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Different approaches in Critical Point Theory for entire Schrödinger equations and one for curl-curl problems

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Contents

List of notations

- Given X a Banach space and a (nonlinear) functional $\mathcal{J} : X \to \mathbb{R}$ we denote by $\mathcal{J}'(u)(v)$ the Frechét derivative of the functional at a point $u \in X$ along $v \in X$. We use $\mathcal{J}''(u)[v, w]$ for the second Frechét derivative of $\mathcal{J}$ at $u \in X$ along $v, w \in X$.
- $\widetilde{\mathbb{R}} = \mathbb{R} \cup \{+\infty\}$.
- $\mathbb{R}^+ = (0, +\infty)$ and $\mathbb{R}_0^+ = [0, \infty)$.
- We denote by $(\cdot, \cdot)$ the inner product in an Hilbert space H.
- We denote by $\langle \cdot, \cdot \rangle_{X^*,X}$ the dual product of a Banach space X, where X^* is the dual of X.
- We denote by $a \cdot b$ the inner product in $\mathbb{R}^N$, for every $a, b \in \mathbb{R}^N$.
- The Laplace operator will be denoted by Δ, defined for every regular function $u : \mathbb{R}^N \to \mathbb{R}$ as
$$\Delta u = \sum_{i=1}^{N} \frac{\partial^2 u}{\partial x_i^2}.$$
- For every $x \in \mathbb{R}^N$ and $h \in \mathbb{R}^N$, we set $(\tau_h f)(x) = f(x + h)$ the *shift function.*
- The *fractional* Sobolev critical exponent is denoted by
$$p_s^* = \begin{cases} \dfrac{Np}{N - sp}, & \text{if } N > sp; \\ +\infty, & \text{if } N = sp. \end{cases}$$
For $s = 1$ we recover the classical Sobolev critical exponent.
- We denote by "$\rightharpoonup$" the weak-convergence and by "$\to$" the strong convergence.
- Given a metric space (M, d), the ball centered in x of radius R will be denote by
$$B(x, R) := \{y \in M : d(x, y) \leq R\}$$
while
$$S(x, R) := \{y \in M : d(x, y) = R\}$$
will denotes the sphere centered in x of radius R. We also use B_R and S_R if they are centered in 0.
- The space of rapidly decaying function, or Schwartz space, will be denoted by
$$\mathcal{S}(\mathbb{R}^N) := \left\{ f \in \mathbb{R}^N : \sup_{x \in \mathbb{R}^N} |x^\alpha \partial^\beta f(x)| < +\infty \right\},$$
where α, β are multi-indeces.

- Given $f \in \mathcal{S}(\mathbb{R}^N)$, we denote by $\mathcal{F}(f)$ the *Fourier transform* of f, that is

$$\mathcal{F}(f)(\xi) = \hat{f}(\xi) := \int_{\mathbb{R}^N} f(x) e^{-2\pi i x \cdot \xi} \, dx$$

  and by $\mathcal{F}^{-1}(f)$ the *inverse Fourier transform*

$$\mathcal{F}^{-1}(f)(x) := \int_{\mathbb{R}^N} f(\xi) e^{2\pi i x \cdot \xi} \, d\xi.$$

- i is the imaginary unit.
- $\hbar = \frac{h}{2\pi}$ denotes the reduced Planck's constant.
- $o(\varepsilon)$ will denote the Landau symbol, meaning that a function $f(\varepsilon) = o(\varepsilon)$ if $\lim_{\varepsilon \to 0} \frac{f(\varepsilon)}{\varepsilon} = 0$.

Introduction

A partial differental equation (PDE) is an equation where the unknown is given by a function that depends from two or more variables and some of its partial derivatives, namely given an integer $k \geq 1$, a subset $U \subset \mathbb{R}^N$ and a function

$$F : \mathbb{R}^{N^k} \times \mathbb{R}^{N^{k-1}} \times \cdots \times \mathbb{R} \times U \to \mathbb{R}$$

we call $k-th$ *order partial differential equation* an expression of the form

$$F(D^k u(x), D^{k-1}u(x), \ldots, Du(x), u(x)) = 0. \tag{1}$$

where

$$u : U \to \mathbb{R}$$

is the unknown.

Compared to the *ordinary differential equations* (ODE), to solve explicitely a PDE like (1), that is to find a solution with a simple expression that solves the equation, can be a very hard task. For this reason, very often we are "satisfied" in deducing the existence of one (or more) solutions and some properties of them, but that almost becomes the only way forward as the difficulty of the PDE increases.

During the last decades, many fields were developed with the aim of facing this problems, like topological methods (e.g. Bifurcation Theory, Perturbation Theory, Degree Theory) and variational methods (Critical Point Theory) regarding the existence, while the wide area of regularity provides properties results for the solutions. In this thesis, we are interesting in Critical Point Theory.

Many partial differential equations can be expressed in the form

$$Lu = 0 \tag{2}$$

where $L : X \to Y$ is a map between two Banach spaces X and Y: in this cases, it is interesting asking if equation (2) admits a variational structure, namely one asks if there exists a functional $\mathcal{J} : X \to \mathbb{R}$ such that

$$L(u)(v) = \lim_{t \to 0} \frac{\mathcal{J}(u + tv) - \mathcal{J}(u)}{t}.$$

If the limit above is finite, then we can write $L = \mathcal{J}'$ and equation (2) becomes

$$\mathcal{J}'(u)(v) = 0 \text{ for every } v \in X. \tag{3}$$

In this way, we have expressed equation (2) in a weak (distributional) form; so, we are saying that the (weak) solutions of (2) are critical points of functional $\mathcal{J}$.

Therefore, we move the problem of finding solutions for the equation to finding critical points of a suitable functional. If the functional were bounded from below (or from above) then obvious candidates would be global minima (or maxima). However, the situation changes when we have to face unbounded functionals, both from above and from below (this can happen very often when considering functional that comes from Quantum Physics). As an example, consider the functional $\mathcal{J} : H_0^1(0, \pi) \to \mathbb{R}$ defined as

$$\mathcal{J}(u) := \int_0^\pi \left(\frac{1}{2}|u'|^2 - \frac{1}{4}|u|^4 \right) dx. \tag{4}$$

This functional is Frechét differentiable, in fact

$$\lim_{t \to 0} \frac{\mathcal{J}(u + tv) - \mathcal{J}(u)}{t} = |u'||v'| - |u|^3|v|$$

for every $u, v \in H_0^1(0, \pi)$ and $t \geq 0$. Moreover, we observe that $u \equiv 0$ is a critical point for the functional (4), but $\mathcal{J}$ is unbounded both from above and from below: to see that, let $t > 0$ and $u \in H_0^1(0, \pi)$, then

$$\mathcal{J}(tu) = \int_0^\pi \left(\frac{t^2}{2}|u'|^2 - \frac{t^4}{4}|u|^4 \right) dx \to -\infty$$

as $t \to +\infty$, while for every $k \in \mathbb{N}$

$$\mathcal{J}(\sin(kx)) = \int_0^\pi \left(\frac{1}{2}|(\sin(kx))'|^2 - \frac{1}{4}|\sin(kx)|^4 \right) dx = \frac{k^2}{4}\pi - \frac{\pi}{4} \to +\infty$$

as $k \to +\infty$. So, from this analysis it is not clear if $u \equiv 0$ is the only critical point or there exist other ones.

Looking for other critical points other than the trivial one can be useful both from a mathematical point of view, because it makes the problem more interesting, and because some "special" critical points can have an important application feedback: for example, in Quantum Physics it is very useful to know the critical point corresponding to the minimal energy level of a system, the so called *ground-state* (see Chapter 2).

For this reasons, new tools that allow to find (possibly) all the critical points, so saddle points too, of a given functional are needed; in this direction, min-max method are very useful. But how does this method works? We define

$$c := \inf_{A \in \mathcal{A}} \sup_{u \in A} \mathcal{J}(u)$$

where $\mathcal{A}$ is a collection of subsets X. Aim of the Critical Point Theory is to show that the set of the critical points of $\mathcal{J}$ of value $c \in \mathbb{R}$, that is

$$K_c := \{u \in X : \mathcal{J}(u) = c, \mathcal{J}'(u) = 0\} \tag{5}$$

is non-empty. Hence, the problem is to choose the right family $\mathcal{A}$ and the right conditions on $\mathcal{J}$. A milestone result in this direction is the *Mountain-Pass Theorem*, proved by Ambrosetti and Rabinowitz in 1972 in [14]. The idea of the Theorem is the following: consider a Banach space X, a functional $\mathcal{J} : X \to \mathbb{R}$ and two points 0_X and e belonging to the space (0_X denotes the origin of X). We can think to the points $(0_X, \mathcal{J}(0))$ and $(e, \mathcal{J}(e))$ as two villages in two distinct valleys separated by a mountain ridge and suppose we want to walk from $(0_X, \mathcal{J}(0))$ to $(e, \mathcal{J}(e))$ climbing as least as possible. To achieve our goal, we should find, between all the (smooth) paths connecting the two points, the one corresponding to the mountain pass with the lowest altitude. Intuitively, the ridge of this mountain pass should be a critical point, namely if

$$\Gamma := \{\gamma \in C([0, 1], X) : \gamma(0) = 0_X, \gamma(1) = e\}$$

is the set of all the possible paths from 0_X to e then, for what we said above, we should try to minimize, with respect to all the possible paths $\gamma(t)$, the functional $\max_{t\in[0,1]} \mathcal{J}(\gamma(t))$: so we are saying that

$$c := \inf_{\gamma(t)\in\Gamma} \sup_{t\in[0,1]} \mathcal{J}(\gamma(t))$$

is a critical point for the functional $\mathcal{J}$. However, although intuitively we are sure that c is a critical point, this is not true in general. In fact, consider in $\mathbb{R}^2$ the following functional

$$J(x,y) = x^2 - (x-1)^3 y^2.$$

We can observe that (see Figure 1 below) this functional admits a local minima on the origin and, definig c as above, we have that for every $e \neq 0$, the point c is positive. In particular, choosing $e = (2,2)$ we get that $c \geq 1$ is a non-trivial critical point. However, it can be shown that does not exists a path $\gamma(t)$ such that $J(\gamma(t)) \leq 1$.

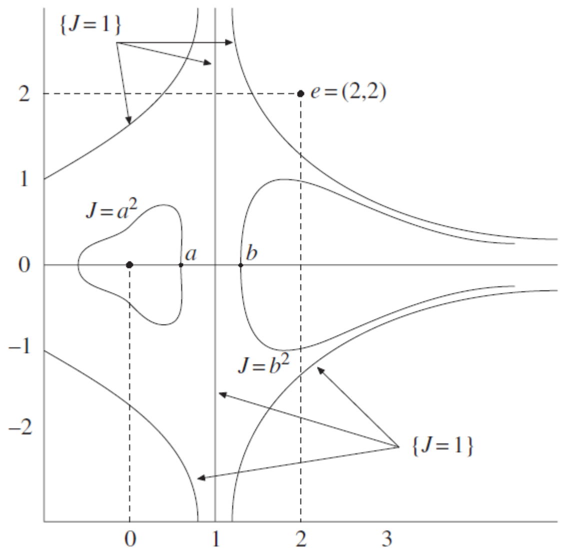


Figure 1: The energy functional J.

To be sure that c is indeed a critical point, we need to assume that the mountain that separates the villages $(0_X, J(0))$ and $(e, J(e))$ actually exists, in the sense that it has an higher altitude than the ones of villages themselves. That is, we need to assume that given $R > 0$

$$\inf_{S(0,R)} J > \max\{J(0), J(e)\}. \tag{6}$$

We want to remark that, some years later the work of Ambrosetti and Rabinowitz, Pucci and Serrin in [116] studied the case where equality holds in (6) (namely the case of a mountain of "altitude 0"): they were able to prove that there exists a critical point in the sphere $S(0,R)$. Some other generalizations were made during the year, and we remind to [71, 115, 117].

The proof of the Mountain-Pass Theorem relies on some very deep and abstract notions due to Palais and Smale (see [107, 109]), that generalized the Morse theory to infinite-dimensional space. Roughly speaking, Palais and Smale introduces a new notion of *compactness* for this spaces, weaker than the usual one, but that turns out to be crucial when applying Critical Point Theory. This compactness notion is called *Palais-Smale condition* and it is defined as follows.

Definition (Palais-Smale condition)**.** *Let X be a Banach space and let $\mathcal{J}$ be a functional of class C^1 on X. We say that $\mathcal{J}$ satisfies the Palais-Smale condition, (PS) for short, if every sequence $(u_n)_n$ on X such that*

$$\mathcal{J}(u_n) \to 0 \quad \textit{and} \quad \mathcal{J}'(u_n) \to 0 \textit{ in } X^* \tag{7}$$

admits a convergent subsequence.

A sequence satisfying (7) is called *Palais-Smale sequence.* We observe that, thanks to the Ekeland's Variational Principle of 1974 (see [59] and Theorem 2.4 in [139]), this is a sufficient condition for the existence of minimizer for functional that are bounded from below.

Moreover, condition above automatically implies that the set K_c defined in (5) is compact for every $c \in \mathbb{R}$ and, thanks to this, the typical strategy in Critical Point Theory is more or less the following:

i) we try to prove the existence of a Palais-Smale sequence;

ii) we show that this sequence is bounded;

iii) we prove that this sequence strongly converges to a nontrivial value.

An important tool that is used when proving the compactness property is a compact embedding result for Sobolev spaces due to Rellich and Kondrachov (see [2], Theorem 6.3). In particular, this Theorem is fundamental for the proof of point iii) above. In fact, suppose we have the following problem

$$\begin{cases} -\Delta u = |u|^{p-2}u, & \text{in } \Omega, \\ u = 0, & \text{in } \partial\Omega \end{cases}$$

where Ω is a bounded domain of $\mathbb{R}^N$, with $N \geq 1$ and $p \in (2, 2^*)$. The associated functional is $J : H_0^1(\Omega) \to \mathbb{R}$ defined as

$$J(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 \, dx - \frac{1}{p}\int_\Omega |u|^p \, dx$$

and we look for critical points of J. We want to prove that Palais-Smale condition holds for J, and to do that we will need the Rellich-Kondrachov embedding. So, let $(u_n)_n \subset H_0^1(\Omega)$ be a Palais-Smale sequence: the first Gâteaux derivative of J is

$$J'(u)(v) = \int_\Omega \nabla u \cdot \nabla v \, dx - \int_\Omega |u|^{p-2}uv \, dx$$

and we compute

$$\begin{aligned} pJ(u_n) - J'(u_n)u_n &= \frac{p}{2}\int_\Omega |\nabla u|^2 \, dx - \int_\Omega |u|^p \, dx - \int_\Omega |\nabla u|^2 \, dx + \int_\Omega |u|^p \, dx \\ &= \left(\frac{p}{2} - 1\right)\int_\Omega |\nabla u|^2 \, dx = \left(\frac{p}{2} - 1\right)\|u_n\|^2_{H_0^1(\Omega)}. \end{aligned} \tag{8}$$

However, by (7) and Cauchy-Schwarz inequality, it is also true that there exist $M, N > 0$ such that

$$pJ(u_n) - J'(u_n)u_n \leq M + \|J'(u_n)\|_{H^{-1}(\Omega)}\|u_n\|_{H_0^1(\Omega)} \leq M + N\|u_n\|_{H_0^1(\Omega)}. \tag{9}$$

Hence, from (8) and (9) we obtain

$$\left(\frac{p}{2} - 1\right)\|u_n\|^2_{H_0^1(\Omega)} \leq M + N\|u_n\|_{H_0^1(\Omega)},$$

and so that $(u_n)_n$ is bounded in $H_0^1(\Omega)$: so, up to a subsequence, we have that

$$u_n \rightharpoonup u \text{ in } H_0^1(\Omega)$$

and, since we are in a bounded domain and $p < 2^*$, then by Rellich-Kondrachov embedding

$$u_n \to u \text{ in } L^(\Omega), \tag{10}$$

for $p \in (2, 2^*)$. Thanks to (10), we can now prove that the sequence $(u_n)_n$ strongly converges to a nontrivial point in $H_0^1(\Omega)$: to see that, it suffices to compute

$$J'(u_n)(u_n - u) = \int_\Omega |\nabla u|^2 \, dx - \int_\Omega \nabla u_n \cdot \nabla u \, dx - \int_\Omega |u_n|^{p-2} u_n (u_n - u) \, dx.$$

Now, by Hölder's inequality the boundedness of the sequence, we get that there exists $C > 0$

$$\begin{aligned} \int_\Omega |u_n|^{p-2} u_n (u_n - u) \, dx &\le \left(\int_\Omega |u_n|^p \, dx \right)^{\frac{p-1}{p}} \left(\int_\Omega |u_n - u|^p \, dx \right)^{\frac{1}{p}} \\ &\le C \|u_n - u\|_{L^p(\Omega)} \to 0 \end{aligned}$$

as $n \to +\infty$. Therefore, we proved that $\int_\Omega |\nabla u|^2 \, dx \to \int_\Omega |\nabla u|^2 \, dx$, namely

$$\|u_n\|^2_{H_0^1(\Omega)} \to \|u\|^2_{H_0^1(\Omega)}$$

and since $u_n \rightharpoonup u$ in $H_0^1(\Omega)$, by Proposition 3.32 in [38], we finally obtain

$$u_n \to u \text{ in } H_0^1(\Omega).$$

Although Rellich-Kondrachov embedding is a powerful tool to prove compactness, it has a "big" limit. In fact, it can only be used if we are in bounded domains (as in the example just concluded): this means that the method used above does not work if we want to study an equation defined, for instance, in the whole space, and it is therefore necessary to find a new way to obtain the compactness property. In these cases, we talk about loss of compactness and this can happen for various reasons, the most common is the impossibility of using the compact embedding result. This is due mainly for two reasons: we are considering a *critical problem*, that is a problem where the exponent of nonlinearity is equal to or greater than the critical Sobolev exponent, or we find ourselves in an unlimited domains, like the whole space. In the latter case, we will talk about *entire problems*. Obviously, and unfortunately, there are many other reasons that lead to the loss of compactness, but in this thesis we will focus on entire problems.

To overcome this problem, several methods have been developed over the years: the most "natural" was developed by Strauss in 1977 (see [132]), who proved a compact embedding Theorem for unbounded domains, provided that we restrict to the subset of the radial functions of the space. In order to use this result, it is essential to prove that the radially symmetric functions has a suitable decay at infinity (see Section 3 in [132]). At this point, Palais's Principle of Symmetric Criticality comes in handy (see [108]), which states that a critical point for a functional constrained to the subset of radial functions is actually a "free" critical point: so radial symmetry is a sort of natural constraint for the problem. The weakness of this strategy is that we do not always manage to obtain a Principle of Symmetric Criticality, so very often the critical points (and therefore the solutions) are bound to a condition of symmetry.

A second approach consists of a perturbation technique developed in the early 1990s by Ambrosetti and Badiale (see [7], [8]) who generalized a previous result by Ambrosetti, Coti Zelati and Ekeland (see [12]): this technique, by exploiting the perturbative nature of the problem, allows to find the critical points (and therefore solutions to the problem) through the analysis of a suitable associated functional. We anticipate that we will use this theory in the course of this thesis, and in Chapter 1, Section 1.5 we recall this technique.

Another approach consists in generalizing the Palais-Smale condition, with the aim of simplifying either the existence or the boundedness of the minimizing sequence. In the last decades many conditions have been introduced for this purpose, but the most famous and used (in fact we will also use this condition in Chapter 2 and Chapter 3) is certainly the *Cerami-condition* (see [43]).

Definition (Cerami condition)**.** *Let X be a Banach space and let $\mathcal{J}$ be a functional of class C^1 on X. We say that $\mathcal{J}$ satisfies the Palais-Smale condition, (PS) for short, if every sequence $(u_n)_n$ on X such that*

$$\mathcal{J}(u_n) \to 0 \quad \text{and} \quad (1 + \|u_n\|_X)\mathcal{J}'(u_n) \to 0 \text{ in } X^* \tag{11}$$

admits a convergent subsequence.

Cerami-condition is weaker than Palais-Smale condition, but often is easier to obtain. The price to pay is that is harder to show the boundedness.

Hence, the hard task in Critical Point Theory is to show the three points i), ii) and iii) and the difficulty in proving these points, as we will see later on, depends on the problem.

Let's now give a small preview of what awaits us in this thesis.

In Chapter 1 we will study a Schrödinger type equation, defined in the whole space $\mathbb{R}^2$ where a convolutive potential (which is none other than Newton's kernel) and a perturbed pure-power nonlinearity, weighted with a real function (see [29]), are considered. Since we are in the plane, Newton's kernel will be of logarithmic type, which will create many problems in applying the Critical Point Theory: in fact, as we will see, first of all we will have to face the ill-posedness of the associated functional in what should be the natural ambient space, that is $H^1(\mathbb{R}^2)$. To solve this problem, we will make use of a variational setting (see Section 1.6) introduced by Stubbe in [135] and then generalized by Cingolani and Weth in [49]. Once this is done, we will use the perturbation technique, which as mentioned we will recall in Section 1.5, thanks to which we will be able to provide two existence results, Theorem 1.22 and Theorem 1.26, depending on the summability conditions assumed on the weight function.

In Chapter 2 we will deal with an equation driven by a semirelativistic Schrödinger operator, where the potential has a singular part and the nonlinearity is of general type and sign-changing (see [27]). The first challenge of this problem is given precisely by the joint presence of the fractional operator and singular potential, which made it difficult to prove that the quadratic form associated with the problem generates a norm equivalent to the standard one. The challenge was solved by exploiting the representation through the Fourier transform of the operator and considering an appropriate hypothesis on the nonsingular part of the potential (see Section 2.5). Then, we concentrated on the existence, boundedness and convergence of a Cerami-sequence (Section 2.6 and Section 2.7). We emphasize that in this case, the "easy" part was proving the boundedness of the sequence. The main results of this Chapter are given by Theorem 2.21 and Theorem 2.23, which together provide an almost-characterization for the existence of a ground-state solution. Finally, in Theorem 2.27, we prove a compactness result for a sequence of ground-state solutions.

In Chapter 3, we state and prove a generalized linking-type Theorem that provides the existence of a Cerami-sequence, bounded away from the origin, for strongly indefinite problems and with sign-changing nonlinearities (see [26]). As an application of this result, we will provide the existence of a non-trivial solution for a singular Schrödinger equation, defined in the whole space $\mathbb{R}^N$ with sign-changing nonlinearity and where we assume that a part of the spectrum of the operator lies below 0 (see Section 3.4). The main result is Theorem 3.22. Moreover, thanks to an equivalence result (see Section 3.3 and Theorem 3.23 in Section 3.5.1), we prove the existence of a non-trivial solution even for a curl-curl problem (see Theorem 3.25). This type of equations are strongly related to Maxwell's equations (see Sections 3.2 and 3.3), which are of recent interest from a variational point of view (see [20–22]).

To conclude this Introduction, a few words about the presentation method. Since the problems studied here are "similar but different" to each other, we have chosen a line for which in each Chapter there will be a short Introduction more related to the problem: then, we move on to explain some of the reasons that led to its study. When entering into the details of mathematics, we first describe the result or the technique used, then we present it in a formal way and finally we apply it to the problem. The ultimate idea is to guide the reader through the difficulties encountered during the study of the individual problems towards their resolutions, in a sort of mathematical tale.

Chapter 1

A perturbed nonlinearity for a planar Schrödinger equation

In Quantum Mechanics, every particle (atom) of a gas is able to moves freely and has its own energy: in particular, bosons[1] have the feature to admit the same energy at the same time. In this situation, if the gas is cooled to a temperature very close to the absolute zero (-273,15 °C), the energy of these atoms decrease and they reduced their velocities. However, because of their quantum nature, it was observed that these atoms behave like waves that increase in size as the temperature decreases and, at a very low temperature, the size of the waves is larger than the average of the distance between two atoms. Therefore, at this very low temperature, all the bosons are in the same quantum state with the very same energy, forming a single collective quantum wave called *Bose-Einstein condensate*.

This phenomenon is described by the Nonlinear Schrödinger equation

$$i\hbar\frac{\partial\psi}{\partial t} = -\hbar^2\Delta\psi + (V(x)+a)\psi + b|\psi|^{p-1}\psi, \text{ in } \mathbb{R}\times\mathbb{R}^N, \tag{1.1}$$

where $\psi : \mathbb{R}\times\mathbb{R}^N \to \mathbb{C}$ is a complex-valued function and $p \geq 3$.

In many situation, it is interesting to look for particular type of solutions of (1.1), called *standing wave* (or stationary) solutions, that is

$$\psi(t,x) = e^{i\alpha\hbar^{-1}t}u(x), \quad u > 0.$$

This ansatz leads to the time-independent equation

$$-\hbar^2\Delta u + (\alpha + a + V(x))u = u^p.$$

Equation above is a good example of *perturbed* problem, but what do we mean with perturbed problems? These are equations where a (small) parameter, say $\varepsilon > 0$, appears as multiplication constant on the operator or in the nonlinearity, and one look for solutions as $\varepsilon \to 0$: literature referes to these as *semiclassical problems*. This kind of problems can be perturbative in nature, presenting a natural parameter (like the Planck's constant $\hbar$ above or in (1.11) below) or this parameter can be added for obtaining different results or change perspective of the problem (this actually will be the case we are going to treat in this Chapter).

For such a problem, a theory that exploit this nature was developed in order to find solutions as critical points of the associated energy functional. We recall this technique in Section 1.5.

[1]Subatomic particle that has an integer spin quantum number.

In this Chapter we present the results obtained in [29], where we deal with the study of a Choquard equation with a perturbed nonlinearity, that is

$$-\Delta u(x) + au(x) - \frac{1}{2\pi}\left(\log\frac{1}{|\cdot|} \star u^2(x)\right)u(x) = \varepsilon h(x)|u(x)|^{p-1}u(x) \tag{1.2}$$

For this problem, we are able to give two results for the existence of solutions depending on the summabilty assumptions considered on the function h (see Section 1.8).

1.1 Deriving the equation

Consider the time-dependent Schrödinger-Poisson (or Schrödinger-Newton) system

$$\begin{cases} i\Psi_t - \Delta\Psi + E(x)\Psi + \gamma\Xi\Psi = 0, & \text{in } \mathbb{R}^N \times \mathbb{R} \\ -\Delta\Xi = |\Psi|^2, & \text{in } \mathbb{R}^N, \end{cases} \tag{1.3}$$

where $\Psi : \mathbb{R}^N \times \mathbb{R} \to \mathbb{C}$ is the wave-function, $\Xi : \mathbb{R}^N \to \mathbb{R}$ is the harmonic solution of the Poisson equation, $E : \mathbb{R} \to \mathbb{R}$ is an external-potential and $\gamma \in \mathbb{R}$.

It is interesting to look for a particular type of solutions of this system that drop-off the dependence on time, the so-called *standing (or stationary) wave solutions*, i.e. solutions of the form

$$\Psi(x,t) = e^{-i\lambda t}u(x), \tag{1.4}$$

where $u : \mathbb{R}^N \to \mathbb{R}$. Indeed, we would like to solve the time-dependent Schrödinger equations looking for the particle's wave function. This particular states are predicted and allowed by the Schrödinger equation and they are very important since denote the energy of the particle considered. Indeed, from the Heisenberg's uncertainty principle it is impossibile to know for a particle both its position and its energy, but knowing energy is much more important than the position because it allows both to solve a posteriori the time-dependent Schrödinger equation for any state and for applications reasons (e.g. if we consider electrons, then its energy is a necessary condition for predicting the chemical reactivity of an atom).

Hence, using the ansatz (1.4) in system (1.3), we obtain the system

$$\begin{cases} e^{-i\lambda t}\left(\lambda u - \Delta u + E(x)u + \gamma\Xi u\right) = 0, & \text{in } \mathbb{R}^N \times \mathbb{R}, \\ \Delta\Xi = |e^{-i\lambda t}u|^2, & \text{in } \mathbb{R}^N. \end{cases} \tag{1.5}$$

and, since $|e^{iy}|^2 = 1$ for every $y \in \mathbb{R}$, system above become

$$\begin{cases} -\Delta u + V(x)u + \gamma\Xi u = 0, & \text{in } \mathbb{R}^N, \\ \Delta\Xi = |u|^2, & \text{in } \mathbb{R}^N, \end{cases} \tag{1.6}$$

where we set $V(x) = E(x) + \lambda$.

A solution for the Poisson equation is known (see Appendix A) and is given up to harmonic functions, i.e. solutions of the Laplace equation $\Delta\Xi = 0$: the formula for the solution of the Poisson equation is

$$\Xi(x) = (\Phi_N \star u^2)(x) = \int_{\mathbb{R}^N} \Phi_N(x-y)u^2(y)\,dy \tag{1.7}$$

where

$$\Phi_N(x) = \begin{cases} \frac{1}{2\pi}\log|x|, & \text{if } N = 2, \\ \frac{1}{N(N-2)\omega(N)}\frac{1}{|x|^{N-2}}, & \text{if } N \geq 3 \end{cases} \tag{1.8}$$

is the solution of the Laplace equation and $\omega(N) = \frac{\pi^{N/2}}{\Gamma(\frac{N}{2}+1)}$ is the volume of the unit ball in $\mathbb{R}^N$.

Substituting (1.7) in the Schrödinger equation in (1.6), we can observe that system (1.6) is equivalent to the single equation

$$-\Delta u + V(x)u + \gamma \left[\Phi_N \star u^2\right] u = 0, \text{ in } \mathbb{R}^N. \tag{1.9}$$

We want to remark that, depending on the dimension of the space, (1.8) takes different forms, and so (1.9): therefore, one can consider two cases and of course, both of them carry on some difficulties.

1.2 The $N \geq 3-$case

In dimension three or higer, the literature concerning the studies of (1.6) or (1.9) is very wide: we report here some of the milestones, and give some references, to which we suggest to give a look for the complete story. We focus here in works that mainly deals with problems in dimension three because, physically speaking, this is the more interesting setting, while the generalization in dimensione higher than three is often just a matter of computations. Indeed, system (1.6), as far as the time-dependent version (1.3), is used to model various phenomena from Physics, especially from Quantum Physics (we refer to [73] for a survey on this topic).

In 1954 Pekar introduced equation (1.9) with $V(x) \equiv \lambda > 0$ and $\gamma > 0$ to study the Quantum Physics of electrons at rest in an ionic crystals ([111]). Later, in 1976 at the Symposium on Coulomb System, Choquard proposed the same equation as an approximation to Hartree-Fock theory for one component plasma to describe an electron trapped in its own hole (see Chapter 2 for a quick recall of this theory). A year later, for $V \equiv 0$ and $\gamma = 1$ (hence for protons), Lieb shows the existence and the uniqueness of a minimum for the associate functional, using symmetric decreasing rearrangement inequalities: for the uniqueness, he employed a strict form of them.

In 1987, Lions ([92]) proved the existence of infinitely many distinct spherically symmetric solutions, provided the potential $V(x)$ is non-negative and radially symmetric.

After that, lots of papers appears, like ([4, 10, 11, 24, 53, 54, 82, 83, 124, 129, 130]).

From a Physical point of view, we cite the works of Penrose. In 1996, he derived system (1.6), for $N = 3$, to describe the self-gravitational collapse of a quantum mechanical system. In fact, he suggested that a superposition of two quantum states corresponding to two separated "lumps" of matter (a piece or mass of solid matter without regular shape or of no particular shape) has a total energy which is proportional to the gravitational self-energy of the gravitational field generated by the difference of the two mass distributions. Penrose suggested that there are preferred quantum states which, in the non-relativistic limit, are stationary states of the Schrödinger - Newton equations ([112–114]).

Roughly speaking, solutions of (1.6) are *basic stationary states* in which a superposition of such states must decay within a certain timescale ([46, 101, 102]).

1.3 The planar case

Why dimension two deserves an entire Section for itself? Contrary to what intuition suggests, going into a lower dimension makes the problem more difficult and some carefuls are needed. Because the use of Variational Methods in this case is not straightforward, some results were obtained via a numerical approach: in this direction, we mention the work Choquard-Stubbe-Vuffray ([45]) that, inspired by [77], proved the existence of a unique positive radially symmetric solution by applying a shooting method.

When trying to apply Variational Methods, we immediately observe that we have to face an ill-posedness problem. We try to make some order: as we say, for $N = 2$ the fundamental solution (1.7) has the form

$$\Phi_2(x) = \frac{1}{2\pi} \log |x|,$$

therefore, in $\mathbb{R}^2$, equation (1.9) become

$$-\Delta u + V(x)u + \gamma \left[\Phi_2 \star u^2\right] u = -\Delta u + V(x)u + \frac{\gamma}{2\pi} \left[\log |x| \star u^2\right] u = 0$$

and developing the convolution we obtain

$$-\Delta u + V(x)u + \frac{\gamma}{2\pi} \left(\int_{\mathbb{R}^2} \log |x - y| u^2(y)\, dy \right) u = 0.$$

We observe that, at least formally, this equation has a variational structure given by the energy functional $\mathcal{J} : H^1(\mathbb{R}^2) \to \mathbb{R}$ defined as

$$u \mapsto \mathcal{J}(u) := \frac{1}{2} \int_{\mathbb{R}^2} \left(|\nabla u|^2 + V(x)u^2\right) dx + \frac{\gamma}{8\pi} \int_{\mathbb{R}^2\times\mathbb{R}^2} \log |x - y| u^2(x)u^2(y)\, dx\, dy. \tag{1.10}$$

However, the presence of the logarithm in the Newton kernel gives rise to some difficulties that are hard to handle with: in fact, being sing-changing and unbounded, the logarithm function makes the functional $\mathcal{J}$ not well-defined in the natural Sobolev space $H^1(\mathbb{R}^2)$. This problem was faced by Stubbe in [135], who first derived the right variational setting (see Section 1.6 below) and then considered by Cingolani and Weth in [49] and by Du and Weth in [58].

1.4 Semiclassical problems

To treat this kind of problem, a Perturbation Theory that exploits this feature of the problems was developed by Ambrosetti et al. (see [7–9, 12]). We report in the next Section the main ideas of the theory, but first we recall some papers that make use of this theory.

In 1997, Ambrosetti-Badiale-Cingolani studied the following equation

$$\begin{cases} -\hbar\Delta u + \lambda u + V(x)u = |u|^{p-1}u, \text{ in } \mathbb{R}^N, \\ \lim_{|x|\to+\infty} u(x) = 0 \end{cases} \tag{1.11}$$

with $N > 2$ and $p \in (1, 2^* - 1)$. They also considered a $C^\infty(\mathbb{R}^N, \mathbb{R})$ potential V that admits a critical point at some $x_0 \in \mathbb{R}^N$.

Setting $\hbar = \varepsilon$ and making the change of variables $x \to \varepsilon x$, equation (1.11) become

$$-\Delta u + \lambda u + V(\varepsilon x)u = |u|^{p-1}u, \text{ in } \mathbb{R}^N. \tag{1.12}$$

For this problem, they used the techniques proved in [12] and they showed the existence of a semiclassical state that corresponds to the unique (up to translation) positive radial solution of

$$-\Delta u + \lambda u = |u|^{p-1}u, \text{ in } \mathbb{R}^N,$$

which can be seen as the *unperturbed* version of (1.12).

Perturbation theory has been then generalized in [7,8], and it was used in many other papers, where also differents elliptic operators are considered too. We cite [126], where the author dealed with the fractional elliptic nonlinear equation with a perturbed nonlinearity, that is

$$\sqrt{-\Delta + m^2}u + \mu u = \left(\frac{1}{|x|} \star u^2\right)(1 + \varepsilon g(x))\, u, \text{ in } \mathbb{R}^3, \tag{1.13}$$

where $\mu > 0$ is a parameter and the non-local operator $\sqrt{-\Delta + m}$ is known as *semirelativistic Schrödinger operator* (see Chapter 2 for (one of) its formal definition).

After embedded the problem in the perturbation setting, the author showed the existence of a solution for (1.13), provided $|\varepsilon|$ is small enough.

For more papers on this topic, see at [46, 48] and the references therein and also at [15], Chapter 17.

We conclude this Section pointing out that in Spring 2021, Bonheure, Cingolani and Secchi published a work [36] on the *singularly perturbed Schrödinger-Poisson system in* $\mathbb{R}^2$, that is where the pertubation is considered in the elliptic operator. The system studied is the following

$$\begin{cases} -\varepsilon^2 \Delta\psi + V(x)\psi = \Xi\psi, & \text{in } \mathbb{R}^2, \\ -\Delta\Xi = |\psi|^2, & \text{in } \mathbb{R}^2, \end{cases} \tag{1.14}$$

and they were able to prove the existence of a semiclassical solution. This work is the natural generalization of [29].

1.5 The perturbation technique

Aim of this Section is to briefly report the Perturbation Theory we will use in next Sections in order to handle our problem. For a precise treatement we remind to the seminal works of Ambrosetti-Coti Zelati-Ekeland [12], Ambrosetti-Badiale-Cingolani [9], Ambrosetti-Badiale [7, 8]. We also cite the monographs [13] and [6] (for this last, cf. Section 11.C).

Let H be an Hilbert space, $\varepsilon > 0$ and suppose to have a functional $\mathcal{J} : H \to \mathbb{R}$ of the form

$$\mathcal{J}_\varepsilon(u) = \mathcal{J}_0(u) + \varepsilon G(u), \tag{1.15}$$

where $\mathcal{J}_0 \in C^2(H, \mathbb{R})$ is called the *unperturbed functional* and $G \in C^2(H, \mathbb{R})$ is the *perturbation.*

We assume that there exists a noncompact manifold of critical point Z for $\mathcal{J}_0$, called the *critical manifold of* $\mathcal{J}_0$, of dimension $0 < d = dim(Z) < \infty$ and of class C^2. Usually in the application, the existence of such a manifold is given by the invariance of the unperturbed functional under the action of a symmetry group (e.g. the group of translation, as it will be in our case). We denote by $T_z Z$ the tangent space of the manifold Z at the point z. Since every $z \in Z$ is a critical point for $\mathcal{J}_0$, then $\mathcal{J}_0'(z)v = 0$ for every $v \in T_z Z$. Differentiating this equation, we obtain that

$$\mathcal{J}_0''(z)[v, w] = 0$$

for every $v \in T_z Z$ and $w \in H$. Hence, we are saying that $v \in T_z Z$ is a solution of the linearized equation $\mathcal{J}_0''(z)[v] = 0$, that is $v \in \ker(\mathcal{J}_0''(z))$ and it follows that

$$T_z Z \subseteq \ker(\mathcal{J}_0''(z)). \tag{1.16}$$

Therefore, $\ker(\mathcal{J}_0''(z))$ is nontrivial and its dimension is at least d[2].

We introduce the following assumption for the functional $\mathcal{J}_0''(z)$:

[2] $dim(T_z Z) = d$ since Z is of class C^2: actually, C^1 was enough.

(ND) $T_z Z = \ker(\mathcal{J}_0''(z))$, for all $z \in Z$.

(Fr) for all $z \in Z$, $\mathcal{J}_0''(z)$ is a Fredholm map of index 0.

Definition 1.1. *We say that a critical manifold is nondegenerate if it satisfies both* (ND) *and* (Fr).

A brief comments on those conditions: by (1.16) it follows that all the critical points in Z are *degenerate*[3] critical points of $\mathcal{J}_0$. Condition (ND) tells us that, since $Z = \{z - \mathcal{J}_0'(z) = 0\}$, then $w - \mathcal{J}_0''(z)[w] = 0$ for any tangent vector $w \in T_z Z$. Hence, (ND) is equivalent to ask the following:

$$\text{if } w - \mathcal{J}_0''(z)[w] = 0, \text{ then } w \in T_z Z.$$

Therefore, (ND) is a sort of *nondegeneracy* condition.

Assumption (Fr) is equivalent to ask that the linear operator $\mathcal{J}_0''(z)$ is compact (see Proposition 1.18 below).

Now, without entering in many details, what is the main idea of this theory? We supposed that the unperturbed functional admits a whole (smooth) manifold of critical points. So, in order to find critical points of (1.15) we use a sort of *bifurcation argument*: that is, we look for *nontrivial solutions* $(\varepsilon, u) \in \mathbb{R} \times H$ of (1.15), with $\varepsilon \neq 0$ and $\mathcal{J}_\varepsilon'(u) = 0$. Hence, if $z \in Z$ is the bifurcation parameter, then

$$\{0\} \times Z \subset \mathbb{R} \times H$$

is the set of the *trivial solutions*. Then, we need conditions on the functional G that allow us to find solutions that branching off from some $z \in Z$.

To do that, we use a method coming from the Theory of Bifurcation, namely a suitable adapted finite dimensional reduction in the sense of Lyapunov-Schmidt.

Let $W = (T_z Z)^\perp$, then we are looking for solutions for (1.15) of the form $u = z + w$ with $z \in Z$ and $w \in W$. We set $P : H \to W$ the *orthogonal projection*, then we can rewrite equation $\mathcal{J}_\varepsilon' = 0$ as

$$\begin{cases} P\mathcal{J}_\varepsilon'(z + w) = 0, \\ (I - P)\mathcal{J}_\varepsilon'(z + w) = 0, \end{cases} \tag{1.17}$$

where with I we are meaning the identity operator. The first equation is called the *auxiliary equation*, while the second one is called the *bifurcation equation*.

Now, the auxiliary equation can be solved by means of the Implicit Function Theorem, finding a unique solution $w(\varepsilon, z) \in W$, of class C^1 with respect $z \in Z$ and such that

$$w(\varepsilon, z) \to 0 \tag{1.18}$$

as $|\varepsilon| \to 0$, uniformly with respect $z \in Z$.

Now, we can define the perturbed critical manifold

$$Z_\varepsilon = \{u \in H : u = z_\xi + w(\varepsilon, \xi)\}$$

and this turns out to be diffeomorphic to Z. Moreover, it is (easy) to show that Z_ε is a *natural constraint* for $\mathcal{J}_\varepsilon$, namely that if $u \in Z_\varepsilon$ is a constrained critical point of $\mathcal{J}_\varepsilon$ (i.e. $\mathcal{J}'_{\varepsilon|Z_\varepsilon}(u) = 0$), then u is actually a free critical point, that is $\mathcal{J}_\varepsilon'(u) = 0$. In fact, the following result holds.

Lemma 1.2. *For ε small, Z_ε is a $d-$dimensional manifold, locally diffeomorphic to Z, with the property that if $u \in Z_\varepsilon$ is such that $\mathcal{J}'_{\varepsilon|Z_\varepsilon}(u) = 0$, then $\mathcal{J}_\varepsilon'(u) = 0$.*

[3]Critical points for which the Hessian is zero.

The proof of this Lemma can be found [9], Lemma 3.3.

Let $\Phi_\varepsilon : Z \to \mathbb{R}$, defined as

$$\Phi_\varepsilon(z) = \mathcal{J}_\varepsilon(z + w(\varepsilon, z)),$$

be the so-called *reduced functional*. Now, we can give the existence result of the Perturbation Theory.

Theorem 1.3 (see [13], Theorem 2.21)**.** *Let $\mathcal{J}_0, G \in C^2(H, \mathbb{R})$ and suppose $\mathcal{J}_0$ has a smooth critical manifold Z which is non-degenerate in the sense of Definition 1.1. Given a compact subset Z_c of Z, let us assume that Φ_ε has a critical point $z_\varepsilon \in Z_c$, provided $|\varepsilon|$ is sufficiently small.*

Then $u_\varepsilon = z_\varepsilon + w(\varepsilon, z_\varepsilon)$ is a critical point for the perturbed functional $\mathcal{J}_\varepsilon = \mathcal{J}_0 + \varepsilon G$.

The above Theorem can be applied in different ways, depending on how we treat the *reduced functional*. Next Theorem is useful if we expand Φ_ε in powers of ε (see Lemma 2.15 in [13]).

Theorem 1.4 (see [13], Theorem 2.16)**.** *Let $\mathcal{J}_0, G \in C^2(H, \mathbb{R})$ and suppose that $\mathcal{J}_0$ has a smooth critical manifold Z which is non-degenerate. Let $\bar{z} \in Z$ be a strict local maximum or minimum of $\Gamma := G_{|Z}$.*

Then, for every $|\varepsilon|$ small, the functional $\mathcal{J}_\varepsilon$ has a critical point u_ε and if $\bar{z}$ is isolated, then $u_\varepsilon \to \bar{z}$ as $\varepsilon \to 0$.

Otherwise, it is possible to study the asymptotic behaviour of $\Phi_\varepsilon(z)$ in the following sense:

$$\lim_{|\xi| \to +\infty} \Phi_\varepsilon(z) = \text{const}.$$

In this last case, we first need a technical Lemma that is the global counterpart of the existence of a unique solution for the auxiliary equation.

Lemma 1.5 (see [13], Lemma 2.21)**.** *Suppose that:*

(i) the operator $P\mathcal{J}_0''(z_\xi)$ is invertible on $W = (T_{z_\xi} Z)^\perp$ uniformly with respect to $\xi \in \mathbb{R}^2$, in the sense that there exists $C > 0$ such that

$$\left\| (P\mathcal{J}_0''(z_\xi))^{-1} \right\|_{\mathcal{L}(W,W)} \le C,$$

for every $\xi \in \mathbb{R}^2$;

(ii) the remainder $R_\xi(w) = \mathcal{J}_0'(z_\xi + w) - \mathcal{J}_0''(z_\xi)[w]$ is such that

$$R_\xi(w) = o(\|w\|)$$

as $\|w\| \to 0$, uniformly with respect to $\xi \in \mathbb{R}^2$;

(iii) there exists a constant $C_1 > 0$ such that

$$\|PG'(z_\xi + w)\| \le C_1$$

for every $\xi \in R^2$ and for every $w \in W$ such that $\|w\| \le 1$.

Then there exists $\bar{\varepsilon} > 0$ such that for every $|\varepsilon| \le \bar{\varepsilon}$ and for every $\xi \in \mathbb{R}^2$, the auxiliary equation in (1.17) has a unique solution $w_\varepsilon(z_\xi)$ and $w_\varepsilon(z_\xi) \to 0$, uniformly with respect to $\xi \in \mathbb{R}^2$.

Theorem 1.6 (see [13], Theorem 2.23). *Let $\mathcal{J}_0, G \in C^2(H,\mathbb{R})$ and assume that $\mathcal{J}_0$ has a smooth critical manifold Z which is non-degenerate. Suppose also that the assumptions of Lemma 1.5 hold and that there exists a constant $C_0 > 0$ such that*

$$\lim_{|\xi|\to+\infty} \Phi_\varepsilon(z_\xi) = C_0,$$

uniformly with respect to $|\varepsilon|$ small.

Then, for $|\varepsilon|$ small, the perturbed functional $\mathcal{J}_\varepsilon = \mathcal{J}_0 + \varepsilon G$ has a critical point.

1.6 Variational framework

Now, we are ready to begin the study of our equation: we recall that we are dealing with the Choquard equation

$$-\Delta u + V(x)u + \frac{\gamma}{2\pi}\left(\int_{\mathbb{R}^2} \log|x-y|u^2(y)\,dy\right)u = 0, \text{ in } \mathbb{R}^2.$$

As told at the end of Section 1.3, we need to face a ill-posedness problem for the associated functional (1.10) because of the presence of the logarithm in the convolution term. To overcome this difficulty, Stubbe in [135] derives a suitable variational framework for the homogeneous equation

$$-\Delta u + au - \frac{1}{2\pi}\left(\log\frac{1}{|\cdot|} \star u^2\right)u = 0, \text{ in } \mathbb{R}^2. \tag{1.19}$$

that has been then developed by Cingolani-Weth (see [49]). Given its importance in the treatment of this problem, we recall here the variational setting for the homogeneous equation (1.19), but the same framework will also fit for the inhomogeneous case, with slightly changes, considering "good" assumptions on the source term (see next Section). We prefer to present it in this way to set notation for the upcoming Sections of this Chapter.

We recall the, up to now, formally energy functional associated to (1.19) is

$$\mathcal{J}(u) := \frac{1}{2}\int_{\mathbb{R}^2}\left(|\nabla u|^2 + V(x)u^2\right)dx + \frac{\gamma}{8\pi}\int_{\mathbb{R}^2\times\mathbb{R}^2}\log|x-y|u^2(x)u^2(y)\,dx\,dy.$$

Let $V:\mathbb{R}\to\mathbb{R}$ such that $\inf_{\mathbb{R}} V > 0$ and consider the inner product

$$(u,v) := \int_{\mathbb{R}^2}\left(\nabla u(x)\cdot\nabla v(x) + V(x)u(x)v(x)\right)dx$$

for every $u, v \in H^1(\mathbb{R}^2)$ and the corresponding norm

$$\|u\|^2_{H^1(\mathbb{R}^2)} := \int_{\mathbb{R}^2}\left(|\nabla u(x)|^2 + V(x)u^2\right)dx$$

for every $u \in H^1(\mathbb{R}^2)$. Moreover, for every measurable function $u:\mathbb{R}^2\to\mathbb{R}$, we set the seminorm

$$|u|_*^2 := \int_{\mathbb{R}^2}\log(1+|x|)u^2(x)\,dx.$$

Now, we consider the Hilbert space

$$X := \left\{u \in H^1(\mathbb{R}^2) : |u|_*^2 < \infty\right\}$$

endowed with the norm

$$\|u\|_X^2 := \|u\|^2_{H^1(\mathbb{R}^2)} + |u|_*^2.$$

This is an equivalent norm with respect to the standard one in $H^1(\mathbb{R}^2)$, i.e. there exists $0 < C_1 \leq C_2$ such that

$$C_1 \int_{\mathbb{R}^2} \left(|\nabla u|^2 + u^2\right) dx \leq \|u\|_X^2 \leq C_2 \int_{\mathbb{R}^2} \left(|\nabla u|^2 + u^2\right) dx,$$

for every $u \in X$.

When a new space is setted, then it would be nice to have an embedding property, preferably in some Lebesuge spaces. Luckily, the following Proposition holds.

Proposition 1.7. *The space X is compactly embedded in $L^s(\mathbb{R}^2)$ for all $s \in [2, \infty)$.*

Proof. We report the proof from Proposition 2.1 in [28] (see also [49]), that relies on the Riesz criterion (see Theorem A.2 in Appendix A).

Let S be a bounded subset of X and because of the continuous injection $X \subset L^t(\mathbb{R}^2)$, for every $t \in [2, +\infty)$ (see [38], Corollary 9.10), we have that S is bounded also in every $L^t(\mathbb{R}^2)$ with $t \in [2, +\infty)$.

We take $R > 0$, $p \in [2, +\infty)$ and $u \in S$ and by Hölder inequality (with $s = s' = 2$) we have that there exists $C > 0$ such that

$$\begin{aligned}
\int_{\{|x|>R\}} |u(x)|^p \, dx &= \int_{\{|x|>R\}} |u(x)|^{p-1} |u(x)| \, dx \\
&\leq \left(\int_{\{|x|>R\}} |u(x)|^{(p-1)2} \, dx \right)^{\frac{1}{2}} \left(\int_{\{|x|>R\}} u^2(x) \, dx \right)^{\frac{1}{2}} \\
&\leq \|u\|_{L^{2p-2}(\mathbb{R}^2)}^{p-1} \left(\int_{\{|x|>R\}} u^2(x) \, dx \right)^{\frac{1}{2}} \leq C \left(\int_{\{|x|>R\}} u^2(x) \, dx \right)^{\frac{1}{2}},
\end{aligned}$$

since $2p - 2 \geq 2$ and for some $C > 0$.

We compute separately

$$\int_{\{|x|>R\}} u^2(x) \, dx \leq \frac{1}{\ln(1+R)} \int_{\{|x|>R\}} \log(1+|x|) u^2(x) \, dx = \frac{1}{\ln(1+R)} |u|_* \leq \frac{C}{\ln(1+R)}.$$

Setting $\widetilde{C} := \widetilde{C}(R) = \frac{C}{\ln(1+R)}$, we get

$$\int_{\{|x|>R\}} |u(x)|^p \, dx \leq \widetilde{C}.$$

It remains to prove that

$$\lim_{|h|\to 0} \|\tau_h u - u\|_{L^2(\mathbb{R}^2)} = 0 \text{ uniformly in } u \in B(0,1) \subset \mathbb{R}^2. \tag{1.20}$$

We recall that $u \in H^1(\mathbb{R}^2)$, so the following inequality holds (Proposition 9.3 in [38])

$$\|\tau_h u - u\|_{L^2(\mathbb{R}^2)} \leq |h| \|\nabla u\|_{L^2(\mathbb{R}^2)};$$

but since $u \in H^1(\mathbb{R}^2)$ we have that $\|\nabla u\|_{L^2(\mathbb{R}^2)}$ if finite, i.e. there exists a constant $C > 0$ such that

$$\|\tau_h u - u\|_{L^2(\mathbb{R}^2)} \leq C|h|,$$

with C independent from $B(0,1)$.

Letting $|h| \to 0$, we obtain (1.20). □

For the sake of simplicity (especially in future computations) we define the following bilinear forms:

$$B_1(u,v) = \int_{\mathbb{R}^2\times\mathbb{R}^2} \log(1+|x-y|)u(x)v(y)\,dx\,dy \tag{1.21}$$

$$B_2(u,v) = \int_{\mathbb{R}^2\times\mathbb{R}^2} \log\left(1+\frac{1}{|x-y|}\right)u(x)v(y)\,dx\,dy. \tag{1.22}$$

Since for every $r>0$ the following relations hold

$$\log r = \log(1+r) - \log\left(1+\frac{1}{r}\right), \tag{1.23}$$

we also define

$$B(u,v) = B_1(u,v) - B_2(u,v) = \int_{\mathbb{R}^2\times\mathbb{R}^2} \log|x-y|u(x)v(y)\,dx\,dy. \tag{1.24}$$

We point out that the bilinear forms are symmetric and defined only for measurable functions $u,v:\mathbb{R}^2\to\mathbb{R}$ such that the integrals are Lebesgue well-defined.

In the next Proposition, we recall some useful estimates for the bilinear forms in terms of the H^1-norm and the seminorm.

