attain (2.4.24). □

Exploiting Lemma 2.4.14 and the fact that the domains $\tilde{B}_{r,n}$ (defined as in (2.4.15)) are star-shaped with respect to the origin, we deduct the following inequality.

Corollary 2.4.15. *Let $0 < r < r_0$. Then there exists $n_0 = n_0(r) \in \mathbb{N} \setminus \{0\}$ such that, for all $n \geq n_0$,*

$$-\frac{N-1}{2}\int_{\tilde{B}_{r,n}} |\nabla u_n|^2\, dz + \frac{r}{2}\int_{\tilde{S}_{r,n}} |\nabla u_n|^2\, dS - r\int_{\tilde{S}_{r,n}} \left|\frac{\partial u_n}{\partial \nu}\right|^2 dS - \mathcal{R}(r, u_n) \geq 0, \quad (2.4.33)$$

where u_n is a weak solution to problem (2.4.18) *for each fixed $n \in \mathbb{N} \setminus \{0\}$.*

Proof. In view of (2.4.24), the left-hand side of (2.4.33) is equal to $\frac{1}{2}\int_{\tilde{\gamma}_{r,n}} \left|\frac{\partial u_n}{\partial \nu}\right|^2 z\cdot\nu\, dS$, which is in fact non-negative since $z \cdot \nu \geq 0$ on $\tilde{\gamma}_{r,n}$ by Lemma 2.4.8. □

Passing to the limit into (2.4.33) as $n \to \infty$, a similar inequality can be derived for a weak solution to (1.2.20).

Proposition 2.4.16. *Let u be a weak solution to* (1.2.20), *with f satisfying either* (H1-1)-(H1-3) *or* (H2-1)-(H2-5). *Then, for a.e. $r \in (0, r_0)$, we have that*

$$-\frac{N-1}{2}\int_{B_r} |\nabla u|^2\, dz + \frac{r}{2}\int_{\partial B_r} |\nabla u|^2\, dS - r\int_{\partial B_r} \left|\frac{\partial u}{\partial \nu}\right|^2 dS - \mathcal{R}(r, u) \geq 0 \quad (2.4.34)$$

and

$$\int_{B_r} |\nabla u|^2\, dz = \int_{B_r} f u^2\, dz + \int_{\partial B_r} u\frac{\partial u}{\partial \nu}\, dS. \quad (2.4.35)$$

Proof. In order to prove (2.4.34), we pass to the limit inside inequality (2.4.33). As regards the first term, it is sufficient to observe that

$$\int_{\tilde{B}_{r,n}} |\nabla u_n|^2 dz = \int_{B_r} |\nabla u_n|^2 dz \to \int_{B_r} |\nabla u|^2 dz \quad \text{as } n \to \infty,$$

for each fixed $r \in (0, r_0)$, as a consequence of Proposition 2.4.13. Dealing with the second term, we observe that, by strong H^1-convergence of u_n to u,

$$\lim_{n\to+\infty} \int_0^{r_0} \left(\int_{\partial B_r} |\nabla(u_n - u)|^2\, dS\right) dr = 0. \quad (2.4.36)$$

Letting

$$F_n(r) = \int_{\partial B_r} |\nabla(u_n - u)|^2 \, dS,$$

(2.4.36) implies that $F_n \to 0$ in $L^1(0, r_0)$. Then we can deduce that there exists a subsequence F_{n_k} such that $F_{n_k}(r) \to 0$ for a.e. $r \in (0, r_0)$, hence having that

$$\int_{\tilde{S}_{r,n_k}} |\nabla u_{n_k}|^2 \, dS = \int_{\partial B_r} |\nabla u_{n_k}|^2 \, dS \to \int_{\partial B_r} |\nabla u|^2 \, dS \quad \text{as } k \to \infty$$

for a.e. $r \in (0, r_0)$. Arguing in a similar way, we obtain that

$$\int_{\tilde{S}_{r,n_k}} \left| \frac{\partial u_{n_k}}{\partial \nu} \right|^2 dS \to \int_{\partial B_r} \left| \frac{\partial u}{\partial \nu} \right|^2 dS \quad \text{as } k \to \infty$$

for a.e. $r \in (0, r_0)$. It remains to prove the convergence of $\mathcal{R}(r, u_n)$ to $\mathcal{R}(r, u)$. Under the set of assumptions (H1-1)-(H1-3), we notice that

$$\begin{aligned} \int_{B_r} |f u_n (z \cdot \nabla u_n) - f u (z \cdot \nabla u)| \, dz &= \int_{B_r} |f(u_n - u)(z \cdot \nabla u_n) - f u z \cdot \nabla(u - u_n)| \, dz \\ &\leq \int_{B_r} |f(u_n - u)(z \cdot \nabla u_n)| dz + \int_{B_r} |f u z \cdot \nabla(u - u_n)| \, dz. \end{aligned} \tag{2.4.37}$$

The Hölder inequality, (2.4.1), and Proposition 2.4.13 imply that

$$\begin{aligned} &\int_{B_r} |f(u_n - u)(z \cdot \nabla u_n)| \, dz \leq \xi_f(r) \left(\int_{B_r} \frac{|u_n - u|^2}{|z|^2} \, dz \right)^{1/2} \left(\int_{B_r} |\nabla u_n|^2 \, dz \right)^{1/2} \\ &\leq \frac{2}{N-1} \xi_f(r) \left(\int_{B_r} |\nabla(u_n - u)|^2 \, dz + \frac{N-1}{2r} \int_{\partial B_r} |u_n - u|^2 dS \right)^{1/2} \left(\int_{B_r} |\nabla u_n|^2 \, dz \right)^{1/2} \to 0 \end{aligned}$$

and

$$\begin{aligned} &\int_{B_r} |f u z \cdot \nabla(u_n - u)| \, dz \leq \xi_f(r) \left(\int_{B_r} \frac{|u(z)|^2}{|z|^2} \, dz \right)^{1/2} \left(\int_{B_r} |\nabla(u_n - u)|^2 \, dz \right)^{1/2} \\ &\leq \frac{2}{N-1} \xi_f(r) \left(\int_{B_r} |\nabla u|^2 \, dz + \frac{N-1}{2r} \int_{\partial B_r} |u|^2 dS \right)^{1/2} \left(\int_{B_r} |\nabla(u_n - u)|^2 \, dz \right)^{1/2} \to 0 \end{aligned}$$

as $n \to \infty$, for all $r \in (0, r_0)$, since $\xi_f(r)$ is bounded thus finite for sufficiently small r, as a consequence of assumption (H1-1). Hence, from (2.4.37) we deduce that

$$\lim_{n \to \infty} \mathcal{R}(r, u_n) = \mathcal{R}(r, u) \tag{2.4.38}$$

under assumptions (H1-1)-(H1-3). In order to prove (2.4.38) under assumptions (H2-1)-(H2-5), we first use Proposition 2.4.13 and the Hölder inequality to observe that

$$\begin{aligned}
&\left| \int_{B_r} [\nabla f \cdot z + (N+1)f](u_n^2 - u^2)\, dz \right| \\
&\leq \left(\int_{B_r} (|\nabla f \cdot z| + (N+1)|f|)|u_n - u|^2\, dz \right)^{1/2} \left(\int_{B_r} (|\nabla f \cdot z| + (N+1)|f|)|u_n + u|^2\, dz \right)^{1/2} \\
&\leq (\eta(r, \nabla f \cdot z) + (N+1)\eta(r,f)) \left(\int_{B_r} |\nabla(u_n - u)|^2\, dz + \frac{N-1}{2r} \int_{\partial B_r} |u_n - u|^2\, dS \right)^{1/2} \\
&\quad \cdot \left(\int_{B_r} |\nabla(u_n + u)|^2\, dz + \frac{N-1}{2r} \int_{\partial B_r} |u_n + u|^2\, dS \right)^{1/2} \to 0,
\end{aligned}$$

as $n \to \infty$, for a.e. $r \in (0, r_0)$, since $\eta(r, \nabla f \cdot z)$ is finite a.e. by assumption (H2-4), $\eta(r, f)$ is finite for sufficiently small r in virtue of (H2-2) and $\{u_n + u\}_n$ is bounded in $H^1(B_r)$ for every $r \in (0, r_0)$. Furthermore, by the fact that f is bounded far from the origin and using the compactness of the trace map from $H^1(B_r)$ to $L^2(\partial B_r)$, it follows that

$$\int_{\partial B_r} f u_n^2\, dS \to \int_{\partial B_r} f u^2\, dS,$$

for a.e. $r \in (0, r_0)$. Hence, passing to the limit in $\mathcal{R}(r, u_n)$ we conclude the first part of the proof.

Finally (2.4.35) follows by testing (2.4.18) with u_n itself and passing to the limit arguing as above. □

2.4.3 The Almgren frequency function

Let $u \in H^1_\Gamma(B_{\hat{R}})$ be a non trivial solution to (1.2.20). For every $r \in (0, \hat{R})$ we define

$$\mathcal{D}(r) = r^{1-N} \int_{B_r} \left(|\nabla u|^2 - f u^2 \right) dz \tag{2.4.39}$$

and

$$\mathcal{H}(r) = r^{-N} \int_{\partial B_r} u^2\, dS. \tag{2.4.40}$$

In the following lemma we compute the derivative of the function $\mathcal{H}$.

Lemma 2.4.17. *It holds that $\mathcal{H} \in W^{1,1}_{\mathrm{loc}}(0, \hat{R})$ and*

$$\mathcal{H}'(r) = 2\, r^{-N} \int_{\partial B_r} u \frac{\partial u}{\partial \nu}\, dS \tag{2.4.41}$$

in a distributional sense and for a.e. $r \in (0, \hat{R})$. Furthermore

$$\mathcal{H}'(r) = \frac{2}{r} \mathcal{D}(r) \quad \text{for a.e. } r \in (0, \hat{R}). \tag{2.4.42}$$

Proof. First we observe that

$$\mathcal{H}(r) = \int_{\mathbb{S}^N} |u(r\theta)|^2 \, dS. \tag{2.4.43}$$

Let $\phi \in C_c^\infty(0, \hat{R})$. If we set $\phi(r) = v(r\theta)$ with $\theta \in \mathbb{S}^N$, we deduce that $\phi'(r) = \nabla v(r\theta) \cdot \theta$ and $v(\hat{R}\theta) = 0$ since ϕ is null at 0 and $\hat{R}$. Then, exploiting all these information

$$\begin{aligned}
-\int_0^{\hat{R}} \mathcal{H}(r)\phi'(r)\, dr = -\int_0^{\hat{R}} \phi'(r) \int_{\partial B_1} u^2(r\theta)\, dS\, dr = -\int_0^{\hat{R}} \int_{\mathbb{S}^N} u^2(r\theta) \nabla v(r\theta) \cdot \theta \, dS\, dr \\
= -\int_{B_{\hat{R}}} |z|^{-N-1} u^2(z) \nabla v(z) \cdot z \, dz \\
= -\int_{\partial B_{\hat{R}}} \big(v(z)|z|^{-N-1}u^2(z)z\big) \cdot \nu \, dS + 2\int_{B_{\hat{R}}} v(z)u(z)|z|^{-N-1} \nabla u(z) \cdot z\, dz \\
= 2\int_{B_{\hat{R}}} v(z)u(z)|z|^{-N-1} \nabla u(z) \cdot z\, dz \\
= 2\int_0^{\hat{R}} \phi(r) \left(\int_{\mathbb{S}^N} u(r\theta) \nabla u(r\theta) \cdot \theta \, dS \right) dr,
\end{aligned}$$

by the divergence Theorem. Thus we proved (2.4.41) in a distributional sense and a.e. Furthermore, using that $u, \nabla u \in L^2(B_{\hat{R}})$, we easily obtain that $\mathcal{H} \in W^{1,1}_{\text{loc}}(0, \hat{R})$. Identity (2.4.42) follows from (2.4.41) and (2.4.35). ☐

In order to define a suitable Almgren-type frequency function we show that the function $\mathcal{H}$ is strictly positive in a neighbourhood of 0.

Lemma 2.4.18. *For any $r \in (0, r_0]$ it holds that $\mathcal{H}(r) > 0$.*

Proof. Assume by contradiction that there exists $r_1 \in (0, r_0]$ such that $\mathcal{H}(r_1) = 0$, thus the trace of u on ∂B_{r_1} is null and hence $u \in H^1_0(B_{r_1} \setminus \Gamma)$. Then, testing (1.2.20) with u, we obtain that

$$\int_{B_{r_1}} |\nabla u|^2 \, dz - \int_{B_{r_1}} f u^2 \, dz = 0. \tag{2.4.44}$$

Therefore, from Lemma 2.4.6 and (2.4.44) it follows that

$$0 = \int_{B_{r_1}} [|\nabla u|^2 - f u^2]\, dz \geq \frac{1}{2} \int_{B_{r_1}} |\nabla u|^2 \, dz,$$

which, together with Lemma 2.4.2, implies that $u \equiv 0$ in B_{r_1}. From classical unique continuation principles for second order elliptic equations with locally bounded coefficients (see e.g. [79]), we can conclude that $u = 0$ a.e. in $B_{\hat{R}}$, a contradiction. ☐

The following lemma contains an estimate from below for the derivative of the function $\mathcal{D}$, making use of the Pohozaev-type inequality found in Section 2.4.2.

Lemma 2.4.19. *The function $\mathcal{D}$ defined in* (3.3.1) *belongs to* $W^{1,1}_{\mathrm{loc}}(0,\hat{R})$ *and*

$$\begin{aligned}\mathcal{D}'(r) \geq 2r^{1-N}\int_{\partial B_r}\left|\frac{\partial u}{\partial \nu}\right|^2 dS + (N-1)r^{-N}\int_{B_r} fu^2\,dz + 2r^{-N}\mathcal{R}(r,u)\\ - r^{1-N}\int_{\partial B_r} fu^2\,dS\end{aligned} \tag{2.4.45}$$

for a.e. $r \in (0, r_0)$.

Proof. By direct calculations, we deduce that

$$\mathcal{D}'(r) = (1-N)r^{-N}\int_{B_r}\left(|\nabla u|^2 - fu^2\right)dz + r^{1-N}\int_{\partial B_r}\left(|\nabla u|^2 - fu^2\right)dS \tag{2.4.46}$$

in the distributional sense and for a.e. $r \in (0,\hat{R})$. This allows us to conclude that $\mathcal{D} \in W^{1,1}_{\mathrm{loc}}(0,\hat{R})$. Inserting (2.4.34) into (2.4.46), we obtain (2.4.45). □

Thanks to Lemma 2.4.18, the Almgren frequency function

$$\mathcal{N}\colon (0,r_0] \to \mathbb{R}, \quad \mathcal{N}(r) = \frac{\mathcal{D}(r)}{\mathcal{H}(r)} \tag{2.4.47}$$

is well defined. As a consequence of Lemmas 2.4.6, 2.4.17 and 2.4.19, we provide the following estimates from below of the Almgren function $\mathcal{N}$ and its derivative.

Lemma 2.4.20. *The function $\mathcal{N}$ defined in* (2.4.47) *belongs to* $W^{1,1}_{\mathrm{loc}}((0,r_0])$ *and*

$$\mathcal{N}'(r) \geq \nu_1(r) + \nu_2(r) \tag{2.4.48}$$

for a.e. $r \in (0,r_0)$, *where*

$$\nu_1(r) = \frac{2r\left[\left(\int_{\partial B_r}\left|\frac{\partial u}{\partial \nu}\right|^2 dS\right)\left(\int_{\partial B_r}|u|^2\,dS\right) - \left(\int_{\partial B_r} u\frac{\partial u}{\partial \nu}\,dS\right)^2\right]}{\left(\int_{\partial B_r}|u|^2\,dS\right)^2}$$

and

$$\nu_2(r) = \frac{2\left[\frac{N-1}{2}\int_{B_r} fu^2\,dz + \mathcal{R}(r,u) - \frac{r}{2}\int_{\partial B_r} fu^2\,dS\right]}{\int_{\partial B_r}|u|^2\,dS}. \tag{2.4.49}$$

Furthermore,

$$\mathcal{N}(r) > -\frac{N-1}{4} \quad \text{for every } r \in (0,r_0) \tag{2.4.50}$$

and, for every $\varepsilon > 0$, there exists $r_\varepsilon > 0$ such that

$$\mathcal{N}(r) > -\varepsilon \quad \text{for every } r \in (0,r_\varepsilon), \tag{2.4.51}$$

i.e. $\liminf\limits_{r\to 0^+}\mathcal{N}(r) \geq 0$.

Proof. We can easily obtain that $\mathcal{N} \in W^{1,1}_{\rm loc}((0,r_0])$, arguing as in Lemma 2.3.17, by Lemmas 2.4.17, 2.4.18, and 2.4.19. Using (2.4.42) we have that

$$\mathcal{N}'(r) = \frac{\mathcal{D}'(r)\mathcal{H}(r) - \mathcal{D}(r)\mathcal{H}'(r)}{[\mathcal{H}(r)]^2} = \frac{\mathcal{D}'(r)\mathcal{H}(r) - \frac{r}{2}[\mathcal{H}'(r)]^2}{[\mathcal{H}(r)]^2}$$

for a.e. $r \in (0,r_0)$ and the proof of (2.4.48) easily follows from (2.4.41) and (2.4.45). To prove (2.4.50) and (2.4.51), we observe that (3.3.1) and (2.4.40), together with Lemma 2.4.6, imply that

$$\mathcal{N}(r) = \frac{\mathcal{D}(r)}{\mathcal{H}(r)} \geq \frac{r\left[\frac{1}{2}\int_{B_r}|\nabla u|^2\,dz - \omega(r)\int_{\partial B_r}|u|^2\,dS\right]}{\int_{\partial B_r}|u|^2\,dS} \geq -r\omega(r) \tag{2.4.52}$$

for every $r \in (0,r_0)$, where ω is defined in (2.4.7). Then (2.4.50) follows directly from (2.4.8). From either assumption (H1-1) or (H2-1) it follows that $\lim_{r\to 0^+} r\omega(r) = 0$; hence (2.4.52) implies (2.4.51). □

Lemma 2.4.21. *Let ν_2 be as in* (2.4.49)*. There exists a positive constant $C_1 > 0$ such that*

$$|\nu_2(r)| \leq C_1\alpha(r)\left[\mathcal{N}(r) + \frac{N-1}{2}\right] \tag{2.4.53}$$

for all $r \in (0,r_0)$, where

$$\alpha(r) = \begin{cases} r^{-1}\xi_f(r), & \textit{under assumptions } \text{(H1-1)-(H1-3)},\\ r^{-1}\left(\eta(r,f) + \eta(r,\nabla f\cdot z)\right), & \textit{under assumptions } \text{(H2-1)-(H2-5)}. \end{cases} \tag{2.4.54}$$

Proof. From Lemma 2.4.6 we deduce that for all $r \in (0,r_0)$,

$$\int_{B_r}|\nabla u|^2\,dz \leq 2\big(r^{N-1}\mathcal{D}(r) + \omega(r)r^N\mathcal{H}(r)\big), \tag{2.4.55}$$

where $\omega(r)$ is defined in (2.4.7).

Let us first suppose to be under assumptions (H1-1)-(H1-3). Estimating the first term in the numerator of $\nu_2(r)$ we obtain that

$$\begin{aligned}\left|\int_{B_r} fu^2\,dz\right| &\leq \xi_f(r)\int_{B_r}\frac{|u(z)|^2}{|z|^2}\,dz \leq \xi_f(r)\frac{4}{(N-1)^2}\left[\int_{B_r}|\nabla u|^2\,dz + \frac{N-1}{2r}\int_{\partial B_r}u^2\,dS\right]\\ &\leq \frac{8}{(N-1)^2}r^{N-1}\xi_f(r)\mathcal{D}(r) + \frac{16}{(N-1)^3}r^{N-1}\left(\xi_f(r)\right)^2\mathcal{H}(r) + \frac{2}{N-1}r^{N-1}\xi_f(r)\mathcal{H}(r)\\ &\leq \frac{8}{(N-1)^2}r^{N-1}\xi_f(r)\mathcal{D}(r) + \frac{4}{N-1}r^{N-1}\xi_f(r)\mathcal{H}(r)\\ &= \frac{8}{(N-1)^2}r^{N-1}\xi_f(r)\left(\mathcal{D}(r) + \frac{N-1}{2}\mathcal{H}(r)\right),\end{aligned} \tag{2.4.56}$$

where we used (H1-3), Lemma 2.4.2, (2.4.55) and (2.4.11). Using Hölder inequality, (2.4.56), (2.4.11), and (2.4.55), the second term can be estimated as follows

$$\begin{aligned}\left|\int_{B_r} fuz\cdot\nabla u\,dz\right| &\le \xi_f(r)\Big(\int_{B_r}\frac{|u(z)|^2}{|z|^2}\,dz\Big)^{1/2}\Big(\int_{B_r}|\nabla u|^2\,dz\Big)^{1/2}\\ &\le \xi_f(r)\frac{4}{N-1}r^{N-1}\left(\mathcal{D}(r)+\frac{N-1}{2}\mathcal{H}(r)\right)^{1/2}\left(\mathcal{D}(r)+\frac{2}{N-1}\xi_f(r)\mathcal{H}(r)\right)^{1/2}\\ &\le \xi_f(r)\frac{4}{N-1}r^{N-1}\left(\mathcal{D}(r)+\frac{N-1}{2}\mathcal{H}(r)\right).\end{aligned} \tag{2.4.57}$$

For the last term we have that

$$r\left|\int_{\partial B_r} fu^2\,dS\right|\le\frac{\xi_f(r)}{r}\int_{\partial B_r}u^2\,dS=\xi_f(r)r^{N-1}\mathcal{H}(r). \tag{2.4.58}$$

Combining (2.4.56), (2.4.57), and (2.4.58), we obtain that, for all $r\in(0,r_0)$,

$$|\nu_2(r)|\le C_1\,\xi_f(r)r^{-1}\left[\mathcal{N}(r)+\frac{N-1}{2}\right]$$

for some positive constant $C_1>0$ which does not depend on r.

Now let us suppose to be under assumptions (H2-1)-(H2-5). In this case, the definition of $\mathcal{R}(r,u)$ allows us to rewrite ν_2 as

$$\nu_2(r)=-\frac{\int_{B_r}(2f+\nabla f\cdot z)u^2\,dz}{\int_{\partial B_r}u^2\,dS}.$$

From (H2-5), (2.4.55) and (2.4.13) it follows that

$$\begin{aligned}\left|\int_{B_r}(2f+\nabla f\cdot z)u^2\,dz\right| &\le (2\eta(r,f)+\eta(r,\nabla f\cdot x))\Big(\int_{B_r}|\nabla u|^2\,dz+\frac{N-1}{2r}\int_{\partial B_r}|u|^2\,dS\Big)\\ &\le 2(2\eta(r,f)+\eta(r,\nabla f\cdot x))r^{N-1}\left(\mathcal{D}(r)+\frac{N-1}{2}\eta(r,f)\mathcal{H}(r)+\frac{N-1}{4}\mathcal{H}(r)\right)\\ &\le 2(2\eta(r,f)+\eta(r,\nabla f\cdot x))r^{N-1}\left(\mathcal{D}(r)+\frac{N-1}{2}\mathcal{H}(r)\right).\end{aligned}$$

Therefore, we have that

$$|\nu_2(r)|\le\frac{2(2\eta(r,f)+\eta(r,\nabla f\cdot x))}{r}\left(\mathcal{N}(r)+\frac{N-1}{2}\right)$$

and estimate (2.4.53) is proved also under assumptions (H2-1)-(H2-5), with $C_1=4$. □

Lemma 2.4.22. *Letting r_0 be as in Lemma 2.4.6 and $\mathcal{N}$ as in* (2.4.47)*, there exists a positive constant $C_2>0$ such that*

$$\mathcal{N}(r)\le C_2\quad\text{for all } r\in(0,r_0). \tag{2.4.59}$$

Proof. By Lemma 2.4.20, Schwarz's inequality, and Lemma 2.4.21, we obtain

$$\left(\mathcal{N} + \frac{N-1}{2}\right)'(r) \geq \nu_2(r) \geq -C_1\alpha(r)\left[\mathcal{N}(r) + \frac{N-1}{2}\right] \tag{2.4.60}$$

for a.e. $r \in (0, r_0)$, where α is defined in (2.4.54). Taking into account that $\mathcal{N}(r)+\frac{N-1}{2} > 0$ for all $r \in (0, r_0)$ in view of (2.4.50) and $\alpha \in L^1(0, r_0)$ thanks to assumptions (H1-2), (H2-2) and (H2-4), after integration over (r, r_0) it follows that

$$\mathcal{N}(r) \leq -\frac{N-1}{2} + \left(\mathcal{N}(r_0) + \frac{N-1}{2}\right)\exp\left(C_1\int_0^{r_0}\alpha(s)ds\right)$$

for any $r \in (0, r_0)$, thus proving estimate (2.4.59). □

Lemma 2.4.23. *The limit*

$$\gamma := \lim_{r\to 0^+} \mathcal{N}(r)$$

exists and is finite. Moreover $\gamma \geq 0$.

Proof. Since $\mathcal{N}'(r) \geq -C_1\alpha(r)\big[\mathcal{N}(r) + \frac{N-1}{2}\big]$ for a.e. $r \in (0, r_0)$ in view of (2.4.60) and $\alpha \in L^1(0, r_0)$ by assumptions (H1-2), (H2-2) and (H2-4), we have that

$$\frac{d}{dr}\left[e^{C_1\int_0^r \alpha(s)\,ds}\left(\mathcal{N}(r) + \frac{N-1}{2}\right)\right] \geq 0 \quad \text{for a.e. } r \in (0, r_0),$$

therefore the limit of $r \mapsto e^{C_1\int_0^r \alpha(s)\,ds}\big(\mathcal{N}(r) + \frac{N-1}{2}\big)$ as $r \to 0^+$ exists; hence the function $\mathcal{N}$ has a limit as $r \to 0^+$.

From (2.4.59) and (2.4.51) it follows that $C_2 \geq \gamma := \lim_{r\to 0^+} \mathcal{N}(r) = \liminf_{r\to 0^+} \mathcal{N}(r) \geq 0$; in particular γ is finite. □

A first consequence of the above analysis on the Almgren's frequency function is the following estimate of $\mathcal{H}(r)$.

Lemma 2.4.24. *Let* γ *be as in Lemma 2.4.23 and* r_0 *be as in Lemma 2.4.6. Then there exists a constant* $K_1 > 0$ *such that*

$$\mathcal{H}(r) \leq K_1 r^{2\gamma} \quad \text{for all } r \in (0, r_0). \tag{2.4.61}$$

On the other hand, for any $\sigma > 0$ *there exists a constant* $K_2(\sigma) > 0$ *depending on* σ *such that*

$$\mathcal{H}(r) \geq K_2(\sigma) r^{2\gamma+\sigma} \quad \text{for all } r \in (0, r_0). \tag{2.4.62}$$

Proof. By (2.4.60) and (2.4.59) we have that

$$\mathcal{N}'(r) \geq -C_1\left(C_2 + \frac{N-1}{2}\right)\alpha(r) \quad \text{a.e. in } (0, r_0). \tag{2.4.63}$$

Moreover, it holds that for all $r \in (0, r_0)$

$$\mathcal{N}(r) - \gamma = \int_0^r \mathcal{N}'(s)\, ds. \tag{2.4.64}$$

Indeed, for any fixed $r \in (0, r_0)$

$$\begin{aligned} \mathcal{N}(r) - \mathcal{N}(\varepsilon) &= \int_\varepsilon^r \mathcal{N}'(s)\, ds = \int_\varepsilon^r \alpha_1(s)\, ds + \int_\varepsilon^r \alpha_2(s)\, ds \\ &= \int_0^r \chi_{(\varepsilon,r)}\alpha_1(s)\, ds + \int_0^r \chi_{(\varepsilon,r)}\alpha_2(s)\, ds, \end{aligned} \tag{2.4.65}$$

with

$$\alpha_1(s) := \mathcal{N}' + C_1\left(C_2 + \frac{N-1}{2}\right)\alpha(s) \tag{2.4.66}$$

and

$$\alpha_2(s) := -C_1\left(C_2 + \frac{N-1}{2}\right)\alpha(s). \tag{2.4.67}$$

Then, passing to the limit as $\varepsilon \to 0$ into (2.4.65), we obtain that the left hand side tends to $\mathcal{N}(r) - \gamma$ by Lemma 2.4.23 and, on the right hand side, the first term tends to $\int_0^r \alpha_1(s)\, ds$ as a consequence of the monotone convergence theorem since $\alpha_1(s) \geq 0$ a.e. in $s \in (0, r)$ by (2.4.63) and the second term goes to $\int_0^r \alpha_2(s)\, ds$ by the Lebesgue's dominated convergence theorem, using that $\alpha_2 \in L^1(0, r_0)$ since $\alpha \in L^1(0, r_0)$ due to assumptions (H1-2), (H2-2) and (H2-4). Therefore from (2.4.64) and (2.4.63), it follows that

$$\mathcal{N}(r) - \gamma \geq -C_1\left(C_2 + \frac{N-1}{2}\right)\int_0^r \alpha(s)\, ds = -C_3 r F(r), \tag{2.4.68}$$

where $C_3 := C_1\big(C_2 + \frac{N-1}{2}\big)$ and

$$F(r) := \frac{1}{r}\int_0^r \alpha(s)\, ds.$$

We observe that, thanks to assumptions (H1-2), (H2-2) and (H2-4),

$$F \in L^1(0, r_0). \tag{2.4.69}$$

From (2.4.42) and (2.4.68) we deduce that, for a.e. $r \in (0, r_0)$,

$$\frac{\mathcal{H}'(r)}{\mathcal{H}(r)} = \frac{2\mathcal{N}(r)}{r} \geq \frac{2\gamma}{r} - 2C_3 F(r),$$

which, thanks to (2.4.69), after integration over the interval (r, r_0), yields (2.4.61).

Let us prove (2.4.62). Since $\gamma := \lim_{r\to 0^+} \mathcal{N}(r)$, for any $\sigma > 0$ there exists $r_\sigma > 0$ such that $\mathcal{N}(r) < \gamma + \sigma/2$ for any $r \in (0, r_\sigma)$ and hence

$$\frac{\mathcal{H}'(r)}{\mathcal{H}(r)} = \frac{2\mathcal{N}(r)}{r} < \frac{2\gamma+\sigma}{r} \quad \text{for all } r \in (0, r_\sigma).$$

Integrating over the interval (r, r_σ), we then obtain that

$$\frac{\mathcal{H}(r)}{r^{2\gamma+\sigma}} \geq \frac{\mathcal{H}(r_\sigma)}{r_\sigma^{2\gamma+\sigma}} \quad \text{for all } r \in (0, r_\sigma). \tag{2.4.70}$$

Neverthless, by the continuity of $\mathcal{H}$ outside 0, we can assert that

$$\frac{\mathcal{H}(r)}{r^{2\gamma+\sigma}} \geq \min_{r\in[r_\sigma, r_0]} \frac{\mathcal{H}(r)}{r^{2\gamma+\sigma}} > 0 \quad \text{for all } r \in [r_\sigma, r_0]. \tag{2.4.71}$$

Combining (2.4.70) and (2.4.71), we derive (2.4.62) for some positive constant $K_2(\sigma) > 0$ depending on σ. □

2.4.4 The blow-up argument

let u be a non trivial weak $H^1(B_{\hat{R}})$-solution to equation (1.2.20) with f satisfying either (H1-1)-(H1-3) or (H2-1)-(H2-5). Let $\mathcal{D}$ and $\mathcal{H}$ be the functions defined in (3.3.1) and (2.4.40) and r_0 be as in Lemma 2.4.6. We define the following scaled function

$$w^\lambda(z) = \frac{u(\lambda z)}{\sqrt{\mathcal{H}(\lambda)}}, \tag{2.4.72}$$

with $\lambda \in (0, r_0)$. We notice that $w^\lambda \in H^1_{\Gamma_\lambda}(B_{\hat{R}/\lambda})$, where

$$\Gamma_\lambda := \Gamma/\lambda = \{x \in \mathbb{R}^N : \lambda x \in \Gamma\} = \left\{x = (x', x_N) \in \mathbb{R}^N : x_N \geq \frac{g(\lambda x')}{\lambda}\right\},$$

and

$$\int_{B_{\hat{R}/\lambda}} \nabla w^\lambda(z) \cdot \nabla v(z)\, dz - \lambda^2 \int_{B_{\hat{R}/\lambda}} f(\lambda z) w^\lambda(z) v(z)\, dz = 0 \quad \text{for all } v \in C_c^\infty(B_{\hat{R}/\lambda} \setminus \Gamma_\lambda),$$

i.e. w^λ weakly solves

$$\begin{cases} -\Delta w^\lambda(z) = \lambda^2 f(\lambda z)\, w^\lambda(z) & \text{in } B_{\hat{R}/\lambda} \setminus \Gamma_\lambda, \\ w^\lambda = 0 & \text{on } \Gamma_\lambda. \end{cases} \tag{2.4.73}$$

Remark 2.4.25. *From assumptions* (1.2.17) *we easily deduce that* $\mathbb{R}^{N+1} \setminus \Gamma_\lambda$ *converges in the sense of Mosco (see [19, 61]) to the set* $\mathbb{R}^{N+1} \setminus \tilde{\Gamma}$, *where*

$$\tilde{\Gamma} = \{(x', x_N) \in \mathbb{R}^N : x_N \geq 0\}. \tag{2.4.74}$$

In particular, for every $R > 0$, *the weak limit points in* $H^1(B_R)$ *as* $\lambda \to 0^+$ *of the family of functions* $\{w^\lambda\}_\lambda$ *belong to* $H^1_{\tilde{\Gamma}}(B_R)$.

Lemma 2.4.26. *Let w^λ be defined in* (2.4.72) *with* $\lambda \in (0, r_0)$. *Then* $\{w^\lambda\}_{\lambda\in(0,r_0)}$ *is bounded in* $H^1(B_1)$.

Proof. From (2.4.43) we deduce that

$$\int_{\partial B_1} |w^\lambda|^2 dS = 1. \tag{2.4.75}$$

By scaling and using (2.4.6) we have that

$$\mathcal{N}(\lambda) \geq \frac{\lambda^{1-N}}{\mathcal{H}(\lambda)} \left(\frac{1}{2} \int_{B_\lambda} |\nabla u|^2 \, dz - \omega(\lambda) \int_{\partial B_\lambda} u^2 \, dS \right) = \frac{1}{2} \int_{B_1} |\nabla w^\lambda(z)|^2 \, dz - \lambda\, \omega(\lambda). \tag{2.4.76}$$

Combining (2.4.76), (2.4.59), and (2.4.8) we infer that for every $\lambda \in (0, r_0)$

$$\frac{1}{2} \int_{B_1} |\nabla w^\lambda(z)|^2 \, dz \leq C_2 + \frac{N-1}{4}. \tag{2.4.77}$$

After applying suitable changes of variable to inequality (2.4.1), we obtain that

$$\begin{aligned} \left(\frac{N-1}{2}\right)^2 \lambda^{N-1}\mathcal{H}(\lambda) \int_{B_1} |w^\lambda(z)|^2 \, dz \leq{}& \frac{N-1}{2} \lambda^{N-1} \mathcal{H}(\lambda) \\ &+ \lambda^{N-1}\mathcal{H}(\lambda) \int_{B_1} |\nabla w^\lambda(z)|^2 \, dz, \end{aligned} \tag{2.4.78}$$

where we used (2.4.75). Dividing each member of (2.4.78) by $\lambda^{N-1}\mathcal{H}(\lambda)$ and exploiting (2.4.77), we achieve

$$\int_{B_1} |w^\lambda(z)|^2 \, dz \leq \frac{N-1}{2} + \int_{B_1} |\nabla w^\lambda(z)|^2 \, dz \leq 2C_2 + N - 1, \tag{2.4.79}$$

thus concluding the proof. □

In the following we exhibit a *doubling* type result.

Lemma 2.4.27. *There exists a positive constant* $C_4 > 0$ *such that*

$$\frac{1}{C_4}\mathcal{H}(\lambda) \leq \mathcal{H}(R\lambda) \leq C_4 \mathcal{H}(\lambda) \quad \text{for any } \lambda \in (0, r_0/2) \text{ and } R \in [1,2], \tag{2.4.80}$$

$$\int_{B_R} |\nabla w^\lambda(z)|^2 \, dz \leq 2^{N-1} C_4 \int_{B_1} |\nabla w^{R\lambda}(z)|^2 \, dz \quad \text{for any } \lambda \in (0, r_0/2) \text{ and } R \in [1,2], \tag{2.4.81}$$

and

$$\int_{B_R} |w^\lambda(z)|^2 \, dx \leq 2^{N+1} C_4 \int_{B_1} |w^{R\lambda}(z)|^2 \, dz \quad \text{for any } \lambda \in (0, r_0/2) \text{ and } R \in [1,2], \tag{2.4.82}$$

where w^λ is defined in (2.4.72).

Proof. By (2.4.42), (2.4.50) and (2.4.59), it follows that

$$-\frac{N-1}{2r} \leq \frac{\mathcal{H}'(r)}{\mathcal{H}(r)} = \frac{2\mathcal{N}(r)}{r} \leq \frac{2C_2}{r} \quad \text{for a.e. } r \in (0, r_0).$$

Let $R \in (1, 2]$. Integrating over $(\lambda, R\lambda)$ for $\lambda < r_0/R$ the above inequality and taking into account that $R \leq 2$, we obtain

$$2^{(1-N)/2}\mathcal{H}(\lambda) \leq \mathcal{H}(R\lambda) \leq 4^{C_2}\mathcal{H}(\lambda) \quad \text{for every } \lambda \in (0, r_0/R).$$

The above estimates trivially hold also for $R = 1$, hence (2.4.80) with

$$C_4 := \max\{4^{C_2}, 2^{(N-1)/2}\}$$

is established. For every $\lambda \in (0, r_0/2)$ and $R \in [1, 2]$, (2.4.80) yields

$$\begin{aligned}\int_{B_R} |\nabla w^\lambda(z)|^2 \, dz &= \frac{\lambda^{1-N}}{\mathcal{H}(\lambda)} \int_{B_{R\lambda}} |\nabla u(z)|^2 \, dz \\ &= R^{N-1} \frac{\mathcal{H}(R\lambda)}{\mathcal{H}(\lambda)} \int_{B_1} |\nabla w^{R\lambda}(z)|^2 \, dz \leq R^{N-1} C_4 \int_{B_1} |\nabla w^{R\lambda}(z)|^2 \, dz,\end{aligned}$$

thus proving (2.4.81). A similar argument allows deducing (2.4.82) from (2.4.80). □

Lemma 2.4.28. *For every $\lambda \in (0, r_0)$, let w^λ be as in* (2.4.72). *Then there exist $M > 0$ and $\lambda_0 > 0$ such that, for any $\lambda \in (0, \lambda_0)$, there exists $R_\lambda \in [1, 2]$ such that*

$$\int_{\partial B_{R_\lambda}} |\nabla w^\lambda|^2 \, dS \leq M \int_{B_{R_\lambda}} |\nabla w^\lambda(z)|^2 \, dz.$$

Proof. From Lemma 2.4.26 we know that the family $\{w^\lambda\}_{\lambda \in (0, r_0)}$ is bounded in $H^1(B_1)$. Moreover Lemma 2.4.27 implies that the set $\{w^\lambda\}_{\lambda \in (0, r_0/2)}$ is bounded in $H^1(B_2)$ and hence

$$\limsup_{\lambda \to 0^+} \int_{B_2} |\nabla w^\lambda(z)|^2 \, dz < +\infty. \tag{2.4.83}$$

For every $\lambda \in (0, r_0/2)$ the function $f_\lambda(r) = \displaystyle\int_{B_r} |\nabla w^\lambda(z)|^2 \, dz$ is absolutely continuous in $[0, 2]$ and its distributional derivative is given by

$$f'_\lambda(r) = \int_{\partial B_r} |\nabla w^\lambda|^2 dS \quad \text{for a.e. } r \in (0, 2).$$

We argue by contradiction and assume that for any $M > 0$ there exists a sequence $\lambda_n \to 0^+$ such that

$$\int_{\partial B_r} |\nabla w^{\lambda_n}|^2 dS > M \int_{B_r} |\nabla w^{\lambda_n}(z)|^2 dz \quad \text{for all } r \in [1, 2] \text{ and } n \in \mathbb{N},$$

i.e.

$$f'_{\lambda_n}(r) > M f_{\lambda_n}(r) \quad \text{for a.e. } r \in [1,2] \text{ and for every } n \in \mathbb{N}. \tag{2.4.84}$$

Integration of (2.4.84) over $[1,2]$ yields $f_{\lambda_n}(2) > e^M f_{\lambda_n}(1)$ for every $n \in \mathbb{N}$ and consequently

$$\limsup_{n\to+\infty} f_{\lambda_n}(1) \le e^{-M} \cdot \limsup_{n\to+\infty} f_{\lambda_n}(2).$$

It follows that

$$\liminf_{\lambda\to 0^+} f_\lambda(1) \le e^{-M} \cdot \limsup_{\lambda\to 0^+} f_\lambda(2) \quad \text{for all } M > 0.$$

Therefore, letting $M \to +\infty$ and taking into account (2.4.83), we obtain that $\liminf_{\lambda\to 0^+} f_\lambda(1) = 0$ i.e.

$$\liminf_{\lambda\to 0^+} \int_{B_1} |\nabla w^\lambda(z)|^2 \, dz = 0. \tag{2.4.85}$$

From (2.4.85) and boundedness of $\{w^\lambda\}_{\lambda\in(0,r_0)}$ in $H^1(B_1)$ we have that there exist a sequence $\tilde\lambda_n \to 0$ and some $w \in H^1(B_1)$ such that $w^{\tilde\lambda_n} \rightharpoonup w$ in $H^1(B_1)$ and

$$\lim_{n\to+\infty} \int_{B_1} |\nabla w^{\tilde\lambda_n}(z)|^2 \, dz = 0. \tag{2.4.86}$$

The compactness of the trace map from $H^1(B_1)$ to $L^2(\partial B_1)$ and (2.4.75) imply that

$$\int_{\partial B_1} |w|^2 dS = 1. \tag{2.4.87}$$

Moreover, by weak lower semicontinuity and (2.4.86),

$$\int_{B_1} |\nabla w(z)|^2 \, dz \le \lim_{n\to+\infty} \int_{B_1} |\nabla w^{\tilde\lambda_n}(z)|^2 \, dz = 0.$$

Hence $w \equiv$ const in B_1. On the other hand, in view of Remark 2.4.25, $w \in H^1_{\tilde\Gamma}(B_1)$ so that $w \equiv 0$ in B_1, thus contradicting (2.4.87). □

In the following lemma we show that the L^2-norm of the gradient of $w^{\lambda R_\lambda}$ on the boundary of the unit ball is bounded from above. It will be crucial to prove a convergence result for scaled solutions (2.4.72).

Lemma 2.4.29. *Let w^λ be as in (2.4.72) and R_λ be as in Lemma 2.4.28. Then there exists $\overline{M}$ such that*

$$\int_{\partial B_1} |\nabla w^{\lambda R_\lambda}|^2 \, dS \le \overline{M} \quad \textit{for any } 0 < \lambda < \min\Big\{\lambda_0, \frac{r_0}{2}\Big\}.$$

Proof. Since

$$\int_{\partial B_1} |\nabla w^{\lambda R_\lambda}|^2 \, dS = \frac{\lambda^2 R_\lambda^{2-N}}{\mathcal{H}(\lambda R_\lambda)} \int_{\partial B_{R_\lambda}} |\nabla u(\lambda z)|^2 \, dS(z) = \frac{R_\lambda^{2-N}\mathcal{H}(\lambda)}{\mathcal{H}(\lambda R_\lambda)} \int_{\partial B_{R_\lambda}} |\nabla w^\lambda|^2 \, dS,$$

from (2.4.80), (2.4.81), Lemma 2.4.28, Lemma 2.4.26, and the fact that $1 \le R_\lambda \le 2$, we deduce that for every $0 < \lambda < \min\{\lambda_0, \frac{r_0}{2}\}$,

$$\int_{\partial B_1} |\nabla w^{\lambda R_\lambda}|^2 \, dS \le C_4 M \int_{B_{R_\lambda}} |\nabla w^\lambda(z)|^2 \, dz \le 2^{N-1} C_4^2 M \int_{B_1} |\nabla w^{\lambda R_\lambda}(z)|^2 \, dz \le \overline{M} < +\infty,$$

thus completing the proof. □

In the following lemma, we derive the explicit formula (1.2.24) for the eigenvalues of problem (1.2.23).

Lemma 2.4.30. *The set of all eigenvalues of problem* (1.2.23) *is*

$$\left\{ \frac{k(k+2N-2)}{4} : \, k \in \mathbb{N} \setminus \{0\} \right\}$$

and all eigenfunctions belong to $L^\infty(\mathbb{S}^N)$.

Proof. Let us start by observing that, if μ is an eigenvalue of (1.2.23) with an associated eigenfunction ψ, then, letting

$$\sigma = -\frac{N-1}{2} + \sqrt{\left(\frac{N-1}{2}\right)^2 + \mu},$$

the function $W(\rho\theta) = \rho^\sigma \psi(\theta)$ belongs to $H^1_{\tilde\Gamma}(B_1)$ and is harmonic in $B_1 \setminus \tilde\Gamma$. From [15] it follows that there exists $k \in \mathbb{N} \setminus \{0\}$ such that $\sigma = \frac{k}{2}$, so that $\mu = \frac{k}{4}(k+2N-2)$. Moreover, from [15] we also deduce that $W \in L^\infty(B_1)$, thus implying that $\psi \in L^\infty(\mathbb{S}^N)$.

Viceversa, let us prove that all numbers of the form $\mu = \frac{k}{4}(k+2N-2)$ with $k \in \mathbb{N} \setminus \{0\}$ are eigenvalues of (1.2.23). Let us fix $k \in \mathbb{N} \setminus \{0\}$ and consider the function W defined, in cylindrical coordinates, as

$$W(x', r\cos t, r\sin t) = r^{k/2} \sin\left(\frac{k}{2} t\right), \quad x' \in \mathbb{R}^{N-1}, \; r \ge 0, \; t \in [0, 2\pi].$$

We have that W belongs to $H^1_{\tilde\Gamma}(B_1)$ and is harmonic in $B_1 \setminus \tilde\Gamma$; furthermore W is homogeneous of degree $k/2$, so that, letting $\psi := W\big|_{\mathbb{S}^N}$, we have that $\psi \in H^1_0(\mathbb{S}^N \setminus S_1^+)$, $\psi \not\equiv 0$, and

$$W(\rho\theta) = \rho^{k/2} \psi(\theta), \quad \rho \ge 0, \; \theta \in \mathbb{S}^N. \tag{2.4.88}$$

Plugging (2.4.88) into the equation $\Delta W = 0$ in $B_1 \setminus \tilde\Gamma$, we obtain that

$$\rho^{\frac{k}{2}-2}\Big(\tfrac{k}{2}\big(\tfrac{k}{2} - 1 + N\big)\psi(\theta) + \Delta_{\mathbb{S}^N}\psi\Big) = 0, \quad \rho > 0, \; \theta \in \mathbb{S}^N \setminus S_1^+,$$

so that $\frac{k}{4}(k+2N-2)$ is an eigenvalue of (1.2.23).

The lemma is thereby proved. □

Lemma 2.4.31. *Let $u \in H^1(B_{\hat{R}}) \setminus \{0\}$ be a non-trivial weak solution to* (1.2.20) *with f satisfying either* (H1-1)-(H1-3) *or* (H2-1)-(H2-5)*. Let γ be as in Lemma 2.4.23. Then*

(*i*) *there exists $k_0 \in \mathbb{N} \setminus \{0\}$ such that $\gamma = \frac{k_0}{2}$;*

(*ii*) *for every sequence $\lambda_n \to 0^+$, there exist a subsequence $\{\lambda_{n_k}\}_{k\in\mathbb{N}}$ and an eigenfunction ψ of problem* (1.2.23) *associated with the eigenvalue μ_{k_0} such that $\|\psi\|_{L^2(\mathbb{S}^N)} = 1$ and*

$$\frac{u(\lambda_{n_k} z)}{\sqrt{\mathcal{H}(\lambda_{n_k})}} \to |z|^\gamma \psi\left(\frac{z}{|z|}\right) \quad \textit{strongly in } H^1(B_1). \tag{2.4.89}$$

Proof. For $\lambda \in (0, \min\{r_0, \lambda_0\})$, let w^λ be as in (2.4.72) and R_λ be as in Lemma 2.4.28. Let $\lambda_n \to 0^+$. By Lemma 2.4.26, we have that the set $\{w^{\lambda R_\lambda} : \lambda \in (0, \min\{r_0/2, \lambda_0\})\}$ is bounded in $H^1(B_1)$. Then there exists a subsequence $\{\lambda_{n_k}\}_k$ such that $w^{\lambda_{n_k} R_{\lambda_{n_k}}} \rightharpoonup w$ weakly in $H^1(B_1)$ for some function $w \in H^1(B_1)$. The compactness of the trace map from $H^1(B_1)$ into $L^2(\partial B_1)$ and (2.4.75) ensure that

$$\int_{\partial B_1} |w|^2 dS = 1 \tag{2.4.90}$$

and, consequently, $w \not\equiv 0$. Furthermore, in view of Remark 2.4.25 we infer that $w \in H^1_{\tilde{\Gamma}}(B_1)$, where $\tilde{\Gamma}$ is the set defined in (2.4.74).

Let $\phi \in C_c^\infty(B_1 \setminus \tilde{\Gamma})$. It is easy to verify that $\phi \in C_c^\infty(B_1 \setminus \Gamma_\lambda)$ provided λ is sufficiently small. Indeed, we notice that a neighbourhood of $\tilde{\Gamma}$ is of type

$$U_\varepsilon := \{z = (x', x_N, t) \in \mathbb{R}^{N+1} \mid x_N > -\varepsilon\},$$

with $\varepsilon > 0$. By assumption ϕ vanishes on $B_1 \cap U_\varepsilon$ for some $\varepsilon > 0$. Then, it is sufficient to show that $B_1 \cap U_\varepsilon$ is a neighbourhood of $B_1 \cap \Gamma_\lambda$ for sufficiently small λ. To this aim, we observe that by (1.2.18), for λ sufficiently small $\left|\frac{g(\lambda x')}{\lambda}\right| \le \text{const}\,\lambda$. Hence $\frac{g(\lambda x')}{\lambda} \ge -\text{const}\,\lambda > -\varepsilon$ if we choose λ sufficiently small, thus having that $B_1 \cap \Gamma_\lambda \subseteq B_1 \cap U_\varepsilon$ and $\phi \in C_c^\infty(B_1 \setminus \Gamma_\lambda)$ provided λ is sufficiently small.

Therefore, since $w^{\lambda_{n_k} R_{\lambda_{n_k}}}$ weakly satisfies equation (2.4.73) with $\lambda = \lambda_{n_k} R_{\lambda_{n_k}}$ and, for sufficiently large k, $B_1 \subset B_{\hat{R}/(\lambda_{n_k} R_{\lambda_{n_k}})}$, we have that

$$\int_{B_1} \nabla w^{\lambda_{n_k} R_{\lambda_{n_k}}} \cdot \nabla \phi \, dz - (\lambda_{n_k} R_{\lambda_{n_k}})^2 \int_{B_1} f(\lambda_{n_k} R_{\lambda_{n_k}} z) w^{\lambda_{n_k} R_{\lambda_{n_k}}} \phi \, dz = 0 \tag{2.4.91}$$

for k sufficiently large. Under the set of assumptions (H1-1)-(H1-3), from (2.4.1) it follows that

$$
\begin{aligned}
&\lambda^2 \left| \int_{B_1} f(\lambda z) w^\lambda(z) \phi(z)\, dz \right| \\
&\le \left(\lambda^2 \int_{B_1} |f(\lambda z)||w^\lambda(z)|^2\, dz \right)^{1/2} \left(\lambda^2 \int_{B_1} |f(\lambda z)||\phi(z)|^2\, dz \right)^{1/2} \\
&\le \xi_f(\lambda) \left(\int_{B_1} \frac{|w^\lambda(z)|^2}{|z|^2}\, dz \right)^{1/2} \left(\int_{B_1} \frac{|\phi(z)|^2}{|z|^2} dz \right)^{1/2} \\
&\le \frac{4\xi_f(\lambda)}{(N-1)^2} \left(\int_{B_1} |\nabla w^\lambda|^2\, dz + \frac{N-1}{2} \right)^{1/2} \left(\int_{B_1} |\nabla \phi|^2\, dz + \frac{N-1}{2} \int_{\partial B_1} \phi^2\, dS \right)^{1/2} \\
&= o(1)
\end{aligned}
\tag{2.4.92}
$$

as $\lambda \to 0^+$, using (2.4.75) and Lemma 2.4.26.

In order to make a similar estimate in the case where f satisfies (H2-1)-(H2-5), we notice that from (H2-5), for any $r \in (0, \hat{R})$ and $u \in H^1(B_r)$

$$
\int_{B_r} |f| u^2\, dz \le \eta(r, f) \left(\int_{B_r} |\nabla u|^2\, dz + \frac{N-1}{2r} \int_{\partial B_r} |u|^2\, dS \right).
$$

Then by the change of variable $z' = \lambda z$, setting $w(z') = u(\lambda z)$ and, consequently, taking into account that $\nabla w(z') = \lambda \nabla u(\lambda z)$, it holds that

$$
\begin{aligned}
&\lambda^{N+1} \int_{B_{r/\lambda}} |f(\lambda z')||w(z')|^2\, dz' \\
&\qquad \le \eta(r, f) \left[\lambda^{N-1} \int_{B_{r/\lambda}} |\nabla w(z')|^2\, dz' + \frac{N-1}{2} \lambda^{N-1} \int_{\partial B_{r/\lambda}} |w(z')|^2\, dS \right],
\end{aligned}
$$

with $w \in H^1(B_{r/\lambda})$. Dividing each member by λ^{N-1}, we obtain that for any $r \in (0, \hat{R})$, $\lambda > 0$ and $w \in H^1(B_{r/\lambda})$

$$
\begin{aligned}
&\lambda^2 \int_{B_{r/\lambda}} |f(\lambda z')||w(z')|^2\, dz' \\
&\qquad \le \eta(r, f) \left[\int_{B_{r/\lambda}} |\nabla w(z')|^2\, dz' + \frac{N-1}{2} \int_{\partial B_{r/\lambda}} |w(z')|^2\, dS \right].
\end{aligned}
\tag{2.4.93}
$$

Thus, under assumptions (H2-1)-(H2-5), applying estimate (2.4.93) to w^λ and ϕ with

$r = \lambda$, using (2.4.75) and Lemma 2.4.26, we deduce that as $\lambda \to 0^+$,

$$\begin{aligned}&\lambda^2\left|\int_{B_1} f(\lambda z)w^\lambda(z)\phi(z)\,dz\right|\\&\leq \left(\lambda^2\int_{B_1}|f(\lambda z)||w^\lambda(z)|^2\,dz\right)^{1/2}\left(\lambda^2\int_{B_1}|f(\lambda z)||\phi(z)|^2\,dz\right)^{1/2}\\&\leq \eta(\lambda,f)\left(\int_{B_1}|\nabla w^\lambda|^2\,dz+\frac{N-1}{2}\right)^{1/2}\left(\int_{B_1}|\nabla\phi|^2\,dz+\frac{N-1}{2}\int_{\partial B_1}\phi^2\,dS\right)^{1/2}\\&= o(1).\end{aligned}\tag{2.4.94}$$

The weak convergence of $w^{\lambda_{n_k}R_{\lambda_{n_k}}}$ to w in $H^1(B_1)$, (2.4.92) and (2.4.94) allow passing to the limit in (2.4.91) thus yielding that $w \in H^1_{\tilde\Gamma}(B_1)$ satisfies the equation

$$\int_{B_1}\nabla w(z)\cdot\nabla\phi(z)\,dz = 0 \quad \text{for all } \phi\in C_c^\infty(B_1\setminus\tilde\Gamma),$$

i.e. w weakly solves

$$\begin{cases}-\Delta w(z) = 0 & \text{in } B_1\setminus\tilde\Gamma,\\ w = 0 & \text{on } \tilde\Gamma.\end{cases}\tag{2.4.95}$$

We observe that, by classical regularity theory, w is smooth in $B_1\setminus\tilde\Gamma$. From Lemma 2.4.29 and the density of $C^\infty(\overline{B_1}\setminus\tilde\Gamma)$ in $H^1_{\tilde\Gamma}(B_1)$, it follows that

$$\int_{B_1}\nabla w^{\lambda_{n_k}R_{\lambda_{n_k}}}\cdot\nabla\phi\,dz = \lambda_{n_k}^2R_{\lambda_{n_k}}^2\int_{B_1}f(\lambda_{n_k}R_{\lambda_{n_k}}z)w^{\lambda_{n_k}R_{\lambda_{n_k}}}\phi\,dz+\int_{\partial B_1}\frac{\partial w^{\lambda_{n_k}R_{\lambda_{n_k}}}}{\partial\nu}\phi\,dS\tag{2.4.96}$$

for every $\phi\in H^1_{\tilde\Gamma}(B_1)$ as well as for every $\phi\in H^1_{\Gamma_{\lambda_{n_k}R_{\lambda_{n_k}}}}(B_1)$. From Lemma 2.4.29 it follows that, up to a subsequence still denoted as $\{\lambda_{n_k}\}$, there exists $g\in L^2(\partial B_1)$ such that

$$\frac{\partial w^{\lambda_{n_k}R_{\lambda_{n_k}}}}{\partial\nu}\rightharpoonup g \quad \text{weakly in } L^2(\partial B_1).\tag{2.4.97}$$

Passing to the limit in (2.4.96) and taking into account (2.4.92)-(2.4.94), we then obtain that

$$\int_{B_1}\nabla w\cdot\nabla\phi\,dz = \int_{\partial B_1} g\,\phi\,dS \quad \text{for every } \phi\in H^1_{\tilde\Gamma}(B_1).$$

In particular, taking $\phi = w$ above, we have that

$$\int_{B_1}|\nabla w|^2\,dz = \int_{\partial B_1} g\,w\,dS.\tag{2.4.98}$$

On the other hand, from (2.4.96) with $\phi = w^{\lambda_{n_k}R_{\lambda_{n_k}}}$, (2.4.92)-(2.4.94), (2.4.97), the weak convergence of $w^{\lambda_{n_k}R_{\lambda_{n_k}}}$ to w in $H^1(B_1)$ (which implies the strong convergence of the traces in $L^2(\partial B_1)$ by compactness of the trace map from $H^1(B_1)$ into $L^2(\partial B_1)$), and (2.4.98) it follows that

$$\begin{aligned}\lim_{k\to+\infty}\int_{B_1}|\nabla w^{\lambda_{n_k}R_{\lambda_{n_k}}}|^2\,dz &= \lim_{k\to+\infty}\bigg(\lambda_{n_k}^2R_{\lambda_{n_k}}^2\int_{B_1}f(\lambda_{n_k}R_{\lambda_{n_k}}z)|w^{\lambda_{n_k}R_{\lambda_{n_k}}}|^2\,dz\\ &\qquad+\int_{\partial B_1}\frac{\partial w^{\lambda_{n_k}R_{\lambda_{n_k}}}}{\partial\nu}w^{\lambda_{n_k}R_{\lambda_{n_k}}}\,dS\bigg)\\ &=\int_{\partial B_1}g\,w\,dS=\int_{B_1}|\nabla w|^2\,dx\end{aligned}$$

which implies that

$$w^{\lambda_{n_k}R_{\lambda_{n_k}}}\to w\quad\text{strongly in }H^1(B_1). \tag{2.4.99}$$

For every $k\in\mathbb{N}$ and $r\in(0,1]$, let

$$\mathcal{D}_k(r)=r^{1-N}\int_{B_r}\Big(|\nabla w^{\lambda_{n_k}R_{\lambda_{n_k}}}(z)|^2-\lambda_{n_k}^2R_{\lambda_{n_k}}^2f(\lambda_{n_k}R_{\lambda_{n_k}}z)|w^{\lambda_{n_k}R_{\lambda_{n_k}}}(z)|^2\Big)\,dz$$

and

$$\mathcal{H}_k(r)=r^{-N}\int_{\partial B_r}|w^{\lambda_{n_k}R_{\lambda_{n_k}}}|^2\,dS.$$

We also define, for all $r\in(0,1]$,

$$\mathcal{D}_w(r)=r^{1-N}\int_{B_r}|\nabla w|^2\,dz\quad\text{and}\qquad\mathcal{H}_w(r)=r^{-N}\int_{\partial B_r}|w|^2\,dS.$$

A change of variables directly gives

$$\mathcal{N}_k(r):=\frac{\mathcal{D}_k(r)}{\mathcal{H}_k(r)}=\frac{\mathcal{D}(\lambda_{n_k}R_{\lambda_{n_k}}r)}{\mathcal{H}(\lambda_{n_k}R_{\lambda_{n_k}}r)}=\mathcal{N}(\lambda_{n_k}R_{\lambda_{n_k}}r)\quad\text{for all }r\in(0,1]. \tag{2.4.100}$$

From (2.4.99), (2.4.92)-(2.4.94) and compactness of the trace map from $H^1(B_r)$ into $L^2(\partial B_r)$, it follows that, for every fixed $r\in(0,1]$,

$$\mathcal{D}_k(r)\to\mathcal{D}_w(r)\quad\text{and}\quad\mathcal{H}_k(r)\to\mathcal{H}_w(r). \tag{2.4.101}$$

We observe that $\mathcal{H}_w(r)>0$ for all $r\in(0,1]$; indeed if, for some $r\in(0,1]$, $\mathcal{H}_w(r)=0$, then $w=0$ on ∂B_r and, testing (2.4.95) with $w\in H_0^1(B_r\setminus\tilde\Gamma)$, we would obtain $\int_{B_r}|\nabla w|^2\,dz=0$ and hence $w\equiv0$ in B_r, thus contradicting classical unique continuation principles for second order elliptic equations (see e.g. [79]). Therefore the function

$$\mathcal{N}_w:(0,1]\to\mathbb{R},\quad\mathcal{N}_w(r):=\frac{\mathcal{D}_w(r)}{\mathcal{H}_w(r)}$$

is well defined. Moreover (2.4.100), (2.4.101), and Lemma 2.4.23, imply that, for all $r\in(0,1]$,

$$\mathcal{N}_w(r)=\lim_{k\to+\infty}\mathcal{N}(\lambda_{n_k}R_{\lambda_{n_k}}r)=\gamma. \tag{2.4.102}$$

Therefore $\mathcal{N}_w$ is constant in $(0,1]$ and hence $\mathcal{N}_w'(r) = 0$ for any $r \in (0,1)$. Hence, from (2.4.95) and Lemma 2.4.20 with $f \equiv 0$, we deduce that, for a.e. $r \in (0,1)$,

$$0 = \mathcal{N}_w'(r) \geq \nu_1(r) = \frac{2r\big[\big(\int_{\partial B_r} \big|\frac{\partial w}{\partial \nu}\big|^2 \, dS\big)\big(\int_{\partial B_r} |w|^2 \, dS\big) - \big(\int_{\partial B_r} w \frac{\partial w}{\partial \nu} \, dS\big)^2\big]}{\big(\int_{\partial B_r} |w|^2 \, dS\big)^2} \geq 0$$

so that $\big(\int_{\partial B_r} \big|\frac{\partial w}{\partial \nu}\big|^2 \, dS\big)\big(\int_{\partial B_r} |w|^2 \, dS\big) - \big(\int_{\partial B_r} w \frac{\partial w}{\partial \nu} \, dS\big)^2 = 0$. This implies that w and $\frac{\partial w}{\partial \nu}$ have the same direction as vectors in $L^2(\partial B_r)$ for a.e. $r \in (0,1)$. Then there exists a function $\zeta = \zeta(r)$, defined a.e. in $(0,1)$, such that

$$\frac{\partial w}{\partial \nu}(r\theta) = \zeta(r) w(r\theta) \tag{2.4.103}$$

for a.e. $r \in (0,1)$ and for all $\theta \in \mathbb{S}^N \setminus S_1^+$. Multiplying by $w(r\theta)$ and integrating over $\mathbb{S}^N$ we obtain that

$$\int_{\mathbb{S}^N} \frac{\partial w}{\partial \nu}(r\theta)\, w(r\theta) \, dS = \zeta(r) \int_{\mathbb{S}^N} w^2(r\theta) \, dS$$

and hence, in view of the definition of $\mathcal{H}_w$, (2.4.41) and (2.4.43), $\zeta(r) = \frac{\mathcal{H}_w'(r)}{2\mathcal{H}_w(r)}$ for a.e $r \in (0,1)$. This in particular implies that $\zeta \in L^1_{\rm loc}((0,1])$, exploiting 1)-2) of Lemma 2.3.17, using that $\mathcal{H}_w(r) > 0$ and $\mathcal{H}_w \in W^{1,1}((0,1])$. Moreover, after integrating (2.4.103), we obtain

$$w(r\theta) = e^{\int_1^r \zeta(s)\, ds} w(1\theta) = \varphi(r)\psi(\theta) \quad \text{for all } r \in (0,1),\, \theta \in \mathbb{S}^N \setminus S_1^+,$$

where $\varphi(r) = e^{\int_1^r \zeta(s)\, ds}$ and $\psi = w\big|_{\mathbb{S}^N}$. The fact that $w \in H^1_\Gamma(B_1)$ implies that $\psi \in H^1_0(\mathbb{S}^N \setminus S_1^+)$; moreover (2.4.90) yields that

$$\int_{\mathbb{S}^N} \psi^2(\theta) \, dS = 1. \tag{2.4.104}$$

Equation (2.4.95) rewritten in polar coordinates r, θ becomes

$$\Big(-\varphi''(r) - \frac{N}{r}\varphi'(r)\Big)\psi(\theta) - \frac{\varphi(r)}{r^2}\Delta_{\mathbb{S}^N}\psi(\theta) = 0 \quad \text{on } \mathbb{S}^N \setminus S_1^+.$$

The above equation for a fixed r implies that ψ is an eigenfunction of problem (1.2.23). Letting $\mu_{k_0} = \frac{k_0(k_0+2N-2)}{4}$ be the corresponding eigenvalue, φ solves

$$-\varphi''(r) - \frac{N}{r}\varphi'(r) + \frac{\mu_{k_0}}{r^2}\varphi(r) = 0.$$

Integrating the last equation we obtain that there exist $c_1, c_2 \in \mathbb{R}$ such that

$$\varphi(r) = c_1 r^{\sigma_{k_0}^+} + c_2 r^{\sigma_{k_0}^-},$$

where

$$\sigma_{k_0}^+ = -\frac{N-1}{2} + \sqrt{\Big(\frac{N-1}{2}\Big)^2 + \mu_{k_0}} = \frac{k_0}{2}$$

and

$$\sigma_{k_0}^- = -\frac{N-1}{2} - \sqrt{\left(\frac{N-1}{2}\right)^2 + \mu_{k_0}} = -\big(N - 1 + \tfrac{k_0}{2}\big).$$

Since the function $|z|^{\sigma_{k_0}^-}\psi\big(\frac{z}{|z|}\big) \notin L^{2^*}(B_1)$ (where $2^* = 2(N+1)/(N-1)$), we have that $|z|^{\sigma_{k_0}^-}\psi\big(\frac{z}{|z|}\big)$ does not belong to $H^1(B_1)$; then necessarily $c_2 = 0$ and $\varphi(r) = c_1 r^{k_0/2}$. Since $\varphi(1) = 1$, we obtain that $c_1 = 1$ and then

$$w(r\theta) = r^{k_0/2}\psi(\theta), \quad \text{for all } r \in (0,1) \text{ and } \theta \in \mathbb{S}^N \setminus S_1^+. \tag{2.4.105}$$

Let us now consider the sequence $\{w^{\lambda_{n_k}}\}_k$. Up to a further subsequence still denoted by $w^{\lambda_{n_k}}$, we may suppose that $w^{\lambda_{n_k}} \rightharpoonup \overline{w}$ weakly in $H^1(B_1)$ for some $\overline{w} \in H^1(B_1)$ and that $R_{\lambda_{n_k}} \to \overline{R}$ for some $\overline{R} \in [1,2]$. Strong convergence of $w^{\lambda_{n_k}R_{\lambda_{n_k}}}$ in $H^1(B_1)$ implies that, up to a subsequence, both $w^{\lambda_{n_k}R_{\lambda_{n_k}}}$ and $|\nabla w^{\lambda_{n_k}R_{\lambda_{n_k}}}|$ are dominated a.e. by a $L^2(B_1)$-function uniformly with respect to k. Furthermore, in view of (2.4.80), up to a subsequence we can assume that the limit

$$\ell := \lim_{k\to+\infty} \frac{\mathcal{H}(\lambda_{n_k}R_{\lambda_{n_k}})}{\mathcal{H}(\lambda_{n_k})}$$

exists and is finite. The Dominated Convergence Theorem then implies

$$\begin{aligned}
&\lim_{k\to+\infty}\int_{B_1} w^{\lambda_{n_k}}(z)v(z)\,dz = \lim_{k\to+\infty} R_{\lambda_{n_k}}^{N+1}\int_{B_{1/R_{\lambda_{n_k}}}} w^{\lambda_{n_k}}(R_{\lambda_{n_k}}z)v(R_{\lambda_{n_k}}z)\,dz\\
&= \lim_{k\to+\infty} R_{\lambda_{n_k}}^{N+1}\sqrt{\frac{\mathcal{H}(\lambda_{n_k}R_{\lambda_{n_k}})}{\mathcal{H}(\lambda_{n_k})}}\int_{B_1}\chi_{B_{1/R_{\lambda_{n_k}}}}(z)w^{\lambda_{n_k}R_{\lambda_{n_k}}}(z)v(R_{\lambda_{n_k}}z)\,dz\\
&= \overline{R}^{N+1}\sqrt{\ell}\int_{B_1}\chi_{B_{1/\overline{R}}}(z)w(z)v(\overline{R}z)\,dz = \overline{R}^{N+1}\sqrt{\ell}\int_{B_{1/\overline{R}}} w(z)v(\overline{R}z)\,dz\\
&= \sqrt{\ell}\int_{B_1} w(z/\overline{R})v(z)\,dz
\end{aligned}$$

for any $v \in C_c^\infty(B_1)$. By density it is easy to verify that the previous convergence also holds for all $v \in L^2(B_1)$. We conclude that $w^{\lambda_{n_k}} \rightharpoonup \sqrt{\ell}\,w(\cdot/\overline{R})$ weakly in $L^2(B_1)$; as a consequence we have that $\overline{w} = \sqrt{\ell}\,w\big(\frac{\cdot}{\overline{R}}\big)$ and $w^{\lambda_{n_k}} \rightharpoonup \sqrt{\ell}\,w(\cdot/\overline{R})$ weakly in $H^1(B_1)$. Moreover

$$\begin{aligned}
&\lim_{k\to+\infty}\int_{B_1}|\nabla w^{\lambda_{n_k}}(z)|^2\,dz = \lim_{k\to+\infty} R_{\lambda_{n_k}}^{N+1}\int_{B_{1/R_{\lambda_{n_k}}}}|\nabla w^{\lambda_{n_k}}(R_{\lambda_{n_k}}z)|^2\,dz\\
&= \lim_{k\to+\infty} R_{\lambda_{n_k}}^{N-1}\frac{\mathcal{H}(\lambda_{n_k}R_{\lambda_{n_k}})}{\mathcal{H}(\lambda_{n_k})}\int_{B_1}\chi_{B_{1/R_{\lambda_{n_k}}}}(z)|\nabla w^{\lambda_{n_k}R_{\lambda_{n_k}}}(z)|^2\,dz\\
&= \overline{R}^{N-1}\ell\int_{B_1}\chi_{B_{1/\overline{R}}}(z)|\nabla w(z)|^2\,dz = \overline{R}^{N-1}\ell\int_{B_{1/\overline{R}}}|\nabla w(z)|^2\,dz = \int_{B_1}|\sqrt{\ell}\,\nabla(w(z/\overline{R}))|^2\,dz.
\end{aligned}$$

Therefore we conclude that $w^{\lambda_{n_k}} \to \overline{w} = \sqrt{\ell}w(\cdot/\overline{R})$ strongly in $H^1(B_1)$. Furthermore, by (2.4.105) and the fact that $\int_{\partial B_1} |\overline{w}|^2 \, dS = \int_{\partial B_1} |w|^2 \, dS = 1$, we deduce that $\overline{w} = w$.

It remains to prove part (i). From (2.4.105) and (2.4.104) it follows that $\mathcal{H}_w(r) = r^{k_0}$. Therefore (2.4.102) and Lemma 2.4.17 applied to w imply that

$$\gamma = \frac{r}{2}\frac{\mathcal{H}_w'(r)}{\mathcal{H}_w(r)} = \frac{r}{2}\frac{k_0\, r^{k_0-1}}{r^{k_0}} = \frac{k_0}{2},$$

thus completing the proof. □

In order to make more explicit the blow-up result proved in Lemma 2.4.31, we describe the asymptotic behavior of $\mathcal{H}(r)$ as $r \to 0^+$.

Lemma 2.4.32. *Let γ be as in Lemma 2.4.23. The limit $\lim_{r\to 0^+} r^{-2\gamma}\mathcal{H}(r)$ exists and is finite.*

Proof. Thanks to estimate (2.4.61), it is enough to prove that the limit exists. By (2.4.42) and (2.4.64) we have

$$\frac{d}{dr}\frac{\mathcal{H}(r)}{r^{2\gamma}} = 2r^{-2\gamma-1}\left(\mathcal{D}(r) - \gamma\mathcal{H}(r)\right) = 2r^{-2\gamma-1}\mathcal{H}(r)\int_0^r \mathcal{N}'(s)\, ds. \tag{2.4.106}$$

Let us write $\mathcal{N}' = \alpha_1 + \alpha_2$, with α_1 and α_2 defined as in (2.4.66) and (2.4.67) respectively.

From (2.4.63) it holds that

$$\alpha_1(r) \geq 0 \quad \text{for a.e. } r \in (0, r_0). \tag{2.4.107}$$

Moreover assumptions (H1-2), (H2-2) and (H2-4) ensure that not only $\alpha_2 \in L^1(0, r_0)$, but also

$$\frac{1}{s}\int_0^s \alpha_2(t)\, dt \in L^1(0, r_0). \tag{2.4.108}$$

Integration of (2.4.106) over (r, r_0) yields

$$\frac{\mathcal{H}(r_0)}{r_0^{2\gamma}} - \frac{\mathcal{H}(r)}{r^{2\gamma}} = \int_r^{r_0} 2s^{-2\gamma-1}\mathcal{H}(s)\left(\int_0^s \alpha_1(t)dt\right) ds + \int_r^{r_0} 2s^{-2\gamma-1}\mathcal{H}(s)\left(\int_0^s \alpha_2(t)dt\right) ds. \tag{2.4.109}$$

In virtue of (2.4.107) we deduce that $\lim_{r\to 0^+}\int_r^{r_0} 2s^{-2\gamma-1}\mathcal{H}(s)\left(\int_0^s \alpha_1(t)dt\right) ds$ exists. On the other hand, (2.4.61) and (2.4.108) imply that

$$\left| s^{-2\gamma-1}\mathcal{H}(s)\left(\int_0^s \alpha_2(t)dt\right) ds\right| \leq K_1 s^{-1}\int_0^s \alpha_2(t)\, dt \in L^1(0, r_0),$$

for all $s \in (0, r_0)$, thus proving that $s^{-2\gamma-1}\mathcal{H}(s)\left(\int_0^s \alpha_2(t)\, dt\right) \in L^1(0, r_0)$. Then we may conclude that both terms on the right hand side of (2.4.109) admit a limit as $r \to 0^+$ and at least one of such limits is finite, thus completing the proof of the lemma. □

2.4.5 Local asymptotics

In order to detect the sharp vanishing order of the function $\mathcal{H}$ and to give a more explicit blow-up result, in this subsection we construct an auxiliary equivalent problem by a diffeomorphic deformation of the domain, inspired by [38], see also [3] and [77]. The purpose of such deformation is to straighten the crack; the advantage of working in a domain with a straight crack will then rely in the possibility of separating radial and angular coordinates in the Fourier expansion of solutions (see (2.4.141)).

Lemma 2.4.33. *There exists $\bar{r} \in (0, r_0)$ such that the function*

$$\Xi(z) = \Xi(x', x_N, t) = \frac{(x', x_N - g(x'), t)}{\sqrt{1 + \frac{g^2(x') - 2g(x')x_N}{|x'|^2 + x_N^2 + t^2}}},$$

is invertible from $B_{\bar{r}}$ to $B_{\bar{r}}$. Furthermore, setting $\Phi = \Xi^{-1}$, we have that

$$\Phi^{-1}(z) = z + O(|z|^2), \quad \operatorname{Jac}\Phi^{-1}(z) = \mathrm{Id}_{N+1} + O(|z|) \quad \textit{as } |z| \to 0, \tag{2.4.110}$$

$$\det \operatorname{Jac}\Phi^{-1}(z) = 1 + O(|z|) \quad \textit{as } |z| \to 0, \tag{2.4.111}$$

$$\Phi(y) = y + O(|y|^2), \quad \operatorname{Jac}\Phi(y) = \mathrm{Id}_{N+1} + O(|y|) \quad \textit{as } |y| \to 0, \tag{2.4.112}$$

$$\det \operatorname{Jac}\Phi(y) = 1 + O(|y|) \quad \textit{as } |y| \to 0, \tag{2.4.113}$$

$$\Phi(B_r \setminus \tilde{\Gamma}) = B_r \setminus \Gamma, \quad \Phi^{-1}(B_r \setminus \Gamma) = B_r \setminus \tilde{\Gamma} \quad \textit{for all } r \in (0, \bar{r}]. \tag{2.4.114}$$

Proof. We can immediately deduce (2.4.110) and (2.4.111) from (1.2.17) and (1.2.18). In particular, $\det \operatorname{Jac}\Xi(0) = 1 \neq 0$, then by the local inversion theorem, there exists a suitable $0 < \bar{r} < r_0$ such that Ξ is invertible from $B_{\bar{r}}$ to itself. Thus, setting $\Phi = \Xi^{-1}$, by (2.4.110) and (2.4.111) we obtain (2.4.112) and (2.4.113). To conclude, properties (2.4.114) hold true since $|\Xi(z)|^2 = |z|^2$ and if $z \in \Gamma^c$, i.e. $x_N < g(x')$, then, setting $y = \Xi(z)$, we have that $y_N = x_N - g(x') < 0$, which is equivalent to prove that $y \in \tilde{\Gamma}^c$. □

Let $u \in H^1(B_{\hat{R}})$ be a weak solution to (1.2.20). Then

$$v = u \circ \Phi \in H^1(B_{\bar{r}}) \tag{2.4.115}$$

is a weak solution to

$$\begin{cases} -\operatorname{div}(A(y)\nabla v(y)) = \tilde{f}(y)v(y) & \text{in } B_{\bar{r}} \setminus \tilde{\Gamma}, \\ v = 0 & \text{on } \tilde{\Gamma}, \end{cases} \tag{2.4.116}$$

with

$$\begin{aligned} A(y) &= |\det \operatorname{Jac}\Phi(y)|(\operatorname{Jac}\Phi(y))^{-1}((\operatorname{Jac}\Phi(y))^{-1})^T, \\ \tilde{f}(y) &= |\det \operatorname{Jac}\Phi(y)| f(\Phi(y)). \end{aligned} \tag{2.4.117}$$

Indeed $v \in H^1_{\tilde{\Gamma}}(B_{\bar{r}})$ thanks to (1.2.22) and (2.4.114). Moreover it holds that

$$\int_{B_{\bar{r}}} A(y)\nabla v(y) \cdot \nabla\psi(y)\, dy - \int_{B_{\bar{r}}} \tilde{f}(y)v(y)\psi(y)\, dy = 0 \quad \text{for any } \psi \in C_c^\infty(B_{\bar{r}} \setminus \tilde{\Gamma}). \tag{2.4.118}$$

Indeed, by (1.2.22) we have that

$$\int_{B_{\bar r}} \nabla u(z)\cdot\nabla\varphi(z)\,dz - \int_{B_{\bar r}} f(z)u(z)\varphi(z)\,dz = 0 \quad \text{for any } \varphi\in C_c^\infty(B_{\bar r}\setminus\Gamma).$$

Thus, setting $z=\Phi(y)$, we obtain that for any $\varphi\in C_c^\infty(B_{\bar r}\setminus\Gamma)$

$$\int_{B_{\bar r}} \nabla u(\Phi(y))\cdot\nabla\varphi(\Phi(y))|\det \mathrm{Jac}\,\Phi(y)|\,dy \\ - \int_{B_{\bar r}} f(\Phi(y))u(\Phi(y))\varphi(\Phi(y))|\det \mathrm{Jac}\,\Phi(y)|\,dy = 0.$$

From this, by (2.4.115), letting $\psi=\varphi\circ\Phi$ and taking into account (2.4.114), we deduce that for any $\psi\in C_c^\infty(B_{\bar r}\setminus\tilde\Gamma)$

$$\int_{B_{\bar r}} \nabla v(y)(\mathrm{Jac}\,\Phi(y))^{-1}\cdot\nabla\psi(y)(\mathrm{Jac}\,\Phi(y))^{-1}|\det \mathrm{Jac}\,\Phi(y)|\,dy \\ - \int_{B_{\bar r}} f(\Phi(y))v(y)\psi(y)|\det \mathrm{Jac}\,\Phi(y)|\,dy = 0,$$

thus obtaining (2.4.118) with $A(y)$ and $\tilde f(y)$ as in (2.4.117).

By Lemma 2.4.33, (2.4.117) and direct calculations, we obtain that

$$A(y) = \mathrm{Id}_{N+1} + O(|y|) \quad \text{as } |y|\to 0. \tag{2.4.119}$$

Lemma 2.4.34. *Letting* $\mathcal H$ *be as in* (2.4.40) *and* $v=u\circ\Phi$ *as in* (2.4.115)*, we have that*

$$\mathcal H(\lambda) = (1+O(\lambda))\int_{\mathbb S^N} v^2(\lambda\theta)\,dS \quad \text{as } \lambda\to 0^+, \tag{2.4.120}$$

$$\frac{\int_{B_1}|\hat v^\lambda(y)|^2\,dy}{\mathcal H(\lambda)} = (1+O(\lambda))\int_{B_1}|w^\lambda(z)|^2\,dz = O(1) \quad \text{as } \lambda\to 0^+, \tag{2.4.121}$$

and

$$\frac{\int_{B_1}|\nabla\hat v^\lambda(y)|^2\,dy}{\mathcal H(\lambda)} = (1+O(\lambda))\int_{B_1}|\nabla w^\lambda(z)|^2\,dz = O(1) \quad \text{as } \lambda\to 0^+, \tag{2.4.122}$$

where w^λ *is defined in* (2.4.72) *and* $\hat v^\lambda(y) := v(\lambda y)$.