Proposition 1.8 (see [49]). *The following estimates hold:*

- *for every $u,v,w,z\in L^2(\mathbb{R}^2)$*

$$\begin{aligned} B_1(uv,wz) &\le \int_{\mathbb{R}^2\times\mathbb{R}^2} \left(\log(1+|x|)+\log(1+|y|)\right)|u(x)v(x)||w(y)z(y)|\,dx\,dy \\ &\le |u|_*|v|_*\|w\|_{L^2(\mathbb{R}^2)}\|z\|_{L^2(\mathbb{R}^2)} + \|u\|_{L^2(\mathbb{R}^2)}\|v\|_{L^2(\mathbb{R}^2)}|w|_*|z|_*; \end{aligned} \tag{1.25}$$

- *for every $u,v\in L^{\frac{4}{3}}(\mathbb{R}^2)$ there exists a constant $C>0$*

$$|B_2(u,v)| \le \int_{\mathbb{R}^2\times\mathbb{R}^2} \frac{1}{|x-y|}|u(x)v(y)|\,dx\,dy \le C\|u\|_{L^{\frac{4}{3}}(\mathbb{R}^2)}\|v\|_{L^{\frac{4}{3}}(\mathbb{R}^2)}. \tag{1.26}$$

Proof. We start with B_1. We observe that, since $1+|x|+|y| \le (1+|x|)(1+|y|)$ for every $x,y\in\mathbb{R}^2$, the following chain of inequalities holds:

$$0 \le \log(1+|x-y|) \le \log(1+|x|+|y|) \le \log(1+|x|)+\log(1+|y|).$$

Then, we have

$$\begin{aligned} |B_1(uv,wz)| &\le \int_{\mathbb{R}^2\times\mathbb{R}^2} \log(1+|x-y|)|u(x)v(x)||w(y)z(y)|\,dx\,dy \\ &\le \int_{\mathbb{R}^2\times\mathbb{R}^2} \left(\log(1+|x|)+\log(1+|x|)\right)|u(x)v(x)||w(y)z(y)|\,dx\,dy \\ &\le \int_{\mathbb{R}^2\times\mathbb{R}^2} \log(1+|x|)|u(x)v(x)||w(y)z(y)|\,dx\,dy \\ &\quad + \int_{\mathbb{R}^2\times\mathbb{R}^2} \log(1+|y|)|w(y)z(y)||u(x)v(x)|\,dx\,dy \\ &= \int_{\mathbb{R}^2} \log(1+|x|)|u(x)v(x)|\,dx \int_{\mathbb{R}^2}|w(y)z(y)|\,dy \\ &\quad + \int_{\mathbb{R}^2} \log(1+|y|)|w(y)z(y)|\,dy \int_{\mathbb{R}^2}|u(x)v(x)|\,dx \end{aligned}$$

$$= |uv|_*(w,z)_{L^2(\mathbb{R}^2)} + |wz|_*(u,v)_{L^2(\mathbb{R}^2)}$$
$$\leq |u|_*|v|_*\|w\|_{L^2(\mathbb{R}^2)}\|z\|_{L^2(\mathbb{R}^2)} + |w|_*|z|_*\|u\|_{L^2(\mathbb{R}^2)}\|v\|_{L^2(\mathbb{R}^2)},$$

for $u, v, w, z \in L^2(\mathbb{R}^2)$, where in the last inequality we also used the Cauchy-Schwarz inequality.

Now, we estimate B_2. We make use of the following inequality: for every $r > 0$

$$0 \leq \log(1 + r) \leq r.$$

Hence, we apply the Hardy-Littlewood-Sobolev inequality (see Appendix A, Theorem A.3) with $p = q$, $\lambda = 1$ and $N = 2$; so, there exists a constant $C := C(N) > 0$ such that

$$|B_2(u,v)| \leq \int_{\mathbb{R}^2\times\mathbb{R}^2} \log\frac{1}{|x-y|}|u(x)v(y)|\,dx\,dy \leq \int_{\mathbb{R}^2\times\mathbb{R}^2} \frac{1}{|x-y|}|u(x)||v(y)|\,dx\,dy$$
$$\leq C\|u\|_{L^{\frac{4}{3}}(\mathbb{R}^2)}\|v\|_{L^{\frac{4}{3}}(\mathbb{R}^2)},$$

for $u, v \in L^{\frac{4}{3}}(\mathbb{R}^2)$. □

Remark 1.9. *In (1.25) we are using the conventions* $\infty \cdot 0 = 0$ *and* $\infty \cdot s = \infty$ *for every* $s > 0$.

In the proof of (1.26), we obtain a posteriori that B_2 *takes value only in* $L^{\frac{4}{3}}(\mathbb{R}^2)$. *Consequently, we anticipate that* F_2 *will take values only in* $L^{\frac{8}{3}}(\mathbb{R}^2)$ *(see below).*

As a particular case we set the functionals $F_1 : H^1(\mathbb{R}^2) \to \mathbb{R}$ and $F_2 : L^{\frac{8}{3}}(\mathbb{R}^2) \to \mathbb{R}$, defined as

$$F_1(u) = B_1(u^2, u^2) = \int_{\mathbb{R}^2\times\mathbb{R}^2} \log(1 + |x-y|)u^2(x)u^2(y)\,dx\,dy \tag{1.27}$$
$$F_2(u) = B_2(u^2, u^2) = \int_{\mathbb{R}^2\times\mathbb{R}^2} \log\left(1 + \frac{1}{|x-y|}\right)u^2(x)u^2(y)\,dx\,dy \tag{1.28}$$

and of course

$$F(u) = F_1(u) - F_2(u) = \int_{\mathbb{R}^2\times\mathbb{R}^2} \log|x-y|u^2(x)u^2(y)\,dx\,dy. \tag{1.29}$$

Then, we have the following Corollary.

Corollary 1.10 (see [49]). *The following estimates hold:*

- *for every* $u \in L^2(\mathbb{R}^2)$
$$|F_1(u)| \leq 2|u|_*^2\|u\|^2_{L^2(\mathbb{R}^2)}; \tag{1.30}$$
- *for every* $u \in L^{\frac{8}{3}}(\mathbb{R}^2)$
$$|F_2(u)| \leq C\|u\|^4_{L^{\frac{8}{3}}(\mathbb{R}^2)}. \tag{1.31}$$

Proof. From (1.25) we have that

$$|F_1(u)| = |B_1(u^2,u^2)| \leq |u|_*^2\|u\|^2_{L^2(\mathbb{R}^2)} + \|u\|^2_{L^2(\mathbb{R}^2)}|u|_*^2 = 2|u|_*^2\|u\|^2_{L^2(\mathbb{R}^2)}.$$

To obtain (1.31), we observe that from (1.26) it follows that there exists a constant $C > 0$ such that

$$|F_2(u)| = |B_2(u^2,u^2)| \leq C\|u^2\|_{L^{\frac{4}{3}}(\mathbb{R}^2)}\|u^2\|_{L^{\frac{4}{3}}(\mathbb{R}^2)} = C\|u^2\|^2_{L^{\frac{4}{3}}(\mathbb{R}^2)} = C\left[\left(\int_{\mathbb{R}^2}|u^2(x)|^{\frac{4}{3}}\,dx\right)^{\frac{3}{4}}\right]^2$$
$$= C\left[\left(\int_{\mathbb{R}^2}|u(x)|^{\frac{8}{3}}\,dx\right)^{\frac{3}{8}}\right]^4 = C\|u\|^4_{L^{\frac{8}{3}}(\mathbb{R}^2)}.$$

□

The following Proposition will be used below to show the regularity of the functional $\mathcal{J}$.

Proposition 1.11. *The functionals $F_1 : H^1(\mathbb{R}^2) \to \mathbb{R}_0^+$, $F_2 : L^{\frac{8}{3}}(\mathbb{R}^2) \to \mathbb{R}_0^+$ and $F : H^1(\mathbb{R}^2) \to \widetilde{\mathbb{R}}$ are of class C^2 on X.*

Moreover, we have that for any $u, v, w \in X$

$$\begin{aligned}
F_1'(u)v &= 4B_1(u^2, uv) & F_1''(u)[v,w] &= 4B_1(u^2, vw) + 8B_1(uv, uw);\\
F_2'(u)v &= 4B_2(u^2, uv) & F_2''(u)[v,w] &= 4B_2(u^2, vw) + 8B_2(uv, uw);\\
F'(u)v &= 4B(u^2, uv) & F''(u)[v,w] &= 4B(u^2, vw) + 8B(uv, uw).
\end{aligned}$$

Proof. *F_1 is continuous on X.* Let $u_n, u \in X$ be such that $u_n \to u$. We need to estimate,

$$|F_1(u_n) - F_1(u)| = \left| \int_{\mathbb{R}^2\times\mathbb{R}^2} \log(1 + |x-y|) \left(u_n^2(x)u_n^2(y) - u^2(x)u^2(y) \right) dx\, dy \right|.$$

We add and subtract $\int_{\mathbb{R}^2\times\mathbb{R}^2} \log(1+|x-y|)u_n^2(x)u^2(y)\,dx\,dy$ and by (1.25) and Proposition 1.7 we obtain

$$\begin{aligned}
|F_1(u_n) - F_1(u)| &\le \int_{\mathbb{R}^2\times\mathbb{R}^2} \log(1+|x-y|)u_n^2(x) \left| u_n^2(y) - u^2(y) \right| dx\, dy\\
&\quad + \int_{\mathbb{R}^2\times\mathbb{R}^2} \log(1+|x-y|) \left| u_n^2(x) - u^2(x) \right| u^2(y)\, dx\, dy\\
&\le \int_{\mathbb{R}^2\times\mathbb{R}^2} [\log(1+|x|) + \log(1+|y|)]\, u_n^2(x)\, |u_n(y) - u(y)|\, |u_n(y) + u(y)|\, dx\, dy\\
&\quad + \int_{\mathbb{R}^2\times\mathbb{R}^2} [\log(1+|x|) + \log(1+|y|)]\, |u_n(y) - u(y)|\, |u_n(y) + u(y)|\, u^2(y)\, dx\, dy\\
&\le |u_n|_*^2 \|u_n - u\|_{L^2(\mathbb{R}^2)} \|u_n + u\|_{L^2(\mathbb{R}^2)} + \|u_n\|_{L^2(\mathbb{R}^2)}^2 |u_n - u|_* |u_n + u|_*\\
&\quad + |u|_*^2 \|u_n - u\|_{L^2(\mathbb{R}^2)} \|u_n + u\|_{L^2(\mathbb{R}^2)} + \|u\|_{L^2(\mathbb{R}^2)}^2 |u_n - u|_* |u_n + u|_*\\
&\le C\|u_n - u\|_X
\end{aligned}$$

and this goes to 0 as $n \to \infty$.

F_1 is of class $C^1(X)$. The first Gâteaux derivative of F_1 at $u \in X$ along direction $v \in X$ is

$$F_1'(u)v = 4\int_{\mathbb{R}^2\times\mathbb{R}^2} \log(1+|x-y|)u^2(x)u(y)v(y)\, dx\, dy = 4B_1(u^2, uv)$$

and we observe that, by (1.25)

$$\begin{aligned}
|F_1'(u)v| &\le 4\int_{\mathbb{R}^2\times\mathbb{R}^2} \log(1+|x-y|)u^2(x)|u(y)v(y)|\, dx\, dy\\
&\le |u|_*^2 \|u\|_{L^2(\mathbb{R}^2)} \|v\|_{L^2(\mathbb{R}^2)} + |u|_* |v|_* \|u\|_{L^2(\mathbb{R}^2)}^2\\
&\le 2\|u\|_X^3 \|v\|_X < +\infty.
\end{aligned}$$

As a consequence, $F_1'(u) \in X^*$ and

$$\|F_1'(u)\|_{X^*} = \sup_{v\in X:\|v\|\le 1} |F_1'(u)v| \le C\|u\|_X^3.$$

Now, we show that F_1' is continuous, i.e. $F_1'(u_n) \to F'(u)$ in X^* if $u_n \to u$ in X, where $u_n, u \in X$. Let $v \in X$, we add and subtract

$$\int_{\mathbb{R}^2\times\mathbb{R}^2} \log(1+|x-y|)u^2(x)u_n(y)v(y)\, dx\, dy$$

and using again (1.25) we have

$$\begin{aligned}
\frac{|F_1'(u_n)v - F_1'(u)v|}{4} &= \int_{\mathbb{R}^2\times\mathbb{R}^2} \log(1+|x-y|)\left(u_n^2(x)u_n(y) - u^2(x)u(y)\right)v(y)\,dx\,dy \\
&= \int_{\mathbb{R}^2\times\mathbb{R}^2} \log(1+|x-y|)\left(u_n^2(x) - u^2(x)\right)u_n(y)v(y)\,dx\,dy \\
&\quad + \int_{\mathbb{R}^2\times\mathbb{R}^2} \log(1+|x-y|)\,(u_n(y)-u(y))\,u^2(x)v(y)\,dx\,dy \\
&= \int_{\mathbb{R}^2\times\mathbb{R}^2} [\log(1+|x|)+\log(1+|x|)]\,|u_n(x)-u(x)||u_n(x)+u(x)||u_n(y)||v(y)|\,dx\,dy \\
&\quad + \int_{\mathbb{R}^2\times\mathbb{R}^2} [\log(1+|x|)+\log(1+|x|)]\,|u^2(x)||u_n(y)-u(y)||v(y)|\,dx\,dy \\
&\le |u_n-u|_*|u_n+u|_*\|u_n\|_{L^2(\mathbb{R}^2)}\|v\|_{L^2(\mathbb{R}^2)} + |u_n|_*|v|_*\|u_n-u\|_{L^2(\mathbb{R}^2)}\|u_n+u\|_{L^2(\mathbb{R}^2)} \\
&\quad + |u|_*^2\|u_n-u\|_{L^2(\mathbb{R}^2)}\|v\|_{L^2(\mathbb{R}^2)} + |u_n-u|_*|v|_*\|u_n\|^2_{L^2(\mathbb{R}^2)} \\
&\le C\|u_n-u\|_X\|v\|_X,
\end{aligned}$$

and this goes to 0 as n diverges.

F_1 *is of class* $C^2(X)$. The second Gâteaux derivative of F_1 at $u \in X$ in the direction $v, w \in X$ is

$$\begin{aligned}
F_1''(u)[v,w] &= 8\int_{\mathbb{R}^2\times\mathbb{R}^2} \log(1+|x-y|)u(x)v(x)u(y)w(y)\,dx\,dy \\
&\quad + 4\int_{\mathbb{R}^2\times\mathbb{R}^2} \log(1+|x-y|)u^2(x)v(y)w(y)\,dx\,dy \\
&= 4B_1(u^2, vw) + 8B_1(uv, uw)
\end{aligned}$$

and, as for the first derivative, we observe that

$$\begin{aligned}
F_1''(u)[v,w] &\le 4\int_{\mathbb{R}^2\times\mathbb{R}^2} \log(1+|x-y|)u^2(x)|v(y)w(y)|\,dx\,dy \\
&\quad + 8\int_{\mathbb{R}^2\times\mathbb{R}^2} \log(1+|x-y|)|u(x)v(x)||u(y)w(y)|\,dx\,dy \\
&\le |u|_*^2\|v\|_{L^2(\mathbb{R}^2)}\|w\|_{L^2(\mathbb{R}^2)} + |v|_*|w|_*\|u\|^2_{L^2(\mathbb{R}^2)} \\
&\quad + |u|_*|v|_*\|u\|_{L^2(\mathbb{R}^2)}\|w\|_{L^2(\mathbb{R}^2)} + |u|_*|w|_*\|u\|_{L^2(\mathbb{R}^2)}\|v\|_{L^2(\mathbb{R}^2)} \\
&= 4\|u\|_X^2\|v\|_X\|w\|_X < +\infty.
\end{aligned}$$

Therefore, $F_1''(u)v \in X^*$ and

$$\|F_1''(u)v\|_{X^*} = \sup_{w\in X:\|w\|\le 1} |F_1''(u)[v,w]| \le 4\|u\|_X^2\|v\|_X.$$

Let $u_n, u \in X$ such that $u_n \to u$ in X and compute

$$\begin{aligned}
&|F_1''(u_n)[v,w] - F_1''(u)[v,w]| \\
&\qquad = 4\int_{\mathbb{R}^2\times\mathbb{R}^2} \log(1+|x-y|)\left(u_n^2(x)-u^2(x)\right)v(y)w(y)\,dx\,dy \\
&\qquad\quad + 8\int_{\mathbb{R}^2\times\mathbb{R}^2} \log(1+|x-y|)\,(u_n(x)u_n(y)-u(x)u(y))\,v(x)w(y)\,dx\,dy;
\end{aligned}$$

adding and subtract $8\int_{\mathbb{R}^2\times\mathbb{R}^2}\log(1+|x-y|)u(x)v(x)u_n(y)w(y)\,dx\,dy$ we obtain

$$\begin{aligned}
&|F_1''(u_n)[v,w] - F_1''(u)[v,w]| \\
&= 4\int_{\mathbb{R}^2\times\mathbb{R}^2} \log(1+|x-y|)u_n^2(x)v(y)w(y)\,dx\,dy \\
&\quad + 8\int_{\mathbb{R}^2\times\mathbb{R}^2} \log(1+|x-y|)\,[u_n(x)-u(x)]\,v(x)u_n(y)w(y)\,dx\,dy \\
&\quad - 8\int_{\mathbb{R}^2\times\mathbb{R}^2} \log(1+|x-y|)u(x)v(x)\,[u_n(y)-u(y)]\,w(y)\,dx\,dy \\
&\le 4\int_{\mathbb{R}^2\times\mathbb{R}^2} (\log(1+|x|)+\log(1+|y|))\,u_n^2(x)v(y)w(y)\,dx\,dy \\
&\quad + 8\int_{\mathbb{R}^2\times\mathbb{R}^2} (\log(1+|x|)+\log(1+|y|))\,[u_n(x)-u(x)]\,v(x)u_n(y)w(y)\,dx\,dy \\
&\quad - 8\int_{\mathbb{R}^2\times\mathbb{R}^2} (\log(1+|x|)+\log(1+|y|))\,u(x)v(x)\,[u_n(y)-u(y)]\,w(y)\,dx\,dy \\
&\le |u_n|_*^2\|v\|_{L^2(\mathbb{R}^2)}\|w\|_{L^2(\mathbb{R}^2)} + |v|_*|w|_*\|u_n\|^2_{L^2(\mathbb{R}^2)} \\
&\quad + |u_n-u|_*|v|_*\|u_n\|_{L^2(\mathbb{R}^2)}\|w\|_{L^2(\mathbb{R}^2)} + |u_n|_*|w|_*\|u_n-u\|_{L^2(\mathbb{R}^2)}\|v\|_{L^2(\mathbb{R}^2)} \\
&\quad + |u|_*|v|_*\|u_n-u\|_{L^2(\mathbb{R}^2)}\|w\|_{L^2(\mathbb{R}^2)} + |u_n-u|_*|w|_*\|u\|_{L^2(\mathbb{R}^2)}\|v\|_{L^2(\mathbb{R}^2)}
\end{aligned}$$

hence,

$$\begin{aligned}
&|F_1''(u_n)[v,w] - F_1''(u)[v,w]| \le \|u_n\|_X^2\|v\|_X\|w\|_X + \|v\|_X\|w\|_X\|u_n\|_X^2 \\
&\quad + \|u_n-u\|_X\|v\|_X\|u_n\|_X\|w\|_X + \|u_n\|_X\|w\|_X\|u_n-u\|_X\|v\|_X \\
&\quad + \|u\|_X\|v\|_X\|u_n-u\|_X\|w\|_X + \|u_n-u\|_X\|w\|_X\|u\|_X\|v\|_X \\
&= \|u_n\|_X^2\|v\|_X\|w\|_X + \|v\|_X\|w\|_X\|u_n\|_X^2 \\
&\quad + \|u_n-u\|_X\,[2\|u_n\|_X\|v\|_X\|w\|_X + 2\|u\|_X\|v\|_X\|w\|_X]
\end{aligned}$$

that tends to 0 as $n \to +\infty$.

This ends the proof that F_1 is of class C^2 on X.

In the same way, heavily using (1.26), we prove the regularity for F_2.

F_2 is continuous on X. Let $u_n, u \in X$ be such that $u_n \to u$ in X, then

$$|F_2(u_n) - F_2(u)| = \left|\int_{\mathbb{R}^2\times\mathbb{R}^2} \log\left(1+\frac{1}{|x-y|}\right)\left(u_n^2(x)u_n^2(y) - u^2(x)u^2(y)\right)\,dx\,dy\right|.$$

We add and subtract $\int_{\mathbb{R}^2\times\mathbb{R}^2} \log\left(1+\frac{1}{|x-y|}\right)u_n^2(x)u^2(y)\,dx\,dy$, by (1.26) and the fact that $0 \le \log(1+r) \le r$ for $r>0$, we have

$$\begin{aligned}
&|F_2(u_n) - F_2(u)| \le \int_{\mathbb{R}^2\times\mathbb{R}^2} \log\left|1+\frac{1}{|x-y|}\right| u_n^2(x)\left(u_n^2(y)-u^2(y)\right)\,dx\,dy \\
&\quad + \int_{\mathbb{R}^2\times\mathbb{R}^2} \log\left(1+\frac{1}{|x-y|}\right)\left|u_n^2(x)-u^2(x)\right| u^2(y)\,dx\,dy \\
&\le \int_{\mathbb{R}^2\times\mathbb{R}^2} \frac{1}{|x-y|} u_n^2(x)|u_n(y)-u(y)||u_n(y)+u(y)|\,dx\,dy \\
&\quad + \int_{\mathbb{R}^2\times\mathbb{R}^2} \frac{1}{|x-y|}|u_n(x)-u(x)||u_n(x)+u(x)|u^2(y)\,dx\,dy \\
&\le C_1\|u_n\|^2_{L^{\frac{8}{3}}(\mathbb{R}^2)}\left(\int_{\mathbb{R}^2}(|u_n(y)-u(y)||u_n(y)+u(y)|)^{\frac{4}{3}}\,dy\right)^{\frac{3}{4}}
\end{aligned}$$

$$
\begin{aligned}
&+ C_2 \left(\int_{\mathbb{R}^2} \left(|u_n(y) - u(y)||u_n(y) + u(y)| \right)^{\frac{4}{3}} \, dy \right)^{\frac{3}{4}} \|u\|^2_{L^{\frac{8}{3}}(\mathbb{R}^2)} \\
&\leq C_1 \|u_n\|^2_{L^{\frac{8}{3}}(\mathbb{R}^2)} \left[\left(\int_{\mathbb{R}^2} \left| u_n(y) - u(y) |^{\frac{4}{3}} \right)^2 dy \right)^{\frac{1}{2}} \left(\int_{\mathbb{R}^2} \left| u_n(y) + u(y) |^{\frac{4}{3}} \right)^2 dy \right)^{\frac{1}{2}} \right]^{\frac{3}{4}} \\
&+ C_2 \left[\left(\int_{\mathbb{R}^2} \left| u_n(y) - u(y) |^{\frac{4}{3}} \right)^2 dy \right)^{\frac{1}{2}} \left(\int_{\mathbb{R}^2} \left| u_n(y) + u(y) |^{\frac{4}{3}} \right)^2 dy \right)^{\frac{1}{2}} \right]^{\frac{3}{4}} \|u\|^2_{L^{\frac{8}{3}}(\mathbb{R}^2)} \\
&\leq C_1 \|u_n\|^2_{L^{\frac{8}{3}}(\mathbb{R}^2)} \|u_n - u\|_{L^{\frac{8}{3}}(\mathbb{R}^2)} \|u_n + u\|_{L^{\frac{8}{3}}(\mathbb{R}^2)} \\
&+ C_2 \|u_n - u\|_{L^{\frac{8}{3}}(\mathbb{R}^2)} \|u_n + u\|_{L^{\frac{8}{3}}(\mathbb{R}^2)} \|u\|^2_{L^{\frac{8}{3}}(\mathbb{R}^2)} \\
&\leq C_3 \|u_n - u\|_{L^{\frac{8}{3}}(\mathbb{R}^2)} \|u_n + u\|_{L^{\frac{8}{3}}(\mathbb{R}^2)} \left(\|u_n\|^2_{L^{\frac{8}{3}}(\mathbb{R}^2)} + \|u\|^2_{L^{\frac{8}{3}}(\mathbb{R}^2)} \right) \\
&\leq C_3 \|u_n - u\|_{L^{\frac{8}{3}}(\mathbb{R}^2)} \left(\|u_n\|_{L^{\frac{8}{3}}(\mathbb{R}^2)} + \|u\|_{L^{\frac{8}{3}}(\mathbb{R}^2)} \right) \left(\|u_n\|^2_{L^{\frac{8}{3}}(\mathbb{R}^2)} + \|u\|^2_{L^{\frac{8}{3}}(\mathbb{R}^2)} \right),
\end{aligned}
$$

where we also used Hölder inequality, and this tends to 0 as $n \to +\infty$.

F_2 is of class $C^1(X)$. The first Gâteaux derivative of F_2 at point $u \in X$ along $v \in X$ is

$$
F_2'(u)v = 4 \int_{\mathbb{R}^2 \times \mathbb{R}^2} \log \left(1 + \frac{1}{|x-y|} \right) u^2(x) u(y) v(y) \, dx \, dy = 4B_2(u^2, uv).
$$

Observe that, by (1.26), there exists a positive constant C such that

$$
\begin{aligned}
|F_2'(u)v| &\leq 4 \int_{\mathbb{R}^2 \times \mathbb{R}^2} \frac{1}{|x-y|} u^2(x) u(y) v(y) \, dx \, dy \\
&\leq C \|u\|^2_{L^{\frac{8}{3}}(\mathbb{R}^2)} \|u\|_{L^{\frac{4}{3}}(\mathbb{R}^2)} \|v\|_{L^{\frac{4}{3}}(\mathbb{R}^2)} \\
&\leq C \|u\|^3_X \|v\|_X < +\infty.
\end{aligned}
$$

As for F_1, we can deduce that $F_2'(u) \in X^*$ and

$$
\|F_2'(u)\|_{X^*} = \sup_{v \in X : \|v\| \leq 1} |F_2'(u)v| \leq C \|u\|^3_X.
$$

Now, let $u_n, u \in X$ be such that $u_n \to u$ in X, we add and subtract $\int_{\mathbb{R}^2 \times \mathbb{R}^2} \log \left(1 + \frac{1}{|x-y|} \right) u^2(x) u_n(y) v(y) \, dx \, dy$ and we estimate

$$
\begin{aligned}
&\frac{|F_2'(u_n)v - F_2'(u)v|}{4} \\
&= \int_{\mathbb{R}^2 \times \mathbb{R}^2} \log \left(1 + \frac{1}{|x-y|} \right) \left(u_n^2(x) u(y) - u^2(x) u(y) \right) v(y) \, dx \, dy \\
&= \int_{\mathbb{R}^2 \times \mathbb{R}^2} \log \left(1 + \frac{1}{|x-y|} \right) \left(u_n^2(x) - u^2(x) \right) u_n(y) v(y) \, dx \, dy \\
&\quad + \int_{\mathbb{R}^2 \times \mathbb{R}^2} \log \left(1 + \frac{1}{|x-y|} \right) u^2(x) \left(u_n(y) - u(y) \right) v(y) \, dx \, dy \\
&\leq \int_{\mathbb{R}^2 \times \mathbb{R}^2} \frac{1}{|x-y|} \left| u_n^2(x) - u^2(x) \right| |u_n(y)||v(y)| \, dx \, dy \\
&\quad + \int_{\mathbb{R}^2 \times \mathbb{R}^2} \frac{1}{|x-y|} u^2(x) \left| u_n(y) - u(y) \right| |v(y)| \, dx \, dy \\
&\leq C_1 \|u_n^2 - u^2\|_{L^{\frac{4}{3}}(\mathbb{R}^2)} \|u_n\|_{L^{\frac{8}{3}}(\mathbb{R}^2)} \|v\|_{L^{\frac{8}{3}}(\mathbb{R}^2)}
\end{aligned}
$$

$$
\begin{aligned}
&\quad + C_2\|u\|^2_{L^{\frac{8}{3}}(\mathbb{R}^2)}\|u_n-u\|_{L^{\frac{8}{3}}(\mathbb{R}^2)}\|v\|_{L^{\frac{8}{3}}(\mathbb{R}^2)}\\
&\le C_1\|u_n-u\|_{L^{\frac{8}{3}}(\mathbb{R}^2)}\|u_n+u\|_{L^{\frac{8}{3}}(\mathbb{R}^2)}\|u_n\|_{L^{\frac{8}{3}}(\mathbb{R}^2)}\|v\|_{L^{\frac{8}{3}}(\mathbb{R}^2)}\\
&\quad + C_2\|u\|^2_{L^{\frac{8}{3}}(\mathbb{R}^2)}\|u_n-u\|_{L^{\frac{8}{3}}(\mathbb{R}^2)}\|v\|_{L^{\frac{8}{3}}(\mathbb{R}^2)}\\
&= C_3\|u_n-u\|_{L^{\frac{8}{3}}(\mathbb{R}^2)}\|v\|_{L^{\frac{8}{3}}(\mathbb{R}^2)}\left(\|u_n+u\|_{L^{\frac{8}{3}}(\mathbb{R}^2)}\|u_n\|_{L^{\frac{8}{3}}(\mathbb{R}^2)}+\|u\|^2_{L^{\frac{8}{3}}(\mathbb{R}^2)}\right).
\end{aligned}
$$

F_2 is of class $C^2(X)$. The second Gâteaux derivate of F_2 at point $u \in X$ along direction $v, w \in X$ is given by

$$
\begin{aligned}
F_2''(u)[v,w] &= 8\int_{\mathbb{R}^2\times\mathbb{R}^2}\log\left(1+\frac{1}{|x-y|}\right)u(x)v(x)u(y)w(y)\,dx\,dy\\
&\quad + 4\int_{\mathbb{R}^2\times\mathbb{R}^2}\log\left(1+\frac{1}{|x-y|}\right)u^2(x)v(y)w(y)\,dx\,dy\\
&= 4B_2(u^2,vw)+8B_2(uv,uw)
\end{aligned}
$$

and we immediately observe that

$$
\begin{aligned}
|F_2''(u)[v,w]| &\le 8\int_{\mathbb{R}^2\times\mathbb{R}^2}\log\left(1+\frac{1}{|x-y|}\right)|u(x)v(x)||u(y)w(y)|\,dx\,dy\\
&\quad + 4\int_{\mathbb{R}^2\times\mathbb{R}^2}\log\left(1+\frac{1}{|x-y|}\right)u^2(x)|v(y)w(y)|\,dx\,dy\\
&\le 8\int_{\mathbb{R}^2\times\mathbb{R}^2}\frac{1}{|x-y|}|u(x)v(x)||u(y)w(y)|\,dx\,dy\\
&\quad + 4\int_{\mathbb{R}^2\times\mathbb{R}^2}\frac{1}{|x-y|}u^2(x)|v(y)w(y)|\,dx\,dy\\
&\le C_1\|u\|_{L^{\frac{4}{3}}(\mathbb{R}^2)}\|v\|_{L^{\frac{4}{3}}(\mathbb{R}^2)}\|u\|_{L^{\frac{4}{3}}(\mathbb{R}^2)}\|w\|_{L^{\frac{4}{3}}(\mathbb{R}^2)}\\
&\quad + C_2\|u\|^2_{L^{\frac{8}{3}}(\mathbb{R}^2)}\|v\|_{L^{\frac{4}{3}}(\mathbb{R}^2)}\|w\|_{L^{\frac{4}{3}}(\mathbb{R}^2)}\\
&\le C_3\|u\|_X^2\|v\|_X\|w\|_X < +\infty.
\end{aligned}
$$

Hence, $F_2''(u)v \in X^*$ and

$$
\|F_2''(u)v\|_{X^*} = \sup_{w\in X:\|w\|\le 1}|F_2''(u)[v,w]| \le C_3\|u\|_X^2\|v\|_X.
$$

Let $u_n, u \in X$ be such that $u_n \to u$ in X, then

$$
\begin{aligned}
|F_2''(u_n)[v,w]-F_2''(u)[v,w]| &= 8\int_{\mathbb{R}^2\times\mathbb{R}^2}\log\left(1+\frac{1}{|x-y|}\right)\left(u_n^2(x)-u^2(x)\right)v(y)w(y)\,dx\,dy\\
&\quad + 4\int_{\mathbb{R}^2\times\mathbb{R}^2}\log\left(1+\frac{1}{|x-y|}\right)\left(u_n(x)u_n(y)-u(x)u(y)\right)v(x)w(y)\,dx\,dy.
\end{aligned}
$$

We add and subtract $4\int_{\mathbb{R}^2\times\mathbb{R}^2}\log\left(1+\frac{1}{x-y|}\right)u(x)v(x)u_n(y)w(y)\,dx\,dy$ and obtain

$$
\begin{aligned}
&|F_2''(u_n)[v,w]-F_2''(u)[v,w]|\\
&= 8\int_{\mathbb{R}^2\times\mathbb{R}^2}\log\left(1+\frac{1}{|x-y|}\right)\left(u_n^2(x)-u^2(x)\right)v(y)w(y)\,dx\,dy\\
&\quad + 4\int_{\mathbb{R}^2\times\mathbb{R}^2}\log\left(1+\frac{1}{|x-y|}\right)\left(u_n(x)-u(x)\right)v(x)u_n(y)w(y)\,dx\,dy\\
&\quad + 4\int_{\mathbb{R}^2\times\mathbb{R}^2}\log\left(1+\frac{1}{|x-y|}\right)u(x)v(x)\left(u_n(y)-u(y)\right)w(y)\,dx\,dy
\end{aligned}
$$

$$\begin{aligned}
&\leq 8\int_{\mathbb{R}^2\times\mathbb{R}^2}\frac{1}{|x-y|}|u_n^2(x)-u^2(x)||v(y)||w(y)|\,dx\,dy\\
&\quad+4\int_{\mathbb{R}^2\times\mathbb{R}^2}\frac{1}{|x-y|}|u_n(x)-u(x)||v(x)||u_n(y)||w(y)|\,dx\,dy\\
&\quad+4\int_{\mathbb{R}^2\times\mathbb{R}^2}\frac{1}{|x-y|}|u(x)||v(x)||u_n(y)-u(y)||w(y)|\,dx\,dy\\
&\leq C_1\|u_n^2-u^2\|_{L^{\frac{4}{3}}(\mathbb{R}^2)}\|v\|_{L^{\frac{8}{3}}(\mathbb{R}^2)}\|w\|_{L^{\frac{8}{3}}(\mathbb{R}^2)}\\
&\quad+C_2\|u_n-u\|_{L^{\frac{8}{3}}(\mathbb{R}^2)}\|v\|_{L^{\frac{8}{3}}(\mathbb{R}^2)}\|u_n\|_{L^{\frac{8}{3}}(\mathbb{R}^2)}\|w\|_{L^{\frac{8}{3}}(\mathbb{R}^2)}\\
&\quad+C_3\|u\|_{L^{\frac{8}{3}}(\mathbb{R}^2)}\|v\|_{L^{\frac{8}{3}}(\mathbb{R}^2)}\|u_n-u\|_{L^{\frac{8}{3}}(\mathbb{R}^2)}\|w\|_{L^{\frac{8}{3}}(\mathbb{R}^2)}\\
&\leq C_1\|u_n-u\|_{L^{\frac{8}{3}}(\mathbb{R}^2)}\|u_n+u\|_{L^{\frac{8}{3}}(\mathbb{R}^2)}\|v\|_{L^{\frac{8}{3}}(\mathbb{R}^2)}\|w\|_{L^{\frac{8}{3}}(\mathbb{R}^2)}\\
&\quad+C_2\|u_n-u\|_{L^{\frac{8}{3}}(\mathbb{R}^2)}\|v\|_{L^{\frac{8}{3}}(\mathbb{R}^2)}\|u_n\|_{L^{\frac{8}{3}}(\mathbb{R}^2)}\|w\|_{L^{\frac{8}{3}}(\mathbb{R}^2)}\\
&\quad+C_3\|u\|_{L^{\frac{8}{3}}(\mathbb{R}^2)}\|v\|_{L^{\frac{8}{3}}(\mathbb{R}^2)}\|u_n-u\|_{L^{\frac{8}{3}}(\mathbb{R}^2)}\|w\|_{L^{\frac{8}{3}}(\mathbb{R}^2)}\\
&\leq C_4\|u_n-u\|_{L^{\frac{8}{3}}(\mathbb{R}^2)}\|v\|_{L^{\frac{8}{3}}(\mathbb{R}^2)}\|w\|_{L^{\frac{8}{3}}(\mathbb{R}^2)}\left(\|u_n+u\|_{L^{\frac{8}{3}}(\mathbb{R}^2)}+\|u_n\|_{L^{\frac{8}{3}}(\mathbb{R}^2)}+\|u\|_{L^{\frac{8}{3}}(\mathbb{R}^2)}\right)\\
&\leq C_5\|u_n-u\|_X\|v\|_X\|w\|_X\left(\|u_n+u\|_X+\|u_n\|_X+\|u\|_X\right)
\end{aligned}$$

and this goes to 0 as n diverges.

To conclude, observe that $F=F_1-F_2$ is of class C^2 on X because difference of C^2 functions: moreover,

$$F'(u)v=F_1'(u)-F_2'(u)=4B(u^2,uv)$$

for every $u,v\in X$ and

$$F''(u)[v,w]=F_1''(u)[v,w]-F_2''(u)[v,w]=4B(u^2,vw)+8B(uv,uw)$$

for every $u,v,w\in X$. □

We recall that we are dealing with the problem

$$-\Delta u+V(x)u+\frac{\gamma}{2\pi}\left(\int_{\mathbb{R}^2}\log|x-y|u^2(y)\,dy\right)u=0,\ \text{ in }\mathbb{R}^2$$

and the, formally, associated functional is the one defined in (1.10): thanks to (1.27) and (1.28), $\mathcal{J}$ can be rewrite as

$$\mathcal{J}(u)=\frac{1}{2}\|u\|^2_{H^1(\mathbb{R}^2)}+\frac{1}{4}F(u)=\frac{1}{2}\|u\|^2_{H^1(\mathbb{R}^2)}+\frac{\gamma}{8\pi}F_1(u)-\frac{\gamma}{8\pi}F_2(u).$$

Using Proposition 1.11, we can finally solve the ill-posedness of the problem, by giving the following regularity result for the functional $\mathcal{J}$.

Proposition 1.12 (see Lemma 2.2 in [49])**.** *The functional $\mathcal{J}:X\to\mathbb{R}$ is of class C^2 on X.*

Proof. The proof descends easily from Proposition 1.11. Indeed, the norm-function

$$\|\cdot\|_{H^1(\mathbb{R}^2)}:H^1(\mathbb{R}^2)\to[0,+\infty)$$

is of class C^2, hence $\mathcal{J}:X\to\mathbb{R}$ is the sum of C^2 terms on X. □

Hence, thanks to Proposition 1.12, it makes sense to look for solutions of (1.19) as critical points of the functional (1.10); thus, we have the following

Definition 1.13. *We say that $u \in X$ is a weak solution of the equation*(1.19) *if*

$$\mathcal{J}'(u)v = 0 \text{ for all } v \in X, \tag{1.32}$$

that is if u is a critical point of $\mathcal{J}$.

1.7 A perturbation from the source

After recalling (a part of) the wide literature, the variational framework and the pertubation technique, we are finally ready to formalize the problem we are dealing with.

Let $h \in L^\infty(\mathbb{R}^2)$, $p > 1$ and we suppose that the external potential is constant and positive, say $V \equiv a > 0$, and consider

$$-\Delta u(x) + au(x) + \frac{\gamma}{2\pi}\left(\int_{\mathbb{R}^2} \log|x-y|u^2(y)\,dy\right)u = \varepsilon h(x)|u(x)|^{p-1}u(x), \text{ in } \mathbb{R}^2. \tag{1.33}$$

The function h is a *weight* for the pure-power nonlinearity.

Semiclassical analysis for the two dimensional case is of recent interest: in fact, the very first approach was made by Masaki in 2009 ([93]), but the author used the WKB-approximation[4] technique. Later on, a variational approach has been developed in [37] and [49] and this papers, together with the now classical Perturbation Theory, inspired our work.

We associate to (1.33) the energy functional $\mathcal{J} : X \to \mathbb{R}$ defined as

$$\begin{aligned}\mathcal{J}_\varepsilon(u) := &\frac{1}{2}\int_{\mathbb{R}^2}\left(|\nabla u(x)|^2 + au^2(x)\right)\,dx \\ &- \frac{1}{8\pi}\int_{\mathbb{R}^2\times\mathbb{R}^2} \log\frac{1}{|x-y|}u^2(x)u^2(y)\,dx\,dy - \frac{1}{p+1}\int_{\mathbb{R}^2} h(x)|u(x)|^{p+1}\,dx\end{aligned} \tag{1.34}$$

and according to the perturbed theory we call

$$\mathcal{J}_0(u) = \frac{1}{2}\int_{\mathbb{R}^2}\left(|\nabla u(x)|^2 + au^2(x)\right)\,dx - \frac{1}{8\pi}\int_{\mathbb{R}^2\times\mathbb{R}^2} \log\frac{1}{|x-y|}u^2(x)u^2(y)\,dx\,dy \tag{1.35}$$

the unperturbed functional and

$$G(u) := -\frac{1}{p+1}\int_{\mathbb{R}^2} h(x)|u(x)|^{p+1}\,dx \tag{1.36}$$

the perturbation, so that

$$\mathcal{J}_\varepsilon(u) = \mathcal{J}_0(u) + \varepsilon G(u).$$

We give immeadiately the regularity result for the perturbed functional.

Proposition 1.14. *The functional $\mathcal{J}_\varepsilon$ is of class C^2 on X.*

Proof. By Proposition 1.12 we have that $\mathcal{J}_0$ is of class C^2 on X. Then, it is enough to show that G is of class C^2 on X.

From the regularity of h and the Proposition 1.7, we have that

$$|G(u)| \le \frac{\varepsilon}{p+1}\int_{\mathbb{R}^2} |h(x)||u(x)|^{p+1}\,dx \le \|h\|_{L^\infty(\mathbb{R}^2)}\|u\|^{p+1}_{L^{p+1}(\mathbb{R}^2)} \le C\|u\|^{p+1}_X < \infty.$$

[4]From Wentzel-Kramers-Brillouin: it is a mathematical physics method for approximating solutions of a PDE whose highest derivative is multiplied by a small parameter.

Let $u_n, u \in X$ be such that $u_n \to u$ in X, then

$$\begin{aligned}|G(u_n) - G(u)| &= \frac{\varepsilon}{p+1} \left| \int_{\mathbb{R}^2} h(x) \left(|u_n|^{p+1} - |u|^{p+1} \right) dx \right| \\ &\leq \frac{\varepsilon}{p+1} \int_{\mathbb{R}^2} |h(x)| \left| |u_n|^{p+1} - |u|^{p+1} \right| dx \\ &\leq C \|h\|_{L^\infty(\mathbb{R}^2)} \int_{\mathbb{R}^2} \left| |u_n|^{p+1} - |u|^{p+1} \right| dx.\end{aligned}$$

By using the Lebesgue dominated convergence Theorem, we have that

$$\int_{\mathbb{R}^2} \left| |u_n(x)|^{p+1} - |u(x)|^{p+1} \right| dx \to 0 \tag{1.37}$$

as $n \to +\infty$ (see also Theorem A.2 in [139]) and the desired continuity is proved.

The first Gâteaux derivative of G at $u \in X$ along $v \in X$ is

$$G'(u)v = \varepsilon \int_{\mathbb{R}^2} h(x)|u(x)|^{p-1} uv \, dx$$

and by Hölder's inequality (with $s = p$ and $s' = p' = \frac{p}{p-1}$) we have

$$|G'(u)v| \leq C \|h\|_{L^\infty(\mathbb{R}^2)} \|u\|^p_{L^{pp'}(\mathbb{R}^2)} \|v\|_{L^p(\mathbb{R}^2)} < \infty.$$

Now, let $u_n, u \in X$ be such that $u_n \to u$ in X: we have

$$\begin{aligned}|G'(u_n)v - G'(u)v| &\leq \varepsilon \int_{\mathbb{R}^2} |h(x)| \, ||u_n(x)|^p - |u(x)|^p| \, |v(x)| \, dx \\ &\leq C \|h\|_{L^\infty(\mathbb{R}^2)} \int_{\mathbb{R}^2} ||u_n(x)|^p - |u(x)|^p| \, |v(x)| \, dx\end{aligned}$$

and, reasoning as for (1.37), this goes to 0 as n diverges.

The second Gâteaux derivative of G at $u \in X$ along $v, w \in X$ is

$$G''(u)[v,w] = -(p-1)\varepsilon \int_{\mathbb{R}^2} h(x)|u(x)|^{p-1} w(x) \, dx - \varepsilon \int_{\mathbb{R}^2} h(x)|u(x)|^{p-1} v(x) w(x) \, dx$$

and

$$\begin{aligned}|G''(u)[v,w]| &\leq (p-1)\varepsilon \int_{\mathbb{R}^2} |h(x)||u(x)|^{p-1}|w(x)| \, dx + \varepsilon \int_{\mathbb{R}^2} |h(x)||u(x)|^{p-1}|v(x)||w(x)| \, dx \\ &\leq (p-1)\varepsilon \|h\|_{L^\infty(\mathbb{R}^2)} \|u\|^{p-1}_{L^{(p-1)p'}(\mathbb{R}^2)} \|w\|_{L^p(\mathbb{R}^2)} \\ &\quad + \varepsilon \|h\|_{L^\infty(\mathbb{R}^2)} \|u\|^{p-1}_{L^{(p-1)p'}(\mathbb{R}^2)} \|vw\|_{L^p(\mathbb{R}^2)} < \infty.\end{aligned}$$

Again, let $u_n, u \in X$ be such that $u_n \to u$ in X and compute

$$\begin{aligned}&|G''(u_n)[v,w] - G''(u)[v,w]| \\ &\leq \varepsilon(p-1) \int_{\mathbb{R}^2} |h(x)| \left(|u_n|^{p-1} - |u|^{p-1} \right) w(x) \, dx \\ &\quad + \varepsilon \int_{R^2} |h(x)| \left(|u_n|^{p-1} - |u|^{p-1} \right) v(x) w(x) \, dx \\ &\leq \varepsilon(p-1) \|h\|_{L^\infty(\mathbb{R}^2)} \int_{\mathbb{R}^2} \left(|u_n(x)|^{p-1} - |u(x)|^{p-1} \right) |v(x)| \, dx \\ &\quad + \varepsilon \int_{\mathbb{R}^2} \left(|u_n(x)|^{p-1} - |u(x)|^{p-1} \right) |v(x)||w(x)| \, dx.\end{aligned}$$

Proceeding as for G and G', the conclusion follows and the Proposition is proved. □

In order to use the Perturbation Theory recalled in Section 1.5, we firstly observe, in view of Definition 1.32, that critical points of the unperturbed functional $\mathcal{J}_0$ are solutions of the homogeneous equation (1.19), that is $\mathcal{J}_0'(u) = 0$ if and only if

$$-\Delta u + au - \frac{1}{2\pi}\left(\log\frac{1}{|\cdot|} \star u^2\right) u = 0, \text{ in } \mathbb{R}^2.$$

This equation admits, up to translations and for every $a > 0$, a unique radially symmetric solutions $u \in X$ and the equation itself is invariant under translations (Theorem 1.3 in [49]); hence we can consider the *critical manifold* Z for $\mathcal{J}_0$ defined as

$$Z := \left\{z_\xi = u(x-\xi) : \xi \in \mathbb{R}^2\right\}. \tag{1.38}$$

We want to show that Z is non-degenerate in the sense of Definition 1.1: we start on showing that (ND) holds. The proof relies on the following asymptotic behaviour of the unique, up to translations, solution of the homogeneous equation (1.19).

Theorem 1.15 ([37], Theorem 2)**.** *If $a > 0$ and if $u \in X$ is a radially symmetric positive solution of* (1.19)*, then there exists $\mu > 0$ such that*

$$u(x) = \frac{\mu + o(1)}{\sqrt{|x|}(\log|x|)^{\frac{1}{4}}} \exp\left(-\sqrt{M} e^{-\frac{a}{M}} \int_1^{e^{a/M}|x|} \sqrt{\log s}\, ds\right), \tag{1.39}$$

as $|x| \to +\infty$ and where

$$M = \frac{1}{2\pi}\int_{\mathbb{R}^2} u^2(x)\, dx.$$

The above result is a key ingredient in showing condition (ND). Indeed, consider the space

$$\widetilde{X} := \Bigg\{\varphi \in X : \text{ there exists } f \in L^2(\mathbb{R}^2) \text{ such that}$$

$$\text{for every } \psi \in C_c^\infty(\mathbb{R}^2) \int_{\mathbb{R}^2} \varphi \mathcal{L}(u)\psi = \int_{\mathbb{R}^2} f\psi\Bigg\}$$

and the linear operator $\mathcal{L}(u) : \widetilde{X} \to L^2(\mathbb{R}^2)$ defined as

$$\mathcal{L}(z_\xi) : \varphi \mapsto -\Delta\varphi + (a-w)\varphi + 2z_\xi\left(\frac{\log}{2\pi} \star (z_\xi\varphi)\right), \tag{1.40}$$

where $w : \mathbb{R}^2 \to \mathbb{R}$ is given by

$$w(x) = \frac{1}{2\pi}\int_{\mathbb{R}^2} \log\frac{1}{|x-y|} z_\xi^2(x)\, dx, \quad x \in \mathbb{R}^2. \tag{1.41}$$

Then, the following Theorem holds.

Theorem 1.16 (see [37], Theorem 3)**.** *If $a > 0$ and $u \in X$ is a positive solution of* (1.19)*, then*

$$\ker \mathcal{L}(u) = \left\{\gamma \cdot \nabla u : \gamma \in \mathbb{R}^2\right\}.$$

The proof of this result is quite technical and can be found in Section 4 of [37], and relies on the angular splitting of the operator $\mathcal{L}$.

Therefore, we are saying that the kernel of the operator $\mathcal{L}(u)$ is spanned by the partial derivatives of u. Hence, the positive solution u is nondegenerate and the following result holds.

Proposition 1.17. *The manifold Z satisfies assumptions (ND).*

To show (Fr), we use the following useful characterization of Fredholm operators (this characterization directly descends from the Fredholm Alternative, Theorem 6.6 in [38]).

Proposition 1.18 (Compact perturbation of the identity)**.** *Let E, F be two Banach spaces. Let $I \in \mathcal{L}(E,F)$ be a continuous invertible operator and $K \in \mathcal{K}(E,F)$ be a continuous linear compact operator and $T : E \to F$ be an operator. Then, T is a Fredholm map and $ind(T) = 0$ if and only if $T = I + K$.*

Using this result, we can now show the following Proposition (see Proposition 2.5 in [29]).

Proposition 1.19. *The operator $\mathcal{J}''(z_\xi) : X \to X*$ is a compact perturbation of the identity operator, i.e. there exist a continuous invertible opeartor L and a continuous compact operator K in X such that*

$$\mathcal{J}_0''(z_\xi) = L - K.$$

Proof. We recall that, for every $z_\xi \in Z$ and $v, w \in X$ we have

$$\begin{aligned}\mathcal{J}_0''(z_\xi)[v,w] = &\int_{\mathbb{R}^2} (\nabla v(x) \cdot \nabla w(x) + a v(x) w(x))\, dx \\ &- \frac{1}{2\pi} \int_{\mathbb{R}^2\times\mathbb{R}^2} \log \frac{1}{|x-y|} z_\xi^2(x) v(y) w(y)\, dx\, dy \\ &- \frac{1}{\pi} \int_{\mathbb{R}^2\times\mathbb{R}^2} \log \frac{1}{|x-y|} z_\xi(x) v(x) z_\xi(y) w(y)\, dx\, dy,\end{aligned}$$

therefore

$$\begin{aligned}\mathcal{J}_0''(z_\xi)[v,w] = (v,w)_{H^1(\mathbb{R}^2)} &- \frac{1}{2\pi} \int_{\mathbb{R}^2\times\mathbb{R}^2} \log \frac{1}{|x-y|} z_\xi^2(x) v(y) w(y)\, dx\, dy \\ &- \frac{1}{\pi} \int_{\mathbb{R}^2\times\mathbb{R}^2} \log \frac{1}{|x-y|} z_\xi(x) v(x) z_\xi(y) w(y)\, dx\, dy.\end{aligned}$$

that is, the second Gâteaux derivative can be seen as the linear operator (1.40).

Now, we take care of the operator

$$\varphi \mapsto -\Delta\varphi + (a - w)\varphi,$$

but we don't have a control on $a - w$ (they can change sign), so this operator need not to be invertible on X. We set

$$c^2 = \frac{1}{2\pi} \int_{\mathbb{R}^2} z_\xi^2(x)\, dx$$

and add and subtract this quantity in (1.40): so we can rewrite $\mathcal{L}(z_\xi)$ as

$$\mathcal{L}(z_\xi) : \varphi \mapsto -\Delta\varphi + (a + c^2 \log(1+|x|))\varphi - (c^2 \log(1+|x|+w)\varphi + 2z_\xi \left(\frac{\log}{2\pi} \star (z_\xi \varphi)\right). \quad (1.42)$$

The first two summands are nothing that the scalar product on X, so it defines a norm that is coercive: hence, the operator

$$\varphi \mapsto -\Delta\varphi + (a + c^2 \log(1+|x|))\varphi$$

is invertible.

By Proposition 2.3 in [49] follows that

$$c^2 \log(1+|x|+w(x)) \to 0$$

as $\|x\| \to +\infty$. Indeed, let $|x| \leq 1$ and we compute (recalling (1.41))

$$|w(x)| \leq \int_{B_2(x)} \left|\log \frac{1}{|x-y|}\right| u^2(y)\,dy + \int_{\mathbb{R}^2\setminus B_2(x)} \left|\log \frac{1}{|x-y|}\right| u^2(y)\,dy$$
$$= \int_{B_2(x)} |\log|x-y||\,u^2(y)\,dy + \int_{\mathbb{R}^2\setminus B_2(x)} |\log|x-y||\,u^2(y)\,dy.$$

Since $1 \leq |x-y| \leq 1+|y|$ for every $y \in \mathbb{R}^2 \setminus B_2(x)$, we have

$$\int_{\mathbb{R}^2\setminus B_2(x)} |\log|x-y||\,u^2(y)\,dx \leq \int_{\mathbb{R}^2\setminus B_2(x)} |\log(1+|y|)|\,u^2(y)\,dx \leq \|u\|_X^2.$$

Now, using Hölder's inequality (with $s = s' = 2$), we can claim that

$$\int_{B_2(x)} |\log|x-y||\,u^2(y)\,dx \leq \left(\int_{B_2(x)} |\log|x-y||^2\ dy\right)^{\frac{1}{2}} \left(\int_{B_2(x)} |u^2(y)|^2\,dy\right)^{\frac{1}{2}}$$
$$\leq \left(\int_{B_2(0)} |\log|y||^2\ dy\right)^{\frac{1}{2}} \|u\|^2_{L^4(B_2(x))}$$
$$\leq C_0\|u\|^2_{L^4(B_2(x))},$$

where C_0 is a positive constant, and by Proposition 1.7, there exists a constant $C_1 > 0$ such that

$$\int_{B_2(x)} |\log|x-y||\,u^2(y)\,dx \leq C_1\|u\|_X^2.$$

Hence, we just proved that $w \in L^\infty(B_1(x))$. Now, let $x \in \mathbb{R}^2$ such that $|x| \geq 1$ and observe that

$$w(x) - \|u\|^2_{L^2(\mathbb{R}^2)} \log|x| = \int_{\mathbb{R}^2} a(x,y)u^2(y)\,dy$$

where $a(x,y) = \log|x-y| - \log|x| = \log\frac{|x-y|}{|x|}$ and

$$a(x,y) \to 0$$

as $|x| \to +\infty$, for every $y \in \mathbb{R}^2$.

We claim that

$$\int_{|x-y|\geq\frac{1}{2}} a(x,y)u^2(y)\,dy \to 0 \tag{1.43}$$

as $|x| \to +\infty$. Indeed, the following estimate holds

$$\log\frac{1}{2} \leq a(x,y)\chi_{|x-y|\geq\frac{1}{2}}(y) \leq \log(1+|y|)$$

for all $x, y \in \mathbb{R}^2$, with $|x| \geq 1$, where χ denotes the characteristic function. The functions $\log\frac{1}{2}u^2$ and $\log(1+|\cdot|)$ belong to the space $L^1(\mathbb{R}^2)$. Reassuming, we have:

$$a(x,y) \leq \log(1+|y|) \in L^1(\mathbb{R}^2);$$
$$a(x,y) \geq \log\frac{1}{2} \in L^1(\mathbb{R}^2);$$
$$a(x,y) \to 0 \text{ as } |x| \to +\infty.$$

So, we can use the Lebesgue Dominated Convergence Theorem and the claim is proved.

It reamins to estimate

$$\int_{|x-y|\leq\frac{1}{2}} a(x,y)u^2(y)\,dy.$$

Let $u \in X$, then

$$0 \leq \log|x| \int_{|x-y|\leq\frac{1}{2}} u^2(y)\,dy \leq \int_{|y|\geq\frac{|x|}{2}} \log(2(1+|y|))u^2(y)\,dy$$

goes to 0 as $|x| \to \infty$, since $|x| \leq 2|y|$ in $|y| \geq \frac{|x|}{2}$. Therefore, using also Hölder inequality (with $s = s' = 2$), there exists a positive constant $C_3 > 0$ such that

$$\begin{aligned}\int_{|x-y|\leq\frac{1}{2}} &\log(2(1+|y|))u^2(y)\,dy \\ &\leq \left(\int_{|x-y|\leq\frac{1}{2}} |\log(2(1+|y|)|^2\,dy\right)^{\frac{1}{2}} \left(\int_{|x-y|\leq\frac{1}{2}} |u^2(y)|^2\,dy\right)^{\frac{1}{2}} \\ &\leq C_3\|u\|^2_{L^4\left(B_{\frac{1}{2}}(x)\right)}. \end{aligned} \tag{1.44}$$

Finally, combining (1.43) and (1.44) we obtain

$$w(x) - \|u\|^2_{L^2(\mathbb{R}^2)} \log|x| \to 0$$

as $|x| \to +\infty$.

Therefore, the multiplication operator

$$\varphi \mapsto (c^2\log(1+|x|) + w)\varphi$$

is compact.

Now, we take care of the last summand of (1.42). As in [125], Lemma 15, let $(v_n)_n$ and $(w_n)_n$ be two seqeunces in X such that

$$\|v_n\|_X \leq 1, \quad \|w_n\|_X \leq 1$$

and there exist $v_0, w_0 \in X$ such that

$$v_n \rightharpoonup v_0 \quad \text{and} \quad w_n \rightharpoonup w_0,$$

and without loss of generality, we can suppose that $v_0 = w_0 = 0$. From the compact embedding $X \hookrightarrow L^s(\mathbb{R}^2)$, for $s \in [2, +\infty)$ (see Proposition 1.7), we can recover the strong convergences

$$v_n \to 0 \quad \text{and} \quad w_n \to 0. \tag{1.45}$$

Now, using the Hardy-Littlewood-Sobolev inequality (see Theorem A.3 in Appendix A), we can say that there exists a constant $C > 0$ such that

$$\int_{\mathbb{R}^2\times\mathbb{R}^2} \log\frac{1}{|x-y|} z_\xi(x)v_n(x)z_\xi(y)w_n(y)\,dx\,dy \leq C\|z_\xi v_n\|_{L^{\frac{4}{3}}(\mathbb{R}^2)}\|z_\xi w_n\|_{L^{\frac{4}{3}}(\mathbb{R}^2)}. \tag{1.46}$$

We want to estimate the two norms above: to do that, we use Hölder inequality and we have

$$\|z_\xi v_n\|^{\frac{4}{3}}_{L^{\frac{4}{3}}(\mathbb{R}^2)} \leq \left(\int_{\mathbb{R}^2} |z_\xi(x)|^4\,dx\right)^{\frac{1}{3}} \left(\int_{\mathbb{R}^2} |v_n(x)|^2\right)^{\frac{2}{3}} \tag{1.47}$$

and similarly

$$\|z_\xi w_n\|^{\frac{4}{3}}_{L^{\frac{4}{3}}(\mathbb{R}^2)} \leq \left(\int_{\mathbb{R}^2} |z_\xi(x)|^4\,dx\right)^{\frac{1}{3}} \left(\int_{\mathbb{R}^2} |w_n(x)|^2\right)^{\frac{2}{3}} \tag{1.48}$$

and both (1.47) and (1.48) goes to 0 as $|x| \to +\infty$ thanks to (1.39) and (1.45).

Hence, we have proved the compactness of the multiplication operator

$$\varphi \mapsto 2z_\xi \left(\frac{\log}{2\pi} \star (z_\xi \varphi) \right).$$

Finally, we may assert that the functional $\mathcal{J}_0''(z_\xi)$ is of the form *identity - compact*, that is $\mathcal{J}_0''(z_\xi) = I - K$, where

$$I\varphi := \left(-\Delta + (a + c^2 \log(1+|x|)) \right) \varphi$$

is a linear, continuous and invertible operator and

$$K\varphi = (c^2 \log(1+|x|) + w)\varphi + 2z_\xi \left(\frac{\log}{2\pi} \star (z_\xi \varphi) \right)$$

is a linear, continuous and compact operator. □

1.8 Existence results

In this Section we present the two main Theorems of [29]. The Perturbation method of Section 1.5 that we are going to use, heavily depends on the behaviour of the function h. Indeed, the presence of the function h produces the effect to break the invariance under translations of the unperturbed functional $\mathcal{J}_0$. For this reason, we are able to provide two difference existence results, depending on the assumption considered on the function h: in Theorem 1.22 we consider a bounded function sufficiently integrable in $\mathbb{R}^2$. We refer to this as *local existence*. On the other hand, in Theorem 1.26 we dropped the summability assumption, supposing that the weight function h is bounded and vanishes at infinity: in this case, we talk of *global existence*. We point out that in the global existence Theorem, a more more delicate analysis of the implicit function w that describes the reduced functional is required because of the lack of the summability assumption.

1.8.1 Local existence

We already saw in Section 1.7 that assumptions (ND) and (Fr) are satisfied, therefore (see Lemma 2.11 in [13]) we can say that the reduced functional is of the form

$$\Phi_\varepsilon(z_\xi) = \mathcal{J}_\varepsilon(z_\xi + w_\varepsilon(\xi)) = c_0 + \varepsilon G(z_\xi) + o(\varepsilon), \tag{1.49}$$

where we set

$$c_0 := \mathcal{J}_0(z_\xi).$$

To see that, it is enough to use Taylor expansion on $\mathcal{J}_0$ and on G and to use (1.18). Now, we define the fucntion $\Gamma : \mathbb{R}^2 \to \mathbb{R}$ defined as

$$\Gamma(\xi) = G(z_\xi) = -\frac{1}{p+1} \int_{\mathbb{R}^2} h(x)|z_\xi|^{p+1}\, dx, \quad \xi \in \mathbb{R}^2, \tag{1.50}$$

and prove the following

Lemma 1.20. *Suppose that $h \in L^\infty(\mathbb{R}^2) \cap L^q(\mathbb{R}^2)$ for some $q > 1$. Then*

$$\lim_{|\xi| \to +\infty} \Gamma(\xi) = 0. \tag{1.51}$$

Proof. We make use of Hölder inequality with $s = q$, so $s' = \frac{q}{q-1}$, obtaining

$$|\Gamma(\xi)| \leq \frac{1}{p+1} \int_{\mathbb{R}^2} |h(x)||z_\xi|^{p+1}\, dx \leq \frac{1}{p+1} \left(\int_{\mathbb{R}^2} |h(x)|^q\, dx \right)^{\frac{1}{q}} \left(\int_{\mathbb{R}^2} |z_\xi|^{(p+1)\frac{q}{q-1}}\, dx \right)^{\frac{q-1}{q}}.$$

By the exponential decay of z_ξ as $|\xi| \to +\infty$ (remember Proposition 1.39) and the regularity of h, we have that there exists a constant $C > 0$ such that

$$|\Gamma(\xi)| \leq C \left(\int_{\mathbb{R}^2} |z_\xi|^{(p+1)\frac{q}{q-1}}\, dx \right)^{\frac{q-1}{q}} \to 0$$

as $|\xi| \to +\infty$, completing the proof. □

Remark 1.21. *Function Γ is known as the Poincaré function, while its derivative is the Melnikov function (cf. [13]).*

We are now ready to show the existence of a *local* solution for (1.33).

Theorem 1.22 (see [29], Theorem 1.3)**.** *Let $p > 1$ and $h \in L^\infty(\mathbb{R}^2) \cap L^q(\mathbb{R}^2)$, for some $q > 1$. Moreover, suppose that*

(h1) $\int_{\mathbb{R}^2} h(x)|z_0|^{p+1}\, dx \neq 0$.

Then, equation (1.33) has a solution, provided $|\varepsilon|$ is small enough.

Proof. The hypotheses of the previous Lemma are satisfied, hence (1.51) holds. Moreover, from *(h1)* follows that

$$\Gamma(0) = -\frac{1}{p+1} \int_{\mathbb{R}^2} h(x)|z_0|^{p+1}\, dx \neq 0.$$

Hence, function $\Gamma \not\equiv 0$ and so it has a maximum or a minimum on $\mathbb{R}^2$.

Therefore, we can apply Theorem 1.4 to obtain the *local* solution. □

1.8.2 Global existence

In this Section, we prove the existence of a *global* solution. The proof relies on Theorem 1.6, but before applying it, we need some Lemmas, which proofs are a bit technical.

Let $W_\xi = \left(T_{z_\xi}\right)^\perp$ and $\widetilde{W_\xi} = \langle z_\xi \rangle \oplus \left(T_{z_\xi} Z\right)$. Also, we set $P = P_\xi : X \to W_\xi$ the orthogonal projection and $R_\xi(w) = \mathcal{J}_0'(z_\xi + w) - \mathcal{J}_0''(z_\xi)[w]$.

Remark 1.23. *In [49] Theorem 1.1, the authors showed that the restriction of $\mathcal{J}$ to the associated Nehari manifold*

$$\mathcal{N} = \{u \in X : \mathcal{J}'(u)u = 0\}$$

attains global minimum and that every minimizer $u \in X$ of the restricted functional is a solution of (1.19) with constant sign and with the variational characterization

$$\mathcal{J}(u) = \inf_{u \in X} \sup_{t \in \mathbb{R}} \mathcal{J}(tu).$$

Hence, the spectrum of $\mathcal{J}_0''(u)$ has exactly one negative simple eigenvalue with eigenspace $\langle u \rangle$. Moreover, $\lambda = 0$ is an eigenvalue with multiplicity N and eigenspace

$$span\left(\{D_i u, i = 1 \ldots N\}\right) = T_u Z :$$

then, the rest of the spectrum is positive, that is there exists $\kappa > 0$ *such that*

$$\mathcal{J}_0''(u)[v,v] \geq \kappa \|v\|_X^2, \tag{1.52}$$

for all $v \perp \langle u \rangle \oplus T_u Z$*.*

The observations made in the previous remark will help us in proving the following Lemma.

Lemma 1.24.

(i) There is $C > 0$ *such that*

$$\left\| \left(P\mathcal{J}_0''(z_\xi) \right)^{-1} \right\|_{\mathcal{L}(W_\xi, W_\xi)} \leq C,$$

for every $\xi \in \mathbb{R}^2$*.*

(ii) $R_\xi(w) = o(\|w\|)$*, uniformly with respect to* $\xi \in \mathbb{R}^2$*.*

Proof. (i) Let $z_\xi \in Z$ be a solution of (1.33): in particular, it is a Mountain-Pass solution, so Remark 1.23, z_ξ holds. Hence, it sufficies to show that there exists $\kappa > 0$ such that

$$P\mathcal{J}_0''(z_\xi)[v,v] \geq \kappa \|v\|^2, \tag{1.53}$$

for every $\xi \in \mathbb{R}^2$ and $v \perp \widetilde{W_\xi}$. We observe that for any fixed $\xi \in \mathbb{R}^2$, in particular for $\xi = 0$, $P\mathcal{J}_0''(z_0) = P\mathcal{J}_0''(u)$ is invertible and there exists $\kappa > 0$ such that (1.53) holds for every $v \in \widetilde{W} := \langle u \rangle \oplus T_u Z$.

Now, we set $v^\xi(x) = v(x+\xi)$ and we compute

$$\begin{aligned} P\mathcal{J}_0''(z_\xi)[v,v] = P\Bigg[& \int_{\mathbb{R}^2} \left(|\nabla v(x)|^2 + a v^2(x) \right) dx \\ & - \frac{1}{2\pi} \int_{\mathbb{R}^2 \times \mathbb{R}^2} \log \frac{1}{|x-y|} z_\xi^2(x) v^2(y) \, dx \, dy \\ & - \frac{1}{\pi} \int_{\mathbb{R}^2 \times \mathbb{R}^2} \log \frac{1}{|x-y|} z_\xi(x) v(x) z_\xi(y) v(y) \, dx \, dy \Bigg]. \end{aligned}$$

Thanks to change of variables $x = s + \xi$ and $y = t + \xi$ we obtain

$$\begin{aligned} P\mathcal{J}_0''(z_\xi)[v,v] = P\Bigg[& \int_{\mathbb{R}^2} \left(|\nabla v(s+\xi)|^2 + a v^2(s+\xi) \right) ds \\ & - \frac{1}{2\pi} \int_{\mathbb{R}^2 \times \mathbb{R}^2} \log \frac{1}{|s-t|} z_\xi^2(s+\xi) v^2(t+\xi) \, ds \, dt \\ & - \frac{1}{\pi} \int_{\mathbb{R}^2 \times \mathbb{R}^2} \log \frac{1}{|s-t|} z_\xi(s+\xi) v(s+\xi) z_\xi(t+\xi) v(t+\xi) \, ds \, dt \Bigg] \\ = P\Bigg[& \int_{\mathbb{R}^2} \left(|\nabla v^\xi(s)|^2 + a \left(v^\xi \right)^2 (s) \right) ds \\ & - \frac{1}{2\pi} \int_{\mathbb{R}^2 \times \mathbb{R}^2} \log \frac{1}{|s-t|} z_0^2(s) |v^\xi(t)|^2 \, ds \, dt \\ & - \frac{1}{\pi} \int_{\mathbb{R}^2 \times \mathbb{R}^2} \log \frac{1}{|s-t|} z_0(s) v^\xi(s) z_0(t) v^\xi(t) \, ds \, dt \Bigg] \\ = {} & P\mathcal{J}_0''(u)[v^\xi, v^\xi]. \end{aligned}$$

To conclude the proof of the first point, it is enough to observe that $v^\xi \perp \widetilde{W}$ whenever $v \perp \widetilde{W_\xi}$: indeed, we have

$$P\mathcal{J}_0''(z_\xi)[v,v] = P\mathcal{J}_0''(u)[v^\xi, v^\xi] \geq \kappa \|v^\xi\|^2 = \kappa \|v\|^2$$

for every $\xi \in \mathbb{R}^2$ and for every $v \perp \widetilde{W_\xi}$.

(ii) We start by computing $R_\xi(w)$: we have

$$\begin{aligned}
R_\xi(w) &= \mathcal{J}_0'(z_\xi + w)v - \mathcal{J}_0''(z_\xi)(w,v) \\
&= \int_{\mathbb{R}^2} \left[\nabla(z_\xi + w)(x)\cdot\nabla v(x) + a(z_\xi + w)(x)v(x)\right] dx \\
&\quad - \frac{1}{2\pi}\int_{\mathbb{R}^2\times\mathbb{R}^2} \log\frac{1}{|x-y|}(z_\xi + w)^2(x)(z_\xi + w)(y)v(y)\,dx\,dy \\
&\quad - \int_{\mathbb{R}^2} \nabla w(x)\cdot\nabla v(x)\,dx + \frac{1}{2\pi}\int_{\mathbb{R}^2\times\mathbb{R}^2} \log\frac{1}{|x-y|} z_\xi^2(x)w(y)v(y)\,dx\,dy \\
&\quad + \frac{1}{\pi}\int_{\mathbb{R}^2\times\mathbb{R}^2} \log\frac{1}{|x-y|} z_\xi(x)w(x)z_\xi(y)v(y)\,dx\,dy.
\end{aligned}$$

After some computations, we obtain

$$\begin{aligned}
R_\xi(w) = &\int_{\mathbb{R}^2} \left[\nabla z_\xi(x)\cdot\nabla v(x) + a z_\xi(x)v(x)\right] dx - \frac{1}{2\pi}\int_{\mathbb{R}^2\times\mathbb{R}^2} \log\frac{1}{|x-y|} z_\xi^2(x)z_\xi(y)v(y)\,dx\,dy \\
&- \frac{1}{2\pi}\int_{\mathbb{R}^2\times\mathbb{R}^2} \log\frac{1}{|x-y|} z_\xi^2(x)w(y)v(y)\,dx\,dy \\
&- \frac{1}{\pi}\int_{\mathbb{R}^2\times\mathbb{R}^2} \log\frac{1}{|x-y|} z_\xi(x)w(x)z_\xi(y)v(y)\,dx\,dy \\
&- \frac{1}{\pi}\int_{\mathbb{R}^2\times\mathbb{R}^2} \log\frac{1}{|x-y|} z_\xi(x)w(x)w(y)v(y)\,dx\,dy \\
&- \frac{1}{2\pi}\int_{\mathbb{R}^2\times\mathbb{R}^2} \log\frac{1}{|x-y|} w^2(x)z_\xi(y)v(y)\,dx\,dy \\
&- \frac{1}{2\pi}\int_{\mathbb{R}^2\times\mathbb{R}^2} \log\frac{1}{|x-y|} w^2(x)w(y)v(y)\,dx\,dy \\
&+ \frac{1}{2\pi}\int_{\mathbb{R}^2\times\mathbb{R}^2} \log\frac{1}{|x-y|} z_\xi^2(x)w(y)v(y)\,dx\,dy \\
&+ \frac{1}{\pi}\int_{\mathbb{R}^2\times\mathbb{R}^2} \log\frac{1}{|x-y|} z_\xi(x)w(x)z_\xi(y)v(y)\,dx\,dy,
\end{aligned}$$

so that

$$\begin{aligned}
R_\xi(w) = \mathcal{J}_0'(z_\xi)v &- \frac{1}{\pi}\int_{\mathbb{R}^2\times\mathbb{R}^2} \log\frac{1}{|x-y|} z_\xi(x)w(x)w(y)v(y)\,dx\,dy \\
&- \frac{1}{2\pi}\int_{\mathbb{R}^2\times\mathbb{R}^2} \log\frac{1}{|x-y|} w^2(x)z_\xi(y)v(y)\,dx\,dy \\
&- \frac{1}{2\pi}\int_{\mathbb{R}^2\times\mathbb{R}^2} \log\frac{1}{|x-y|} w^2(x)w(y)v(y)\,dx\,dy,
\end{aligned}$$

but since z_ξ is a solution for (1.33), hence a critical point of (1.35), the first summand on the right-hand side is zero, so that

$$\begin{aligned}
R_\xi(w) = &-\frac{1}{\pi}\int_{\mathbb{R}^2\times\mathbb{R}^2} \log\frac{1}{|x-y|} z_\xi(x)w(x)w(y)v(y)\,dx\,dy \\
&- \frac{1}{2\pi}\int_{\mathbb{R}^2\times\mathbb{R}^2} \log\frac{1}{|x-y|} w^2(x)z_\xi(y)v(y)\,dx\,dy \\
&- \frac{1}{2\pi}\int_{\mathbb{R}^2\times\mathbb{R}^2} \log\frac{1}{|x-y|} w^2(x)w(y)v(y)\,dx\,dy.
\end{aligned}$$

We need to estimate all these integral: from (1.30) and (1.31) we have that there exist three constants $C_1, C_2, C_3 > 0$ such that

$$\begin{aligned}&\left|\int_{\mathbb{R}^2\times\mathbb{R}^2} \log\frac{1}{|x-y|} z_\xi(x)w(x)w(y)v(y)\,dx\,dy\right| \\ &\le C_1\|z_\xi\|_{L^{\frac{4}{3}}(\mathbb{R}^2)}\|w\|^2_{L^{\frac{4}{3}}(\mathbb{R}^2)}\|v\|_{L^{\frac{4}{3}}(\mathbb{R}^2)} \\ &\quad + |z_\xi|_*|w|_*\|w\|_{L^2(\mathbb{R}^2)}\|v\|_{L^2(\mathbb{R}^2)} + |w|_*|v|_*\|z_\xi\|_{L^2(\mathbb{R}^2)}\|w\|_{L^2(\mathbb{R}^2)},\end{aligned} \tag{1.54}$$

and

$$\begin{aligned}&\left|\int_{\mathbb{R}^2\times\mathbb{R}^2} \log\frac{1}{|x-y|} w^2(x)z_\xi(y)v(y)\,dx\,dy\right| \\ &\le C_2\|w\|^2_{L^{\frac{8}{3}}(\mathbb{R}^2)}\|z_\xi\|_{L^{\frac{4}{3}}(\mathbb{R}^2)}\|v\|_{L^{\frac{4}{3}}(\mathbb{R}^2)} + |w|^2_*\|z_\xi\|_{L^2(\mathbb{R}^2)}\|v\|_{L^2(\mathbb{R}^2)} + |z_\xi|_*|v|_*\|w\|^2_{L^2(\mathbb{R}^2)}\end{aligned} \tag{1.55}$$

and

$$\begin{aligned}&\left|\int_{\mathbb{R}^2\times\mathbb{R}^2} \log\frac{1}{|x-y|} w^2(x)w(y)v(y)\,dx\,dy\right| \\ &\le C_3\|w\|^2_{L^{\frac{8}{3}}(\mathbb{R}^2)}\|w\|_{L^{\frac{4}{3}}(\mathbb{R}^2)}\|v\|_{L^{\frac{4}{3}}(\mathbb{R}^2)} + |w|^2_*\|w\|_{L^2(\mathbb{R}^2)}\|v\|_{L^2(\mathbb{R}^2)} + |w|_*|v|_*\|w\|^2_{L^2(\mathbb{R}^2)}.\end{aligned} \tag{1.56}$$

Therefore, from (1.54),(1.55) and (1.56) we obtain

$$\begin{aligned}|R_\xi(w)| \le \frac{1}{\pi}&\Big(C_1\|z_\xi\|_{L^{\frac{4}{3}}(\mathbb{R}^2)}\|w\|^2_{L^{\frac{4}{3}}(\mathbb{R}^2)}\|v\|_{L^{\frac{4}{3}}(\mathbb{R}^2)} \\ &+|z_\xi|_*|w|_*\|w\|_{L^2(\mathbb{R}^2)}\|v\|_{L^2(\mathbb{R}^2)} + |w|_*|v|_*\|z_\xi\|_{L^2(\mathbb{R}^2)}\|w\|_{L^2(\mathbb{R}^2)}\Big) \\ + \frac{1}{2\pi}&\Big(C_2\|w\|^2_{L^{\frac{8}{3}}(\mathbb{R}^2)}\|z_\xi\|_{L^{\frac{4}{3}}(\mathbb{R}^2)}\|v\|_{L^{\frac{4}{3}}(\mathbb{R}^2)} \\ &+|w|^2_*\|z_\xi\|_{L^2(\mathbb{R}^2)}\|v\|_{L^2(\mathbb{R}^2)} + |z_\xi|_*|v|_*\|w\|^2_{L^2(\mathbb{R}^2)}\Big) \\ + \frac{1}{2\pi}&\Big(C_3\|w\|^2_{L^{\frac{8}{3}}(\mathbb{R}^2)}\|w\|_{L^{\frac{4}{3}}(\mathbb{R}^2)}\|v\|_{L^{\frac{4}{3}}(\mathbb{R}^2)} \\ &+|w|^2_*\|w\|_{L^2(\mathbb{R}^2)}\|v\|_{L^2(\mathbb{R}^2)} + |w|_*|v|_*\|w\|^2_{L^2(\mathbb{R}^2)}\Big).\end{aligned}$$

Now, from Proposition 1.7 we deduce that there exist some embedding constants (that for the sake of simplicity we denote always with C) such that

$$\begin{aligned}|R_\xi(w)| \le \frac{1}{\pi}&\left(C\|z_\xi\|_X\|w\|^2_X\|v\|_X + C\|z_\xi\|_X\|w\|_X\|w\|_X\|v\|_X + C\|w\|_X\|v\|_X\|z_\xi\|_X\|w\|_X\right) \\ + \frac{1}{2\pi}&\left(C\|w\|^2_X\|z_\xi\|_X\|v\|_X + C\|w\|^2_X\|z_\xi\|_X\|v\|_X + C\|z_\xi\|_X\|v\|_X\|w\|^2_X\right) \\ + \frac{1}{2\pi}&\left(C\|w\|^2_X\|w\|_X\|v\|_X + C\|w\|^2_X\|w\|_X\|v\|_X + C\|w\|_X\|v\|_X\|w\|^2_X\right)\end{aligned}$$

and arranging a bit, we can say that there exist $C_4, C_5, C_6 > 0$ such that

$$|R_\xi(w)| \le C_4\|z_\xi\|_X\|w\|^2_X\|v\|_X + C_5\|w\|^2_X\|z_\xi\|_X\|v\|_X + C_6\|w\|^3_X\|v\|_X. \tag{1.57}$$

Finally,

$$\frac{|R_\xi(w)|}{\|w\|_X} \le C_4\|z_\xi\|_X\|w\|_X\|v\|_X + C_5\|w\|_X\|z_\xi\|_X\|v\|_X + C_6\|w\|^2_X\|v\|_X$$

and the proof is conclude by going to the limit for $\|w\|_X \to 0$ uniformly with respect to $\xi \in \mathbb{R}^2$. □

As a consequence of the previous result, we can apply Lemma 1.5, so there is an $\varepsilon_0 > 0$ such that for every $|\varepsilon| \le \varepsilon_0$ and all $\xi \in \mathbb{R}^2$ the auxiliary equation $P\mathcal{J}'_\varepsilon(z_\xi + w) = 0$ admits a unique solution $w_{\varepsilon,\xi} := w_\varepsilon(z_\xi)$ such that

$$\lim_{\varepsilon\to 0} \|w_\varepsilon(z_\xi)\| = 0 \tag{1.58}$$

uniformly with respect to $\xi \in \mathbb{R}^2$.

Now, we prove the second technical Lemma that deals with vanishing of this unique solution in the strong sense.

Lemma 1.25. *There exixts $\varepsilon_1 > 0$ such that for all $|\varepsilon| \le \varepsilon_1$ it holds*

$$\lim_{|\xi|\to+\infty} w_\varepsilon(z_\xi) = 0$$

strongly in X.

Proof. We divide the proof in several steps.

Step 1: $w_{\varepsilon,\xi} \rightharpoonup w_{\varepsilon,\infty}$ in X. From (1.58) we have that the sequence $(w_{\varepsilon,\xi})_\xi$ is bounded; so, there exists (see [38], Theorem 3.18) a subsequence, for convenience still denoted with $(w_{\varepsilon,\xi})_\xi$, and a limit point $w_{\varepsilon,\infty}$ such that

$$w_{\varepsilon,\xi} \rightharpoonup w_{\varepsilon,\infty}$$

as $|\xi| \to +\infty$.

Step 2: the weak limit $w_{\varepsilon,\infty}$ is a weak solution of

$$-\Delta w_{\varepsilon,\infty} + a w_{\varepsilon,\infty} - \frac{1}{2\pi}\left(\log\frac{1}{|\cdot|} \star w_{\varepsilon,\infty}^2\right) w_{\varepsilon,\infty} = \varepsilon h(x)|w_{\varepsilon,\infty}|^{p-1} w_{\varepsilon,\infty}. \tag{1.59}$$

We recall that $w_\xi(z_\xi)$ solves the auxiliary equation, that is

$$\begin{aligned}
&-\Delta w_{\varepsilon,\xi} + a w_{\varepsilon,\xi} - \frac{1}{2\pi}\left(\log\frac{1}{|\cdot|} \star w_{\varepsilon,\xi}^2\right) w_{\varepsilon,\xi} \\
&\quad - \frac{1}{2\pi}\left(\log\frac{1}{|\cdot|} \star z_\xi^2\right) w_{\varepsilon,\xi} - \frac{1}{\pi}\left(\log\frac{1}{|\cdot|} \star z_\xi^2\right)(z_\xi + w_{\varepsilon,\xi}) - \frac{1}{2\pi}\left(\log\frac{1}{|\cdot|} \star w_{\varepsilon,\xi}^2\right) z_\xi \\
&\qquad = \varepsilon h(x)|z_\xi + w_{\varepsilon,\xi}|^{p-1}(z_\xi + w_{\varepsilon,\xi}) - z_\xi^{p-1} - \sum_{i=1}^{2} a_i D_i z_\xi
\end{aligned}$$

where

$$a_i = \int_{\mathbb{R}^2} \left(\varepsilon h(x)|z_\xi + w_{\varepsilon,\xi}|^{p-1} - z_\xi^{p-1}\right) D_i z_\xi \, dx.$$

Let $v \in X$ be any test function, integrating by parts the equation above we obtain

$$\begin{aligned}
&\int_{\mathbb{R}^2} \left(\nabla w_{\varepsilon,\xi} \cdot \nabla v + a w_{\varepsilon,\xi} v\right) dx - \frac{1}{2\pi}\int_{\mathbb{R}^2\times\mathbb{R}^2} \log\frac{1}{|x-y|} w_{\varepsilon,\xi}^2(x) w_{\varepsilon,\xi}(y) v(y)\, dx\, dy \\
&= \frac{1}{2\pi}\int_{\mathbb{R}^2\times\mathbb{R}^2} \log\frac{1}{|x-y|} z_\xi^2(x) w_{\varepsilon,\xi}(y) v(y)\, dx\, dy \\
&\quad + \frac{1}{\pi}\int_{\mathbb{R}^2\times\mathbb{R}^2} \log\frac{1}{|x-y|} z_\xi^2(x) \left(z_\xi + w_{\varepsilon,\xi}\right)(y) v(y)\, dx\, dy \\
&\quad + \frac{1}{2\pi}\int_{\mathbb{R}^2\times\mathbb{R}^2} \log\frac{1}{|x-y|} w_{\varepsilon,\xi}^2(x) z_\xi(y) v(y)\, dx\, dy \\
&\quad + \int_{\mathbb{R}^2} \varepsilon h(x)|z_\xi + w_{\varepsilon,\xi}|^{p-1} v(x)\, dx - \int_{\mathbb{R}^2} z_\xi^{p-1}(x) v(x)\, dx \\
&\quad - \sum_{i=1}^{2} a_i \int_{\mathbb{R}^2} D_i z_\xi(x) v(x)\, dx.
\end{aligned}$$

Following the same computations as in Proposition 1.19 and using the exponential decay (1.39) we can observe that, letting $|\xi| \to +\infty$,

$$\int_{\mathbb{R}^2\times\mathbb{R}^2} \log\frac{1}{|x-y|} z_\xi^2(x) w_{\varepsilon,\xi}(y) v(y)\,dx\,dy \to 0,$$
$$\int_{\mathbb{R}^2\times\mathbb{R}^2} \log\frac{1}{|x-y|} z_\xi^2(x) \left(z_\xi + w_{\varepsilon,\xi}\right)(y) v(y)\,dx\,dy \to 0,$$
$$\int_{\mathbb{R}^2\times\mathbb{R}^2} \log\frac{1}{|x-y|} w_{\varepsilon,\xi}^2(x) z_\xi(y) v(y)\,dx\,dy \to 0.$$

Now, we take care of the term

$$\int_{\mathbb{R}^2} \varepsilon h(x)|z_\xi + w_{\varepsilon,\xi}|^{p-1} v(x)\,dx. \tag{1.60}$$

We would like to pass to the limit in this integral and to do that, we claim that

$$\lim_{\xi\to 0}\int_{\mathbb{R}^2} z_\xi^{p-1-k}(x) w_{\varepsilon,\xi}^k(x) v(x)\,dx = 0 \tag{1.61}$$

for all $k \in [0, p-1)$.

We fix $\rho > 0$ and we split the integral as

$$\int_{\mathbb{R}^2} z_\xi^{p-1-k}(x) w_{\varepsilon,\xi}^k(x) v(x)\,dx$$
$$= \int_{|x|\le\rho} z_\xi^{p-1-k}(x) w_{\varepsilon,\xi}^k(x) v(x)\,dx + \int_{|x|>\rho} z_\xi^{p-1-k}(x) w_{\varepsilon,\xi}^k(x) v(x)\,dx.$$

We make us of the Hölder inequality with $s = k+1$ and $s' = \frac{p}{p-1-k}$ for both the integrals in the right-hand side: in the first case we obtain

$$\begin{aligned}
&\left|\int_{|x|\le\rho} z_\xi^{p-1-k}(x) w_{\varepsilon,\xi}^k(x) v(x)\,dx\right| \\
&\le \left(\int_{|x|\le\rho} |z_\xi(x)|^{(p-1-k)s'}\,dx\right)^{\frac{1}{s'}} \left(\int_{|x|\le\rho} |w_{\varepsilon,\xi}(x)|^{ks} |v(x)|^s\,dx\right)^{\frac{1}{s}} \\
&\le \left(\int_{|x|\le\rho} |z_\xi(x)|^p\,dx\right)^{\frac{p-k-1}{p}} \left(\int_{|x|\le\rho} |w_{\varepsilon,\xi}(x)|^{k(k+1)} |v(x)|^{k+1}\,dx\right)^{\frac{1}{k+1}} \\
&\le C\left(\int_{|x|\le\rho} |z_\xi(x)|^p\,dx\right)^{\frac{p-k-1}{p}}
\end{aligned} \tag{1.62}$$

where $C > 0$ follows from (1.39) and, as ρ goes to infinity, the latter term goes to 0.

For the second integral, we have

$$\begin{aligned}
&\left|\int_{|x|>\rho} z_\xi^{p-1-k}(x) w_{\varepsilon,\xi}^k(x) v(x)\,dx\right| \\
&\le \left(\int_{|x|>\rho} |z_\xi(x)|^{(p-1-k)s'} |w_{\varepsilon,\xi}(x)|^{ks'}\,dx\right)^{\frac{1}{s'}} \left(\int_{|x|>\rho} |v(x)|^s\,dx\right)^{\frac{1}{s}} \\
&= \left(\int_{|x|>\rho} |z_\xi(x)|^p |w_{\varepsilon,\xi}(x)|^{k\frac{p}{p-k-1}}\,dx\right)^{\frac{p-k-1}{p}} \left(\int_{|x|>\rho} |v(x)|^{k+1}\,dx\right)^{\frac{1}{k+1}}
\end{aligned} \tag{1.63}$$

and since v is a test function, this goes to 0 as ρ diverges.

From (1.62) and (1.63) we prove the claim, therefore we can pass to the limit in (1.60), so

$$\int_{\mathbb{R}^2} \varepsilon h(x)|z_\xi + w_{\varepsilon,\xi}|^{p-1} v(x)\, dx \to \int_{\mathbb{R}^2} \varepsilon h(x)|w_{\varepsilon,\infty}|^{p-1} v(x)\, dx, \tag{1.64}$$

$$\int_{\mathbb{R}^2} |w_{\varepsilon,\xi}(x)|^{p-1} v(x)\, dx \to \int_{\mathbb{R}^2} |w_{\varepsilon,\infty}(x)|^{p-1} v(x)\, dx, \tag{1.65}$$

$$\int_{\mathbb{R}^2} \varepsilon h(x)|w_{\varepsilon,\xi}(x)|^{p-1} v(x)\, dx \to \int_{\mathbb{R}^2} \varepsilon h(x)|w_{\varepsilon,\infty}(x)|^{p-1} v(x)\, dx \tag{1.66}$$

as $|\xi| \to +\infty$.

Finally, by the exponential decay (1.39), we have

$$\int_{\mathbb{R}^2} z_\xi^{p-1}(x) v(x)\, dx \to 0, \tag{1.67}$$

$$\int_{\mathbb{R}^2} D_i z_\xi(x) v(x)\, dx \to 0 \tag{1.68}$$

as $|\xi|$ diverges.

Hence, letting $|\xi| \to +\infty$ and by (1.60), (1.64) and (1.67), we obtain

$$\begin{aligned}\int_{\mathbb{R}^2} \left(\nabla w_{\varepsilon,\infty}(x) \cdot \nabla v(x) + a w_{\varepsilon,\infty}(x) v(x)\right)\, dx \\ - \frac{1}{2\pi}\int_{\mathbb{R}^2\times\mathbb{R}^2} \log\frac{1}{|x-y|} w_{\varepsilon,\infty}^2(x) w_{\varepsilon,\infty}(y) v(y)\, dx\, dy \\ = \int_{\mathbb{R}^2} \varepsilon h(x)|w_{\varepsilon,\infty}(x)|^{p-1} v(x)\, dx,\end{aligned}$$

that is, $w_{\varepsilon,\infty}$ is a weak solution of (1.59).

Step 3: $w_{\varepsilon,\infty} = 0$. First, we observe that from (1.58) follows that

$$\lim_{|\varepsilon|\to 0} w_{\varepsilon,\infty} = 0.$$

We claim the following: since the unique solution $w \in X$ of

$$-\Delta w(x) + a w(x) - \frac{1}{2\pi}\left(\log\frac{1}{|\cdot|} \star w^2(x)\right) w(x) = \varepsilon h(x)|w(x)|^{p-1} w(x)$$

with small norm is $w = 0$, then we infer that $w_{\varepsilon,\infty} = 0$, provided $|\varepsilon|$ is small.

To prove this claim, we show that there exists a constant $C > 0$ such that

$$\|w\|_X \geq C.$$

Considering the first Gâteaux derivative of (1.34) evaluated at $w \in X$ along $w \in X$ we have

$$\begin{aligned}0 = \mathcal{J}_\varepsilon'(w)[w] = \int_{\mathbb{R}^2} \left(|\nabla w(x)|^2 + a w^2(x)\right)\, dx \\ - \frac{1}{2\pi}\int_{\mathbb{R}^2\times\mathbb{R}^2} \log\frac{1}{|x-y|} w^2(x) w^2(y)\, dx\, dy - \varepsilon \int_{\mathbb{R}^2} h(x)|w(x)|^{p+1}\, dx\end{aligned}$$

that is

$$\begin{aligned}\|w\|_{H^1(\mathbb{R}^2)}^2 &= \frac{1}{2\pi}\int_{\mathbb{R}^2\times\mathbb{R}^2} \log\frac{1}{|x-y|} w^2(x) w^2(y)\, dx\, dy + \varepsilon \int_{\mathbb{R}^2} h(x)|w(x)|^{p+1}\, dx \\ &\leq \frac{1}{2\pi}\int_{\mathbb{R}^2\times\mathbb{R}^2} \log\frac{1}{|x-y|} w^2(x) w^2(y)\, dx\, dy + \varepsilon \int_{\mathbb{R}^2} |h(x)||w(x)|^{p+1}\, dx.\end{aligned} \tag{1.69}$$

Observe that, from (1.23), we have

$$\int_{\mathbb{R}^2} \log\left(1+\frac{1}{|x-y|}\right) w^2(x)w^2(y)\,dx\,dy$$
$$\leq \int_{\mathbb{R}^2} \log\left(1+|x-y|\right) w^2(x)w^2(y)\,dx\,dy + \int_{\mathbb{R}^2} \log\frac{1}{|x-y|} w^2(x)w^2(y)\,dx\,dy$$

and it follows that

$$\int_{\mathbb{R}^2} \log\left(1+\frac{1}{|x-y|}\right) w^2(x)w^2(y)\,dx\,dy \leq \int_{\mathbb{R}^2} \log\frac{1}{|x-y|} w^2(x)w^2(y)\,dx\,dy.$$

Morover, from the assumpion on h and the Sobolev embedding we have that there exist $C_1, C_2 > 0$ such that

$$\varepsilon \int_{\mathbb{R}^2} |h(x)||w(x)|^{p+1}\,dx \leq C_1 \|w\|^{p+1}_{L^{p+1}(\mathbb{R}^2)} \leq C_2 \|w\|^{p+1}_{H^1(\mathbb{R}^2)},$$

and from (1.69), (1.31) and Sobolev embedding, it follows that there exists a constant $C > 0$ such that

$$\begin{aligned}\|w\|^2_{H^1(\mathbb{R}^2)} &\leq \int_{\mathbb{R}^2\times\mathbb{R}^2} \log\frac{1}{|x-y|} w^2(x)w^2(y)\,dx\,dy + \varepsilon \int_{\mathbb{R}^2} |h(x)||w(x)|^{p+1}\,dx \\ &\leq C\|w\|^4_{L^{\frac{8}{3}}(\mathbb{R}^2)} + C_1\|w\|^{p+1}_{H^1(\mathbb{R}^2)} \leq C_3 \|w\|^{\eta}_{H^1(\mathbb{R}^2)}\end{aligned}$$

where C_3 is a positive constant and $\eta = \max\{p+1, 4\}$: notice that $\eta > 2$ since $p > 1$. We also observe that

$$\|w\|^{\eta-2}_{H^1(\mathbb{R}^2)} \leq \|w\|^{\eta-2}_X,$$

hence, there exists a constant $C_5 > 0$ such that.

$$\|w\|^{\eta-2}_X \geq \frac{1}{C_3} > 0$$

and the claim is proved.

Step 4: $\|w_{\varepsilon,\xi}\|_X \to 0$ as $|\xi| \to +\infty$.

We observe that we can rewrite the auxiliary equation

$$J'_\varepsilon(z_\xi + w_{\varepsilon,\xi}) = \mathcal{J}'_0(z_\xi + w_{\varepsilon,\xi}) + \varepsilon G'(z_\xi + w_{\varepsilon,\xi}) = 0$$

as

$$\mathcal{J}''_0(z_\xi)[w_{\varepsilon,\xi}] + R_\xi(w_{\varepsilon,\xi}) + \varepsilon G'(z_\xi + w_{\varepsilon,\xi}) = 0,$$

where

$$G'(z_\xi + w_{\varepsilon,\xi}) = -\int_{\mathbb{R}^2} h(x)|z_\xi(x) + w_{\varepsilon,\xi}(x)|^{p-1}(z_\xi(x) + w_{\varepsilon,\xi}(x))\,dx$$

and

$$\begin{aligned}R_\xi(w_{\varepsilon,\xi}) = &-\frac{1}{\pi}\int_{\mathbb{R}^2\times\mathbb{R}^2} \log\frac{1}{|x-y|} z_\xi(x)w_{\varepsilon,\xi}(x)w_{\varepsilon,\xi}(y)v(y)\,dx\,dy \\ &-\frac{1}{2\pi}\int_{\mathbb{R}^2\times\mathbb{R}^2} \log\frac{1}{|x-y|} w^2_{\varepsilon,\xi}(x)z_\xi(y)v(y)\,dx\,dy \\ &-\frac{1}{2\pi}\int_{\mathbb{R}^2\times\mathbb{R}^2} \log\frac{1}{|x-y|} w^2_{\varepsilon,\xi}(x)w_{\varepsilon,\xi}(y)v(y)\,dx.\,dy\end{aligned}$$

Then,

$$P\mathcal{J}''_0(z_\xi)[w_{\varepsilon,\xi}] + PR_\xi(w_{\varepsilon,\xi}) + P\varepsilon G'(z_\xi + w_{\varepsilon,\xi}) = 0$$

and since $P\mathcal{J}_0''(z_\xi)$ is invertible, we obtain the fixed point equation

$$w_{\varepsilon,\xi} = N_{\varepsilon,\xi}(w_{\varepsilon,\xi}) \tag{1.70}$$

with $N_{\varepsilon,\xi}(w_{\varepsilon,\xi}) := (P\mathcal{J}_0''(z_\xi))^{-1}\left[-PR_\xi(w_{\varepsilon,\xi}) - P\varepsilon G'(z_\xi + w_{\varepsilon,\xi})\right].$

From Theorem 1.24 (i) and (1.70) it follows that there exists a constant $C > 0$ such that

$$\|w_{\varepsilon,\xi}\|^2 \leq C\left(|\varepsilon|\,|G'(z_\xi + w_{\varepsilon,\xi})w_{\varepsilon,\xi}| + |R_\xi(w_{\varepsilon,\xi})w_{\varepsilon,\xi}|\right). \tag{1.71}$$

By the compact embedding 1.7, Step 1 and Step 2 we have that

$$\begin{aligned}
|G'(z_\xi + w_{\varepsilon,\xi})w_{\varepsilon,\xi}| &\leq \int_{\mathbb{R}^2} |h(x)||z_\xi(x) + w_{\varepsilon,\xi}(x)|^p |w_{\varepsilon,\xi}(x)\,dx \\
&\leq \|h\|_{L^\infty(\mathbb{R}^2)}\left(\int_{\mathbb{R}^2} |z_\xi(x)|^p |w_{\varepsilon,\xi}(x)|\,dx + \int_{\mathbb{R}^2} |w_{\varepsilon,\xi}(x)|^{p+1}\,dx\right) \\
&\leq \|h\|_{L^\infty(\mathbb{R}^2)}\left[\left(\int_{\mathbb{R}^2} |z_\xi(x)|^{2p}\,dx\right)^{\frac{1}{2}}\left(\int_{\mathbb{R}^2} |w_{\varepsilon,\xi}(x)|^2\,dx\right)^{\frac{1}{2}} + \|w_{\varepsilon,\xi}\|^{p+1}_{L^{p+1}(\mathbb{R}^2)}\right] \\
&\leq C\left(\|w_{\varepsilon,\xi}\|^2_X + \|w_{\varepsilon,\xi}\|^{p+1}_X\right)
\end{aligned}$$

goes to 0 as $|\xi| \to +\infty$.

For the second addendum of (1.71), we have that from (1.57),

$$|R_\xi(w_{\varepsilon,\xi})w_{\varepsilon,\xi}| \leq C_4\|z_\xi\|_X\|w_{\varepsilon,\xi}\|^3_X + C_5\|w_{\varepsilon,\xi}\|^3_X\|z_\xi\|_X + C_6\|w_{\varepsilon,\xi}\|^4_X,$$

that is

$$|R_\xi(w_{\varepsilon,\xi})w_{\varepsilon,\xi}| \leq C_6\|w_{\varepsilon,\xi}\|^4_X + o(\varepsilon),$$

as $|\xi| \to +\infty$.

Putting this inequality in (1.71), we obtain

$$\|w_{\varepsilon,\xi}\|^2_X \leq C_6\|w_{\varepsilon,\xi}\|^4_X + o(\varepsilon),$$

as $|\xi| \to +\infty$ and passing to the limit we have

$$\lim_{|\xi|\to+\infty} \|w_{\varepsilon,\xi}\|^2_X \leq C_6 \lim_{|\xi|\to+\infty} \|w_{\varepsilon,\xi}\|^4_X.$$

We recall that $w_{\varepsilon,\xi}$ is small in X, provided $|\varepsilon| \to 0$; hence,

$$\lim_{|\xi|\to+\infty} \|w_{\varepsilon,\xi}\|_X = 0$$

priveded $|\varepsilon| << 1$. The proof is concluded. □

Finally, we give the main result of this subsection, i.e. the existence of a *global* solution for (1.33).

Theorem 1.26. *Let $p > 2$ and suppose that h satisfies the following assumption:*

(h2) $h \in L^\infty(\mathbb{R}^2)$ and $\lim_{|x|\to+\infty} h(x) = 0$.

Then, for every $|\varepsilon|$ small, the equation (1.33) has a solution.

Proof. Let $\Phi_\varepsilon(\xi) = \mathcal{J}_\varepsilon(z_\xi + w_{\varepsilon,\xi})$ be the reduced functional, i.e.

$$\Phi_\varepsilon(\xi) = \frac{1}{2}\|z_\xi + w_{\varepsilon,\xi}\|^2_{H^1(\mathbb{R}^2)} - \frac{1}{8\pi}\int_{\mathbb{R}^2\times\mathbb{R}^2} \log\frac{1}{|x-y|}(z_\xi + w_{\varepsilon,\xi})^2(x)(z_\xi + w_{\varepsilon,\xi})^2(y)\,dx\,dy \quad (1.72)$$

$$- \frac{\varepsilon}{p+1}\int_{\mathbb{R}^2} h(x)|z_\xi + w_{\varepsilon,\xi}|^{p+1}\,dx. \quad (1.73)$$

We set

$$c_0 = \mathcal{J}_0(z_\xi) = \frac{1}{2}\|z_\xi\|^2_{H^1(\mathbb{R}^2)} - \frac{1}{8\pi}\int_{\mathbb{R}^2\times\mathbb{R}^2} \log\frac{1}{|x-y|} z_\xi^2(x)z_\xi^2(y)\,dx\,dy,$$

therefore

$$\frac{1}{2}\|z_\xi\|^2_{H^1(\mathbb{R}^2)} = c_0 + \frac{1}{8\pi}\int_{\mathbb{R}^2\times\mathbb{R}^2} \log\frac{1}{|x-y|} z_\xi^2(x)z_\xi^2(y)\,dx\,dy. \quad (1.74)$$

Moreover, from

$$-\Delta z_\xi + a z_\xi - \frac{1}{2\pi}\left(\log\frac{1}{|\cdot|} \star z_\xi^2\right) z_\xi = 0$$

follows

$$(z_\xi, w_{\varepsilon,\xi})_{H^1(\mathbb{R}^2)} = \frac{1}{2\pi}\int_{\mathbb{R}^2\times\mathbb{R}^2} \log\frac{1}{|x-y|} z_\xi^2(x)z_\xi^2(y)\,dx\,dy. \quad (1.75)$$

Putting (1.74) and (1.75) into (1.72) we obtain

$$\begin{aligned}
\Phi_\varepsilon(\xi) &= \frac{1}{2}\|z_\xi\|^2_{H^1(\mathbb{R}^2)} + \frac{1}{2}\|w_{\varepsilon,\xi}\|^2_{H^1(\mathbb{R}^2)} + (z_\xi, w_{\varepsilon,\xi})_{H^1(\mathbb{R}^2)} \\
&\quad - \frac{1}{8\pi}\int_{\mathbb{R}^2\times\mathbb{R}^2} \log\frac{1}{|x-y|}(z_\xi + w_{\varepsilon,\xi})^2(x)(z_\xi + w_{\varepsilon,\xi})^2(y)\,dx\,dy \\
&\quad - \frac{\varepsilon}{p+1}\int_{\mathbb{R}^2} h(x)|z_\xi + w_{\varepsilon,\xi}|^{p+1}\,dx \\
&= c_0 + \frac{1}{8\pi}\int_{\mathbb{R}^2\times\mathbb{R}^2} \log\frac{1}{|x-y|} z_\xi^2(x)z_\xi^2(y)\,dx\,dy \\
&\quad + \frac{1}{2}\|w_{\varepsilon,\xi}\|^2_{H^1(\mathbb{R}^2)} + \frac{1}{2\pi}\int_{\mathbb{R}^2\times\mathbb{R}^2} \log\frac{1}{|x-y|} z_\xi^2(x)z_\xi(y)w_{\varepsilon,\xi}(y)\,dx\,dy \\
&\quad - \frac{1}{8\pi}\int_{\mathbb{R}^2\times\mathbb{R}^2} \log\frac{1}{|x-y|}(z_\xi + w_{\varepsilon,\xi})^2(x)(z_\xi + w_{\varepsilon,\xi})^2(y)\,dx\,dy \\
&\quad - \frac{\varepsilon}{p+1}\int_{\mathbb{R}^2} h(x)|z_\xi + w_{\varepsilon,\xi}|^{p+1}\,dx.
\end{aligned}$$

Making some computations, the reduced functional become

$$\begin{aligned}
\Phi_\varepsilon(\xi) &= c_0 + \frac{1}{8\pi}\int_{\mathbb{R}^2\times\mathbb{R}^2} \log\frac{1}{|x-y|} z_\xi^2(x)z_\xi^2(y)\,dx\,dy \\
&\quad + \frac{1}{2}\|w_{\varepsilon,\xi}\|^2_{H^1(\mathbb{R}^2)} + \frac{1}{2\pi}\int_{\mathbb{R}^2\times\mathbb{R}^2} \log\frac{1}{|x-y|} z_\xi^2(x)z_\xi(y)w_{\varepsilon,\xi}(y)\,dx\,dy \\
&\quad - \frac{1}{8\pi}\int_{\mathbb{R}^2\times\mathbb{R}^2} \log\frac{1}{|x-y|} z_\xi^2(x)z_\xi^2(y)\,dx\,dy \\
&\quad - \frac{1}{4\pi}\int_{\mathbb{R}^2\times\mathbb{R}^2} \log\frac{1}{|x-y|} z_\xi^2(x)z_\xi(y)w_{\varepsilon,\xi}(y)\,dx\,dy \\
&\quad - \frac{1}{8\pi}\int_{\mathbb{R}^2\times\mathbb{R}^2} \log\frac{1}{|x-y|} z_\xi^2(x)w_{\varepsilon,\xi}^2(y)\,dx\,dy \\
&\quad - \frac{1}{4\pi}\int_{\mathbb{R}^2\times\mathbb{R}^2} \log\frac{1}{|x-y|} z_\xi(x)w_{\varepsilon,\xi}(x)z_\xi^2(y)\,dx\,dy \\
&\quad - \frac{1}{2\pi}\int_{\mathbb{R}^2\times\mathbb{R}^2} \log\frac{1}{|x-y|} z_\xi(x)w_{\varepsilon,\xi}(x)z_\xi(y)w_{\varepsilon,\xi}(y)\,dx\,dy
\end{aligned}$$

$$
\begin{aligned}
&- \frac{1}{4\pi}\int_{\mathbb{R}^2\times\mathbb{R}^2} \log\frac{1}{|x-y|} z_\xi(x) w_{\varepsilon,\xi}(x) w_{\varepsilon,\xi}^2(y)\,dx\,dy \\
&- \frac{1}{8\pi}\int_{\mathbb{R}^2\times\mathbb{R}^2} \log\frac{1}{|x-y|} w_{\varepsilon,\xi}^2(x) z_\xi^2(y)\,dx\,dy \\
&- \frac{1}{4\pi}\int_{\mathbb{R}^2\times\mathbb{R}^2} \log\frac{1}{|x-y|} w_{\varepsilon,\xi}^2(x) z_\xi(y) w_{\varepsilon,\xi}(y)\,dx\,dy \\
&- \frac{1}{8\pi}\int_{\mathbb{R}^2\times\mathbb{R}^2} \log\frac{1}{|x-y|} w_{\varepsilon,\xi}^2(x) w_{\varepsilon,\xi}^2(y)\,dx\,dy \\
&- \frac{\varepsilon}{p+1}\int_{\mathbb{R}^2} h(x)|z_\xi(x)+w_{\varepsilon,\xi}(x)|^{p+1}\,dx,
\end{aligned}
$$

and after some simplifications, we finally obtain

$$
\begin{aligned}
\Phi_\varepsilon(\xi) = c_0 &+ \frac{1}{2}\|w_{\varepsilon,\xi}\|^2_{H^1(\mathbb{R}^2)} \\
&+ \frac{1}{2\pi}\int_{\mathbb{R}^2\times\mathbb{R}^2} \log\frac{1}{|x-y|} z_\xi^2(x) z_\xi(y) w_{\varepsilon,\xi}(y)\,dx\,dy \\
&- \frac{1}{4\pi}\int_{\mathbb{R}^2\times\mathbb{R}^2} \log\frac{1}{|x-y|} z_\xi^2(x) z_\xi(y) w_{\varepsilon,\xi}(y)\,dx\,dy \\
&- \frac{1}{8\pi}\int_{\mathbb{R}^2\times\mathbb{R}^2} \log\frac{1}{|x-y|} z_\xi^2(x) w_{\varepsilon,\xi}^2(y)\,dx\,dy \\
&- \frac{1}{4\pi}\int_{\mathbb{R}^2\times\mathbb{R}^2} \log\frac{1}{|x-y|} z_\xi(x) w_{\varepsilon,\xi}(x) z_\xi^2(y)\,dx\,dy \\
&- \frac{1}{2\pi}\int_{\mathbb{R}^2\times\mathbb{R}^2} \log\frac{1}{|x-y|} z_\xi(x) w_{\varepsilon,\xi}(x) z_\xi(y) w_{\varepsilon,\xi}(y)\,dx\,dy \\
&- \frac{1}{4\pi}\int_{\mathbb{R}^2\times\mathbb{R}^2} \log\frac{1}{|x-y|} z_\xi(x) w_{\varepsilon,\xi}(x) w_{\varepsilon,\xi}^2(y)\,dx\,dy \\
&- \frac{1}{8\pi}\int_{\mathbb{R}^2\times\mathbb{R}^2} \log\frac{1}{|x-y|} w_{\varepsilon,\xi}^2(x) z_\xi^2(y)\,dx\,dy \\
&- \frac{1}{4\pi}\int_{\mathbb{R}^2\times\mathbb{R}^2} \log\frac{1}{|x-y|} w_{\varepsilon,\xi}^2(x) z_\xi(y) w_{\varepsilon,\xi}(y)\,dx\,dy \\
&- \frac{1}{8\pi}\int_{\mathbb{R}^2\times\mathbb{R}^2} \log\frac{1}{|x-y|} w_{\varepsilon,\xi}^2(x) w_{\varepsilon,\xi}^2(y)\,dx\,dy \\
&- \frac{\varepsilon}{p+1}\int_{\mathbb{R}^2} h(x)|z_\xi(x)+w_{\varepsilon,\xi}(x)|^{p+1}\,dx.
\end{aligned}
$$

Now, as in Proposition 1.19, all the double integrals vanishes as $|\xi|$ diverges to infinity, while by Lemma 1.25 follows that

$$\|w_{\varepsilon,\xi}\|^2_{H^1(\mathbb{R}^2)} \to 0$$

as $|\xi| \to +\infty$. Finally, by Minkowski inequality, Lemma 1.25, Proposition 1.39 and assumption (h2)

$$
\begin{aligned}
&\int_{\mathbb{R}^2} h(x)|z_\xi(x)+w_{\varepsilon,\xi}(x)|^{p+1}\,dx \\
&\qquad \le \|h\|_{L^\infty(\mathbb{R}^2)} \int_{\mathbb{R}^2} |z_\xi(x)+w_{\varepsilon,\xi}(x)|^{p+1}\,dx \\
&\qquad\qquad \le \|h\|_{L^\infty(\mathbb{R}^2)} \left(\int_{\mathbb{R}^2} |z_\xi(x)|^{p+1}\,dx + \int_{\mathbb{R}^2} |w_{\varepsilon,\xi}(x)|^{p+1}\,dx\right);
\end{aligned}
$$

hence

$$\lim_{|\xi|\to+\infty} \int_{\mathbb{R}^2} h(x)|z_\xi(x)+w_{\varepsilon,\xi}(x)|^{p+1}\,dx = 0.$$

from which follows that

$$\lim_{|\xi|\to+\infty} \Phi_\varepsilon(\xi) = c_0.$$

This last equality tells us that the reduced functional Φ_ε is either constant or it has a critical point (maximum or minimum). Whatever the case, we can then apply Theorem 1.6 to find a solution for equation (1.33). □

Chapter 2

Semirelativistic Choquard equation

Laplace operator has attracted (and still does) the attention of many mathematicians who are focused their efforts in the study of PDEs since many decades. Although the interest on this operator is still very high, in the last years some attention moved on problems driven by non-integer powers of the Laplacian, i.e. the so called *fractional Laplacian*. As often happens, the reasons come both from theoretical motivations and "concrete" ones. In fact, from a mathematical point of view, the study of problems where these operators appears can be interesting because we move from a local operator, the Laplacian, to a non-local one. With *local* we are saying that, given a point of the space, the operator gives back us information of what happen in a neighborhood of that point. Conversely, a *non-local* operator takes into account also the interaction that the points can have with other points, even at infinite distance. To justify this claim, consider a function $u \in C_c^\infty(B_2)$ such that $u = 1 \in B_1$ and $u \geq 0$. We take a point $x \in B_4$ and compute,

$$-(-\Delta)^s u(x) = \int_{\mathbb{R}^N} \frac{u(y) - u(x)}{|x-y|^{N+2s}}\, dy \geq \int_{B_1} \frac{dy}{|x-y|^{N+2s}} \geq \int_{B_1} \frac{dy}{2|x|^{N+2s}} = \frac{C}{|x|^{N+2}}$$

where in the last step we used the fact that in B_1 it holds $|x-y| \leq |x| + |y| \leq 2|x|$. However, $-\Delta u(x) = 0$ because $u = 0$ in a neighborhood of x.