Proof. From (2.4.114) and a change of variable it follows that

$$\int_{B_\lambda} u^2(z)\,dz = \int_{B_\lambda} v^2(y)|\det \mathrm{Jac}\,\Phi(y)|\,dy \quad \text{for all } \lambda\in(0,\bar r).$$

Differentiating the above identity with respect to λ we obtain that

$$\int_{\partial B_\lambda} u^2\,dS = \int_{\partial B_\lambda} v^2|\det \mathrm{Jac}\,\Phi|\,dS \quad \text{for a.e. } \lambda\in(0,\bar r).$$

Hence, by the continuity of $\mathcal{H}$, we deduce that

$$\mathcal{H}(\lambda) = \lambda^{-N} \int_{\partial B_\lambda} v^2 |\det \operatorname{Jac} \Phi| \, dS = \int_{\mathbb{S}^N} v^2(\lambda\theta) |\det \operatorname{Jac} \Phi(\lambda\theta)| dS \quad \text{for all } \lambda \in (0, \bar{r}),$$

which yields (2.4.120) in view of (2.4.113). Furthermore, from (2.4.114) and a change of variable it also follows that

$$\begin{aligned} \frac{\int_{B_1} |\hat{v}^\lambda(y)|^2 \, dy}{\mathcal{H}(\lambda)} &= \frac{\int_{B_1} |u(\Phi(\lambda y))|^2 \, dy}{\mathcal{H}(\lambda)} = \frac{\int_{B_1} |u(\lambda z)|^2 |\det \operatorname{Jac} \Phi^{-1}(\lambda z)| \, dz}{\mathcal{H}(\lambda)} \\ &= \int_{B_1} |w^\lambda(z)|^2 |\det \operatorname{Jac} \Phi^{-1}(\lambda z)| \, dz \end{aligned}$$

and

$$\begin{aligned} \frac{\int_{B_1} |\nabla \hat{v}^\lambda(y)|^2 \, dy}{\mathcal{H}(\lambda)} &= \frac{\int_{B_1} \lambda^2 |\nabla u(\Phi(\lambda y)) \operatorname{Jac} \Phi(\lambda y)|^2 \, dy}{\mathcal{H}(\lambda)} \\ &= \frac{\int_{B_1} \lambda^2 |\nabla u(\lambda z) \operatorname{Jac} \Phi(\Phi^{-1}(\lambda z))|^2 |\det \operatorname{Jac} \Phi^{-1}(\lambda z)| \, dz}{\mathcal{H}(\lambda)} \\ &= \int_{B_1} |\nabla w^\lambda(z) \operatorname{Jac} \Phi(\Phi^{-1}(\lambda z))|^2 |\det \operatorname{Jac} \Phi^{-1}(\lambda z)| \, dz \end{aligned}$$

for all $\lambda \in (0, \bar{r})$. The above identities, together with (2.4.110), (2.4.111), (2.4.112) and the boundedness in $H^1(B_1)$ of $\{w^\lambda\}$ established in Lemma 2.4.26, imply respectively estimates (2.4.121) and (2.4.122). □

Lemma 2.4.35. *Let $v = u \circ \Phi$ be as in* (2.4.115) *and let k_0 and γ be as in Lemma 2.4.31 (i). Then, for every sequence $\lambda_n \to 0^+$, there exist a subsequence $\{\lambda_{n_k}\}_{k \in \mathbb{N}}$ and an eigenfunction ψ of problem* (1.2.23) *associated with the eigenvalue μ_{k_0} such that $\|\psi\|_{L^2(\mathbb{S}^N)} = 1$, the convergence* (2.4.89) *holds and*

$$\frac{v(\lambda_{n_k} \cdot)}{\sqrt{\int_{\mathbb{S}^N} v^2(\lambda_{n_k}\theta) \, dS}} \to \psi \quad \textit{strongly in } L^2(\mathbb{S}^N).$$

Proof. From Lemma 2.4.31, there exist a subsequence λ_{n_k} and an eigenfunction ψ of problem (1.2.23) associated with the eigenvalue μ_{k_0} such that $\|\psi\|_{L^2(\mathbb{S}^N)} = 1$ and (2.4.89) holds. From (2.4.89) it follows that, up to passing to a further subsequence, $w^{\lambda_{n_k}}\big|_{\partial B_1}$ converges to ψ in $L^2(\mathbb{S}^N)$ and almost everywhere on $\mathbb{S}^N$, where w^λ is defined in (2.4.72). From Lemma 2.4.34 it follows that $\{\hat{v}^\lambda / \sqrt{\mathcal{H}(\lambda)}\}_\lambda$ is bounded in $H^1(B_1)$ and hence, up to a further subsequence still denoted by λ_{n_k}, there exists $\tilde{\psi} \in H^1(B_1)$ such that $\{\hat{v}^{\lambda_{n_k}} / \sqrt{\mathcal{H}(\lambda_{n_k})}\}_k$ weakly converges to $\tilde{\psi}$ in $H^1(B_1)$. From this, in view of (2.4.120), we have that up to a further subsequence,

$$\frac{v(\lambda_{n_k} \cdot)}{\sqrt{\int_{\mathbb{S}^N} v^2(\lambda_{n_k}\theta) \, dS}} \to \tilde{\psi} \quad \text{strongly in } L^2(\mathbb{S}^N) \text{ and almost everywhere on } \mathbb{S}^N. \tag{2.4.123}$$

To conclude it is enough to show that $\tilde{\psi} = \psi$. To this aim we observe that, for every $\varphi \in C_c^\infty(\mathbb{S}^N)$, from (2.4.115), (2.4.120), and a change of variable, arguing as in the proof of Lemma 2.4.34, it follows that

$$\int_{\mathbb{S}^N} \frac{v(\lambda_{n_k}\theta)}{\sqrt{\int_{\mathbb{S}^N} v^2(\lambda_{n_k}\cdot)\,dS}}\varphi(\theta)\,dS$$
$$= (1 + O(\lambda_{n_k})) \int_{\mathbb{S}^N} w^{\lambda_{n_k}}(\theta)\varphi\left(\tfrac{\Phi^{-1}(\lambda_{n_k}\theta)}{\lambda_{n_k}}\right)|\det \mathrm{Jac}\,\Phi^{-1}(\lambda_{n_k}\theta)|\,dS. \quad (2.4.124)$$

In view of (2.4.110) and (2.4.111) we have that, for all $\theta \in \mathbb{S}^N$,

$$\lim_{k\to+\infty} \varphi\left(\tfrac{\Phi^{-1}(\lambda_{n_k}\theta)}{\lambda_{n_k}}\right)|\det \mathrm{Jac}\,\Phi^{-1}(\lambda_{n_k}\theta)| = \varphi(\theta),$$

so that, by the Dominated Convergence Theorem, the right hand side of (2.4.124) converges to $\int_{\mathbb{S}^N} \psi(\theta)\varphi(\theta)\,dS$. On the other hand (2.4.123) implies that the left hand side of (2.4.124) converges to $\int_{\mathbb{S}^N} \tilde{\psi}(\theta)\varphi(\theta)\,dS$. Therefore, passing to the limit in (2.4.124), we obtain that

$$\int_{\mathbb{S}^N} \psi(\theta)\varphi(\theta)\,dS = \int_{\mathbb{S}^N} \tilde{\psi}(\theta)\varphi(\theta)\,dS \quad \text{for all } \varphi \in C_c^\infty(\mathbb{S}^N)$$

thus implying that $\psi = \tilde{\psi}$. □

Lemma 2.4.36. *Let k_0 be as in Lemma 2.4.31 and let $M_{k_0} \in \mathbb{N} \setminus \{0\}$ be the multiplicity of μ_{k_0} as an eigenvalue of* (2.3.18). *Let $\{Y_{k_0,m}\}_{m=1,2,\dots,M_{k_0}}$ be as in* (2.3.131). *Then, for any sequence $\lambda_n \to 0^+$, there exists $m \in \{1, 2, \dots, M_{k_0}\}$ such that*

$$\limsup_{n\to+\infty} \frac{\left|\int_{\mathbb{S}^N} v(\lambda_n\theta)Y_{k_0,m}(\theta)\,dS\right|}{\sqrt{\mathcal{H}(\lambda_n)}} > 0.$$

Proof. We argue by contradiction and assume that, along a sequence $\lambda_n \to 0^+$,

$$\limsup_{n\to+\infty} \frac{\left|\int_{\mathbb{S}^N} v(\lambda_n\theta)Y_{k_0,m}(\theta)\,dS\right|}{\sqrt{\mathcal{H}(\lambda_n)}} = 0 \quad (2.4.125)$$

for all $m \in \{1, 2, \dots, M_{k_0}\}$. From Lemma 2.4.35 and (2.4.120) it follows that there exist a subsequence $\{\lambda_{n_k}\}$ and an eigenfunction ψ of problem (2.3.18) associated with the eigenvalue μ_{k_0} such that $\|\psi\|_{L^2(\mathbb{S}^N)} = 1$ and

$$\frac{v(\lambda_{n_k}\theta)}{\sqrt{\mathcal{H}(\lambda_{n_k})}} \to \psi(\theta) \quad \text{strongly in } L^2(\mathbb{S}^N).$$

Furthermore, from (2.4.125) we have that, for every $m \in \{1, 2, \dots, M_{k_0}\}$

$$\lim_{k\to+\infty} \int_{\mathbb{S}^N} \frac{v(\lambda_{n_k}\theta)}{\sqrt{\mathcal{H}(\lambda_{n_k})}} Y_{k_0,m}(\theta)\,dS = 0.$$

Therefore $\int_{\mathbb{S}^N} \psi\, Y_{k_0,m}\, dS = 0$ for all $m \in \{1, 2, \dots, M_{k_0}\}$, thus implying that $\psi \equiv 0$ and giving rise to a contradiction. □

For all $k \in \mathbb{N} \setminus \{0\}$, $m \in \{1, 2, \dots, M_k\}$, and $\lambda \in (0, \bar r)$, we define

$$\varphi_{k,m}(\lambda) := \int_{\mathbb{S}^N} v(\lambda\theta) Y_{k,m}(\theta)\, dS \tag{2.4.126}$$

and

$$\begin{aligned}\Upsilon_{k,m}(\lambda) = &-\int_{B_\lambda} (A - \mathrm{Id}_{N+1})\nabla v(y) \cdot \frac{\nabla_{\mathbb{S}^N} Y_{k,m}(y/|y|)}{|y|}\, dy + \int_{B_\lambda} \tilde f(y) v(y) Y_{k,m}(y/|y|)\, dy \\ &+ \int_{\partial B_\lambda} (A - \mathrm{Id}_{N+1})\nabla v(y) \cdot \frac{y}{|y|} Y_{k,m}(y/|y|)\, dS,\end{aligned} \tag{2.4.127}$$

where the functions $\{Y_{k,m}\}_{m=1,2,\dots,M_k}$ are introduced in (1.2.25).

In the following lemma we provide an asymptotic expansion as $\lambda \to 0^+$ for the Fourier coefficients associated with v.

Lemma 2.4.37. *Let k_0 be as in Lemma 2.4.31. For all $m \in \{1, 2, \dots, M_{k_0}\}$ and $R \in (0, \bar r]$*

$$\begin{aligned}\varphi_{k_0,m}(\lambda) = \lambda^{\frac{k_0}{2}} \Bigg(&R^{-\frac{k_0}{2}} \varphi_{k_0,m}(R) + \frac{2N + k_0 - 2}{2(N + k_0 - 1)} \int_\lambda^R s^{-N-\frac{k_0}{2}} \Upsilon_{k_0,m}(s)\, ds \\ &+ \frac{k_0\, R^{-N+1-k_0}}{2(N + k_0 - 1)} \int_0^R s^{\frac{k_0}{2}-1} \Upsilon_{k_0,m}(s)\, ds \Bigg) + o(\lambda^{\frac{k_0}{2}})\end{aligned} \tag{2.4.128}$$

as $\lambda \to 0^+$.

Proof. For all $k \in \mathbb{N} \setminus \{0\}$ and $m \in \{1, 2, \dots, M_k\}$, we consider the distribution $\zeta_{k,m}$ on $(0, \bar r)$ defined as

$$\begin{aligned}{}_{\mathcal{D}'(0,\bar r)}\langle \zeta_{k,m}, \omega \rangle_{\mathcal{D}(0,\bar r)} = &\int_0^{\bar r} \omega(\lambda) \left(\int_{\mathbb{S}^N} \tilde f(\lambda\theta) v(\lambda\theta) Y_{k,m}(\theta)\, dS \right) d\lambda \\ &+ {}_{H^{-1}(B_{\bar r})} \big\langle \mathrm{div}((A - \mathrm{Id}_{N+1})\nabla v), |y|^{-N} \omega(|y|) Y_{k,m}(y/|y|) \big\rangle_{H^1_0(B_{\bar r})}\end{aligned}$$

for all $\omega \in \mathcal{D}(0, \bar r)$, where

$${}_{H^{-1}(B_{\bar r})}\big\langle \mathrm{div}((A - \mathrm{Id}_{N+1})\nabla v), \phi \big\rangle_{H^1_0(B_{\bar r})} = -\int_{B_{\bar r}} (A - \mathrm{Id}_{N+1})\nabla v \cdot \nabla\phi\, dy$$

for all $\phi \in H^1_0(B_{\bar r})$. Letting $\Upsilon_{k,m}$ as in (2.4.127), we observe that $\Upsilon_{k,m} \in L^1_{\mathrm{loc}}(0, \bar r)$ and, by direct calculations,

$$\Upsilon'_{k,m}(\lambda) = \lambda^N \zeta_{k,m}(\lambda) \quad \text{in } \mathcal{D}'(0, \bar r). \tag{2.4.129}$$

From the definition of $\zeta_{k,m}$, (2.4.116), and the fact that $Y_{k,m}$ is an eigenfunction of (1.2.23) associated to the eigenvalue μ_k, it follows that, for all $k \in \mathbb{N}\setminus\{0\}$ and $m \in \{1,2,\dots,M_k\}$, the function $\varphi_{k,m}$ defined in (2.4.126) solves

$$-\varphi''_{k,m}(\lambda) - \frac{N}{\lambda}\varphi'_{k,m}(\lambda) + \frac{\mu_k}{\lambda^2}\varphi_{k,m}(\lambda) = \zeta_{k,m}(\lambda)$$

in the sense of distributions in $(0,\bar r)$, which, in view of (1.2.24), can be also written as

$$-(\lambda^{N+k}(\lambda^{-\frac{k}{2}}\varphi_{k,m}(\lambda))')' = \lambda^{N+\frac{k}{2}}\zeta_{k,m}(\lambda)$$

in the sense of distributions in $(0,\bar r)$. Integrating by parts and taking into account (2.4.129), we obtain that, for every $k \in \mathbb{N}\setminus\{0\}$, $m \in \{1,2,\dots,M_k\}$, and $R \in (0,\bar r]$, there exists $c_{k,m}(R) \in \mathbb{R}$ such that

$$(\lambda^{-\frac{k}{2}}\varphi_{k,m}(\lambda))' = -\lambda^{-N-\frac{k}{2}}\Upsilon_{k,m}(\lambda) - \frac{k}{2}\lambda^{-N-k}\left(c_{k,m}(R) + \int_\lambda^R s^{\frac{k}{2}-1}\Upsilon_{k,m}(s)\,ds\right)$$

in the sense of distributions in $(0,\bar r)$. In particular, $\varphi_{k,m} \in W^{1,1}_{\rm loc}(0,\bar r)$ and, by a further integration,

$$\begin{aligned}\varphi_{k,m}(\lambda) &= \lambda^{\frac{k}{2}}\left(R^{-\frac{k}{2}}\varphi_{k,m}(R) + \int_\lambda^R s^{-N-\frac{k}{2}}\Upsilon_{k,m}(s)\,ds\right)\\
&\quad + \frac{k}{2}\lambda^{\frac{k}{2}}\int_\lambda^R s^{-N-k}\left(c_{k,m}(R) + \int_s^R t^{\frac{k}{2}-1}\Upsilon_{k,m}(t)\,dt\right)ds\\
&= \lambda^{\frac{k}{2}}\left(R^{-\frac{k}{2}}\varphi_{k,m}(R) + \frac{2N+k-2}{2(N+k-1)}\int_\lambda^R s^{-N-\frac{k}{2}}\Upsilon_{k,m}(s)\,ds - \frac{k\,c_{k,m}(R)R^{-N+1-k}}{2(N+k-1)}\right)\\
&\quad + \frac{k\,\lambda^{-N+1-\frac{k}{2}}}{2(N-1+k)}\left(c_{k,m}(R) + \int_\lambda^R t^{\frac{k}{2}-1}\Upsilon_{k,m}(t)\,dt\right).\end{aligned} \tag{2.4.130}$$

Let now k_0 be as in Lemma 2.4.31. We claim that

$$\text{the function } s \mapsto s^{-N-\frac{k_0}{2}}\Upsilon_{k_0,m}(s) \text{ belongs to } L^1(0,\bar r) \text{ for any } m \in \{1,2,\dots,M_{k_0}\}. \tag{2.4.131}$$

To this purpose, let us estimate each term in (2.4.127). By (2.4.119), (2.4.122), the Hölder inequality and a change of variable we obtain that, for all $s \in (0,\bar r)$,

$$\begin{aligned}&\left|\int_{B_s}(A(y)-\mathrm{Id}_{N+1})\nabla v(y)\cdot\frac{\nabla_{\mathbb{S}^N}Y_{k_0,m}\big(\frac{y}{|y|}\big)}{|y|}\,dy\right|\\
&\qquad\le \text{const}\sqrt{\int_{B_s}|\nabla v(y)|^2\,dy}\sqrt{\int_{B_s}\left|\nabla_{\mathbb{S}^N}Y_{k_0,m}\left(\frac{y}{|y|}\right)\right|^2 dy}\\
&\qquad\le \text{const}\, s^{\frac{N-1}{2}}\, s^{\frac{N+1}{2}}\sqrt{\mathcal{H}(s)}\sqrt{\int_{B_1}\frac{|\nabla\hat v^s(y)|^2}{\mathcal{H}(s)}\,dy}\\
&\qquad\le \text{const}\, s^N\sqrt{\mathcal{H}(s)},\end{aligned} \tag{2.4.132}$$

taking into account that $\nabla v(sy) = s^{-1}\nabla \hat{v}^s(y)$ and that

$$\int_{B_s} \left|\nabla_{\mathbb{S}^N} Y_{k_0,m}\left(\frac{y}{|y|}\right)\right|^2 dy = \int_0^s t^N \left(\int_{\mathbb{S}^N} \left|\nabla_{\mathbb{S}^N} Y_{k_0,m}\left(\frac{y}{|y|}\right)\right|^2 dS\right) dt$$
$$= \mu_{k_0} \int_0^s t^N \left(\int_{\mathbb{S}^N} \left|Y_{k_0,m}\left(\frac{y}{|y|}\right)\right|^2 dS\right) dt$$
$$= \frac{s^{N+1}}{N+1}.$$

By the Hölder inequality, (2.4.115), (2.4.114), and the definition of $\tilde{f}$ in (2.4.117) we have that,

$$\left|\int_{B_s} \tilde{f}(y)v(y)Y_{k_0,m}\big(\tfrac{y}{|y|}\big)\, dy\right| \leq \sqrt{\int_{B_s} |\tilde{f}(y)|v^2(y)\, dy} \cdot \sqrt{\int_{B_s} |\tilde{f}(y)|Y^2_{k_0,m}\big(\tfrac{y}{|y|}\big)\, dy}$$
$$= \sqrt{\int_{B_s} |f(z)|u^2(z)\, dz} \cdot \sqrt{\int_{B_s} |f(z)|Y^2_{k_0,m}\big(\tfrac{\Phi^{-1}(z)}{|\Phi^{-1}(z)|}\big)\, dz}.$$

Using (H2-5), (2.4.55), (2.4.13), (2.4.59) under assumptions (H2-1)-(H2-5), and (2.4.56) under assumptions (H1-1)-(H1-3), it follows that

$$\int_{B_s} |f|u^2\, dz \leq \text{const}\, \beta(s,f)s^{N-1}\mathcal{H}(s)$$

where $\beta(s,f) = \eta(s,f)$ under assumptions (H2-1)-(H2-5) and $\beta(s,f) = \xi_f(s)$ under assumptions (H1-1)-(H1-3). Moreover, by (H2-5) under assumptions (H2-1)-(H2-5), and from (2.4.10) under assumptions (H1-1)-(H1-3), we also have that

$$\int_{B_s} |f(z)|Y^2_{k_0,m}\big(\tfrac{\Phi^{-1}(z)}{|\Phi^{-1}(z)|}\big)\, dz \leq \text{const}\, \beta(s,f)s^{N-1}.$$

Therefore we conclude that, for all $s \in (0,\bar{r})$,

$$\left|\int_{B_s} \tilde{f}(y)v(y)Y_{k_0,m}\big(\tfrac{y}{|y|}\big)\, dy\right| \leq \text{const}\, \beta(s,f)s^{N-1}\sqrt{\mathcal{H}(s)}. \tag{2.4.133}$$

As regards the last term in (2.4.127), we observe that, for a.e. $s \in (0,\bar{r})$,

$$\left|\int_{\partial B_s} (A - \mathrm{Id}_{N+1})\nabla v(y) \cdot \frac{y}{|y|} Y_{k_0,m}\big(\tfrac{y}{|y|}\big)\, dS\right| \leq \text{const}\, s \int_{\partial B_s} |\nabla v|\big|Y_{k_0,m}\big(\tfrac{y}{|y|}\big)\big| dS, \tag{2.4.134}$$

as a consequence of (2.4.119). Integrating by parts and using (2.4.122), Lemma 2.4.26, the Hölder inequality and a change of variable we have that, for every $R \in (0, \bar{r}]$,

$$\int_0^R s^{-N-\frac{k_0}{2}+1}\Big(\int_{\partial B_s} |\nabla v||Y_{k_0,m}\big(\tfrac{y}{|y|}\big)|dS\Big)ds = R^{-N-\frac{k_0}{2}+1}\int_{B_R} |\nabla v|\big|Y_{k_0,m}\big(\tfrac{y}{|y|}\big)\big|dy$$
$$+\big(N+\tfrac{k_0}{2}-1\big)\int_0^R s^{-N-\frac{k_0}{2}}\Big(\int_{B_s} |\nabla v|\big|Y_{k_0,m}\big(\tfrac{y}{|y|}\big)\big|\,dy\Big)ds$$
$$\le \text{const}\left(R^{-\frac{k_0}{2}+1}\sqrt{\mathcal{H}(R)}+\int_0^R s^{-\frac{k_0}{2}}\sqrt{\mathcal{H}(s)}ds\right), \tag{2.4.135}$$

as a consequence of (2.4.132). From (2.4.127), (2.4.132), (2.4.133), and (2.4.135) we deduce that, for all $m \in \{1, 2, \dots, M_{k_0}\}$ and $R \in (0, \bar{r}]$,

$$\int_0^R s^{-N-\frac{k_0}{2}}|\Upsilon_{k_0,m}(s)|\,ds \le \text{const}\,R^{-\frac{k_0}{2}+1}\sqrt{\mathcal{H}(R)}+\int_0^R s^{-\frac{k_0}{2}}\sqrt{\mathcal{H}(s)}\,\big(1+s^{-1}\beta(s,f)\big)\,ds. \tag{2.4.136}$$

Thus claim (2.4.131) follows from (2.4.136), (2.4.61) and assumptions (H1-2) and (H2-2).

From (2.4.131) we deduce that, for every fixed $R \in (0, \bar{r}]$,

$$\lambda^{\frac{k_0}{2}}\left(R^{-\frac{k_0}{2}}\varphi_{k_0,m}(R)+\frac{2N+k_0-2}{2(N+k_0-1)}\int_\lambda^R s^{-N-\frac{k_0}{2}}\Upsilon_{k_0,m}(s)ds-\frac{k_0\,c_{k_0,m}(R)R^{-N+1-k_0}}{2(N+k_0-1)}\right)$$
$$= O(\lambda^{\frac{k_0}{2}}) = o(\lambda^{-N+1-\frac{k_0}{2}}) \quad \text{as } \lambda \to 0^+. \tag{2.4.137}$$

On the other hand, (2.4.131) also implies that $t \mapsto t^{\frac{k_0}{2}-1}\Upsilon_{k_0,m}(t) \in L^1(0, \bar{r})$. We claim that, for every $R \in (0, \bar{r}]$,

$$c_{k_0,m}(R)+\int_0^R t^{\frac{k_0}{2}-1}\Upsilon_{k_0,m}(t)dt = 0. \tag{2.4.138}$$

Suppose by contradiction that (2.4.138) is not true for some $R \in (0, \bar{r}]$. Then, from (2.4.130) and (2.4.137) we infer that

$$\varphi_{k_0,m}(\lambda) \sim \frac{k_0\,\lambda^{-N+1-\frac{k_0}{2}}}{2(N-1+k_0)}\left(c_{k_0,m}(R)+\int_0^R t^{\frac{k_0}{2}-1}\Upsilon_{k_0,m}(t)dt\right) \quad \text{as } \lambda \to 0^+. \tag{2.4.139}$$

Lemma 2.4.2 and the fact that $v \in H^1(B_{\bar{r}})$ imply that

$$\int_0^{\bar{r}} \lambda^{N-2}|\varphi_{k_0,m}(\lambda)|^2\,d\lambda \le \int_0^{\bar{r}} \lambda^{N-2}\Big(\int_{\mathbb{S}^N}|v(\lambda\theta)|^2dS\Big)d\lambda = \int_{B_{\bar{r}}}\frac{|v(y)|^2}{|y|^2}dy < +\infty,$$

thus contradicting (2.4.139), since $N-1+k_0/2 \ge 1$. Claim (2.4.138) is thereby proved.

From (2.4.131) and (2.4.138) it follows that, for every $R \in (0, \bar r]$,

$$\left|\lambda^{-N+1-\frac{k_0}{2}}\left(c_{k_0,m}(R) + \int_\lambda^R t^{\frac{k_0}{2}-1}\Upsilon_{k_0,m}(t)dt\right)\right| = \lambda^{-N+1-\frac{k_0}{2}}\left|\int_0^\lambda t^{\frac{k_0}{2}-1}\Upsilon_{k_0,m}(t)dt\right|$$
$$\leq \lambda^{-N+1-\frac{k_0}{2}}\int_0^\lambda t^{N+k_0-1}\left|t^{-N-\frac{k_0}{2}}\Upsilon_{k_0,m}(t)\right|dt \leq \lambda^{\frac{k_0}{2}}\int_0^\lambda \left|t^{-N-\frac{k_0}{2}}\Upsilon_{k_0,m}(t)\right|dt = o(\lambda^{\frac{k_0}{2}}) \tag{2.4.140}$$

as $\lambda \to 0^+$.

The conclusion follows by (2.4.130), (2.4.140) and (2.4.138). □

Lemma 2.4.38. *Let γ be as in Lemma 2.4.23. Then* $\lim_{r\to 0^+} r^{-2\gamma}\mathcal{H}(r) > 0$.

Proof. For any $\lambda \in (0, \bar r)$, we expand $\theta \mapsto v(\lambda\theta) \in L^2(\mathbb{S}^N)$ in Fourier series with respect to the orthonormal basis $\{Y_{k,m}\}_{m=1,2,\dots,M_k}$ introduced in (1.2.25), i.e.

$$v(\lambda\theta) = \sum_{k=1}^{\infty}\sum_{m=1}^{M_k}\varphi_{k,m}(\lambda)Y_{k,m}(\theta) \quad \text{in } L^2(\mathbb{S}^N), \tag{2.4.141}$$

where $m \in \{1, 2, \dots, M_k\}$ for all $k \in \mathbb{N}\setminus\{0\}$, $\lambda \in (0, \bar r)$ and $\varphi_{k,m}(\lambda)$ is defined in (2.4.126).

Let $k_0 \in \mathbb{N}$, $k_0 \geq 1$, be as in Lemma 2.4.31, so that

$$\gamma = \lim_{r\to 0^+}\mathcal{N}(r) = \frac{k_0}{2}. \tag{2.4.142}$$

From (2.4.120) and the Parseval identity we deduce that

$$\mathcal{H}(\lambda) = (1 + O(\lambda))\int_{\mathbb{S}^N} v^2(\lambda\theta)\,dS = (1 + O(\lambda))\sum_{k=1}^{\infty}\sum_{m=1}^{M_k}\varphi_{k,m}^2(\lambda), \tag{2.4.143}$$

for all $0 < \lambda < \bar r$. Let us assume by contradiction that $\lim_{\lambda\to 0^+}\lambda^{-2\gamma}\mathcal{H}(\lambda) = 0$. Then, (2.4.142) and (2.4.143) imply that

$$\lim_{\lambda\to 0^+}\lambda^{-k_0/2}\varphi_{k_0,m}(\lambda) = 0 \quad \text{for any } m \in \{1, 2, \dots, M_{k_0}\}. \tag{2.4.144}$$

From (2.4.128) and (2.4.144) we obtain that

$$R^{-\frac{k_0}{2}}\varphi_{k_0,m}(R) + \frac{2N + k_0 - 2}{2(N + k_0 - 1)}\int_0^R s^{-N-\frac{k_0}{2}}\Upsilon_{k_0,m}(s)ds + \frac{k_0\,R^{-N+1-k_0}}{2(N + k_0 - 1)}\int_0^R s^{\frac{k_0}{2}-1}\Upsilon_{k_0,m}(s)\,ds = 0 \tag{2.4.145}$$

for all $R \in (0, \bar r]$ and $m \in \{1, 2, \dots, M_{k_0}\}$.

Since we are assuming by contradiction that $\lim_{\lambda\to 0^+} \lambda^{-2\gamma}\mathcal{H}(\lambda) = 0$, there exists a sequence $\{R_n\}_{n\in\mathbb{N}} \subset (0,\bar{r})$ such that $R_{n+1} < R_n$, $\lim_{n\to\infty} R_n = 0$ and

$$R_n^{-k_0/2}\sqrt{\mathcal{H}(R_n)} = \max_{s\in[0,R_n]} \left(s^{-k_0/2}\sqrt{\mathcal{H}(s)}\right).$$

By Lemma 2.4.36 with $\lambda_n = R_n$, there exists $m_0 \in \{1,2,\dots,M_{k_0}\}$ such that, up to a subsequence,

$$\lim_{n\to+\infty} \frac{\varphi_{k_0,m_0}(R_n)}{\sqrt{\mathcal{H}(R_n)}} \neq 0. \tag{2.4.146}$$

By (2.4.145), (2.4.136), (2.4.146), (H1-2) and (H2-2), we have

$$\begin{aligned}
&\left|R_n^{-\frac{k_0}{2}}\varphi_{k_0,m_0}(R_n) + \frac{k_0\, R_n^{-N+1-k_0}}{2(N+k_0-1)}\int_0^{R_n} s^{\frac{k_0}{2}-1}\Upsilon_{k_0,m_0}(s)\,ds\right| \\
&= \left|\frac{2N+k_0-2}{2(N+k_0-1)}\int_0^{R_n} s^{-N-\frac{k_0}{2}}\Upsilon_{k_0,m_0}(s)\,ds\right| \\
&\le \frac{2N+k_0-2}{2(N+k_0-1)}\int_0^{R_n} s^{-N-\frac{k_0}{2}}|\Upsilon_{k_0,m_0}(s)|\,ds \\
&\le \text{const}\left(R_n^{-\frac{k_0}{2}+1}\sqrt{\mathcal{H}(R_n)} + \int_0^{R_n} s^{-\frac{k_0}{2}}\sqrt{\mathcal{H}(s)}\Big(1+s^{-1}\beta(s,f)\Big)\,ds\right) \\
&\le \text{const}\left(R_n^{-\frac{k_0}{2}}\sqrt{\mathcal{H}(R_n)}R_n + R_n^{-\frac{k_0}{2}}\sqrt{\mathcal{H}(R_n)}\int_0^{R_n}\frac{\beta(s,f)}{s}\,ds\right) \\
&\le \text{const}\left(\left|\frac{\sqrt{\mathcal{H}(R_n)}}{\varphi_{k_0,m_0}(R_n)}\right|\left|\frac{\varphi_{k_0,m_0}(R_n)}{R_n^{k_0/2}}\right|R_n + \left|\frac{\sqrt{\mathcal{H}(R_n)}}{\varphi_{k_0,m_0}(R_n)}\right|\left|\frac{\varphi_{k_0,m_0}(R_n)}{R_n^{k_0/2}}\right|\int_0^{R_n}\frac{\beta(s,f)}{s}\,ds\right) \\
&= o\left(\frac{\varphi_{k_0,m_0}(R_n)}{R_n^{k_0/2}}\right)
\end{aligned} \tag{2.4.147}$$

as $n\to+\infty$. On the other hand, by (2.4.147) we also have that

$$\begin{aligned}
&\frac{k_0\, R_n^{-N+1-k_0}}{2(N+k_0-1)}\left|\int_0^{R_n} t^{\frac{k_0}{2}-1}\Upsilon_{k_0,m_0}(t)dt\right| \\
&\qquad = \frac{k_0\, R_n^{-N+1-k_0}}{2(N+k_0-1)}\left|\int_0^{R_n} t^{N+k_0-1}t^{-N-\frac{k_0}{2}}\Upsilon_{k_0,m_0}(t)dt\right| \\
&\qquad \le \frac{k_0}{2(N+k_0-1)}\int_0^{R_n} t^{-N-\frac{k_0}{2}}|\Upsilon_{k_0,m_0}(t)|dt = o\left(\frac{\varphi_{k_0,m_0}(R_n)}{R_n^{k_0/2}}\right)
\end{aligned} \tag{2.4.148}$$

as $n\to+\infty$. Combining (2.4.147) with (2.4.148) we obtain that

$$R_n^{-\frac{k_0}{2}}\varphi_{k_0,m_0}(R_n) = o\Big(R_n^{-\frac{k_0}{2}}\varphi_{k_0,m_0}(R_n)\Big) \quad \text{as } n\to+\infty,$$

which is a contradiction. □

Combining Lemma 2.4.31, Lemma 2.4.35 and Lemma 2.4.38, we can now prove the following theorem which gives a more precise description of the limit angular profile ψ of Lemma 2.4.31.

Theorem 2.4.39. *Let $N \geq 2$ and $u \in H^1(B_{\hat{R}}) \setminus \{0\}$ be a non-trivial weak solution to* (1.2.20)*, with f satisfying either assumptions* (H1-1)-(H1-3) *or* (H2-1)-(H2-5)*. Then, letting $\mathcal{N}(r)$ be as in* (2.4.47)*, there exists $k_0 \in \mathbb{N}$, $k_0 \geq 1$, such that*

$$\lim_{r \to 0^+} \mathcal{N}(r) = \frac{k_0}{2}. \tag{2.4.149}$$

Furthermore, if $M_{k_0} \in \mathbb{N} \setminus \{0\}$ is the multiplicity of μ_{k_0} as an eigenvalue of problem (1.2.23) *and $\{Y_{k_0,i} : 1 \leq i \leq M_{k_0}\}$ is a $L^2(\mathbb{S}^N)$-orthonormal basis of the eigenspace associated to μ_{k_0}, then*

$$\frac{u(\lambda z)}{\lambda^{k_0/2}} \to |z|^{k_0/2} \sum_{m=1}^{M_{k_0}} \beta_m Y_{k_0,m}\left(\frac{z}{|z|}\right) \quad \text{in } H^1(B_1) \quad \text{as } \lambda \to 0^+, \tag{2.4.150}$$

where $(\beta_1, \beta_2, \ldots, \beta_{M_{k_0}}) \neq (0, 0, \ldots, 0)$ and

$$\begin{aligned} \beta_m = &\int_{\mathbb{S}^N} R^{-k_0/2} u(\Phi(R\theta)) Y_{k_0,m}(\theta)\, dS \\ &+ \frac{1}{1 - N - k_0} \int_0^R \left(\frac{1 - N - \frac{k_0}{2}}{s^{N + \frac{k_0}{2}}} - \frac{k_0\, s^{\frac{k_0}{2} - 1}}{2R^{N-1+k_0}} \right) \Upsilon_{k_0,m}(s)\, ds \end{aligned} \tag{2.4.151}$$

for all $R \in (0, \bar{r}]$ for some $\bar{r} > 0$, where $\Upsilon_{k_0,m}$ is defined in (2.4.127) *and Φ is the diffeomorphism introduced in Lemma 2.4.33.*

Proof. Identity (2.4.149) follows immediately from Lemma 2.4.31.

In order to prove (2.4.150), let $\{\lambda_n\}_{n \in \mathbb{N}} \subset (0, \infty)$ be such that $\lambda_n \to 0^+$ as $n \to +\infty$. By Lemmas 2.4.31, 2.4.32, 2.4.35, 2.4.38 and (2.4.120), there exist a subsequence $\{\lambda_{n_j}\}_{j \in \mathbb{N}}$ and constants $\beta_1, \beta_2, \ldots, \beta_{M_{k_0}} \in \mathbb{R}$ such that $(\beta_1, \beta_2, \ldots, \beta_{M_{k_0}}) \neq (0, 0, \ldots, 0)$,

$$\lambda_{n_j}^{-\frac{k_0}{2}} u(\lambda_{n_j} z) \to |z|^{\frac{k_0}{2}} \sum_{m=1}^{M_{k_0}} \beta_m Y_{k_0,m}\left(\frac{z}{|z|}\right) \quad \text{in } H^1(B_1) \quad \text{as } j \to +\infty \tag{2.4.152}$$

and

$$\lambda_{n_j}^{-\frac{k_0}{2}} v(\lambda_{n_j} \cdot) \to \sum_{m=1}^{M_{k_0}} \beta_m Y_{k_0,m} \quad \text{in } L^2(\mathbb{S}^N) \quad \text{as } j \to +\infty, \tag{2.4.153}$$

where $\{Y_{k_0,i} : 1 \leq i \leq M_{k_0}\}$ is a $L^2(\mathbb{S}^N)$-orthonormal basis of the eigenspace associated to μ_{k_0}. We will now prove that the β_m's depend neither on the sequence $\{\lambda_n\}_{n \in \mathbb{N}}$ nor on its subsequence $\{\lambda_{n_j}\}_{j\in\mathbb{N}}$. Let us fix $R \in (0, \bar r]$, with $\bar r$ as in Lemma 2.4.33, and define $\varphi_{k_0,m}$ as in (2.4.126). From (2.4.153) it follows that, for any $m = 1, 2, \dots, M_{k_0}$,

$$\lim_{j\to+\infty} \lambda_{n_j}^{-\frac{k_0}{2}} \varphi_{k_0,m}(\lambda_{n_j}) = \lim_{j\to+\infty} \int_{\mathbb{S}^N} \frac{v(\lambda_{n_j}\theta)}{\lambda_{n_j}^{k_0/2}} Y_{k_0,m}(\theta) dS = \sum_{i=1}^{M_{k_0}} \beta_i \int_{\mathbb{S}^N} Y_{k_0,i} Y_{k_0,m} dS = \beta_m. \tag{2.4.154}$$

On the other hand, (2.4.128) implies that, for any $m = 1, 2, \dots, M_{k_0}$,

$$\lim_{\lambda\to0^+} \lambda^{-\frac{k_0}{2}} \varphi_{k_0,m}(\lambda) = R^{-\frac{k_0}{2}} \varphi_{k_0,m}(R) + \frac{2N + k_0 - 2}{2(N + k_0 - 1)} \int_0^R s^{-N-\frac{k_0}{2}} \Upsilon_{k_0,m}(s) ds \\ + \frac{k_0\, R^{-N+1-k_0}}{2(N + k_0 - 1)} \int_0^R s^{\frac{k_0}{2}-1} \Upsilon_{k_0,m}(s)\, ds,$$

with $\Upsilon_{k_0,m}$ as in (2.4.127), and therefore from (2.4.154) we deduce that

$$\beta_m = R^{-\frac{k_0}{2}} \varphi_{k_0,m}(R) + \frac{2N + k_0 - 2}{2(N + k_0 - 1)} \int_0^R s^{-N-\frac{k_0}{2}} \Upsilon_{k_0,m}(s) ds \\ + \frac{k_0\, R^{-N+1-k_0}}{2(N + k_0 - 1)} \int_0^R s^{\frac{k_0}{2}-1} \Upsilon_{k_0,m}(s)\, ds$$

for any $m = 1, 2, \dots, M_{k_0}$. In particular the β_m's depend neither on the sequence $\{\lambda_n\}_{n\in\mathbb{N}}$ nor on its subsequence $\{\lambda_{n_j}\}_{j\in\mathbb{N}}$.

Thanks to Lemma 2.3.9 we obtain that the convergence in (2.4.152) actually holds as $\lambda \to 0^+$, thus proving the theorem. □

As a direct consequence, we deduce the following strong unique continuation principle.

Theorem 2.4.40. *Under the same assumptions as in Theorem 2.4.39, let $u \in H^1(B_{\hat R})$ be a weak solution to* (1.2.15) *such that $u(z) = O(|z|^k)$ as $|z| \to 0$ for any $k \in \mathbb{N}$. Then $u \equiv 0$ in $B_{\hat R}$.*

Proof. Let $u \in H^1(B_{\hat R})$ be a non-trivial weak solution to (1.2.15). By assumption, for every $k \in \mathbb{N}$

$$\left|\frac{u(\lambda z)}{\lambda^{k_0/2}}\right| \le \text{const}\lambda^{k-k_0/2} \tag{2.4.155}$$

for λ sufficiently small. In particular if $k > k_0/2$ then $\frac{u(\lambda z)}{\lambda^{k_0/2}}$ tends to 0 in $L^2(B_1)$ as $\lambda \to 0$, as a consequence of (2.4.155). This contradicts (2.4.150). □

Appendix A

A.1 Some boundary regularity results at edges of cylinders

Let us consider the following local problem: $\Omega \subset \mathbb{R}^N$ is a $C^{1,1}$ domain, $x_0 \in \partial\Omega$, $R, T > 0$ and U is a weak solution to

$$\begin{cases} \operatorname{div}\left(t^{1-2s}\nabla U\right) = 0 & \text{in } C_{R,T}(x_0), \\ U = 0 & \text{in } D_{R,T}(x_0), \\ \lim\limits_{t\to 0} t^{1-2s}\partial_t U = 0 & \text{in } \sigma_{R,T}(x_0), \end{cases} \tag{A.1.1}$$

where

$$C_{R,T}(x_0) := (B'_R(x_0) \cap \Omega) \times (0,T), \quad D_{R,T}(x_0) := (B'_R(x_0) \cap \partial\Omega) \times (0,T),$$
$$\sigma_{R,T}(x_0) = (B'_R(x_0) \cap \Omega) \times \{0\};$$

i.e. U belongs to the space $\mathcal{H}$ defined as the closure of the set

$$\{v \in C^\infty(\overline{C_{R,T}(x_0)}) : v = 0 \text{ in a neighbourhood of } D_{R,T}(x_0)\}$$

in $H^1(C_{R,T}(x_0), t^{1-2s}\, dz)$, and

$$\int_{C_{R,T}(x_0)} t^{1-2s}\nabla U \cdot \nabla\Phi \, dz = 0 \quad \text{for all } \Phi \in C_c^\infty(C_{R,T}(x_0) \cup \sigma_{R,T}(x_0)).$$

The following regularity result holds true.