Regarding the concrete field, it has been observed that fractional Laplacian finds several applications in many sectors, including quantum physics, finance, chemistry and fluid dynamics geophysics, describing models of many phenomena, such as dislocation of crystals, local and non-local phase transition and, especially, non-local Schrödinger equation (see [40] for a detailed treamtent of these models).

Another important field where non-local operator finds applications is the probabilistic theory: in fact, if we consider a probabilistic process in which a particle is allowed to move randomly even with "long jumps" then, for small jumps and times, this model leads to the fractional heat equation. Fractional Laplacian can also be used to describe a payoff model (see [40], Chapter 2). For a more in-depth story and further details about fractional Laplacian, and in general to non-local operators, we remind to [15, 99] and, obviously, to [56].

Aim of this Chapter is to show the results contained in [27], where we deal with the study of a semirelativistic Choquard equation where a singular potential and a general nonlinearity are considered. With *semirelativistic* we are meaning the operator

$$u \mapsto \left((-\Delta + m^2)^s - m^{2s} \right) u$$

where $s \in (0,1)$. If $s = 1/2$, this operator has a remarkable response in Quantum Physics (see Section 2.2 and 2.3 below). In this Chapter, we consider exactly the case with $s = 1/2$, that is we are dealing with the operator

$$u \mapsto \left(\sqrt{-\Delta + m^2} - m\right) u.$$

The main problem when dealing with these operators is given by the fact that, compared to the local case, they do not enjoy a scaling property. To overcome this problem, many approaches were introduced. One of the most famous and used is the Caffarelli-Silvestre extension (see [41]), or one can define the fractional operator via the modified Bessel function (see Appendix B in [62], Chapter 4 in [15]). However, we are not going to use none of the methods above, using instead the representation of the operator via its Fourier transform (in Section 2.5 below we will give the formal definition).

Going in the detail of this Chapter, in the next Section we will present the equation we are dealing with, deducing it form a Schrödinger-Poisson system. As in the first Chapter, we then present some physical motivations that lead to the study of this problem and then we will present the *state of art*. From Section 2.5 we will formally and mathematically enter in the problem, starting from the assumptions and giving some remarks on them. Afterwards, we will state and explain the reason of every technique or tools we needed to face our problem.

The last two Sections are devoted to the statements and the proofs of the main results. In particular, we give conditions for which a *ground-state* solution, i.e. a solution that correspond to the lowest energy level, does exists or not and we prove a compactness-results for this ground states.

2.1 Deriving the equation: act II

Consider the following Schrödinger-Poisson system

$$\begin{cases} i\Psi_t = \left(\sqrt{-\Delta + m^2} - m\right)\Psi + \left(V(x) - \dfrac{\mu}{|x|} + \lambda\right)\Psi - \Xi \widetilde{f}(x, |\Psi|) + K(x)|\Psi|^{q-2}\Psi, \\ (-\Delta)^{\frac{\alpha}{2}}\Xi = C_{N,\alpha} F(x, |\Psi|), \end{cases} \tag{2.1}$$

where $\Psi : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ is the wave function, $V : \mathbb{R} \to \mathbb{R}$ is an external potential, $-\frac{\mu}{|x|}$ is the singular part of the potential, with $\mu > 0$, $\widetilde{f}$ is a general nonlinearity and $K : \mathbb{R} \to \mathbb{R}$ is a general bounded and non-negative function. As we saw in Chapter 1, system (2.1) can be written as a single equation, solving the Poisson equation, and in this case it has the form

$$i\Psi_t = \left(\sqrt{-\Delta + m^2} - m\right)\Psi + \left(V(x) - \frac{\mu}{|x|} + \lambda\right)\Psi \\ - \left[I_\alpha \star \widetilde{F}(x, |\Psi|)\right] \widetilde{f}(x, |\Psi|) + K(x)|\Psi|^{q-2}\Psi \tag{2.2}$$

where I_α denotes the Riesz potential (see Appendix B, Definition B.1) and

$$\widetilde{F}(x, t) = \int_0^t \widetilde{f}(x, s)\, ds.$$

Considering the *ansatz*

$$\Psi(t, x) = e^{-i\lambda t} u(x) \tag{2.3}$$

and subsituting in (2.2), we obtain

$$\left(\sqrt{-\Delta + m^2} - m\right) u(x) + \left(V(x) - \frac{\mu}{|x|}\right) u(x) \\ - \left[I_\alpha \star F(x, u(x))\right] f(x, u(x)) + K(x)|u(x)|^{q-2}u(x) = 0 \tag{2.4}$$

where $f(x, t) = e^{i\lambda t}\widetilde{f}(x, t)$.

2.2 Some physical (and others) motivation

In this Section, we would like to provide some motivations that led us to the study of equation (2.4). In particular, we mention a physic technique used to study many-particles systems: these models, can be often derived by computational chemistry problems. It is not the aim of this thesis to enter in any technicism, referring to [120] for a more detailed treatment.

2.2.1 Hartree approximation

Consider a molecular orbital of many electrons (say N). We are interested in finding a wavefunction[1] for this system but, in this situation, the task is harder. Electrons have a negative charge, so they are led to reject themselves and this repulsion vary from moment to moment, depending also on the distance between the particles: in particular, it will be maximum if two particles "share" the same space. These electrons will try to minimize the repulsive force and to stabilize the whole system. From here, the difficulties on finding a wavefunction, that is not only a mathematical or physical problem, but also a computational chemistry one.

For this reasons, as often happens in this cases, a semplification of the model is introduced: one suppose that particles can still interact between each other, but this interaction is not instant, in the sense that the interaction for one particle to the other will change only when the electrons move, which, however, complicates its motions.

Despite this complication, in this way the wavefunction we are looking at is the product of many wavefunctions, one for every particle.

This can seems a strong semplification, but these individuals wavefunctions are often useful to describe a big amount of information on the chemical behaviour of a molecule. This kind of approximation is called Hartree Method.

2.2.2 Hartree-Fock theory

The Hartree-Fock theory is a generalization of the Hartree method above: it is based on the fact that the many-electrons functions is described by an antisymmetrized product (called the *Slater determinant*) of one-electron function. This antisymmetrization is considered with respect the spin of a particle.

However, this method suited perfectly for an electrons orbital model (and so for our case, see below). In fact, electrons are *fermions*, i.e. particle with half-integer spin (in particular, electrons have spin $\pm 1/2$): but, for this kind of particles Pauli's exclusion property holds, saying that two particles can not occupy the same quantum state. Therefore, two interacting particles must have opposite spin, hence they are antisymmetric and the Slater determinant can be applied to deal with the system. We refer to [60, 66–69] for more details of these Physical models.

2.3 Some recent history

But how is our problem related to the Hartree-Fock approximation? Consider a model with an atom with N electrons and nuclear charge Z. The kinematic energy of the electrons is described by the expression

$$\sqrt{(|\mathbf{p}|c)^2 + (mc^2)^2} - mc^2$$

[1] a mathematical description of the quantum state of an isolated quantum system.

and takes into account some kinematic relativist effects. Normalizing some constants, $\hbar = m = e = 1$, the Hamiltonian associated to the model become

$$\mathcal{H} = \sum_{j=1}^{N} \left\{ \sqrt{-\alpha^{-2}\Delta_j + \alpha^{-4}} - \alpha^{-2} - \frac{Z}{|x_j|} \right\} + \sum_{1 \le i < j \le N} \frac{1}{|x_i - x_j|}. \tag{2.5}$$

Here, α is the Sommerfeld's fine structure constant[2], and physically $\alpha = \frac{ke^2}{\hbar c} \approx 1/137,036$ (standing on the last CODATA[3] recommendend values of the fundamental Physical constant), where k is the Coulomb constant.

In the particular case of one-electron model (like an hydrogen atom), i.e. $N = 1$, (2.5) become

$$\mathcal{H} = \sqrt{-\alpha^{-2}\Delta + \alpha^{-4}} - \alpha^{-2} - \frac{Z}{|x|} \tag{2.6}$$

and it gives rise to the Hartree-Fock equation.

We also remark that system (2.1) was studied by Fefferman and de la Llave in [63] where they showed how a system governed by the operator $\mathcal{H}$ can implode: in fact, this is happen for a single quantized electron attracted to a single nucleus of charge Z fixed at the origin. In [90] Lieb and Yau studied the quantum mechanical many-body problem where they consider the problem where electrons and fixed nuclei interact via Coulomb forces with a relativistic kinetic energy. They proved that stability of relativistic matters occurs for suitable values Z and α. In [89], the same authors consider operator $\mathcal{H}$ with $Z = 0$, that is electrically neutral gravitating particles (e.g. fermions or bosons) and they showed that the ground state of stars can be obtained in the limit as G (the gravitation constant) goes to zero and n (the number of particles) goes to infinity.

So, rescaling some unit measures in (2.6), we are mathematically interested in studying equations driven by the elliptic fractional operator

$$u \mapsto \sqrt{-\Delta + m^2}u - mu - \frac{\mu}{|x|}u. \tag{2.7}$$

This kind of operator was used in the recent past by many authors: in [47], the authors consider the equation

$$\sqrt{-\Delta + m^2}u + V(x)u = (W \star |u|^p)\,|u|^{p-2}u, \text{ in } \mathbb{R}^N$$

where $N \ge 3$, $m > 0$, $u \in H^{\frac{1}{2}}(\mathbb{R}^N)$, V is a bounded potential and W is a radially symmetric convolution potential satisfying suitable assumptions: in [48], the same authors provided a semiclassical analysis.

Our work, was inspired by the paper [34], where the authors considered the following equation:

$$\sqrt{-\Delta + m^2}u - mu + V(x)u = (I_\alpha \star |u|^p)\,|u|^{p-2}u - K(x)|u|^{q-2}u, \text{ in } \mathbb{R}^N, \tag{2.8}$$

with I_α denoting the Riesz potential.

For this equation, they give some criteria on the potential V for which the ground state exists or not: they also show a compactness result on the potential K. To handle the elliptic operator (2.7), they made use of the Caffarelli-Silvestre extension [41].

Problem with a singular potential has been studied, for example, by Guo and Mederski in [74] in a strong-indefinite setting (i.e. 0 does not belong to the spectrum of the operator, see

[2]coupling constant of the electromagnetic interaction.
[3]Commitee on Data.

Chapter 3 where we will study a problem of this type) proving the existence of a ground state solution, provide the parameter μ is sufficiently small. Equation (2.8) was studied in many other papers with different assumption and we refer to [1,23,50–52,55,82,85,87,88,98,103,104,127,128] and the references therein.

2.4 Back to our problem: the hypotheses

We are ready to enter in the details of our work and we start by presenting the assumptions we considered. We prefer to dedicate an entire Section for the hypotheses to concentrate all the comments and the implications that descends from them, hoping to give a better readabilty. In particular, since our work is a generalization of [34], we will compare our hypotheses with their ones.

We start giving the assumption on the dimension of the space and on the exponents.

(N) $N \geq 2, (N-1)p - N < \alpha < N, 2 < q < \min\left\{2p, \frac{2N}{N-1}\right\}, p > 2.$

We observe that, from $N \geq 2, (N-1)p - N < \alpha < N$ follows that $p < \frac{2N}{N-1}$, hence we are implicitely asking that p is strictly smaller than the *critical Sobolev exponent* for the space $H^{\frac{1}{2}}(\mathbb{R}^N)$ (look at assumption **(F1)** below).

On the potential V we assume:

(V1) $V = V_p + V_l$, where $V_p \in L^\infty(\mathbb{R}^N)$ is $\mathbb{Z}^N-$periodic and $V_l \in L^\infty(\mathbb{R}^N) \cap L^N(\mathbb{R}^N)$ satisfies

$$\lim_{|x|\to+\infty} V_l(x) = 0.$$

(V2) $\operatorname*{ess\,inf}_{x\in\mathbb{R}^N} V(x) > m.$

Assumptions on the potential are very important in order to show that the elliptic operator $\sqrt{-\Delta + m^2} + V(x) - m$ is positive definite and they are slightly different with respect to the ones in [34]: indeed, if we assume that $\operatorname*{ess\,inf}_{x\in\mathbb{R}^N} V(x) > 0$ as in [34], then we could show that the operator $\sqrt{-\Delta + m^2} + V(x) - m$ is positive definite, but we don't have any information about our operator, that is $\sqrt{-\Delta + m^2} + V(x) - m - \frac{\mu}{|x|}$. Assumption **(V2)** is crucial to show, see Lemma 2.9 in the next Section, that for small values of $\mu > 0$, the operator is positive definite.

We want also to remark that, since the potential is not necessarily $\mathbb{Z}^N-$periodic, application of Lions' concentration-compactness principle is not straightforward.

Now, we list the assumption on the nonlinearity f, where we call $F(x, u) = \int_0^u f(x, s)\, ds$.

(F1) $f : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function, $\mathbb{Z}^N-$periodic in $x \in \mathbb{R}^N$ and there is a constant $C > 0$ such that

$$|f(x, u)| \leq C\left(|u|^{\frac{\alpha}{N}} + |u|^{p-1}\right). \tag{2.9}$$

(F2) $\lim_{|x|\to 0} \frac{f(x,u)}{|u|^{\frac{\alpha}{N}}} = 0$, uniformly with respect to $x \in \mathbb{R}^N$.

(F3) $\lim_{|x|\to+\infty} \frac{F(x,u)}{|u|^{\frac{q}{2}}} = +\infty$, uniformly with respect to $x \in \mathbb{R}^N$. Moreover, $F(x, u) \geq 0$ for $x \in \mathbb{R}^N$ and $u \in \mathbb{R}$.

(F4) The function $u \mapsto \frac{f(x,u)}{|u|^{\frac{q-2}{2}}}$ is non-decreasing on $(-\infty, 0)$ and on $(0, +\infty)$.

These assumption are quite standard when dealing with general nonlinearity (see for example [25] where there were introduced the so-called Berestycki-Lions conditions), since they allow to recover some important useful estimates. First of all, we recall the following definition.

Definition 2.1 (Carathéodory function). *We say that $f : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function if*

- *$f(\cdot, u)$ is measurable for every $u \in \mathbb{R}$;*
- *$f(x, \cdot)$ is continuous for almost every $x \in \mathbb{R}^N$.*

Thanks to the assumptions listed above, we can deduce some important information on the function f (and of course on F too). In fact, from hypothesis **(F2)** we have that for every $\varepsilon > 0$ there exists a $\delta := \delta(\varepsilon) > 0$ such that for every $|u| < \delta$

$$|f(x,u)| \leq \varepsilon |u|^{\frac{\alpha}{N}}. \tag{2.10}$$

The growth assumption in **(F1)** holds for every $u \in \mathbb{R}$, but we consider it only for $|u| \geq \delta$, i.e. for every $\varepsilon > 0$ there exists $C > 0$ such that

$$|f(x,u)| \leq C\left(|u|^{\frac{\alpha}{N}} + |u|^{p-1}\right)$$

for all $|u| \geq \delta$. Hence, we can make the following computations:

$$\begin{aligned} |f(x,u)| &\leq C\left(|u|^{\frac{\alpha}{N}} + |u|^{p-1}\right) \leq C\left(|u|^{\frac{\alpha}{N}} \frac{\delta^{p-\frac{\alpha}{N}-1}}{\delta^{p-\frac{\alpha}{N}-1}} + |u|^{p-1}\right) \\ &\leq C\left(|u|^{\frac{\alpha}{N}} \frac{|u|^{p-\frac{\alpha}{N}-1}}{\delta^{p-\frac{\alpha}{N}-1}} + |u|^{p-1}\right) = C\left(\frac{|u|^{p-1}}{\delta^{p-\frac{\alpha}{N}-1}} + |u|^{p-1}\right) \\ &= \left(\frac{C}{\delta^{p-\frac{\alpha}{N}-1}} + C\right)|u|^{p-1} = C_\varepsilon |u|^{p-1}. \end{aligned} \tag{2.11}$$

Therefore, combining (2.10) and (2.11) we obtain that for every $\varepsilon > 0$ there exists a constant $C_\varepsilon > 0$ such that, for every $u \in \mathbb{R}$,

$$|f(x,u)| \leq \varepsilon |u|^{\frac{\alpha}{N}} + C_\varepsilon |u|^{p-1}. \tag{2.12}$$

Integrating (2.12) we can recover a growth assumption for the primitive, i.e.

$$|F(x,u)| \leq \varepsilon |u|^{\frac{\alpha}{N}+1} + C_\varepsilon |u|^{p}. \tag{2.13}$$

Morover, assumption **(F4)** leads to the following condition. Let $u \geq s > 0$, then

$$\frac{f(x,s)}{|s|^{\frac{q-2}{2}}} \leq \frac{f(x,u)}{|u|^{\frac{q-2}{2}}};$$

integrating from 0 to u, that is

$$\int_0^u \frac{f(x,s)}{|s|^{\frac{q-2}{2}}}\,ds \leq \int_0^u \frac{f(x,u)}{|u|^{\frac{q-2}{2}}}\,ds$$

and making some computations we obtain

$$0 \leq F(x,s) = \int_0^u f(x,s)\,ds \leq \frac{f(x,u)}{|u|^{\frac{q-2}{2}}} \int_0^u |s|^{\frac{q-2}{2}}\,ds = \frac{f(x,u)}{|u|^{\frac{q-2}{2}}} \frac{|u|^{\frac{q-2}{2}} u}{\frac{q}{2}} = \frac{f(x,u)u}{\frac{q}{2}}.$$

Therefore, **(F4)** implies

$$0 \leq \frac{q}{2} F(x,u) \leq f(x,u)u \tag{2.14}$$

for almost every $x \in \mathbb{R}^N$ and $u \in \mathbb{R}$.

Condition (2.14) is weaker version of the well-known Ambrosetti-Rabinowitz condition introduced in [14] (see also [105]) and it is usually used to recover a compactness property. Here we recall it here for the reader's convenience.

Definition 2.2 (Ambrosetti-Rabinowitz condition)**.** *Let $\theta > 2$ and $R \geq 0$ such that*

$$0 \leq \theta F(x,s) \leq f(x,s)s$$

for any $s \in \mathbb{R}$ and almost every $x \in \mathbb{R}^N$.

Remark 2.3. *Assumption **(F2)**, that is $f(x,u) = o\left(|u|^{\frac{\alpha}{N}}\right)$, will be crucial in the proof of the decomposition of the Cerami-sequence (see Lemma 2.20, Step 4 in Section 2.7).*

We need just one last hypothesis.

(K) $K \in L^\infty(\mathbb{R}^N)$ is $\mathbb{Z}^N-$periodic and non-negative.

We end this Section giving three examples of functions that satisfy our hypothesis.

Example 2.4 (Example 1)**.** *Consider the function $f : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ defined as*

$$u \mapsto f(x,u) = |u|^{p-2}u,$$

*that is a pure-power nonlinearity and we can easily see that, if **(N)** holds then f satisfies hypoteses **(F1)**-**(F4)**.*

Indeed,

$$|f(x,u)| = |u|^{p-1} \leq C\left(|u|^{\frac{\alpha}{N}} + |u|^{p-1}\right),$$

*so **(F1)** holds and from*

$$\lim_{|u|\to 0} \frac{|f(x,u)|}{|u|^{\frac{\alpha}{N}}} = \lim_{|u|\to 0} \frac{|u|^{p-1}}{|u|^{\frac{\alpha}{N}}} = \lim_{|u|\to 0} |u|^{p-\frac{\alpha}{N}-1}u = 0$$

*follows that also **(F2)** holds, since $p > 2$.*

Computing the primitive of f, we have that

$$F(x,u) = \int_0^u f(x,s)\,ds = \int_0^u |s|^{p-2}s\,ds = \frac{|u|^p}{p} \geq 0,$$

for every $u \in \mathbb{R}$. Hence,

$$\lim_{|u|\to+\infty} \frac{F(x,u)}{|u|^{\frac{q}{2}}} = \frac{1}{p} \lim_{|u|\to+\infty} \frac{|u|^p}{|u|^{\frac{q}{2}}} = \frac{1}{p} \lim_{|u|\to+\infty} |u|^{p-\frac{q}{2}} = +\infty$$

*since $2p > q$ and **(F3)** holds.*

Finally, let $u \in (0,+\infty)$ and compute

$$\frac{f(x,u)}{|u|^{\frac{q-2}{2}}} = \frac{|u|^{p-2}u}{|u|^{\frac{q-2}{2}}} = u^{p-1-\frac{q-2}{2}} = u^{\frac{2p-q}{2}}$$

*and this is non-decreasing since $2p-q > 0$. The same computations can be done for $u \in (-\infty,0)$. This proves **(F4)**.*

Example 2.5 (Example 2)**.** *Consider the function $f : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ defined by the law*

$$f(x, u) = L(x)u \log(1 + |u|^{p-2}),$$

where $L : \mathbb{R}^N \to \mathbb{R}$ is Z^N-periodic, $L \in L^\infty(\mathbb{R}^N)$ and $\inf_{x\in\mathbb{R}^N} L(x) > 0$.

We observe that, from the assumption on L, there exists a constant $C > 0$ such that

$$|f(x, u)| = L(x)|u| \log(1 + |u|^{p-2}) \leq C|u||u|^{p-2} = C|u|^{p-1} \leq C\left(|u|^{\frac{\alpha}{N}} + |u|^{p-1}\right),$$

*where we also used the fact that $\log(1 + x) \leq x$ for every $x > 0$, that is **(F1)** holds.*

Now, we compute

$$\begin{aligned}\lim_{|u|\to 0} \frac{|f(x, u)|}{|u|^{\frac{\alpha}{N}}} &= \lim_{|u|\to 0} \frac{L(x)|u| \log(1 + |u|^{p-2})}{|u|^{\frac{\alpha}{N}}} \\ &= \lim_{|u|\to 0} \frac{L(x)|u| \log(1 + |u|^{p-2})}{|u|^{\frac{\alpha}{N}}} \frac{|u|^{p-2}}{|u|^{p-2}} \\ &= L(x) \lim_{|u|\to 0} \frac{\log(1 + |u|^{p-2})}{|u|^{p-2}} |u|^{p-1-\frac{\alpha}{N}} = 0\end{aligned}$$

*by use of limit $\lim_{x\to 0} \frac{\log(1+x)}{x} = 1$ and since $p - 1 - \frac{\alpha}{N} > 0$. Therefore, **(F2)** holds.*

*We show that **(F3)** holds by using the De L'Hôpital rule, that is*

$$\begin{aligned}&\lim_{|u|\to\infty} \frac{F(x, u)}{|u|^{\frac{q}{2}}} = \lim_{|u|\to+\infty} \frac{f(x, u)}{\frac{q}{2}|u|^{\frac{q}{2}-1}\frac{u}{|u|}} = \lim_{|u|\to+\infty} \frac{2L(x)u \log(1 + |u|^{p-2})}{q|u|^{\frac{q}{2}-2}u} \\ &= \frac{2L(x)}{q} \lim_{|u|\to+\infty} \frac{\log(1 + |u|^{p-2})}{|u|^{\frac{q-4}{2}}} = \frac{2L(x)}{q} \lim_{|u|\to+\infty} |u|^{\frac{4-q}{2}} \log(1 + |u|^{p-2}) = +\infty\end{aligned}$$

*since from **(N)** follows*

$$q < \frac{2N}{N-1} \leq 4.$$

Let $u \geq 0$, integrating by part we have

$$F(x, u) = L(x) \int_0^u s \log(1 + s^{p-2})\, ds \geq 0,$$

but this is true also for $u < 0$ since f is odd in u, so it follows that $F(x, u) = F(x, -u) \geq 0$.

Finally, we observe that, for $u \in (0, +\infty)$ it holds that

$$\frac{f(x, u)}{|u|} = L(x) \log(1 + u^{p-2})$$

is non-decreasing and non-negative. Hence,

$$\frac{f(x, u)}{|u|^{\frac{q-2}{2}}} = \frac{f(x, u)}{|u|^{\frac{q-2}{2}}} \frac{|u|}{|u|} = \frac{f(x, u)}{|u|} |u|^{\frac{4-q}{2}}$$

*is non-decreasing in $(0, +\infty)$. Similar computations yields that the map $u \mapsto \frac{f(x,u)}{|u|^{\frac{q-2}{2}}}$ is non-decreasing also in $(-\infty, 0)$, so **(F4)** holds.*

Example 2.6 (Example 3)**.** *Suppose that $\widetilde{f}$ satisfies **(F1)-(F4)**. It follows then that $\widetilde{f}(x, u) > 0$ in $(0, +\infty)$. Let $M > 1$ and define, for $u \geq 0$, the function*

$$f(x, u) = \begin{cases} L(x)\widetilde{f}(x, u), & \text{if } u < 1, \\ L(x)\widetilde{f}(x, u)u^{\frac{q-2}{2}}, & \text{if } 1 \leq u \leq M, \\ L(x)\dfrac{M^{\frac{q-2}{2}}\widetilde{f}(x, 1)}{\widetilde{f}(x, M)}\widetilde{f}(x, u), & \text{if } u > M, \end{cases}$$

with $f(x,u) = f(x,-u)$ *for* $u < 0$ *and* $L : \mathbb{R}^N \to \mathbb{R}$ *a* $\mathbb{Z}^N$*−periodic function such that* $\inf_{x\in\mathbb{R}^N} L(x) > 0$.

It can be proved that f *satisfies* ***(F1)****-****(F4)*** *and that on* $[1, M]$ *is sublinear, that is* $\frac{q-2}{2} \leq 1$. *We omit the details.*

2.5 Variational framework

In order to deal with equation (2.4), we make use of the variational approach, that is we associate to the equation a suitable energy functional (the *Euler-Lagrange* equation) and then we will look for critical points of that functional. The critical points turn out to be the (weak) solutions of the equation.

First of all, we recall that our equation is

$$\left(\sqrt{-\Delta+m^2}-m\right)u(x)+\left(V(x)-\frac{\mu}{|x|}\right)u(x) \\ -\left[I_\alpha\star F(x,u(x))\right]f(x,u(x))+K(x)|u(x)|^{q-2}u(x)=0,$$

and expanding the convolution it can be rewrite as

$$\left(\sqrt{-\Delta+m^2}-m\right)u(x)+\left(V(x)-\frac{\mu}{|x|}\right)u(x) \\ -\int_{\mathbb{R}^N\times\mathbb{R}^N}\frac{F(x,u(x))F(y,u(y))}{|x-y|^{N-\alpha}}\,dx\,dy+K(x)|u(x)|^{q-2}u(x)=0.$$

Before defining the energy functional, some comments on the elliptic operator $\left(\sqrt{-\Delta+m^2}\right)$ are due. As we anticipated in the Introduction of this Chapter, there are many ways to define this operator, for instance one can use the extension result introduced by Caffarelli-Silvestre in [41] (as done in [34]) or by means of the Bessel functions (see [62] Appendix B).

By the way, here we don't want to use this methods, both for giving a different approach with respect to the recent literature and mainly because the approach we chosen is more appropriate for our purpose. Indeed, we have choosen to use the Fourier transform (e.g. see [15], Chapter 4 for a survey on this operator): let u be a function belonging to the Schwartz space $\mathcal{S}(\mathbb{R}^N)$. Then, we can define the semirelativistic operator $\left(\sqrt{-\Delta+m^2}\right)$ as

$$\mathcal{F}\left(\sqrt{-\Delta+m^2}\right)=\sqrt{|\xi|^2+m^2}\mathcal{F}(u)=\sqrt{|\xi|^2+m^2}\hat{u}(\xi).$$

Hence, the energy functional $\mathcal{E} : H^{\frac{1}{2}}(\mathbb{R}^N) \to \mathbb{R}$ associated to (2.4) is

$$\mathcal{E}(u) :=\frac{1}{2}\int_{\mathbb{R}^N}\sqrt{|\xi|^2+m^2}|\hat{u}(x)|^2\,d\xi+\frac{1}{2}\int_{\mathbb{R}^N}\left(V(x)-m\right)|u(x)|^2\,dx-\frac{\mu}{2}\int_{\mathbb{R}^N}\frac{|u(x)|^2}{|x|}\,dx \\ -\frac{1}{2}\int_{\mathbb{R}^N\times\mathbb{R}^N}\frac{F(x,u(x))F(y,u(y))}{|x-y|^{N-\alpha}}\,dx\,dy+\frac{1}{q}\int_{\mathbb{R}^N}K(x)|u(x)|^q\,dx. \tag{2.15}$$

Our aim is to prove some norm-equivalence in the Sobolev space $H^{\frac{1}{2}}(\mathbb{R}^N)$, that is the natural space where to set our problem.

We start by setting the quadratic form $Q(u) : H^{\frac{1}{2}}(\mathbb{R}^N) \to \mathbb{R}$ defined by the law

$$u\mapsto Q(u):=\int_{\mathbb{R}^N}\sqrt{|\xi|^2+m^2}|\hat{u}(\xi)|^2\,d\xi+\int_{\mathbb{R}^N}(V(x)-m)|u(x)|^2\,dx.$$

Then, we can prove the following result.

Lemma 2.7. *The quadratic form $Q(u)$ is positive-definite and generates a norm on $H^{\frac{1}{2}}(\mathbb{R}^N)$ that is equivalent to the standard one. In particular, there exist two positive constants $C(N,m) \leq C\left(N, m, \|V\|_{L^\infty(\mathbb{R}^N)}\right)$ such that*

$$C(N,m)\left(\|u\|^2_{L^2(\mathbb{R}^2)} + [u]^2\right) \leq Q(u) \leq C\left(N, m, \|V\|_{L^\infty(\mathbb{R}^N)}\right)\left(\|u\|^2_{L^2(\mathbb{R}^2)} + [u]^2\right).$$

Proof. By the Plancherel's identity (see Appendix B, Theorem B.2) and assumption **(V)** we have

$$\begin{aligned} Q(u) &= \int_{\mathbb{R}^N} \sqrt{|\xi|^2 + m^2}|\hat{u}(\xi)|^2\, d\xi + \int_{\mathbb{R}^N} (V(x) - m)|u(x)|^2\, dx \\ &\leq \int_{\mathbb{R}^N} (|\xi| + m)\,|\hat{u}(\xi)|^2\, d\xi + \int_{\mathbb{R}^N} \left(\|V\|_{L^\infty(\mathbb{R}^N)} - m\right)|u(x)|^2\, dx \\ &= \int_{\mathbb{R}^N} |\xi||\hat{u}(\xi)|^2\, d\xi + \int_{\mathbb{R}^N} \|V\|_{L^\infty(\mathbb{R}^N)}|u(x)|^2\, dx. \end{aligned}$$

and thanks to Proposition 3.4 in [56] it follows

$$\begin{aligned} Q(u) &= \frac{1}{2}C\left(N, \frac{1}{2}\right)[u]^2 + \int_{\mathbb{R}^N} \|V\|_{L^\infty(\mathbb{R}^N)}|u(x)|^2\, dx \\ &\leq \max\left\{\frac{1}{2}C\left(N, \frac{1}{2}\right), \|V\|_{L^\infty(\mathbb{R}^N)}\right\}\left(\|u\|^2_{L^2(\mathbb{R}^N)} + [u]^2\right), \end{aligned}$$

where the constant (see [56])

$$C\left(N, \frac{1}{2}\right) = \left(\int_{\mathbb{R}^N} \frac{1 - cos\zeta_1}{|\zeta|^{N+1}}\, d\zeta\right)^{-1}.$$

Now, by assumption **(V)** and by the fact that $\sqrt{a^2 + b^2} \geq a^2$ for all $a, b \in \mathbb{R}$, we obtain

$$\begin{aligned} Q(u) &= \int_{\mathbb{R}^N} \sqrt{|\xi|^2 + m^2}|\hat{u}(\xi)|^2\, d\xi + \int_{\mathbb{R}^N} (V(x) - m)|u(x)|^2\, dx \\ &\geq \int_{\mathbb{R}^N} |\xi||\hat{u}(\xi)|^2\, d\xi + \int_{\mathbb{R}^N} \left(\operatorname*{ess\,inf}_{\mathbb{R}^N} V - m\right)|u(x)|^2\, dx \end{aligned}$$

and again by Proposition 3.4 in [56]

$$\begin{aligned} Q(u) &= \frac{1}{2}C\left(N, \frac{1}{2}\right)[u]^2 + \left(\operatorname*{ess\,inf}_{\mathbb{R}^N} V - m\right)\|u\|^2_{L^2(\mathbb{R}^N)} \\ &\geq \min\left\{\frac{1}{2}C\left(N, \frac{1}{2}\right), \operatorname*{ess\,inf}_{\mathbb{R}^N} V - m\right\}\left(\|u\|^2_{L^2(\mathbb{R}^N)} + [u]^2\right). \end{aligned}$$

□

Remark 2.8. *Following [56], Section 4, we report here the resolution of the integral in the constant $C\left(N, \frac{1}{2}\right)$. Consider the change of variable $\eta' = \frac{\zeta'}{|\zeta_1|}$, where $\zeta = (\zeta_1, \zeta') \in \mathbb{R} \times \mathbb{R}^N$, so we have*

$$\begin{aligned} C\left(N, \frac{1}{2}\right)^{-1} &= \int_{\mathbb{R}^N} \frac{1 - \cos\zeta_1}{|\zeta|^{N+1}}\, d\zeta = \int_{\mathbb{R}}\left(\int_{\mathbb{R}^{N-1}} \frac{1 - \cos(\zeta_1)}{|\zeta_1|^{N+2s}} \frac{1}{\left(1 + \frac{|\zeta'|^2|}{|\zeta_1|^2}\right)^{\frac{N+1}{2}}}\, d\zeta'\right) d\zeta_1 \\ &= \int_{\mathbb{R}}\left(\int_{\mathbb{R}^{N-1}} \frac{1 - \cos(\zeta_1)}{|\zeta_1|^{N+2s}} \frac{1}{(1 + |\eta'|^2)^{\frac{N+1}{2}}}\, |\zeta_1|^{N-1} d\eta'\right) d\zeta_1 \end{aligned}$$

$$= \int_{\mathbb{R}} \left(\int_{\mathbb{R}^{N-1}} \frac{1-\cos(\zeta_1)}{|\zeta_1|^{1+2s}} \frac{1}{(1+|\eta'|^2)^{\frac{N+1}{2}}} \, d\eta' \right) d\zeta_1$$
$$= \int_{\mathbb{R}} \frac{1-\cos(\zeta_1)}{|\zeta_1|^{1+2s}} \, d\zeta_1 \int_{\mathbb{R}^{N-1}} \frac{1}{(1+|\eta'|^2)^{\frac{N+1}{2}}} \, d\eta'.$$

Now, by contour integration it follows that

$$\int_{\mathbb{R}} \frac{1-\cos(\zeta_1)}{|\zeta_1|^{1+2s}} \, d\zeta_1 = \pi. \tag{2.16}$$

In Appendix B, Proposition B.6, are reported the computations of (2.16).

We take care of the second integral, and we divide the computations in two cases. If $N = 2$ we get

$$\int_{\mathbb{R}} \frac{1}{(1+|\eta'|^2)^{\frac{3}{2}}} \, d\eta' = \frac{\eta}{\sqrt{1+\eta^2}} \Big|_{-\infty}^{+\infty} = 2. \tag{2.17}$$

If $N \geq 3$, we use the polar coordinates and we obtain

$$\begin{aligned} \int_{\mathbb{R}^{N-1}} \frac{1}{(1+|\eta'|^2)^{\frac{N+1}{2}}} \, d\eta' &= \omega_{N-2} \int_0^{+\infty} \frac{r^{N-2}}{(1+r^2)^{\frac{N+1}{2}}} \, dr \\ &= \omega_{N-2} \frac{r^{N-1}}{(N-1)(1+r^2)^{\frac{N-1}{2}}} \Big|_0^{+\infty} \\ &= \frac{\omega_{N-2}}{N-1} = \frac{2}{(N-1)} \frac{\pi^{\frac{N-1}{2}}}{\Gamma\left(\frac{N-1}{2}\right)} \\ &= \frac{1}{\frac{(N-1)}{2}} \frac{\pi^{\frac{N-1}{2}}}{\Gamma\left(\frac{N-1}{2}\right)} = \frac{\pi^{\frac{N+1}{2}}}{\Gamma\left(\frac{N+1}{2}\right)}. \end{aligned} \tag{2.18}$$

where we used that $\omega_{N-2} = 2\frac{\pi^{\frac{N-1}{2}}}{\Gamma(\frac{N-1}{2})}$ and that $\Gamma(N+1) = N\Gamma(N)$.

Putting together (2.16), (2.17) and (2.18) we obtain

$$C\left(N, \frac{1}{2}\right)^{-1} = \begin{cases} 2\pi, & \text{if } N = 2, \\ \frac{\pi^{\frac{N+1}{2}}}{\Gamma\left(\frac{N+1}{2}\right)}, & \text{if } N \geq 3. \end{cases} \tag{2.19}$$

Now, we define the following quadratic form: $Q_\mu : H^{\frac{1}{2}}(\mathbb{R}^N) \to \mathbb{R}$ defined as

$$Q_\mu(u) := Q(u) - \mu \int_{\mathbb{R}^N} \frac{u^2(x)}{|x|} \, dx \tag{2.20}$$

and we can prove the following Lemma, that gives us desired norm-equivalence, provided $\mu > 0$ is small enough.

Lemma 2.9. *There exists $\mu^* > 0$ such that for any $0 < \mu < \mu^*$ the quadratic form Q_μ is positive-definite and generates a norm on $H^{\frac{1}{2}}(\mathbb{R}^N)$ that is equivalent to the standard one.*

Morever, the constant μ^* is explicit and depends only on N, that is

$$\mu^* = \mu^*(N) = 2\frac{\Gamma\left(\frac{N+1}{4}\right)^2}{\Gamma\left(\frac{N-1}{4}\right)^2}.$$

Proof. We immeadiately observe that one inequality is trivial: indeed, we have that for any $u \in H^{\frac{1}{2}}(\mathbb{R}^N)$ it holds

$$Q_\mu(u) \le Q(u).$$

So, we need to prove only the other inequality. The computations are similar to the ones in Lemma 2.7, that is

$$\begin{aligned}
Q_\mu(u) &= Q(u) - \mu \int_{\mathbb{R}^N} \frac{u^2(x)}{|x|}\, dx \\
&\ge \int_{\mathbb{R}^N} |\xi||\hat{u}(\xi)|^2 \, d\xi + \left(\operatorname*{ess\,inf}_{\mathbb{R}^N} V - m\right) \|u\|^2_{L^2(\mathbb{R}^N)} - \mu \int_{\mathbb{R}^N} \frac{u^2(x)}{|x|}\, dx \\
&\ge \frac{1}{2} C\left(N, \frac{1}{2}\right)[u]^2 + \left(\operatorname*{ess\,inf}_{\mathbb{R}^N} V - m\right) \|u\|^2_{L^2(\mathbb{R}^N)} - \mu \int_{\mathbb{R}^N} \frac{u^2(x)}{|x|}\, dx.
\end{aligned}$$

At this point, we use the fractional version of the Hardy inequality for singular potential (see Appendix B, Lemma B.3), so we obtain

$$\begin{aligned}
Q_\mu(u) &\ge \frac{1}{2} C\left(N, \frac{1}{2}\right)[u]^2 + \left(\operatorname*{ess\,inf}_{\mathbb{R}^N} V - m\right) \|u\|^2_{L^2(\mathbb{R}^N)} - \frac{\mu}{C\left(N, \frac{1}{2}, \frac{1}{2}\right)}[u]^2 \\
&= \left(\frac{1}{2} C\left(N, \frac{1}{2}\right) - \frac{\mu}{C\left(N, \frac{1}{2}, \frac{1}{2}\right)}\right)[u]^2 + \left(\operatorname*{ess\,inf}_{\mathbb{R}^N} V - m\right) \|u\|^2_{L^2(\mathbb{R}^N)} \\
&\ge \left\{\frac{1}{2} C\left(N, \frac{1}{2}\right) - \frac{\mu}{C\left(N, \frac{1}{2}, \frac{1}{2}\right)}, \left(\operatorname*{ess\,inf}_{\mathbb{R}^N} V - m\right)\right\} \left(\|u\|^2_{L^2(\mathbb{R}^N)} + [u]^2\right)
\end{aligned}$$

and the equivalence is proved for $\mu < \frac{1}{2} C\left(N, \frac{1}{2}\right) C\left(N, \frac{1}{2}, \frac{1}{2}\right)$.

To obtain the value of μ^*, we use (2.19): if $N = 2$ then

$$\mu < \frac{1}{2} C\left(2, \frac{1}{2}\right) C\left(2, \frac{1}{2}, \frac{1}{2}\right) = \frac{1}{2} \frac{\Gamma\left(\frac{3}{4}\right)^2}{\Gamma\left(\frac{1}{4}\right)^2} \frac{2\sqrt{\pi}}{\Gamma\left(\frac{3}{2}\right)} = 2 \frac{\Gamma\left(\frac{3}{4}\right)^2}{\Gamma\left(\frac{1}{4}\right)^2}.$$

If $N \ge 3$ then

$$\begin{aligned}
\mu &< \frac{1}{2} C\left(N, \frac{1}{2}\right) C\left(N, \frac{1}{2}, \frac{1}{2}\right) \\
&= \frac{1}{2} 2\pi^{\frac{N}{2}} \frac{\Gamma\left(\frac{N+1}{4}\right)^2}{\Gamma\left(\frac{N-1}{4}\right)^2} \frac{\left|\Gamma\left(-\frac{1}{2}\right)\right|}{\Gamma\left(\frac{N+1}{2}\right)} \frac{(N-1)}{\pi} \frac{\Gamma\left(\frac{N-1}{2}\right)}{2\pi^{\frac{N-1}{2}}} \\
&= 2 \frac{1}{2\sqrt{\pi}} \frac{\Gamma\left(\frac{N+1}{4}\right)^2}{\Gamma\left(\frac{N-1}{4}\right)^2} \frac{\left|\Gamma\left(-\frac{1}{2}\right)\right|}{\Gamma\left(\frac{N+1}{2}\right)} \frac{(N-1)}{2} \Gamma\left(\frac{N-1}{2}\right).
\end{aligned}$$

We recall that

$$\begin{aligned}
\Gamma\left(-\frac{1}{2}\right) &= -2\sqrt{\pi} \\
\frac{N-1}{2} \Gamma\left(\frac{N-1}{2}\right) &= \Gamma\left(\frac{N+1}{2}\right),
\end{aligned}$$

hence,

$$\mu < 2 \frac{\Gamma\left(\frac{N+1}{4}\right)^2}{\Gamma\left(\frac{N-1}{4}\right)^2}.$$

To conclude, we define

$$\mu^* = \mu^*(N) = \begin{cases} 2\dfrac{\Gamma\left(\frac{3}{4}\right)^2}{\Gamma\left(\frac{1}{4}\right)^2}, & \text{if } N = 2, \\ 2\dfrac{\Gamma\left(\frac{N+1}{4}\right)^2}{\Gamma\left(\frac{N-1}{4}\right)^2}, & \text{if } N \geq 3. \end{cases}$$

□

Hence, by the previous Lemma, we introduce the norm $\|u\|_\mu := \sqrt{Q_\mu}$ on $H^{\frac{1}{2}}(\mathbb{R}^N)$, for $0 < \mu < \mu^*$. Morever, we set by $(\cdot,\cdot)$ the scalar product corresponding to $Q(u)$ and by $(\cdot,\cdot)_\mu$ the ones corresponding to $Q_\mu(u)$.

For the sake of simplicity, we also define the functional $\mathcal{D} : H^{\frac{1}{2}}(\mathbb{R}^N) \to \mathbb{R}$ as

$$\mathcal{D}(u) := \int_{\mathbb{R}^N\times\mathbb{R}^N} \frac{F(x,u(x))F(y,u(y))}{|x-y|^{N-\alpha}}\, dx\, dy.$$

Thanks to (2.13), **(N)** and the Hardy-Littlewood-Sobolev inequality (see Appendix A, Theorem A.3), this is well-defined on $H^{\frac{1}{2}}(\mathbb{R}^N)$. Indeed, the following result holds.

Lemma 2.10. *Suppose* (2.13), ***(N)*** *hold, then the functional $\mathcal{D} : H^{\frac{1}{2}}(\mathbb{R}^N) \to \mathbb{R}$ is well-defined in $H^{\frac{1}{2}}(\mathbb{R}^N)$. Moreover, there exists a constant $C > 0$ such that*

$$\mathcal{D}(u) \leq C\left(\|u\|_\mu^{2\left(\frac{\alpha}{N}+1\right)} + \|u\|_\mu^{p+\frac{\alpha}{N}-1} + \|u\|_\mu^{2p}\right) \tag{2.21}$$

Proof. Let $\varepsilon > 0$, then by (2.13) there exists $C_\varepsilon > 0$ such that

$$\begin{aligned}\mathcal{D}(u) &\leq \int_{\mathbb{R}^N\times\mathbb{R}^N} \frac{\left(\varepsilon|u(x)|^{\frac{\alpha}{N}+1} + C_\varepsilon|u(x)|^p\right)\left(\varepsilon|u(y)|^{\frac{\alpha}{N}+1} + C_\varepsilon|u(y)|^p\right)}{|x-y|^{N-\alpha}}\, dx\, dy \\ &\leq C(N)\|\varepsilon|u(x)|^{\frac{\alpha}{N}+1} + C_\varepsilon|u(x)|^p\|_{L^r(\mathbb{R}^N)}\|\varepsilon|u(y)|^{\frac{\alpha}{N}+1} + C_\varepsilon|u(y)|^p\|_{L^r(\mathbb{R}^N)} \\ &= C(N)\|\varepsilon|u(x)|^{\frac{\alpha}{N}+1} + C_\varepsilon|u(x)|^p\|^2_{L^r(\mathbb{R}^N)}\end{aligned}$$

where we used Hardy-Littlewood-Sobolev inequality (Theorem A.3 in Appendix A) with $r = \frac{2N}{N+\alpha}$. By Minkowski inequality we have

$$\begin{aligned}\mathcal{D}(u) &\leq C(N)\left(\varepsilon\|u^{\frac{\alpha}{N}+1}\|_{L^r(\mathbb{R}^N)} + C_\varepsilon\|u^p\|_{L^r(\mathbb{R}^N)}\right)^2 \\ &\leq C(N)\left(\varepsilon\|u\|^{\frac{\alpha}{N}+1}_{L^{\left(\frac{\alpha}{N}+1\right)r}(\mathbb{R}^N)} + C_\varepsilon\|u\|^p_{L^{pr}(\mathbb{R}^N)}\right)^2 \\ &= C(N)\left(\varepsilon^2\|u\|^{2\left(\frac{\alpha}{N}+1\right)}_{L^{\left(\frac{\alpha}{N}+1\right)r}(\mathbb{R}^N)} + 2\varepsilon C_\varepsilon\|u\|^{\frac{\alpha}{N}+1}_{L^{\left(\frac{\alpha}{N}+1\right)r}(\mathbb{R}^N)}\|u\|^p_{L^{pr}(\mathbb{R}^N)} + C_\varepsilon^2\|u\|^{2p}_{L^{pr}(\mathbb{R}^N)}\right).\end{aligned}$$

From assumption **(N)** follows that $pr < \frac{2N}{N-1}$. Moreover,

$$\left(\frac{\alpha}{N}+1\right)r = \left(\frac{\alpha}{N}+1\right)\frac{2N}{N+\alpha} = \frac{2\alpha}{N+\alpha} + \frac{2N}{N+\alpha} = 2 \leq \frac{2N}{N-1},$$

so we can use the Sobolev embeddings to obtain

$$\mathcal{D}(u) \leq C(N)\left(\varepsilon^2\|u\|_\mu^{2\left(\frac{\alpha}{N}+1\right)} + 2\varepsilon C_\varepsilon\|u\|_\mu^{\frac{\alpha}{N}+1}\|u\|_\mu^p + C_\varepsilon^2\|u\|_\mu^{2p}\right)$$

$$= C(N)\left(\varepsilon^2\|u\|_\mu^{2\left(\frac{\alpha}{N}+1\right)} + 2\varepsilon C_\varepsilon\|u\|_\mu^{p+\frac{\alpha}{N}+1} + C_\varepsilon^2\|u\|_\mu^{2p}\right),$$

therefore, calling $\widetilde{C} = \max\{\varepsilon^2, 2\varepsilon C_\varepsilon, C_\varepsilon^2\}$, we get

$$\mathcal{D}(u) \le \widetilde{C}\left(\|u\|_\mu^{2\left(\frac{\alpha}{N}+1\right)} + \|u\|_\mu^{p+\frac{\alpha}{N}+1} + \|u\|_\mu^{2p}\right)$$

and the proof is concluded. □

We can rewrite the energy functional (2.15) as

$$\mathcal{E}(u) = \frac{1}{2}\|u\|_\mu^2 - \frac{1}{2}\mathcal{D}(u) + \frac{1}{q}\int_{\mathbb{R}^N} K(x)|u(x)|^q\,dx. \tag{2.22}$$

We immeadiately give a regularity result for $\mathcal{E}$.

Proposition 2.11. *The energy functional $\mathcal{E} : H^{\frac{1}{2}}(\mathbb{R}^N) \to \mathbb{R}$ defined in (2.22) is of class C^1 on $H^{\frac{1}{2}}(\mathbb{R}^N)$.*

Proof. It is enough to check that $\mathcal{I}(u) = \frac{1}{2}\mathcal{D}(u) - \frac{1}{q}\int_{\mathbb{R}^N} K(x)|u(x)|^q\,dx$ is of class C^1 on $H^{\frac{1}{2}}(\mathbb{R}^N)$.

Let $u \in H^{\frac{1}{2}}(\mathbb{R}^N)$. By Lemma 2.10, **(K)** and the Sobolev embedding we have that

$$\begin{aligned}
|\mathcal{I}(u)| &\le \frac{1}{2}|\mathcal{D}(u)| + \frac{1}{q}\int_{\mathbb{R}^N} K(x)|u(x)|^q\,dx \\
&\le C\left(\|u\|_\mu^{2\left(\frac{\alpha}{N}+1\right)} + \|u\|_\mu^{p+\frac{\alpha}{N}+1} + \|u\|_\mu^{2p}\right) + \frac{1}{q}\|K\|_{L^\infty(\mathbb{R}^N)}\|u\|_{L^q(\mathbb{R}^N)}^q \\
&\le C\left(\|u\|_\mu^{2\left(\frac{\alpha}{N}+1\right)} + \|u\|_\mu^{p+\frac{\alpha}{N}+1} + \|u\|_\mu^{2p}\right) + \frac{C}{q}\|K\|_{L^\infty(\mathbb{R}^N)}\|u\|_\mu^q < +\infty.
\end{aligned}$$

Now, let $u_n, u \in H^{\frac{1}{2}}(\mathbb{R}^N)$ such that $u_n \to u$ in $H^{\frac{1}{2}}(\mathbb{R}^N)$. Then,

$$\begin{aligned}
|\mathcal{I}(u_n) - \mathcal{I}(u)| &= \left|-\frac{1}{2}\mathcal{D}(u_n) + \frac{1}{q}\int_{\mathbb{R}^N} K(x)|u_n(x)|^q\,dx + \frac{1}{2}\mathcal{D}(u) - \frac{1}{q}\int_{\mathbb{R}^N} K(x)|u(x)|^q\,dx\right| \\
&\le \frac{1}{2}\left|\mathcal{D}(u_n) - \mathcal{D}(u)\right| + \frac{1}{q}\int_{\mathbb{R}^N} K(x)\left(|u_n(x)|^q - |u(x)|^q\right)\,dx \\
&\le \frac{1}{2}\left|\mathcal{D}(u_n) - \mathcal{D}(u)\right| + \frac{1}{q}\|K\|_{L^\infty(\mathbb{R}^N)}\int_{\mathbb{R}^N}\left(|u_n(x)|^q - |u(x)|^q\right)\,dx.
\end{aligned}$$

Adding and subtract $F(x, u_n(x))F(y, u(y))$ on the numerator of the first summand we have

$$\begin{aligned}
|\mathcal{D}(u_n) - \mathcal{D}(u)| \le &\int_{\mathbb{R}^N\times\mathbb{R}^N} \frac{F(x, u_n(x))[F(y, u_n(y)) - F(y, u(y))]}{|x-y|^{N-\alpha}}\,dx\,dy \\
&+ \int_{\mathbb{R}^N\times\mathbb{R}^N} \frac{F(y, u(y))[F(x, u_n(x)) - F(x, u(x))]}{|x-y|^{N-\alpha}}\,dx\,dy
\end{aligned}$$

and both the integrals goes to 0 as $n \to +\infty$ thanks to Theorem A.4 in [139].

For the same reason,

$$\int_{\mathbb{R}^N}\left(|u_n(x)|^q - |u(x)|^q\right)\,dx \to 0$$

as $n \to +\infty$. Therefore, $\mathcal{I} : H^{\frac{1}{2}}(\mathbb{R}^N) \to \mathbb{R}$ is continuous, and so also $\mathcal{E}$.

Now, the first Gâteaux derivative of $\mathcal{I}$ at $u \in H^{\frac{1}{2}}(\mathbb{R}^N)$ along $v \in H^{\frac{1}{2}}(\mathbb{R}^N)$ is

$$\mathcal{I}'(u)(v) = \mathcal{D}'(u)(v) - \int_{\mathbb{R}^N} K(x)|u(x)|^{q-2}u(x)v(x)\,dx.$$

Proceeding as for the continuity part, but with the suitable modifications, we obtain the C^1 regularity. □

Hence, it makes sense to give the following Definition.

Definition 2.12. *Critical points of the functional $\mathcal{E} : H^{\frac{1}{2}}(\mathbb{R}^N) \to \mathbb{R}$ are weak solutions of (2.4).*

2.6 Abstract setting and application

When Critical Points theory is used to prove the existence of solutions for a PDE (and, as in our case, the existence of a ground state) it is shown that a compactness property is satisfied by the associated functional. Here, we state the abstract Theorem we will use in order to recover compactness: in particular, with the help of the results of this Section, we will be able to provide the existence of a bounded Cerami-sequence. The proof relies on the Nehari-manifold technique and allows to deal with functionals that may change sign. This technique was introduced in [136] as an extension of the well-known Nehari-Pankov method and developed by [20] for positive definite functionals. In [32] they extend this technique for sign-changing nonlinearities, and this will be the ones that fitted for us and also allows to give equivalent min-max type characterization of the level of the Cerami-sequcene. For the sake of completness, we recall also the works [5, 33, 64, 137].

Let $(H; \|\cdot\|)$ a general Hilbert space and $\mathcal{E}: H \to \mathbb{R}$ be nonlinear functional of the form

$$\mathcal{E}(u) = \frac{1}{2}\|u\|^2 - \mathcal{I}(u),$$

where $\mathcal{I}: H \to \mathbb{R}$ is of class C^1 on H and $\mathcal{I}(0) = 0$. We introduce the *Nehari manifold*

$$\mathcal{N} := \{u \in H : \mathcal{E}'(u)(u) = 0\},$$

that contains all the nontrivial critical points of $\mathcal{E}$.

Theorem 2.13. *Suppose that:*

(J1) there is $r > 0$ such that

$$\inf_{\|u\|=r} \mathcal{E}(u) > 0;$$

(J2) $\frac{\mathcal{I}(t_n u_n)}{t_n^2} \to +\infty$ for $t_n \to +\infty$ and $u_n \to u \neq 0$;

(J3) for all $t > 0$ and $u \in \mathcal{N}$ there holds

$$\frac{t^2 - 1}{2}\mathcal{I}'(u)(u) - \mathcal{I}(tu) + \mathcal{I}(u) \leq 0.$$

Then, $\mathcal{N} \neq \emptyset$ and

$$c = \inf_{\mathcal{N}} \mathcal{E} = \inf_{\gamma\in\Gamma} \sup t \in [0,1]\mathcal{E}(\gamma(t)) = \inf_{u\in H\setminus\{0\}} \sup_{t\geq 0} \mathcal{E}(tu) > 0,$$

where

$$\Gamma := \{\gamma \in C([0,1], H) : \gamma(0) = 0_H, \|\gamma(1)\| > r, \mathcal{E}(\gamma(1)) < 0\} \neq \emptyset.$$

Moreover, there is a Cerami sequence for $\mathcal{E}$ at level $c \in \mathbb{R}$, i.e. a sequence $(u_n)_n \subset H$ such that

$$\mathcal{E}(u_n) \to c \text{ in } H$$
$$(1 + \|u_n\|)\mathcal{E}'(u_n) \to 0 \text{ in } H' = H.$$

We end this Section applying this Theorem to our problem, proving the existence of a Cerami-sequence for (2.22) and then showing its boundedness. In our case, the Hilbert space H is $H^{\frac{1}{2}}(\mathbb{R}^N)$, (2.22) is the nonlinear functional and

$$\mathcal{I}(u) = \frac{1}{2}\mathcal{D}(u) - \frac{1}{q}\int_{\mathbb{R}^N} K(x)|u(x)|^q\,dx.$$

We observe that from Lemma 2.11 we have that $\mathcal{I}$ is of class C^1 on $H^{\frac{1}{2}}(\mathbb{R}^N)$ and $\mathcal{I}(0) = 0$.

Lemma 2.14. *Suppose that* **(N)**, **(F1)**-**(F4)** *and* **(K)** *are satisfied. Then $\mathcal{E}$ satisfies (J1)-(J3).*

Proof. (J1). By the definition of $\mathcal{I}$, we see that

$$\mathcal{I}(u) \leq \frac{1}{2}\mathcal{D}(u)$$

and by (2.21) we have

$$\begin{aligned}\mathcal{I}(u) &\leq C\left(\|u\|_\mu^{2(\frac{\alpha}{N}+1)} + \|u\|_\mu^{p+\frac{\alpha}{N}-1} + \|u\|_\mu^{2p}\right)\\ &= C\|u\|_\mu^2\left(\|u\|_\mu^{\frac{\alpha}{N}} + \|u\|_\mu^{p+\frac{\alpha}{N}-1} + \|u\|_\mu^{2p-2}\right).\end{aligned}$$

Let $r > 0$, then if $\|u\|_\mu \leq r$ then

$$\mathcal{I}(u) \leq C\|u\|_\mu^2\left(r^{\frac{\alpha}{N}} + r^{p+\frac{\alpha}{N}-1} + r^{2p-2}\right).$$

We call $A(r) := C\left(r^{\frac{\alpha}{N}} + r^{p+\frac{\alpha}{N}-1} + r^{2p-2}\right)$, hence

$$\mathcal{I}(u) \leq A(r)\|u\|_\mu^2.$$

We observe that $A : [0,+\infty) \to [0,+\infty)$ is a continuous function, $A(0) = 0$ and

$$\lim_{r\to+\infty} A(r) = +\infty,$$

so there exists $r \in [0,+\infty)$ such that $A(r) = \frac{1}{4}$. Therefore,

$$\mathcal{I}(u) \leq \frac{1}{4}\|u\|_\mu^2.$$

Finally, if $\|u\|_\mu^2 = r$ we have

$$\mathcal{E}(u) = \frac{1}{2}\|u\|_\mu^2 - \mathcal{I}(u) \geq \frac{1}{2}\|u\|_\mu^2 - \frac{1}{4}\|u\|_\mu^2 = \frac{1}{4}\|u\|_\mu^2 = \frac{1}{4}r^2 > 0.$$

(J2). Let $(t_n)_n \subset (0,+\infty)$ be a seqeunce such that $t_n \to +\infty$. Then, we can assume that there exists a $\bar{n}$ such that $t_n \geq 1$ for all $n \geq \bar{n}$. Hence, since $q > 2$,

$$\begin{aligned}\frac{\mathcal{I}(t_nu_n)}{t_n^2} \geq \frac{\mathcal{I}(t_nu_n)}{t_n^q} &= \frac{1}{2}D(t_nu_n) - \frac{1}{qt_n^q}\int_{\mathbb{R}^N} K(x)|t_nu_n(x)|^q\,dx\\ &= \frac{1}{2}\int_{\mathbb{R}^N\times\mathbb{R}^N} \frac{F(x,t_nu_n(x))F(y,t_nu_n(y))}{t_n^q|x-y|^{N-\alpha}}\,dx\,dy - \frac{1}{q}\int_{\mathbb{R}^N} K(x)|u_n(x)|^q\,dx\\ &= \frac{1}{2}\int_{\mathbb{R}^N\times\mathbb{R}^N} \frac{F(x,t_nu_n(x))F(y,t_nu_n(y))}{t_n^{\frac{q}{2}}|x-y|^{N-\alpha}t_n^{\frac{q}{2}}}\,dx\,dy - \frac{1}{q}\int_{\mathbb{R}^N} K(x)|u_n(x)|^q\,dx.\end{aligned}$$

Passing to the limit and thanks to the Fatou's Lemma and assumption **(F3)** we get that

$$\frac{\mathcal{I}(t_nu_n)}{t_n^2} \to +\infty.$$

(J3). Let $u \in \mathcal{N}$ and $t \geq 0$, we define

$$\varphi(t) = \frac{t^2-1}{2}\mathcal{I}'(u)(u) - \mathcal{I}(tu) + \mathcal{I}(u)$$

and we note that $\varphi(1) = 0$. Since $u \in \mathcal{N}$ we have the following identity,

$$\|u\|_\mu^2 = \mathcal{I}'(u)(u) > 0,$$

that is

$$\int_{\mathbb{R}^N\times\mathbb{R}^N} \frac{F(x,u(x))f(y,u(y))u(y)}{|x-y|^{N-\alpha}}\,dx\,dy > \int_{\mathbb{R}^N} K(x)|u(x)|^q\,dx. \tag{2.23}$$

Now, we compute

$$\begin{aligned}
\frac{d\varphi(t)}{dt} &= t\mathcal{I}'(u)(u) - \mathcal{I}'(tu)(u)\\
&= t\int_{\mathbb{R}^N\times\mathbb{R}^N} \frac{F(x,u(x))f(y,u(y))u(y)}{|x-y|^{N-\alpha}}\,dx\,dy - t\int_{\mathbb{R}^N} K(x)|u(x)|^q\,dx\\
&\quad - \int_{\mathbb{R}^N\times\mathbb{R}^N} \frac{F(x,tu(x))f(y,tu(y))u(y)}{|x-y|^{N-\alpha}}\,dx\,dy + \int_{\mathbb{R}^N} K(x)|tu(x)|^q\,dx\\
&= \int_{\mathbb{R}^N\times\mathbb{R}^N} \left[\frac{F(x,u(x))f(y,u(y))tu(y)}{|x-y|^{N-\alpha}} - \frac{F(x,tu(x))f(y,tu(y))u(y)}{|x-y|^{N-\alpha}}\right]dx\,dy\\
&\quad + \left(t^{q-1}-t\right)\int_{\mathbb{R}^N} K(x)|tu(x)|^q\,dx.
\end{aligned}$$

We define the map $\psi : (0,+\infty) \to \mathbb{R}$ as

$$\psi(t) := \psi_{(x,y)}(t) := \frac{F(x,tu(x))f(y,tu(y))u(y)}{t^{q-1}} \tag{2.24}$$

for almost all $x, y \in \mathbb{R}^N$ fixed. Consider $t < 1$, then $t^{q-1} - t < 0$ and by (2.23) and (2.24) we have

$$\begin{aligned}
&\frac{d\varphi(t)}{dt} = t\mathcal{I}'(u)(u) - \mathcal{I}'(tu)(u)\\
&\geq \int_{\mathbb{R}^N\times\mathbb{R}^N} \left[\frac{F(x,u(x))f(y,u(y))tu(y)}{|x-y|^{N-\alpha}} - \frac{F(x,tu(x))f(y,tu(y))u(y)}{|x-y|^{N-\alpha}}\right]dx\,dy\\
&\quad + \left(t^{q-1}-t\right)\int_{\mathbb{R}^N\times\mathbb{R}^N} \frac{F(x,u(x))f(y,u(y))u(y)}{|x-y|^{N-\alpha}}\,dx\,dy\\
&= \int_{\mathbb{R}^N\times\mathbb{R}^N} \left[\frac{F(x,u(x))f(y,u(y))tu(y)}{|x-y|^{N-\alpha}} - \frac{F(x,tu(x))f(y,tu(y))u(y)}{|x-y|^{N-\alpha}}\right]dx\,dy\\
&\quad + t^{q-1}\int_{\mathbb{R}^N\times\mathbb{R}^N} \frac{F(x,u(x))f(y,u(y))u(y)}{|x-y|^{N-\alpha}}\,dx\,dy - \int_{\mathbb{R}^N\times\mathbb{R}^N} \frac{F(x,u(x))f(y,u(y))tu(y)}{|x-y|^{N-\alpha}}\,dx\,dy\\
&= \int_{\mathbb{R}^N\times\mathbb{R}^N} \left[\frac{F(x,u(x))f(y,u(y))tu(y)}{|x-y|^{N-\alpha}} - t^{q-1}\frac{F(x,u(x))f(y,u(y))u(y)}{|x-y|^{N-\alpha}}\right]dx\,dy\\
&= t^{q-1}\int_{\mathbb{R}^N\times\mathbb{R}^N} \frac{\psi(1)-\psi(t)}{|x-y|^{N-\alpha}}\,dx\,dy.
\end{aligned}$$

We show that the function ψ has the following properties:

- $\psi(t) \geq 0$;
- ψ is non-decreasing on $(0, 1]$.

The non-negativity easily comes from **(F3)** and (2.14): for the non-decreasing, we rewrite ψ as

$$\psi(t) = \frac{F(x,tu(x))}{t^{\frac{q}{2}}}\,\frac{f(y,yu(y))u(y)}{t^{\frac{q}{2}-1}}.$$

Using (2.14) we observe that

$$\frac{d}{dt}\left(\frac{F(x,tu(x))}{t^{\frac{q}{2}}}\right)=\frac{f(x,tu(x))tu(x)-\frac{q}{2}F(x,tu(x))}{t^{\frac{q}{2}+1}}\geq 0,$$

so the function $t\mapsto \frac{F(x,tu(x))}{t^{\frac{q}{2}}}$ is non-decreasing on $(0,+\infty)$. Now, we observe that by **(F4)**,

$$\frac{f(y,tu(y))u(y)}{t^{\frac{q}{2}}-1}=\frac{f(y,tu(y))u(y)}{|tu(y)|^{\frac{q}{2}}-1}|u(y)|^{\frac{q}{2}-1}$$

is non-decreasing, provided $u(y)\neq 0$.

Hence, the function ψ is the product of non-negative and non-decreasing functions, so it is non-decreasing itself. Therefore,

$$\frac{d\varphi}{dt}=t\mathcal{I}'(u)(u)-\mathcal{I}'(tu)(u)\geq t^{q-1}\int_{\mathbb{R}^N\times\mathbb{R}^N}\frac{\psi(1)-\psi(t)}{|x-y|^{N-\alpha}}\,dx\,dy\geq 0,$$

and it follows that $\varphi(t)\leq\varphi(1)$ for $t\in(0,1]$.

Following the same ideas as above (the only difference is that $t^{q-1}-t>0$), we can prove that if $t\in(1,+\infty)$ then

$$\frac{d\varphi(t)}{dt}=t\mathcal{I}'(u)(u)-\mathcal{I}'(tu)(u)\leq 0,$$

therefore, $\varphi(t)\leq\varphi(1)=0$ for $t\in(1,+\infty)$. □

Hence, by Theorem 2.13, there exists a Cerami-sequence $(u_n)_n\subset H^{\frac{1}{2}}(\mathbb{R}^N)$ at level $c\in\mathbb{R}$. In the following Lemma, we show that this is actually a bounded Cerami-sequence.

Lemma 2.15. *Any Cerami-sequence $(u_n)_n$ for $\mathcal{E}$ is bounded.*

Proof. Exploiting the properties of the Cerami-sequence, we have

$$\begin{aligned}\limsup_{n\to+\infty}\mathcal{E}(u_n)&=\limsup_{n\to+\infty}\left(\mathcal{E}(u_n)-\frac{1}{q}\mathcal{E}'(u_n)(u_n)\right)\\&=\limsup_{n\to+\infty}\left[\frac{1}{2}\|u_n\|_\mu^2-\frac{1}{2}\int_{\mathbb{R}^N\times\mathbb{R}^N}\frac{F(x,u_n(x))F(y,u_n(y))}{|x-y|^{N-\alpha}}\,dx\,dy\right.\\&\quad+\frac{1}{q}\int_{\mathbb{R}^N}K(x)|u(x)|^q\,dx\\&\quad-\frac{1}{q}\|u_n\|_\mu^2+\frac{1}{q}\int_{\mathbb{R}^N\times\mathbb{R}^N}\frac{F(x,u_n(x))f(y,u_n(y))u_n(y)}{|x-y|^{N-\alpha}}\,dx\,dy\\&\quad\left.-\frac{1}{q}\int_{\mathbb{R}^N}K(x)|u(x)|^q\,dx\right]\\&=\limsup_{n\to+\infty}\left[\left(\frac{1}{2}-\frac{1}{q}\right)\|u_n\|_\mu^2\right.\\&\quad\left.+\frac{1}{q}\int_{\mathbb{R}^N\times\mathbb{R}^N}\frac{F(x,u_n(x))\left[f(y,u_n(y))u_n(y)-\frac{q}{2}F(y,u_n(y))\right]}{|x-y|^{N-\alpha}}\,dx\,dy\right]\end{aligned}$$

and by (2.14) we obtain

$$\limsup_{n\to+\infty}\mathcal{E}(u_n)\geq\limsup_{n\to+\infty}\left(\frac{1}{2}-\frac{1}{q}\right)\|u_n\|_\mu^2,$$

hence, since the functional $\mathcal{E}$ is bounded,

$$\|u_n\|_\mu<+\infty.$$

□

2.7 A decomposition argument

Now that we have found a bounded Cerami-sequence, we need that this sequence converges (up to a subsequence, if necessary) to a critical point of the functional, that is we would like to obtain the compactness property we are looking for. In this way, since the functional was introduced such that its critical points are (weak) solutions of the equation, see Definition 2.12, we obtain a solutions for (2.4). However, in our setting this task is not easy to solve and we need to do some work in order to obtain it: in fact, we can not face the problem with the "classical" methods like the Mountain Pass ([14]) or Ekelend's Variational Principle ([59]) becuase the functional is unbounded and we are working in the whole space $\mathbb{R}^N$, therefore we can not use the Rellich-Kondrachov embedding (see the seminal work of [122] and [2], Theorem 6.3). However, exploiting the periodicity, one can build a finite number of limit problems, say m, in the whole space, proving that a bounded Palais-Smale sequence converges, up to a subsequence, to the sum of a critical point in a bounded domain plus m-critical points of the infinite problems (see Proposition II.1 in [18]).

This strategy, together with a Brezis-Lieb argument (see Lemma 2.16 and Lemma 2.17 below), is presented here, suitable adapted to our problem.

Lemma 2.16. *Suppose that $(u_n)_n \subset H^{\frac{1}{2}}(\mathbb{R}^N)$ is a bounded sequence such that $u_n \rightharpoonup u_0$ in $H^{\frac{1}{2}}(\mathbb{R}^N)$. Then*

$$\mathcal{D}(u_n - u_0) - \mathcal{D}(u_n) + \mathcal{D}(u_0) \to 0$$

as $n \to +\infty$.

Proof. The proof follows the same spirit of Lemma 2.2 in [42], so we refer to this paper and we omit it. □

Lemma 2.17. *Suppose that $(u_n)_n \subset H^{\frac{1}{2}}(\mathbb{R}^N)$ and there are a number $l \geq 0$, a sequence $(z_n^k)_n \subset \mathbb{Z}^N$ and $w^k \in H^{\frac{1}{2}}(\mathbb{R}^N)$, $k = 1, \dots, l$ such that*

$$u_n\left(\cdot - z_n^k\right) \rightharpoonup w^k \text{ in } x \in H^{\frac{1}{2}}(\mathbb{R}^N)$$

and

$$\left\| u_n - u_0 - \sum_{k=1}^{l} w^k\left(\cdot - z_n^k\right) \right\| \to 0.$$

Then,

$$\mathcal{D}(u_n) \to \mathcal{D}(u_0) + \sum_{k=1}^{l} \mathcal{D}(w^k).$$

Proof. For $m = 1, \dots, l$ we introduce

$$a_m^n := u_n - u_0 - \sum_{k=1}^{m} w^k\left(\cdot - z_n^k\right). \tag{2.25}$$

If $m = 0$ we can apply the previous Lemma to obtain

$$\mathcal{D}(a_0^n) - \mathcal{D}(u_n) + \mathcal{D}(u_0) \to 0$$

as $n \to +\infty$. Now, if $m = 1$ we take $a_0^n\left(\cdot + z_n^1\right)$ as u_n and w^1 as u_0 in Lemma 2.16, hence

$$\mathcal{D}\left(a_0^n\left(\cdot + z_n^1\right) - w^1\right) - \mathcal{D}\left(a_0^n\left(\cdot + z_n^1\right)\right) + \mathcal{D}(w^1) \to 0,$$

or, using (2.25)

$$\mathcal{D}(a_1^n) - \mathcal{D}(a_0^n) + \mathcal{D}(w^1) \to 0, \tag{2.26}$$

as $n \to +\infty$. If $m = 2$, we take $a_1^n\left(\cdot + z_n^2\right)$ as u_n and w^2 as u_0 in Lemma 2.16, so

$$\mathcal{D}\left(a_1^n\left(\cdot + z_n^2\right) - w^2\right) - \mathcal{D}\left(a_1^n\left(\cdot + z_n^2\right)\right) + \mathcal{D}(w^2) \to 0,$$

or, again by (2.25),

$$\mathcal{D}(a_2^n) - \mathcal{D}(a_1^n) + \mathcal{D}(w^2) \to 0, \tag{2.27}$$

as $n \to +\infty$. Combining (2.26) and (2.27), we have

$$\mathcal{D}(a_2^n) + \mathcal{D}(w^2) + \mathcal{D}(w^1) - \mathcal{D}(a_0^n) \to 0,$$

as $n \to +\infty$.

Using the same reasoning, we can iterate to obtain

$$\mathcal{D}(a_l^n) + \sum_{k=1}^{l} \mathcal{D}(w^k) - \mathcal{D}(a_0^n) = \mathcal{D}(a_l^n) + \sum_{k=1}^{l} \mathcal{D}(w^k) - \mathcal{D}(u_n - u_0) \to 0.$$

By the hypotheses of the Lemma, we have that

$$\mathcal{D}(a_l^n) \to 0$$

as $n \to +\infty$, therefore

$$\mathcal{D}(u_n - u_0) \to \sum_{k=1}^{l} \mathcal{D}(w^k)$$

as $n \to +\infty$, but we can conclude using Lemma 2.16: indeed,

$$\mathcal{D}(u_n) \to \mathcal{D}(u_0) + \sum_{k=1}^{l} \mathcal{D}(w^k)$$

as $n \to +\infty$. □

Now, we give a continuity result for the convolution part of the energy functional.

Lemma 2.18. *The functional $\mathcal{D}' : H^{\frac{1}{2}}(\mathbb{R}^N) \to \left(H^{\frac{1}{2}}(\mathbb{R}^N)\right)^*$ is weak-to-weak* continuous, i.e. if $(u_n)_n \subset H^{\frac{1}{2}}(\mathbb{R}^N)$ is bounded and $u_n \rightharpoonup u \in H^{\frac{1}{2}}(\mathbb{R}^N)$ and $\varphi \in H^{\frac{1}{2}}(\mathbb{R}^N)$ then*

$$\mathcal{D}'(u_n)(\varphi) \to \mathcal{D}'(u)(\varphi).$$

Proof. Let $(u_n)_n \in H^{\frac{1}{2}}(\mathbb{R}^N)$ and $\varphi \in H^{\frac{1}{2}}(\mathbb{R}^N)$, we recall

$$\mathcal{D}'(u_n)(\varphi) = \int_{\mathbb{R}^N \times \mathbb{R}^N} \left(I_\alpha \star \left(F(\cdot, u_n(\cdot))\right)(x) f(x, u_n(x))\varphi(x)\, dx.$$

Since, by assumption, $(u_n)_n \subset H^{\frac{1}{2}}(\mathbb{R}^N)$ is bounded, by Sobolev embedding we have that the sequence is also bounded in $L^2(\mathbb{R}^N) \cap L^{\frac{2N}{N-1}}(\mathbb{R}^N)$. Now, from (2.13) it follows that

$$\begin{aligned} |F(x, u_n(x)|^{\frac{2N}{N+\alpha}} &\le \left(\varepsilon |u_n|^{\frac{\alpha}{N}+1} + C_\varepsilon |u_n|^p\right)\right)^{\frac{2N}{N+\alpha}} \\ &\le \varepsilon^{\frac{2N}{N+\alpha}} |u_n|^2 + C_\varepsilon^{\frac{2N}{N+\alpha}} |u_n|^{p\frac{2N}{N+\alpha}}. \end{aligned}$$

From the weak convergence, we deduce that $u_n(x) \to u(x)$ for almost every $x \in \mathbb{R}^N$ and since F is a Carathéodory function, it follows that

$$F(x, u_n(x)) \to F(x, u(x))$$

for almost every $x \in \mathbb{R}^N$, hence

$$F(\cdot, u_n(\cdot) \rightharpoonup F(\cdot, u(\cdot))$$

in $L^{\frac{2N}{N+\alpha}}(\mathbb{R}^N)$. From the Hardy-Littlewood-Sobolev inequality (Theorem A.3 in Appendix A) we have that

$$\left(I_\alpha \star (F(\cdot, u_n(\cdot))\right)(x) \rightharpoonup \left(I_\alpha \star (F(\cdot, u(\cdot))\right)(x)$$

in $L^{\frac{2N}{N-\alpha}}(\mathbb{R}^N)$.

From **(F1)**, $f(x, u_n(x)) \to f(x, u(x))$ in $L^{\frac{2N}{N+\alpha}}_{loc}(\mathbb{R}^N)$. Therefore, for any $\varphi \in C_c^\infty(\mathbb{R}^N)$, by Hölder inequality (with $s = \frac{2N}{N-\alpha}$ and $s' = \frac{2N}{N+\alpha}$) there holds

$$\begin{aligned}
&\int_{\mathbb{R}^N} \left(I_\alpha \star (F(y, u_n(y))\right)(x) f(x, u_n(x))\varphi(x)\, dx \\
&\qquad \le \left(\int_{\mathbb{R}^N} |I_\alpha \star (F(\cdot, u_n(\cdot))|^s\, dx\right)^{\frac{1}{s}} \left(\int_{\mathbb{R}^N} |f(x, u_n(x))\varphi(x)|^{s'}\, dx\right)^{\frac{1}{s'}} \\
&\qquad \le \|I_\alpha \star (F(\cdot, u_n(\cdot))\|_{L^{\frac{2N}{N-\alpha}}(\mathbb{R}^N)} \|f(\cdot, u_n(\cdot))\varphi(x)\|_{L^{\frac{2N}{N+\alpha}}(\mathbb{R}^N)} \\
&\qquad\qquad \le \|I_\alpha \star (F(\cdot, u_n(\cdot))\|_{L^{\frac{2N}{N-\alpha}}(\mathbb{R}^N)} \|f(\cdot, u_n(\cdot))\|_{L^{\frac{2N}{N+\alpha}}(\mathbb{R}^N)} \|\varphi(x)\|_{L^{\frac{2N}{N+\alpha}}(\mathbb{R}^N)}
\end{aligned}$$

and, since $u_n(x) \to u(x)$ for almost every $x \in \mathbb{R}^N$, we obtain

$$\left(I_\alpha \star (F(\cdot, u_n(\cdot))\right)(x) f(x, u_n(x))\varphi(x) \to \left(I_\alpha \star (F(\cdot, u(\cdot))\right)(x) f(x, u(x))\varphi(x)$$

in $L^1(\mathbb{R}^N)$. This implies that

$$\mathcal{D}'(u_n)(\varphi) \to \mathcal{D}'(u)(\varphi)$$

and the proof is complete. □

From the above result, it follows immediately the continuity property for the whole functional $\mathcal{E}$.

Corollary 2.19. *The functional $\mathcal{E}' : H^{\frac{1}{2}}(\mathbb{R}^N) \to \left(H^{\frac{1}{2}}(\mathbb{R}^N)\right)^*$ is weak-to-weak* continuous.*

Proof. Let $(u_n)_n \subset H^{\frac{1}{2}}(\mathbb{R}^N)$ such that $u_n \rightharpoonup u$ and let $\varphi \in C_c^\infty(\mathbb{R}^N)$. The first Gâteaux derivative of the energy functional (2.22) is

$$\mathcal{E}'(u_n)(\varphi) = (u_n, \varphi)_\mu - \frac{1}{2}\mathcal{D}'(u_n)(\varphi) + \int_{\mathbb{R}^N} K(x)|u_n(x)|^{q-2} u_n(x)\varphi(x)\, dx.$$

From the weak convergence, up to subsequence (but for the sake of simplicity we continue to denote the subsequence as the original sequence), we have that $u_n(x) \to u(x)$ for every $x \in \mathbb{R}^N$ and it follows that

$$(u_n, \varphi)_\mu \to (u, \varphi)_\mu.$$

Morover, from Lemma 2.18 we have that

$$\mathcal{D}'(u_n)(\varphi) \to \mathcal{D}'(u)(\varphi).$$

So, it remains to take care only of the integral term. For any measurable set $E \subset \operatorname{supp}\varphi$, by **(K)** and the Hölder inequality, we have

$$\left|\int_{\mathbb{R}^N} K(x)|u_n(x)|^{q-2}u_n(x)\varphi(x)\,dx\right| \le \|K\|_{L^\infty(\mathbb{R}^N)}\|u_n\|^{q-1}_{L^q(\mathbb{R}^N)}\|\varphi\chi_E\|_{L^q(\mathbb{R}^N)}$$

and we can conlude by means of the Vitali convergence Theorem (see Theorem B.5 in Appendix B), that is

$$\int_{\mathbb{R}^N} K(x)|u_n(x)|^{q-2}u_n(x)\varphi(x)\,dx \to \int_{\mathbb{R}^N} K(x)|u(x)|^{q-2}u(x)\varphi(x)\,dx.$$

□

Before entering in the decomposition argument, we need to introduce the periodic version of the energy functional in the following way:

$$\mathcal{E}_{per}(u) := \mathcal{E}(u) - \frac{1}{2}\int_{\mathbb{R}^N} V_l(x)u^2(x) + \frac{\mu}{2}\int_{\mathbb{R}^N}\frac{u^2(x)}{|x|}\,dx \tag{2.28}$$

and we note that $\mathcal{E}_{per}(u(\cdot - z)) = \mathcal{E}_{per}(u)$ for any $z \in \mathbb{Z}^N$.

Theorem 2.20 (Decomposition Lemma). *Let $(u_n)_n \subset H^{\frac{1}{2}}(\mathbb{R}^N)$ be a bounded Palais-Smale sequence. Then, up to a subsequence, there is an integer $l \ge 0$ and sequences $(z_n^k)_n \subset \mathbb{Z}^N$, $w^k \in H^{\frac{1}{2}}(\mathbb{R}^N)$, $k = 1, \dots, l$ such that*

- *(i)* $u_n \rightharpoonup u_0$ *and* $\mathcal{E}'(u_0) = 0$;
- *(ii)* $|z_n^k| \to +\infty$ *and* $|z_n^k - z_n^{k'}| \to +\infty$ *for* $k \ne k'$;
- *(iii)* $w^k \ne 0$ *and* $\mathcal{E}'_{per}(w^k) = 0$ *for* $1 \le k \le l$;
- *(iv)* $u_n - u_0 - \sum_{k=1}^l w^k(\cdot - z_n^k) \to 0$;
- *(v)* $\mathcal{E}(u_n) \to \mathcal{E}(u_0) + \sum_{k=1}^l \mathcal{E}_{per}(w^k)$.

Proof. We follow the ideas in [79] (also used in [30, 32, 34]), dividing the proof in several steps.

Step 1. *$u_n \rightharpoonup u_0$, up to a subsequence, and $\mathcal{E}'(u_0) = 0$.*

The weak convergence follows from the boundedness of the sequence (see [38], Theorem 3.18). From Corollary 2.19 we then have that $\mathcal{E}'(u_n)(\varphi) \to \mathcal{E}'(u_0)(\varphi)$ for every $\varphi \in C_c^\infty(\mathbb{R}^N)$. But, $(u_n)_n$ is a Palais-Smale sequence, so we know that $\mathcal{E}'(u_n) \to 0$ in $\left(H^{\frac{1}{2}}(\mathbb{R}^N)\right)^*$. From the uniqueness of the limit, it follows that

$$\mathcal{E}'(u_0) = 0.$$

Step 2. *Let $v_n^1 := u_n - u_0$ and suppose that*

$$\lim_{n\to+\infty}\sup_{x\in\mathbb{R}^N}\int_{B(z,1)} |v_n^1(x)|^2\,dx = 0. \tag{2.29}$$

Then $u_n \to u_0$ and the Theorem is true for $l = 0$.

We compute

$$\mathcal{E}'(u_n)(v_n^1) = \mathcal{E}'(u_n)(u_n - u) = (u_n, u_n - u)_\mu - \frac{1}{2}\mathcal{D}'(u_n)(u_n - u_0)$$

$$
\begin{aligned}
&+\int_{\mathbb{R}^N} K(x)|u_n(x)|^{q-2}u_n(x)(u_n-u_0)(x)\,dx\\
&=(u_n-u_0+u_0,u_n-u)_\mu-\frac{1}{2}\mathcal{D}'(u_n)(v_n^1)\\
&+\int_{\mathbb{R}^N} K(x)|u_n(x)|^{q-2}u_n(x)v_n^1(x)\,dx,
\end{aligned}
$$

obtaining in the end,

$$
\mathcal{E}'(u_n)(v_n^1)=\|v_n^1\|_\mu^2+(u_0,u_n-u_0)_\mu-\frac{1}{2}\mathcal{D}'(u_n)(v_n^1)+\int_{\mathbb{R}^N} K(x)|u_n(x)|^{q-2}u_n(x)v_n^1(x)\,dx. \tag{2.30}
$$

From **Step 1.**, we know that, choosing v_n^1 as φ,

$$
0=\mathcal{E}'(u_0)(v_n^1)=(u_0,u_n-u_0)_\mu-\frac{1}{2}\mathcal{D}'(u_0)(v_n^1)+\int_{\mathbb{R}^N} K(x)|u_0(x)|^{q-2}u_0(x)v_n^1(x)\,dx
$$

that is,

$$
(u_0,u_n-u_0)_\mu=\frac{1}{2}\mathcal{D}'(u_0)(v_n^1)-\int_{\mathbb{R}^N} K(x)|u_0(x)|^{q-2}u_0(x)v_n^1(x)\,dx.
$$

Putting this identity in (2.30) we obtain

$$
\begin{aligned}
\|v_n^1\|_\mu^2&=\mathcal{E}'(u_n)(v_n^1)-(u_0,u_n-u_0)_\mu+\frac{1}{2}\mathcal{D}'(u_n)(v_n^1)-\int_{\mathbb{R}^N} K(x)|u_n(x)|^{q-2}u_n(x)v_n^1(x)\,dx\\
&=\mathcal{E}'(u_n)(v_n^1)-\frac{1}{2}\mathcal{D}'(u_0)(v_n^1)+\int_{\mathbb{R}^N} K(x)|u_0(x)|^{q-2}u_0(x)v_n^1(x)\,dx\\
&\quad+\frac{1}{2}\mathcal{D}'(u_n)(v_n^1)-\int_{\mathbb{R}^N} K(x)|u_n(x)|^{q-2}u_n(x)v_n^1(x)\,dx.
\end{aligned} \tag{2.31}
$$

Since $(u_n)_n$ is a Palais-Smale sequence and $(v_n^1)_n$ is bounded we deduce that

$$
\mathcal{E}'(u_n)(v_n^1)=\longrightarrow 0. \tag{2.32}
$$

Now, by (2.29) we have that

$$
v_n^1\to 0
$$

in $L^t(\mathbb{R}^N)$ for $t\in\left(2,\frac{2N}{N-1}\right)$ and using also Hölder inequality (with $s=q$ and $s'=\frac{q}{q-1}$) and **(K)** we obtain

$$
\begin{aligned}
&\left|\int_{\mathbb{R}^N} K(x)|u(x)|^{q-2}u(x)v(x)\,dx\right|\le\|K\|_{L^\infty(\mathbb{R}^N)}\int_{\mathbb{R}^N}|u(x)|^{q-1}v(x)\,dx\\
&\le\|K\|_{L^\infty(\mathbb{R}^N)}\left(\int_{\mathbb{R}^N}|u(x)|^{(q-1)s'}\,dx\right)^{\frac{1}{s'}}\left(\int_{\mathbb{R}^N}|v_n^1(x)|^s\,dx\right)^{\frac{1}{s}}\\
&\le\|K\|_{L^\infty(\mathbb{R}^N)}\left(\int_{\mathbb{R}^N}|u(x)|^q\,dx\right)^{\frac{q-1}{q}}\left(\int_{\mathbb{R}^N}|v_n^1(x)|^q\,dx\right)^{\frac{1}{q}}\\
&\le\|K\|_{L^\infty(\mathbb{R}^N)}\|u\|_{L^q(\mathbb{R}^N)}^{q-1}\|v_n^1\|_{L^q(\mathbb{R}^N)}.
\end{aligned} \tag{2.33}
$$

With the same computations, we have that

$$
\left|\int_{\mathbb{R}^N} K(x)|u_n(x)|^{q-2}u(x)v(x)\,dx\right|\le\|K\|_{L^\infty(\mathbb{R}^N)}\|u_n\|_{L^q(\mathbb{R}^N)}^{q-1}\|v_n^1\|_{L^q(\mathbb{R}^N)}, \tag{2.34}
$$

and both (2.33) and (2.34) goes to 0 as $n\to+\infty$.

To conclude this Step, we observe that from the Hardy-Littlewood-Sobolev inequality (see (A.1) in Appendix A), (2.12) and (2.13), we have

$$
\mathcal{D}'(u_n)(v_n^1)=\mathcal{D}'(u_n)(u_n-u_0)
$$

$$
\begin{aligned}
&= \int_{\mathbb{R}^N\times\mathbb{R}^N} \frac{F(x,u_n(x))f(y,u_n(y))(u_n-u_0)(y)}{|x-y|^{N-\alpha}}\,dx\,dy\\
&\le \int_{\mathbb{R}^N\times\mathbb{R}^N} \frac{\left(\varepsilon|u_n(x)|^{\frac{\alpha}{N}+1}+C_\varepsilon|u_n(x)|^p\right)\left(\varepsilon|u_n(y)|^{\frac{\alpha}{N}}+C_\varepsilon|u_n(y)|^{p-1}\right)(u_n-u_0)(y)}{|x-y|^{N-\alpha}}\,dx\,dy\\
&\le C(N)\left\|\varepsilon|u_n(x)|^{\frac{\alpha}{N}+1}+C_\varepsilon|u_n(x)|^p\right\|_{L^r(\mathbb{R}^N)}\left\|\left(\varepsilon|u_n(y)|^{\frac{\alpha}{N}}+C_\varepsilon|u_n(y)|^{p-1}\right)(u_n-u_0)(y)\right\|_{L^r(\mathbb{R}^N)}\\
&\le C(N)\left\|\varepsilon|u_n(x)|^{\frac{\alpha}{N}+1}+C_\varepsilon|u_n(x)|^p\right\|_{L^r(\mathbb{R}^N)}\left\|\varepsilon|u_n(y)|^{\frac{\alpha}{N}}+C_\varepsilon|u_n(y)|^{p-1}\right\|_{L^r(\mathbb{R}^N)}\|u_n-u_0\|_{L^r(\mathbb{R}^N)}\\
&\le C(N)\left(\|u_n\|^{\frac{\alpha}{N}+1}_{L^2(\mathbb{R}^N)}+\|u_n\|^p_{L^{pr}(\mathbb{R}^N)}\right)\left(\|u_n\|^{\frac{\alpha}{N}}_{L^{r\frac{\alpha}{N}}(\mathbb{R}^N)}+\|u_n\|^{p-1}_{L^{(p-1)r}(\mathbb{R}^N)}\right)\|u_n-u_0\|_{L^r(\mathbb{R}^N)}\\
&\le C(N)\left(\|u_n\|^{\frac{\alpha}{N}+1}_\mu+\|u_n\|^p_\mu\right)\left(\|u_n\|^{\frac{\alpha}{N}}_\mu+\|u_n\|^{p-1}_\mu\right)\|u_n-u_0\|_{L^r(\mathbb{R}^N)} \qquad (2.35)
\end{aligned}
$$

that goes to 0 as $n\to+\infty$, because $r=\frac{2N}{N+\alpha}<\frac{2N}{N-1}$.

Similarly, also

$$\mathcal{D}'(u_0)(v_n^1)\to 0 \qquad (2.36)$$

as $n\to+\infty$.

Hence, putting (2.32), (2.33), (2.34), (2.35) and (2.36) in (2.31) we obtain that

$$\|v_n^1\|_\mu^2\to 0,$$

as $n\to+\infty$, hence

$$u_n\to u_0$$

as $n\to+\infty$. Therefore, $\mathcal{E}(u_n)\to\mathcal{E}(u_0)$ as $n\to+\infty$ and the **Step 2** is proved.

Step 3. *Suppose that there is a sequence $(z_n)_n\subset\mathbb{Z}^N$ such that*

$$\liminf_{n\to+\infty}\int_{B\left(z_n,1+\sqrt{N}\right)}|v_n^1(x)|^2\,dx>0. \qquad (2.37)$$

Then, there is $w\in H^{\frac12}(\mathbb{R}^N)$ such that, up to a subsequence,

(i) $|z_n|\to+\infty$;

(ii) $u_n(\cdot+z_n)\rightharpoonup w\neq 0$;

(iii) $\mathcal{E}'_{per}(w)=0$.

Clearly, (i) and (ii) hold.

Let $v_n:=u_n(\cdot+z_n)$. As in Step 1, we have that

$$\mathcal{E}'_{per}(v_n)(\varphi)\to\mathcal{E}'_{per}(w)(\varphi)$$

for every $\varphi\in C_c^\infty(\mathbb{R}^N)$. Moreover,

$$
\begin{aligned}
o(1)=\mathcal{E}'(u_n)(\varphi(\cdot-z_n))&=\mathcal{E}'_{per}(v_n)(\varphi)+\int_{\mathbb{R}^N}V_l(x+z_n)v_n(x)\varphi(\cdot-z_n)\,dx\\
&\quad-\mu\int_{\mathbb{R}^N}\frac{v_n(x)\varphi(\cdot-z_n)}{|x|}\,dx\\
&=\mathcal{E}'_{per}(w)(\varphi)+\int_{\operatorname{supp}\varphi}V_l(x+z_n)v_n(x)\varphi(\cdot-z_n)\,dx\\
&\quad-\mu\int_{\mathbb{R}^N}\frac{v_n(x)\varphi(\cdot-z_n)}{|x|}\,dx+o(1),
\end{aligned}
$$

that is,

$$\mathcal{E}'_{per}(w)(\varphi) = -\int_{\operatorname{supp}\varphi} V_l(x+z_n)v_n(x)\varphi(\cdot - z_n)\,dx + \mu\int_{\mathbb{R}^N} \frac{v_n(x)\varphi(\cdot - z_n)}{|x|}\,dx + o(1).$$

From Lemma B.3 (see Appendix B) it follows that the sequence $(u_n)_n$ is bounded in $L^2(\mathbb{R}^N;|x|^{-1}\,dx)$, hence we may assume that $v_n \rightharpoonup u_0$ in $L^2(\mathbb{R}^N;|x|^{-1}\,dx)$ and, following Lemma 2.5 in [30] we have that,

$$\left|\int_{\mathbb{R}^N} \frac{v_n(x)\varphi(x-z_n)}{|x|}\,dx\right| \le \left(\int_{\mathbb{R}^N} \frac{v_n^2(x)}{|x|}\,dx\right)^{\frac{1}{2}} \left(\int_{\mathbb{R}^N} \frac{|\varphi(x-z_n)|^2}{|x|}\,dx\right)^{\frac{1}{2}}$$

and this goes to 0 as $|z_n| \to +\infty$.

Claim: $\int_{\operatorname{supp}\varphi} V_l(x+z_n)v_n(x)\varphi(\cdot - z_n)\,dx \to 0$.

Fix any measurable set $E \subset \operatorname{supp}\varphi$. By Hölder inequality (with $s = s' = 2$) we have

$$\begin{aligned}
&\left|\int_E V_l(x+z_n)v_n(x)\varphi(\cdot - z_n)\,dx\right| \le \int_E |V_l(x+z_n)v_n(x)\varphi(\cdot - z_n)|\,dx \\
&\le \|V_l\|_{L^\infty(\mathbb{R}^N)} \int_E |v_n(x)\varphi(\cdot - z_n)|\,dx \\
&\le \|V_l\|_{L^\infty(\mathbb{R}^N)} \left(\int_E |v_n(x)|^2\,dx\right)^{\frac{1}{2}} \left(\int_E |\varphi(x-z_n)|^2\,dx\right)^{\frac{1}{2}} \\
&\le \|V_l\|_{L^\infty(\mathbb{R}^N)} \|v_n\|_{L^2(\mathbb{R}^N)} \|\varphi\chi_E\|_{L^2(\mathbb{R}^N)}.
\end{aligned}$$

Remembering that the sequence $(v_n)_n \subset L^2(\mathbb{R}^N)$ is bounded, we deduce that the family $(V_l(\cdot + z_n)v_n\varphi)_n$ is uniformly integrable on $\operatorname{supp}\varphi$: we obtain the claim using the Vitali convergence Theorem. Step 3 is completed.

Step 4. *Suppose that there are $m \ge 1$, $(z_n^k)_n \subset \mathbb{Z}^N$, $w^k \in H^{\frac{1}{2}}(\mathbb{R}^N)$ for $k = 1,\dots,m$ such that*

(i) $|z_n^k| \to +\infty$ and $|z_n^k - z_n^{k'}| \to +\infty$ as $n \to +\infty$, for $1 \le k < k' \le m$;

(ii) $u_n(\cdot + z_n^k) \to w^k \ne 0$ for $1 \le k \le m$;

(iii) $\mathcal{E}'_{per}(w^k) = 0$ for $1 \le k \le m$.

Then,

(1) if

$$\sup_{z\in\mathbb{R}^N} \int_{B(z,1)} \left|u_n - u_0 - \sum_{k=1}^m w^k(\cdot - z_n^k)\right|^2 dx \to 0 \text{ as } n \to +\infty \tag{2.38}$$

then

$$u_n - u_0 - \sum_{k=1}^m w^k(\cdot - z_n^k) \to 0 \text{ as } n \to +\infty.$$

(2) if there is $(z_n^{m+1})_n \subset \mathbb{Z}^N$ such that

$$\liminf_{n\to+\infty} \int_{B\left(z_n^{m+1},1+\sqrt{N}\right)} \left|u_n - u_0 - \sum_{k=1}^m w^k(\cdot - z_n^k)\right|^2 dx > 0 \tag{2.39}$$

then there exists $w^{m+1} \in H^{\frac{1}{2}}(\mathbb{R}^N)$ such that, up to a subsequence, *(2.1)* $|z_n^{m+1}| \to +\infty$ *and* $|z_n^{m+1} - z_n^k| \to +\infty$ *as* $n \to +\infty$*, for* $1 \le k \le m$*;*

(2.2) $u_n(\cdot + z_n^{m+1}) \to w^{m+1} \neq 0$*;*

(2.3) $\mathcal{E}'_{per}(w^{m+1}) = 0$*.*

For the sake of simplicity, we set

$$\xi_n = u_n - u_0 - \sum_{k=1}^{m} w^k(\cdot - z_n^k).$$

Now, suppose that (2.38) holds, then by Lions' Concentration-Compactness principle (see [91]) we have

$$\xi_n \to 0$$

in $L^t(\mathbb{R}^N)$ for $t \in \left(2, \frac{2N}{N-1}\right)$. We compute, adding and subtracting $u_0 + \sum_{k=1}^m w^k(\cdot - z_n^k)$ in the scalar product,

$$\begin{aligned}
\mathcal{E}'(u_n)(\xi_n) &= (u_n, \xi_n)_\mu - \frac{1}{2}\mathcal{D}'(u_n)(\xi_n) - \int_{\mathbb{R}^N} K(x)|u_n(x)|^{q-2}u_n(x)\xi_n(x)\,dx \\
&= \left(u_n - u_0 - \sum_{k=1}^m w^k(\cdot - z_n^k) + u_0 + \sum_{k=1}^m w^k(\cdot - z_n^k), \xi_n\right)_\mu \\
&\quad - \frac{1}{2}\mathcal{D}'(u_n)(\xi_n) - \int_{\mathbb{R}^N} K(x)|u_n(x)|^{q-2}u_n(x)\xi_n(x)\,dx \\
&= \|\xi_n\|_\mu^2 + \left(u_0 + \sum_{k=1}^m w^k(\cdot - z_n^k), \xi_n\right)_\mu \\
&\quad - \frac{1}{2}\mathcal{D}'(u_n)(\xi_n) + \int_{\mathbb{R}^N} K(x)|u_n(x)|^{q-2}u_n(x)\xi_n(x)\,dx \\
&= \|\xi_n\|_\mu^2 + (u_0, \xi_n)_\mu + \sum_{k=1}^m (w^k(\cdot - z_n^k), \xi_n)_\mu \\
&\quad - \frac{1}{2}\mathcal{D}'(u_n)(\xi_n) + \int_{\mathbb{R}^N} K(x)|u_n(x)|^{q-2}u_n(x)\xi_n(x)\,dx.
\end{aligned} \tag{2.40}$$

We recall that $\mathcal{E}'(u_0)(\varphi) = 0$ for every $\varphi \in C_c^\infty(\mathbb{R}^N)$, so in particular for $\varphi = \xi_n$, that is

$$0 = \mathcal{E}'(u_n)(\xi_n) = (u_0, \xi_n)_\mu - \frac{1}{2}\mathcal{D}'(u_0)(\xi_n) + \int_{\mathbb{R}^N} K(x)|u_0(x)|^{q-2}u_0(x)\xi_n(x)\,dx$$

from which

$$-(u_0, \xi_n)_\mu = -\frac{1}{2}\mathcal{D}'(u_0)(\xi_n) + \int_{\mathbb{R}^N} K(x)|u_0(x)|^{q-2}u_0(x)\xi_n(x)\,dx \tag{2.41}$$

Hence, substituting (2.41) in (2.40) we obtain

$$\begin{aligned}
\|\xi_n\|_\mu^2 &= -\frac{1}{2}\mathcal{D}'(u_0)(\xi_n) + \int_{\mathbb{R}^N} K(x)|u_0(x)|^{q-2}u_0(x)\xi_n(x)\,dx \\
&\quad - \sum_{k=1}^m (w^k(\cdot - z_n^k), \xi_n)_\mu \\
&\quad + \frac{1}{2}\mathcal{D}'(u_n)(\xi_n) - \int_{\mathbb{R}^N} K(x)|u_n(x)|^{q-2}u_n(x)\xi_n(x)\,dx + o(1).
\end{aligned} \tag{2.42}$$

From assumption *(iii)* of Step 4, we have that w^k is a critical point per $\mathcal{E}_{per}$, i.e.

$$0 = \mathcal{E}'_{per}(w^k)(\xi_n) = (w^k, \xi_n)_\mu - \frac{1}{2}\mathcal{D}'(w^k)(\xi_n)$$

$$
\begin{aligned}
&+\int_{\mathbb{R}^N} K(x)|w^k(x-z_n^k)|^{q-2}w^k(x)\xi_n(x)\,dx\\
&-\int_{\mathbb{R}^N} V_l(x)w^k(x-z_n^k)\xi_n(x)\,dx-\mu\int_{\mathbb{R}^N}\frac{w^k(x-z_n^k)\xi_n(x)}{|x|}\,dx
\end{aligned}
$$

therefore,

$$
\begin{aligned}
-(w^k,\xi_n)_\mu=&-\frac{1}{2}\mathcal{D}'(w^k(x-z_n^k)(\xi_n)\\
&+\int_{\mathbb{R}^N} K(x)|w^k(x-z_n^k)|^{q-2}w^k(x)\xi_n(x)\,dx\\
&-\int_{\mathbb{R}^N} V_l(x)w^k(x-z_n^k)\xi_n(x)\,dx-\mu\int_{\mathbb{R}^N}\frac{w^k(x-z_n^k)\xi_n(x)}{|x|}\,dx.
\end{aligned}
\tag{2.43}
$$

Putting (2.43) in (2.42) we have

$$
\begin{aligned}
\|\xi_n\|_\mu^2=&-\frac{1}{2}\mathcal{D}'(u_0)(\xi_n)+\int_{\mathbb{R}^N} K(x)|u_0(x)|^{q-2}u_0(x)\xi_n(x)\,dx\\
&-\sum_{k=1}^m\frac{1}{2}\mathcal{D}'(w^k(x-z_n^k)(\xi_n)\\
&+\frac{1}{2}\mathcal{D}'(u_n)(\xi_n)-\int_{\mathbb{R}^N} K(x)|u_n(x)|^{q-2}u_n(x)\xi_n(x)\,dx\\
&+\sum_{k=1}^m\int_{\mathbb{R}^N} K(x)|w^k(x-z_n^k)|^{q-2}w^k(x-z_n^k)\xi_n(x)\,dx\\
&-\sum_{k=1}^m\int_{\mathbb{R}^N} V_l(x)w^k(x-z_n^k)\xi_n(x)\,dx+\sum_{k=1}^m\mu\int_{\mathbb{R}^N}\frac{w^k(x-z_n^k)\xi_n(x)}{|x|}\,dx.
\end{aligned}
$$

Making some order, we obtain

$$
\begin{aligned}
\|\xi_n\|_\mu^2=&\frac{1}{2}\mathcal{D}'(u_n)(\xi_n)-\frac{1}{2}\mathcal{D}'(u_0)(\xi_n)-\sum_{k=1}^m\frac{1}{2}\mathcal{D}'(w^k(x-z_n^k))(\xi_n)\\
&-\int_{\mathbb{R}^N} K(x)\left(|u_n(x)|^{q-2}u_n(x)-|u_0(x)|^{q-2}u_0(x)-\sum_{k=1}^m|w^k(x-z_n^k)|^{q-2}w^k(x-z_n^k)\right)\xi_n\,dx\\
&-\sum_{k=1}^m\int_{\mathbb{R}^N} V_l(x)w^k(x-z_n^k)\xi_n(x)\,dx+\sum_{k=1}^m\mu\int_{\mathbb{R}^N}\frac{w^k(x-z_n^k)\xi_n(x)}{|x|}\,dx+o(1).
\end{aligned}
$$

As in Step 3, we have that

$$
\int_{\mathbb{R}^N}\frac{w^k(x-z_n^k)\xi_n(x)}{|x|}\,dx\le\left(\int_{\mathbb{R}^N}\frac{|w^k(x-z_n^k)|^2}{|x|}\,dx\right)^{\frac{1}{2}}\left(\int_{\mathbb{R}^N}\frac{|\xi_n(x)|^2}{|x|}\,dx\right)^{\frac{1}{2}}
$$

that goes to 0 as $|z_n^k|\to+\infty$.