Lemma A.1.1. *Let $\alpha \in (0,1)$, $\beta \in (0,1) \cap (0, 2-2s)$, $r < R$, and $\tau < T$. Then there exists a positive constant $C > 0$ such that, for every weak solution U to* (A.1.1)*,*

$$\|U\|_{C^{1,\alpha}(C_{r,\tau}(x_0))} + \|t^{1-2s}\partial_t U\|_{C^{0,\beta}(C_{r,\tau}(x_0))} \le C\|U\|_{L^2(C_{R,T}(x_0), t^{1-2s}dz)}.$$

Proof. We denote the total variable $z = (x,t) \in \mathbb{R}^N \times (0,+\infty)$, with $x = (x', x_N) \in \mathbb{R}^{N-1} \times \mathbb{R}$, and we consider $g \in C^{1,1}(\mathbb{R}^{N-1})$ such that

$$B'_R(x_0) \cap \Omega = \{x = (x', x_N) \in B'_R(x_0) : x_N < g(x')\}.$$

Without loss of generality we can assume that $x_0 = 0$, $g(0) = 0$ and $\nabla g(0) = 0$. Starting from this function g, we can argue as in Subsection 2.3.1 and construct a function F as in (2.3.5), which turns out to be a diffeomorphism in a neighbourhood of 0. Hence there exist $0 < r_0 < R$ and $0 < \tau_0 < T$ such that the composition $W = U \circ F$ weakly solves the following straightened problem

$$\begin{cases} \operatorname{div}\left(t^{1-2s} A \nabla W\right) = 0 & \text{in } \Gamma^-_{r_0} \times (0, \tau_0), \\ W = 0 & \text{in } (B'_{r_0} \cap \{y_N = 0\}) \times (0, \tau_0), \\ \lim_{t\to 0} t^{1-2s} A \nabla W \cdot \nu = 0 & \text{in } \Gamma^-_{r_0}, \end{cases}$$

with $A = A(y)$ being as in (2.3.12); in particular the matrix $A(y)$ does not depend on the vertical variable t, is symmetric, uniformly elliptic, and possesses $C^{0,1}$ coefficients.

Let us consider the odd reflection of W (which we still denote as W) through the hyperplane $\{y_N = 0\}$ in $B'_{r_0} \times (0, \tau_0)$, i.e. we set $W(y', y_N, t) = -W(y', -y_N, t)$ for $y_N < 0$; it is easy to verify that W weakly satisfies

$$\begin{cases} \operatorname{div}(t^{1-2s} \widetilde{A} \nabla W) = 0 & \text{in } B'_{r_0} \times (0, \tau_0), \\ \lim_{t\to 0} t^{1-2s} \widetilde{A} \nabla W \cdot \nu = 0 & \text{in } B'_{r_0}, \end{cases}$$

where

$$\widetilde{A}(y) = \widetilde{A}(y', y_N) := \begin{cases} A(y', y_N), & \text{if } y_N \le 0, \\ SA(y', -y_N)S, & \text{if } y_N > 0, \end{cases}$$

with

$$S := \left(\begin{array}{c|c|c} \mathrm{Id}_{N-1} & \mathbf{0} & \mathbf{0} \\ \hline \mathbf{0}^T & -1 & 0 \\ \hline \mathbf{0}^T & 0 & 1 \end{array}\right).$$

We observe that no discontinuities appear in the coefficients of the matrix $\widetilde{A}$ since, denoting as (a_{ij}) the entries of the matrix A,

$$a_{i,N}(y', 0, t) = 0 \quad \text{for all } i < N$$

thanks to (2.3.18) and (2.3.19). Then the matrix $\widetilde{A}$ has Lipschitz continuous coefficients. Thus we consider the even reflection of W (which we still denote as W) through the hyperplane $\{t = 0\}$ in $B'_{r_0} \times (-\tau_0, \tau_0)$, i.e. we set $W(y', y_N, t) = W(y', y_N, -t)$ for $t < 0$; due to the homogeneous Neumann type boundary condition satisfied by W on B'_{r_0} and the fact that the matrix A is independent on t, we obtain that such even reflection through $\{t = 0\}$ weakly solves

$$\operatorname{div}(|t|^{1-2s} \widetilde{A} \nabla W) = 0 \quad \text{in } B'_{r_0} \times (-\tau_0, \tau_0).$$

From [75, Lemma 7.1] it follows that $V = |t|^{1-2s} \partial_t W \in H^1_{\mathrm{loc}}(B'_{r_0} \times (-\tau_0, \tau_0), |t|^{2s-1}\, dz)$ is a weak solution to

$$\operatorname{div}(|t|^{2s-1} \widetilde{A} \nabla V) = 0 \quad \text{in } B'_{r_0} \times (-\tau_0, \tau_0)$$

and such V is odd with respect to $\{t = 0\}$, i.e. $V(y', y_N, -t) = -V(y', y_N, -t)$.

From [75, Theorem 1.2] it follows that, for all $r \in (0, r_0)$ and $\tau \in (0, \tau_0)$,

$$W \in C^{1,\alpha}(B'_r \times (-\tau, \tau)),$$

and

$$\|W\|_{C^{1,\alpha}(B'_r \times(-\tau,\tau))} \leq \text{const}\|W\|_{L^2(B'_{r_0}\times(-\tau_0,\tau_0),|t|^{1-2s}\,dz)}$$

for some const > 0 (independent of W). Furthermore, [35] ensures that V is locally Hölder continous. More precisely, [74, Proposition 2.10] yields that the function

$$\Phi(x, t) = \frac{V(x, t)}{t|t|^{1-2s}},$$

which is even in the variable t, belongs to the weighted Sobolev space

$$H^1_{\text{loc}}(B'_{r_0} \times (-\tau_0, \tau_0), |t|^{3-2s}\, dz),$$

and weakly solves

$$\text{div}(|t|^{3-2s}\widetilde{A}\nabla\Phi) = 0 \quad \text{in } B'_{r_0} \times (-\tau_0, \tau_0),$$

thanks to the fact that the matrix $\widetilde{A}$ is independent of t.

From [75, Theorem 1.2] we have that $\Phi \in C^{0,\gamma}(B'_r \times (-\tau, \tau))$ for all $\gamma \in (0, 1)$, $r \in (0, r_0)$ and $\tau \in (0, \tau_0)$, and

$$\|\Phi\|_{C^{0,\gamma}(B'_r\times(-\tau,\tau))} = \left\|\frac{V}{t|t|^{1-2s}}\right\|_{C^{0,\gamma}(B'_r\times(-\tau,\tau))} \leq \text{const}\,\|V\|_{L^2(B'_{r_0}\times(-\tau_0,\tau_0),|t|^{2s-1}\,dz)}$$

for some const > 0 (independent of V).

Therefore $V \in C^{0,\delta}(B'_r \times (-\tau, \tau))$ with $\delta = \min\{2 - 2s, \gamma\}$ and

$$\|V\|_{C^{0,\delta}(B'_r\times(-\tau,\tau))} \leq \text{const}\,\|V\|_{L^2(B'_{r_0}\times(-\tau_0,\tau_0),|t|^{2s-1}\,dz)}.$$

The conclusion follows by recalling that $U = W \circ F^{-1}$ with F^{-1} being of class $C^{1,1}$ and taking into account the particular form of the matrix in (2.3.6). □

A.2 Homogeneity degrees and eigenvalues of the spherical problem

We derive an explicit formula for the eigenvalues of problem (1.2.11), which follows from a complete classification of possible homogeneity degrees of homogeneous weak solutions to the problem

$$\begin{cases} -\text{div}\left(t^{1-2s}\nabla\Psi\right) = 0 & \text{in } \mathbb{R}^{N+1}_+, \\ \lim\limits_{t\to0^+}\left(t^{1-2s}\nabla\Psi\cdot\nu\right) = 0 & \text{in } \Gamma^-, \\ \Psi = 0 & \text{in } \Gamma^+, \end{cases} \tag{A.2.1}$$

where $\Gamma^- := \{(y', y_N, 0) \in \mathbb{R}^N\times\{0\} : y_N < 0\}$ and $\Gamma^+ := \{(y', y_N, 0) \in \mathbb{R}^N\times\{0\} : y_N \geq 0\}$.

Proposition A.2.1. *Let* $\Psi \in \bigcap_{r>0} H^1_{\Gamma_r^+}(B_r^+, t^{1-2s}\,dz)$ *be a weak solution to* (A.2.1)*, i.e.*

$$\int_{\mathbb{R}^{N+1}_+} t^{1-2s} \nabla\Psi \cdot \nabla\Phi \, dz = 0, \quad \textit{for all } \Phi \in C_c^\infty(\overline{\mathbb{R}^{N+1}_+} \setminus \Gamma^+).$$

If, for some $\gamma \geq 0$*,* $\Psi(z) = |z|^\gamma \Psi(\frac{z}{|z|})$*, then there exists* $j \in \mathbb{N}$ *such that* $\gamma = j + s$*.*

The proof of Proposition A.2.1 requires a polynomial Liouville type theorem for even solutions to degenerate equations with a weight which is possibly out of the A_2-Muckenhoupt class. To this aim, Lemma A.2.2 below provides a generalization of Lemma 2.7 in [10].

For all $a \in (-1, +\infty)$ and $r > 0$, we define $H^1(B_r, |t|^a\,dz)$ as the completion of $C^\infty(\overline{B_r})$ with respect to the norm

$$\sqrt{\int_{B_r} |t|^a \left(|\Psi|^2 + |\nabla\Psi|^2\right) \, dz}$$

and

$$H^{1,a}_{\mathrm{loc}}(\mathbb{R}^{N+1}) := \{\Psi \in L^2_{\mathrm{loc}}(\mathbb{R}^{N+1}, |t|^a\,dz) : \Psi \in H^1(B_r, |t|^a\,dz) \text{ for all } r > 0\}.$$

We also define

$$H^{1,a}_{\mathrm{loc}}(\overline{\mathbb{R}^{N+1}_+}) = \{\Psi \in L^2_{\mathrm{loc}}(\overline{\mathbb{R}^{N+1}_+}, t^a\,dz) : \Psi \in H^1(B_r^+, t^a\,dz) \text{ for all } r > 0\}.$$

Lemma A.2.2. *Let* $a \in (-1, +\infty)$ *and* $v \in H^{1,a}_{\mathrm{loc}}(\mathbb{R}^{N+1})$ *be a weak solution to*

$$\operatorname{div}(|t|^a \nabla v) = 0 \quad \textit{in } \mathbb{R}^{N+1} \tag{A.2.2}$$

which is even in t, i.e.

$$v(x, -t) = v(x, t) \quad \textit{a.e. in } \mathbb{R}^{N+1}.$$

If there exist $k \in \mathbb{N}$ *and* $c > 0$ *such that*

$$|v(z)| \leq c(1 + |z|^k) \qquad \text{for all } z \in \mathbb{R}^{N+1},$$

then v is a polynomial.

Proof. Let $a > -1$ and $v \in H^{1,a}_{\mathrm{loc}}(\mathbb{R}^{N+1})$ be a weak solution to (A.2.2) even in t. For $\alpha \in (0, 1)$ and $k \in \mathbb{N}$, let $D_x^{\beta_k} v$ be the partial derivative with respect to the variables $x = (x_1, ..., x_N)$ of order $k = |\beta_k|$, with $\beta_k \in \mathbb{N}^N$ multiindex. Then, there exists a positive constant $C > 0$ depending only on N, α, a, k such that

$$\sup_{B_{r/2}} |D_x^{\beta_k} v| \leq \frac{C}{r^k} \sup_{B_r} |v| \tag{A.2.3}$$

and

$$[D_x^{\beta_k} v]_{C^{0,\alpha}(B_{r/2})} \leq \frac{C}{r^{k+\alpha}} \sup_{B_r} |v|, \tag{A.2.4}$$

where $[w]_{C^{0,\alpha}(\Lambda)} := \sup_{z,z'\in\Lambda} |z-z'|^{-\alpha}|w(z)-w(z')|$. In order to prove the previous inequalities we apply some local regularity estimates for even solutions contained in [75]. If $k=0$, then the inequalities follow by scaling

$$\|v\|_{C^{0,\alpha}(B_{1/2})} \leq C\|v\|_{L^\infty(B_1)}$$

proved in [75, Theorem 1.2 part *i*)]. If $k \geq 1$, we remark that any partial derivation in variables x_i for $i \in \{1, ..., N\}$ commutes with the operator $\mathrm{div}(|t|^a\nabla\cdot)$ and $D_x^{\beta_k}v$ are actually even solutions to the same equation, (see [75, Section 7] for details). Hence, inequalities (A.2.3) and (A.2.4) follow by scaling and iterating the estimate

$$\|v\|_{C^{1,\alpha}(B_{1/2})} \leq C\|v\|_{L^\infty(B_1)}$$

proved in [75, Theorem 1.2 part *ii*)]. Indeed, after fixing a multiindex β_k, we can choose

$$r_k = 1/2 < r_{k-1} < ... < r_0 = 1,$$

then

$$\begin{aligned}\|D_x^{\beta_k}v\|_{C^{0,\alpha}(B_{1/2})} &\leq C_{k-1}\sup_{B_{r_{k-1}}} |D_x^{\beta_{k-1}}v| \leq C_{k-1}C_{k-2}\sup_{B_{r_{k-2}}} |D_x^{\beta_{k-2}}v| \\ &\leq ... \leq \left(\prod_{i=0}^{k-1} C_i\right)\sup_{B_1}|v|.\end{aligned}$$

Once we have (A.2.3) and (A.2.4), we can proceed exactly as in proof of [10, Lemma 2.7]. We have only to remark that for any $a \in (-1, +\infty)$, given an even solution to (A.2.2) v, then $\partial_{tt}^2 v + \frac{a}{t}\partial_t v = -\Delta_x v$ is also an even solution to (A.2.3). □

Now we are able to prove Proposition A.2.1.

Proof of Proposition A.2.1. Let $\Psi \in H^{1,1-2s}_{\mathrm{loc}}(\overline{\mathbb{R}^{N+1}_+})$ be a weak solution to (A.2.1), such that

$$\Psi(z) = |z|^\gamma \Psi\left(\frac{z}{|z|}\right) \quad \text{in } \mathbb{R}^{N+1}_+,$$

for some $\gamma \geq 0$. The homogeneity condition trivially implies a polynomial global bound on the growth of Ψ. The same bound is inherited by the trace $\phi = \mathrm{Tr}\,\Psi$ on $\mathbb{R}^N = \partial\mathbb{R}^{N+1}_+$, which is also γ-homogeneous. Moreover, $\phi \in C^\infty(\Gamma^-)$ by [75, Theorem 1.1] and $\phi \in C^0(\mathbb{R}^N)$ by [62, Proposition 5.3]. With these premises, we can define the extension V of ϕ in the sense of [1, Lemma 3.3]. Actually, we introduce a minor change in the definition of the extension given in [1]; that is, for every $R > 0$ we define

$$\phi_R = \phi\eta_R \tag{A.2.5}$$

(instead of $\phi_R = \phi\chi_{B'_R}$), where $\eta_R \in C^\infty_c(B'_{2R})$ is a radially decreasing cut-off function with $|\eta_R| \leq 1$ and $\eta_R \equiv 1$ in B'_R. We remark that the adjusted family of functions ϕ_R

convoluted with the usual Poisson kernel of the upper half-space converge in a suitable way to the same extension V obtained by Abatangelo and Ros-Oton in [1]. Moreover, defining the extension starting from (A.2.5), we can easily ensure that $V \in H^{1,1-2s}_{\mathrm{loc}}(\overline{\mathbb{R}^{N+1}_+})$ and that it is weak solution to (A.2.1). Nevertheless, also V inherits from ϕ an at most polynomial growth. Let us consider $W = V - \Psi \in H^{1,1-2s}_{\mathrm{loc}}(\overline{\mathbb{R}^{N+1}_+})$, which weakly solves

$$\begin{cases} \operatorname{div}(t^{1-2s}\nabla W) = 0 & \text{in } \mathbb{R}^{N+1}_+, \\ \operatorname{Tr} W = 0 & \text{on } \mathbb{R}^N = \partial\mathbb{R}^{N+1}_+. \end{cases}$$

Then, denoting as $\widetilde{W}$ the odd reflection of W through $\mathbb{R}^N = \partial\mathbb{R}^{N+1}_+$, by [74, Proposition 2.10]

$$\overline{W} = \frac{\widetilde{W}}{t|t|^{2s-1}} \in H^{1,1+2s}_{\mathrm{loc}}(\mathbb{R}^{N+1})$$

is an even entire weak solution to (A.2.2) with $a = 1+2s \in (1,3)$. We have that $\overline{W}$ satisfies the assumptions of Lemma A.2.2, being a polynomial bound on its growth ensured by the polynomial bounds of Ψ and V. From Lemma A.2.2 we can promptly conclude that $\overline{W}$ is a polynomial. We also have that

$$t^{1-2s}\partial_t V = t^{1-2s}\partial_t \Psi + t^{1-2s}\partial_t(t^{2s}\overline{W}) = t^{1-2s}\partial_t\Psi + P_k$$

for some polynomial P_k of degree $k \in \mathbb{N}$. Hence, passing to the trace of the weighted derivative above, by [1, Lemma 3.3] it follows that

$$(-\Delta)^s\phi \overset{k+1}{=} 0 \quad \text{in } \Gamma^-$$

and $\phi = 0$ in Γ^+, where the above identity is meant in the sense of the notion of "fractional Laplacian modulus polynomials of degree at most k" given in [1, Definition 3.1], see also [30]. Hence, by [1, Theorem 3.10], we have that

$$\phi(x) = p(x)(x_N)^s_-,$$

for some polynomial p. By homogeneity of ϕ, this implies that necessary there exists $j \in \mathbb{N}$ such that $\gamma = j + s$. □

We are now going to derive from Proposition A.2.1 the explicit formula (1.2.13) for the eigenvalues of problem (1.2.11). We first observe that, if μ is an eigenvalue of (1.2.11) with an associated eigenfunction ψ, then the function $\Psi(\rho\theta) = \rho^\sigma\psi(\theta)$ with

$$\sigma = -\frac{N-2s}{2} + \sqrt{\left(\frac{N-2s}{2}\right)^2 + \mu}$$

belongs to $H^{1,1-2s}_{\mathrm{loc}}(\overline{\mathbb{R}^{N+1}_+})$ and is a weak solution to (A.2.1). From Proposition A.2.1 we then deduce that there exists $j \in \mathbb{N}$ such that $\sigma = j + s$ and hence

$$\mu = (j+s)(j+N-s).$$

Viceversa, we prove now that all numbers of the form $\mu = (j+s)(j+N-s)$ with $j \in \mathbb{N}$ are eigenvalues of (1.2.11). For any fixed $j \in \mathbb{N}$, we consider the function Ψ defined, in cylindrical coordinates, as

$$\Psi(x', r\cos\tau, r\sin\tau) = r^{s+j} \left|\sin\left(\frac{\tau}{2}\right)\right|^{2s} {}_2F_1\left(-j, j+1; 1-s; \frac{1+\cos\tau}{2}\right), \ r \geq 0, \ \tau \in [0, 2\pi],$$

where ${}_2F_1$ is the hypergeometric function. From [65] we have that $\Psi \in H^{1,1-2s}_{\rm loc}(\overline{\mathbb{R}^{N+1}_+})$ is a weak solution to (A.2.1). Furthermore Ψ is homogeneous of degree $s+j$ and therefore the function $\psi := \Psi\big|_{\mathbb{S}^N_+}$ belongs to $\mathcal{H}_0$, $\psi \not\equiv 0$, and

$$\Psi(\rho\theta) = \rho^{s+j}\psi(\theta), \quad \rho \geq 0, \ \theta \in \mathbb{S}^N_+.$$

Plugging the above characterization of Ψ into (A.2.1), we obtain that

$$\rho^{j-1-s}\Big((j+s)(j+N-s)\theta_{N+1}^{1-2s}\psi(\theta) + \mathrm{div}_{\mathbb{S}^N}\left(\theta_{N+1}^{1-2s}\nabla_{\mathbb{S}^N}\psi\right)\Big) = 0, \quad \rho > 0, \ \theta \in \mathbb{S}^N_+,$$

so that $(j+s)(j+N-s)$ is an eigenvalue of (1.2.11).

We then conclude that the set of all eigenvalues of problem (1.2.11) is

$$\{(j+s)(j+N-s) : j \in \mathbb{N}\}.$$

Chapter 3

A nonlocal capillarity problem

In the present chapter we discuss the results contained in [23], namely we perform the study of a nonlocal capillarity problem with interaction kernels that are possibly anisotropic and not necessarily invariant under scaling. In particular, the lack of scale invariance will be modeled via two different fractional exponents $s_1, s_2 \in (0,1)$ which take into account the possibility that the container and the environment present different features with respect to particle interactions.

In detail, given an open set $\Omega \subseteq \mathbb{R}^n$ ($n \geq 2$), s_1, $s_2 \in (0,1)$ and $\sigma \in \mathbb{R}$, for every $K_1 \in \mathbf{K}(n, s_1, \lambda, \varrho)$ and $K_2 \in \mathbf{K}(n, s_2, \lambda, \varrho)$ (see Section 1.2 for the definition of the space $\mathbf{K}(n, s, \lambda, \varrho)$ with $s \in (0,1)$) and every set $E \subseteq \Omega$, we consider a functional $\mathcal{E}(E)$ defined as in (1.2.33), where $I_1 := I_{K_1}$ and $I_2 := I_{K_2}$ according to (1.2.32). From this, we consider the functional $\mathcal{C}$ as in (1.2.34) with $g \in L^\infty(\Omega)$.

Our aim is to investigate the existence of minimizers of the nonlocal capillarity functional $\mathcal{C}$ among all the sets E with a given volume and to find the equation prescribing the contact angle between the droplet and the container.

Before to dive into the technicalities, we introduce the following notations that will be used throughout all this chapter:

- given a set $F \subseteq \mathbb{R}^n$, $x_0 \in \mathbb{R}^n$ and $r > 0$, we let

$$F^{x_0,r} := \frac{F - x_0}{r}; \tag{3.0.1}$$

- for any two angles ϑ_1, $\vartheta_2 \in [0, 2\pi)$, with $\vartheta_1 < \vartheta_2$, we define

$$J_{\vartheta_1,\vartheta_2} := \Big\{ x \in \mathbb{R}^n \,:\, \exists\, \beta \in (\vartheta_1, \vartheta_2),\ \rho > 0 \text{ such that } (x_1, x_n) = \rho(\cos\beta, \sin\beta) \Big\}; \tag{3.0.2}$$

- for any angle α, we set

$$e(\alpha) := \cos\alpha\, e_1 + \sin\alpha\, e_n. \tag{3.0.3}$$

In particular, in our analysis we consider $K_1 \in \mathbf{K}^2(n, s_1, \lambda, \varrho)$ and $K_2 \in \mathbf{K}^2(n, s_2, \lambda, \varrho)$ such that the associated blow-up kernels defined as in (1.2.31) are well-defined and given by

$$K_1^*(\zeta) = \frac{a_1(\overrightarrow{\zeta})}{|\zeta|^{n+s_1}} \qquad \text{and} \qquad K_2^*(\zeta) = \frac{a_2(\overrightarrow{\zeta})}{|\zeta|^{n+s_2}}, \tag{3.0.4}$$

where $\overrightarrow{\zeta} := \frac{\zeta}{|\zeta|}$ and a_1, a_2 are continuous functions on ∂B_1, bounded from above and below by two positive constants and satisfying

$$a_i(\omega) = a_i(-\omega) \tag{3.0.5}$$

for all $\omega \in \partial B_1$ and $i \in \{1, 2\}$.

3.1 Existence of minimizers

In this section we prove the existence of minimizers for the functional $\mathcal{C}$ defined in (1.2.34), which is based on a semicontinuity argument and on a direct minimization procedure.

For this we first provide the following lower semicontinuity lemma.

Lemma 3.1.1 (Semicontinuity of the energy). *If $I_2(\Omega, \Omega^c) < +\infty$, $E_j \subseteq \Omega$ and $E_j \to E$ in $L^1(\Omega)$, then*

$$\liminf_{j \to +\infty} \mathcal{E}(E_j) \geq \mathcal{E}(E).$$

Proof. If $\sigma \geq 0$, the proof follows by Fatou's Lemma. If instead $\sigma < 0$, then we observe that

$$I_2(\Omega, \Omega^c) = I_2(E, \Omega^c) + I_2(E^c \cap \Omega, \Omega^c),$$

and therefore, using that $\sigma = -|\sigma|$, we can write

$$\begin{aligned} \mathcal{E}(E) &= I_1(E, E^c \cap \Omega) - |\sigma| I_2(E, \Omega^c) + (|\sigma| + 1) I_2(\Omega, \Omega^c) - (|\sigma| + 1) I_2(\Omega, \Omega^c) \\ &= I_1(E, E^c \cap \Omega) + I_2(E, \Omega^c) + (|\sigma| + 1) I_2(E^c \cap \Omega, \Omega^c) - (|\sigma| + 1) I_2(\Omega, \Omega^c). \end{aligned}$$

As a consequence, we can exploit Fatou's Lemma and obtain the desired result. □

With this we are able to prove the following result.

Proposition 3.1.2. *Let $K_1 \in \mathbf{K}(n, s_1, \lambda, \varrho)$ and $K_2 \in \mathbf{K}(n, s_2, \lambda, \varrho)$. Let Ω be an open and bounded set with $I_1(\Omega, \Omega^c) + I_2(\Omega, \Omega^c) < +\infty$.*

Let $m \in (0, |\Omega|)$ and $g \in L^\infty(\Omega)$.

Then, there exists a minimizer for the functional $\mathcal{C}$ in (1.2.34) among all the sets E with Lebesgue measure equal to m.

Moreover, $I_1(E, E^c \cap \Omega) < +\infty$ for every minimizer E.

Proof. We observe that, if $K_1 \in \mathbf{K}(n, s_1, \lambda, \varrho)$, then, for any $p \in \mathbb{R}^n$,

$$I_1(F, F^c) \geq \frac{1}{\lambda} I_{s_1}(F \cap B_{\varrho/2}(p), F^c \cap B_{\varrho/2}(p)) \quad \text{for every } F \subseteq \mathbb{R}^n. \tag{3.1.1}$$

To prove it, we notice that if $x, y \in B_{\varrho/2}(p)$, then $|x-y| \le |x-p| + |p-y| < \varrho$, and therefore, recalling (1.2.29),

$$I_1(F, F^c) \ge \int_{F\cap B_{\varrho/2}(p)} \int_{F^c\cap B_{\varrho/2}(p)} K_1(x-y)\,dx\,dy \ge \frac{1}{\lambda} \int_{F\cap B_{\varrho/2}(p)} \int_{F^c\cap B_{\varrho/2}(p)} \frac{dx\,dy}{|x-y|^{n+s_1}},$$

which establishes (3.1.1).

Now, if H is a half-space such that $|H \cap \Omega| = m$ and $R > 0$ is such that $\Omega \subseteq B_R$, then, using (1.2.29), we see that

$$I_1(H \cap B_R, H^c \cap B_R) = C\, R^{n-s_1},$$

for some $C > 0$ depending only on n and s_1, and therefore

$$\begin{aligned} \mathcal{E}(H\cap\Omega) &= I_1(H\cap\Omega, (H\cap\Omega)^c\cap\Omega) + \sigma I_2(H\cap\Omega, \Omega^c) \\ &= I_1(H\cap\Omega, H^c\cap\Omega) + \sigma I_2(H\cap\Omega, \Omega^c) \\ &\le I_1(H\cap B_R, H^c\cap B_R) + |\sigma|\, I_2(\Omega, \Omega^c) \\ &< +\infty. \end{aligned}$$

As a consequence, we find that

$$\gamma := \inf \{\mathcal{C}(E) \;:\; E \subseteq \Omega,\ |E| = m\} < +\infty.$$

Let now $E_j \subseteq \Omega$ be such that $|E_j| = m$ and $\mathcal{C}(E_j) = \mathcal{E}(E_j) + \int_{E_j} g \to \gamma$ as $j \to +\infty$. Then, if j is large enough, we have that

$$\gamma + 1 + \int_\Omega |g| \ge \mathcal{E}(E_j) = I_1(E_j, E_j^c\cap\Omega) + \sigma I_2(E_j, \Omega^c) \ge I_1(E_j, E_j^c\cap\Omega) - |\sigma|\, I_2(\Omega, \Omega^c).$$

Consequently

$$I_1(E_j, E_j^c) = I_1(E_j, E_j^c\cap\Omega) + I_1(E_j, E_j^c\cap\Omega^c) \le \gamma + 1 + \int_\Omega |g| + I_1(\Omega, \Omega^c) + |\sigma| I_2(\Omega, \Omega^c).$$

Since $E_j \subseteq B_R$, using (3.1.1) and the fact that the space $W^{s_1,2}(B_R)$ is compactly embedded in $L^1(B_R)$, we find that, up to a subsequence, $E_j \to E$ in $L^1_{\rm loc}(\mathbb{R}^n)$ for some $E \subseteq \Omega$ with $|E| = m$. Hence, using the semicontinuity property in Lemma 3.1.1, we conclude that E is a minimizer.

We also remark that

$$I_2(E, \Omega^c) \le I_2(\Omega, \Omega^c) < +\infty,$$

and therefore, since $\mathcal{E}(E) < +\infty$, we conclude that

$$I_1(E, E^c\cap\Omega) < +\infty,$$

as desired. □

3.2 The Euler-Lagrange equation

In this section we present one of the main basic features of the capillarity energy functional in (1.2.34). More explicitly, the volume constrained minimizers (and, more generally, the volume constrained critical points) obtained in Proposition 3.1.2 satisfy (under reasonable regularity assumptions on the domain and on the interaction kernels) a suitable Euler-Lagrange equation, according to the following result. To state it precisely, it is convenient to denote by Reg_E the collection of all those points $x_0 \in \overline{\Omega \cap \partial E}$ for which there exists $\rho > 0$ and $\alpha \in (s_1, 1)$ such that $B_\rho(x_0) \cap \partial E$ is a manifold of class $C^{1,\alpha}$ possibly with boundary, and the boundary (if any) is contained in $\partial\Omega$, see Figure 3.1.

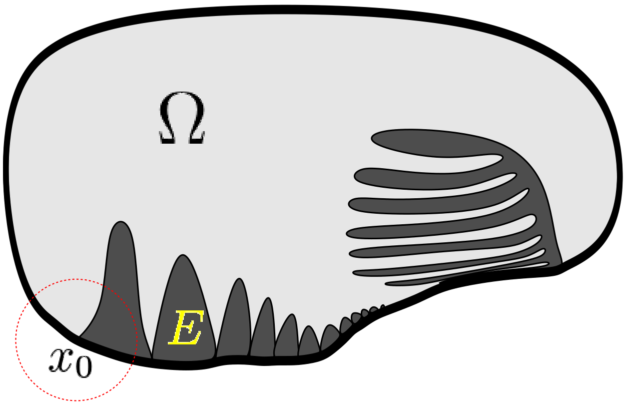


Figure 3.1: The geometry involved in the definition of Reg_E.

Given a kernel $K \in \mathbf{K}(n, s_1, \lambda, \varrho)$, it is also convenient to recall the notion of K-mean curvature, that is defined, for all $x \in \Omega \cap \mathrm{Reg}_E$, as

$$\mathbf{H}^K_{\partial E}(x) := \text{p.v.} \int_{\mathbb{R}^n} K(x-y)\big(\chi_{E^c}(y) - \chi_E(y)\big)\, dy. \tag{3.2.1}$$

Here p.v. stands for the principal value, that we omit from now on for the sake of simplicity of notation. We also say that $E \subseteq \Omega$ is a critical point of $\mathcal{C}$ among sets with prescribed Lebesgue measure if

$$\frac{d}{dt}\Big|_{t=0} \mathcal{C}\big(f_t(E)\big) = 0,$$

for every family of diffeomorphisms $\{f_t\}_{|t|<\delta}$ such that, for each $|t| < \delta$, one has that $f_0 = \mathrm{Id}$, the support of $f_t - \mathrm{Id}$ is a compact set, $f_t(\Omega) = \Omega$ and $|f_t(E)| = |E|$.

With this notation, we have the following result:

Proposition 3.2.1. *Let $K_1 \in \mathbf{K}^1(n, s_1, \lambda, \varrho)$ and $K_2 \in \mathbf{K}^1(n, s_2, \lambda, \varrho)$. Let Ω be an open bounded set with C^1-boundary and $g \in C^1(\mathbb{R}^n)$. Let E be a critical point of $\mathcal{C}$ in* (1.2.34) *among all the sets with Lebesgue measure equal to m. Then, there exists $c \in \mathbb{R}$ such that*

$$\begin{aligned}&\iint_{E\times(E^c\cap\Omega)} \mathrm{div}_{(x,y)}\Big(K_1(x-y)\big(T(x),T(y)\big)\Big)\,dx\,dy\\&+\sigma\iint_{E\times\Omega^c} \mathrm{div}_{(x,y)}\Big(K_2(x-y)\big(T(x),T(y)\big)\Big)\,dx\,dy+\int_E \mathrm{div}(g\,T)=c\int_E \mathrm{div}T\end{aligned} \tag{3.2.2}$$

for every $T \in C_c^\infty(\mathbb{R}^n;\mathbb{R}^n)$ with

$$T\cdot\nu_\Omega = 0 \quad \textit{on } \partial\Omega.$$

Moreover, if $K_1 \in \mathbf{K}^2(n,s_1,\lambda,\varrho)$ and $K_2 \in \mathbf{K}^2(n,s_2,\lambda,\varrho)$, then

$$\mathbf{H}^{K_1}_{\partial E}(x) - \int_{\Omega^c} K_1(x-y)\,dy + \sigma\int_{\Omega^c} K_2(x-y)\,dy + g(x) = c \tag{3.2.3}$$

for all $x \in \Omega\cap \mathrm{Reg}_E$.

The proof of Proposition 3.2.1 relies on a modification of techniques previously exploited in [9, 45, 60]. We omit the proof here since one can follow precisely the proof of Theorem 1.3 in [60] with obvious modifications due to the presence of different kernels.

3.3 The cancellation property in the anisotropic setting

In this section we exhibit the proof of the cancellation property in the anisotropic setting. The argument relies on a delicate analysis of the geometric properties of the integrals involved in the definition of the function (3.3.1).

Proposition 3.3.1. *Given $\vartheta \in (0,\pi)$, for every $\bar\vartheta \in (0,2\pi)$ let*

$$\mathcal{D}_\vartheta(\bar\vartheta) := \int_{J_{\vartheta,\vartheta+\bar\vartheta}} \frac{a_1(\overrightarrow{x-e(\vartheta)})}{|x-e(\vartheta)|^{n+s_1}}\,dx - \int_{J_{0,\vartheta}} \frac{a_1(\overrightarrow{x-e(\vartheta)})}{|x-e(\vartheta)|^{n+s_1}}\,dx. \tag{3.3.1}$$

Then,

$$\mathcal{D}_\vartheta \textit{ is well-defined in the principal value sense;} \tag{3.3.2}$$

$$\mathcal{D}_\vartheta \textit{ is continuous in } (0,2\pi); \tag{3.3.3}$$

$$\lim_{\bar\vartheta\searrow 0} \mathcal{D}_\vartheta(\bar\vartheta) = -\infty; \tag{3.3.4}$$

$$\lim_{\bar\vartheta\nearrow 2\pi} \mathcal{D}_\vartheta(\bar\vartheta) = +\infty. \tag{3.3.5}$$

Moreover, for every $c \in \mathbb{R}$ and every angle $\vartheta \in (0,\pi)$, there exists a unique angle $\widehat{\vartheta} \in (0,2\pi)$ such that

$$\mathcal{D}_\vartheta(\widehat{\vartheta}) = c. \tag{3.3.6}$$

Proof. We focus on the proof of (3.3.2), (3.3.3), (3.3.4) and (3.3.5): once these statements are proved, we can conclude that there exists an angle $\widehat{\vartheta} \in (0, 2\pi)$ such that $\mathcal{D}_\vartheta(\widehat{\vartheta}) = 0$, and this angle is unique since $\mathcal{D}_\vartheta$ is strictly increasing, thus establishing (3.3.6).

We start by proving (3.3.2). For this, we observe that the definition in (3.3.1) has to be interpreted in the principal-value sense, namely

$$\mathcal{D}_\vartheta(\bar{\vartheta}) = \lim_{\rho \searrow 0} \left(\int_{J_{\vartheta,\vartheta+\bar{\vartheta}} \setminus B_\rho(e(\vartheta))} \frac{a_1(\overrightarrow{x - e(\vartheta)})}{|x - e(\vartheta)|^{n+s_1}} \, dx - \int_{J_{0,\vartheta} \setminus B_\rho(e(\vartheta))} \frac{a_1(\overrightarrow{x - e(\vartheta)})}{|x - e(\vartheta)|^{n+s_1}} \, dx \right). \tag{3.3.7}$$

Hence, to establish (3.3.2), we want to show that the limit in (3.3.7) does exist and is finite. To this end, given $\bar{\vartheta} \in (0, 2\pi)$, we let $\delta := \min\{\sin \bar{\vartheta}, \sin \vartheta\}$ and we note that $B_\delta(e(\vartheta))$ is contained in $J_{0,\vartheta+\bar{\vartheta}}$. Then, for every $\rho \in (0, \delta]$ we set

$$f(\rho) := \int_{J_{\vartheta,\vartheta+\bar{\vartheta}} \setminus B_\rho(e(\vartheta))} \frac{a_1(\overrightarrow{x - e(\vartheta)})}{|x - e(\vartheta)|^{n+s_1}} \, dx - \int_{J_{0,\vartheta} \setminus B_\rho(e(\vartheta))} \frac{a_1(\overrightarrow{x - e(\vartheta)})}{|x - e(\vartheta)|^{n+s_1}} \, dx.$$

We also define $A_{\delta,\rho}(e(\vartheta)) := B_\delta(e(\vartheta)) \setminus B_\rho(e(\vartheta))$, see Figure 3.2. By the change of variable $x \mapsto 2e(\vartheta) - x$, we see that

$$\begin{aligned} &\int_{J_{\vartheta,\vartheta+\bar{\vartheta}} \cap A_{\delta,\rho}(e(\vartheta))} \frac{a_1(\overrightarrow{x - e(\vartheta)})}{|x - e(\vartheta)|^{n+s_1}} \, dx - \int_{J_{0,\vartheta} \cap A_{\delta,\rho}(e(\vartheta))} \frac{a_1(\overrightarrow{x - e(\vartheta)})}{|x - e(\vartheta)|^{n+s_1}} \, dx \\ &= \int_{J_{0,\vartheta} \cap A_{\delta,\rho}(e(\vartheta))} \left[\frac{a_1(\overrightarrow{e(\vartheta) - x})}{|e(\vartheta) - x|^{n+s_1}} - \frac{a_1(\overrightarrow{x - e(\vartheta)})}{|x - e(\vartheta)|^{n+s_1}} \right] dx = 0, \end{aligned}$$

since a_1 is symmetric. From this, we deduce that for every $\rho \in (0, \delta]$

$$f(\rho) - f(\delta) = \int_{J_{\vartheta,\vartheta+\bar{\vartheta}} \cap A_{\delta,\rho}(e(\vartheta))} \frac{a_1(\overrightarrow{x - e(\vartheta)})}{|x - e(\vartheta)|^{n+s_1}} \, dx - \int_{J_{0,\vartheta} \cap A_{\delta,\rho}(e(\vartheta))} \frac{a_1(\overrightarrow{x - e(\vartheta)})}{|x - e(\vartheta)|^{n+s_1}} \, dx = 0.$$

Hence we conclude that

$$\lim_{\rho \searrow 0} f(\rho) = f(\delta), \tag{3.3.8}$$

thus proving the existence and finiteness of the limit in (3.3.7).

This completes the proof of (3.3.2) and we now focus on the proof of (3.3.3).

For this, we notice that, if $\widetilde{\vartheta}$, $\bar{\vartheta} \in (0, 2\pi)$ with $\bar{\vartheta} \geq \widetilde{\vartheta}$,

$$\begin{aligned} \mathcal{D}_\vartheta(\bar{\vartheta}) - \mathcal{D}_\vartheta(\widetilde{\vartheta}) &= \int_{J_{\vartheta,\vartheta+\bar{\vartheta}}} \frac{a_1(\overrightarrow{x - e(\vartheta)})}{|x - e(\vartheta)|^{n+s_1}} \, dx - \int_{J_{\vartheta,\vartheta+\widetilde{\vartheta}}} \frac{a_1(\overrightarrow{x - e(\vartheta)})}{|x - e(\vartheta)|^{n+s_1}} \, dx \\ &= \int_{J_{\vartheta+\widetilde{\vartheta},\vartheta+\bar{\vartheta}}} \frac{a_1(\overrightarrow{x - e(\vartheta)})}{|x - e(\vartheta)|^{n+s_1}} \, dx. \end{aligned}$$

Since the denominator in the latter integral is bounded from below by a positive constant (depending on $\widetilde{\vartheta}$), the claim in (3.3.3) follows from the Dominated Convergence Theorem.

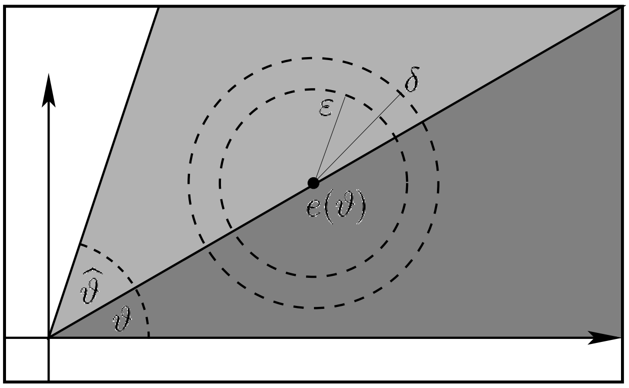


Figure 3.2: The geometry involved in the proof of the existence and finiteness of the limit in (3.3.7).

We now deal with the proof of (3.3.4) and (3.3.5). To this end, we first prove that

$$\begin{aligned} &\lim_{\varepsilon\searrow 0}\left(\int_{J_{\vartheta-\varepsilon,\vartheta}} \frac{a_1(\overrightarrow{x-e(\vartheta)})}{|x-e(\vartheta)|^{n+s_1}}\,dx - \int_{J_{\vartheta,\vartheta+2\varepsilon}} \frac{a_1(\overrightarrow{x-e(\vartheta)})}{|x-e(\vartheta)|^{n+s_1}}\,dx\right) = -\infty \\ \text{and}\qquad &\lim_{\varepsilon\searrow 0}\left(\int_{J_{\vartheta-2\varepsilon,\vartheta}} \frac{a_1(\overrightarrow{x-e(\vartheta)})}{|x-e(\vartheta)|^{n+s_1}}\,dx - \int_{J_{\vartheta,\vartheta+\varepsilon}} \frac{a_1(\overrightarrow{x-e(\vartheta)})}{|x-e(\vartheta)|^{n+s_1}}\,dx\right) = +\infty. \end{aligned} \tag{3.3.9}$$

We focus on the proof of the first claim in (3.3.9) since a similar argument would take care of the second one. For this, let Ξ be the first limit in (3.3.9) and $\mathcal{R}$ be the rotation by an angle ϑ in the (x_1, x_n) plane that sends $e(\vartheta)$ in $e_1 = (1, 0, \dots, 0)$. Let also $a_{1,\vartheta} := a_1 \circ \mathcal{R}$ and notice that $a_{1,\vartheta}$ inherits the properties of a_1, that is $a_{1,\vartheta}$ is a continuous functions on ∂B_1, bounded from above and below by two positive constants and satisfying $a_{1,\vartheta}(\omega) = a_{1,\vartheta}(-\omega)$ for all $\omega \in \partial B_1$.

With this notation, we have

$$\Xi = \lim_{\varepsilon\searrow 0}\left(\int_{J_{-\varepsilon,0}} \frac{a_{1,\vartheta}(\overrightarrow{x-e_1})}{|x-e_1|^{n+s_1}}\,dx - \int_{J_{0,2\varepsilon}} \frac{a_{1,\vartheta}(\overrightarrow{x-e_1})}{|x-e_1|^{n+s_1}}\,dx\right). \tag{3.3.10}$$

We also remark that, in view of the boundedness of $a_{1,\vartheta}$,

$$\int_{J_{-2\varepsilon,2\varepsilon}\setminus B_2} \frac{a_{1,\vartheta}(\overrightarrow{x-e_1})}{|x-e_1|^{n+s_1}}\,dx \le \int_{\mathbb{R}^n\setminus B_1} \frac{C}{|y|^{n+s_1}}\,dy \le C,$$

for a suitable constant $C \ge 1$ possibly varying from step to step.

Combining this information with (3.3.10) we find that

$$\Xi \le \lim_{\varepsilon\searrow 0}\left(\int_{J_{-\varepsilon,0}\cap B_2} \frac{a_{1,\vartheta}(\overrightarrow{x-e_1})}{|x-e_1|^{n+s_1}}\,dx - \int_{J_{0,2\varepsilon}\cap B_2} \frac{a_{1,\vartheta}(\overrightarrow{x-e_1})}{|x-e_1|^{n+s_1}}\,dx + C\right). \tag{3.3.11}$$

Now we claim that, if ε is sufficiently small,

$$B_{\varepsilon/10}\left(e_1+\frac{3\varepsilon}{2}e_n\right)\subseteq J_{\varepsilon,2\varepsilon}\cap B_2. \tag{3.3.12}$$

To check this, let $y\in B_{\varepsilon/10}\left(e_1+\frac{3\varepsilon}{2}e_n\right)$. Then,

$$\frac{\varepsilon^2}{100}\geq |y_1-1|^2+\left|y_n-\frac{3\varepsilon}{2}\right|^2$$

and accordingly $y_1\in\left[1-\frac{\varepsilon}{10},1+\frac{\varepsilon}{10}\right]$ and $y_n\in\left[\frac{7\varepsilon}{5},\frac{8\varepsilon}{5}\right]$. As a consequence, if ε is conveniently small,

$$\frac{y_n}{y_1}\in\left[\frac{\frac{7\varepsilon}{5}}{1+\frac{\varepsilon}{10}},\frac{\frac{8\varepsilon}{5}}{1-\frac{\varepsilon}{10}}\right]\subseteq\left[\frac{6\varepsilon}{5},\frac{9\varepsilon}{5}\right]\subseteq[\tan\varepsilon,\tan(2\varepsilon)],$$

which, recalling (3.0.2), establishes (3.3.12).

Using (3.3.12) and the assumption that $a_{1,\vartheta}$ is bounded from below away from zero, we obtain that

$$\int_{J_{\varepsilon,2\varepsilon}\cap B_2}\frac{a_{1,\vartheta}(\overrightarrow{x-e_1})}{|x-e_1|^{n+s_1}}\,dx\geq\frac{1}{C}\int_{B_{\varepsilon/10}\left(e_1+\frac{3\varepsilon}{2}e_n\right)}\frac{dx}{|x-e_1|^{n+s_1}}\geq\frac{1}{C\varepsilon^{s_1}}.$$

This and (3.3.11) entail that

$$\Xi\leq\lim_{\varepsilon\searrow0}\left(\int_{J_{-\varepsilon,0}\cap B_2}\frac{a_{1,\vartheta}(\overrightarrow{x-e_1})}{|x-e_1|^{n+s_1}}\,dx-\int_{J_{0,\varepsilon}\cap B_2}\frac{a_{1,\vartheta}(\overrightarrow{x-e_1})}{|x-e_1|^{n+s_1}}\,dx-\frac{1}{C\varepsilon^{s_1}}+C\right). \tag{3.3.13}$$

Now we observe that

$$J_{-\varepsilon,\varepsilon}=\left\{x\in\mathbb{R}^n\;:\;|x_n|<\tan\varepsilon\,x_1\right\}$$

and we define

$$\begin{aligned}
&J^\sharp_\varepsilon:=2e_1-J_{-\varepsilon,\varepsilon},\\
&R_\varepsilon:=J_{-\varepsilon,\varepsilon}\cap J^\sharp_\varepsilon,\\
&J^\star_\varepsilon:=J_{0,\varepsilon}\setminus R_\varepsilon\\
\text{and}\qquad&\mathcal{B}_\varepsilon:=\left\{x\in J^\star_\varepsilon\;:\;x_n>\frac{x_1-1}{|\log\varepsilon|}\right\},
\end{aligned}$$

see Figure 3.3.

The intuition behind this set decomposition is that, on the one hand, the set R_ε accounts for the cancellations due to the symmetry of $a_{1,\vartheta}$ (corresponding to the reflection through e_1, namely $x\mapsto 2e_1-x$); on the other hand, the remaining integral contributions in $J^\star_\varepsilon$ cancel exactly when $a_{1,\vartheta}$ is constant, thanks to the reflection through the horizontal hyperplane $x\mapsto(x',-x_n)$, but they may provide additional terms when $a_{1,\vartheta}$ is not constant. To overcome this difficulty, our idea is to exploit the continuity of $a_{1,\vartheta}$ together

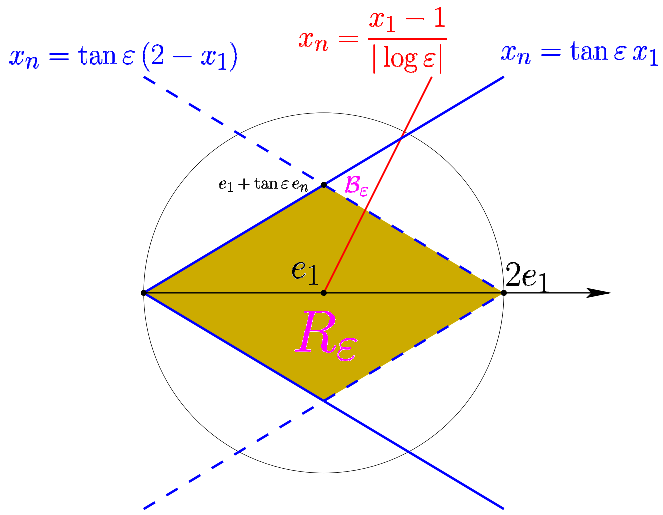


Figure 3.3: The set decomposition involved in the proof of (3.3.9).

with the reflection through the horizontal hyperplane in order to "approximately cancel" as many contributions as possible.

This idea by itself however does not exhaust the complexity of the problem, because two adjacent points can end up being projected far away from each other on the sphere (for instance, if a point is close to $e_1 + \tan\varepsilon\, e_n$ and the other to $e_1 - \tan\varepsilon\, e_n$). To overcome this additional complication, we exploit the set $\mathcal{B}_\varepsilon$: roughly speaking, points outside $\mathcal{B}_\varepsilon$ remain sufficiently close after they get projected on the sphere (and here we can take advantage of the continuity of $a_{1,\vartheta}$), while the points in $\mathcal{B}_\varepsilon$ provide an additional, but small, correction, in view of the location of $\mathcal{B}_\varepsilon$ and of its measure.

The details of the quantitative computation needed to implement this combination of ideas go as follows.

We stress that

$$\text{if } x \text{ belongs to } R_\varepsilon, \text{ then so does } 2e_1 - x. \tag{3.3.14}$$

Indeed, if $x \in R_\varepsilon$ then $x \in J_{-\varepsilon,\varepsilon}$ and $x \in 2e_1 - J_{-\varepsilon,\varepsilon}$, and consequently $2e_1 - x \in 2e_1 - J_{-\varepsilon,\varepsilon}$ and $2e_1 - x \in J_{-\varepsilon,\varepsilon}$, which gives (3.3.14).

Also, we see that $J_{-\varepsilon,0} \cap R_\varepsilon = R_\varepsilon \cap \{x_n < 0\}$ and $J_{0,\varepsilon} \cap R_\varepsilon = R_\varepsilon \cap \{x_n > 0\}$. Thus, using (3.3.14), the change of variable $x \mapsto 2e_1 - x$ and the symmetry of $a_{1,\vartheta}$, taking into account that under this transformation some vectors end up outside the ball B_2,

$$\begin{aligned}
\int_{J_{-\varepsilon,0}\cap B_2\cap R_\varepsilon} \frac{a_{1,\vartheta}(\overrightarrow{x-e_1})}{|x-e_1|^{n+s_1}}\,dx &= \int_{R_\varepsilon\cap\{x_n<0\}\cap B_2} \frac{a_{1,\vartheta}(\overrightarrow{x-e_1})}{|x-e_1|^{n+s_1}}\,dx \\
&\le \int_{R_\varepsilon\cap\{x_n>0\}\cap B_2} \frac{a_{1,\vartheta}(\overrightarrow{e_1-x})}{|e_1-x|^{n+s_1}}\,dx + C\int_{\mathbb{R}^n\setminus B_2} \frac{dx}{|x-e_1|^{n+s_1}} \\
&\le \int_{J_{0,\varepsilon}\cap B_2\cap R_\varepsilon} \frac{a_{1,\vartheta}(\overrightarrow{x-e_1})}{|x-e_1|^{n+s_1}}\,dx + C.
\end{aligned}$$

Plugging this cancellation into (3.3.13), we conclude that

$$\Xi \le \lim_{\varepsilon\searrow 0}\left(\int_{(J_{-\varepsilon,0}\cap B_2)\setminus R_\varepsilon}\frac{a_{1,\vartheta}(\overrightarrow{x-e_1})}{|x-e_1|^{n+s_1}}\,dx-\int_{(J_{0,\varepsilon}\cap B_2)\setminus R_\varepsilon}\frac{a_{1,\vartheta}(\overrightarrow{x-e_1})}{|x-e_1|^{n+s_1}}\,dx-\frac{1}{C\varepsilon^{s_1}}+C\right).$$

Using the change of variable $x\mapsto(x',-x_n)$ and noticing that $|(x',-x_n)-e_1|=|x-e_1|$, we thus find that

$$\begin{aligned}\Xi &\le \lim_{\varepsilon\searrow 0}\left(\int_{(J_{0,\varepsilon}\cap B_2)\setminus R_\varepsilon}\frac{a_{1,\vartheta}(\overrightarrow{(x',-x_n)-e_1})-a_{1,\vartheta}(\overrightarrow{x-e_1})}{|x-e_1|^{n+s_1}}\,dx-\frac{1}{C\varepsilon^{s_1}}+C\right)\\ &= \lim_{\varepsilon\searrow 0}\left(\int_{J^\star_\varepsilon\cap B_2}\frac{a_{1,\vartheta}(\overrightarrow{(x',-x_n)-e_1})-a_{1,\vartheta}(\overrightarrow{x-e_1})}{|x-e_1|^{n+s_1}}\,dx-\frac{1}{C\varepsilon^{s_1}}+C\right).\end{aligned}\tag{3.3.15}$$

We point out that

$$J^\star_\varepsilon\subseteq\{x_1\ge 1\}.\tag{3.3.16}$$

Indeed, if $x\in J^\star_\varepsilon$, then $x\in J_{0,\varepsilon}$, whence

$$x_n\in(0,\tan\varepsilon x_1)\,.\tag{3.3.17}$$

Also, we have that $x\notin R_\varepsilon$ and therefore either $x\notin J_{-\varepsilon,\varepsilon}$ or $x\notin J^\sharp_\varepsilon$. In fact, since $J_{0,\varepsilon}\subseteq J_{-\varepsilon,\varepsilon}$, we have that necessarily $x\notin J^\sharp_\varepsilon$, and, as a result, $2e_1-x\notin J_{-\varepsilon,\varepsilon}$. This gives that $|x_n|\ge\tan\varepsilon\,(2-x_1)$. Therefore, by (3.3.17),

$$2-x_1\le\frac{|x_n|}{\tan\varepsilon}=\frac{x_n}{\tan\varepsilon}\le x_1,\tag{3.3.18}$$

and this entails (3.3.16).

Now we claim that

$$\mathcal{B}_\varepsilon\subseteq\left\{x\in\mathbb{R}^n\;:\;|x_1-1|\le 2\varepsilon\,|\log\varepsilon|,\;|x_n-\tan\varepsilon|\le 2\varepsilon^2\,|\log\varepsilon|\right\}.\tag{3.3.19}$$

To check this, let $x\in\mathcal{B}_\varepsilon$. Then,

$$\frac{x_1-1}{|\log\varepsilon|}\le x_n\le\tan\varepsilon\,x_1=\tan\varepsilon\,(x_1-1)+\tan\varepsilon.\tag{3.3.20}$$

Recalling (3.3.16), we thus find that

$$\left(\frac{1}{|\log\varepsilon|}-\tan\varepsilon\right)|x_1-1|=\left(\frac{1}{|\log\varepsilon|}-\tan\varepsilon\right)(x_1-1)\le\tan\varepsilon.$$

Consequently, if ε is conveniently small,

$$\frac{9}{10}\,|x_1-1|\le(1-\tan\varepsilon\,|\log\varepsilon|)\,|x_1-1|\le\tan\varepsilon\,|\log\varepsilon|\le\frac{11}{10}\,\varepsilon\,|\log\varepsilon|.\tag{3.3.21}$$

Furthermore, by the second inequality in (3.3.20) and (3.3.21),

$$x_n - \tan\varepsilon \le \tan\varepsilon\,(x_1-1) \le \tan\varepsilon\,|x_1-1| \le \frac{11}{9}\,\varepsilon\,\tan\varepsilon\,|\log\varepsilon| \le 2\varepsilon^2\,|\log\varepsilon|. \tag{3.3.22}$$

Moreover, from (3.3.18),

$$x_n \ge \tan\varepsilon\,(2-x_1),$$

whence, utilizing again (3.3.21),

$$\tan\varepsilon - x_n \le \tan\varepsilon + \tan\varepsilon\,(x_1-2) \le \tan\varepsilon\,|x_1-1| \le \frac{11}{9}\,\varepsilon\,\tan\varepsilon\,|\log\varepsilon| \le 2\varepsilon^2\,|\log\varepsilon|.$$

From this, (3.3.21) and (3.3.22) we obtain (3.3.19), as desired.

Now, using (3.3.19) and the changes of variable $y := \frac{x-e_1}{\tan\varepsilon}$ and $z := \left(\frac{y'}{|y_n|}, y_n\right)$, we see that

$$\begin{aligned}
\int_{\mathcal{B}_\varepsilon} \frac{dx}{|x-e_1|^{n+s_1}} &\le \int_{\substack{\{|x_1-1|\le 2\varepsilon\,|\log\varepsilon|\}\\ \{|x_n-\tan\varepsilon|\le 2\varepsilon^2\,|\log\varepsilon|\}}} \frac{dx}{|x-e_1|^{n+s_1}} \\
&= \frac{1}{(\tan\varepsilon)^{s_1}} \int_{\substack{\{|y_1|\le 2\varepsilon\,|\log\varepsilon|/\tan\varepsilon\}\\ \{|y_n-1|\le 2\varepsilon^2\,|\log\varepsilon|/\tan\varepsilon\}}} \frac{dy}{|y|^{n+s_1}} \\
&\le \frac{2}{\varepsilon^{s_1}} \int_{\{|y_n-1|\le 4\varepsilon\,|\log\varepsilon|\}} \frac{dy}{|y|^{n+s_1}} \\
&= \frac{2}{\varepsilon^{s_1}} \int_{\{|z_n-1|\le 4\varepsilon\,|\log\varepsilon|\}} \frac{dz}{|z_n|^{1+s_1}\big(|z'|^2+1\big)^{\frac{n+s_1}{2}}} \\
&\le \frac{C}{\varepsilon^{s_1}} \int_{1-4\varepsilon\,|\log\varepsilon|}^{1+4\varepsilon\,|\log\varepsilon|} \frac{dz_n}{z_n^{1+s_1}} \\
&\le C\varepsilon^{1-s_1}|\log\varepsilon|,
\end{aligned} \tag{3.3.23}$$

up to renaming the positive constant C line after line.

We also recall that $|(x',-x_n)-e_1| = |x-e_1|$ and accordingly

$$\left|\overrightarrow{(x',-x_n)-e_1} - \overrightarrow{x-e_1}\right| = \frac{|((x',-x_n)-e_1)-(x-e_1)|}{|x-e_1|} = \frac{2|x_n|}{|x-e_1|}. \tag{3.3.24}$$

As a result, recalling (3.3.16) and (3.3.17), if $x \in J_\varepsilon^\star \setminus \mathcal{B}_\varepsilon$ then

$$|x_n| = x_n \le \frac{x_1-1}{|\log\varepsilon|} = \frac{|x_1-1|}{|\log\varepsilon|}.$$

This and (3.3.24) give that

$$\left|\overrightarrow{(x',-x_n)-e_1} - \overrightarrow{x-e_1}\right| \le \frac{2}{|\log\varepsilon|}.$$

Consequently, if we consider the modulus of continuity of $a_{1,\vartheta}$, namely

$$\sigma(t) := \sup_{\substack{v,w\in\partial B_1\\ |v-w|\le t}} |a_{1,\vartheta}(v) - a_{1,\vartheta}(w)|,$$

we deduce that if $x \in J^\star_\varepsilon \setminus \mathcal{B}_\varepsilon$ then

$$\left|a_{1,\vartheta}(\overrightarrow{(x',-x_n)-e_1}) - a_{1,\vartheta}(\overrightarrow{x-e_1})\right| \le \sigma\left(\frac{2}{|\log\varepsilon|}\right)$$

and thus

$$\int_{J^\star_\varepsilon\setminus\mathcal{B}_\varepsilon} \frac{a_{1,\vartheta}(\overrightarrow{(x',-x_n)-e_1}) - a_{1,\vartheta}(\overrightarrow{x-e_1})}{|x-e_1|^{n+s_1}}\,dx \le \sigma\left(\frac{2}{|\log\varepsilon|}\right)\int_{J^\star_\varepsilon\setminus\mathcal{B}_\varepsilon}\frac{dx}{|x-e_1|^{n+s_1}}. \quad (3.3.25)$$

Notice also that

$$J^\star_\varepsilon \subseteq \mathbb{R}^n \setminus B_{\varepsilon/100}(e_1). \quad (3.3.26)$$

Indeed, if $x \in J^\star_\varepsilon$, from (3.3.18) we have that

$$|x_n| = x_n \ge \tan\varepsilon(2-x_1). \quad (3.3.27)$$

Now, if $x_1 \ge \frac{19}{10}$, then $|x-e_1| \ge |x_1-1| \ge \frac{9}{10}$ and (3.3.26) plainly follows; if instead $x_1 < \frac{19}{10}$ we deduce from (3.3.27) that

$$|x-e_1| \ge |x_n| \ge \frac{\tan\varepsilon}{10},$$

from which (3.3.26) follows in this case too.

By combining (3.3.25) and (3.3.26) we deduce that

$$\begin{aligned}\int_{J^\star_\varepsilon\setminus\mathcal{B}_\varepsilon} \frac{a_{1,\vartheta}(\overrightarrow{(x',-x_n)-e_1}) - a_{1,\vartheta}(\overrightarrow{x-e_1})}{|x-e_1|^{n+s_1}}\,dx &\le \sigma\left(\frac{2}{|\log\varepsilon|}\right)\int_{\mathbb{R}^n\setminus B_{\varepsilon/100}}\frac{dy}{|y|^{n+s_1}}\\ &\le \frac{C}{\varepsilon^{s_1}}\,\sigma\left(\frac{2}{|\log\varepsilon|}\right),\end{aligned}$$

which together with (3.3.23) leads to

$$\int_{J^\star_\varepsilon} \frac{a_{1,\vartheta}(\overrightarrow{(x',-x_n)-e_1}) - a_{1,\vartheta}(\overrightarrow{x-e_1})}{|x-e_1|^{n+s_1}}\,dx \le \frac{C}{\varepsilon^{s_1}}\,\sigma\left(\frac{2}{|\log\varepsilon|}\right) + C\varepsilon^{1-s_1}|\log\varepsilon|.$$

Joining this information with (3.3.15) we find that

$$\begin{aligned}\Xi &\le \lim_{\varepsilon\searrow 0}\left[\frac{C}{\varepsilon^{s_1}}\,\sigma\left(\frac{2}{|\log\varepsilon|}\right) + C\varepsilon^{1-s_1}|\log\varepsilon| - \frac{1}{C\varepsilon^{s_1}} + C\right]\\ &= \lim_{\varepsilon\searrow 0}\frac{1}{\varepsilon^{s_1}}\left[C\sigma\left(\frac{2}{|\log\varepsilon|}\right) + C\varepsilon\,|\log\varepsilon| - \frac{1}{C} + C\varepsilon^{s_1}\right]\\ &\le \lim_{\varepsilon\searrow 0}\left(-\frac{1}{2C\varepsilon^{s_1}}\right)\\ &= -\infty\end{aligned}$$

This completes the proof of (3.3.9). Now, using (3.3.9),

$$
\begin{aligned}
&\lim_{\bar\vartheta\searrow 0}\mathcal{D}_\vartheta(\bar\vartheta)\\
&=\lim_{\bar\vartheta\searrow 0}\left(\int_{J_{\vartheta,\vartheta+\bar\vartheta}}\frac{a_1(\overrightarrow{x-e(\vartheta)})}{|x-e(\vartheta)|^{n+s_1}}\,dx-\int_{J_{\vartheta-2\bar\vartheta,\vartheta}}\frac{a_1(\overrightarrow{x-e(\vartheta)})}{|x-e(\vartheta)|^{n+s_1}}\,dx-\int_{J_{0,\vartheta-2\bar\vartheta}}\frac{a_1(\overrightarrow{x-e(\vartheta)})}{|x-e(\vartheta)|^{n+s_1}}\,dx\right)\\
&\le\lim_{\bar\vartheta\searrow 0}\left(\int_{J_{\vartheta,\vartheta+\bar\vartheta}}\frac{a_1(\overrightarrow{x-e(\vartheta)})}{|x-e(\vartheta)|^{n+s_1}}\,dx-\int_{J_{\vartheta-2\bar\vartheta,\vartheta}}\frac{a_1(\overrightarrow{x-e(\vartheta)})}{|x-e(\vartheta)|^{n+s_1}}\,dx\right)=-\infty,
\end{aligned}
$$

which proves (3.3.4), and

$$
\begin{aligned}
\lim_{\bar\vartheta\nearrow 2\pi}\mathcal{D}_\vartheta(\bar\vartheta)&=\lim_{\alpha\searrow 0}\mathcal{D}_\vartheta(2\pi-\alpha)\\
&=\lim_{\alpha\searrow 0}\left(\int_{J_{\vartheta,\vartheta+2\pi-\alpha}}\frac{a_1(\overrightarrow{x-e(\vartheta)})}{|x-e(\vartheta)|^{n+s_1}}\,dx-\int_{J_{0,\vartheta}}\frac{a_1(\overrightarrow{x-e(\vartheta)})}{|x-e(\vartheta)|^{n+s_1}}\,dx\right)\\
&=\lim_{\alpha\searrow 0}\left(\int_{J_{\vartheta,2\pi}}\frac{a_1(\overrightarrow{x-e(\vartheta)})}{|x-e(\vartheta)|^{n+s_1}}\,dx-\int_{J_{\vartheta-\alpha,\vartheta}}\frac{a_1(\overrightarrow{x-e(\vartheta)})}{|x-e(\vartheta)|^{n+s_1}}\,dx\right)\\
&\ge\lim_{\alpha\searrow 0}\left(\int_{J_{\vartheta,\vartheta+2\alpha}}\frac{a_1(\overrightarrow{x-e(\vartheta)})}{|x-e(\vartheta)|^{n+s_1}}\,dx-\int_{J_{\vartheta-\alpha,\vartheta}}\frac{a_1(\overrightarrow{x-e(\vartheta)})}{|x-e(\vartheta)|^{n+s_1}}\,dx\right)=+\infty,
\end{aligned}
$$

which proves (3.3.5). □

3.4 Nonlocal Young's law

One of the pivotal steps of any capillarity theory is the determination of the contact angle between the droplet and the container (in jargon, the Young's law) that we are going to treat in the present section.

We showcase below a first version of the nonlocal Young's law corresponding to the case $s_1 \neq s_2$.

Theorem 3.4.1. *Let $K_1 \in \mathbf{K}^2(n, s_1, \lambda, \varrho)$ and $K_2 \in \mathbf{K}^2(n, s_2, \lambda, \varrho)$. Suppose that K_1, K_2 admit blow-up limits K_1^*, K_2^* (according to (1.2.31)) that satisfy assumption (3.0.4).*

Let $g \in C^1(\mathbb{R}^n)$. Let Ω be an open bounded set with C^1-boundary and E be a volume-constrained critical set of $\mathcal{C}$.

Let $x_0 \in \mathrm{Reg}_E \cap \partial\Omega$ and suppose that H and V are open half-spaces such that

$$\Omega^{x_0,r} \to H \qquad \text{and} \qquad E^{x_0,r} \to H \cap V \quad \text{in } L^1_{\mathrm{loc}}(\mathbb{R}^n) \text{ as } r \to 0^+. \tag{3.4.1}$$

Let also $\vartheta \in [0, \pi]$ be the angle between the half-spaces H and V, that is $H \cap V = J_{0,\vartheta}$ in the notation of (3.0.2).

Then, the following statements hold true.

1) If $s_1 < s_2$ and $\sigma < 0$ then $\vartheta = 0$.

2) If $s_1 < s_2$ and $\sigma > 0$ then $\vartheta = \pi$.

3) If

$$\text{either } s_1 < s_2 \text{ and } \sigma = 0, \text{ or } s_1 > s_2, \tag{3.4.2}$$

then $\vartheta \in (0, \pi)$. Also, letting $\widehat{\vartheta} \in (0, 2\pi)$ be as in (3.3.6) with $c = 0$, we have that $\widehat{\vartheta} = \pi - \vartheta$. Moreover, for all $v \in H \cap \partial V$,

$$\mathbf{H}^{K_1^*}_{\partial(H\cap V)}(v) - \int_{H^c} K_1^*(v - y)\, dy = 0. \tag{3.4.3}$$

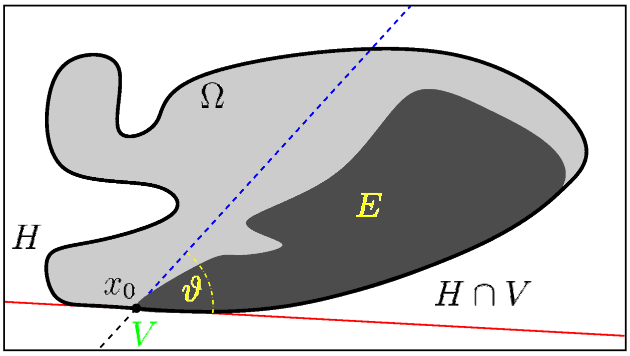


Figure 3.4: The geometry involved in the asymptotics in (3.4.1).

The asymptotics in (3.4.1) are depicted in Figure 3.4. As a particular case of Theorem 3.4.1, we single out the special situation in which the kernel K_1^* is isotropic. In this setting, the cancellation condition in (3.3.6) boils down to an explicit condition for the contact angle, and we have:

Corollary 3.4.2. *Under the same assumptions of Theorem 3.4.1, we additionally suppose that $a_1 \equiv \text{const}$.*

Then, the following statements hold true.

1) If $s_1 < s_2$ and $\sigma < 0$ then $\vartheta = 0$.

2) If $s_1 < s_2$ and $\sigma > 0$ then $\vartheta = \pi$.

3) If either $s_1 < s_2$ and $\sigma = 0$, or $s_1 > s_2$, then $\vartheta = \frac{\pi}{2}$.

We exhibit below the nonlocal Young's law in the case $s_1 = s_2$, which was left out of Theorem 3.4.1.

Theorem 3.4.3. *Let $s \in (0, 1)$ and $K_1, K_2 \in \mathbf{K}^2(n, s, \lambda, \varrho)$. Suppose that K_1, K_2 admit blow-up limits K_1^*, K_2^* (according to (1.2.31)) that satisfy assumption (3.0.4). Assume that there exists $\varepsilon_0 \in (0, 1)$ such that*

$$|\sigma|\, K_2(\zeta) \le (1 - \varepsilon_0)\, K_1(\zeta) \qquad \text{for all } \zeta \in B_{\varepsilon_0} \setminus \{0\}. \tag{3.4.4}$$

Let $g \in C^1(\mathbb{R}^n)$. Let Ω be an open bounded set with C^1-boundary and E be a volume-constrained critical set of $\mathcal{C}$.

Let $x_0 \in \mathrm{Reg}_E \cap \partial\Omega$ and suppose that H and V are open half-spaces such that

$$\Omega^{x_0,r} \to H \qquad \text{and} \qquad E^{x_0,r} \to H \cap V \quad \text{in } L^1_{\mathrm{loc}}(\mathbb{R}^n) \text{ as } r \to 0^+.$$

Let also $\vartheta \in [0,\pi]$ be the angle between the half-spaces H and V, that is $H \cap V = J_{0,\vartheta}$ in the notation of (3.0.2)*, and let $\nu_E(x_0) := \nu_V(0)$.*

Then, we have that $\vartheta \in (0,\pi)$ and, for all $v \in H \cap \partial V$,

$$\mathbf{H}^{K_1^*}_{\partial(H\cap V)}(v) - \int_{H^c} K_1^*(v-z)\,dz + \sigma \int_{H^c} K_2^*(v-z)\,dz = 0. \tag{3.4.5}$$

Even in the very special situation in which $K_1(\zeta) = K_2(\zeta) = \frac{1}{|\zeta|^{n+s}}$, Theorem 3.4.3 here can be seen as a strengthening of Theorem 1.4 in [60] (and, in particular, of formula (1.24) there): indeed, the result here establishes explicitly the nondegeneracy of the contact angle ϑ by proving that $\vartheta \in (0,\pi)$.

We point out that the case $\sigma = 0$ makes indistinguishable the setting $s_1 = s_2$ from that of $s_1 \neq s_2$: consistently with this, we observe that the contact angle prescription when $s_1 = s_2$, as given in (3.4.5), reduces to (3.4.3) when additionally $\sigma = 0$.

Also, we remark that when $\sigma = 0$ condition (3.4.4) is automatically satisfied. Furthermore, when $K_1 = K_2$, condition (3.4.4) reduces to $\sigma \in (-1,1)$, which is precisely the assumption taken in [60].

We now reformulate the condition of contact angle according to the following result:

Proposition 3.4.4. *Let K_1^* and K_2^* be as in* (3.0.4)*. Let $\sigma \in \mathbb{R}$. Assume that*

$$\text{either } s_1 = s_2, \text{ or } \sigma = 0. \tag{3.4.6}$$

Let H and V be open half-spaces and let $\vartheta \in (0,\pi)$ be the angle between H and V, that is $H \cap V = J_{0,\vartheta}$ in the notation of (3.0.2)*. Let also $\widehat{\vartheta} \in (0,2\pi)$ be as in* (3.3.6) *with $c := 0$*

Suppose that there exists $v \in H \cap \partial V$ such that

$$\mathbf{H}^{K_1^*}_{\partial(H\cap V)}(v) - \int_{H^c} K_1^*(v-z)\,dz + \sigma \int_{H^c} K_2^*(v-z)\,dz = 0. \tag{3.4.7}$$

Then, we have that ϑ and σ satisfy the relation

$$\int_{J_{\vartheta,\pi}} \frac{a_1(\overrightarrow{e(\vartheta)-x})}{|e(\vartheta)-x|^{n+s_1}}\,dx - \int_{J_{0,\vartheta}} \frac{a_1(\overrightarrow{e(\vartheta)-x})}{|e(\vartheta)-x|^{n+s_1}}\,dx + \sigma \int_{H^c} \frac{a_2(\overrightarrow{e(\vartheta)-x})}{|e(\vartheta)-x|^{n+s_1}}\,dx = 0. \tag{3.4.8}$$

In order to prove Theorems 3.4.1 and 3.4.3, Corollary 3.4.2 and Proposition 3.4.4, we first recall an ancillary result on the continuity of the nonlocal K-mean curvature defined in (3.2.1) (for the usual fractional mean curvature, that is when the kernel K is as in (1.2.28), similar continuity results were presented in [17, 45]).

From now on, we denote points $x \in \mathbb{R}^n$ as $x = (x', x_n) \in \mathbb{R}^{n-1} \times \mathbb{R}$ and we set

$$\mathbf{C} := \{x = (x', x_n) \in \mathbb{R}^n : |x'| < 1, |x_n| < 1\}$$

$$\text{and} \quad \mathbf{D} := \{z \in \mathbb{R}^{n-1} : |z| < 1\}.$$

Lemma 3.4.5. *Let $\lambda \geq 1$, $s \in (0,1)$ and $\alpha \in (s,1)$. Let $\{F_k\}_{k\in\mathbb{N}}$ be a sequence of Borel sets in $\mathbb{R}^n$ such that $0 \in \partial F_k$ and*

$$F_k \to F \text{ in } L^1_{\mathrm{loc}}(\mathbb{R}^n) \text{ for some } F \subseteq \mathbb{R}^n.$$

and u_k, $u \in C^{1,\alpha}(\mathbb{R}^{n-1})$ be such that

$$\mathbf{C} \cap F_k = \{x \in \mathbf{C} : x_n \leq u_k(x')\}$$

and

$$\lim_{k\to+\infty} \|u_k - u\|_{C^{1,\alpha}(\mathbf{D})} = 0.$$

Let K_k, $K \in \mathbf{K}(n,s,\lambda,0)$ be such that $K_k \to K$ pointwise in $\mathbb{R}^n \setminus \{0\}$ as $k \to +\infty$.

Then

$$\lim_{k\to+\infty} \mathbf{H}^{K_k}_{\partial F_k}(0) = \mathbf{H}^K_{\partial F}(0).$$

For the proof of Lemma 3.4.5 here, see Lemma 4.1 in [60].

We will also need a technical lemma to distinguish between the nondegenerate case $\vartheta \in (0,\pi)$ and the particular cases in which $\vartheta \in \{0,\pi\}$.

Lemma 3.4.6. *Let $K_1 \in \mathbf{K}^2(n,s_1,\lambda,\varrho)$ be such that it admits a blow-up limit K_1^* (according to (1.2.31)). Let Ω be an open bounded set with C^1-boundary and E be a volume-constrained critical set of $\mathcal{C}$.*

Let $x_0 \in \mathrm{Reg}_E \cap \partial\Omega$, $x_k \in \mathrm{Reg}_E \cap \Omega$ such that $x_k \to x_0$ as $k \to +\infty$ and $r_k > 0$ such that $r_k \to 0$ as $k \to +\infty$.

Suppose that H and V are open half-spaces such that

$$\Omega^{x_0,r_k} \to H \qquad \text{and} \qquad E^{x_0,r_k} \to H \cap V \quad \text{in } L^1_{\mathrm{loc}}(\mathbb{R}^n) \text{ as } k \to +\infty. \tag{3.4.9}$$

Set $v_k := \frac{x_k - x_0}{r_k}$ and suppose that there exists $v \in H \cap \partial V$ such that $v_k \to v$ as $k \to +\infty$.

Let $\vartheta \in [0,\pi]$ be the angle between the half-spaces H and V, that is $H \cap V = J_{0,\vartheta}$ in the notation of (3.0.2).

Then,

i) if $\vartheta = 0$ then

$$\lim_{k\to+\infty} r_k^{s_1} \left[\mathbf{H}^{K_1}_{\partial E}(x_k) - \int_{\Omega^c} K_1(x_k - y)\,dy \right] = +\infty;$$

ii) if $\vartheta = \pi$ then

$$\lim_{k\to+\infty} r_k^{s_1} \left[\mathbf{H}^{K_1}_{\partial E}(x_k) - \int_{\Omega^c} K_1(x_k - y)\,dy \right] = -\infty;$$

iii) if $\vartheta \in (0,\pi)$ then

$$\lim_{k\to+\infty} r_k^{s_1} \left[\mathbf{H}^{K_1}_{\partial E}(x_k) - \int_{\Omega^c} K_1(x_k - y)\,dy \right] = \mathbf{H}^{K_1^*}_{\partial(H\cap V)}(v) - \int_{H^c} K_1^*(v-y)\,dy \in \mathbb{R}.$$

Proof. We start by proving i). For this, we notice that

$$\begin{aligned}
\Xi_k &:= r_k^{s_1}\left[\mathbf{H}^{K_1}_{\partial E}(x_k) - \int_{\Omega^c} K_1(x_k - y)\,dy\right] \\
&= r_k^{s_1}\left[\int_{E^c\cap\Omega} K_1(x_k - y)\,dy - \int_E K_1(x_k - y)\,dy\right] \\
&= r_k^{n+s_1}\left[\int_{(E^{x_0,r_k})^c\cap\Omega^{x_0,r_k}} K_1(x_k - x_0 - r_k z)\,dz - \int_{E^{x_0,r_k}} K_1(x_k - x_0 - r_k z)\,dz\right] \\
&= r_k^{n+s_1}\left[\int_{(E^{x_0,r_k})^c\cap\Omega^{x_0,r_k}} K_1\big(r_k(v_k - z)\big)\,dz - \int_{E^{x_0,r_k}} K_1\big(r_k(v_k - z)\big)\,dz\right],
\end{aligned}$$

where the change of variable $z = \frac{y-x_0}{r_k}$ has been used.

Now we point out that

$$r_k^{n+s_1}\int_{\mathbb{R}^n\setminus B_{1/2}(v_k)} K_1\big(r_k(v_k - z)\big)\,dz \le \lambda \int_{\mathbb{R}^n\setminus B_{1/2}(v_k)} \frac{dz}{|v_k - z|^{n+s_1}} \le C,$$

thanks to (1.2.29), for some positive constant C, depending only on n, s_1 and λ.