Let $E\subset\mathbb{R}^N$ be a measurable set. By means of Hölder inequality (with $s=s'=2$) we have

$$
\begin{aligned}
&\left|\int_{\mathbb{R}^N} V_l(x)w^k(x-z_n^k)\xi_n(x)\,dx\right|\le\int_{\mathbb{R}^N}|V_l(x)w^k(x-z_n^k)\xi_n(x)|\,dx\\
&\le\|V_l\|_{L^\infty(\mathbb{R}^N)}\int_{\mathbb{R}^N}|w^k(x-z_n^k)\xi_n(x)|\,dx\\
&\le\|V_l\|_{L^\infty(\mathbb{R}^N)}\left(\int_E|w^k(x-z_n^k)|^2\,dx\right)^{\frac{1}{2}}\left(\int_E|\xi_n|^2\,dx\right)^{\frac{1}{2}}
\end{aligned}
$$

$$\leq \|V_l\|_{L^\infty(\mathbb{R}^N)}\|w^k\|_{L^2(\mathbb{R}^N)}\|\xi_n\|_{L^2(\mathbb{R}^N)}.$$

Hence, the family $\left(V_l(x)w^k(x-z_n^k)\xi_n(x)\right)_n$ is uniformly integrable on any measurable set E of $\mathbb{R}^N$, so by Vitali's convergence Theorem we deduce that

$$\int_{\mathbb{R}^N} V_l(x)w^k(x-z_n^k)\xi_n(x)\,dx \to 0$$

as $n \to +\infty$.

Now, from **(K)** and assumption and the fact that $\xi_n \to 0$ in $L^t(\mathbb{R}^N)$ for $t \in \left(2, \frac{2N}{N-1}\right)$, we obtain

$$\begin{aligned}
&\left|\int_{\mathbb{R}^N} K(x)\left(|u_n(x)|^{q-2}u_n(x) - |u_0(x)|^{q-2}u_0(x) - \sum_{k=1}^m |w^k(x-z_n^k)|^{q-2}w^k(x)\right)\xi_n\,dx\right| \\
&\leq \|K\|_{L^\infty(\mathbb{R}^N)}\int_{\mathbb{R}^N}\left(|u_n(x)|^{q-1} - |u_0(x)|^{q-1} - \sum_{k=1}^m |w^k(x-z_n^k)|^{q-1}\right)|\xi_n|\,dx \\
&\leq \|K\|_{L^\infty(\mathbb{R}^N)}\left(\int_{\mathbb{R}^N}\left(|u_n(x)|^{q-1} - |u_0(x)|^{q-1} - \sum_{k=1}^m |w^k(x-z_n^k)|^{q-1}\right)^2 dx\right)^{\frac{1}{2}}\left(\int_{\mathbb{R}^N}|\xi_n|^2\,dx\right)^{\frac{1}{2}} \\
&\leq \|K\|_{L^\infty(\mathbb{R}^N)}\left(\int_{\mathbb{R}^N}\left(|u_n(x)|^{q-1} - |u_0(x)|^{q-1} - \sum_{k=1}^m |w^k(x-z_n^k)|^{q-1}\right)^2 dx\right)^{\frac{1}{2}}\|\xi_n\|_{L^2(\mathbb{R}^N)}
\end{aligned}$$

and this goes to 0 as $n \to +\infty$.

Hence, until now we have that

$$\|\xi_n\|_\mu^2 = \frac{1}{2}\mathcal{D}'(u_n)(\xi_n) - \frac{1}{2}\mathcal{D}'(u_0)(\xi_n) - \sum_{k=1}^m \frac{1}{2}\mathcal{D}'(w^k(x-z_n^k))(\xi_n) + o(1).$$

Now, from **(F1)**, (2.13) and making use of the Hardy-Littlewood-Sobolev inequality (see (A.1) in Appendix A) and Minkowski inequality, we can prove

$$\begin{aligned}
&|\mathcal{D}'(u_n)(\xi_n)| \leq 2\int_{\mathbb{R}^N\times\mathbb{R}^N}\frac{|F(x,u_n(x))f(y,u_n(y))\xi_n(y)}{|x-y|^{N-\alpha}}\,dx\,dy \\
&\leq C_1(N)\int \mathbb{R}^N\times\mathbb{R}^N\frac{\left(\varepsilon|u_n(x)|^{\frac{\alpha}{N}+1} + C_\varepsilon|u_n(x)|^p\right)\left(\varepsilon|u_n(y)|^{\frac{\alpha}{N}} + C_\varepsilon|u_n(y)|^{p-1}\right)|\xi_n(y)|}{|x-y|^{N-\alpha}}\,dx\,dy \\
&\leq C_2(N)\left\||u_n(x)|^{\frac{\alpha}{N}+1} + |u_n(x)|^p\right\|_{L^r(\mathbb{R}^N)}\left\|\varepsilon|u_n(y)|^{\frac{\alpha}{N}}|\xi_n| + C_\varepsilon|u_n(y)|^{p-1}|\xi_n|\right\|_{L^r(\mathbb{R}^N)} \\
&\leq C_3(N)\left(\|u_n\|^2_{L^{\left(\frac{\alpha}{N}+1\right)r}(\mathbb{R}^N)} + \|u_n\|^r_{L^{pr}(\mathbb{R}^N)}\right)\left(\varepsilon\left\|u_n\|^{\frac{\alpha}{N}}|\xi_n|\right\|_{L^r(\mathbb{R}^N)} + C_\varepsilon\left\|u_n\|^{p-1}|\xi_n|\right\|_{L^{pr}(\mathbb{R}^N)}\right)
\end{aligned}$$

where $r = \frac{2N}{N+\alpha} < \frac{2N}{N-1}$. Moreover, we note that

$$\left(\frac{\alpha}{N}+1\right)r = \left(\frac{\alpha}{N}+1\right)\frac{2N}{N+\alpha} = \frac{2\alpha}{N+\alpha} + \frac{2N}{N+\alpha} = 2$$

and by **(N)**, $pr < \frac{2N}{N-1}$.

Hence, by the continuous Sobolev embedding and Hölder ineqaulity (with $s = s' = 2$), we deduce that

$$|\mathcal{D}'(u_n)(\xi_n)| \leq C_4(N)\left(\varepsilon\left\|u_n\|^{\frac{\alpha}{N}}|\xi_n|\right\|_{L^r(\mathbb{R}^N)} + C_\varepsilon\left\|u_n\|^{p-1}|\xi_n|\right\|_{L^{pr}(\mathbb{R}^N)}\right)$$

$$\leq C_4(N)\left(\||u_n|^{\frac{\alpha}{N}}\|_{L^{\frac{2N}{\alpha}}(\mathbb{R}^N)}\|\xi_n\|_{L^2(\mathbb{R}^N)}+\mathbb{C}_\varepsilon\|u_n\|^{p-1}_{L^{pr}(\mathbb{R}^N)}\|xi\|_{L^{pr}(\mathbb{R}^N)}\right)$$
$$\leq o(1)+C_5(N)\varepsilon\limsup_{n\to+\infty}\||u_n|^{\frac{\alpha}{N}}\|_{L^{\frac{2N}{\alpha}}(\mathbb{R}^N)}\|\xi_n\|_{L^2(\mathbb{R}^N)}$$

that goes to 0 as $\varepsilon\to 0^+$, since $\|\xi_n\|_{L^{pr}(\mathbb{R}^N)}\to$ and $\|\xi_n\|_{L^2(\mathbb{R}^N)}$ is bounded. With very similar computations, we can also show that

$$\mathcal{D}'(u_0)(\xi_n)\to 0; \mathcal{D}'(w^k(\cdot-z_n^k))(\xi_n)\to 0.$$

Therefore, we finally obtain that

$$\|\xi_n\|^2_\mu\to 0$$

that is

$$\xi_n=u_n-u_0-\sum_{k=1}^m w^k(\cdot-z_n^k)\to 0.$$

Now, suppose that there exists a sequence $(z_n^{m+1})_n$ such that (2.39) holds. Then, let $v_n(x):=u_n(x+z_n^{m+1})$ for almost every $x\in\mathbb{R}^N$: reasoning as in Step 3, we can deduce that

$$\mathcal{E}'_{per}(v_n)(\varphi)\to 0$$

for every $\varphi\in C_c^\infty(\mathbb{R}^N)$ and Step 4 is concluded.

Step 5. *Statements (i)-(iv)are true.*

Iterating Step 4, we can create functions $w^k\neq 0$ and sequences $(z_n^k)_n\subset\mathbb{Z}^N$. The functions w^k are critical points for $\mathcal{E}_{per}$, hence there exists $\rho>0$ such that $\|w^k\|\geq\rho$.

Moreover, from the properties of the weak convergence and the fact that we are in an Hilbert space, we have

$$\begin{aligned}0&\leq\lim_{n\to+\infty}\left\|u_n-u_0-\sum_{k=1}^l w^k(\cdot-z_n^k)\right\|^2\\&=\lim_{n\to+\infty}\left(\|u_n\|^2-\|u_0\|^2-\sum_{k=1}^l\|w^k\|^2\right)\\&\leq\limsup_{n\to+\infty}\|u_n\|^2-\|u_0\|^2-m\rho^2,\end{aligned}$$

that is

$$m\leq\frac{1}{\rho^2}\limsup_{n\to+\infty}\|u_n\|^2-\|u_0\|^2<+\infty.$$

This last inequality tells us that the procedure finishes in a finite number of steps, say l.

Step 6. *(v) holds.*

To show (v) we still need to make some computations. First of all, we rewrite the energy functional

$$\mathcal{E}(u_n)=\frac{1}{2}(u_n,u_n)_\mu-\frac{1}{2}\mathcal{D}(u_n)+\frac{1}{q}\int_{\mathbb{R}^N}K(x)|u_n(x)|^q\,dx$$

as

$$\begin{aligned}\mathcal{E}(u_n)&=\frac{1}{2}(u_n-u_0+u_0,u_n-u_0+u_0)_\mu-\frac{1}{2}\mathcal{D}(u_n)+\frac{1}{q}\int_{\mathbb{R}^N}K(x)|u_n(x)|^q\,dx\\&=\frac{1}{2}(u_0,u_0)_\mu+\frac{1}{2}(u_n-u_0,u_n-u_0)_\mu+(u_n,u_n-u_0)_\mu\end{aligned}$$

$$-\frac{1}{2}\mathcal{D}(u_n)+\frac{1}{q}\int_{\mathbb{R}^N}K(x)|u_n(x)|^q\,dx.$$

Now, we add and subtract the following quantities:

$$\frac{1}{2}\mathcal{D}(u_0); \qquad \frac{1}{q}\int_{\mathbb{R}^N}K(x)|u_0(x)|^q\,dx;$$
$$\frac{1}{2}\mathcal{D}(u_n-u_0); \qquad \frac{1}{q}\int_{\mathbb{R}^N}K(x)|u_n(x)-u_0(x)|^q\,dx.$$

Then, recalling the definition of the periodic energy functional $\mathcal{E}_{per}$ and ordering the terms, we have

$$\begin{aligned}\mathcal{E}(u_n)&=\mathcal{E}(u_0)+\mathcal{E}_{per}(u_n-u_0)+(u_n,u_n-u_0)_\mu-\frac{1}{2}\mathcal{D}(u_n-u_0)\\&-\frac{1}{2}\mathcal{D}(u_n)+\frac{1}{2}\mathcal{D}(u_0)-\frac{1}{q}\int_{\mathbb{R}^N}K(x)\left(|u_n-u_0|^q+|u_0|^q-|u_n|^q\right)dx\\&-\frac{1}{2}\int_{\mathbb{R}^N}V_l(x)(u_n-u_0)^2(x)\,dx-\frac{\mu}{2}\int_{\mathbb{R}^N}\frac{(u_n-u_0)^2(x)}{|x|}\,dx.\end{aligned}\tag{2.44}$$

By the weak convergence, it follows

$$(u_0,u_n-u_0)_\mu\to 0.\tag{2.45}$$

By Lemma 2.16 we have

$$\mathcal{D}(u_n-u_0)-\mathcal{D}(u_n)+\mathcal{D}(u_0)\to 0.\tag{2.46}$$

From the classical Brezis-Lieb decomposition (see [39]) and **(K)**, we obtain

$$\begin{aligned}&\int_{\mathbb{R}^N}K(x)\left(|u_n-u_0|^q+|u_0|^q-|u_n|^q\right)dx\\&\le\|K\|_{L^\infty(\mathbb{R}^N)}\left(\|u_n-u_0\|^q_{L^q(\mathbb{R}^N)}+\|u_0\|^q_{L^q(\mathbb{R}^N)}-\|u_n\|^q_{L^q(\mathbb{R}^N)}\right)\to 0.\end{aligned}\tag{2.47}$$

Now, let $E\subset\mathbb{R}^N$ be a measurable set, by **(V1)** and Hölder inequality (with $s=N$ and $s'=\frac{N}{N-1}$) we have

$$\begin{aligned}\left|\int_E V_l(x)(u_n-u_0)^2(x)\,dx\right|&\le\int_E|V_l(x)||(u_n-u_0)(x)|^2\,dx\\&\le\left(\int_E|V_l(x)|^N\,dx\right)^{\frac{1}{N}}\left(\int_E|(u_n-u_0)(x)|^{2\frac{N}{N-1}}\,dx\right)^{\frac{N-1}{N}}\\&\le\|V_l\chi_E\|_{L^N(\mathbb{R}^N)}\|u_n-u_0\|^2_{L^{\frac{2N}{N-1}}(\mathbb{R}^N)}.\end{aligned}$$

Since $(u_n-u_0)_n$ is bounded in $H^{\frac{1}{2}}(\mathbb{R}^N)$, by Vitali convergence Theorem we obtain

$$\int_{\mathbb{R}^N}V_l(x)(u_n-u_0)^2(x)\,dx\to 0\tag{2.48}$$

as $n\to+\infty$.

We observe that

$$\int_{\mathbb{R}^N}\frac{(u_n-u_0)^2(x)}{|x|}\,dx=\int_{\mathbb{R}^N}\frac{(u_n-u_0)(x)u_n(x)}{|x|}\,dx-\int_{\mathbb{R}^N}\frac{(u_n-u_0)(x)u_0(x)}{|x|}\,dx.$$

As in previous Steps, by Hardy inequality (see (B.1) in Appendix B) we deduce

$$\int_{\mathbb{R}^N}\frac{(u_n-u_0)(x)u_0(x)}{|x|}\,dx\to 0$$

as n goes to $+\infty$. We can rewrite the firt integral on the right-hand side as

$$\int_{\mathbb{R}^N} \frac{(u_n-u_0)(x)u_n(x)}{|x|}\,dx = \int_{\mathbb{R}^N} \frac{\left(u_n-u_0-\sum_{k=1}^{l} w^k(\cdot-z_n^k)\right)u_n(x)}{|x|}\,dx + \int_{\mathbb{R}^N}\sum_{k=1}^{l}\frac{w^k(\cdot-z_n^k)}{|x|}\,dx$$

Then, again by Hardy inequality (see (B.1) in Appendix B)

$$\left|\int_{\mathbb{R}^N} \frac{w^k(\cdot-z_n^k)}{|x|}\,dx\right| \to 0 \tag{2.49}$$

as n diverges to $+\infty$, and by Hölder inequaliy (with $s=s'=2$),

$$\begin{aligned}
&\left|\int_{\mathbb{R}^N} \frac{\left(u_n-u_0-\sum_{k=1}^{l} w^k(\cdot-z_n^k)\right)u_n(x)}{|x|}\,dx\right| \\
&\le \left(\int_{\mathbb{R}^N} \frac{\left(u_n-u_0-\sum_{k=1}^{l} w^k(\cdot-z_n^k)\right)^2}{|x|}\,dx\right)^{\frac12}\left(\int_{\mathbb{R}^N}\frac{u_n^2(x)}{|x|}\,dx\right)^{\frac12} \\
&\le C\left\|u_n-u_0-\sum_{k=1}^{l} w^k(\cdot-z_n^k)\right\|\left(\int_{\mathbb{R}^N}\frac{u_n^2(x)}{|x|}\,dx\right)^{\frac12}
\end{aligned} \tag{2.50}$$

and this vanishes as $n\to+\infty$, since $\int_{\mathbb{R}^N}\frac{u_n^2(x)}{|x|}\,dx$ is bounded by the fractional Hardy inequality (B.1) in Appendix B.

Hence, from (2.49) and (2.50) we have that

$$\int_{\mathbb{R}^N} \frac{(u_n-u_0)(x)u_n(x)}{|x|}\,dx \to 0 \tag{2.51}$$

as n goes to $+\infty$. Putting (2.45), (2.46), (2.47), (2.48) and (2.51) in (2.44) we obtain

$$\mathcal{E}(u_n) = \mathcal{E}(u_0) + \mathcal{E}_{per}(u_n-u_0) + o(1).$$

Hence, if we show that

$$\mathcal{E}_{per}(u_n-u_0) \to \sum_{k=1}^{l}\mathcal{E}_{per}(w^k)$$

we complete this Step and the proof. To show that, we compute

$$\begin{aligned}
\mathcal{E}_{per}(u_n-u_0) = {}& \frac12\|u_n-u_0\|_\mu^2 - \frac12\mathcal{D}(u_n-u_0) + \frac1q\int_{\mathbb{R}^N} K(x)|u_n-u_0|^q\,dx \\
&- \frac12\int_{\mathbb{R}^N} V_l(x)(u_n-u_0)^2(x)\,dx + \frac{\mu}{2}\int_{\mathbb{R}^N}\frac{(u_n-u_0)^2(x)}{|x|}\,dx.
\end{aligned}$$

We add and subtract $\sum_{k=1}^{l} w^k(\cdot-z_n^k)$ and we recall that

$$\int_{\mathbb{R}^N}\frac{(u_n-u_0)^2(x)}{|x|}\,dx \to 0,$$
$$\int_{\mathbb{R}^N} V_l(x)(u_n-u_0)^2(x)\,dx$$

as $n \to +\infty$; exploting also the fact that we are in an Hilbert space, we have

$$\begin{aligned}
\mathcal{E}_{per}(u_n - u_0) &= \frac{1}{2}\left\| u_n - u_0 \sum_{k=1}^{l} w^k(\cdot - z_n^k)\right\|_\mu^2 - \frac{1}{2}\mathcal{D}(u_n - u_0) \\
&\quad + \frac{1}{q}\int_{\mathbb{R}^N} K(x)|u_n - u_0|^q\,dx + \frac{1}{2}\sum_{k=1}^{l}\left\| w^k(\cdot - z_n^k)\right\|_\mu + o(1) \\
&= \sum_{k=1}^{l}\mathcal{E}_{per}(w^k) + \frac{1}{2}\sum_{k=1}^{l}\mathcal{D}(w^k(\cdot - z_n^k)) \\
&\quad - \frac{1}{q}\int_{\mathbb{R}^N} K(x)|w^k(\cdot - z_n^k)|^q\,dx - \frac{1}{2}\mathcal{D}(u_n - u_0) \\
&\quad + \frac{1}{q}\int_{\mathbb{R}^N} K(x)|u_n - u_0|^q\,dx + o(1).
\end{aligned}$$

Iterating Lemma 2.16 and by Lemma 2.18 we have

$$\mathcal{D}(u_n - u_0) - \sum_{k=1}^{l}\mathcal{D}(w^k(\cdot - z_n^k)) \to 0,$$
$$\int_{\mathbb{R}^N} K(x)|u_n - u_0|^q\,dx - \int_{\mathbb{R}^N} K(x)|w^k(\cdot - z_n^k)|^q\,dx \to 0$$

as $n \to +\infty$. Therefore,

$$\mathcal{E}_{per}(u_n - u_0) \to \sum_{k=1}^{l}\mathcal{E}_{per}(w^k)$$

as n diverges to $+\infty$ and the proof is complete. □

2.8 Existence of ground state: an almost characterization

In this Section, we provide the existence results for equation (2.4). We will prove two Theorems, where the first one gives a criterion on the local part of the potential for the existence of a ground state solution, while the second Theorem is a sort of counterpart of the first one, in the sense that to prove the non-existence part we need to ask a modified versione of hypothesis **(V2)**.

The proofs follow from the ones in [34], of course with some suitable modifications.

Theorem 2.21 (Existence of a ground state solution). *Suppose that **(N)**, **(V1)**, **(V2)**, **(F1)**-**(F4)**, **(K)** are satisfied. There exists $\mu^* > 0$ such that for all $\mu \in (0, \mu^*)$ and any V_l satisfying*

$$V_l(x) < \frac{\mu}{|x|} \tag{2.52}$$

for almost every $x \in \mathbb{R}^N \setminus \{0\}$, there is a ground state solution $u \in H^{\frac{1}{2}}(\mathbb{R}^N)$ of (2.4). Moreover, the constant

$$\mu^* := \mu^*(N) = 2\frac{\Gamma\left(\frac{N+1}{4}\right)^2}{\Gamma\left(\frac{N-1}{4}\right)^2},$$

depends on the dimension N, but is independent of the potential V or of the nonlinearity f.

Remark 2.22. *We want to remark that we do not require $V_l(x) < 0$ for almost every $x \in \mathbb{R}^N$: indeed, the local part of the potential may be positive in some neighborhood of the origin.*

Proof of Theorem 2.21. Let $c_{per} := \inf_{\mathcal{N}_{per}} \mathcal{E}_{per}$, where $\mathcal{E}_{per}$ is given in (2.28) and

$$\mathcal{N}_{per} := \left\{ u \in H^{\frac{1}{2}}(\mathbb{R}^N) : \mathcal{E}'_{per}(u)(u) = 0 \right\}$$

is the corresponding Nehari manifold.

From Theorem 2.20 (iii) and (v) we have that

$$c = \lim_{n\to+\infty} \mathcal{E}(u_n) = \mathcal{E}(u_0) + \sum_{k=1}^{l} \mathcal{E}_{per}(w^k) \geq \mathcal{E}(u_0) + lc_{per}.$$

We know that there exists $u \in \mathcal{N}_{per}$ such that $\mathcal{E}_{per}(u) = c_{per}$. From (2.52) there follows that, for almost every $x \in \mathbb{R}^N$,

$$V(x) - \frac{\mu}{|x|} = V_p(x) + V_l(x) - \frac{\mu}{|x|} < V_p(x)$$

and this implies that, giving a number $t_p > 0$ such that $t_p u \in \mathcal{N}$, it holds

$$c_{per} = \mathcal{E}_{per}(u) \geq \mathcal{E}_{per}(t_p u) > \mathcal{E}(t_p u) \geq \inf_{\mathcal{N}} \mathcal{E} = c > 0.$$

We argue by contradiction: suppose that $u_0 = 0$. Then,

$$c = \mathcal{E}(u_0) + \sum_{k=1}^{l} \mathcal{E}_{per}(w^k) = \sum_{k=1}^{l} \mathcal{E}_{per}(w^k) \geq lc_{per}.$$

Now, if $l \geq 1$ we obtain $c \geq lc_{per} > lc$, but this is a contradiction. Hence, $l = 0$ and

$$0 < c = \mathcal{E}(u_0) = \mathcal{E}(0) = 0,$$

but this is also a contradiction. Hence, $u_0 \neq 0$ is a nontrivial ground state solution. □

We give now the non-existence result, with a slightly modification on assumption **(V2)**.

Theorem 2.23 (Non-existence of ground state solution). *Suppose that **(N)**, **(V1)**, **(F1)-(F4)**, **(K)** are satisfied and that*

(V2') $\operatorname{ess\,inf}_{x\in\mathbb{R}^N} V_p(x) > m$.

If $\mu < 0$ and

$$V_l(x) > \frac{\mu}{|x|} \tag{2.53}$$

for almost every $x \in \mathbb{R}^N \setminus \{0\}$, Then there is no ground state solution of (2.4).

Proof. We argue by contradiction, supposing that u_0 is a ground state for $\mathcal{E}$. In particular, it holds

$$c = \inf_{\mathcal{N}} \mathcal{E} = \mathcal{E}(u_0) > 0.$$

From (2.53) we deduce

$$V(x) - \frac{\mu}{|x|} = V_p(x) + V_l(x) - \frac{\mu}{|x|} > V_p(x)$$

for almost every $x \in \mathbb{R}^N$; hence, let $t_p > 0$ be a number such that $t_p u_0 \in \mathcal{N}_{per}$, then

$$c = \inf_{\mathcal{N}} \mathcal{E}(u_0) \geq \mathcal{E}(t_p u_0) \geq \mathcal{E}_{per}(t_p u_0) \geq \inf_{\mathcal{N}_{per}} \mathcal{E}_{per} = c_{per}. \tag{2.54}$$

Now, fix $u \in \mathcal{N}_{per}$, for any $z \in \mathbb{Z}^N$ we choose $t_z > 0$ such that $t_z u(\cdot - z) \in \mathcal{N}$. Then

$$\begin{aligned}\mathcal{E}_{per}(u) &= \mathcal{E}_{per}(u(\cdot - z)) \geq \mathcal{E}_{per}(t_z u(\cdot - z)) \\ &= \mathcal{E}(t_z u(\cdot - z)) - \frac{1}{2}\int_{\mathbb{R}^N} V_l(x)|t_z u(\cdot - z)|^2 \, dx + \frac{\mu}{2}\int_{\mathbb{R}^N} \frac{|t_z u(\cdot - z)|^2}{|x|} \, dx \\ &\geq c - \frac{1}{2}\int_{\mathbb{R}^N} V_l(x)|t_z u(\cdot - z)|^2 \, dz + \frac{\mu}{2}\int_{\mathbb{R}^N} \frac{|t_z u(\cdot - z)|^2}{|x|} \, dx.\end{aligned}$$

We remark that the functional $\mathcal{E}_{per}$ is coercive on $\mathcal{N}_{per}$, so from the inequality

$$\mathcal{E}_{per}(t_n u(\cdot - z)) = \mathcal{E}_{per}(t_z u) \leq c_{per}$$

it follows that

$$\sup_{z \in \mathbb{Z}^N} t_z < +\infty.$$

Therefore,

$$\int_{\mathbb{R}^N} V_l(x)|t_z u(\cdot - z)|^2 \, dx = t_z^2 \int_{\mathbb{R}^N} V_l(x + z) u^2(x) \, dx \to 0$$

as $|z| \to +\infty$. By Hardy inequality (see (B.1) in Appendix B), we deduce

$$\int_{\mathbb{R}^N} \frac{|t_z u(\cdot - z)|^2}{|x|} \, dx = t_z^2 \int_{\mathbb{R}^N} \frac{|u(\cdot - z)|^2}{|x|} \, dx \to 0$$

as $|z| \to +\infty$. Hence,

$$\mathcal{E}_{per}(u) \geq c + o(1);$$

we take the infimum over $u \in \mathcal{N}_{per}$ to obtain

$$c_{per} = \inf_{\mathcal{N}_{per}} \mathcal{E}_{per} \geq c,$$

but this contradicts (2.54). □

2.9 Compactness for the ground state

This Section is devoted to the compactness result for ground states when the parameter μ tends to 0^+.

Before proceding in the proof of the main result, we need some technical Lemmas and to fix some notation.

Let $(\mu_n)_n \subset (0, \mu^*)$ be a sequence such that $\mu_n \to 0^+$ as $n \to +\infty$ and let $\mathcal{E}_n$ be the Euler functional (2.15) for $\mu = \mu_n$, that is

$$\begin{aligned}\mathcal{E}_n(u) :&= \frac{1}{2}\int_{\mathbb{R}^N} \sqrt{|\xi|^2 + m^2}|\hat{u}(x)|^2 \, d\xi + \frac{1}{2}\int_{\mathbb{R}^N} (V(x) - m)\,|u(x)|^2 \, dx - \frac{\mu_n}{2}\int_{\mathbb{R}^N} \frac{|u(x)|^2}{|x|} \, dx \\ &\quad - \frac{1}{2}\int_{\mathbb{R}^N \times \mathbb{R}^N} \frac{F(x, u(x))F(y, u(y))}{|x - y|^{N-\alpha}} \, dx\, dy + \frac{1}{q}\int_{\mathbb{R}^N} K(x)|u(x)|^q \, dx \\ &= \frac{1}{2}\|u\|_{\mu_n}^2 - \frac{1}{2}\mathcal{D}(u) + \frac{1}{q}\int_{\mathbb{R}^N} K(x)|u(x)|^q \, dx.\end{aligned} \tag{2.55}$$

We denote by $\mathcal{E}_0$ the energy functional (2.15) for $\mu = 0$, that is

$$\begin{aligned}\mathcal{E}_0(u) :&= \frac{1}{2}\int_{\mathbb{R}^N}\sqrt{|\xi|^2+m^2}|\hat{u}(x)|^2\,d\xi + \frac{1}{2}\int_{\mathbb{R}^N}(V(x)-m)\,|u(x)|^2\,dx \\ &\quad - \frac{1}{2}\int_{\mathbb{R}^N\times\mathbb{R}^N}\frac{F(x,u(x))F(y,u(y))}{|x-y|^{N-\alpha}}\,dx\,dy + \frac{1}{q}\int_{\mathbb{R}^N}K(x)|u(x)|^q\,dx \\ &= \frac{1}{2}\|u\|^2_{H^{\frac{1}{2}}(\mathbb{R}^N)} - \frac{1}{2}\mathcal{D}(u) + \frac{1}{q}\int_{\mathbb{R}^N}K(x)|u(x)|^q\,dx\end{aligned} \tag{2.56}$$

and by $\mathcal{N}_0$ the corresponding Nehari manifold. Finally, we set

$$c_n := \mathcal{E}_n(u_n) = \inf_{\mathcal{N}_n}\mathcal{E}_n, \quad c_0 := \mathcal{E}_0(u_0) = \inf_{\mathcal{N}_0}\mathcal{E}_0,$$

where $u_n \in \mathcal{N}_n$ is the ground state solution for $\mathcal{E}_n$ and u_0 is the ground state solution for $\mathcal{E}_0$.

We are ready to prove the Lemmas we need in the proof of the compactness Theorem below.

Lemma 2.24. *There exists a positive radius $r > 0$ such that*

$$\inf_{n\geq 1}\inf_{\|u\|_{\mu_n}=r}\mathcal{E}_n(u) > 0.$$

Proof. Fix $n \geq 1$. As in Lemma 2.14, we call $\mathcal{I}(u) = \mathcal{D}(u) - \int_{\mathbb{R}^N}K(x)|u(x)|^q\,dx$. Hence,

$$\mathcal{I}(u) \leq \frac{1}{2}\mathcal{D}(u)$$

and by (2.21) we have

$$\begin{aligned}\mathcal{I}(u) &\leq C\left(\|u\|_\mu^{2\left(\frac{\alpha}{N}+1\right)} + \|u\|_\mu^{p+\frac{\alpha}{N}-1} + \|u\|_\mu^{2p}\right) \\ &= C\|u\|_\mu^2\left(\|u\|_\mu^{\frac{\alpha}{N}} + \|u\|_\mu^{p+\frac{\alpha}{N}-1} + \|u\|_\mu^{2p-2}\right).\end{aligned}$$

Let $r > 0$, then if $\|u\|_\mu \leq r$ then

$$\mathcal{I}(u) \leq C\|u\|_\mu^2\left(r^{\frac{\alpha}{N}} + r^{p+\frac{\alpha}{N}-1} + r^{2p-2}\right).$$

We call $A(r) := C\left(r^{\frac{\alpha}{N}} + r^{p+\frac{\alpha}{N}-1} + r^{2p-2}\right)$, hence

$$\mathcal{I}(u) \leq A(r)\|u\|_\mu^2.$$

We observe that $A : [0,+\infty) \to [0,+\infty)$ is a continuous function, $A(0) = 0$ and

$$\lim_{r\to+\infty}A(r) = +\infty,$$

so there exists $r \in [0,+\infty)$ such that $A(r) = \frac{1}{4}$. Therefore,

$$\mathcal{I}(u) \leq \frac{1}{4}\|u\|_\mu^2.$$

Finally, if $\|u\|_\mu^2 = r$ we have

$$\mathcal{E}_n(u) = \frac{1}{2}\|u\|_{\mu_n}^2 - \mathcal{I}(u) \geq \frac{1}{2}\|u\|_{\mu_n}^2 - \frac{1}{4}\|u\|_{\mu_n}^2 = \frac{1}{4}\|u\|_{\mu_n}^2 = \frac{1}{4}r^2 > 0.$$

□

Lemma 2.25. *The sequence $(u_n)_n$ is bounded in $H^{\frac{1}{2}}(\mathbb{R}^N)$.*

Proof. We recall that (see (2.20))

$$\begin{aligned} Q_{\mu_n}(u_n) &= Q(u) - \mu_n \int_{\mathbb{R}^N} \frac{u^2(x)}{|x|}\,dx \\ &= \int_{\mathbb{R}^N} \sqrt{|\xi|^2+m^2}|\hat{u}(\xi)|^2\,d\xi + \int_{\mathbb{R}^N} (V(x)-m)|u(x)|^2\,dx \\ &\quad - \mu_n \int_{\mathbb{R}^N} \frac{u^2(x)}{|x|}\,dx. \end{aligned}$$

and suppose by contradiction that $Q(u_n) \to +\infty$ as $n \to +\infty$. By Lemma 2.9, there exists a constant $C > 0$, that does not depend on μ_n, such that

$$\begin{aligned} Q_{\mu_n} &\geq \min\left\{\frac{1}{2}C\left(N,\frac{1}{2}\right) - \frac{\mu_n}{C\left(N,\frac{1}{2},\frac{1}{2}\right)}, \operatorname*{ess\,inf}_{\mathbb{R}^N} V - m\right\}\left(\|u_n\|^2_{L^2(\mathbb{R}^N)} + [u_n]^2\right) \\ &\geq CQ(u_n). \end{aligned}$$

Therefore, letting $n \to +\infty$, we have that $\|u_n\|_{\mu_n} \to +\infty$. Now, since u_n is a ground state for $\mathcal{E}_n$, it follows that $\mathcal{E}_n'(u_n) = 0$, hence we compute

$$\begin{aligned} c_0 &= \lim_{n\to+\infty} \mathcal{E}_n(u_n) = \lim_{n\to+\infty}\left(\mathcal{E}_n(u_n) - \frac{1}{q}\mathcal{E}_n'(u_n)(u_n)\right) \\ &= \lim_{n\to+\infty}\left[\frac{1}{2}\|u_n\|^2_{\mu_n} - \frac{1}{2}\mathcal{D}(u_n) + \frac{1}{q}\int_{\mathbb{R}^N} K(x)|u_n(x)|^q\,dx \right. \\ &\quad \left. -\frac{1}{q}\|u_n\|^2_{\mu_n} + \frac{1}{q}\mathcal{D}'(u_n)(u_n) - \frac{1}{q}\int_{\mathbb{R}^N} K(x)|u_n(x)|^q\,dx\right] \\ &= \lim_{n\to+\infty}\left[\left(\frac{1}{2}-\frac{1}{q}\right)\|u_n\|^2_{\mu_n} + \frac{1}{q}\int_{\mathbb{R}^N\times\mathbb{R}^N} \frac{F(x,u_n(x))f(y,u_n(y))u_n(y)}{|x-y|^{N-\alpha}}\,dx\,dy \right. \\ &\quad \left. -\frac{1}{2}\int_{\mathbb{R}^N\times\mathbb{R}^N} \frac{F(x,u_n(x))F(y,u_n(y))}{|x-y|^{N-\alpha}}\,dx\,dy\right]. \end{aligned}$$

Calling

$$\varphi(y,u_n(y)) = \frac{1}{q}f(y,u_n(y)) - \frac{1}{2}F(y,u_n(y))$$

we can rewrite c_0 as

$$c_0 = \lim_{n\to+\infty}\left[\left(\frac{1}{2}-\frac{1}{q}\right)\|u_n\|^2_{\mu_n} + \int_{\mathbb{R}^N\times\mathbb{R}^N} \frac{F(x,u_n(x))\varphi(y,u_n(y))}{|x-y|^{N-\alpha}}\,dx\,dy\right].$$

We observe that, by (2.14), $\varphi(y,u_n(y)) \geq 0$, hence

$$c_0 \geq \lim_{n\to+\infty}\left(\frac{1}{2}-\frac{1}{q}\right)\|u_n\|^2_{\mu_n} = +\infty,$$

but this is a contradiction. □

Lemma 2.26. *There holds*

$$c_0 = \lim_{n\to+\infty} c_n.$$

Proof. Consider $t_n > 0$ such that $t_n u_n \in \mathcal{N}_0$ and note that

$$\begin{aligned} c_n = \mathcal{E}_n(u_n) \geq cE_n(t_n u_n) &= \mathcal{E}_0(t_n u_n) - \frac{\mu_n t_n^2}{2}\int_{\mathbb{R}^N} \frac{u_n^2(x)}{|x|}\,dx \\ &\geq c_0 - \frac{\mu_n t_n^2}{2}\int_{\mathbb{R}^N} \frac{u_n^2(x)}{|x|}\,dx. \end{aligned} \tag{2.57}$$

Now, let $s_n > 0$ such that $s_n u_0 \in \mathcal{N}_n$, then

$$\begin{aligned} c_0 = \mathcal{E}_0(u_0) \geq \mathcal{E}_0(s_n u_n) &= \mathcal{E}_n(s_n u_0) + \frac{\mu_n s_n^2}{2} \int_{\mathbb{R}^N} \frac{u_0^2(x)}{|x|}\, dx \\ &\geq c_n + \frac{\mu_n s_n^2}{2} \int_{\mathbb{R}^N} \frac{u_0^2(x)}{|x|}\, dx. \end{aligned} \tag{2.58}$$

From (2.57) and (2.58) we obtain

$$c_0 \geq c_n + \frac{\mu_n s_n^2}{2} \int_{\mathbb{R}^N} \frac{u_0^2(x)}{|x|}\, dx \geq c_n \geq c_0 - \frac{\mu_n t_n^2}{2} \int_{\mathbb{R}^N} \frac{u_n^2(x)}{|x|}\, dx,$$

that is

$$c_0 - \frac{\mu_n t_n^2}{2} \int_{\mathbb{R}^N} \frac{u_n^2(x)}{|x|}\, dx \leq c_n \leq c_0.$$

By Lemma 2.25, the sequence $(u_n)_n$ is bounded in $H^{\frac{1}{2}}(\mathbb{R}^N)$ and by Hardy inequality (see (B.1) in Appendix B) it follows that

$$\int_{\mathbb{R}^N} \frac{u_n^2(x)}{|x|}\, dx < +\infty.$$

So, if we show that also the sequence $(t_n)_n$ is bounded, we complete the proof. We argue by contradiction, that is suppose that $t_n \to +\infty$ as n diverges to $+\infty$. Then, since $t_n u_n \in \mathcal{N}_0$ we get

$$\begin{aligned} \mathcal{E}_0'(t_n u_n)(t_n u_n) = \frac{t_n^2}{2} \|u_n\|^2_{H^{\frac{1}{2}}(\mathbb{R}^N)} &- \int_{\mathbb{R}^N \times \mathbb{R}^N} \frac{F(x, t_n u_n(x)) f(y, t_n u_n(y)) t_n u_n(y))}{|x - yN - \alpha}\, dx\, dy \\ &- t_n^q \int_{\mathbb{R}^N} K(x)|u_n(x)|^q\, dx = 0. \end{aligned}$$

Recalling that $Q(u_n) = \|u_n\|^2_{H^{\frac{1}{2}}(\mathbb{R}^N)}$ and dividing by t_n^q we obtain

$$\frac{Q(u_n)}{t_n^{q-2}} = \frac{1}{2} \frac{\mathcal{D}'(t_n u_n)(t_n u_n)}{t_n^q} - \int_{\mathbb{R}^N} K(x)|u_n(x)|^q\, dx. \tag{2.59}$$

By Lemma 2.25, Sobolev embedding and assumption **(K)** we have

$$\int_{\mathbb{R}^N} K(x)|u_n(x)|^q\, dx \leq \|K\|_{L^\infty(\mathbb{R}^N)} \|u_n\|_{L^q(\mathbb{R}^N)} \leq \|K\|_{L^\infty(\mathbb{R}^N)} \|u_n\|_{H^{\frac{1}{2}}(\mathbb{R}^N)} < +\infty.$$

Now, again by Lemma 2.25 and since $q > 2$, we have

$$\frac{Q(u_n)}{t_n^{q-2}} \to 0$$

as $n \to +\infty$. Therefore, going back to (2.59), we obtain that also

$$\frac{\mathcal{D}'(t_n u_n)(t_n u_n)}{t_n^q} < +\infty. \tag{2.60}$$

But, from (2.14), **(F3)** and Fatou's Lemma it follows

$$\begin{aligned} \frac{\mathcal{D}'(t_n u_n)(t_n u_n)}{t_n^q} &= 2 \frac{1}{t_n^q} \int_{\mathbb{R}^N \times \mathbb{R}^N} \frac{F(x, t_n u_n(x)) f(y, t_n u_n(y)) t_n u_n(y)}{|x - y|^{N-\alpha}}\, dx\, dy \\ &\geq 2 \frac{q}{2} \frac{1}{t_n^q} \int_{\mathbb{R}^N \times \mathbb{R}^N} \frac{F(x, t_n u_n(x)) F(y, t_n u_n(y))}{|x - y|^{N-\alpha}}\, dx\, dy \\ &= q \int_{\mathbb{R}^N \times \mathbb{R}^N} \frac{1}{|x - y|^{N-\alpha}} \frac{F(x, t_n u_n(x))}{t_n^{\frac{q}{2}}} \frac{F(y, t_n u_n(y))}{t_n^{\frac{q}{2}}}\, dx\, dy \end{aligned}$$

and this goes to $+\infty$ as $n \to +\infty$, contradicting (2.60). □

We are ready to state and prove the last result of this Chapter.

Theorem 2.27 (Compactness of gound states)**.** *Suppose that **(N)**, **(V1)**, **(V2)**, **(F1)-(F4)**, **(K)** are satisfied and* $V_l \equiv 0$. *Let* $(\mu_n)_n \subset (0, \mu^*)$ *be a sequence such that* $\mu_n \to 0^+$. *Then, for any choice of ground states* $u_n \in H^{\frac{1}{2}}(\mathbb{R}^N)$ *of* (2.4)*, with* $\mu = \mu_n$*, there is a sequence of translations* $(z_n)_n \subset \mathbb{Z}^N$ *such that, up to a subsequence,*

$$u_n(\cdot - z_n) \rightharpoonup u_0 \text{ in } H^{\frac{1}{2}}(\mathbb{R}^N),$$

where $u_0 \in H^{\frac{1}{2}}(\mathbb{R}^N)$ *is a ground state solution for* (2.4) *with* $\mu = 0$. *Moreover,* $c_n \to c$ *where* $c_n = \mathcal{E}(u_n)$ *and* $c = \mathcal{E}(u)$.

Proof. Suppose that

$$\lim_{n\to+\infty} \sup_{z\in\mathbb{R}^N} \int_{B(z,1)} |u_n(x)|^2 \, dx = 0,$$

then, from Lion's Concentration-Compactness principle, we obtain

$$u_n \to 0$$

in $L^t(\mathbb{R}^N)$ for all $t \in \left(2, \frac{2N}{N-1}\right)$. By hypothesis, u_n is a ground state for $\mathcal{E}_n$, that is

$$0 = \mathcal{E}_n'(u_n)(u_n) = \|u_n\|^2_{\mu_n} - \frac{1}{2}\mathcal{D}'(u_n)(u_n) + \int_{\mathbb{R}^N} K(x)|u_n(x)|^q \, dx,$$

from which

$$\|u_n\|^2_{\mu_n} = \frac{1}{2}\mathcal{D}'(u_n)(u_n) - \int_{\mathbb{R}^N} K(x)|u_n(x)|^q \, dx.$$

Reasoning as in Lemma 2.20 Step 4, we have that

$$\mathcal{D}'(u_n)(u_n) \to 0$$

and by **(K)**

$$\int_{\mathbb{R}^N} K(x)|u_n(x)|^q \, dx \to 0$$

as $n \to +\infty$. Hence, also

$$\|u_n\|^2_{\mu_n} \to 0$$

as $n \to +\infty$, and we get that

$$\begin{aligned} 0 \le \left(\|u_n\|^2_{L^2(\mathbb{R}^N)} + [u_n]^2\right) &\le \frac{Q_{\mu_n}(u_n)}{\min\left\{\frac{1}{2}C\left(N, \frac{1}{2}\right) - \frac{\mu_n}{C\left(N,\frac{1}{2},\frac{1}{2}\right)}, \operatorname{ess\,inf}_{\mathbb{R}^N} V - m\right\}} \\ &\to \frac{0}{\min\left\{\frac{1}{2}C\left(N, \frac{1}{2}\right), \operatorname{ess\,inf}_{\mathbb{R}^N} V - m\right\}} = 0, \end{aligned}$$

that is $u_n \to 0$ in $H^{\frac{1}{2}}(\mathbb{R}^N)$.

By Lemma 2.24, there exists $\beta > 0$ such that

$$\mathcal{E}_n(u_n) \ge \mathcal{E}_n\left(r\frac{u_n}{\|u_n\|}\right) > \beta > 0$$

and by Lemma 2.25

$$\limsup_{n\to+\infty} \mathcal{E}_n(u_n) = \limsup_{n\to+\infty} \left(-\frac{1}{2}\mathcal{D}(u_n)\right) \le 0$$

and so we reach a contradiction. Therefore, there exists a sequence $(z_n)_n \in \mathbb{Z}^N$ such that

$$\liminf_{n\to+\infty} \int_{B\left(z_n,1+\sqrt{N}\right)} |u_n(x)|^2\, dx > 0.$$

Again, by Lemma 2.25, there exists $u \in H^{\frac{1}{2}}(\mathbb{R}^N)$, with $u \neq 0$, such that

$$u_n(\cdot + z_n) \rightharpoonup u \text{ in } H^{\frac{1}{2}}(\mathbb{R}^N);$$

hence,

$$u_n(\cdot + z_n) \to u \text{ in } L^2_{loc}(\mathbb{R}^N);$$
$$u_n(x + z_n) \to u(x) \text{ for almost every } x \in \mathbb{R}^N.$$

Now, let $w_n = u_n(\cdot + z_n)$ and fix any $\varphi \in C_c^\infty(\mathbb{R}^N)$. We note that

$$\begin{aligned}
\mathcal{E}_0'(w_n)(\varphi) &= \mathcal{E}_n'(u_n)(\varphi(\cdot - z_n)) + \mu_n \int_{\mathbb{R}^N} \frac{u_n\varphi(\cdot - z_n)}{|x|}\, dx \\
&= \mu_n \int_{\mathbb{R}^N} \frac{u_n\varphi(\cdot - z_n)}{|x|}\, dx.
\end{aligned}$$

By Hardy inequality (see (B.1) in Appendix B) and Hölder inequality, we have that

$$\mu_n \int_{\mathbb{R}^N} \frac{|u_n(x)||\varphi(x - z_n)|}{|x|}\, dx \to 0$$

as $n \to +\infty$, hence

$$\mathcal{E}_0'(w_n)(\varphi) \to 0$$

and from Corollary 2.19

$$\mathcal{E}_0'(w_n)(\varphi) \to \mathcal{E}_0'(u)(\varphi)$$

as $n \to +\infty$. Therefore, u is a nontrivial critical point of $\mathcal{E}_0$ and in particular $u \in \mathcal{N}_0$. Now, by (2.14), Lemma 2.26 and Fatou's Lemma, and recalling that u_n is a gound state of $\mathcal{E}_n$, we compute

$$\begin{aligned}
c_0 &= \liminf_{n\to+\infty} \mathcal{E}_n(u_n) = \liminf_{n\to+\infty} \left(\mathcal{E}_n(u_n) - \frac{1}{q}\mathcal{E}_n'(u_n)(u_n) \right) \\
&= \liminf_{n\to+\infty} \Bigg[\left(\frac{1}{2} - \frac{1}{q}\right) Q(u_n) + \left(\frac{1}{q} - \frac{1}{2}\right) \mu_n \int_{\mathbb{R}^N} \frac{u_n^2(x)}{|x|}\, dx \\
&\quad + \frac{1}{q} \int_{\mathbb{R}^N\times\mathbb{R}^N} \frac{F(x,u_n(x))\left(f(y,u_n(y))u_n(y) - \frac{q}{2}F(y,u_n(y))\right)}{|x-y|^{N-\alpha}}\, dx\, dy \Bigg] \\
&= \liminf_{n\to+\infty} \Bigg[\left(\frac{1}{2} - \frac{1}{q}\right) Q(w_n) + \left(\frac{1}{q} - \frac{1}{2}\right) \mu_n \int_{\mathbb{R}^N} \frac{u_n^2(x)}{|x|}\, dx \\
&\quad + \frac{1}{q} \int_{\mathbb{R}^N\times\mathbb{R}^N} \frac{F(x,w_n(x))\left(f(y,w_n(y))w_n(y) - \frac{q}{2}F(y,w_n(y))\right)}{|x-y|^{N-\alpha}}\, dx\, dy \Bigg] \\
&\geq \left(\frac{1}{2} - \frac{1}{q}\right) Q(u) \\
&\quad + \frac{1}{q} \int_{\mathbb{R}^N\times\mathbb{R}^N} \frac{F(x,u(x))\left(f(y,u(y))u(y) - \frac{q}{2}F(y,u(y))\right)}{|x-y|^{N-\alpha}}\, dx\, dy \\
&= \mathcal{E}_0(u) - \frac{1}{q}\mathcal{E}'(u)(u) = \mathcal{E}_0(u) \geq c_0,
\end{aligned}$$

where we also used the weak lower semicontinuity of the norm $Q(\cdot)$ and the fact that

$$\mu \int_{\mathbb{R}^N} \frac{u_n^2(x)}{|x|}\, dx \to 0$$

as $n \to +\infty$ by the Hardy inequality (see (B.1) in Appendix B).

Therefore, $\mathcal{E}_0(u) = c$ and $u \in H^{\frac{1}{2}}(\mathbb{R}^N)$ is a ground state solution for $\mathcal{E}_0$ and the proof is complete.

□

Chapter 3

A Linking-type approach for a curl-curl problem

The two problems treated until now share a common feature, that is 0 is a local minimum of the associated energy functional. So the next question arises naturally: what happened if 0 is not anymore a local minimum? As an example, we can think at the following problem

$$-\Delta u = \lambda u + |u|^{p-2}u, \text{ in } \mathbb{R}^N$$

where $p \in (2, 2^*)$ and $\lambda \in \mathbb{R}$. The associated energy functional is $\mathcal{J} : H^1(\mathbb{R}^N) \to \mathbb{R}$ defined as

$$\mathcal{J}(u) := \frac{1}{2}\int_{\mathbb{R}^N} |\nabla u(x)|^2\, dx - \frac{\lambda}{2}\int_{\mathbb{R}^N} u^2(x)\, dx - \frac{1}{p}\int_{\mathbb{R}^N} |u(x)|^p\, dx.$$

We can easily observe that $\mathcal{J}$ is of class C^1 on $H^1(\mathbb{R}^N)$ and $\mathcal{J}(0) = 0$. Morever, evaluating the functional along any direction (say e_1 for simplicity) we get that

$$\mathcal{J}(te_1) = \frac{t^2}{2}\lambda_1 - \frac{t^2\lambda}{2} - \frac{t^p}{p}\int_{\mathbb{R}^N} |e_1|^p\, dx = \frac{\lambda_1 - \lambda}{2}t^2 - \frac{t^p}{p}\int_{\mathbb{R}^N} |e_1|^p\, dx, \tag{3.1}$$

where we used the characterization of the first eigenvalue of $-\Delta$ via the Rayleigh quotient, that is

$$\lambda_1 := \frac{\int_{\mathbb{R}^N} |\nabla u|^2\, dx}{\int_{\mathbb{R}^N} u^2\, dx}.$$

If $\lambda < \lambda_1$ then the first summands of (3.1) is positive and can be taken as a norm on the space, hence the functional $\mathcal{J}$ will be positive until some points and then the greatest power will dominate letting the functional diverges to minus infinity. So, in this case, $\mathcal{J}$ has the so-called Mountain-Pass geometry (see [14] and Figure 3.1 for an idea of the functional geometry). Conversely, if $\lambda \geq \lambda_1$, then the first summands of (3.1) is non-positive and so $\mathcal{J}$ is negative: hence, the functional $\mathcal{J}$ has a maximum in 0 (see Figure 3.2) and this fact requires a more careful treatment. Problems of this type are known in literature as *strongly indefinite problems* and can also appear when a part of the spectrum of the operator lies below zero (this last case will be exactly the one that we will treat in this Chapter: we will give more details in the next Sections).

A very useful, though not so trivial, tool that allows to deal with strong indefinite problems is the *Linking Theorem*, proved by Rabinowitz in 1978, see [118]. Indeed, if 0 belongs to a spectral gap of the spectrum (an equivalent way to say that we are facing a strong indefinite problem) then we can say that the associated functional joins the Linking geometry (much could be said

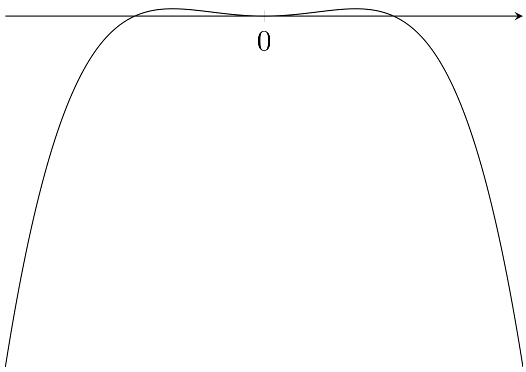


Figure 3.1: Example of functional with 0 as local minimum

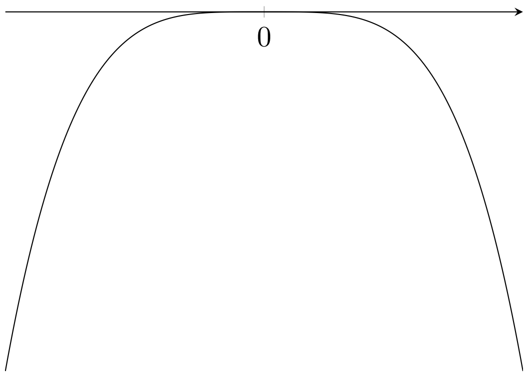


Figure 3.2: Example of functional with 0 as global maximum

about this fascinating argument, but in order not to load this thesis with too much detail, we refer to the book [119] for a treatment of this and other related Minimax tools). Briefly, the *classical* Linking Theorem asks for a decomposition of the ambient space into two subspaces, where at least one them is finite-dimensional (the case where both of them are of finite dimension is actually trivial). Therefore, what if both the subspaces are of infinite dimension? This is, in fact, the case we are going to treat and showing that linking geometry holds becomes a very hard task, since the strategy used by Rabinowitz (a Leray-Schauder degree argument) does not work anymore. To solve this problem, in 1997 Kryszewski and Szulkin in [81] provided a generalized Linking Theorem introducing a weak-strong topology and a topological degree theory for a suitable class of maps.

In 2016, Mederski in [95] used this setting to show the existence of a ground state solution for a system of nonlinear Schrödinger equations. Recently, Chen and Wang (see [44]) obtained an infinite Linking-type theorem replacing one hypothesis from the original version of Kryszewki and Szulkin, by another one that allows to deal also with sing-changing nonlinearities: in particular, they substitute the upper semicontuinuity (with respect to the weak-strong topology) with an additional request on the functional (see Remark 3.9 for the details). Although Chen and Wang result can be applied to a larger class of nonlinearities, it covers only the case of pure-power sign-changing nonlinearities.

Inspired by those works, we present here a new result contained in the work [26] and recently submitted, where we state a generalized Linking-type Theorem which further expands the class of nonlinearities, including also general sing-changing nonlinearities. Moreover, this result allows to consider operators with a singularity. In fact, we will show also an application of our Theorem to a singular Schrödinger equation with general sign-changing nonlinearities.

At very end of the Chapter, thanks to the connection between Schrödinger equations and curl-curl problems, and exploiting a Theorem on the equivalence of solutions, we are able to show the existence of a nontrivial solution for a curl-curl problem, that has a strong relation with the Maxwell equations.

We have a lot on, but we will try to shed some light during this Chapter.

3.1 Abstract result

In this Section, we state and prove our generalized Linking Theorem. To do that, we first need to introduce the weak-strong topology and the concept of admissible map. Then, we list the assumptions of the Theorem and make some comments on them, which will be the last step before diving into the Main Theorem.

3.1.1 Weak-strong topology

Let $(E, \|\cdot\|)$ be an Hilbert space and assume that it has an orthogonal splitting $E = E^+ \oplus E^-$: hence, every element $u \in E$ admits a unique decomposition $u = u^+ + u^-$, where $u^+ \in E^+$ and $u^- \in E^-$.

Following [81], we introduce a topology on E. Let $(e_n)_{n=1}^{+\infty} \subset E^-$ be a complete[1] orthonormal sequence for E^-. We define the norm $|||\cdot||| : E \to [0, +\infty)$ by

$$|||u||| := \max\left\{ \|u^+\|_{E^+}, \sum_{k=1}^{+\infty} \frac{1}{2^{k+1}} \left|(u^-, e_k)\right| \right\}. \tag{3.2}$$

We denote by τ the topology generates by $|||\cdot|||$ and we note that τ is a weaker topology than the one generated by the norm $\|\cdot\|$ on E: moreover, the following inequalities hold:

$$\|u^+\| \leq |||u||| \leq \|u\| \tag{3.3}$$

for every $u \in E$. Some remarks on this topology are needed.

Remark 3.1. *The space E endowed with the triple norm (3.2) is not complete. Indeed, it sufficies to consider the sequence $(u_n)_n \in E^-$ defined as*

$$u_n = \sum_{j=1}^{n} j e_j.$$

This is a Cauchy sequence, but it does not converge to any element in E^-.

Remark 3.2. *The $\tau-$topology is a weak-strong topology in E. Indeed, let $(u_n)_n \subset E$ be a bounded sequence, then*

$$u_n \xrightarrow{\tau} u \Leftrightarrow u_n^- \rightharpoonup u^- \text{ and } u_n^+ \to u^+.$$

Now, let $\mathcal{J} : E \to \mathbb{R}$ be a nonlinear functional. For every $u \in E \setminus E^-$ and $R(u) > r > 0$ we introduce the following sets:

$$\mathcal{S}_r^+ := \left\{ u^+ \in E^+ : \|u\| = r \right\}, \tag{3.4}$$

and

$$M(u) := \left\{ tu + v^- : v^- \in E^-, t \geq 0, \|tu + v^-\| \leq R \right\}. \tag{3.5}$$

The set (3.5) is a submanifold of $\mathbb{R}^+ u^+ \oplus E^-$ with boundary

$$\partial M(u) := \left\{ v^- \in E^- : \|v^-\| \leq R \right\} \cup \left\{ tu + v^- : v^- \in E^-, t > 0, \|tu + v^-\| = R \right\}. \tag{3.6}$$

We need also the following sets: the sub-level and the upper-level sets for the functional $\mathcal{J}$, defined for every $\alpha \leq \beta$ by

$$\mathcal{J}_\alpha := \left\{ u \in E : \alpha < \mathcal{J}(u) \right\},$$

[1] A sequence such that $(e_j, e_k) = \delta_{jk}$ and $\lim_{k\to+\infty} \left\| x - \sum_{i=1}^{k} (x, e_i)_E e_i \right\| = 0$ for every $x \in E$.

$$\mathcal{J}^{\beta} := \{u \in E : \mathcal{J}(u) \leq \beta\},$$

and the strip between them,

$$\mathcal{J}_{\alpha}^{\beta} = \mathcal{J}_{\alpha} \cap \mathcal{J}^{\beta}.$$

Moreover, let $\mathcal{P} \subset E \setminus E^{-}$ be a nonempty set and let $\mathcal{N}$ be the Nehari-Pankov manifold of $\mathcal{J}$ (introduced in [110], see [20, 97, 136] for some applications of this method), that is

$$\mathcal{N} := \{u \in E \setminus E^{-} : \mathcal{J}'(u)(u) = 0, \mathcal{J}'(u)(v) = 0 \text{ for every } v \in E^{-}\}. \tag{3.7}$$

Remark 3.3. *If $E^{-} = \emptyset$, then the Nehari-Pankov manifold (3.7) coincides with the classical Nehari manifold (see Section 2.6).*

3.1.2 Admissible maps and a new degree

We said in the Introduction of this Chapter that the use of the Leray-Schauder degree, as used by Rabinowitz in his original work, does not work in this case. However, the "old road" is still viable somehow, though the weak-strong topology alone is not enough. For this reason, Kryszewski and Szulkin needed to construct a new degree based on a new set of admissible maps, and the weak-strong topology will play a key role. We recall here only the definitions of degree and admissible maps, starting by the latter one, remainding to [81] for the details (see also [139], Chapter 6).

Let $A \in E$, $I \subset [0, +\infty)$ with $0 \in I$ and let $h : A \times I \to E$: we consider the following assumptions.

(h1) h is $\tau-$continuous, that is $h(v_n, t_n) \xrightarrow{\tau} h(v, t)$ for $v_n \xrightarrow{\tau} v$ and $t_n \to t$;

(h2) $h(u, 0) = h(u)$ for $u \in A$;

(h3) $\mathcal{J}(u) \geq \mathcal{J}(h(u, t))$ for $(u, t) \in A \times I$;

(h4) for every $(u, t) \in A \times I$ there is an open neighborhood $W \subset E \times I$ of (u, t) (in the sense of the product topology (E, τ) and $(I, \|\cdot\|)$) such that $\{v - h(v, s) : (v, s) \in W \cap (A \times I)\}$ is contained in a finite dimensional subspace of E.

Definition 3.4 (Admissible map)**.** *We say that $g : A \to E$ is admissible if it is $\tau-$continuous and the map $I - g$ it $\tau-$locally finite-dimensional.*

Definition 3.5 (Admissible homotopy)**.** *If a map $h : A \times I \to E$ satisfy **(h1)** and **(h4)**, then we say that h is admissible.*

Now, we give the definition of the degree.

Definition 3.6. *Let Z be a finite-dimensional subspace of E^{+} and U an open subset of the space $E_0 := Z \oplus E^{-}$. Suppose that*

(a) $g : \bar{U} \to E_0$ is admissible;

(b) $g^{-1}(0) \cap \partial U = \emptyset$ (closedness and boundedness are considered with respect the original topology of E_0);

(c) $g^{-1}(0)$ is $\tau-$compact.

For every $u \in g^{-1}(0)$, let W_u be a $\tau-$neighborhood of u such that $(I-g)(W_u \cap U)$ is contained if a finite-dimensional subspace of E. Thus, there are points $u_1, \dots, u_m \in g^{-1}(0)$ such that $g^{-1}(0) \subset W := \bigcup_{i=1}^n (W_{u_i} \cap U)$. The set W is open and there is a finite-dimensional subspace $L \subset E_0$ such that $(I-g)(W) \subset L$ and let $W_L := W \cap L$. Consider the map $g_L := g|_{W_L} : W_L \to L$: clearly, $g^{-1}(0) = g_L^{-1}(0)$, so that $g_L^{-1}(0)$ is compact in L. Therefore,

$$\deg(g, U, 0) = \deg_B(g_L, W_L, 0),$$

where $\deg_B$ denotes the classical Brouwer degree.

It can be shown that the degree is well-defined, that is it does not depend on the choice of W and L (see Proposition 6.4 in [139]). We conclude this part stating some properties of the degree that we will use in the proof of the main Theorem.

Proposition 3.7. *Let $U \in E$.*

(a) (Existence.) *If g is admissibile and $\deg(g, U, 0) \neq 0$, then $0 \in g(U)$.*

(b) *If $g(u) = u - u_0$, where $u_0 \in U$, then $\deg(g, U, 0) = 1$.*

(c) (Homotopy invariance.) *If h is an admissibile homotopy, then $\deg(h(t, \cdot), U, 0)$ is independent of $t \in I$.*

3.1.3 Main Theorem with proof

We are now ready to state and prove our version of the generalized Linking Theorem.

Theorem 3.8. *Suppose that the functional $\mathcal{J} : E \to \mathbb{R}$ satisfies*

(A1) *$\mathcal{J}$ is of class C^1 on E and $\mathcal{J}(0) = 0$;*

(A2) *$\mathcal{J}'$ is sequentially weak-to-weak* continuous[2];*

(A3) *there are $\delta > 0$ and $r > 0$ such that for every $u \in \mathcal{P}$ there is a radius $R = R(u) > r$ with*

$$\inf_{\mathcal{S}_r^+} \mathcal{J} > \max \left\{ \sup_{\partial M(u)} \mathcal{J}, \sup_{\|\|v\|\| \leq \delta} \mathcal{J}(v) \right\}.$$

(A4) *for every $u \in \mathcal{N}_\mathcal{P}, v \in E^-$ and $t \geq 0$ there holds*

$$\mathcal{J}(u) \geq \mathcal{J}(tu + v).$$

Then there exists a Cerami-sequence $(u_n)_n \subset E$ bounded away from zero, i.e. a sequence such that

$$\sup_n \mathcal{J} \leq c, \quad (1 + \|u_n\|)\mathcal{J}'(u_n) \to 0 \text{ in } E^*, \quad \inf_n \|\|u_n\|\| \geq \frac{\delta}{2}, \tag{3.8}$$

where

$$c := \inf_{u \in \mathcal{P}} \inf_{\gamma \in \Gamma(u)} \sup_{u' \in M(u)} \mathcal{J}(h(u', 1)) \geq \inf_{S_r^+} \mathcal{J} > 0$$

and

$$\Gamma(u) := \{h \in C(M(u) \times [0,1]) : h \text{ satisfy } \textbf{(h1)-(h4)}\}.$$

*If additionally **(A4)** holds, then*

$$c \leq \inf_{\mathcal{N}_\mathcal{P}} \mathcal{J}.$$

[2]See the definition in Lemma 2.18 in Chapter 2.

Before starting the proof, some comments on the assumptions are in order.

Remark 3.9. *Assumption* ***(A4)*** *is needed to obtain a comparison between the energy level obtained in Theorem 3.8 with the infimum on the* $\mathcal{N}_{\mathcal{P}}$.

In ***(A3)****, the assumption* $\inf_{\mathcal{S}_r^+} \mathcal{J} > \sup_{\|u\| \leq \delta} \mathcal{J}(u)$ *is the ones introduced by Chen and Wang that replaces the original assumption of Kryszewski and Szulkin on the* $\tau-$*upper-semicontinuity of* $\mathcal{J}$.

Proof of the Theorem. The proof is divided in several steps.

Step 1. *The family* $\Gamma(u)$ *is nonempty for every* $u \in \mathcal{P}$.

Fix $u \in \mathcal{P}$ and consider $h : M(u) \times [0,1] \to E$, defined by $h(v,t) = v$. Trivially, **(h1), (h2)** and **(h3)** are satisfied. To show **(h4)** it sufficies to observe that $v - h(v,s) = 0$ for every $v \in E$ and $s \in [0,1]$, so we take $W = E \times I$. Hence $h \in \Gamma(u)$ and the set is nonempty.

Step 2. $c \geq \inf_{\mathcal{S}_r^+} \mathcal{J}$.

Fix $u \in \mathcal{P}$ and $h \in \Gamma(u)$. We define the map

$$H : M(u) \times [0,1] \to \mathbb{R}u^+ \oplus E^- \subset E$$

by

$$H(v,t) := (\|h(v,t)^+\| - r)\frac{u^+}{\|u^+\|} + h(v,t)^-$$

and we claim that this map is admissible. In fact, **(h1)** is clear by definition. To get **(h4)**, we fix a point (v', t) and take the neighborhood W for h at this point. Then,

$$\begin{aligned} v - H(v,s) &= v - (\|h(v,s)^+\| - r)\frac{u^+}{\|u^+\|} - h(v,s)^- \\ &= v^+ - (\|h(v,s)^+\| - r)\frac{u^+}{\|u^+\|} + (v - h(v,s)^-) \end{aligned}$$

and the set $\{v - H(v,s) : (v,s) \in W \cap (A \times I)\}$ is contained in a finite-dimensional subspace. We observe that $H(v,t) = 0$ if and only if $h(v,t)^- = 0$ and $\|h(v,t)^+\| = r$, that is if $h(v,t) \in S_r^+$. Then, **(h3)** implies that

$$\sup_{\partial M(u)} \mathcal{J} \geq \mathcal{J}(v) \geq \mathcal{J}(h(v,t)) \geq \inf_{\mathcal{S}_r^+} \mathcal{J},$$

which contradicts **(A3)**. Hence, $0 \notin H(\partial M(u) \times [0,1])$ and

$$H(v,0) = v - r\frac{u^+}{\|u^+\|}.$$

Therfore, exploiting the homotopy invariance and existence property of the degree (see Proposition 3.7), we have

$$\deg(H(\cdot,1), M(u), 0) = \deg(H(\cdot,0), M(u), 0) = \deg\left(I - r\frac{u^+}{\|u^+\|}, M(u), 0\right) = 1.$$

Therefore, $\deg(H(\cdot,1), M(u), 0) \neq 0$ and this means that there exists $v \in M(u)$ with $H(v,1) = 0$, that is $h(v,1) \in \mathcal{S}_r^+$ and

$$\sup_{u' \in M(u)} \mathcal{J}(h(u',1)) \geq \mathcal{J}(h(v,1)) \geq \inf_{\mathcal{S}_r^+} \mathcal{J} > 0.$$

This completes the Step 2.

Hereafter, our aim is to show the existence of a Cerami-sequence satisfying (3.8). We argue by contradiction, that is there exists $\varepsilon > 0$ such that

$$(1 + \|u\|)\|\mathcal{J}'(u)\| \geq \varepsilon$$

for every $u \in \mathcal{J}^{c+\varepsilon} \cap \left\{u \in E : |||u||| \geq \frac{\delta}{2}\right\}$. We can suppose, without loss of generality, that $\varepsilon < \inf_{\mathcal{S}_r^+} \mathcal{J}$.

Step 3. *There exists a vector field in a neighborhood of* $\mathcal{J}^{c+\varepsilon} \cap \left\{u \in E : |||u||| \geq \frac{\delta}{2}\right\}$ *and we can construct a flow* η*.*

For the sake of simplicity, we set $Y = \mathcal{J}^{c+\varepsilon} \cap \left\{u \in E : |||u||| \geq \frac{\delta}{2}\right\}$. We fix $\rho > 0$ and for $u \in Y \cap B_\rho$ we define

$$w(u) := \frac{2\nabla\mathcal{J}(u)}{\|\mathcal{J}'(u)\|^2},$$

there follows that

$$(\nabla\mathcal{J}(u), w(u)) = 2$$

and

$$\|w(u)\| = \frac{2}{\|\mathcal{J}'(u)\|} \leq \frac{2}{\varepsilon}(1 + \|u\|).$$

Then, we claim the existence of a $\tau-$open neighborhood U_u of u with

$$(\nabla\mathcal{J}(u), w(u)) > 1,$$
$$\|w(u)\| \leq \frac{4}{\varepsilon}(1 + \|v\|), \text{ for } v \in U_u.$$

Indeed, let $(u_n)_n \subset Y \cap B_\rho$ be a sequence such that $u_n \xrightarrow{\tau} u$, then $u_n \rightharpoonup u$ in E and by (A2) $\mathcal{J}'(u_n)(\varphi) \to \mathcal{J}'(u)(\varphi)$ for every $\varphi \in E$, so it follows that $\mathcal{J}'$ is sequentially $\tau-$continuous in $Y \cap B_\rho$. This fact, together with the weakly lower semi-continuity of the norm in E, gives us the existence of the neighborhood U_u (see also [81], Proposition 3.2 and Remark 2.1(iii)).

We recall that B_ρ is closed, bounded and convex, so it is also $\tau-$closed. Hence, the set $U_0 := E \setminus B_\rho$ is $\tau-$open and the family

$$\mathcal{F} := \{U_u\}_{u\in Y\cap B_\rho} \cup \{U_0\}$$

is a $\tau-$open covering of Y and we set

$$\mathcal{U} := \bigcup \mathcal{F}.$$

The family $\mathcal{F}$ is the $\tau-$open covering of a metric space $\mathcal{U}$, which is paracompact (see for example [123]). Hence, we can find a $\tau-$locally finite $\tau-$open refinement $\{\widetilde{N}_j\}_{j\in J}$ of the covering $\mathcal{F}$ of $\mathcal{U}$: it follows that

$$Y \subset \mathcal{U} \subset \widetilde{N} := \bigcup_{j\in J} \widetilde{N}_j$$

and, of course, $\widetilde{N}$ is a $\tau-$open set.

Now, since $\mathcal{U}$ is paracompact, let $\{\lambda_j\}_{j\in J}$ denote the $\tau-$Lipschitzian partition of the unity subordinated to $\{\widetilde{N}_j\}_{j\in J}$; we set $w_j := w(u_j)$ if $\widetilde{N}_j \subset U_{u_j}$ for some u_j, otherwise we set $w_j = 0$ if $\widetilde{N}_j \subset U_0$ and we put

$$\widetilde{V}(u) := \sum_{j\in J} \lambda_j(u) w_j, \quad u \in \widetilde{N}.$$

The sum above si finite for every $u \in \widetilde{N}$, so there exists a $\tau-$open neighborhood $U_u \subset \widetilde{N}$ of u and L_u such that $\widetilde{V}(U_u)$ is contained in a finite-dimensional subspace of E and

$$\|\widetilde{V}(v) - \widetilde{V}(w)\| \leq L_u |||v - w|||$$

for $v, w \in U_u$.

Moreover, we have that

$$\begin{aligned} &\left(\nabla \mathcal{J}(u), \widetilde{V}(u)\right) \geq 0, \quad u \in \widetilde{N}, \\ &\left(\nabla \mathcal{J}(u), \widetilde{V}(u)\right) > 1, \quad u \in Y \cap B_\rho, \\ &\left\|\widetilde{V}(u)\right\| \leq \frac{4}{\varepsilon}(1 + \|u\|), \quad u \in \widetilde{N}. \end{aligned} \tag{3.9}$$

Now, we choose a function $\chi : \mathbb{R} \to [0, 1]$ such that

$$\chi(t) = \begin{cases} 0, & \text{for } t \leq \dfrac{2\delta}{3}, \\ 1, & \text{for } t \geq \delta, \end{cases}$$

and we set

$$V(u) := \begin{cases} \chi(|||u|||)\widetilde{V}(u), & \text{if } u \in \widetilde{N}, \\ 0, & \text{if } |||u||| \leq \dfrac{2\delta}{3}. \end{cases}$$

We call $N := \mathcal{U} \cup \{u \in E : |||u||| < \delta\}$ and observe that it is a $\tau-$neighborhood of $\mathcal{J}^{c+\varepsilon} \cup (E \setminus B_\rho)$. The function V is locally Lipschitz and $\tau-$locally Lipschitz continuous and by (3.9) there follow

$$\begin{aligned} &(\nabla \mathcal{J}(u), V(u)) \geq 0, \quad u \in N, \\ &(\nabla \mathcal{J}(u), V(u)) > 1, \quad u \in \mathcal{J}^{c+\varepsilon} \cap \{u \in E : |||u||| \geq \delta\} \cap B_\rho, \\ &\|V(u)\| \leq \frac{4}{\varepsilon}(1 + \|u\|), \quad u \in N. \end{aligned} \tag{3.10}$$

To cosntruct the flow η, we consider the following initial value problem

$$\begin{cases} \dfrac{\partial \eta}{\partial t}(u, t) = -V(\eta(u, t)) \\ \eta(u, 0) = u \in N \supset \mathcal{J}^{c+\varepsilon} \cup (E \setminus B_\rho) \end{cases}$$

and, since V is locally Lipschitz continuous, this have a unique solution $\eta(u, \cdot) : [0, T^+(u)) \to E$, where $T^+(u) > 0$ is the maximal time of existence in a positive direction.

Step 4. *Properties of the flow* η. We have that η is $\tau-$continuous. The proof is not straightforward, since the space (E, τ) is not complete (see Remark 3.1), so we will show the $\tau-$continuity here in the spirit of [81]. Let $(u_0, t_0) \in N \times [0, T^+(u))$ and we set $\Lambda := \eta\left\{\{u_0\} \times [0, T^+(u))\right\}$: this is a compact set, so Λ it is also $\tau-$compact, therefore there exist $r > 0$ and $K > 0$ such that

$$B := \{u \in E : |||u - K||| < r\}$$

and for every $u, v \in B$ we have

$$|||V(u) - V(v)||| \leq K|||u - v|||,$$

so $V(B)$ is contained in a finite-dimensional space of E, say E_1.

Now, suppose that $\eta(u,s) \in B$ for every $0 \le s \le t \le T^+(u)$: we get

$$\begin{aligned}\|\!|\eta(u,t) - \eta(u_0,t)|\!\| &\le \|\!|u - u_0|\!\| + \int_0^t \|\!|V(\eta(u,s)) - V(\eta(u_0,s))|\!\|\, ds \\ &\le \|\!|u-u_0|\!\| + K\int_0^t \|\!|\eta(u,s) - \eta(u_0,s)|\!\|\, ds\end{aligned}$$

and by Gronwall's inequality

$$\|\!|\eta(u,t) - \eta(u_0,t)|\!\| \le \|\!|u-u_0|\!\| e^{Kt} \le \|\!|u-u_0|\!\| e^{KT^+(u)}. \tag{3.11}$$

Let $0 < \delta < re^{-KT^+(u)}$, if $\|\!|u - u_0|\!\| \le \delta$ then

$$\|\!|\eta(u,t) - \eta(u_{,0}\,,t)|\!\| \le r,$$

hence $\eta(u,t) \in B$ for each $t \in [0, T^+(u))$. Therefore, if $|t - t_0| < \delta$, then using (3.11),

$$\begin{aligned}\|\!|\eta(u,t) - \eta(u_0,t_0)|\!\| &\le \|\!|\eta(u,t) - \eta(u_0,t)|\!\| + \|\!|\eta(u_0,t) - \eta(u_0,t_0)|\!\| \\ &\le \|\!|\eta(u,t) - \eta(u_0,t_0)|\!\| + \int_{t_0}^t \|\!|V(\eta(u_0,s))|\!\|\, ds \\ &\le \|\!|u-u_0|\!\| e^{KT^+(u)} + m|t-t_0| \le \left(e^{KT^+(u))} + m\right)\delta,\end{aligned}$$

where we call $m := \sup_{u\in N} \|\!|V(u)|\!\|$. Being δ arbitrary, the $\tau-$continuity is proved.

Now, for every $u \in N$, we have by (3.10),

$$\begin{aligned}\frac{d}{dt}\mathcal{J}(\eta(u,t)) &= \mathcal{J}'(\eta(u,t))\frac{\partial}{\partial t}\eta(u,t) \\ &= \mathcal{J}'(\eta(u,t))(-V(\eta(u,t)) = -\left(\nabla\mathcal{J}(\eta(u,t)), V(\eta(u,t))\right) \le 0\end{aligned}$$

hence $\mathcal{J}$ is nonincreasing along the trajectories $t \mapsto \eta(u,t)$. In particular, it follows that if $u \in \mathcal{J}^{c+\varepsilon}$, then

$$\left\{\eta(u,t) : 0 \le t < T^+(u)\right\} \subset \mathcal{J}^{c+\varepsilon}.$$

As long as V is sublinear, then for $u \in \mathcal{J}^{c+\varepsilon}$, we get $T^+(u) = +\infty$ and moreover, using (3.10),

$$\begin{aligned}\|\eta(u,t)\| = \left\|u - \int_0^t V(\eta(u,s))\, ds\right\| &\le \|u\| + \int_0^t \|V(\eta(u,s)\|\ ds \\ &\le \|u\| + \frac{4}{\varepsilon}\int_0^t 1 + \|\eta(u,s)\|\, ds.\end{aligned}$$

From Gronwall's inequality

$$\|\eta(u,t)\| \le (1 + \|u\|)e^{\frac{4t}{\varepsilon}} - 1. \tag{3.12}$$

We call $b := \inf_{\mathcal{S}_r^+}\mathcal{J}$, so

$$\sup_{\|\!|u|\!\|\le\delta} \mathcal{J} < b - \varepsilon$$

and in particular

$$\{u \in E : \|\!|u|\!\| \le \delta\} \subset \mathcal{J}^{b-\varepsilon}.$$

Hence,

$$\mathcal{J}_{b-\varepsilon}^{c+\varepsilon} \cap B_\rho \subset \mathcal{J}^{c+\varepsilon} \cap \{u \in E : \|\!|u|\!\| \ge \delta\} \cap B_\rho,$$

therefore,

$$(\nabla\mathcal{J}(u), V(u)) > 1, \quad u \in \mathcal{J}_{b-\varepsilon}^{c+\varepsilon} \cap B_\rho. \tag{3.13}$$

Step 5. *Conclusion.* We fix $u \in \mathcal{P}$ and $h \in \Gamma(u)$ such that $\sup_{u' \in M(u)} \mathcal{J}(h(u',1)) < c + \varepsilon$ and we claim that

$$\sup_{u' \in M(u)} \|h(u',1)\| < +\infty. \tag{3.14}$$

We argue by contradiction, supposing that there exists a sequence $(u_n)_n \in M(u)$ such that

$$\|h(u_n,1)\| \to +\infty \tag{3.15}$$

as n approaches infinity. Since $(u_n)_n \in M(u)$, then $\|u_n\| \leq R(u)$ and there exists an element $u_0 \in E$ such that, up to a subsequence, $u_n \rightharpoonup u_0$ in E and u_n can be written as $u_n = t_n u^+ + w_n$, where $t_n \geq 0$ and $w_n \in E^-$. On finite-dimensional spaces, the weak-convergence is equivalent to the norm-convergence, hence $u_n \xrightarrow{\tau} u_0$.

By assumption **(h1)** it follows that $h(u_n,1) \xrightarrow{\tau} h(u_0,1)$ and in particular the sequence $(h(u_n,1))_n$ is $\tau-$bounded. From (h4) we can find a neighborhood W in the product topology (E,τ) and $([0,1],|\cdot|)$ of the point $(u_0,1)$ such that the set

$$\{v - h(v,t) : (v,t) \in W \cap (M(u) \times [0,1])\}$$

is contained in a finite dimensional space $\mathbb{V}$. We observe that for sufficiently large n we have that $(u_n,1) \in W$, hence

$$u_n - h(u_n,1) \in \mathbb{V}.$$

But on finite dimensional spaces all the norms are equivalent, so there exist two constants $0 \leq C_1 \leq C_2$ such that

$$\begin{aligned} C_1\|u_n - h(u_n,1)\| &\leq |||u_n - h(u_n,1)||| \leq |||u_n||| + |||h(u_n,1)||| \\ &\leq \|u_n\| + |||h(u_n,1)||| \leq C_2. \end{aligned}$$

Therefore, there exists a constant $C_3 > 0$ such that

$$\|h(u_n,1)\| = \|u_n - h(u_n,1) - u_n\| \leq \|u_n - h(u_n,1)\| + \|u_n\| \leq C_3,$$

but this contradicts (3.15).

We set $\rho(u,h) := \left(1 + \sup_{u' \in M(u)} \|h(u',1)\|\right) e^{\frac{4T_0}{\varepsilon}} - 1$, where $T_0 := 2\varepsilon + c - b$. Choosing $\rho := \rho(u,h)$ we find the flow η of Step 3 with the properties of Step 4.

We remark that for $u' \in M(u)$ we have that $h(u',1) \in \mathcal{J}^{c+\varepsilon}$ and from (3.12) we obtain

$$\|\eta(h(u',1),t)\| \leq (1 + \|h(u',1)\|)e^{\frac{4t}{\varepsilon}} \leq \rho(u,h)$$

for $t \in [0,T_0]$.

Hence, if $t \in [0,T_0]$ then $\eta(h(u',1),t) \in B_\rho$ and, since the flow is nonincreasing, also $h(u',1) \in B_\rho$. From (3.13) it follows that $\eta(h(u',1),T_0) \in \mathcal{J}^{b-\varepsilon}$.

We define the function $g : M(u) \times [0,1] \to E$ as

$$g(u',t) := \begin{cases} h(u',2t), & \text{if } t \in [0,1/2], \\ \eta(h(u',1),T_0(2t-1)), & \text{if } t \in [1/2,1]. \end{cases}$$

Then $g \in \Gamma(u)$ and $\mathcal{J}(g(u',1)) = \mathcal{J}(\eta(h(u',1),T_0)) \leq b - \varepsilon \leq c - \varepsilon$ for any $u' \in M(u)$, but this contradicts the definition of c.

Step 6. *If (A4) holds, then* $c \leq \inf_{\mathcal{N}_\mathcal{P}} \mathcal{J}$. If $\mathcal{N}_\mathcal{P} = \emptyset$ then $\inf_{\mathcal{N}_\mathcal{P}} \mathcal{J} = +\infty$ and inequality is trivial. We suppose then $\mathcal{N}_\mathcal{P} \neq \emptyset$, we take any $u \in \mathcal{N}_\mathcal{P} \subset \mathcal{P}$ and we define $h : M(u) \times [0,1] \to E$ by the formula $h(u',t) = u'$ for $u' \in M(u)$. We observe that h satisfy **(h1)** and **(h4)**.

(h1) is trivial: indeed, let $(u_n)_n \subset M(u)$ and $(t_n)_n \subset [0,1]$ be two sequences such that $u_n \xrightarrow{\tau} u$ and $t_n \to t$. Then

$$h(u'_n, t_n) = u'_n \xrightarrow{\tau} u' = h(u', t).$$

To show **(h4)**, observe that

$$v - h(v, s) = v - v = 0,$$

so it is enough to take a neighborhood $W = M(u) \times [0,1]$.

Then, assumption **(A4)** implies that

$$c \leq \sup_{u' \in M(u)} \mathcal{J}(h(u', 1)) = \sup_{u' \in M(u)} \mathcal{J}(u') = \sup_{tu+v \in M(u)} \mathcal{J}(tu + v) \leq \mathcal{J}(u),$$

where $t \geq 0$ and $v \in E^-$. Computing the infimum over $\mathcal{N}_{\mathcal{P}}$ will complete the proof. □

3.2 A look at Modern Physics

Since the end of the 18th century, electromagnetism was a topic of great interests and many studies, mainly experimental, were conducted by brilliant scientists, such as Coulomb, Ampère, Faraday. However, these studies were carried out individually, obtaining results that could not fully explain the various phenomena. This lack of communication, mainly due to the different historical moments in which the experiments were conducted, left electromagnetism an open problem for many years.

In the early 1870s, Maxwell understood the individual principles and summarized them in a single theory, finally managing to provide a model that would fully explain the phenomena of electromagnetism. However, his model was made up of more than 20 equations, but in 1884 Heaviside, developing vector calculus, managed to simplify Maxwell's model to only 4 equations, that is

$$\begin{cases} \nabla \times \mathcal{H} = \mathcal{J} + \dfrac{\partial \mathcal{D}}{\partial t} & \text{(Ampére's Law)} \\ \operatorname{div}(\mathcal{D}) = \rho & \text{(Gauss' Electric Law)} \\ \dfrac{\partial \mathcal{B}}{\partial t} + \nabla \times \mathcal{E} = 0 & \text{(Faraday's Law of Induction)} \\ \operatorname{div}(\mathcal{B}) = 0 & \text{(Gauss' Magnetic Law)} \end{cases} \tag{3.16}$$

where:

- $\mathcal{E}$ is the electric field;
- $\mathcal{B}$ is the magnetic field;
- $\mathcal{D}$ is the electric displacement field;
- $\mathcal{H}$ is the magnetic induction;
- $\mathcal{J}$ is the electric current intensity;
- ρ electric charge density.

This model is easier to handle and conforms to quantum physics and is called the Maxwell-Heaviside equations, although the literature refers to them only as Maxwell's equations.

System above contains more unknows than equation, so in order to find a solution, some relations between the unkowns are needed. The relations are given by the following constituive relations

$$\begin{cases} \mathcal{D} = \varepsilon\mathcal{E} + \mathcal{P} \\ \mathcal{H} = \dfrac{1}{\mu}\mathcal{B} - \mathcal{M}, \end{cases}$$

where $\mathcal{P}$ is the polarization and $\mathcal{M}$ is the magnetization. Moreover, ε is the permettivity of the medium and μ is permeability. Of course, this is only a brief introduction to this subject, but this is enough for our purpose and we remind to [57, 80, 106] for a more detailed treatment.

If we consider a simplified model, that is a model with absence of charge, currents and magnetization (i.e. $\rho = \mathcal{J} = \mathcal{M} = 0$) and setting the permeability $\mu = 1$, then we obtain that system (3.16) is equivalent to the following time-dependent equation

$$\nabla \times (\nabla \times \mathcal{E}) + \varepsilon \frac{\partial^2 \mathcal{E}}{\partial t^2} = \frac{-\partial^2 \mathcal{P}}{\partial t^2}. \tag{3.17}$$

Looking for time-harmonic fields $\mathcal{E} = \mathbf{E}(x)\cos(\omega t)$, $\mathcal{P} = \mathbf{P}(x)\cos(\omega t)$ where $\mathbf{P}$ depends nonlinearly on $\mathbf{E}$, we obtain

$$\nabla \times (\nabla \times \mathbf{E}) + V(x)\mathbf{E} = h(\mathbf{E}), \quad x \in \mathbb{R}^3, \tag{3.18}$$

where $V(x) = -\omega^2\varepsilon(x)$.

When a time-harmonic field, like $\mathbf{E}$ above, oscillates it generates an oscillating magnetic field (and viceversa). Propagating by the source, it transfers some energy to the objects on its path and an electromagnetic wave tends to store some energy both in the electric and magnetic field. The finitness of this energy is very important in the study of self-guided beams of light in a nonlinear medium (see [3, 94, 133, 134]). As in [94], we are able to prove the finitness and the independence from time of the electromagnetic energy (see Proposition 3.26)

$$\mathcal{L}(t) := \frac{1}{2}\int_{\mathbb{R}^3} (\mathcal{E}\mathcal{D} + \mathcal{B}\mathcal{H})\, dx \tag{3.19}$$

with respect to the solution of (3.18).

Due to the difficulties given by the *curl-curl* operator, literature on this problem is very poor, but some progresses were made in the recent years. In a series of paper (see [20–22]), Bartsch and Mederski studied the Maxwell equation in a bounded domain $\Omega \subset \mathbb{R}^3$ where they considered the metallic boundary condition

$$\nu \times \mathbf{E} = 0, \text{ on } \partial\Omega$$

and $\nu : \partial\Omega \to \mathbb{R}^3$ is the exterior normal vector field. In their papers, they used approaches based on the Helmholz decomposition and the Nehari-Pankov manifold method. The same problem was considered also in [19] in a cylindrically symmetric setting, showing the existence of a ground-state solution. For some other papers in bounded domains we refer to [78, 140], while a numerical-approach was considered in [100].

3.3 From Maxwell to Schrödinger... and back

Dealing with the curl-curl operator is an hard task, mainly for two reasons. First, fixed an element $\varphi \in C_c^\infty(\mathbb{R}^3)$, then $\nabla \times (\nabla\varphi) = 0$; this means that $\ker(\nabla \times \nabla\times)$ has infinite dimension. Hence, the (formal) energy functional associated to (3.18) is strongly indefinite, in fact is unbounded both from above and from below. Moreover, the first Gâteaux derivative of this functional is not weak-to-weak* continuous, a key property in Variational Methods (see for example Chapter 2, Lemma 2.18 and Corollary 2.19), so that every nontrivial critical point has infinite Morse index.

To overcome these problems, we consider particular solutions for (3.18), the so-called *cylindrically symmetric solutions* (see [19, 141]), that is solutions of the form

$$\mathbf{E}(x) = \frac{u(r, x_3)}{r} \begin{pmatrix} -x_2 \\ x_1 \\ 0 \end{pmatrix} \tag{3.20}$$

with $r = \sqrt{x_1^2 + x_2^2}$. With this ansatz, equation (3.18) become the Schrödinger equation

$$-\Delta u + V(x)u + \frac{1}{r^2}u = h(u), \quad x = (y, z) \in \mathbb{R}^2 \times \mathbb{R}, r = |y|, \tag{3.21}$$

where the nonlinearity h is described by the following relation:

$$h(\alpha w) = f(\alpha)w - \lambda g(\alpha)w$$

for $w \in \mathbb{R}^3$, $|w| = 1$, $\alpha \in \mathbb{R}$.

Actually, due to its own interesting, we are going to study a more general version of (3.21), that is we will deal with the following equation:

$$-\Delta u + V(x)u + \frac{a}{r^2}u = f(u) - \lambda g(u), \quad x = (y, z) \in \mathbb{R}^K \times \mathbb{R}^{N-K}, r = |y|, \tag{3.22}$$

where $a > -\frac{(K-2)^2}{4}$ and $N > K \geq 2$ (see below for the assumptions on f and g).

Equation (3.22) is the time-indepentent Schrödinger equation obtained from

$$i\frac{\partial \Psi}{\partial t} = -\Delta\Psi + (V(x) + \frac{a}{r^2} + \omega)\Psi - f(|\Psi|) + \lambda g(|\Psi|)$$

when looking for stading-wave solutions, i.e. if $\Psi(x, t) = e^{-i\omega t}u(x)$.

Problem (3.22) with sign-constant nonlinearities (that is $g \equiv 0$) was considered in [16] with $a = 1$ and $V \equiv 0$ and they found a nontrivial nonnegative solution. In [70] the authors proved existence and multiplicity of solutions for (3.22) for $V \equiv 0$, $a > -\frac{(K-2)^2}{4}$ and f odd, while in [31] the author studied the equation for $V \neq 0$ and sign-changing nonlinearity.

3.4 Application to a Schrödinger equations

We recall that we want to study the following equation

$$-\Delta u(x) + V(x) + \frac{a}{r^2}u(x) = f(u) - \lambda g(u) \tag{3.23}$$

where $x = (y, z) \in \mathbb{R}^K \times \mathbb{R}^{N-K}, r = |y|$. We start by giving the assumption on the potential V:

(V) $V \in L^\infty(\mathbb{R}^N)$ is $O(K) \times \{I_{N-K}\}$ −invariant, Z^{N-K}−periodic in z and

$$0 \notin \sigma\left(-\Delta + \frac{a}{r^2} + V(x)\right) \text{ and } \sigma\left(-\Delta + \frac{a}{r^2} + V(x)\right) \cap (-\infty, 0) \neq 0. \tag{3.24}$$

How does this hypothesis affect the treatment of the problem? Let

$$X = \left\{ u \in H^1(\mathbb{R}^N) : u \text{ is } O(K) \times \{I_{N-K}\} \text{ invariant and } \int_{\mathbb{R}^N} \frac{u^2}{r^2}\, dx < +\infty \right\}.$$

The spectrum of the operator $-\Delta + \frac{a}{r^2} + V$ is purely continuous for $a > \frac{-(K-2)^4}{4}$ and consists of closed disjoint intervals: hence, there exists a maximal open interval $(-\mu_-, \mu_+)$, with $\mu_-, \mu_+ > 0$, free of the spectrum (that is, a gap) and moreover, these gaps are disjoint because of the purely continuity of the spectrum. The second request of assumption (V) implies that μ_- is finite, that is 0 belongs to a finite gap. In fact, if this is not the case, then we would have that 0 belongs to a gap of the form $(-\infty, \mu_+)$, so the spectrum would be positive and the problem would be not strongly indefinite. We point out that the second request on (V) is necessary because on the right-hand side of (3.23) we have a sign-changing nonlinearity. In fact, in the case of a positive nonlinearity, in order to obtain a strongly indefinite problem, it would have been enough to require only that 0 does not belong to the spectrum of the operator (e.g. see [81]).

Hence, under assumption (V), the space X can be orthogonally splitted as $X = X^+ \oplus X^-$ such that the quadratic form

$$\int_{\mathbb{R}^N} \left(|\nabla u(x)|^2 + a\frac{u^2(x)}{r^2} + V(x)u^2(x) \right) dx$$

is positive definite on X^+ and negative definite on X^-. Indeed, let $L : X \to X$ be the self-adjoint operator defined by

$$(Lu, v) = \int_{\mathbb{R}^N} \left(\nabla u(x) \cdot \nabla v(x) + a\frac{u(x)v(x)}{r^2} + V(x)u(x)v(x) \right) dx.$$

So, if $u \in X^+$ we have

$$(Lu, v) \geq 0,$$

while if $u \in X^-$

$$(Lu, v) \leq 0$$

for every $v \in X$.

Hence, we can define a norm both on X^+ and X^- as

$$\|u^\pm\|^2 := \pm \int_{\mathbb{R}^N} \left(|\nabla u(x)|^2 + a\frac{u^2(x)}{r^2} + V(x)u^2(x) \right) dx, \quad u^\pm \in X^\pm$$

and the product topology on X by

$$\|u\|^2 := \|u^+\|^2 + \|u^-\|^2,$$

where $u = u^+ + u^-, u^\pm \in X^\pm$. We also remark that the projections

$$P : X \to X^+, \quad Q : X \to X^-$$

are continuous in $L^q(\mathbb{R}^N)$ (see Proposition 7 in [138]). We denote by $\kappa \geq 1$ the constant such that

$$\|u^\pm\|_{L^q(\mathbb{R}^N)} \leq \kappa \|u\|_{L^q(\mathbb{R}^N} \tag{3.25}$$

for $u \in X$. From hypothesis (V) there follows that there exists a constant $\mu_0 > 0$ such that

$$\mu_0 \|u\|_{L^2(\mathbb{R}^N)} \leq \|u\| \tag{3.26}$$

for $u \in X$.

We remark that, for $K > 2$ the following inequality holds (see [17])

$$\int_{\mathbb{R}^N} \frac{u^2}{r^2}\, dx \le \left(\frac{2}{K-2}\right)^2 \int_{\mathbb{R}^N} |\nabla u|^2\, dx$$

for every $u \in H^1(\mathbb{R}^2)$, so the singular integral is finite for every $u \in H^1(\mathbb{R}^2)$.

Now, we list the assumptions on the nonlinearities f and g:

(F1) $f : \mathbb{R} \to \mathbb{R}$ is odd, continuous, $p \in (2, 2^*)$ and there exists a constant $C > 0$ such that

$$|f(u)| \le C(1 + |u|^{p-1})$$

for all $u \in \mathbb{R}$.

(F2) $f(u) = o(|u|)$ as $u \to 0$.

(F3) There is $q \in (2, p)$ such that $\frac{F(u)}{|u|^q} \to +\infty$ as $|u| \to +\infty$, where $F(u) = \int_0^u f(s)\, ds$ and $F(u) \ge 0$ for all $u \in \mathbb{R}$.

(F4) $u \mapsto \frac{f(u)}{|u|^{q-1}}$ is nondecreasing in $(-\infty, 0) \cup (0, +\infty)$.

(F5) There is $\rho > 0$ and there are $C_1, C_2 > 0$ such that

$$C_1|u|^{p-1} \le |f(u)| \le C_2|u|^{p-1}$$

for $|u| \ge \rho$.

(G1) $g : \mathbb{R} \to \mathbb{R}$ is odd, continuous and there exists $C > 0$ such that

$$|g(u)| \le C(1 + |u|^{q-1})$$

for all $u \in \mathbb{R}$.

(G2) $g(u) = o(|u|)$ as $u \to 0$.

(G3) $u \mapsto \frac{g(u)}{|u|^{q-1}}$ is nonincreasing in $(-\infty, 0) \cup (0, +\infty)$ and there holds

$$g(u)u \ge 0$$

for all $u \in \mathbb{R}$.

3.4.1 Some useful estimates

We collect here some estimates that follow from the hypothesis above. Many of them are now classic, but we report them here together with the proofs for the sake of completeness.

Proposition 3.10. *Suppose (F1)-(F2) and (G1)-(G2) hold, then for every $\varepsilon > 0$ there exists $C_\varepsilon > 0$ and $C_{G,\varepsilon} > 0$ such that*

$$|f(u)| \le \varepsilon|u| + C_\varepsilon|u|^{p-1}; \tag{3.27}$$

$$|g(u)| \le \varepsilon|u| + C_{G,\varepsilon}|u|^{q-1}. \tag{3.28}$$

Proof. We show the proof for f, being the one for g similar. From (F2) we have that for every $\varepsilon > 0$ there exists $\delta := \delta(\varepsilon) > 0$ such that

$$f(u) \leq \varepsilon |u|. \tag{3.29}$$

Now, we have that assumption (F1) holds for every $u \in \mathbb{R}$: for our purpose, we consider it only for u such that $|u| \geq \delta$: therefore,

$$\begin{aligned} f(u) &\leq C(1 + |u|^{p-1}) = C\left(\frac{\delta^{p-1}}{\delta^{p-1}} + |u|^{p-1}\right) \leq C\left(\frac{|u|^{p-1}}{\delta^{p-1}} + |u|^{p-1}\right) \\ &= |u|^{p-1}\left(\frac{C}{\delta^{p-1}} + C\right) = C_\varepsilon |u|^{p-1}. \end{aligned} \tag{3.30}$$

Hence, from (3.29) and (3.30) we get that for every $\varepsilon > 0$ there exists $C_\varepsilon > 0$ such that

$$f(u) \leq \varepsilon |u| + C_\varepsilon |u|^{p-1}.$$

Following the same reasoning, we have that for every $\varepsilon > 0$ there exists $C_{G,\varepsilon} > 0$ such that

$$g(u) \leq \varepsilon |u| + C_{G,\varepsilon} |u|^{q-1}.$$

□

Remark 3.11. *We want to observe that, by a simple integration, we obtain the following estimates for the primitive of f and g:*

$$F(u) \leq \varepsilon |u|^2 + C_\varepsilon |u|^p \tag{3.31}$$

and

$$G(u) \leq \varepsilon |u|^2 + C_{G,\varepsilon} |u|^q. \tag{3.32}$$

Proposition 3.12. *Suppose (F3),(F4) and (G3) hold. Then*

$$0 \leq qF(u) \leq f(u)u; \tag{3.33}$$

$$0 \leq g(u)u \leq qG(u). \tag{3.34}$$

Proof. Let $u \geq s > 0$, then by (F4) we have

$$\frac{f(s)}{|s|^{q-1}} \leq \frac{f(u)}{|u|^{q-1}},$$

that is

$$\int_0^u f(s)\, ds \leq \frac{f(u)}{|u|^{q-1}} \int_0^u |s|^{q-1}\, ds. \tag{3.35}$$

We compute

$$\int_0^u |s|^{q-1}\, ds = |u|^{q-1}u - (q-1)\int_0^u |s|^{q-1}\, ds,$$

therefore

$$\int_0^u |s|^{q-1}\, ds = \frac{|u|^{q-1}u}{q}.$$

Putting this last result in (3.35) we finally have

$$0 \leq F(u) = \int_0^u f(s)\, ds \leq \frac{f(u)}{|u|^{q-1}} \frac{|u|^{q-1}u}{q} = \frac{1}{q} f(u)u.$$

With a similar reasoning, we obtain the inequalities for g. □

Proposition 3.13. *For every $\varepsilon > 0$ there exists $C_{F,\varepsilon} > 0$ such that*

$$F(u) \geq C_{F,\varepsilon}|u|^q - \varepsilon|u|^2, \tag{3.36}$$

for any $u \in \mathbb{R}$, with $q \in (2, p)$ given in (F3).

Proof. Fix $\varepsilon > 0$. Hypothesis (F3) implies that there exists $\zeta_0 > 0$ with $F(\zeta_0) > 0$. As long as F is even, we know that $F(\zeta_0) = -F(\zeta_0)$ and we may set $c_1 := F(\zeta_0) > 0$. Hence, we will show the inequality for $u > 0$, getting the other one from the evenness of F.

Let $u \geq \zeta_0$, then

$$\frac{f(u)}{|u|^{q-1}} \geq \frac{f(\zeta_0)}{\zeta_0^{q-1}} = \frac{f(\zeta_0)\zeta_0}{\zeta_0^q} = \frac{qc_1}{\zeta_0^q}.$$

Hence, $f(u) \geq qc_1\zeta_0^{-q}u^{q-1}$ and by integration

$$F(u) \geq c_1\frac{1}{\zeta_0^q}u^q \geq C_1 u^q - \varepsilon u^2, \tag{3.37}$$

where $C_1 := c_1\zeta_0^{-q} > 0$.

Now, let $0 < u < \zeta_0$, then it follows that $\frac{u}{\zeta_0} < 1$ and, since $q > 2$,

$$\left(\frac{u}{\zeta_0}\right)^q < \left(\frac{u}{\zeta_0}\right)^2.$$

Therefore,

$$F(u) \geq 0 \geq \varepsilon\zeta_0^2\left(\frac{u}{\zeta_0}\right)^q - \varepsilon\zeta_0^2\left(\frac{u}{\zeta_0}\right)^2 = \varepsilon\zeta_0^{2-q} - \varepsilon u^2. \tag{3.38}$$

Putting together (3.37) and (3.38) we get

$$F(u) \geq \max\left\{C_1, \varepsilon\zeta_0^{2-q}\right\} u^q \varepsilon u^2 = C_\varepsilon u^q - \varepsilon u^2,$$

where $C_\varepsilon := \max\left\{C_1, \varepsilon\zeta_0^{2-q}\right\} > 0$. □

Remark 3.14. *Without loss of generality, we may assume that $C_{G,\varepsilon} \geq C_{F,\varepsilon}$. Indeed, suppose that $C_{G,\varepsilon} < C_{F,\varepsilon}$. Then, from (3.32), we can simply write that*

$$G(u) \leq \varepsilon|u|^2 + C_{G,\varepsilon}|u|^q \leq \varepsilon|u|^2 + C_{F,\varepsilon}|u|^q$$

and then constants C_ε are the same. Hence, we may always assume that $C_{G,\varepsilon} \geq C_{F,\varepsilon}$.