From these observations we conclude that

$$\begin{aligned}
\Xi_k \ge r_k^{n+s_1}\Bigg[&\int_{(E^{x_0,r_k})^c\cap\Omega^{x_0,r_k}\cap B_{1/2}(v_k)} K_1\big(r_k(v_k - z)\big)\,dz \\
&- \int_{E^{x_0,r_k}\cap B_{1/2}(v_k)} K_1\big(r_k(v_k - z)\big)\,dz\Bigg] - C.
\end{aligned} \tag{3.4.10}$$

Now we notice that $E^{x_0,r_k}\cap B_{1/2}(v_k)$ can be written as a portion of space included between the graphs of the functions describing $\partial\Omega^{x_0,r_k}$ and $\partial E^{x_0,r_k}$, that we denote respectively by ψ_k and u_k. More precisely, recalling that $x_0 \in \mathrm{Reg}_E \cap \partial\Omega$, in the vicinity of x_0 we can describe $\partial\Omega$ and ∂E by the graphs of two functions ψ and u, respectively, with ψ of class C^1 and u of class $C^{1,\alpha}$ with $\alpha \in (s_1, 1)$, and $\psi(x_0') = u(x_0') = x_{0,n}$. Up to a rotation, we also assume that $\nabla\psi(x_0') = 0$. In this way,

$$\psi_k(x') = \frac{\psi(x_0' + r_k x') - x_{0,n}}{r_k} \qquad \text{and} \qquad u_k(x') = \frac{u(x_0' + r_k x') - x_{0,n}}{r_k}. \tag{3.4.11}$$

Moreover,

$$E^{x_0,r_k}\cap B_{1/2}(v_k) = \Big\{x \in B_{1/2}(v_k) \,:\, x_n \in \big(\psi_k(x'), u_k(x')\big)\Big\}$$

and notice that, since $E \subseteq \Omega$, it follows that $\psi \le u$ and so $\psi_k \le u_k$. As a result,

$$\big\{x \in B_{1/2}(v_k) \,:\, x_n > u_k(x')\big\} \subseteq (E^{x_0,r_k})^c \cap \Omega^{x_0,r_k} \cap B_{1/2}(v_k).$$

Hence, from (3.4.10) we obtain that

$$\begin{aligned}\Xi_k \geq r_k^{n+s_1}\Bigg[&\int_{B_{1/2}(v_k)\cap\{x_n>u_k(x')\}} K_1\big(r_k(v_k-z)\big)\,dz\\&-\int_{B_{1/2}(v_k)\cap\{x_n\in(\psi_k(x'),u_k(x'))\}} K_1\big(r_k(v_k-z)\big)\,dz\Bigg]-C.\end{aligned}\tag{3.4.12}$$

We now define

$$\widetilde{u}_k(x') := u_k(v_k') + \nabla u_k(v_k')\cdot(x'-v_k')$$

and we point out that, if $|x'-v_k'|\leq 3$,

$$\begin{aligned}|u_k(x')-\widetilde{u}_k(x')| &= \left|\frac{u(x_0'+r_kx')-u(x_0'+r_kv_k')}{r_k}-\nabla u(x_0'+r_kv_k')\cdot(x'-v_k')\right|\\&= \left|\frac{u\big(x_k'+r_k(x'-v_k')\big)-u(x_k')}{r_k}-\nabla u(x_k')\cdot(x'-v_k')\right|\\&= \left|\int_0^1 \nabla u\big(x_k'+tr_k(x'-v_k')\big)\cdot(x'-v_k')\,dt-\nabla u(x_k')\cdot(x'-v_k')\right|\\&\leq \|u\|_{C^{1,\alpha}(B_\rho'(x_0'))}\, r_k^\alpha\,|x'-v_k'|^{1+\alpha},\end{aligned}$$

for a suitable $\rho>0$. As a consequence,

$$\begin{aligned}r_k^{n+s_1}&\int_{(\{x_n>u_k(x')\}\Delta\{x_n>\widetilde{u}_k(x')\})\cap B_{1/2}(v_k)} K_1\big(r_k(v_k-z)\big)\,dz\\&\leq \lambda\int_{(\{x_n>u_k(x')\}\Delta\{x_n>\widetilde{u}_k(x')\})\cap B_{1/2}(v_k)}\frac{dz}{|v_k-z|^{n+s_1}}\\&\leq \lambda\|u\|_{C^{1,\alpha}(B_\rho'(x_0'))}\,r_k^\alpha\int_{B_{1/2}'(v_k')}\frac{|v_k'-z'|^{1+\alpha}}{|v_k'-z'|^{n+s_1}}\,dz'\\&\leq C\,r_k^\alpha,\end{aligned}$$

up to renaming C, possibly in dependence of u as well.

Plugging this information into (3.4.12), and possibly renaming C again, we obtain that

$$\begin{aligned}\Xi_k \geq r_k^{n+s_1}\Bigg[&\int_{B_{1/2}(v_k)\cap\{x_n>\widetilde{u}_k(x')\}} K_1\big(r_k(v_k-z)\big)\,dz\\&-\int_{B_{1/2}(v_k)\cap\{x_n\in(\psi_k(x'),\widetilde{u}_k(x'))\}} K_1\big(r_k(v_k-z)\big)\,dz\Bigg]-C.\end{aligned}\tag{3.4.13}$$

Now, from (3.4.11) we see that $\psi_k(x')\to\nabla\psi(x_0')\cdot x'$ and $u_k(x')\to\nabla u(x_0')\cdot x'$ as $k\to+\infty$.

Hence, if $\vartheta = 0$ it follows that $\nabla\psi(x_0') = \nabla u(x_0')$. Consequently, if $x' \in B'_{1/2}(v_k')$ then

$$
\begin{aligned}
&|\widetilde{u}_k(x') - \psi_k(x')| \\
&= \left| u_k(v_k') + \nabla u_k(v_k') \cdot (x' - v_k') - \frac{\psi(x_0' + r_k x') - \psi(x_0')}{r_k} \right| \\
&= \left| \frac{u(x_0' + r_k v_k') - u(x_0')}{r_k} + \nabla u(x_0' + r_k v_k') \cdot (x' - v_k') - \int_0^1 \nabla\psi(x_0' + t r_k x') \cdot x'\, dt \right| \\
&= \left| \int_0^1 \nabla u(x_0' + t r_k v_k') \cdot v_k'\, dt + \nabla u(x_0' + r_k v_k') \cdot (x' - v_k') - \int_0^1 \nabla\psi(x_0' + t r_k x') \cdot x'\, dt \right| \\
&\le \left| \int_0^1 \nabla u(x_0') \cdot v_k'\, dt + \nabla u(x_0') \cdot (x' - v_k') - \int_0^1 \nabla\psi(x_0') \cdot x'\, dt \right| + \delta_k \\
&= \delta_k,
\end{aligned}
\tag{3.4.14}
$$

for a suitable δ_k such that $\delta_k \to 0$ as $k \to +\infty$.

This and (3.4.13) give that

$$
\begin{aligned}
\Xi_k \ge r_k^{n+s_1} \Bigg[&\int_{B_{1/2}(v_k) \cap \{x_n > \widetilde{u}_k(x')\}} K_1\big(r_k(v_k - z)\big)\, dz \\
&- \int_{B_{1/2}(v_k) \cap \{x_n \in (\widetilde{u}_k(x') - \delta_k, \widetilde{u}_k(x'))\}} K_1\big(r_k(v_k - z)\big)\, dz \Bigg] - C.
\end{aligned}
\tag{3.4.15}
$$

Now we define the map $Y(z) := 2v_k - z$ and we show that

$$
Y\Big(B_{1/2}(v_k) \cap \{x_n \in (\widetilde{u}_k(x') - \delta_k, \widetilde{u}_k(x'))\}\Big) \subseteq B_{1/2}(v_k) \cap \{x_n \in (\widetilde{u}_k(x'), \widetilde{u}_k(x') + \delta_k)\}.
\tag{3.4.16}
$$

Indeed, let $z \in B_{1/2}(v_k) \cap \{x_n \in (\widetilde{u}_k(x') - \delta_k, \widetilde{u}_k(x'))\}$ and call $y := Y(z)$. We have that $|y - v_k| = |v_k - z| < 1/2$. Moreover,

$$
\begin{aligned}
y_n - \widetilde{u}_k(y') &= 2v_{k,n} - z_n - \widetilde{u}_k\big(2v_k' - z'\big) \\
&= 2u_k(v_k') - z_n - \widetilde{u}_k\big(2v_k' - z'\big) \\
&\in \Big(2u_k(v_k') - \widetilde{u}_k(z') - \widetilde{u}_k\big(2v_k' - z'\big),\, 2u_k(v_k') - \widetilde{u}_k(z') - \widetilde{u}_k\big(2v_k' - z'\big) + \delta_k\Big) \\
&= \Big(2u_k(v_k') - 2\widetilde{u}_k(v_k'),\, 2u_k(v_k') - 2\widetilde{u}_k(v_k') + \delta_k\Big) \\
&= \big(0,\, \delta_k\big)
\end{aligned}
$$

and the proof of (3.4.16) is thus complete.

Using (3.4.16) and changing variable $y = Y(z)$ we see that

$$\begin{aligned}
&\int_{B_{1/2}(v_k)\cap\{x_n\in(\widetilde{u}_k(x')-\delta_k,\widetilde{u}_k(x'))\}} K_1\big(r_k(v_k-z)\big)\,dz\\
\le\;&\int_{B_{1/2}(v_k)\cap\{x_n\in(\widetilde{u}_k(x'),\widetilde{u}_k(x')+\delta_k)\}} K_1\big(r_k(y-v_k)\big)\,dy\\
=\;&\int_{B_{1/2}(v_k)\cap\{x_n\in(\widetilde{u}_k(x'),\widetilde{u}_k(x')+\delta_k)\}} K_1\big(r_k(v_k-y)\big)\,dy.
\end{aligned}$$

Combining this and (3.4.15), and recalling (1.2.29), we arrive at

$$\begin{aligned}
\Xi_k &\ge r_k^{n+s_1}\int_{B_{1/2}(v_k)\cap\{x_n>\widetilde{u}_k(x')+\delta_k\}} K_1\big(r_k(v_k-z)\big)\,dz - C\\
&\ge \frac{1}{\lambda}\int_{B_{1/2}(v_k)\cap\{x_n>\widetilde{u}_k(x')+\delta_k\}} \frac{dz}{|v_k-z|^{n+s_1}}\,dz - C.
\end{aligned}\tag{3.4.17}$$

Now we define

$$\nu_k := \frac{(-\nabla u_k(v_k'),1)}{\sqrt{1+|\nabla u_k(v_k')|^2}} \qquad\text{and}\qquad \zeta_k := v_k + 3\delta_k\nu_k \tag{3.4.18}$$

and we claim that, if k is sufficiently large,

$$B_{\delta_k}(\zeta_k) \subseteq B_{1/2}(v_k)\cap\{x_n > \widetilde{u}_k(x') + \delta_k\}. \tag{3.4.19}$$

To check this, we observe that

$$\lim_{k\to+\infty}|\nabla u_k(v_k')| = \lim_{k\to+\infty}|\nabla u(x_k')| = |\nabla u(x_0')| = |\nabla\psi(x_0')| = 0$$

and consequently

$$\lim_{k\to+\infty}\frac{3}{\sqrt{1+|\nabla u_k(v_k')|^2}} - 4|\nabla u_k(v_k')| - 2 = 1. \tag{3.4.20}$$

Now, pick $w\in B_{\delta_k}(\zeta_k)$. We have that

$$|w-v_k| \le |w-\zeta_k| + |\zeta_k - v_k| < \delta_k + 3\delta_k = 4\delta_k$$

and thus $w\in B_{1/2}(v_k)$ as long as k is large enough.

Moreover,

$$\begin{aligned}
w_n-\widetilde{u}_k(w')-\delta_k &\geq (\zeta_{k,n}-\delta_k)-u_k(v_k')-\nabla u_k(v_k')(w'-v_k')-\delta_k \\
&= \left(v_{k,n}+\frac{3\delta_k}{\sqrt{1+|\nabla u_k(v_k')|^2}}-\delta_k\right)-v_{k,n}-\nabla u_k(v_k')(w'-v_k')-\delta_k \\
&= \frac{3\delta_k}{\sqrt{1+|\nabla u_k(v_k')|^2}}-\nabla u_k(v_k')(w'-v_k')-2\delta_k \\
&\geq \frac{3\delta_k}{\sqrt{1+|\nabla u_k(v_k')|^2}}-|\nabla u_k(v_k')|\,|w'-v_k'|-2\delta_k \\
&\geq \left(\frac{3}{\sqrt{1+|\nabla u_k(v_k')|^2}}-4|\nabla u_k(v_k')|-2\right)\delta_k \\
&> 0,
\end{aligned}$$

thanks to (3.4.20).

The proof of (3.4.19) is thereby complete.

Thus, exploiting (3.4.17) and (3.4.19), we find that

$$\Xi_k \geq \int_{B_{\delta_k}(\zeta_k)} \frac{dz}{|v_k-z|^{n+s_1}}\,dz - C.$$

Notice also that if $z\in B_{\delta_k}(\zeta_k)$ then $|v_k-z|\leq|v_k-\zeta_k|+|\zeta_k-z|\leq 3\delta_k+\delta_k=4\delta_k$ and accordingly

$$\Xi_k \geq \int_{B_{\delta_k}(\zeta_k)} \frac{dz}{(4\delta_k)^{n+s_1}}\,dz - C = \frac{c}{\delta_k^{s_1}} - C,$$

for some $c>0$. This establishes the claim in i), as desired.

The claim in ii) can be proved similarly.

As for the claim in iii), we suppose that $\vartheta\in(0,\pi)$ and, for every $k\in\mathbb{N}$, we denote by F_k the set obtained by a suitable rigid motion of the set $E^{x_0,r_k}-v_k$ so as to have that $0\in\partial F_k$ and

$$\mathbf{C}\cap F_k=\left\{x\in\mathbf{C} : x_n\leq u_k(x')\right\}, \tag{3.4.21}$$

for some $u_k\in C^{1,\alpha}(\mathbb{R}^{n-1})$. Let also u be the linear function such that $u_k\to u$ in $C^{1,\alpha}(\mathbf{D})$ as $k\to+\infty$. We notice that, by (3.4.9), up to a rigid motion,

$$F_k\to F:=H\cap V-v \quad \text{in } L^1_{\mathrm{loc}}(\mathbb{R}^n) \text{ as } k\to+\infty. \tag{3.4.22}$$

Furthermore, recalling the definition of mean curvature in (3.2.1) and exploiting the change of variable $y=x_0+r_kz$, we see that

$$\begin{aligned}
\mathbf{H}^{K_1}_{\partial E}(x_k) &= \int_{\mathbb{R}^n} K_1(x_k-y)\big(\chi_{E^c}(y)-\chi_E(y)\big)\,dy \\
&= r_k^{-s_1}\int_{\mathbb{R}^n} r_k^{n+s_1}K_1(x_k-x_0-r_kz)\left(\chi_{(E^{x_0,r_k})^c}(z)-\chi_{E^{x_0,r_k}}(z)\right)\,dz.
\end{aligned} \tag{3.4.23}$$

We also introduce, for every $\zeta \in \mathbb{R}^n \setminus \{0\}$, the kernel

$$K_{1,k}(\zeta) := r_k^{n+s_1} K_1(r_k \zeta),$$

and we observe that, in light of (3.4.23),

$$\mathbf{H}^{K_1}_{\partial E}(x_k) = r_k^{-s_1} \mathbf{H}^{K_{1,k}}_{\partial F_k}(0). \tag{3.4.24}$$

Furthermore, we recall that $K_{1,k} \to K_1^*$ pointwise in $\mathbb{R}^n \setminus \{0\}$, hence one can infer from (3.4.21), (3.4.22), (3.4.24) and Lemma 3.4.5 that

$$\lim_{k\to+\infty} r_k^{s_1} \mathbf{H}^{K_1}_{\partial E}(x_k) = \mathbf{H}^{K_1^*}_{\partial(H\cap V)}(v). \tag{3.4.25}$$

Moreover, since $\vartheta \in (0, \pi)$, one can use the Lebesgue's Dominated Convergence Theorem and find that

$$\begin{aligned}\lim_{k\to+\infty} r_k^{s_1} \int_{\Omega^c} K_1(x_k - y)\, dy &= \lim_{k\to+\infty} \int_{(\Omega^{x_0,r_k})^c} r_k^{n+s_1} K_1(r_k(v_k - y))\, dy \\ &= \int_{H^c} K_1^*(v - y)\, dy.\end{aligned}$$

From this and (3.4.25) we obtain the desired result in iii). □

Now we showcase a refinement of Lemma 3.4.6 which will be needed to exclude the degenerate blow-up limits $\vartheta \in \{0, \pi\}$ in the case $s_1 > s_2$.

Lemma 3.4.7. *Let $s_1 > s_2$, $K_1 \in \mathbf{K}^2(n, s_1, \lambda, \varrho)$ and $K_2 \in \mathbf{K}^2(n, s_2, \lambda, \varrho)$. Let Ω be an open bounded set with C^1-boundary and E be a volume-constrained critical set of $\mathcal{C}$.*

Let $x_0 \in \mathrm{Reg}_E \cap \partial\Omega$, $x_k \in \mathrm{Reg}_E \cap \Omega$ such that $x_k \to x_0$ as $k \to +\infty$ and $r_k > 0$ such that $r_k \to 0$ as $k \to +\infty$.

Suppose that H and V are open half-spaces such that

$$\Omega^{x_0,r_k} \to H \qquad \text{and} \qquad E^{x_0,r_k} \to H \cap V \quad \text{in } L^1_{\mathrm{loc}}(\mathbb{R}^n) \text{ as } k \to +\infty.$$

Let $\vartheta \in [0, \pi]$ be the angle between the half-spaces H and V, that is $H \cap V = J_{0,\vartheta}$ in the notation of (3.0.2).

Then,

i) if $\vartheta = 0$ then

$$\lim_{k\to+\infty} r_k^{s_1} \left[\mathbf{H}^{K_1}_{\partial E}(x_k) - \int_{\Omega^c} K_1(x_k - y)\, dy\right] + \sigma\, r_k^{s_1 - s_2}\, r_k^{s_2} \int_{\Omega^c} K_2(x_k - y)\, dy = +\infty;$$

ii) if $\vartheta = \pi$ then

$$\lim_{k\to+\infty} r_k^{s_1} \left[\mathbf{H}^{K_1}_{\partial E}(x_k) - \int_{\Omega^c} K_1(x_k - y)\, dy\right] + \sigma\, r_k^{s_1 - s_2}\, r_k^{s_2} \int_{\Omega^c} K_2(x_k - y)\, dy = -\infty.$$

Proof. We focus on the proof of i), since the proof of ii) is similar, up to sign changes. To this end, we exploit the notation introduced in Lemma 3.4.6, and specifically (3.4.13), and we set $v_k := \frac{x_k - x_0}{r_k}$, to see that

$$\begin{aligned}
\Upsilon_k &:= r_k^{s_1} \left[\mathbf{H}^{K_1}_{\partial E}(x_k) - \int_{\Omega^c} K_1(x_k - y)\, dy \right] + \sigma\, r_k^{s_1 - s_2}\, r_k^{s_2} \int_{\Omega^c} K_2(x_k - y)\, dy \\
&\geq \Xi_k - |\sigma|\, r_k^{s_1 - s_2}\, r_k^{s_2} \int_{\Omega^c} K_2(x_k - y)\, dy \\
&\geq r_k^{n+s_1} \left[\int_{B_{1/2}(v_k) \cap \{x_n > \widetilde{u}_k(x')\}} K_1\big(r_k(v_k - z)\big)\, dz \right. \\
&\qquad \left. - \int_{B_{1/2}(v_k) \cap \{x_n \in (\psi_k(x'), \widetilde{u}_k(x'))\}} K_1\big(r_k(v_k - z)\big)\, dz \right] \\
&\qquad - |\sigma|\, r_k^{s_1 - s_2}\, r_k^{n+s_2} \int_{\mathbb{R}^n \setminus \Omega^{x_0, r_k}} K_2\big(r_k(v_k - z)\big)\, dz - C \\
&\geq r_k^{n+s_1} \left[\int_{B_{1/2}(v_k) \cap \{x_n > \widetilde{u}_k(x')\}} K_1\big(r_k(v_k - z)\big)\, dz \right. \\
&\qquad \left. - \int_{B_{1/2}(v_k) \cap \{x_n \in (\psi_k(x'), \widetilde{u}_k(x'))\}} K_1\big(r_k(v_k - z)\big)\, dz \right] \\
&\qquad - |\sigma|\, r_k^{s_1 - s_2}\, r_k^{n+s_2} \int_{B_{1/2}(v_k) \cap \{x_n < \psi_k(x')\}} K_2\big(r_k(v_k - z)\big)\, dz - C,
\end{aligned} \tag{3.4.26}$$

up to changing $C > 0$ from line to line.

Also, by (3.4.14),

$$\begin{aligned}
\int_{B_{1/2}(v_k) \cap \{x_n < \psi_k(x')\}} K_2\big(r_k(v_k - z)\big)\, dz &= \int_{B_{1/2}(v_k) \cap \{x_n \in (\widetilde{u}_k(x') - \delta_k, \psi_k(x'))\}} K_2\big(r_k(v_k - z)\big)\, dz \\
&\quad + \int_{B_{1/2}(v_k) \cap \{x_n < \widetilde{u}_k(x') - \delta_k\}} K_2\big(r_k(v_k - z)\big)\, dz.
\end{aligned}$$

Therefore, we can write (3.4.26) as

$$\begin{aligned}
\Upsilon_k \geq r_k^{n+s_1} &\left[\int_{B_{1/2}(v_k) \cap \{x_n > \widetilde{u}_k(x')\}} K_1\big(r_k(v_k - z)\big)\, dz \right. \\
&\qquad \left. - \int_{B_{1/2}(v_k) \cap \{x_n \in (\psi_k(x'), \widetilde{u}_k(x'))\}} K_1\big(r_k(v_k - z)\big)\, dz \right] \\
&\qquad - |\sigma|\, r_k^{s_1 - s_2}\, r_k^{n+s_2} \int_{B_{1/2}(v_k) \cap \{x_n \in (\widetilde{u}_k(x') - \delta_k, \psi_k(x'))\}} K_2\big(r_k(v_k - z)\big)\, dz \\
&\qquad - |\sigma|\, r_k^{s_1 - s_2}\, r_k^{n+s_2} \int_{B_{1/2}(v_k) \cap \{x_n < \widetilde{u}_k(x') - \delta_k\}} K_2\big(r_k(v_k - z)\big)\, dz - C.
\end{aligned} \tag{3.4.27}$$

Now we set

$$\mathcal{Z}_k(x) := \max\Big\{ r_k^{n+s_1}\, K_1(x),\ \ |\sigma|\, r_k^{s_1-s_2}\, r_k^{n+s_2}\, K_2(x)\Big\}. \tag{3.4.28}$$

In this way, we deduce from (3.4.27) that

$$\begin{aligned}\Upsilon_k \geq\, & r_k^{n+s_1} \int_{B_{1/2}(v_k)\cap\{x_n>\widetilde{u}_k(x')\}} K_1\big(r_k(v_k-z)\big)\,dz \\ & - \int_{B_{1/2}(v_k)\cap\{x_n\in(\widetilde{u}_k(x')-\delta_k,\widetilde{u}_k(x'))\}} \mathcal{Z}_k\big(r_k(v_k-z)\big)\,dz \\ & - |\sigma|\, r_k^{s_1-s_2}\, r_k^{n+s_2} \int_{B_{1/2}(v_k)\cap\{x_n<\widetilde{u}_k(x')-\delta_k\}} K_2\big(r_k(v_k-z)\big)\,dz - C.\end{aligned} \tag{3.4.29}$$

Let $Y(z) := 2v_k - z$. We also use the short notation

$$\begin{aligned} & \mathcal{P}_k := B_{1/2}(v_k)\cap\{x_n>\widetilde{u}_k(x')\}, \\ & \mathcal{Q}_k := B_{1/2}(v_k)\cap\{x_n\in(\widetilde{u}_k(x')-\delta_k,\widetilde{u}_k(x'))\} \\ \text{and}\qquad & \mathcal{R}_k := B_{1/2}(v_k)\cap\{x_n<\widetilde{u}_k(x')-\delta_k\}.\end{aligned}$$

We know from (3.4.16) that

$$Y(\mathcal{Q}_k)\subseteq B_{1/2}(v_k)\cap\{x_n\in(\widetilde{u}_k(x'),\widetilde{u}_k(x')+\delta_k)\}\subseteq\mathcal{P}_k. \tag{3.4.30}$$

We also claim that

$$Y(\mathcal{R}_k)\subseteq\mathcal{P}_k\setminus Y(\mathcal{Q}_k). \tag{3.4.31}$$

Indeed, if there were a point $y\in Y(\mathcal{Q}_k)\cap Y(\mathcal{R}_k)$ we would have that $y=2v_k-Q=2v_k-R$ for some $Q\in\mathcal{Q}_k$ and $R\in\mathcal{R}_k$, but this would entail that $Q=R\in\mathcal{Q}_k\cap\mathcal{R}_k=\varnothing$, which is a contradiction. This shows that $Y(\mathcal{R}_k)$ lies in the complement of $Y(\mathcal{Q}_k)$, thus, to complete the proof of (3.4.31), it only remains to show that $Y(\mathcal{R}_k)\subseteq\mathcal{P}_k$. To this end, we observe that if $z_n<\widetilde{u}_k(z')-\delta_k$ and $y=Y(z)$, then

$$\begin{aligned} y_n-\widetilde{u}_k(y') &= 2v_{k,n}-z_n-\widetilde{u}_k(y') = 2\widetilde{u}_k(v_k')-z_n-\widetilde{u}_k(2v_k'-z') \\ &> 2\widetilde{u}_k(v_k')-\widetilde{u}_k(z')+\delta_k-\widetilde{u}_k(2v_k'-z')=\delta_k>0.\end{aligned}$$

This completes the proof of (3.4.31).

Hence, by (3.4.29), (3.4.30) and (3.4.31),

$$\begin{aligned}\Upsilon_k \geq\, & r_k^{n+s_1}\int_{\mathcal{P}_k} K_1\big(r_k(v_k-z)\big)\,dz - \int_{\mathcal{Q}_k}\mathcal{Z}_k\big(r_k(v_k-z)\big)\,dz \\ & - |\sigma|\, r_k^{s_1-s_2}\, r_k^{n+s_2}\int_{\mathcal{R}_k} K_2\big(r_k(v_k-z)\big)\,dz - C \\ =\, & r_k^{n+s_1}\int_{\mathcal{P}_k} K_1\big(r_k(v_k-z)\big)\,dz - \int_{Y(\mathcal{Q}_k)}\mathcal{Z}_k\big(r_k(v_k-y)\big)\,dy \\ & - |\sigma|\, r_k^{s_1-s_2}\, r_k^{n+s_2}\int_{Y(\mathcal{R}_k)} K_2\big(r_k(v_k-y)\big)\,dy - C \\ =\, & r_k^{n+s_1}\int_{\mathcal{P}_k\setminus(Y(\mathcal{Q}_k)\cup Y(\mathcal{R}_k))} K_1\big(r_k(v_k-z)\big)\,dz + \int_{Y(\mathcal{Q}_k)}\alpha_k(z)\,dz + \int_{Y(\mathcal{R}_k)}\beta_k(z)\,dz - C,\end{aligned} \tag{3.4.32}$$

where

$$\begin{aligned} &\alpha_k(z) := r_k^{n+s_1}\, K_1\big(r_k(v_k-z)\big) - \mathcal{Z}_k\big(r_k(v_k-z)\big) \\ \text{and} \qquad &\beta_k(z) := r_k^{n+s_1}\, K_1\big(r_k(v_k-z)\big) - |\sigma|\, r_k^{s_1-s_2}\, r_k^{n+s_2}\, K_2\big(r_k(v_k-z)\big). \end{aligned}$$

We stress that up to now the condition $s_1 > s_2$ has not been used. We are going to exploit it now to bound α_k and β_k. For this, we note that, if $z \in B_{1/2}(v_k)$ and k is large enough, then

$$\begin{aligned} |\sigma|\, r_k^{s_1-s_2}\, r_k^{n+s_2}\, K_2\big(r_k(v_k-z)\big) &\le \frac{\lambda\,|\sigma|\, r_k^{s_1-s_2}}{|v_k-z|^{n+s_2}} \le \frac{\lambda\,|\sigma|\, r_k^{s_1-s_2}}{|v_k-z|^{n+s_1}} = \frac{\lambda\,|\sigma|\, r_k^{s_1-s_2}\, r_k^{n+s_1}}{\big|r_k(v_k-z)\big|^{n+s_1}} \\ &\le \lambda^2\,|\sigma|\, r_k^{s_1-s_2}\, r_k^{n+s_1}\, K_1\big(r_k(v_k-z)\big) \le \frac{1}{2}\, r_k^{n+s_1}\, K_1\big(r_k(v_k-z)\big). \end{aligned}$$

This and (3.4.28) entail that if $z \in B_{1/2}(v_k)$ and k is large enough, then $\mathcal{Z}_k\big(r_k(v_k-z)\big) = r_k^{n+s_1}\, K_1\big(r_k(v_k-z)\big)$, and therefore $\alpha_k(z) = 0$. In addition,

$$\beta_k(z) \ge \frac{1}{2}\, r_k^{n+s_1}\, K_1\big(r_k(v_k-z)\big).$$

From these observations and (3.4.32) we arrive at

$$\begin{aligned} \Upsilon_k \;&\ge\; r_k^{n+s_1} \int_{\mathcal{P}_k \setminus (Y(\mathcal{Q}_k)\cup Y(\mathcal{R}_k))} K_1\big(r_k(v_k-z)\big)\,dz + \frac{1}{2}\, r_k^{n+s_1} \int_{Y(\mathcal{R}_k)} K_1\big(r_k(v_k-z)\big)\,dz - C \\ &\ge\; \frac{1}{2}\, r_k^{n+s_1} \int_{\mathcal{P}_k \setminus Y(\mathcal{Q}_k)} K_1\big(r_k(v_k-z)\big)\,dz - C. \end{aligned} \tag{3.4.33}$$

Now we utilize the notation in (3.4.18), the inclusion in (3.4.19) and the first inclusion in (3.4.30) to see that

$$\begin{aligned} \mathcal{P}_k \setminus Y(\mathcal{Q}_k) &\supseteq \mathcal{P}_k \setminus \Big(B_{1/2}(v_k) \cap \{x_n \in (\widetilde{u}_k(x'), \widetilde{u}_k(x')+\delta_k)\}\Big) \\ &= B_{1/2}(v_k) \cap \{x_n \ge \widetilde{u}_k(x')+\delta_k\} \\ &\supseteq B_{\delta_k}(\zeta_k). \end{aligned} \tag{3.4.34}$$

By plugging this information into (3.4.33), we thereby conclude that

$$\begin{aligned} \Upsilon_k &\ge \frac{1}{2}\, r_k^{n+s_1} \int_{B_{\delta_k}(\zeta_k)} K_1\big(r_k(v_k-z)\big)\,dz - C \\ &\ge \frac{1}{2} \int_{B_{\delta_k}(\zeta_k)} \frac{dz}{|v_k-z|^{n+s_1}} - C \\ &= \frac{c}{\delta_k^{s_1}} - C, \end{aligned} \tag{3.4.35}$$

for some $c > 0$. From this, the desired result in i) plainly follows. □

With this, we are in the position of providing the proof of Theorem 3.4.1, where we suppose that a_1 and a_2 are anisotropic functions and then, as a special case, we exhibit the proof of Corollary 3.4.2 where we take $a_1 \equiv \text{const}$.

Proof of Theorem 3.4.1. We fix a point $x_0 \in \partial\Omega \cap \mathrm{Reg}_E$ and a sequence of points $x_k \in \Omega \cap \mathrm{Reg}_E$ such that $x_k \to x_0$ as $k \to +\infty$. We also set $r_k := |x_k - x_0|$ and we observe that $r_k \to 0$ as $k \to +\infty$.

From (3.2.3) evaluated at x_k, we get

$$\mathbf{H}^{K_1}_{\partial E}(x_k) - \int_{\Omega^c} K_1(x_k - y)\, dy + \sigma \int_{\Omega^c} K_2(x_k - y)\, dy + g(x_k) = c,$$

where c does not depend on k. Multiplying both sides by $r_k^{s_1}$, we thereby obtain that

$$r_k^{s_1}\, \mathbf{H}^{K_1}_{\partial E}(x_k) - r_k^{s_1} \int_{\Omega^c} K_1(x_k - y)\, dy + \sigma\, r_k^{s_1 - s_2}\, r_k^{s_2} \int_{\Omega^c} K_2(x_k - y)\, dy + r_k^{s_1}\, g(x_k) = c\, r_k^{s_1}.$$

Notice that, since g is locally bounded, we have that $r_k^{s_1} g(x_k) \to 0$ as $k \to +\infty$. As a consequence,

$$\lim_{k \to +\infty} r_k^{s_1} \left[\mathbf{H}^{K_1}_{\partial E}(x_k) - \int_{\Omega^c} K_1(x_k - y)\, dy \right] + \sigma r_k^{s_1 - s_2}\, r_k^{s_2} \int_{\Omega^c} K_2(x_k - y)\, dy = 0. \tag{3.4.36}$$

Now, we prove the statement in 1) of Theorem 3.4.1. For this, we suppose that $s_1 < s_2$ and $\sigma < 0$. In this case,

$$\sigma\, r_k^{s_1 - s_2}\, r_k^{s_2} \int_{\Omega^c} K_2(x_k - y)\, dy \le 0,$$

and therefore by ii) in Lemma 3.4.6 and (3.4.36) we deduce that $\vartheta \neq \pi$. Hence, to prove 1) it remains to check that

$$\vartheta \notin (0, \pi). \tag{3.4.37}$$

To this end, we suppose by contradiction that $\vartheta \in (0, \pi)$. Then, by the Lebesgue's Dominated Convergence Theorem,

$$\begin{aligned} \lim_{k \to +\infty} r_k^{s_2} \int_{\Omega^c} K_2(x_k - y)\, dy &= \lim_{k \to +\infty} \int_{(\Omega^{x_0, r_k})^c} r_k^{n + s_2} K_2(r_k(v_k - y))\, dy \\ &= \int_{H^c} K_2^*(v - y)\, dy \end{aligned} \tag{3.4.38}$$

and this limit is finite. Consequently,

$$\lim_{k \to +\infty} \sigma\, r_k^{s_1 - s_2}\, r_k^{s_2} \int_{\Omega^c} K_2(x_k - y)\, dy = -\infty.$$

This and iii) in Lemma 3.4.6 contradict (3.4.36), and thus (3.4.37) is proved.

Accordingly, if $s_1 < s_2$ and $\sigma < 0$, then necessarily $\vartheta = 0$, which establishes 1).

We now prove the statement in 2). Namely we consider the case in which $s_1 < s_2$ and $\sigma > 0$, and thus

$$\sigma\, r_k^{s_1-s_2}\, r_k^{s_2} \int_{\Omega^c} K_2(x_k - y)\, dy \geq 0.$$

From this, i) in Lemma 3.4.6 and (3.4.36) we infer that $\vartheta \neq 0$. Hence, to establish 2) we show that

$$\vartheta \notin (0, \pi). \tag{3.4.39}$$

We argue as before and we suppose by contradiction that $\vartheta \in (0, \pi)$. Then, exploiting (3.4.38) we see that

$$\lim_{k\to+\infty} \sigma\, r_k^{s_1-s_2}\, r_k^{s_2} \int_{\Omega^c} K_2(x_k - y)\, dy = +\infty.$$

This and iii) in Lemma 3.4.6 contradict (3.4.36), and thus (3.4.39) is proved.

As a consequence, if $s_1 < s_2$ and $\sigma > 0$, then $\vartheta = \pi$, hence we have established 2) as well. Hence, we now focus on the statement in 3).

For this, we first suppose that $s_1 < s_2$ and $\sigma = 0$. Then, (3.4.36) becomes

$$\lim_{k\to+\infty} r_k^{s_1} \left[\mathbf{H}^{K_1}_{\partial E}(x_k) - \int_{\Omega^c} K_1(x_k - y)\, dy \right] = 0. \tag{3.4.40}$$

This and Lemma 3.4.6 give that $\vartheta \in (0, \pi)$ in this case.

In the case in which $s_1 > s_2$, if $\vartheta \in \{0, \pi\}$ then we would use Lemma 3.4.7 to find a contradiction with (3.4.36), hence we conclude that necessarily $\vartheta \in (0, \pi)$ in this case as well.

Now, in order to prove (3.4.3), we take $v \in H \cap \partial V$, then by (3.4.1) we have that, for every k, there exists $v_k \in \Omega^{x_0,r_k} \cap \partial E^{x_0,r_k}$ such that $v_k \to v$ as $k \to +\infty$, where r_k is an infinitesimal sequence as $k \to +\infty$. As a consequence, for every k, there exists $x_k \in \mathrm{Reg}_E \cap \Omega$ such that $v_k = \frac{x_k - x_0}{r_k}$ and $x_k \to x_0$ as $k \to +\infty$. Then, we are in the position to apply iii) in Lemma 3.4.6 and conclude that

$$\lim_{k\to+\infty} r_k^{s_1} \left[\mathbf{H}^{K_1}_{\partial E}(x_k) - \int_{\Omega^c} K_1(x_k - y)\, dy \right] = \mathbf{H}^{K_1^*}_{\partial(H\cap V)}(v) - \int_{H^c} K_1^*(v - y)\, dy. \tag{3.4.41}$$

Also, if $s_1 > s_2$, we recall that the limit in (3.4.38) is finite (since $\vartheta \in (0, \pi)$) and that r_k is infinitesimal to infer that

$$\lim_{k\to+\infty} r_k^{s_1-s_2}\, r_k^{s_2} \int_{\Omega^c} K_2(x_k - y)\, dy = 0.$$

This, together with (3.4.36), gives that (3.4.40) holds true in this case as well.

Accordingly, from (3.4.40) and (3.4.41) we deduce that

$$\mathbf{H}^{K_1^*}_{\partial(H\cap V)}(v) - \int_{H^c} K_1^*(v - y)\, dy = 0,$$

which establishes (3.4.3).

Hence, to complete the proof of the statement in 3), it remains to check that $\widehat{\vartheta} = \pi - \vartheta$, being $\widehat{\vartheta} \in (0, 2\pi)$ the angle given in (3.3.6) with $c = 0$.

For this, we exploit the notation in (3.0.3), the assumption in (3.0.4) and the change of variable $z = y/|v|$, to see that, for all $v \in H \cap \partial V$, the left hand side of (3.4.3) can be written as

$$\begin{aligned}\mathbf{H}^{K_1^*}_{\partial(H\cap V)}(v) - \int_{H^c} K_1^*(v-y)\,dy &= \int_{\mathbb{R}^n} K_1^*(v-y)\big(\chi_{(H\cap V)^c\cap H}(y) - \chi_{H\cap V}(y)\big)\,dy \\ &= \int_{\mathbb{R}^n} \frac{a_1(\overrightarrow{v-y})}{|v-y|^{n+s_1}}\big(\chi_{(H\cap V)^c\cap H}(y) - \chi_{H\cap V}(y)\big)\,dy \\ &= |v|^{-s_1}\int_{\mathbb{R}^n} \frac{a_1(\overrightarrow{e(\vartheta)-z})\big(\chi_{J^c_{0,\vartheta}\cap H}(z) - \chi_{J_{0,\vartheta}}(z)\big)}{|e(\vartheta)-z|^{n+s_1}}\,dz \\ &= |v|^{-s_1}\int_{J_{\vartheta,\pi}} \frac{a_1(\overrightarrow{e(\vartheta)-z})}{|e(\vartheta)-z|^{n+s_1}}\,dz - |v|^{-s_1}\int_{J_{0,\vartheta}} \frac{a_1(\overrightarrow{e(\vartheta)-z})}{|e(\vartheta)-z|^{n+s_1}}\,dz.\end{aligned}$$

Therefore, by (3.4.3),

$$\int_{J_{\vartheta,\pi}} \frac{a_1(\overrightarrow{e(\vartheta)-z})}{|e(\vartheta)-z|^{n+s_1}}\,dz - \int_{J_{0,\vartheta}} \frac{a_1(\overrightarrow{e(\vartheta)-z})}{|e(\vartheta)-z|^{n+s_1}}\,dz = 0. \tag{3.4.42}$$

Consequently, recalling the notation in (3.3.1) and exploting (3.3.6) with $c = 0$, we have that

$$\mathcal{D}_\vartheta(\pi-\vartheta) = \int_{J_{\vartheta,\pi}} \frac{a_1(\overrightarrow{e(\vartheta)-z})}{|e(\vartheta)-z|^{n+s_1}}\,dz - \int_{J_{0,\vartheta}} \frac{a_1(\overrightarrow{e(\vartheta)-z})}{|e(\vartheta)-z|^{n+s_1}}\,dz = 0 = \mathcal{D}_\vartheta(\widehat{\vartheta}).$$

By the uniqueness claim in Proposition 3.3.1, we conclude that $\pi - \vartheta = \widehat{\vartheta}$, as desired.

This completes the proof of 3), and in turn of Theorem 3.4.1. □

proof of Corollary 3.4.2. We point out that 1) and 2) in Corollary 3.4.2 follow from 1) and 2) in Theorem 3.4.1, respectively.

To prove 3) of Corollary 3.4.2, we first notice that $\vartheta \in (0, \pi)$ in these cases. Also, if $a_1 \equiv$ const, then the cancellation property in (3.3.6) boils down to $\mathcal{D}_\vartheta(\vartheta) = 0$, and therefore, by the uniqueness claim in Proposition 3.3.1 we obtain that $\widehat{\vartheta} = \vartheta$.

Furthermore, we recall that (3.4.3) holds true in this case, thanks to 3) of Theorem 3.4.1, and therefore, using the equivalent formulation of (3.4.3) given in (3.4.42) (with $a_1 \equiv$ const in this case), we find that

$$\mathcal{D}_\vartheta(\pi-\vartheta) = \int_{J_{\vartheta,\pi}} \frac{a_1}{|e(\vartheta)-z|^{n+s_1}}\,dz - \int_{J_{0,\vartheta}} \frac{a_1}{|e(\vartheta)-z|^{n+s_1}}\,dz = 0 = \mathcal{D}_\vartheta(\vartheta).$$

Hence, using again the uniqueness claim in Proposition 3.3.1 we conclude that $\pi - \vartheta = \vartheta$, which gives that $\vartheta = \frac{\pi}{2}$, as desired. □

We now deal with the case $s_1 = s_2$, as given by Theorem 3.4.3. For this, we need a variation of Lemma 3.4.7 that takes into account the situation in which $s_1 = s_2$.

Lemma 3.4.8. *Let $s \in (0,1)$ and $K_1,\ K_2 \in \mathbf{K}^2(n,s,\lambda,\varrho)$. Assume that there exists $\varepsilon_0 \in (0,1)$ such that*

$$|\sigma|\, K_2(\zeta) \le (1-\varepsilon_0)\, K_1(\zeta) \qquad \text{for all } \zeta \in B_{\varepsilon_0} \setminus \{0\}. \tag{3.4.43}$$

Let Ω be an open bounded set with C^1-boundary and E be a volume-constrained critical set of $\mathcal{C}$.

Let $x_0 \in \mathrm{Reg}_E \cap \partial\Omega$, $x_k \in \mathrm{Reg}_E \cap \Omega$ such that $x_k \to x_0$ as $k \to +\infty$ and $r_k > 0$ such that $r_k \to 0$ as $k \to +\infty$.

Suppose that H and V are open half-spaces such that

$$\Omega^{x_0,r_k} \to H \qquad \text{and} \qquad E^{x_0,r_k} \to H \cap V \quad \text{in } L^1_{\mathrm{loc}}(\mathbb{R}^n) \text{ as } k \to +\infty.$$

Let $\vartheta \in [0,\pi]$ be the angle between the half-spaces H and V, that is $H \cap V = J_{0,\vartheta}$ in the notation of (3.0.2).

Then,

i) if $\vartheta = 0$ then

$$\lim_{k\to+\infty} r_k^s \left[\mathbf{H}^{K_1}_{\partial E}(x_k) - \int_{\Omega^c} K_1(x_k-y)\,dy + \sigma \int_{\Omega^c} K_2(x_k-y)\,dy\right] = +\infty;$$

ii) if $\vartheta = \pi$ then

$$\lim_{k\to+\infty} r_k^s \left[\mathbf{H}^{K_1}_{\partial E}(x_k) - \int_{\Omega^c} K_1(x_k-y)\,dy + \sigma \int_{\Omega^c} K_2(x_k-y)\,dy\right] = -\infty.$$

Proof. We establish i), being the proof of ii) analogous. For this, we use the notation introduced in the proof of Lemma 3.4.7, and specifically we recall formula (3.4.32), to be used here with $s_1 = s_2 = s$. In this case, we use (3.4.43) to see that, if k is large enough, for all $z \in B_{1/2}(v_k)$ we have that

$$|\sigma|\, K_2\big(r_k(v_k-z)\big) \le (1-\varepsilon_0)\, K_1\big(r_k(v_k-z)\big). \tag{3.4.44}$$

This and (3.4.28) give that

$$\begin{aligned}\mathcal{Z}_k\big(r_k(v_k-z)\big) &= r_k^{n+s} \max\Big\{K_1\big(r_k(v_k-z)\big),\ |\sigma|\, K_2\big(r_k(v_k-z)\big)\Big\} \\ &= r_k^{n+s} K_1\big(r_k(v_k-z)\big),\end{aligned}$$

which entails that $\alpha_k(z) = 0$.

Also, using again (3.4.44), it follows that

$$\beta_k(z) = r_k^{n+s}\Big(K_1\big(r_k(v_k-z)\big) - |\sigma|\, K_2\big(r_k(v_k-z)\big)\Big) \ge \varepsilon_0\, r_k^{n+s}\, K_1\big(r_k(v_k-z)\big).$$

In light of these observations, (3.4.32) in this framework reduces to

$$\Upsilon_k \geq \varepsilon_0\, r_k^{n+s} \int_{\mathcal{P}_k \setminus Y(\mathcal{Q}_k)} K_1\big(r_k(v_k - z)\big)\, dz - C.$$

We have thus recovered the last inequality in (3.4.33), with $1/2$ replaced by the constant ε_0. Then it suffices to proceed as in (3.4.34) and (3.4.35) to complete the proof. □

With this additional result, we are now in the position of giving the proof of Theorem 3.4.3.

Proof of Theorem 3.4.3. We fix a point $x_0 \in \partial\Omega \cap \mathrm{Reg}_E$ and a sequence of points $x_k \in \Omega \cap \mathrm{Reg}_E$ such that $x_k \to x_0$ as $k \to +\infty$. We also set $r_k := |x_k - x_0|$ and we observe that $r_k \to 0$ as $k \to +\infty$.

From (3.2.3) evaluated at x_k, we get

$$\mathbf{H}^{K_1}_{\partial E}(x_k) - \int_{\Omega^c} K_1(x_k - y)\, dy + \sigma \int_{\Omega^c} K_2(x_k - y)\, dy + g(x_k) = c,$$

where c does not depend on k. Thus, multiplying both sides by r_k^s, we find that

$$r_k^s\, \mathbf{H}^{K_1}_{\partial E}(x_k) - r_k^s \int_{\Omega^c} K_1(x_k - y)\, dy + \sigma\, r_k^{s_1} \int_{\Omega^c} K_2(x_k - y)\, dy + r_k^s\, g(x_k) = c\, r_k^s.$$

Since g is locally bounded, we have that $r_k^s g(x_k) \to 0$ as $k \to +\infty$, and therefore

$$\lim_{k \to +\infty} r_k^s \left[\mathbf{H}^{K_1}_{\partial E}(x_k) - \int_{\Omega^c} K_1(x_k - y)\, dy + \sigma \int_{\Omega^c} K_2(x_k - y)\, dy\right] = 0. \tag{3.4.45}$$

In light of Lemma 3.4.8 (which can be exploited here thanks to assumption (3.4.4)), this gives that the angle ϑ between H and V lies in $(0, \pi)$.

Thus, in order to prove (3.4.5), we can take $v \in H \cap \partial V$ and we see that, for every k, there exists $v_k \in \Omega^{x_0, r_k} \cap \partial E^{x_0, r_k}$ such that $v_k \to v$ as $k \to +\infty$, where r_k is an infinitesimal sequence as $k \to +\infty$. As a consequence, for every k, there exists $x_k \in \mathrm{Reg}_E \cap \Omega$ such that $v_k = \frac{x_k - x_0}{r_k}$ and $x_k \to x_0$ as $k \to +\infty$. Then, we are in the position to apply iii) in Lemma 3.4.6 and conclude that

$$\lim_{k \to +\infty} r_k^{s_1} \left[\mathbf{H}^{K_1}_{\partial E}(x_k) - \int_{\Omega^c} K_1(x_k - y)\, dy\right] = \mathbf{H}^{K_1^*}_{\partial (H \cap V)}(v) - \int_{H^c} K_1^*(v - y)\, dy.$$

Also, by Lebesgue's Dominated Convergence Theorem,

$$\begin{aligned} \lim_{k \to +\infty} r_k^s \int_{\Omega^c} K_2(x_k - y)\, dy &= \lim_{k \to +\infty} \int_{(\Omega^{x_0, r_k})^c} r_k^{n+s} K_2(r_k(v_k - y))\, dy \\ &= \int_{H^c} K_2^*(v - y)\, dy \end{aligned}$$

and this limit is finite.

These considerations and (3.4.45) give the desired result in (3.4.5). □

We are now in the position of proving Proposition 3.4.4.

Proof of Proposition 3.4.4. We exploit the notation in (3.0.3), the assumption in (3.0.4) and the change of variable $z = y/|v|$, to see that (3.4.7) can be written as

$$
\begin{aligned}
0 &= \mathbf{H}^{K_1^*}_{\partial(H\cap V)}(v) - \int_{H^c} K_1^*(v-y)\,dy + \sigma \int_{H^c} K_2^*(v-y)\,dy \\
&= \int_{\mathbb{R}^n} K_1^*(v-y)\big(\chi_{(H\cap V)^c\cap H}(y) - \chi_{H\cap V}(y)\big)\,dy + \sigma \int_{H^c} K_2^*(v-y)\,dy \\
&= \int_{\mathbb{R}^n} \frac{a_1(\overrightarrow{v-y})}{|v-y|^{n+s_1}}\big(\chi_{(H\cap V)^c\cap H}(y) - \chi_{H\cap V}(y)\big)\,dy + \sigma \int_{H^c} \frac{a_2(\overrightarrow{v-y})}{|v-y|^{n+s_2}}\,dy \\
&= |v|^{-s_1} \int_{\mathbb{R}^n} \frac{a_1(\overrightarrow{e(\vartheta)-z})\big(\chi_{J^c_{0,\vartheta}\cap H}(z) - \chi_{J_{0,\vartheta}}(z)\big)}{|e(\vartheta)-z|^{n+s_1}}\,dz + \sigma|v|^{-s_2} \int_{H^c} \frac{a_2(\overrightarrow{e(\vartheta)-z})}{|e(\vartheta)-z|^{n+s_2}}\,dz \\
&= |v|^{-s_1} \int_{J_{\vartheta,\pi}} \frac{a_1(\overrightarrow{e(\vartheta)-z})}{|e(\vartheta)-z|^{n+s_1}}\,dz - |v|^{-s_1} \int_{J_{0,\vartheta}} \frac{a_1(\overrightarrow{e(\vartheta)-z})}{|e(\vartheta)-z|^{n+s_1}}\,dz \\
&\qquad + \sigma|v|^{-s_2} \int_{H^c} \frac{a_2(\overrightarrow{e(\vartheta)-z})}{|e(\vartheta)-z|^{n+s_2}}\,dz.
\end{aligned}
$$

Hence, recalling the assumption in (3.4.6), this gives the desired result in (3.4.8). □

3.5 Possible stickiness or detachment of nonlocal droplets

In this section we investigate the possibly degenerate cases in which the nonlocal droplets either detach from the container or adhere completely to its surfaces. These cases depend on the strong attraction or repulsion of the second kernel and are described in the examples provided in Theorems 3.5.1 and 3.5.2.

Theorem 3.5.1. *Let $\sigma > 0$, $\Omega := B_1$, $g := 0$, $K_1(\xi) := \frac{k_1}{|\xi|^{n+s_1}}$ and $K_2(\xi) := \frac{k_2}{|\xi|^{n+s_2}}$, for some k_1, $k_2 > 0$.*

Let E be a volume-constrained minimizer of $\mathcal{C}$. Assume that there exist $p \in \partial B_1$ and $\varepsilon_0 > 0$ such that $B_{\varepsilon_0}(p) \cap B_1 \subseteq E$. Assume also that $\mathrm{Reg}_E \cap \Omega \neq \varnothing$.

Then, either $s_1 > s_2$, or $s_1 = s_2$ and $k_1 > \sigma k_2$.

Theorem 3.5.2. *Let $\sigma < 0$, $\Omega := B_1$, $g := 0$, $K_1(\xi) := \frac{k_1}{|\xi|^{n+s_1}}$ and $K_2(\xi) := \frac{k_2}{|\xi|^{n+s_2}}$, for some k_1, $k_2 > 0$.*

Let E be a volume-constrained minimizer of $\mathcal{C}$. Assume that there exist $p \in \partial B_1$ and $\varepsilon_0 > 0$ such that $B_{\varepsilon_0}(p) \cap B_1 \subseteq (\Omega \setminus E)$. Assume also that $\mathrm{Reg}_E \cap \Omega \neq \varnothing$.

Then, either $s_1 > s_2$, or $s_1 = s_2$ and $-k_1 < \sigma k_2$.

In order to prove Theorems 3.5.1 and 3.5.2, we need some auxiliary integral estimates to detect the interaction between "thin sets". This is formalized in Lemmata 3.5.3 and 3.5.4 here below.

Lemma 3.5.3. *Let r, $t > 0$, $s \in (0,1)$ and*

$$D := \{x = (x', x_n) \in \mathbb{R}^n \,:\, |x'| < r \text{ and } x_n \in (0,t)\}.$$

Then,

$$\iint_{D\times\{y_n<0\}} \frac{dx\,dy}{|x-y|^{n+s}} = c_\star\, r^{n-1}\, t^{1-s},$$

for a suitable $c_\star > 0$, depending only on n and s.

Proof. We recall that the surface area of the $(n-1)$-dimensional unit sphere is equal to $\frac{2\pi^{\frac{n}{2}}}{\Gamma(\frac{n}{2})}$, where Γ is the Gamma Function. Furthermore,

$$\int_0^{+\infty} \frac{\ell^{n-2}\,d\ell}{\left(\ell^2+1\right)^{\frac{n+s}{2}}} = \frac{\Gamma\left(\frac{n-1}{2}\right)\Gamma\left(\frac{1+s}{2}\right)}{2\Gamma\left(\frac{n+s}{2}\right)}.$$

Hence, we use the substitution $\xi := \frac{y'-x'}{x_n-y_n}$ to see that

$$\begin{aligned}
&\iint_{D\times\{y_n<0\}} \frac{dx\,dy}{|x-y|^{n+s}} \\
&\quad= \int_0^t \left[\int_{\{|x'|<r\}} \left[\int_{-\infty}^0 \left[\int_{\mathbb{R}^{n-1}} \frac{d\xi}{(x_n-y_n)^{1+s}\big(|\xi|^2+1\big)^{\frac{n+s}{2}}}\right] dy_n\right] dx'\right] dx_n \\
&\quad= \frac{2\pi^{\frac{n-1}{2}}}{\Gamma\left(\frac{n-1}{2}\right)} \int_0^t \left[\int_{\{|x'|<r\}} \left[\int_{-\infty}^0 \left[\int_0^{+\infty} \frac{\ell^{n-2}\,d\ell}{(x_n-y_n)^{1+s}\big(\ell^2+1\big)^{\frac{n+s}{2}}}\right] dy_n\right] dx'\right] dx_n \\
&\quad= \frac{\pi^{\frac{n-1}{2}}\Gamma\left(\frac{1+s}{2}\right)}{\Gamma\left(\frac{n+s}{2}\right)} \int_0^t \left[\int_{\{|x'|<r\}} \left[\int_{-\infty}^0 \frac{dy_n}{(x_n-y_n)^{1+s}}\right] dx'\right] dx_n \\
&\quad= \frac{2\pi^{\frac{2n-1}{2}}\Gamma\left(\frac{1+s}{2}\right)}{\Gamma\left(\frac{n}{2}\right)\Gamma\left(\frac{n+s}{2}\right)}\, r^{n-1} \int_0^t \left[\int_{-\infty}^0 \frac{dy_n}{(x_n-y_n)^{1+s}}\right] dx_n \\
&\quad= \frac{2\pi^{\frac{2n-1}{2}}\Gamma\left(\frac{1+s}{2}\right)}{s\,\Gamma\left(\frac{n}{2}\right)\Gamma\left(\frac{n+s}{2}\right)}\, r^{n-1} \int_0^t \frac{dx_n}{x_n^s} \\
&\quad= \frac{2\pi^{\frac{2n-1}{2}}\Gamma\left(\frac{1+s}{2}\right)}{s\,(1-s)\,\Gamma\left(\frac{n}{2}\right)\Gamma\left(\frac{n+s}{2}\right)}\, r^{n-1}\, t^{1-s},
\end{aligned}$$

as desired. □

Lemma 3.5.4. *Let r, $t > 0$, $s \in (0,1)$,*

$$\begin{aligned}
&D := \{x = (x', x_n) \in \mathbb{R}^n \,:\, |x'| < r \text{ and } x_n \in (0,t)\} \\
\text{and}\quad &F := \{x = (x', x_n) \in \mathbb{R}^n \,:\, |x'| > r \text{ and } x_n \in (0,t)\}.
\end{aligned}$$

Then,
$$\iint_{D\times F}\frac{dx\,dy}{|x-y|^{n+s}}\le C\,t\,r^{n-1-s},$$
for some $C>0$ depending only on n and s.

Proof. Differently from the proof of Lemma 3.5.3, here it is convenient to exploit the substitutions $\alpha:=\frac{x_n}{|x'-y'|}$ and $\beta:=\frac{y_n}{|x'-y'|}$. In this way we see that

$$\begin{aligned}
&\iint_{D\times F}\frac{dx\,dy}{|x-y|^{n+s}}\\
&=\int_{\{|x'|<r\}}\left[\int_{\{|y'|>r\}}\left[\int_0^{t/|x'-y'|}\left[\int_0^{t/|x'-y'|}\frac{d\beta}{|x'-y'|^{n+s-2}(1+(\alpha-\beta)^2)^{\frac{n+s}{2}}}\right]d\alpha\right]dy'\right]dx'\\
&\le\int_{\{|x'|<r\}}\left[\int_{\{|y'|>r\}}\left[\int_0^{t/|x'-y'|}\left[\int_0^{+\infty}\frac{d\gamma}{|x'-y'|^{n+s-2}(1+\gamma^2)^{\frac{n+s}{2}}}\right]d\alpha\right]dy'\right]dx'\\
&=C\int_{\{|x'|<r\}}\left[\int_{\{|y'|>r\}}\left[\int_0^{t/|x'-y'|}\frac{d\alpha}{|x'-y'|^{n+s-2}}\right]dy'\right]dx'\\
&=Ct\int_{\{|x'|<r\}}\left[\int_{\{|y'|>r\}}\frac{dy'}{|x'-y'|^{n+s-1}}\right]dx'\\
&=C\,t\,r^{n-1-s}\int_{\{|X'|<1\}}\left[\int_{\{|Y'|>1\}}\frac{dY'}{|X'-Y'|^{n+s-1}}\right]dX'\\
&=C\,t\,r^{n-1-s},
\end{aligned}$$

where, as customary, we took the freedom of renaming C line after line. □

Now, in the forthcoming Lemma 3.5.5 we present a further technical result that detects suitable cancellations involving “thin sets”. This is a pivotal result to account for the nonlocal scenario. Indeed, in the classical capillarity theory, to look for a competitor for a given set, one can dig out a (small deformation of a) cylinder with base radius equal to ε and height $\delta\varepsilon$ and then add a ball with the same volume. A very convenient fact in this scenario is that the surface error produced by the cylinder is of order ε^{n-1}, while the one produced by the balls are of order $(\delta\varepsilon^n)^{\frac{n-1}{n}}=\delta^{\frac{n-1}{n}}\varepsilon^{n-1}$. That is, for δ suitably small, the surface tension produced by the new ball is negligible with respect to the surface tension of the cylinder, thus allowing us to construct competitors in a nice and simple way.

Instead, in the nonlocal setting, for a given value of the fractional parameter, the corresponding nonlocal surface tension produced by cylinders and balls of the same volume are comparable. This makes the idea of “adding a ball to compensate the loss of volume caused by removing a cylinder” not suitable for the nonlocal framework. Instead, as we will see in the proof of Theorem 3.5.1, the volume compensation should occur through the addition of a suitably thin set placed at a regular point of the droplet. The fact that the corresponding nonlocal surface energy produces a negligible contribution will rely on the following result:

Lemma 3.5.5. *Let $s \in (0,1)$, $0 < \varepsilon < \delta < 1$ and $\eta \in (0,1)$. Let $f \in C^{1,\alpha}_0\left(\mathbb{R}^{n-1}, \left(-\frac{\delta}{2}, \frac{\delta}{2}\right)\right)$ for some $\alpha \in (0,1)$ and assume that $f(0) = 0$ and $\partial_i f(0) = 0$ for all $i \in \{1, \dots, n-1\}$. Let $\varphi \in C^\infty(\mathbb{R}^{n-1}, [0, +\infty))$ be such that $\varphi(x') = 0$ whenever $|x'| \geq 1$ and*

$$\int_{\mathbb{R}^{n-1}} \varphi(x')\, dx' = 1.$$

Let

$$\begin{aligned}
&\psi(x') := \frac{\eta}{\varepsilon^{n-1}} \varphi\left(\frac{x'}{\varepsilon}\right), \\
&\mathcal{P} := \big\{ x = (x', x_n) \in \mathbb{R}^n \,:\, |x'| < \delta \text{ and } x_n > f(x') + \psi(x') \big\}, \\
&\mathcal{Q} := \big\{ x = (x', x_n) \in \mathbb{R}^n \,:\, |x'| < \delta \text{ and } x_n \in \big(f(x'),\, f(x') + \psi(x')\big) \big\} \\
\text{and} \quad &\mathcal{R} := \big\{ x = (x', x_n) \in \mathbb{R}^n \,:\, |x'| < \delta \text{ and } x_n < f(x') \big\}.
\end{aligned}$$

Then, there exist $\delta_0 \in (0,1)$ and $C > 0$, depending only on n, s, α, f and φ, such that if $\delta < \delta_0$ and $\eta < \delta_0 \varepsilon^n$ then

$$\left| \iint_{\mathcal{P} \times \mathcal{Q}} \frac{dx\, dy}{|x - y|^{n+s}} - \iint_{\mathcal{R} \times \mathcal{Q}} \frac{dx\, dy}{|x - y|^{n+s}} \right| \leq C \left(\delta^\alpha + \frac{\eta}{\varepsilon^n} \right) \varepsilon^{(n-1)s}\, \eta^{1-s}.$$

Proof. The gist of this proof is to use a suitable reflection to simplify most of the integral contributions. For this, we consider the map

$$T(x) := \big(-x',\, 2f(x') + \psi(x') - x_n \big).$$

We observe that when $|x'| < \delta$ the distance between the Jacobian of T and minus the identity matrix is bounded from above by

$$C \sup_{|x'| < \delta} \big(|\nabla f(x')| + |\nabla \psi(x')| \big) \leq C \sup_{|x'| < \delta} \left(|\nabla f(x') - \nabla f(0)| + \frac{\eta}{\varepsilon^n} \right) \leq C \left(\delta^\alpha + \frac{\eta}{\varepsilon^n} \right),$$

and the latter is a small quantity, as long as δ_0 is chosen sufficiently small.

Moreover, the condition $T(x) \in \mathcal{Q}$ is equivalent to $|x'| < \delta$ and $2f(x') + \psi(x') - x_n \in \big(f(x'),\, f(x') + \psi(x')\big)$, which is in turn equivalent to $x \in \mathcal{Q}$.

Similarly, the condition $T(x) \in \mathcal{P}$ is equivalent to $x \in \mathcal{R}$, as well as the condition $T(x) \in \mathcal{R}$ is equivalent to $x \in \mathcal{P}$.

From these observations and the change of variable $(X, Y) := (T(x), T(y))$ we arrive at

$$\iint_{\mathcal{P} \times \mathcal{Q}} \frac{dx\, dy}{|x - y|^{n+s}} = \left(1 + O\left(\delta^\alpha + \frac{\eta}{\varepsilon^n} \right) \right) \iint_{\mathcal{R} \times \mathcal{Q}} \frac{dX\, dY}{|X - Y|^{n+s}}.$$

As a result,

$$\left| \iint_{\mathcal{P} \times \mathcal{Q}} \frac{dx\, dy}{|x - y|^{n+s}} - \iint_{\mathcal{R} \times \mathcal{Q}} \frac{dx\, dy}{|x - y|^{n+s}} \right| \leq C \left(\delta^\alpha + \frac{\eta}{\varepsilon^n} \right) \iint_{\mathcal{R} \times \mathcal{Q}} \frac{dx\, dy}{|x - y|^{n+s}}. \tag{3.5.1}$$

Now we consider the transformation $S(x) := (x', x_n - f(x'))$. When $|x'| < \delta$ the distance between the Jacobian of S and the identity matrix is bounded from above by

$$C \sup_{|x'|<\delta} |\nabla f(x')| = C \sup_{|x'|<\delta} |\nabla f(x') - \nabla f(0)| \le C\delta^\alpha.$$

Besides, if $x \in \mathcal{R}$ then $S(x) \in \big\{x \in \mathbb{R}^n \,:\, |x'| < \delta \text{ and } x_n < 0\big\}$. Also, if $x \in \mathcal{Q}$ then

$$S(x) \in \{x \in \mathbb{R}^n \,:\, |x'| < \delta \text{ and } x_n \in (0, \psi(x'))\} \subseteq \left\{x \in \mathbb{R}^n \,:\, |x'| < \varepsilon \text{ and } x_n \in \left(0, \frac{C\eta}{\varepsilon^{n-1}}\right)\right\}.$$

We stress that we are using here the fact that $\psi(x') = 0$ when $|x'| \ge \varepsilon$.

From these remarks and (3.5.1), using now the change of variable $(X, Y) := (S(x), S(y))$, it follows that

$$\begin{aligned} &\left| \iint_{\mathcal{P}\times\mathcal{Q}} \frac{dx\,dy}{|x-y|^{n+s}} - \iint_{\mathcal{R}\times\mathcal{Q}} \frac{dx\,dy}{|x-y|^{n+s}} \right| \\ &\quad \le C\left(\delta^\alpha + \frac{\eta}{\varepsilon^n}\right) \iint_{\{X_n<0\}\times\left\{|Y'|<\varepsilon,\, Y_n \in \left(0, \frac{C\eta}{\varepsilon^{n-1}}\right)\right\}} \frac{dX\,dY}{|X-Y|^{n+s}}. \end{aligned}$$

We can thus employ Lemma 3.5.3 with $r := \varepsilon$ and $t := \frac{C\eta}{\varepsilon^{n-1}}$ and conclude that

$$\left| \iint_{\mathcal{P}\times\mathcal{Q}} \frac{dx\,dy}{|x-y|^{n+s}} - \iint_{\mathcal{R}\times\mathcal{Q}} \frac{dx\,dy}{|x-y|^{n+s}} \right| \le C\left(\delta^\alpha + \frac{\eta}{\varepsilon^n}\right) \varepsilon^{n-1} \left(\frac{\eta}{\varepsilon^{n-1}}\right)^{1-s},$$

from which the desired result follows. □

With this preliminary work, we can now prove Theorems 3.5.1 and 3.5.2.

Proof of Theorem 3.5.1. Up to a rigid motion we can suppose that $p = e_n$. We let $\varepsilon > 0$ and $\delta > 0$, to be taken as small as we wish in what follows. We also define

$$\mathcal{B} := \Big\{x = (x', x_n) \in B_1 \setminus B_{1-\delta\varepsilon} \,:\, x_n > 0 \text{ and } |x'| < \varepsilon\Big\}.$$

We stress that $\mathcal{B} \subseteq B_{\varepsilon_0/2}(p) \cap B_1$ as long as ε is small enough. Also, we pick a point $q \in \mathrm{Reg}_E \cap \Omega$ and we modify the surface of ∂E in the normal direction in an ε-neighborhood of q by a set $\mathcal{B}'$ with $|\mathcal{B}'| = |\mathcal{B}|$, see Figure 3.5 and notice that the geometry of Lemma 3.5.5 can be reproduced, up to a rigid motion. We stress that η in Lemma 3.5.5 corresponds to the volume of the perturbation induced by ψ, therefore in this setting we will apply Lemma 3.5.5 with $\eta := |\mathcal{B}'| = |\mathcal{B}| \le C\delta\varepsilon^n$.

We also denote by Θ a cylinder centered at q (oriented by the normal of $\mathcal{B}'$ at q) of height equal to 2δ and radius of the basis equal to δ. In this way, we have that if $x \in \mathcal{B}'$ and $y \in \mathbb{R}^n \setminus \Theta$ then $|x - y| \ge |y - q| - |q - x| \ge \frac{\sqrt{5}\delta}{2} - C\varepsilon \ge \frac{\delta}{4}$, as long as ε is small enough, possibly in dependence of δ, see Figure 3.6, whence

$$I_1(\mathcal{B}', B_1 \setminus \Theta) \le C \int_{\mathcal{B}'\times B_1} \frac{dx\,dy}{\delta^{n+s_1}} \le \frac{C\,|\mathcal{B}'|}{\delta^{n+s_1}}.$$

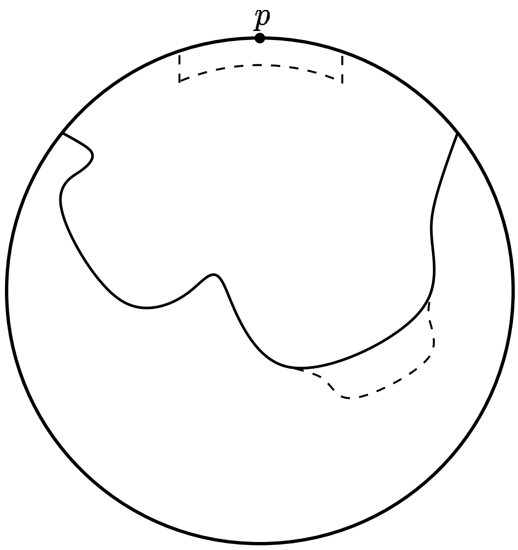


Figure 3.5: Removing the thin set $\mathcal{B}$ to E near p and adding the thin set $\mathcal{B}'$ with the same volume.

Consequently,

$$\begin{aligned} I_1(\mathcal{B}', B_1 \setminus E \setminus \mathcal{B}') - I_1(\mathcal{B}', E) &\leq I_1\big(\mathcal{B}', (B_1 \setminus E \setminus \mathcal{B}') \cap \Theta\big) - I_1(\mathcal{B}', E \cap \Theta) + \frac{C|\mathcal{B}'|}{\delta^{n+s_1}} \\ &\leq I_1\big(\mathcal{B}', (B_1 \setminus E \setminus \mathcal{B}') \cap \Theta\big) - I_1(\mathcal{B}', E \cap \Theta) + \frac{C\varepsilon^n}{\delta^{n-1+s_1}}, \end{aligned} \tag{3.5.2}$$

for some $C > 0$ that, as usual, gets renamed line after line.

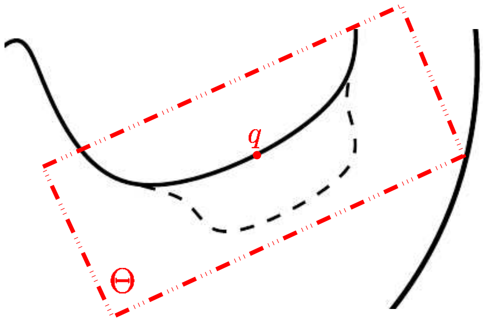


Figure 3.6: Surrounding $\mathcal{B}'$ with a small cylinder Θ.

In view of Lemma 3.5.5, we also know that

$$I_1\big(\mathcal{B}', (B_1 \setminus E \setminus \mathcal{B}') \cap \Theta\big) - I_1(\mathcal{B}', E \cap \Theta) \leq C\delta^\alpha \varepsilon^{(n-1)s_1} \left(\delta\varepsilon^n\right)^{1-s_1} = C\delta^{1-s_1+\alpha}\varepsilon^{n-s_1}.$$

This and (3.5.2) lead to

$$I_1(\mathcal{B}', B_1 \setminus E \setminus \mathcal{B}') - I_1(\mathcal{B}', E) \le C\delta^{1-s_1+\alpha}\varepsilon^{n-s_1} + \frac{C\varepsilon^n}{\delta^{n-1+s_1}}. \tag{3.5.3}$$

Now we consider the set $(E \setminus \mathcal{B}) \cup \mathcal{B}'$ which is a competitor for the minimal set E with the same volume of E. Accordingly, comparing their energies, we have that

$$\begin{aligned} &I_1(\mathcal{B}, B_1 \setminus E) + I_1(\mathcal{B}', E) + \sigma I_2(\mathcal{B}, B_1^c) \\ &\qquad \le I_1(\mathcal{B}, E \cup \mathcal{B}' \setminus \mathcal{B}) + I_1(\mathcal{B}', B_1 \setminus E \setminus \mathcal{B}') + I_1(\mathcal{B}, \mathcal{B}') + \sigma I_2(\mathcal{B}', B_1^c). \end{aligned}$$

By combining this and (3.5.3) we find that

$$\begin{aligned} &I_1(\mathcal{B}, B_1 \setminus E) + \sigma I_2(\mathcal{B}, B_1^c) \\ &\qquad \le I_1(\mathcal{B}, E \cup \mathcal{B}' \setminus \mathcal{B}) + I_1(\mathcal{B}, \mathcal{B}') + \sigma I_2(\mathcal{B}', B_1^c) + C\delta^{1-s_1+\alpha}\varepsilon^{n-s_1} + \frac{C\varepsilon^n}{\delta^{n-1+s_1}}. \end{aligned} \tag{3.5.4}$$

Besides, since the distance between $\mathcal{B}'$ and B_1^c is bounded from below by a uniform quantity, only depending on q and ε_0 (and, in particular, independent of ε), we have that

$$I_2(\mathcal{B}', B_1^c) = k_2 \iint_{\mathcal{B}' \times B_1^c} \frac{dx\, dy}{|x-y|^{n+s_2}} \le C|\mathcal{B}'| = C|\mathcal{B}| \le C\varepsilon^n,$$

for some $C > 0$ depending only on n, s_2, k_2, ε_0, q and the regularity of ∂E in the vicinity of q. This and (3.5.4) yield that

$$\begin{aligned} \sigma I_2(\mathcal{B}, B_1^c) &\le I_1(\mathcal{B}, E \cup \mathcal{B}' \setminus \mathcal{B}) + I_1(\mathcal{B}, \mathcal{B}') + C\varepsilon^n + C\delta^{1-s_1+\alpha}\varepsilon^{n-s_1} + \frac{C\varepsilon^n}{\delta^{n-1+s_1}} \\ &\le I_1(\mathcal{B}, B_1 \setminus \mathcal{B}) + I_1(\mathcal{B}, \mathcal{B}') + C\varepsilon^n + C\delta^{1-s_1+\alpha}\varepsilon^{n-s_1} + \frac{C\varepsilon^n}{\delta^{n-1+s_1}} \\ &\le I_1(\mathcal{B}, B_1 \setminus \mathcal{B}) + C\delta^{1-s_1+\alpha}\varepsilon^{n-s_1} + \frac{C\varepsilon^n}{\delta^{n-1+s_1}}, \end{aligned} \tag{3.5.5}$$

up to renaming C line after line. Now, we use the change of variables $X := \frac{x-e_n}{\varepsilon}$ and $Y := \frac{y-e_n}{\varepsilon}$ to see that

$$\varepsilon^{s_1-n} I_1(\mathcal{B}, B_1 \setminus \mathcal{B}) = k_1\, \varepsilon^{s_1-n} \iint_{\mathcal{B} \times (B_1 \setminus \mathcal{B})} \frac{dx\, dy}{|x-y|^{n+s_1}} = k_1 \iint_{\mathcal{Z}_\varepsilon \times \mathcal{A}_\varepsilon} \frac{dX\, dY}{|X-Y|^{n+s_1}}, \tag{3.5.6}$$

where

$$\mathcal{Z}_\varepsilon := \frac{\mathcal{B} - e_n}{\varepsilon} = \left\{ X \in \mathbb{R}^n \,:\, |X'| < 1,\ X_n > -\frac{1}{\varepsilon} \text{ and } \left| X + \frac{e_n}{\varepsilon} \right| \in \left[\frac{1}{\varepsilon} - \delta, \frac{1}{\varepsilon} \right) \right\}$$

and

$$\mathcal{A}_\varepsilon := \frac{(B_1 \setminus \mathcal{B}) - e_n}{\varepsilon} = \mathcal{L}_\varepsilon \cup \mathcal{M}_\varepsilon \cup \mathcal{N}_\varepsilon,$$

with

$$\begin{aligned}
\mathcal{L}_\varepsilon &:= \left\{ X \in \mathbb{R}^n \,:\, \left| X + \frac{e_n}{\varepsilon} \right| < \frac{1}{\varepsilon} - \delta \right\}, \\
\mathcal{M}_\varepsilon &:= \left\{ X \in \mathbb{R}^n \,:\, |X'| \geq 1 \text{ and } \left| X + \frac{e_n}{\varepsilon} \right| \in \left[\frac{1}{\varepsilon} - \delta, \frac{1}{\varepsilon} \right) \right\} \\
\text{and} \qquad \mathcal{N}_\varepsilon &:= \left\{ X \in \mathbb{R}^n \,:\, |X'| < 1,\ X_n \leq -\frac{1}{\varepsilon} \text{ and } \left| X + \frac{e_n}{\varepsilon} \right| \in \left[\frac{1}{\varepsilon} - \delta, \frac{1}{\varepsilon} \right) \right\}.
\end{aligned}$$

Similarly,

$$\varepsilon^{s_2-n} I_2(\mathcal{B}, B_1^c) = k_2\, \varepsilon^{s_2-n} \iint_{\mathcal{B}\times B_1^c} \frac{dx\, dy}{|x-y|^{n+s_2}} = k_2 \iint_{\mathcal{Z}_\varepsilon \times \mathcal{O}_\varepsilon} \frac{dX\, dY}{|X-Y|^{n+s_2}}, \tag{3.5.7}$$

where

$$\mathcal{O}_\varepsilon := \left\{ X \in \mathbb{R}^n \,:\, \left| X + \frac{e_n}{\varepsilon} \right| \geq \frac{1}{\varepsilon} \right\}.$$

Plugging (3.5.6) and (3.5.7) into (3.5.5), we arrive at

$$\sigma\, \varepsilon^{s_1-s_2}\, k_2 \iint_{\mathcal{Z}_\varepsilon \times \mathcal{O}_\varepsilon} \frac{dX\, dY}{|X-Y|^{n+s_2}} \leq k_1 \iint_{\mathcal{Z}_\varepsilon \times \mathcal{A}_\varepsilon} \frac{dX\, dY}{|X-Y|^{n+s_1}} + C\delta^{1-s_1+\alpha} + \frac{C\varepsilon^{s_1}}{\delta^{n-1+s_1}}. \tag{3.5.8}$$

Now we claim that, if $\varepsilon > 0$ is suitably small, possibly in depedence of δ, then

$$\mathcal{B} \subseteq \big\{ x = (x', x_n) \in \mathbb{R}^n \,:\, |x'| < \varepsilon \ \text{ and } \ x_n \in [1 - (1+\delta)\delta\varepsilon, 1) \big\}. \tag{3.5.9}$$

Indeed, if $x \in \mathcal{B}$ then

$$\begin{aligned}
x_n &= \sqrt{|x|^2 - |x'|^2} \geq \sqrt{(1-\delta\varepsilon)^2 - \varepsilon^2} = \sqrt{1 - 2\delta\varepsilon + \delta^2\varepsilon^2 - \varepsilon^2} \\
&\geq \sqrt{1 - 2(1+\delta)\delta\varepsilon + (1+\delta)^2\delta^2\varepsilon^2} = \sqrt{(1-(1+\delta)\delta\varepsilon)^2} = 1 - (1+\delta)\delta\varepsilon
\end{aligned}$$

taking $\varepsilon \leq (2\delta^2)/(\delta^4 + 2\delta^3 + 1)$, thus establishing (3.5.9). Now from (3.5.9) it follows that

$$\mathcal{Z}_\varepsilon \subseteq \big\{ X = (X', X_n) \in \mathbb{R}^n \,:\, |X'| < 1 \ \text{ and } \ X_n \in [-(1+\delta)\delta, 0) \big\} =: \mathcal{Z}_\delta^\star. \tag{3.5.10}$$

Note also that

$$\mathcal{O}_\varepsilon \supseteq \{Y_n > 0\}. \tag{3.5.11}$$

Indeed, if $Y \in \mathbb{R}^n$ is such that $Y_n > 0$, then

$$\left| Y + \frac{e_n}{\varepsilon} \right| = \sqrt{|Y'|^2 + \left(Y_n + \frac{1}{\varepsilon} \right)^2} = \sqrt{|Y'|^2 + Y_n^2 + \frac{2Y_n}{\varepsilon} + \frac{1}{\varepsilon^2}} \geq \frac{1}{\varepsilon},$$

as desired. We now claim that

$$\mathcal{Z}_\varepsilon \supseteq \left\{ X \in \mathbb{R}^n \,:\, |X'| < 1,\ X_n \in (-\delta, 0) \ \text{ and } \ \left| X + \frac{e_n}{\varepsilon} \right| < \frac{1}{\varepsilon} \right\} =: \mathcal{W}_\varepsilon. \tag{3.5.12}$$

To check this, suppose by contradiction that there exists $X \in \mathcal{W}_\varepsilon$ with $\left|X + \frac{e_n}{\varepsilon}\right| < \frac{1}{\varepsilon} - \delta$. Then, we have that

$$
\begin{aligned}
0 < \left(\frac{1}{\varepsilon} - \delta\right)^2 - \left|X + \frac{e_n}{\varepsilon}\right|^2 &= \frac{1}{\varepsilon^2} + \delta^2 - \frac{2\delta}{\varepsilon} - |X'|^2 - \left(X_n + \frac{1}{\varepsilon}\right)^2 \\
&= \delta^2 - \frac{2\delta}{\varepsilon} - |X'|^2 - X_n^2 - \frac{2X_n}{\varepsilon} \le \delta^2 - |X'|^2 - X_n^2,
\end{aligned}
$$

that is $|X| < \delta$, and thus

$$
\frac{1}{\varepsilon} - \delta > \left|X + \frac{e_n}{\varepsilon}\right| \ge \left|\frac{e_n}{\varepsilon}\right| - |X| = \frac{1}{\varepsilon} - |X| > \frac{1}{\varepsilon} - \delta.
$$

This is a contradiction which establishes (3.5.12). Hence, by (3.5.11) and (3.5.12), we see that

$$
\begin{aligned}
\iint_{\mathcal{Z}_\varepsilon \times \mathcal{O}_\varepsilon} \frac{dX\,dY}{|X-Y|^{n+s_2}} &\ge \iint_{\mathcal{W}_\varepsilon \times \{Y_n > 0\}} \frac{dX\,dY}{|X-Y|^{n+s_2}} \\
&= \iint_{\mathcal{W}^\star_\delta \times \{Y_n > 0\}} \frac{dX\,dY}{|X-Y|^{n+s_2}} - \iint_{(\mathcal{W}^\star_\delta \setminus \mathcal{W}_\varepsilon) \times \{Y_n > 0\}} \frac{dX\,dY}{|X-Y|^{n+s_2}},
\end{aligned}
\tag{3.5.13}
$$

where

$$
\mathcal{W}^\star_\delta := \big\{X \in \mathbb{R}^n \,:\, |X'| < 1 \text{ and } X_n \in (-\delta, 0)\big\}.
$$

We observe that

$$
\iint_{(\mathcal{W}^\star_\delta \setminus \mathcal{W}_\varepsilon) \times \{Y_n > 0\}} \frac{dX\,dY}{|X-Y|^{n+s_2}} \le \iint_{\mathcal{W}^\star_\delta \times \{Y_n > 0\}} \frac{dX\,dY}{|X-Y|^{n+s_2}} < +\infty
$$

and

$$
\lim_{\varepsilon \searrow 0} \big|\mathcal{W}^\star_\delta \setminus \mathcal{W}_\varepsilon\big| = 0,
$$

since as long as ε is taken smaller in dependence on δ, the condition $\left|X + \frac{e_n}{\varepsilon}\right| < 1/\varepsilon$ is satisfied. Hence we can conclude that

$$
\lim_{\varepsilon \searrow 0} \iint_{(\mathcal{W}^\star_\delta \setminus \mathcal{W}_\varepsilon) \times \{Y_n > 0\}} \frac{dX\,dY}{|X-Y|^{n+s_2}} = 0
$$

and, as a consequence, we infer from (3.5.13) that

$$
\liminf_{\varepsilon \searrow 0} \iint_{\mathcal{Z}_\varepsilon \times \mathcal{O}_\varepsilon} \frac{dX\,dY}{|X-Y|^{n+s_2}} \ge \iint_{\mathcal{W}^\star_\delta \times \{Y_n > 0\}} \frac{dX\,dY}{|X-Y|^{n+s_2}}. \tag{3.5.14}
$$