We recall that the energy functional associated to (3.22) (introduced in Subsection 3.5.1) is $\mathcal{J} : X \to \mathbb{R}$ defined as

$$\mathcal{J}(u) := \frac{1}{2}\|u^+\|^2 - \frac{1}{2}\|u^-\|^2. \int_{\mathbb{R}^N} F(u)\,dx + \lambda\int_{\mathbb{R}^N} G(u)\,dx \tag{3.39}$$

and we can prove the following regularity result.

Proposition 3.15. *If (F1) and (G1) hold, then $\mathcal{J}$ is of class C^1 on X.*

Proof. Let $u \in X$, from (3.31) and (3.32) we get

$$\begin{aligned}|\mathcal{J}(u)| &\leq \frac{1}{2}\|u^+\|^2 + \frac{1}{2}\|u^-\|^2 + \int_{\mathbb{R}^N} F(u)\,dx + \lambda\int_{\mathbb{R}^N} G(u)\,dx \\ &\leq \frac{1}{2}\|u^+\|^2 + \frac{1}{2}\|u^-\|^2 + \int_{\mathbb{R}^N}\left(\varepsilon|u|^2 + C_\varepsilon|u|^p\right)dx + \lambda\int_{\mathbb{R}^N}\left(\varepsilon|u|^2 + C_{G,\varepsilon}|u|^p\right)dx\end{aligned}$$

$$
\begin{aligned}
&\leq \frac{1}{2}\|u^+\|^2 + \frac{1}{2}\|u^-\|^2 + \varepsilon\|u\|^2_{L^2(\mathbb{R}^N)} + C_\varepsilon\|u\|^p_{L^p(\mathbb{R}^N)} \\
&\quad + \lambda\varepsilon\|u\|^2_{L^2(\mathbb{R}^N)} + \lambda C_{G,\varepsilon}\|u\|^q_{L^q(\mathbb{R}^N)} < +\infty.
\end{aligned}
$$

Now, let $(u_n)_n \subset X$ be a sequence such that $u_n \to u$ in X, then

$$
\begin{aligned}
|\mathcal{J}(u_n) - \mathcal{J}(u)| &= \Bigg|\frac{1}{2}\|u_n^+\|^2 - \frac{1}{2}\|u_n^-\|^2 - \int_{\mathbb{R}^N} F(u_n)\,dx + \lambda\int_{\mathbb{R}^N} G(u_n)\,dx \\
&\quad -\frac{1}{2}\|u^+\|^2 + \frac{1}{2}\|u^-\|^2 + \int_{\mathbb{R}^N} F(u)\,dx - \lambda\int_{\mathbb{R}^N} G(u)\,dx\Bigg| \\
&= \Bigg|\frac{1}{2}\|u_n^+ - u^+\|^2 - \frac{1}{2}\|u_n^- - u^-\|^2 - \int_{\mathbb{R}^N} \left(F(u_n) - F(u)\right)\,dx \\
&\quad +\lambda\int_{\mathbb{R}^N} \left(G(u_n) - G(u)\right)\,dx\Bigg|
\end{aligned}
$$

and this goes to 0 as $n \to +\infty$ (see Thorem A.4 in [139]). The first Gâteaux derivative of $\mathcal{J}$ at $u \in X$ along $v \in X$ is

$$
\mathcal{J}'(u)(v) = (u^+, v) - (u^-, v) - \int_{\mathbb{R}^N} f(u)v\,dx + \lambda\int_{\mathbb{R}^N} g(u)v\,dx.
$$

Hence, from (3.27) and (3.28) we get

$$
\begin{aligned}
|\mathcal{J}'(u)(v)| &\leq (u^+, v) + (u^-, v) + \int_{\mathbb{R}^N} f(u)v\,dx + \lambda\int_{\mathbb{R}^N} g(u)v\,dx \\
&\leq (u^+, v) + (u^-, v) + \int_{\mathbb{R}^N} \left(\varepsilon|u| + C_\varepsilon|u|^{p-1}\right)v\,dx + \lambda\int_{\mathbb{R}^N} \left(\varepsilon|u| + C_{G,\varepsilon}|u|^{q-1}\right)v\,dx \\
&\leq \int_{\mathbb{R}^N} \left(\varepsilon|u| + C_\varepsilon|u|^{p-1}\right)v\,dx + \varepsilon\|u\|^2_{L^2(\mathbb{R}^N)}\|v\|^2_{L^2(\mathbb{R}^N)} + C_\varepsilon\|u\|^p_{L^p(\mathbb{R}^N)}\|v\|^p_{L^p(\mathbb{R}^N)} \\
&\quad + \varepsilon\|u\|^2_{L^2(\mathbb{R}^N)}\|v\|^2_{L^2(\mathbb{R}^N)} + C_{G,\varepsilon}\|u\|^q_{L^q(\mathbb{R}^N)}\|v\|^q_{L^q(\mathbb{R}^N)} < +\infty,
\end{aligned}
$$

where we also used Hölder inequality many times.

Again, let $(u_n)_n \subset X$ be a sequence such that $u_n \to u$ in X, then

$$
\begin{aligned}
&|\mathcal{J}'(u_n)(v) - \mathcal{J}'(u)(v)| \\
&\quad = \Bigg|(u_n^+, v) - (u_n^-, v) - \int_{\mathbb{R}^N} f(u_n)v\,dx + \lambda\int_{\mathbb{R}^N} g(u_n)v\,dx \\
&\qquad -(u^+, v) + (u^-, v) + \int_{\mathbb{R}^N} f(u)v\,dx - \lambda\int_{\mathbb{R}^N} g(u)v\,dx\Bigg| \\
&\quad = \Bigg|(u_n^+ - u^+, v) - (u_n^- - u^-, v) \\
&\qquad - \int_{\mathbb{R}^N} \left(f(u_n) - f(u)\right)v\,dx + \lambda\int_{\mathbb{R}^N} \left(g(u_n) - g(u)\right)v\,dx\Bigg|
\end{aligned}
$$

and this goes to 0 as $n \to +\infty$ (again by Theorem A.4 in [139]). □

Moreover, we can prove also the following property for $\mathcal{J}$.

Proposition 3.16. *The functional $\mathcal{J}'$ is weak-to-weak* continuous in X.*

Proof. Let $(u_n)_n \subset X$ be a bounded sequence such that $u_n \rightharpoonup u_0$, with $u_0 \in X$. From the boundedness we get that the sequence $(u_n)_n$ is also bounded in $L^2(\mathbb{R}^N) \cap L^{2^*}(\mathbb{R}^N)$, while from the weak convergenge we have that $u_n \to u_0$ in $L^p(\mathbb{R}^N)$, with $p \in (2, 2^*)$ and $u_n(x) \to u_0(x)$ for almost every $x \in \mathbb{R}^N$. Therefore,

$$
(u_n^+, \varphi) \to (u_0^+, \varphi) \quad \text{and} \quad (u_n^-, \varphi) \to (u_0^-, \varphi). \tag{3.40}
$$

We know that $f(u_n(x)) \to f(u_0(x))$ for almost every $x \in \mathbb{R}^N$. For every measurable set $F \subset \operatorname{supp}\varphi$, using (F1),

$$\left|\int_F f(u_n)\varphi\,dx\right| \le \int_F \left(C(1+|u_n|^{p-1})|\varphi|\right)\,dx \le C\|\varphi\|_{L^p(\mathbb{R}^N)}\mu(F)^{p'} + C\|u_n\|_{L^p(\mathbb{R}^n)}^{p-1}\|\varphi\chi_F\|_{L^p(\mathbb{R}^N)},$$

for every $\varphi \in X$. Hence, by Vitali's Convergence Theorem (Appendix B, Theorem B.5),

$$\int_{\mathbb{R}^N} f(u_n)\,dx \to \int_{\mathbb{R}^N} f(u_0)\,dx. \tag{3.41}$$

Similarly, using *(G1)*, we obtain that

$$\int_{\mathbb{R}^N} g(u_n)\varphi\,dx \to \int_{\mathbb{R}^N} g(u_0)\varphi\,dx, \tag{3.42}$$

for every $\varphi \in X$.

From (3.40), (3.41) and (3.42) we conclude that

$$\mathcal{J}'(u_n)(\varphi) \to \mathcal{J}'(u_0)(vp)$$

for every $\varphi \in X$. □

3.4.2 A Cerami-sequence bounded away from zero...

We show that the energy functional (3.39) satisfies assumptions **(A1)**-**(A3)** introduced in Subsection 3.1.3, proving then the existence of a Cerami-sequence bounded away from zero. Later on, we will prove that such a Cerami-sequence is bounded also from above (see Lemma 3.19 below).

Lemma 3.17. *The energy functional* (3.39) *satisfies **(A1)**-**(A3)** for* $\mathcal{P} := X^+ \setminus \{0\}$.

Proof. By definition, $\mathcal{J}(0) = 0$ and by Proposition 3.15 we have the desired regularity, so **(A1)** holds. Moreover, by Proposition 3.16, we have **(A2)**. Hence, it only remains to show **(A3)**: we divide the proof in three steps.

Step 1: *there is* $r > 0$ *such that* $\inf_{\mathcal{S}_r^+} \mathcal{J} > 0$. Fix $u^+ \in X^+$, from (3.33) we have

$$\begin{aligned}\mathcal{J}(u) &\ge \frac{1}{2}\|u^+\|^2 - \int_{\mathbb{R}^N} F(u)\,dx \ge \frac{1}{2}\|u^+\|^2 - \int_{\mathbb{R}^N}\left(\varepsilon|u|^2 + C_\varepsilon|u|^p\right)\,dx\\ &\ge \frac{1}{2}\|u^+\|^2 - \varepsilon\|u\|_{L^2(\mathbb{R}^N)}^2 - C_\varepsilon\|u\|_{L^p(\mathbb{R}^N)}^p\end{aligned}$$

and by Sobolev embedding there exists $C > 0$ such that

$$\mathcal{J}(u) \ge \frac{1}{2}\|u^+\|^2 - \varepsilon C\|u\|^2 - CC_\varepsilon\|u\|^p.$$

Recalling that $\|u^+\|^2 = \|u^+\|^2 + \|u^-\|^2$, there follows that

$$\|u\|^p = \|u\|^{p-2}\|u\|^2 = \|u\|^{p-2}\|u^+\|,$$

where we used the fact that we are in $\mathcal{S}_r^+$. Then, calling $\widetilde{C}_\varepsilon := CC_\varepsilon$,

$$\begin{aligned}\mathcal{J}(u) &\ge \frac{1}{2}\|u^+\|^2 - \varepsilon C\|u^+\|^2 - \widetilde{C}_\varepsilon\|u\|^{p-2}\|u^+\|^2 = \|u^+\|^2\left(\frac{1}{2} - \varepsilon C - \widetilde{C}_\varepsilon\|u\|^{p-2}\right)\\ &= \|u^+\|^2\left(\frac{1}{2} - \varepsilon C - \widetilde{C}_\varepsilon r^{p-2}\right).\end{aligned}$$

Choosing $\varepsilon > 0$ small (that it will be fixed precisely below) and $0 < r < \left(\frac{\frac{1}{2}-\varepsilon C}{\widetilde{C}_\varepsilon}\right)^{2-p}$ we finally obtain

$$\inf_{\mathcal{S}_r^+} \mathcal{J} \geq \frac{r^2}{4} > 0.$$

For Step 2 and Step 3 we need to assume a priori that

$$\lambda < \frac{1}{\kappa 2^q} \frac{C_{F,\mu_0/8}}{C_{G,\mu_0/8}}, \tag{3.43}$$

where $C_{F,\mu_0/8}$ is given in (3.36) and $C_{G,\mu_0/8}$ is given in (3.32). We observe that by Remark 3.14 it follows that $\lambda \leq 1$.

Step 2: *for $u \in \mathcal{P}$ there is a radius $R(u) > r$ such that* $\sup_{\partial M(u)} \mathcal{J} \leq 0$. Fix $u \in \mathcal{P}$ and take $u_n \in \mathbb{R}^+ u \oplus X^-$, hence u_n is of the form $u_n = t_n u + u_n^-$, for some $t_n \geq 0$, $u_n^- \in X^-$. We will show that $\sup_{\partial M(u)} \mathcal{J} \leq 0$ for sufficiently large $R(u)$. By (3.26), (3.28) and (3.36) we have

$$\begin{aligned}
\mathcal{J}(u_n) = \mathcal{J}(t_n u + u_n^-) &= \frac{1}{2}t_n^2 - \frac{1}{2}\|u_n^-\|^2 - \int_{\mathbb{R}^N} F(u_n)\,dx + \lambda \int_{\mathbb{R}^N} G(u_n)\,dx \\
&\leq \frac{1}{2}t_n^2 - \frac{1}{2}\|u_n^-\|^2 - C_{F,\varepsilon}\|u_n\|^q_{L^q(\mathbb{R}^N)} + \varepsilon\|u_n\|^2_{L^2(\mathbb{R}^N)} + \lambda\varepsilon\|u_n\|^2_{L^2(\mathbb{R}^N)} + \lambda C_{G,\varepsilon}\|u_n\|^q_{L^q(\mathbb{R}^N)} \\
&= \frac{1}{2}t_n^2 - \frac{1}{2}\|u_n^-\|^2 - C_{F,\varepsilon}\|t_n u + u_n^-\|^q_{L^q(\mathbb{R}^N)} + \varepsilon\|t_n u + u_n^-\|^2_{L^2(\mathbb{R}^N)} \\
&\quad + \lambda\varepsilon\|t_n u + u_n^-\|^2_{L^2(\mathbb{R}^N)} + \lambda C_{G,\varepsilon}\|t_n u + u_n^-\|^q_{L^q(\mathbb{R}^N)} \\
&\leq \left(-\frac{1}{2} + \frac{\varepsilon + \lambda\varepsilon}{\mu_0}\right)\|u_n^-\|^2 + \left(\frac{1}{2} + \frac{\varepsilon + \lambda\varepsilon}{\mu_0}\right)t_n^2 \\
&\quad - C_{F,\varepsilon}\|t_n u + u_n^-\|^q_{L^q(\mathbb{R}^N)} + \lambda C_{G,\varepsilon}\|t_n u + u_n^-\|^q_{L^q(\mathbb{R}^N)} \\
&\leq \left(-\frac{1}{2} + \frac{2\varepsilon}{\mu_0}\right)\|u_n^-\|^2 + \left(\frac{1}{2} + \frac{2\varepsilon}{\mu_0}\right)t_n^2 - C_{F,\varepsilon}\|t_n u + u_n^-\|^q_{L^q(\mathbb{R}^N)} + \lambda C_{G,\varepsilon}\|t_n u + u_n^-\|^q_{L^q(\mathbb{R}^N)}.
\end{aligned}$$

Now, we choose $\varepsilon := \frac{\mu_0}{8}$. By (3.25) it follows that

$$\|t_n u\|^q_{L^q(\mathbb{R}^N)} \leq \kappa\|t_n u + u_n^-\|^q_{L^q(\mathbb{R}^N)}$$

and

$$\|u_n^-\|^q_{L^q(\mathbb{R}^N)} \leq \kappa\|t_n u + u_n^-\|^q_{L^q(\mathbb{R}^N)}$$

therefore

$$\|t_n u\|^q_{L^q(\mathbb{R}^N)} + \|u_n^-\|^q_{L^q(\mathbb{R}^N)} \leq 2\kappa\|t_n u + u_n^-\|^q_{L^q(\mathbb{R}^N)}.$$

Moreover, using the inequality $(a+b)^p \leq 2^{p-1}(a^p + b^+)$, we have

$$\begin{aligned}
\mathcal{J}(u_n) &\leq -\frac{1}{4}\|u_n^-\|^2 + \frac{3}{4}t_n^2 - C_{F,\mu_0/8}\|t_n u + u_n^-\|^q_{L^q(\mathbb{R}^N)} + \lambda C_{G,\mu_0/8}\|t_n u + u_n^-\|^q_{L^q(\mathbb{R}^N)} \\
&\leq -\frac{1}{4}\|u_n^-\|^2 + \frac{3}{4}t_n^2 - C_{F,\mu_0/8}\|t_n u + u_n^-\|^q_{L^q(\mathbb{R}^N)} + \lambda C_{G,\mu_0/8}\|t_n u + u_n^-\|^q_{L^q(\mathbb{R}^N)} \\
&\leq -\frac{1}{4}\|u_n^-\|^2 + \frac{3}{4}t_n^2 - \frac{C_{F,\mu_0/8}}{2\kappa}t_n^q\|u\|^q_{L^q(\mathbb{R}^N)} - \frac{C_{F,\mu_0/8}}{2\kappa}\|u_n^-\|^q_{L^q(\mathbb{R}^N)} \\
&\quad + 2^{q-1}\lambda C_{G,\mu_0/8}t_n^q\|u\|^q_{L^q(\mathbb{R}^N)} + 2^{q-1}\lambda C_{G,\mu_0/8}\|u_n^-\|^q_{L^q(\mathbb{R}^N)} \\
&= -\frac{1}{4}\|u_n^-\|^2 + \frac{3}{4}t_n^2 + \left(2^{q-1}\lambda C_{G,\mu_0/8} - \frac{C_{F,\mu_0/8}}{2\kappa}\right)t_n^q\|u\|^q_{L^q(\mathbb{R}^N)} \\
&\quad + \left(2^{q-1}\lambda C_{G,\mu_0/8} - \frac{C_{F,\mu_0/8}}{2\kappa}\right)\|u_n^-\|^q_{L^q(\mathbb{R}^N)}.
\end{aligned}$$

Using (3.43), we obtain

$$\mathcal{J}(u_n) \le -\frac{1}{4}\|u_n^-\|^2 + \frac{3}{4}t_n^2 + \left(2^{q-1}\lambda C_{G,\mu_0/8} - \frac{C_{F,\mu_0/8}}{2\kappa}\right)\|u_n^-\|^q_{L^q(\mathbb{R}^N)},$$

so $\mathcal{J}(u_n) \to -\infty$ as $\|t_n u + u_n^-\| \to -\infty$.

Step 3: there is $\delta > 0$ such that $\sup_{|||u||| \le \delta} \mathcal{J}(u) < \inf_{\mathcal{S}_r^+} \mathcal{J}$. As in Step 2, by (3.26), (3.28) and (3.36) we have

$$\begin{aligned}\mathcal{J}(u) &\le \frac{1}{2}\|u^+\|^2 - \frac{1}{2}\|u^-\|^2 - C_{F,\varepsilon}\|u\|^q_{L^q(\mathbb{R}^N)} + \varepsilon\|u\|^2_{L^2(\mathbb{R}^N)} + \lambda\varepsilon\|u\|^2_{L^2(\mathbb{R}^N)} + \lambda C_{G,\varepsilon}\|u\|^q_{L^q(\mathbb{R}^N)} \\ &\le \left(\frac{1}{2} + \frac{\varepsilon + \lambda\varepsilon}{\mu_0}\right)\|u^+\|^2 - \left(\frac{1}{2} + \frac{\varepsilon - \lambda\varepsilon}{\mu_0}\right)\|u^-\|^2 - C_{F,\varepsilon}\|u\|^q_{L^q(\mathbb{R}^N)} + \lambda C_{G,\varepsilon}\|u\|^q_{L^q(\mathbb{R}^N)}.\end{aligned}$$

We observe that (3.43) implies that

$$\lambda < \frac{C_{F,\mu_0/8}}{C_{G,\mu_0/8}} \le 1,$$

hence, for $\varepsilon = \frac{\mu_0}{8}$

$$\mathcal{J}(u) \le \left(\frac{1}{2} + \frac{2}{\mu_0}\right)\|u^+\|^2 - \left(\frac{1}{2} + \frac{2}{\mu_0}\right)\|u^-\|^2 \le \frac{3}{4}\|u^+\|^2 \le \frac{3}{4}\left|\left|\left|u^+\right|\right|\right|^2$$

that goes to zero as $|||u||| \to 0$. This conludes the proof of Step 3 and **(A3)** is proved. $\square$

Therefore, we can conclude that there exists a Cerami-sequence $(u_n)_n \subset X$ at level $c \in \mathbb{R}$ (defined as in Theorem 3.8) bounded away from zero.

Remark 3.18. *If $\lambda = 0$ we can also prove assumption **(A4)**: indeed, it naturally follows from the inequality*

$$\mathcal{J}(u) \ge \mathcal{J}(tu + v) - \mathcal{J}'(u)\left(\frac{t^2 - 1}{2}u + tv\right),$$

for $u \in X$, $v \in X^-$, $t \ge 0$, obtaining an additional estimates of c in terms of the Nehari-Pankov manifold. The previous inequality is impossibile to obtain if $\lambda > 0$ withouth further assumptions.

3.4.3 ... and also from above

Here, we show the boundedness of the Cerami-sequence obtained above. The proof is technical and requires that $\lambda > 0$ and $\rho > 0$ are small enough. Usually the boundedness of a Cerami sequence may be obtained by application of the appropriate Lion's concentration-compactness principle, cf. Lemma 4.7 in [97] and Lemma 5.1 in [32]. This method is unavailable in our case as long as we don't know the sign of the Cerami-sequence level nor the sign of the nonlinear term.

Lemma 3.19. *Suppose that $\lambda > 0$ and $\rho > 0$ in (F5) is sufficiently small. Let $(u_n)_n \subset X$ be a sequence such that*

$$\mathcal{J}(u_n) \le \beta, \quad (1 + \|u_n\|)\mathcal{J}'(u_n) \to 0$$

for some $\beta \in \mathbb{R}$. Then $(u_n)_n$ is bounded in X. In particular, any Cerami-sequence for $\mathcal{J}$ is bounded.

Proof. We argue by contradiction, so we suppose that $\|u_n\| \to +\infty$. We note that, since $(1+\|u_n\|)\mathcal{J}'(u_n)$, then

$$\begin{aligned}\|u_n\|^2 = \|u_n^+\|^2 + \|u_n^-\|^2 &= \int_{\mathbb{R}^N} (f(u_n) - \lambda g(u_n))\, u_n\, dx + o(1)\\ &= \int_{\mathbb{R}^N} (f(u_n) - \lambda g(u_n))\, (u_n^+ - u_n^-)\, dx + o(1)\end{aligned}$$

and

$$\begin{aligned}&\int_{\mathbb{R}^N} (f(u_n) - \lambda g(u_n))\, (u_n^+ - u_n^-)\, dx\\ &= \int_{|u_n|<\rho} (f(u_n) - \lambda g(u_n))\, (u_n^+ - u_n^-)\, dx + \int_{|u_n|\geq\rho} (f(u_n) - \lambda g(u_n))\, (u_n^+ - u_n^-)\, dx\\ &=: I_1 + I_2.\end{aligned}$$

We estimate separately the two integrals and we start with I_1. We fix $\varepsilon > 0$, from (3.27) and (3.28) there exists $C_\varepsilon > 0$ such that

$$\begin{aligned}I_1 = |I_1| &\leq \int_{|u_n|<\rho} |f(u_n) - \lambda g(u_n)|\, |u_n^+ + u_n^-|\, dx = \int_{|u_n|<\rho} |f(u_n) - \lambda g(u_n)|\, u_n\, dx\\ &\leq \int_{|u_n|<\rho} \left(\varepsilon|u_n|^2 + C_\varepsilon|u_n|^p\right)\, dx + \lambda \int_{|u_n|<\rho} \left(\varepsilon|u_n|^2 + C_\varepsilon|u_n|^q\right)\, dx\\ &= \varepsilon(1+\lambda) \int_{|u_n|<\rho} |u_n|^2\, dx + C_\varepsilon \int_{|u_n|<\rho} |u_n|^p\, dx + C_\varepsilon\lambda \int_{|u_n|<\rho} |u_n|^q\, dx\\ &\leq \varepsilon(1+\lambda) \int_{|u_n|<\rho} |u_n|^2\, dx + C_\varepsilon\rho^{p-2} \int_{|u_n|<\rho} |u_n|^2\, dx + C_\varepsilon\lambda\rho^{q-2} \int_{|u_n|<\rho} |u_n|^2\, dx\end{aligned}$$

and from (3.26) we obtain

$$I_1 \leq \frac{1}{\mu_0}\left(\varepsilon(1+\lambda) + C_\varepsilon\rho^2 + \lambda C_\varepsilon\rho^{q-2}\right)\|u_n\|^2. \tag{3.44}$$

Now, we take care of I_2: from (F1), (F4), (F5), (G1), (G3) it follows that the maps

$$\{|u| \geq \rho\} \ni u \mapsto \frac{g(u)}{f(u)} \in \mathbb{R}$$

is well-defined, nondecreasing, nonnegative and even.

Hence,

$$\frac{g(\rho)}{f(\rho)} \geq \left|\frac{g(u)}{f(u)}\right|, \quad |u| \geq \rho$$

and by (F5) and (3.25)

$$\begin{aligned}I_2 &\leq \int_{|u_n|\geq\rho} |f(u_n) - \lambda g(u_n)| \left(|u_n^+| + |u_n^-|\right)\, dx\\ &= \int_{|u_n|\geq\rho} |f(u_n)| \left|1 - \lambda\frac{g(u_n)}{f(u_n)}\right| \left(|u_n^+| + |u_n^-|\right)\, dx\\ &\leq \left(1 + \lambda\frac{g(\rho)}{f(\rho)}\right) \int_{|u_n|\geq\rho} |f(u_n)| \left(|u_n^+| + |u_n^-|\right)\, dx\\ &\leq C\left(1 + \lambda\frac{g(\rho)}{f(\rho)}\right) \int_{|u_n|\geq\rho} |u_n|^{p-1} \left(|u_n^+| + |u_n^-|\right)\, dx\\ &\leq C\left(1 + \lambda\frac{g(\rho)}{f(\rho)}\right) 2\kappa \int_{\mathbb{R}^N} |u_n|^p\, dx.\end{aligned}$$

We need to estimate the L^p-norm of u_n: to do that, we call

$$\Phi(u) := \frac{1}{2}f(u)u - F(u) + \lambda G(u) - \frac{\lambda}{2}g(u)u$$

and we observe that

$$\beta + o(1) \geq \mathcal{J}(u_n) - \frac{1}{2}\mathcal{J}'(u_n)(u_n) = \int_{\mathbb{R}^N} \Phi(u_n)\, dx.$$

Thanks to (F5), (3.33), (3.34) and choosing λ small enough such that

$$1 - \lambda\frac{g(\rho)}{f(\rho)} > 0,$$

we can compute

$$\begin{aligned}
&\beta + o(1) + \int_{|u_n|<\rho} |\Phi(u)|\, dx \\
&\geq \beta + o(1) - \int_{|u_n|<\rho} \Phi(u)\, dx \\
&= \beta + o(1) - \int_{\mathbb{R}^N} \Phi(u_n)\, dx + \int_{|u_n|\geq\rho} \Phi(u_n)\, dx \\
&\geq \int_{|u_n|\geq\rho} \Phi(u_n)\, dx = \int_{|u_n|\geq\rho} \left[\frac{1}{2}f(u_n)u_n - F(u_n) + \lambda G(u_n) - \frac{1}{2}g(u_n)u_n\right] dx \\
&\geq \left(\frac{1}{2} - \frac{1}{q}\right) \int_{|u_n|\geq\rho} \left(f(u_n)u_n - \lambda g(u_n)u_n\right)\, dx \\
&= \left(\frac{1}{2} - \frac{1}{q}\right) \int_{|u_n|\geq\rho} \left(1 - \lambda\frac{g(u_n)}{f(u_n)}\right) f(u_n)u_n\, dx \\
&\geq \left(\frac{1}{2} - \frac{1}{q}\right) \left(1 - \lambda\frac{g(\rho)}{f(\rho)}\right) \int_{|u_n|\geq\rho} f(u_n)u_n\, dx \\
&\geq C\left(\frac{1}{2} - \frac{1}{q}\right) \left(1 - \lambda\frac{g(\rho)}{f(\rho)}\right) \int_{|u_n|\geq\rho} |u_n|^p\, dx.
\end{aligned}$$

Therefore,

$$\int_{|u_n|\geq\rho} |u_n|^p\, dx \leq \widetilde{C}\left(1 - \lambda\frac{g(\rho)}{f(\rho)}\right)^{-1} \left(\beta + \int_{|u_n|<\rho} |\Phi(u_n)|\, dx\right) + o(1), \tag{3.45}$$

where $\widetilde{C} > 0$ is independent of n, λ and p.

Hence, setting $D(\lambda, \rho, \varepsilon) := C\left(1 + \lambda\frac{g(\rho)}{f(\rho)}\right) 2\kappa,$. by (3.45) we have

$$\begin{aligned}
I_2 &\leq D(\lambda, \rho, \varepsilon) \left(\int_{|u_n|<\rho} |u_n|^p\, dx + \int_{|u_n|\geq\rho} |u_n|^p\, dx\right) \\
&\leq D(\lambda, \rho, \varepsilon) \left(\int_{|u_n|<\rho} |u_n|^p\, dx + \widetilde{C}\left(1 - \lambda\frac{g(\rho)}{f(\rho)}\right)^{-1} \left(\beta + \int_{|u_n|<\rho} |\Phi(u_n)|\, dx\right)\right) + o(1) \\
&\leq D(\lambda, \rho, \varepsilon) \left(\rho^{p-2}\|u_n\|^2_{L^2(\mathbb{R}^N)} + \widetilde{C}\left(1 - \lambda\frac{g(\rho)}{f(\rho)}\right)^{-1} \left(\beta + \sup_{|t|\leq\rho}\frac{|\Phi(t)|}{t^2}\|u_n\|^2_{L^2(\mathbb{R}^N)}\right)\right) + o(1) \\
&\leq D(\lambda, \rho, \varepsilon) \left(\frac{\rho^{p-2}}{\mu_0}\|u_n\|^2 + \frac{\widetilde{C}\beta}{\left(1 - \lambda\frac{g(\rho)}{f(\rho)}\right)} + \frac{\widetilde{C}}{\left(1 - \lambda\frac{g(\rho)}{f(\rho)}\right)\mu_0} \sup_{|t|\leq\rho}\frac{|\Phi(t)|}{t^2}\|u_n\|^2\right) + o(1) \\
&\leq D(\lambda, \rho, \varepsilon) \left(\frac{\rho^{p-2}}{\mu_0} + \frac{\widetilde{C}}{\left(1 - \lambda\frac{g(\rho)}{f(\rho)}\right)} \sup_{|t|\leq\rho}\frac{|\Phi(t)|}{t^2}\right) \|u_n\|^2 + \bar{C},
\end{aligned}$$

where $\bar{C} := \bar{C}(\lambda, \rho, \varepsilon) > 0$.

Therefore,

$$\|u_n\|^2 = I_1 + I_2 + o(1) \leq \frac{K}{\mu_0}\|u_n\|^2 + \bar{C}$$

where

$$\begin{aligned} K :&= \varepsilon(1+\lambda) + C_\varepsilon \rho^{p-2} + \lambda C_\varepsilon \rho^{q-2} + D(\lambda,\rho,\varepsilon)\left(\rho^{p-2} + \frac{\widetilde{C}}{1-\lambda\frac{g(\rho)}{f(\rho)}} \sup_{|t|\leq\rho}\frac{|\Phi(t)|}{t^2}\right) \\ &= \varepsilon(1+\lambda) + C_\varepsilon \rho^{p-2} + \lambda C_\varepsilon \rho^{q-2} + \widetilde{C}2\kappa\rho^{p-2} + \widetilde{C}2\kappa\lambda\frac{g(\rho)}{f(\rho)}\rho^{p-2} + C\frac{1+\lambda\frac{g(\rho)}{f(\rho)}}{1-\lambda\frac{g(\rho)}{f(\rho)}}\sup_{|t|\leq\rho}\frac{|\Phi(t)|}{t^2}. \end{aligned}$$

So, to finish the proof is enough to show that $K < \mu_0$. We observe that

$$\lim_{t\to 0}\frac{|\Phi(t)|}{t^2} = 0,$$

hence $\sup_{|t|\leq\rho}\frac{|\Phi(t)|}{t^2}$ can be arbitrarily small for $\rho > 0$ small. Moreover, we already saw that $\lambda \leq 1$, therefore

$$K \leq 2\varepsilon + C_\varepsilon \rho^{p-2} + C_\varepsilon \rho^{q-2} + \widetilde{C}2\kappa\rho^{p-2} + \widetilde{C}2\kappa\lambda\frac{g(\rho)}{f(\rho)}\rho^{p-2} + \widetilde{C}\frac{1+\lambda\frac{g(\rho)}{f(\rho)}}{1-\lambda\frac{g(\rho)}{f(\rho)}}\sup_{|t|\leq\rho}\frac{|\Phi(t)|}{t^2}.$$

We fix $\varepsilon < \frac{\mu_0}{12}$ and we choose $\rho > 0$ small enough such that

$$C_\varepsilon \rho^{p-2} + C_\varepsilon \rho^{q-2} + \widetilde{C}2\kappa\rho^{p-2} + 2\widetilde{C}\sup_{|t|\leq\rho}\frac{|\Phi(t)|}{t^2} < \frac{2\mu_0}{3}.$$

Now, choosing $\lambda > 0$ small such that

$$\widetilde{C}2\kappa\lambda\frac{g(\rho)}{f(\rho)}\rho^{p-2}$$

and

$$0 \leq \frac{1+\lambda\frac{g(\rho)}{f(\rho)}}{1-\lambda\frac{g(\rho)}{f(\rho)}} \leq 2$$

provides $K < \mu_0$ and concludes the proof. □

3.4.4 The Existence result for the singular Schrödinger equation

We are almost ready to state and proof the main Theorem of this Chapter, but we still need a couple of Lemmas before: in particular, we need a concentration-compactness principle result in the spirit of Lions (see [91]). The proofs of the next two Lemmas can be found in [31], Corollary 7.1 and Lemma 7.2 (see also [96], Corollary 3.2 and Remark 3.3).

Lemma 3.20. *Suppose that $(u_n)_n \subset X$ is bounded and for all $R > 0$ the following vanishing conditions*

$$\lim_{n\to+\infty}\sup_{z\in\mathbb{R}^{N-K}}\int_{B(0,z),R)}|u_n|^2\,dx = 0 \tag{3.46}$$

holds. Then

$$\int_{\mathbb{R}^N}|\Psi(u_n)|\,dx \to 0 \text{ as } n\to+\infty$$

for any continuous function $\Psi : \mathbb{R}\to\mathbb{R}$ satisfying

$$\lim_{s\to 0}\frac{\Psi(s)}{s^2} = \lim_{|s|\to+\infty}\frac{\Psi(s)}{s^{2*}} = 0.$$

A direct consequence of Lemma 3.20 is the following.

Proposition 3.21. *Suppose that a bounded sequence* $(u_n)_n \subset X$ *satisfies* (3.46) *for every* $R > 0$. *Then,*

$$\int_{\mathbb{R}^N} \widetilde{f}(u_n)u_n^{\pm}\, dx \to 0,$$

where $\widetilde{f}$ *is defined in* (3.48).

Now, we are ready to state the main Theorem that provides the existence of a solution for the singular Schrödinger equation (3.22).

Theorem 3.22. *Suppose that (V), (F1)-(F5), (G1)-(G3) hold. If* $\lambda > 0$ *and* $\rho > 0$ *in (F5) are sufficiently small, then there is a nontrivial solution to* (3.22).

Proof. By Lemma 3.17 it follows that the energy functional $\mathcal{J}$ satisfies assumption **(A1)-(A3)** of Theorem 3.8. Hence, there exists a sequence $(u_n)_n \subset X$ satisfying (3.8), i.e. such that

$$\sup_n \mathcal{J} \le c, \quad (1 + \|u_n\|)\mathcal{J}'(u_n) \to 0 \text{ in } E^*, \quad \inf_n |||u_n||| \ge \frac{\delta}{2}.$$

By Proposition 3.19 the sequence $(u_n)_n$ is bounded, so, up to a subsequence, there exists $u_0 \in X$ such that $u_n \rightharpoonup u_0 \in X$.

Now, suppose that (3.46) holds for every $R > 0$: then, by Lemma 3.20 we get that $u_n \to 0$ in $L^t(\mathbb{R}^N)$ for every $t \in (2, 2*)$. We note that, from (3.8) and Proposition 3.21, we have

$$o(1) = \mathcal{J}'(u_n)(u_n^+) = \|u_n^+\|^2 - \int_{\mathbb{R}^N} \widetilde{f}(u_n)u_n^+\, dx = \|u_n^+\|^2 + o(1),$$

that is

$$\|u_n^+\| \to 0 \text{ as } n \to +\infty.$$

Reasoning in the same way, we also obtain that

$$\|u_n^-\| \to 0 \text{ as } n \to +\infty,$$

therefore, by (3.3) we obtain

$$|||u_n||| \le \|u_n\| \to 0 \text{ as } n \to +\infty,$$

but this is a contradiction, since $|||u_n||| \ge \frac{\delta}{2}$. Hence, there exists an $R > 0$ and a sequence $(z_n)_n \subset \mathbb{Z}^{N-K}$ such that

$$\liminf_{n\to+\infty} \int_{B(0,R)} |v_n|^2\, dx > 0,$$

where $v_n := u_n(\cdot, \cdot - z_n)$. We note that $\|v_n\| = \|u_n\|$, that is $(v_n)_n$ is also bounded and there exists $v_0 \neq 0$ such that $v_n \rightharpoonup v_0$.

Since $\mathcal{J}$ is $\{I_K\} \times \mathbb{Z}^{N-K}$−invariant, then the sequence $(v_n)_n$ satisfies $(1 + \|v_n\|)\mathcal{J}'(v_n) \to 0$: by Proposition 3.16, $\mathcal{J}'$ is weak-to-weak* continuous, hence $\mathcal{J}'(v_0) = 0$ and the proof is complete. □

3.5 Application to a curl-curl problem

3.5.1 Equivalence of solutions

Why the study of the singular Schrödiner equation (3.22) will helps us in finding a solution of the curl-curl problem (3.18)? The answer is given by the following Theorem (see [31], Theorem 1.1 and [70], Theorem 2.1). Before stating the result, we introduce the C^1 energy functional associated to (3.18), $\mathcal{E} : H^1(\mathbb{R}^3, \mathbb{R}^3) \to \mathbb{R}$ defined as

$$\mathcal{E}(\mathbf{E}) := \frac{1}{2}\int_{\mathbb{R}^3} \left(|\nabla \times \mathbf{E}|^2 + V(x)|\mathbf{E}|^2\right) dx - \int_{\mathbb{R}^3} H(\mathbf{E})\, dx, \tag{3.47}$$

where

$$H(\mathbf{E}) := \int_0^1 h(t\mathbf{E}) \cdot \mathbf{E}\, dt$$

and we recall (3.39), that is $\mathcal{J} : X \to \mathbb{R}$ defined as

$$\mathcal{J}(u) := \frac{1}{2}\int_{\mathbb{R}^N} \left(|\nabla u|^2 + a\frac{u^2}{r^2} + V(x)u^2\right) dx - \int_{\mathbb{R}^N} F(u)\, dx + \lambda \int_{\mathbb{R}^N} G(u)\, dx.$$

with $F(u) = \int_0^u f(s)\, ds$ and $G(u) = \int_0^u g(s)\, ds$.

Moreover, for a better readability, we call

$$\widetilde{f}(u) := f(u) - \lambda g(u). \tag{3.48}$$

Theorem 3.23. *Let $N = 3, K = 2, a = 1$. Suppose that (V) holds, $\widetilde{f}$ is continuous and there exists a constant $C > 0$ such that*

$$|\widetilde{f}(u)| \leq C(|u| + |u|^5), \quad u \in \mathbb{R}.$$

If $\boldsymbol{E} \in H^1(\mathbb{R}^3, \mathbb{R}^3)$ is a weak solution to (3.18) of the form (3.20), where u is cylindrically symmetric, then $u \in H^1(\mathbb{R}^3)$ and u is a weak solution to (3.22).

If $u \in H^1(\mathbb{R}^3)$ is a cylindrically symmetric, weak solution to (3.22), then $\boldsymbol{E}$ given by (3.20) lies in $H^1(\mathbb{R}^3, \mathbb{R}^3)$ and is a weak solution to (3.18).

Moreover, $\operatorname{div} \boldsymbol{E} = 0$ and $\mathcal{E}(\boldsymbol{E}) = \mathcal{J}(u)$.

Remark 3.24. *We want to remark that the proof of this last Theorem is not trivial and quite technical and, as cited above, we refer to [31] and [70] for the proof. We omit it here so as not to weigh down the reading.*

3.5.2 The Existence result for the curl-curl problem

In this Section we will provide the existence of a solution for the curl-curl problem (3.18), i.e.

$$\nabla \times (\nabla \times \mathbf{E}) + V(x)\mathbf{E} = h(\mathbf{E}), \quad x \in \mathbb{R}^3.$$

The proof of the following Theorem is a direct consequence of Theorem 3.22 and Theorem 3.23.

Theorem 3.25. *Suppose that (V), (F1)-(F5), (G1)-(G3) hold. If $\lambda > 0$ and $\rho > 0$ in (F5) are sufficiently small, then there is a nontrivial solution to (3.18).*

Proof. Applying Theorem 3.22 with $N = 3$, $K = 2$ and $a = 1$, there exists a solution to (3.21). Therefore, thanks to Theorem 3.23, there exists a solution to (3.18). □

We conclude, by showing a property for the electromagnetic energy (3.19)

Proposition 3.26. *The total electromagnetic energy $\mathcal{L}(t)$ of the solution $\boldsymbol{E}$ found in Theorem 3.25 is finite and does not depend on t.*

Proof. We recall that the constitutive relations for Maxwell's equations, together with the simplifications considered in Chapter 3.2, are

$$\begin{cases} \mathcal{D} = \varepsilon\mathcal{E} + \mathcal{P} \\ \mathcal{H} = \mathcal{B} \end{cases}$$

Thanks to these relations and the equivalence result from Theorem 3.23, we can compute

$$\begin{aligned} \mathcal{L}(t) &= \frac{1}{2}\int_{\mathbb{R}^3} (\mathcal{E}\mathcal{D} + \mathcal{B}\mathcal{H})\, dx = \frac{1}{2}\int_{\mathbb{R}^3} (\mathcal{E}\mathcal{D} + \mathcal{B}\mathcal{B})\, dx \\ &= \frac{1}{2\omega^2}\int_{\mathbb{R}^3} \Big[\Big(-V(x)|\mathbf{E}|^2 + h(\mathbf{E})\mathbf{E}\Big)\cos^2(\omega t) + |\nabla\times\mathbf{E}|^2\sin^2(\omega t)\Big]\, dx \\ &= \frac{1}{2\omega^2}\int_{\mathbb{R}^3} \left[\Big(-V(x)|u|^2 + \widetilde{f}(u)u\Big)\cos^2(\omega t) + \left(|\nabla u|^2 + \frac{u^2}{r^2}\right)\sin^2(\omega t)\right] dx. \end{aligned}$$

Since $u \in X$ we have that $\mathcal{L}(t) < +\infty$. Now, we observe that

$$\begin{aligned} \frac{d}{dt}\mathcal{L}(t) &= \frac{d}{dt}\left(\frac{1}{2\omega^2}\int_{\mathbb{R}^3} \left[\Big(-V(x)|u|^2 + \widetilde{f}(u)u\Big)\cos^2(\omega t) + \left(|\nabla u|^2 + \frac{u^2}{r^2}\right)\sin^2(\omega t)\right] dx\right) \\ &= \frac{\sin(\omega t)\cos(\omega t)}{\omega}\int_{\mathbb{R}^3}\left(|\nabla u|^2 + \frac{u^2}{r^2} + V(x)|u|^2 - \widetilde{f}(u)u\right) dx = 0, \end{aligned}$$

and this show that $\mathcal{L}(t)$ does not depend on time. □

Appendix A

Theorem A.1 (Solving Poisson's equation, see Theorem 1 in [61]). *Set*

$$u(x) = \int_{\mathbb{R}^N} \Phi(x-y)f(y)\,dy = \begin{cases} -\frac{1}{2\pi}\int_{\mathbb{R}^2} \log|x-y|f(y)\,dy, & \text{if } N=2, \\ \frac{1}{N(N-2)\omega_N}\int_{\mathbb{R}^N} \frac{f(y)}{|x-y|^{N-2}}\,dy, & \text{if } N \geq 3. \end{cases}$$

Then,

(i) $u \in C^2(\mathbb{R}^N)$*;*

(ii) $-\Delta u = f$ *in* $\mathbb{R}^N$.

Theorem A.2 (Riesz's Criterion, see Theorem XIII.66 in [121]). *Let* $p < \infty$ *and* B *the unit ball contained in* $L^p(\mathbb{R}^N)$*. Then the closure of* B *with respect to the* p*-norm is compact if and only if the following conditions hold:*

(1) for every $\varepsilon > 0$ *there exists a bounded set* $K \subset \mathbb{R}^N$ *such that*

$$\int_{K^c} |f(x)|^p\,dx \leq \varepsilon^p$$

for all $f \in B$*;*

(2) for every $\varepsilon > 0$ *there exists* $\delta := \delta(\varepsilon) > 0$ *such that if* $f \in B$ *and* $|y| < \delta$ *then*

$$\int_{\mathbb{R}^N} |\tau_{-y}f(x) - f(x)|^p\,dx \leq \varepsilon^p.$$

Theorem A.3 (Hardy-Littlewood-Sobolev, see Theorem 4.3 in [86]). *Let* $p, q > 1$ *and* $0 < \lambda < N$ *with*

$$\frac{1}{p} + \frac{\lambda}{N} + \frac{1}{q} = 2. \tag{A.1}$$

Let $f \in L^p(\mathbb{R}^2)$ *and* $g \in L^q(\mathbb{R}^2)$.

Then, there exists a sharp constant $C := C(N, \lambda, p) > 0$ *and independent on* f *and* g*, such that*

$$\left|\int_{\mathbb{R}^N \times \mathbb{R}^N} \frac{1}{|x-y|^\lambda} f(x)g(y)\,dx\,dy\right| \leq C\|f\|_{L^p(\mathbb{R}^N)}\|g\|_{L^q(\mathbb{R}^N)}.$$

Remark A.4. *Inequality (A.1) was firstly proved by Hardy-Littlewood in [75, 76], though not in the sharp form. The sharp constant was proved by Lieb in [84].*

Appendix B

Given a function f sufficiently smooth and small at infinity, it is related to its Laplacian by

$$\mathcal{F}\left((-\Delta)^{\frac{\beta}{2}} f\right)(x) = (2\pi|x|)^{\beta} \mathcal{F} f(x),$$

for $-N < \beta < 0$. This operator admits a realization as an integral operator given by

$$I_\alpha = (-\Delta)^{-\frac{\alpha}{2}}(f),$$

for $0 < \alpha < N$. Then,

Definition B.1 (Riesz potential, see [131], Chapter V)**.** *We call Riesz potential the function*

$$I_\alpha(f) := \frac{1}{\gamma(\alpha)} \int_{\mathbb{R}^N} \frac{f(y)}{|x-y|^{N-\alpha}}\, dy,$$

where

$$\gamma(\alpha) := \pi^{\frac{N}{2}} 2^{\alpha} \frac{\Gamma\left(\frac{\alpha}{2}\right)}{\Gamma\left(\frac{N-\alpha}{2}\right)}$$

Theorem B.2 (Plancherel's identity, see [72], Chapter 2.2.2)**.** *Given $f \in \mathcal{S}(\mathbb{R}^N)$ then following identity holds:*

$$\int_{\mathbb{R}^N} |f|^2\, dx = \int_{\mathbb{R}^N} |\mathcal{F}(f)|^2\, dx = \int_{\mathbb{R}^N} |\mathcal{F}^{-1}(f)|\, dx,$$

that is

$$\|f\|_{L^2(\mathbb{R}^N)} = \|\mathcal{F}(f)\|_{L^2(\mathbb{R}^N)} = \|\mathcal{F}^{-1}(f)\|_{L^2(\mathbb{R}^N)}$$

Lemma B.3 (Fractional Hardy inequality, see [65], Theorem 1.1)**.** *There exists a constant $C\left(N, \frac{1}{2}, \frac{1}{2}\right) > 0$, depending only on $N \geq 1$, such that for every $u \in H^{\frac{1}{2}}(\mathbb{R}^N)$ there holds*

$$[u]^2 \geq C\left(N, \frac{1}{2}, \frac{1}{2}\right) \int_{\mathbb{R}^N} \frac{u^2(x)}{|x|}\, dx, \tag{B.1}$$

where

$$C\left(N, \frac{1}{2}, \frac{1}{2}\right) = 2\pi^{\frac{N}{2}} \frac{\Gamma\left(\frac{N+1}{4}\right)^2 \left|\Gamma\left(-\frac{1}{2}\right)\right|}{\Gamma\left(\frac{N-1}{4}\right)^2 \Gamma\left(\frac{N+1}{2}\right)}$$

is the sharp constant of the inequality.

Remark B.4. *We remark that the fractional Hardy inequality (B.1) was proved in [65], Theorem 1.1 and it holds for every $s \in (0,1)$ and*

$$\begin{cases} u \in \dot{W}^s_p(\mathbb{R}^N), & \text{if } p \in \left(1, \frac{N}{s}\right), \\ u \in \dot{W}^s_p(\mathbb{R}^N \setminus \{0\}), & \text{if } p > \frac{N}{s}. \end{cases}$$

We reported here the version for $s = \frac{1}{2}$ and $p = 2$, that is needded for our purpose.

Theorem B.5 (Vitali's Theorem, see Theorem 4.5.4 in [35]). *Let μ be a finite measure. Suppose that f is a $\mu-$measurable function and $(f_n)_n$ is a sequence of $\mu-$integrable functions. Then, the following assertions are equivalent:*

(i) the sequence $(f_n)_n$ converges to f in measure and is uniformly integrable;

(ii) the function f is integrable and the sequence $(f_n)_n$ converges to f in the space $L^1(\mu)$.

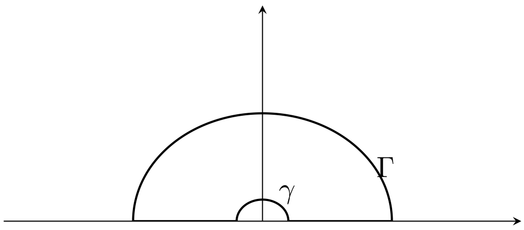
Proposition B.6 (Contour integral).

$$\int_{\mathbb{R}} \frac{1-\cos t}{t^2}\, dt = \pi.$$

Proof. Consider the complex function $f(z) = \frac{1-e^{iz}}{z^2}$. This function has a pole in $z = 0$. Then we evaluate the contour integral

$$0 = \oint_C f(z)\, dz = \int_{-R}^{-\varepsilon} f(z)\, dz + \int_{\gamma} f(z)\, dz + \int_{\varepsilon}^{R} f(z)\, dz + \int_{\Gamma} f(z)\, dz$$

where C is the curve given by the two segments of lenght $R - \varepsilon$, $\gamma : z = Re^{i\theta}, \theta \in [0, \pi]$ and $\Gamma : z = \varepsilon e^{i\theta}, \theta \in [\pi, 0]$ (see Figure below)



We observe that

$$\int_{-R}^{-\varepsilon} f(z)\, dz + \int_{\varepsilon}^{R} f(z)\, dz = \int_{-\infty}^{+\infty} f(z)\, dz.$$

Now, recalling that $|z| = R$,

$$\begin{aligned}\left|\int_{\Gamma} f(z)\, dz\right| &= \left|\int_0^{\pi} \frac{1-e^{iRe^{i\theta}}}{z^2}\, iz\, d\theta\right| \le \left|\int_0^{\pi} \frac{1-e^{iRe^{i\theta}}}{z}\, d\theta\right| = \int_0^{\pi} \frac{\left|1-e^{iRe^{i\theta}}\right|}{R}\, d\theta \\ &\le \frac{1}{R}\int_0^{\pi} \left(1 + \left|e^{iRe^{i\theta}}\right|\right) d\theta = \frac{1}{R}\int_0^{\pi} \left(1 + \left|e^{iR\cos\theta - R\sin\theta}\right|\right) d\theta \\ &\frac{1}{R}\int_0^{\pi} \left(1 + \left|e^{iR\cos\theta}\right|\left|e^{-R\sin\theta}\right|\right) d\theta.\end{aligned}$$

We observe that $R\cos\theta \in \mathbb{R}$, hence $\left|e^{iR\cos\theta}\right| = 1$: morevoer, $\left|e^{-R\sin\theta}\right| = e^{-R\sin\theta}$. Therefore,

$$\begin{aligned}\left|\int_{\Gamma} f(z)\, dz\right| &\le \frac{1}{R}\int_0^{\pi} \left(1 + e^{-R\sin\theta}\right) d\theta = \frac{1}{R}\int_0^{\pi} e^{-R\sin\theta}\, d\theta \\ &\le \frac{2}{R}\int_0^{\frac{\pi}{2}} e^{-\frac{2R}{\pi}\theta}\, d\theta = \frac{2}{R}\left(-\frac{\pi}{2R}e^{-\frac{2\pi}{R}\theta}\right)_0^{\frac{\pi}{2}} = \frac{\pi}{R^2}\left(e^{-R} - 1\right).\end{aligned}$$

Hence,

$$0 \le \lim_{R\to+\infty} \left|\int_{\Gamma} f(z)\, dz\right| \le \lim_{R\to+\infty} \frac{\pi}{R^2}\left(e^{-R} - 1\right) = 0,$$

so

$$\int_{\Gamma} f(z)\, dz = 0.$$

Now, from the Taylor expansion of the exponential function we have

$$\int_\gamma f(z)\,dz = \int_\pi^0 \frac{1-e^{i\varepsilon e^{i\theta}}}{z^2} iz\,d\theta = -i\int_0^\pi \frac{1-e^{i\varepsilon e^{i\theta}}}{z}\,d\theta$$
$$= -i\int_0^\pi \frac{1-\sum_{k=0}^{+\infty}\frac{\left(i\varepsilon e^{i\theta}\right)^k}{k!}}{\varepsilon e^{i\theta}}\,d\theta = i\int_0^\pi \sum_{k=1}^{+\infty}\frac{i^k\left(\varepsilon e^{i\theta}\right)^{k-1}}{k!}\,d\theta,$$

but, since we are intersted in the limit as $\varepsilon \to 0$, the only term that survives is $k = 1$: hence, as $\varepsilon \to 0$

$$\int_\gamma f(z)\,dz \to i\int_0^\pi i\,d\theta = -\int_0^\pi d\theta = -\pi.$$

Finally,

$$0 = \oint_C f(z)\,dz = \int_{-\infty}^{+\infty} f(z)\,dz - \pi$$

that is

$$\int_{-\infty}^{+\infty} f(z)\,dz = \pi.$$

□

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