We also note that if $a \gg b > 0$, then there exists a positive constant $C > 0$ such that

$$
\left|\sqrt{a^2 - b^2} - a - \frac{b^2}{2a}\right| = \left|a\sqrt{1 - \frac{b^2}{a^2}} - a - \frac{b^2}{2a}\right| \le \frac{b^4}{a^3},
$$

whence if $X \in \mathcal{N}_\varepsilon$ then

$$\begin{aligned}
-X_n - \frac{2}{\varepsilon} &= \left|X_n + \frac{1}{\varepsilon}\right| - \frac{1}{\varepsilon} = \sqrt{\left|X + \frac{e_n}{\varepsilon}\right|^2 - |X'|^2} - \frac{1}{\varepsilon} \\
&\in \left[\left|X + \frac{e_n}{\varepsilon}\right| + \frac{|X'|^2}{2\left|X + \frac{e_n}{\varepsilon}\right|} - \frac{|X'|^4}{\left|X + \frac{e_n}{\varepsilon}\right|^3} - \frac{1}{\varepsilon},\ \left|X + \frac{e_n}{\varepsilon}\right| + \frac{|X'|^2}{2\left|X + \frac{e_n}{\varepsilon}\right|} + \frac{|X'|^4}{\left|X + \frac{e_n}{\varepsilon}\right|^3} - \frac{1}{\varepsilon}\right] \\
&\subseteq [-\delta - 2\varepsilon,\ 2\varepsilon] \subseteq [-2\delta, 2\delta],
\end{aligned}$$

as long as ε is sufficiently small, leading to

$$|\mathcal{N}_\varepsilon| \leq \left|\left\{X \in \mathbb{R}^n \,:\, |X'| < 1, \text{ and } X_n \in \left[-\frac{2}{\varepsilon} - 2\delta, -\frac{2}{\varepsilon} + 2\delta\right]\right\}\right| \leq C\delta. \tag{3.5.15}$$

Furthermore, if $X \in \mathcal{Z}_\varepsilon$ then $X_n \geq -(1+\delta)\delta$, thanks to (3.5.9), and therefore if $Y \in \mathcal{N}_\varepsilon$ we have that

$$|X - Y| \geq X_n - Y_n \geq -(1+\delta)\delta + \frac{1}{\varepsilon} \geq \frac{1}{2\varepsilon},$$

choosing ε small enough depending on δ. This yields that

$$\iint_{\mathcal{Z}_\varepsilon \times \mathcal{N}_\varepsilon} \frac{dX\, dY}{|X - Y|^{n+s_1}} \leq C\varepsilon^{n+s_1} |\mathcal{Z}_\varepsilon|\, |\mathcal{N}_\varepsilon| \leq C\varepsilon^{n+s_1}. \tag{3.5.16}$$

Now we set

$$\mathcal{M}'_\varepsilon := \mathcal{M}_\varepsilon \cap B_2 \qquad \text{and} \qquad \mathcal{M}''_\varepsilon := \mathcal{M}_\varepsilon \setminus B_2.$$

We remark that, if $\varepsilon > 0$ is suitably small, possibly in depedence of δ, then

$$\mathcal{M}'_\varepsilon \subseteq \left\{X \in \mathbb{R}^n \,:\, |X'| \in [1,2] \text{ and } X_n \in [-(1+\delta)\delta, 0)\right\} =: \mathcal{M}^\star_\delta. \tag{3.5.17}$$

Indeed, if $X \in \mathcal{M}'_\varepsilon$ then $|X'| \geq 1$ and $|X'| \leq |X| < 2$. Furthermore,

$$1 + \left|X_n + \frac{1}{\varepsilon}\right|^2 \leq |X'|^2 + \left|X_n + \frac{1}{\varepsilon}\right|^2 = \left|X + \frac{e_n}{\varepsilon}\right|^2 \leq \frac{1}{\varepsilon^2},$$

thus

$$\left|X_n + \frac{1}{\varepsilon}\right| \leq \sqrt{\frac{1}{\varepsilon^2} - 1}$$

which in particular gives that

$$X_n \leq \sqrt{\frac{1}{\varepsilon^2} - 1} - \frac{1}{\varepsilon} < 0.$$

Hence $X_n < 0$. Moreover,

$$4 + \left|X_n + \frac{1}{\varepsilon}\right|^2 \geq |X'|^2 + \left|X_n + \frac{1}{\varepsilon}\right|^2 = \left|X + \frac{e_n}{\varepsilon}\right|^2 \geq \left(\frac{1}{\varepsilon} - \delta\right)^2.$$

Since $X_n \geq -|X| \geq -2$, this gives that

$$X_n + \frac{1}{\varepsilon} = \sqrt{\left|X_n + \frac{1}{\varepsilon}\right|^2} \geq \sqrt{\left(\frac{1}{\varepsilon} - \delta\right)^2 - 4} = \sqrt{\frac{1}{\varepsilon^2} - \frac{2\delta}{\varepsilon} + \delta^2 - 4}$$
$$= \frac{1}{\varepsilon}\sqrt{1 - 2\delta\varepsilon + \delta^2\varepsilon^2 - 4\varepsilon^2} \geq \frac{1}{\varepsilon}(1 - (1+\delta)\delta\varepsilon)$$

taking $\varepsilon \leq (2\delta^2)/(\delta^4 + 2\delta^3 + 4)$, and accordingly $X_n \geq -(1+\delta)\delta$. These observations complete the proof of (3.5.17). We now use (3.5.17) in combination with (3.5.10). In this way, we see that

$$\iint_{\mathcal{Z}_\varepsilon \times \mathcal{M}'_\varepsilon} \frac{dX\,dY}{|X-Y|^{n+s_1}} \leq \iint_{\mathcal{Z}^\star_\delta \times \mathcal{M}^\star_\delta} \frac{dX\,dY}{|X-Y|^{n+s_1}}. \tag{3.5.18}$$

Besides we notice that if $X \in \mathcal{Z}_\varepsilon$, then $|X| \leq 3/2$ for sufficiently small δ, indeed by (3.5.10)

$$|X| = \sqrt{|X'|^2 + X_n^2} < \sqrt{1 + (1+\delta)^2\delta^2} = \sqrt{1 + \delta^2 + \delta^4 + 2\delta^3} \leq \frac{3}{2}$$

if δ is taken sufficiently small. Thus, if $X \in \mathcal{Z}_\varepsilon$ and $Y \in \mathcal{M}''_\varepsilon$ then $|X-Y| \geq |Y| - |X| \geq 2 - \frac{3}{2} = \frac{1}{2}$ and, as a result,

$$\iint_{\mathcal{Z}_\varepsilon \times \mathcal{M}''_\varepsilon} \frac{dX\,dY}{|X-Y|^{n+s_1}} = \int_{\mathcal{Z}_\varepsilon} dX \int_{\mathbb{R}^n \setminus B_{1/2}} \frac{dZ}{|Z|^{n+s_1}} \leq |\mathcal{Z}_\varepsilon| \int_{\mathbb{R}^n \setminus B_{1/2}} \frac{dZ}{|Z|^{n+s_1}} \leq C\delta.$$

Combining this and (3.5.18) we conclude that

$$\iint_{\mathcal{Z}_\varepsilon \times \mathcal{M}_\varepsilon} \frac{dX\,dY}{|X-Y|^{n+s_1}} \leq \iint_{\mathcal{Z}^\star_\delta \times \mathcal{M}^\star_\delta} \frac{dX\,dY}{|X-Y|^{n+s_1}} + C\delta.$$

Using the latter inequality and (3.5.16) we obtain that

$$\begin{aligned} &\limsup_{\varepsilon \searrow 0} \iint_{\mathcal{Z}_\varepsilon \times \mathcal{A}_\varepsilon} \frac{dX\,dY}{|X-Y|^{n+s_1}} \\ &\quad \leq \iint_{\mathcal{Z}^\star_\delta \times \mathcal{M}^\star_\delta} \frac{dX\,dY}{|X-Y|^{n+s_1}} + C\delta + \limsup_{\varepsilon \searrow 0} \iint_{\mathcal{Z}_\varepsilon \times \mathcal{L}_\varepsilon} \frac{dX\,dY}{|X-Y|^{n+s_1}}. \end{aligned} \tag{3.5.19}$$

Now we consider the map

$$\left\{\mathbb{R}^n : |X'| < 2\right\} \ni X = (X', X_n) \longmapsto T(X) := \left(X',\, X_n - \sqrt{\left(\frac{1}{\varepsilon} - \delta\right)^2 - |X'|^2} + \frac{1}{\varepsilon}\right)$$

and we observe that if $X \in \mathcal{Z}_\varepsilon$, in particular $|X'| < 2$, then $\underline{X} := T(X)$ satisfies $|\underline{X}'| < 1$ and

$$\begin{aligned} \underline{X}_n &= \left|X_n + \frac{1}{\varepsilon}\right| - \sqrt{\left(\frac{1}{\varepsilon} - \delta\right)^2 - |X'|^2} = \sqrt{\left|X + \frac{e_n}{\varepsilon}\right|^2 - |X'|^2} - \sqrt{\left(\frac{1}{\varepsilon} - \delta\right)^2 - |X'|^2} \\ &\in \left[0,\, \sqrt{\frac{1}{\varepsilon^2} - |X'|^2} - \sqrt{\left(\frac{1}{\varepsilon} - \delta\right)^2 - |X'|^2}\right) \subseteq [0, (1+\delta)\delta]. \end{aligned}$$

In addition, if $Y \in \mathcal{L}'_\varepsilon := \mathcal{L}_\varepsilon \cap B_2$ and $\underline{Y} := T(Y)$, we have that $|\underline{Y}'| < 2$ and

$$\underline{Y}_n \le \left|Y_n + \frac{1}{\varepsilon}\right| - \sqrt{\left(\frac{1}{\varepsilon} - \delta\right)^2 - |Y'|^2} = \sqrt{\left|Y + \frac{e_n}{\varepsilon}\right|^2 - |Y'|^2} - \sqrt{\left(\frac{1}{\varepsilon} - \delta\right)^2 - |Y'|^2} < 0.$$

We also observe that the distance of the Jacobian matrix of T from the identity is bounded from above by

$$C\left|\nabla_{X'}\left(\sqrt{\left(\frac{1}{\varepsilon} - \delta\right)^2 - |X'|^2}\right)\right| \le \frac{C|X'|}{\sqrt{\left(\frac{1}{\varepsilon} - \delta\right)^2 - |X'|^2}} \le C\varepsilon,$$

yielding that, in the above notation, $|\underline{X} - \underline{Y}| \le (1 + C\varepsilon)|X - Y|$, with the freedom, as usual, of renaming C. These observations allow us to conclude that

$$\iint_{\mathcal{Z}_\varepsilon \times \mathcal{L}'_\varepsilon} \frac{dX\, dY}{|X - Y|^{n+s_1}} \le (1 + C\varepsilon) \iint_{\mathcal{X}^\star_\delta \times \mathcal{Y}^\star} \frac{d\underline{X}\, d\underline{Y}}{|\underline{X} - \underline{Y}|^{n+s_1}} \tag{3.5.20}$$

where

$$\mathcal{X}^\star_\delta := \big\{X \in \mathbb{R}^n \,:\, |X'| < 1 \text{ and } X_n \in (0, (1 + \delta)\delta)\big\}$$
$$\text{and} \qquad \mathcal{Y}^\star := \big\{X \in \mathbb{R}^n \,:\, |X'| < 2 \text{ and } X_n < 0\big\}.$$

Also, setting $\mathcal{L}''_\varepsilon := \mathcal{L}_\varepsilon \setminus B_2$, we have that

$$\iint_{\mathcal{Z}_\varepsilon \times \mathcal{L}''_\varepsilon} \frac{dX\, dY}{|X - Y|^{n+s_1}} = \int_{\mathcal{Z}_\varepsilon} dX \int_{\mathbb{R}^n \setminus B_{1/2}} \frac{dZ}{|Z|^{n+s_1}} \le |\mathcal{Z}_\varepsilon| \int_{\mathbb{R}^n \setminus B_{1/2}} \frac{dZ}{|Z|^{n+s_1}} \le C\delta.$$

Combining this inequality and (3.5.20) we find that

$$\iint_{\mathcal{Z}_\varepsilon \times \mathcal{L}_\varepsilon} \frac{dX\, dY}{|X - Y|^{n+s_1}} \le (1 + C\varepsilon) \iint_{\mathcal{X}^\star_\delta \times \mathcal{Y}^\star} \frac{d\underline{X}\, d\underline{Y}}{|\underline{X} - \underline{Y}|^{n+s_1}} + C\delta.$$

From this and (3.5.19) we arrive at

$$\begin{aligned}\limsup_{\varepsilon \searrow 0} &\iint_{\mathcal{Z}_\varepsilon \times \mathcal{A}_\varepsilon} \frac{dX\, dY}{|X - Y|^{n+s_1}} \\ &\le \iint_{\mathcal{Z}^\star_\delta \times \mathcal{M}^\star_\delta} \frac{dX\, dY}{|X - Y|^{n+s_1}} + \limsup_{\varepsilon \searrow 0}(1 + C\varepsilon) \iint_{\mathcal{X}^\star_\delta \times \mathcal{Y}^\star} \frac{d\underline{X}\, d\underline{Y}}{|\underline{X} - \underline{Y}|^{n+s_1}} + C\delta.\end{aligned}$$

Thus, given $\delta > 0$, to be taken conveniently small, we consider the limit $\varepsilon \searrow 0$ and we deduce from the latter inequality, (3.5.8) and (3.5.14) that, as $\varepsilon \searrow 0$,

$$\begin{aligned}\sigma\, \varepsilon^{s_1 - s_2}\, k_2 &\iint_{\mathcal{W}^\star_\delta \times \{Y_n > 0\}} \frac{dX\, dY}{|X - Y|^{n+s_2}} \\ &\le k_1 \left(\iint_{\mathcal{Z}^\star_\delta \times \mathcal{M}^\star_\delta} \frac{dX\, dY}{|X - Y|^{n+s_1}} + (1 + C\varepsilon) \iint_{\mathcal{X}^\star_\delta \times \mathcal{Y}^\star} \frac{d\underline{X}\, d\underline{Y}}{|\underline{X} - \underline{Y}|^{n+s_1}}\right) \\ &\qquad + C\delta + C\delta^{1 - s_1 + \alpha} + \frac{C\varepsilon^{s_1}}{\delta^{n-1+s_1}}.\end{aligned} \tag{3.5.21}$$

This yields that necessarily

$$s_1 \geq s_2. \tag{3.5.22}$$

Furthermore, if $s_1 = s_2$ then we obtain, passing to the limit (3.5.21) as $\varepsilon \searrow 0$, that

$$\begin{aligned} &\sigma\, k_2 \iint_{\mathcal{W}^\star_\delta \times \{Y_n>0\}} \frac{dX\,dY}{|X-Y|^{n+s_1}} \\ &\quad \leq k_1 \left(\iint_{\mathcal{Z}^\star_\delta \times \mathcal{M}^\star_\delta} \frac{dX\,dY}{|X-Y|^{n+s_1}} + \iint_{\mathcal{X}^\star_\delta \times \mathcal{Y}^\star} \frac{d\underline{X}\,d\underline{Y}}{|\underline{X}-\underline{Y}|^{n+s_1}} \right) + C\delta + C\delta^{1-s_1+\alpha}. \end{aligned} \tag{3.5.23}$$

We are now ready to send $\delta \searrow 0$. To this end, we multiply (3.5.23) by δ^{s_1-1} and we make use of Lemmata 3.5.3 and 3.5.4 to find that

$$\begin{aligned} c_\star\, \sigma\, k_2 &= \lim_{\delta\searrow 0} \sigma\, k_2\, \delta^{s_1-1} \iint_{\mathcal{W}^\star_\delta \times \{Y_n>0\}} \frac{dX\,dY}{|X-Y|^{n+s_1}} \\ &\leq \lim_{\delta\searrow 0} \left[k_1\, \delta^{s_1-1} \left(\iint_{\mathcal{Z}^\star_\delta \times \mathcal{M}^\star_\delta} \frac{dX\,dY}{|X-Y|^{n+s_1}} + \iint_{\mathcal{X}^\star_\delta \times \mathcal{Y}^\star} \frac{d\underline{X}\,d\underline{Y}}{|\underline{X}-\underline{Y}|^{n+s_1}} \right) + C\delta^{s_1} + C\delta^{\alpha} \right] \\ &\leq \lim_{\delta\searrow 0} \left[C\delta^{s_1}(1+\delta) + c_\star\, k_1\, (1+\delta)^{1-s_1} + C\delta^{s_1} + C\delta^{\alpha} \right] \\ &= c_\star\, k_1 \end{aligned}$$

and therefore $\sigma k_2 \leq k_1$. Thanks to this, we have that, to complete the proof of Theorem 3.5.1, it only remains to rule out the case $s_1 = s_2$ and $k_1 = \sigma k_2$. In this situation,

$$\mathcal{C}(F) = \mathcal{E}(F) = k_1 \iint_{F\times F^c} \frac{dx\,dy}{|x-y|^{n+s_1}},$$

hence all the minimizers with prescribed volume correspond to balls, thanks to [46]. But this violates the assumptions about the point p in Theorem 3.5.1. □

Proof of Theorem 3.5.2. This can be seen as a counterpart of Theorem 3.5.1 based on complementary sets. For this argument, we denote by $\mathcal{C}_\sigma$, instead of $\mathcal{C}$, the functional in (1.2.34), in order to showcase explicitly its dependence on the relative adhesion coefficient σ. Thus, in the setting of Theorem 3.5.2, if $F \subseteq \Omega$ and $\widetilde{F} := \Omega \setminus F$,

$$\begin{aligned} \mathcal{C}_\sigma(\widetilde{F}) &= I_1\big(\Omega\setminus F, (\Omega\setminus F)^c \cap \Omega\big) + \sigma\, I_2(\Omega\setminus F, \Omega^c) \\ &= I_1(\Omega\setminus F, F) + \sigma\, I_2(\Omega\setminus F, \Omega^c) \\ &= \mathcal{C}_{-\sigma}(F) + \sigma\, I_2(F, \Omega^c) + \sigma\, I_2(\Omega\setminus F, \Omega^c) \\ &= \mathcal{C}_{-\sigma}(F) + \sigma\, I_2(\Omega, \Omega^c). \end{aligned}$$

Since the latter term does not depend on F, we see that if E, as in the statement of Theorem 3.5.2, is a volume-constrained minimizer of $\mathcal{C}_\sigma$, then $\widetilde{E} := \Omega \setminus E$ is a volume-constrained minimizer of $\mathcal{C}_{-\sigma}$. Now, the set $\widetilde{E}$ fulfills the assumptions of Theorem 3.5.1 with σ replaced by $-\sigma$. It follows that either $s_1 > s_2$, or $s_1 = s_2$ and $k_1 > -\sigma k_2$, as desired. □

3.6 Unique determination of the contact angle

A topical question in view of Proposition 3.4.4 is to understand whether or not equation (3.4.8) identifies a unique contact angle ϑ. This is indeed the case, precisely under the natural condition in (3.4.6), according to the following result in Theorem 3.6.3. To state it in full generality, it is convenient to introduce some notation. Indeed, in the forthcoming computations, it comes in handy to reduce the problem to a two-dimensional situation. For this, we revisit the setting in (3.0.2) by defining its two-dimensional projection onto the variables (x_1, x_n), namely one sets

$$J^\star_{\vartheta_1,\vartheta_2} := \Big\{(x_1,x_n)\in\mathbb{R}^2 \,:\, \exists\,\beta\in(\vartheta_1,\vartheta_2),\ \rho>0 \text{ such that } (x_1,x_n)=\rho(\cos\beta,\sin\beta)\Big\}. \tag{3.6.1}$$

Let also $e^\star(\vartheta) := (\cos\vartheta, \sin\vartheta)$ and, for every $x=(x_1,x_2)\in\partial B_1\subseteq\mathbb{R}^2$ and $j\in\{1,2\}$,

$$a^\star_j(x) := \begin{cases} a_j(x) & \text{if } n=2,\\[2mm] \displaystyle\int_{\mathbb{R}^{n-2}} \frac{a_j\Big(\overrightarrow{x_1\,e_1+x_2\,e_n+|x|(0,\bar y,0)}\Big)}{\big(1+|\bar y|^2\big)^{\frac{n+s_j}{2}}}\,d\bar y & \text{if } n\geq 3.\end{cases} \tag{3.6.2}$$

Let also

$$\phi_j(\vartheta) := a^\star_j(\cos\vartheta,\sin\vartheta). \tag{3.6.3}$$

We remark that, as a byproduct of (3.0.5),

$$a^\star_j(x) = a^\star_j(-x) \qquad\text{and}\qquad \phi_j(\vartheta)=\phi_j(\pi+\vartheta). \tag{3.6.4}$$

Before exhibiting the proof of Theorem 3.6.3, it is also convenient to perform some integral computations in order to appropriately rewrite integral interactions involving cones, detecting cancellations, using a dimensional reduction argument and a well designed notation of polar angle with respect to the kernel singularity. The details go as follows.

Lemma 3.6.1. *In the notation of* (3.0.2), (3.6.1), (3.6.2) *and* (3.6.3), *if* $\vartheta\in(0,\pi)$, *then*

$$\begin{aligned}&\int_{J_{\vartheta,\pi}} \frac{a_1(\overrightarrow{x-e(\vartheta)})}{|x-e(\vartheta)|^{n+s_1}}\,dx - \int_{J_{0,\vartheta}} \frac{a_1(\overrightarrow{x-e(\vartheta)})}{|x-e(\vartheta)|^{n+s_1}}\,dx\\ &\qquad = \frac{1}{s_1(\sin\vartheta)^{s_1}}\left(\int_0^\vartheta \phi_1(\alpha)\,(\sin\alpha)^{s_1}\,d\alpha - \int_\vartheta^\pi \phi_1(\alpha)\,(\sin\alpha)^{s_1}\,d\alpha\right).\end{aligned} \tag{3.6.5}$$

Proof. We stress that each of the integrals on the left hand side of (3.6.5) is divergent, hence the two terms have to be considered together, in the principal value sense. However, for typographical convenience, we will formally act on the integrals by omitting the principal value notation and perform the cancellations necessary to have only finite contributions to obtain the desired result. To this end, we recall (3.0.2) and observe

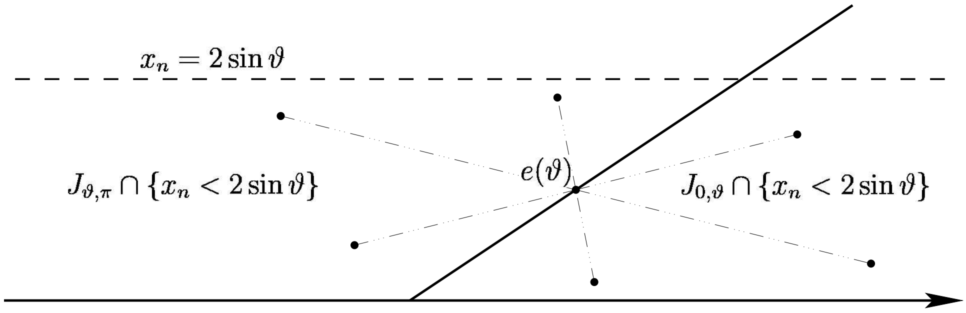


Figure 3.7: A geometric argument involved in the proof of Lemma 3.6.1.

that $x \in J_{0,\vartheta} \cap \{x_n < 2\sin\vartheta\}$ if and only if $z := 2e(\vartheta) - x \in J_{\vartheta,\pi} \cap \{x_n < 2\sin\vartheta\}$, see Figure 3.7. Hence, by the symmetry of a_1,

$$\int_{J_{0,\vartheta}\cap\{x_n<2\sin\vartheta\}} \frac{a_1(\overrightarrow{x-e(\vartheta)})}{|x-e(\vartheta)|^{n+s_1}}\,dx = \int_{J_{\vartheta,\pi}\cap\{z_n<2\sin\vartheta\}} \frac{a_1(\overrightarrow{z-e(\vartheta)})}{|z-e(\vartheta)|^{n+s_1}}\,dz.$$

Consequently, if we denote by Υ the left hand side of (3.6.5), we see after a cancellation that

$$\Upsilon = \int_{J_{\vartheta,\pi}\cap\{x_n>2\sin\vartheta\}} \frac{a_1(\overrightarrow{x-e(\vartheta)})}{|x-e(\vartheta)|^{n+s_1}}\,dx - \int_{J_{0,\vartheta}\cap\{x_n>2\sin\vartheta\}} \frac{a_1(\overrightarrow{x-e(\vartheta)})}{|x-e(\vartheta)|^{n+s_1}}\,dx. \tag{3.6.6}$$

It is useful now to reduce the problem to that in dimension 2. To this end, we adopt the notation in (3.6.1) and (3.6.2) and note that

$$\begin{aligned}
&\int_{J_{\vartheta,\pi}\cap\{x_n>2\sin\vartheta\}} \frac{a_1(\overrightarrow{x-e(\vartheta)})}{|x-e(\vartheta)|^{n+s_1}}\,dx \\
&= \iiint_{\{(x_1,x_n)\in J^\star_{\vartheta,\pi},\ \bar{x}\in\mathbb{R}^{n-2},\ x_n>2\sin\vartheta\}} \frac{a_1\Big(\overrightarrow{(x_1-\cos\vartheta)e_1+(x_n-\sin\vartheta)e_n+(0,\bar{x},0)}\Big)}{\Big((x_1-\cos\vartheta)^2+(x_n-\sin\vartheta)^2+|\bar{x}|^2\Big)^{\frac{n+s_1}{2}}}\,d\bar{x}\,dx_1\,dx_n \\
&= \iint_{\{y=(y_1,y_2)\in J^\star_{\vartheta,\pi},\ \bar{y}\in\mathbb{R}^{n-2},\ y_2>2\sin\vartheta\}} \frac{a_1\Big(\overrightarrow{(y_1-\cos\vartheta)\,e_1+(y_2-\sin\vartheta)\,e_n+|y-e^\star(\vartheta)|(0,\bar{y},0)}\Big)}{|y-e^\star(\vartheta)|^{2+s_1}\,\big(1+|\bar{y}|^2\big)^{\frac{n+s_1}{2}}}\,d\bar{y}\,dy \\
&= \int_{J^\star_{\vartheta,\pi}\cap\{y_2>2\sin\vartheta\}} \frac{a_1^\star(\overrightarrow{y-e^\star(\vartheta)})}{|y-e^\star(\vartheta)|^{2+s_1}}\,dy.
\end{aligned} \tag{3.6.7}$$

Similarly,

$$\int_{J_{0,\vartheta}\cap\{x_n>2\sin\vartheta\}} \frac{a_1(\overrightarrow{x-e(\vartheta)})}{|x-e(\vartheta)|^{n+s_1}}\,dx = \int_{J^\star_{0,\vartheta}\cap\{y_2>2\sin\vartheta\}} \frac{a_1^\star(\overrightarrow{y-e^\star(\vartheta)})}{|y-e^\star(\vartheta)|^{2+s_1}}\,dy.$$

Thanks to these observations, we rewrite (3.6.6) in the form

$$\Upsilon = \int_{J^\star_{\vartheta,\pi}\cap\{x_2>2\sin\vartheta\}} \frac{a_1^\star(\overrightarrow{x-e^\star(\vartheta)})}{|x-e^\star(\vartheta)|^{2+s_1}}\,dx - \int_{J^\star_{0,\vartheta}\cap\{x_2>2\sin\vartheta\}} \frac{a_1^\star(\overrightarrow{x-e^\star(\vartheta)})}{|x-e^\star(\vartheta)|^{2+s_1}}\,dx. \qquad (3.6.8)$$

Now we use polar coordinates centered at $e^\star(\vartheta)$. For this, if $x \in J^\star_{0,\vartheta}\cap\{x_2>2\sin\vartheta\}$, we write $x = (\cos\vartheta,\sin\vartheta)+\rho(\cos\alpha,\sin\alpha)$ with $\alpha\in(0,\vartheta)$ and $\rho > \frac{\sin\vartheta}{\sin\alpha}$. Similarly, if $x\in J^\star_{\vartheta,\pi}\cap\{x_2>2\sin\vartheta\}$, we write $x=(\cos\vartheta,\sin\vartheta)+\rho(\cos\beta,\sin\beta)$ with $\beta\in(\vartheta,\pi)$ and $\rho>\frac{\sin\vartheta}{\sin\beta}$, see Figure 3.8.

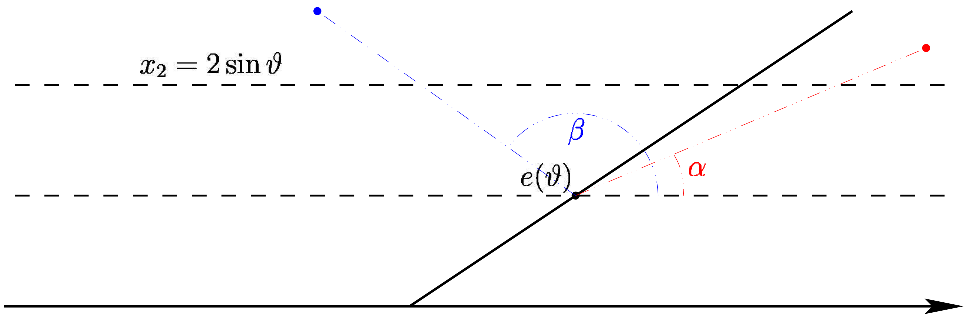


Figure 3.8: Another geometric argument involved in the proof of Lemma 3.6.1.

As a result, using the notation in (3.6.3), we deduce from (3.6.8) that

$$\begin{aligned}\Upsilon &= \iint_{(0,\vartheta)\times\left(\frac{\sin\vartheta}{\sin\alpha},+\infty\right)} \frac{\phi_1(\alpha)}{\rho^{1+s_1}}\,d\alpha\,d\rho - \iint_{(\vartheta,\pi)\times\left(\frac{\sin\vartheta}{\sin\beta},+\infty\right)} \frac{\phi_1(\beta)}{\rho^{1+s_1}}\,d\beta\,d\rho\\ &= \frac{1}{s_1(\sin\vartheta)^{s_1}}\left(\int_0^\vartheta \phi_1(\alpha)\,(\sin\alpha)^{s_1}\,d\alpha - \int_\vartheta^\pi \phi_1(\beta)\,(\sin\beta)^{s_1}\,d\beta\right),\end{aligned}$$

which establishes (3.6.5). □

Lemma 3.6.2. *Let the notation in* (3.0.2)*,* (3.6.1)*,* (3.6.2) *and* (3.6.3) *hold true. Then,*

$$\int_{H^c} \frac{a_2(\overrightarrow{e(\vartheta)-x})}{|e(\vartheta)-x|^{n+s_1}}\,dx = \frac{1}{s_1(\sin\vartheta)^{s_1}}\int_{-\pi}^0 \phi_2(\alpha)\,|\sin\alpha|^{s_1}\,d\alpha. \qquad (3.6.9)$$

Proof. As in (3.6.7), we have that the left hand side of (3.6.9) equals to

$$\Lambda := \int_{\mathbb{R}\times(-\infty,0)} \frac{a_2^\star(\overrightarrow{y-e^\star(\vartheta)})}{|y-e^\star(\vartheta)|^{2+s_1}}\,dy.$$

Now we use polar coordinates centered at $e^\star(\vartheta)$ by considering $y=(\cos\vartheta,\sin\vartheta)+\rho(\cos\alpha,\sin\alpha)$ with $\alpha\in(-\pi,0)$ and $\rho>\frac{\sin\vartheta}{|\sin\alpha|}$, see Figure 3.9. In this way, and recalling (3.6.3), it follows that

$$\Lambda = \iint_{(-\pi,0)\times\left(\frac{\sin\vartheta}{|\sin\alpha|},+\infty\right)} \frac{\phi_2(\alpha)}{\rho^{1+s_1}}\,d\alpha\,d\rho = \frac{1}{s_1(\sin\vartheta)^{s_1}}\int_{-\pi}^0 \phi_2(\alpha)\,|\sin\alpha|^{s_1}\,d\alpha,$$

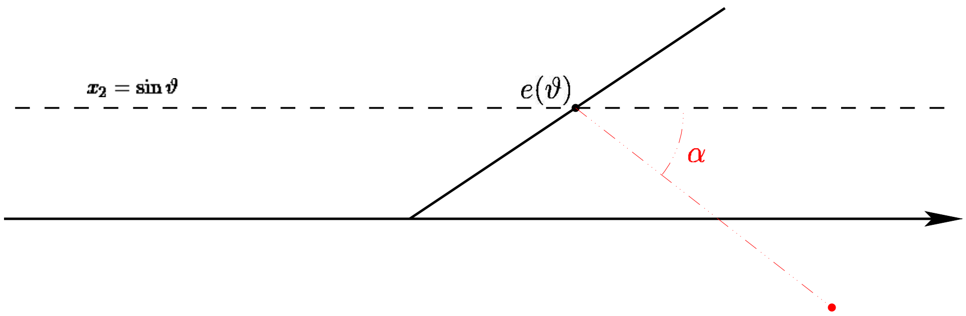


Figure 3.9: A geometric argument involved in the proof of Lemma 3.6.2.

as desired. □

With this, we can uniquely determine the contact angle, as follows.

Theorem 3.6.3. *Let K_1^* and K_2^* be as in* (3.0.4)*. Let $\sigma \in \mathbb{R}$ and assume that* (3.4.6) *holds true. Then, there exists at most one $\vartheta \in (0, \pi)$ satisfying the contact angle condition in* (3.4.8)*.*

Furthermore, if

$$|\sigma| < \frac{\displaystyle\int_0^\pi \phi_1(\alpha)\,(\sin\alpha)^{s_1}\,d\alpha}{\displaystyle\int_0^\pi \phi_2(\alpha)\,(\sin\alpha)^{s_1}\,d\alpha}. \tag{3.6.10}$$

then there exists a unique solution $\vartheta \in (0, \pi)$ of (3.4.8)*.*

We stress once again that when $a_1 = a_2$ (and in particular for constant $a_1 = a_2$), assumption (3.6.10) reduces to the structural assumption $|\sigma| < 1$ that was taken in [60].

Moreover, if $K_1(\xi) := \frac{k_1}{|\xi|^{s_1}}$ and $K_2(\xi) := \frac{k_2}{|\xi|^{s_2}}$ for some k_1, $k_2 > 0$, then assumption (3.6.10) boils down to $|\sigma| < \frac{k_1}{k_2}$, which is precisely the condition for nontrivial minimizers obtained in Theorems 3.5.1 and 3.5.2.

For these reasons, Theorem 3.6.3 showcases the interesting fact that the equation prescribing the contact angle in (3.4.8) admits one and only one solution precisely in the natural range of kernels given by (3.4.6) and (3.6.10).

proof of Theorem 3.6.3. We let

$$\begin{aligned}\mathcal{W}(\vartheta) :=& s_1(\sin\vartheta)^{s_1}\left(\int_{J_{\vartheta,\pi}} \frac{a_1(\overrightarrow{e(\vartheta)-x})}{|e(\vartheta)-x|^{n+s_1}}\,dx - \int_{J_{0,\vartheta}} \frac{a_1(\overrightarrow{e(\vartheta)-x})}{|e(\vartheta)-x|^{n+s_1}}\,dx\right)\\ &- s_1(\sin\vartheta)^{s_1}\sigma\int_{H^c} \frac{a_2(\overrightarrow{e(\vartheta)-x})}{|e(\vartheta)-x|^{n+s_1}}\,dx\end{aligned}$$

and we observe that solutions of (3.4.8) correspond to zeros of $\mathcal{W}$ in $[0, \pi]$.

Also, by Lemmata 3.6.1 and 3.6.2, and recalling (3.6.4),

$$\begin{aligned}\mathcal{W}(\vartheta) &= \int_0^{\vartheta} \phi_1(\alpha)\,(\sin\alpha)^{s_1}\,d\alpha - \int_{\vartheta}^{\pi} \phi_1(\alpha)\,(\sin\alpha)^{s_1}\,d\alpha - \sigma\int_{-\pi}^{0} \phi_2(\alpha)\,|\sin\alpha|^{s_1}\,d\alpha \\ &= \int_0^{\vartheta} \phi_1(\alpha)\,(\sin\alpha)^{s_1}\,d\alpha - \int_{\vartheta}^{\pi} \phi_1(\alpha)\,(\sin\alpha)^{s_1}\,d\alpha - \sigma\int_{-\pi}^{0} \phi_2(\pi+\alpha)\,(\sin(\pi+\alpha))^{s_1}\,d\alpha \\ &= \int_0^{\vartheta} \phi_1(\alpha)\,(\sin\alpha)^{s_1}\,d\alpha - \int_{\vartheta}^{\pi} \phi_1(\alpha)\,(\sin\alpha)^{s_1}\,d\alpha - \sigma\int_{0}^{\pi} \phi_2(\alpha)\,(\sin\alpha)^{s_1}\,d\alpha.\end{aligned} \tag{3.6.11}$$

In particular, $\mathcal{W}$ is continuous in $[0,\pi]$, differentiable in $(0,\pi)$ and, for each $\vartheta\in(0,\pi)$,

$$\mathcal{W}'(\vartheta) = 2\,\phi_1(\vartheta)\,(\sin\vartheta)^{s_1} > 0,$$

which shows that $\mathcal{W}$ admits at most one zero in $(0,\pi)$. This establishes the uniqueness result stated in Theorem 3.6.3.

Now we show the existence result claimed in Theorem 3.6.3 under assumption (3.6.10). To this end, it suffices to notice that, by (3.6.10) and (3.6.11), we have that

$$\mathcal{W}(0) = -\int_0^{\pi} \phi_1(\alpha)\,(\sin\alpha)^{s_1}\,d\alpha - \sigma\int_0^{\pi} \phi_2(\alpha)\,(\sin\alpha)^{s_1}\,d\alpha < 0$$

and

$$\mathcal{W}(\pi) = \int_0^{\pi} \phi_1(\alpha)\,(\sin\alpha)^{s_1}\,d\alpha - \sigma\int_0^{\pi} \phi_2(\alpha)\,(\sin\alpha)^{s_1}\,d\alpha > 0.$$

From this and the continuity of $\mathcal{W}$, we obtain the existence of a zero of $\mathcal{W}$ in $(0,\pi)$. □

Remark 3.6.4. We stress that the strict positivity of the kernel is essential for the uniqueness result in Theorem 3.6.3: indeed, if one allows degenerate kernels in which a_1 is only nonnegative, such a uniqueness claim can be violated. As an example, consider $\sigma := 0$ and pick $\vartheta_0 \in \left(0,\frac{\pi}{2}\right)$. Let $\phi_1 \in C^\infty(\mathbb{R})$ be such that $\phi_1(\alpha) := 0$ for all $\alpha \in [\vartheta_0, \pi-\vartheta_0]$. Assume also that $\phi_1\left(\frac{\pi}{2}+\alpha\right) = \phi_1\left(\frac{\pi}{2}-\alpha\right)$ for all $\alpha\in\left(0,\frac{\pi}{2}\right)$ and that $\phi_1(\alpha+\pi) = \phi_1(\alpha)$ for all $\alpha\in(0,\pi)$. See e.g. Figure 3.10 for a sketch of this function.

Then, by (3.6.11), for every $\bar\vartheta \in \left[\vartheta_0,\frac{\pi}{2}\right]$,

$$\begin{aligned}\mathcal{W}(\bar\vartheta) &= \int_0^{\bar\vartheta} \phi_1(\alpha)\,(\sin\alpha)^{s_1}\,d\alpha - \int_{\bar\vartheta}^{\pi} \phi_1(\alpha)\,(\sin\alpha)^{s_1}\,d\alpha \\ &= \int_0^{\vartheta_0} \phi_1(\alpha)\,(\sin\alpha)^{s_1}\,d\alpha - \int_{\pi-\vartheta_0}^{\pi} \phi_1(\alpha)\,(\sin\alpha)^{s_1}\,d\alpha \\ &= \int_0^{\vartheta_0} \phi_1(\alpha)\,(\sin\alpha)^{s_1}\,d\alpha - \int_0^{\vartheta_0} \phi_1(\pi-\beta)\,(\sin(\pi-\beta))^{s_1}\,d\beta \\ &= \int_0^{\vartheta_0} \phi_1(\alpha)\,(\sin\alpha)^{s_1}\,d\alpha - \int_0^{\vartheta_0} \phi_1\left(\frac{\pi}{2}+\frac{\pi}{2}-\beta\right)\,(\sin\beta)^{s_1}\,d\beta \\ &= \int_0^{\vartheta_0} \phi_1(\beta)\,(\sin\beta)^{s_1}\,d\beta - \int_0^{\vartheta_0} \phi_1\left(\frac{\pi}{2}-\left(\frac{\pi}{2}-\beta\right)\right)\,(\sin\beta)^{s_1}\,d\beta \\ &= 0,\end{aligned}$$

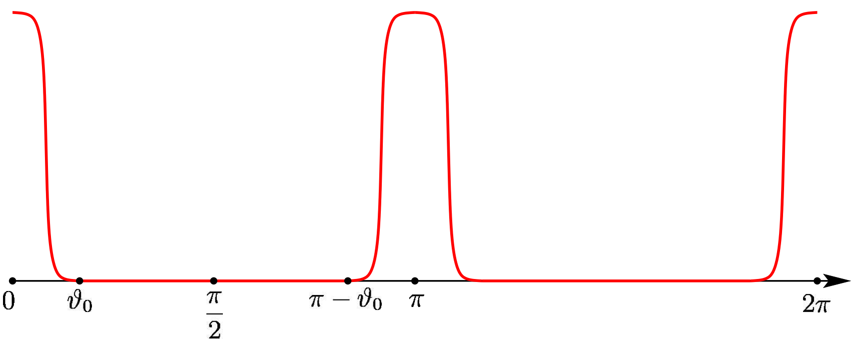


Figure 3.10: A degenerate example of ϕ_1 leading to a multiplicity of the contact angle in (3.4.6).

which shows that in this degenerate case every angle $\bar{\vartheta} \in \left[\vartheta_0, \frac{\pi}{2}\right]$ would be a zero of $\mathcal{W}$, hence a solution of the contact angle equation in (3.4.8). Accordingly, the assumption of strict positivity of the kernel cannot be dropped in Theorem 3.6.3.

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