BV Solutions to 1D Isentropic Euler Equations in the Zero Mach Number Limit

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Abstract

Two compressible immiscible fluids in 1D and in the isentropic approximation are considered. The first fluid is surrounded and in contact with the second one. As the Mach number of the first fluid vanishes, we prove the rigorous convergence for the fully non–linear compressible to incompressible limit of the coupled dynamics of the two fluids. A key role is played by a suitably refined wave front tracking algorithm, which yields precise **BV**, $\mathbf{L^1}$ and weak* convergence estimates, either uniform or explicitly dependent on the Mach number.

Keywords: Incompressible limit, Compressible Euler Equations, Hyperbolic Conservation Laws, Zero Mach Number Limit

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1 Introduction

This paper is devoted to the compressible to incompressible limit in the equations of isentropic gas dynamics, a widely studied subject in the literature, see for instance the well known results [12, 13, 14, 15], the more recent [18], the review [16] with the references therein and the monograph [10] for the Navier Stokes equations. For Euler equations, the usual setting considers regular solutions, whose existence is proved only for a finite time, to the compressible equations in 2 or 3 space dimensions. As the Mach number vanishes, these solutions are proved to converge to the solutions to the incompressible Euler equations.

Consider for instance the isentropic Euler equations in the three dimensional space:

$$\begin{cases} \partial_t \rho + \nabla \cdot (\rho u) = 0 & \overline{P}(\rho) > 0\,, \quad \overline{P}'(\rho) > 0, \\ \partial_t (\rho u) + \nabla \cdot (\rho u \otimes u) + \nabla \overline{P}(\rho) = 0\,, & (t,x) \in [0, +\infty[\times \mathbb{R}^3. \end{cases}$$

where ρ is the fluid density, u is its speed and $\overline{P}(\rho)$ is the pressure. For smooth solutions, this system is equivalent to

$$\begin{cases} \partial_t \rho + u \cdot \nabla \rho + \rho \nabla \cdot u = 0 \\ \partial_t u + u \cdot \nabla u + \frac{1}{\rho} \nabla \overline{P}(\rho) = 0\,. \end{cases} \tag{1.1}$$

The Mach number is the ratio between the speed of the particles and the sound speed; it can be introduced into the equations in at least two different ways [16].

First, following [14], since the incompressible limit can be understood as the limit when the Mach number tends to zero, one begins by rescaling the fluid velocity $u \to \kappa u$ where κ is a small parameter that eventually converges to zero. In order to capture the motion of the particles

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traveling with a small speed of order of κ one needs a space–time rescaling, $\frac{x}{t} \to \kappa\, \frac{x}{t}$, which allows to obtain, in the rescaled variables, the system

$$\begin{cases} \partial_t \rho + u \cdot \nabla \rho + \rho \nabla \cdot u = 0 \\ \partial_t u + u \cdot \nabla u + \frac{1}{\rho} \frac{1}{\kappa^2} \nabla \overline{P}\left(\rho\right) = 0\,. \end{cases} \tag{1.2}$$

Alternatively, the same system is considered in [13], but motivated by the following approach, see [13, 15]. Consider fluids having equations of state $\overline{P}_\kappa(\rho)$, parametrized by κ, such that the speed of sound $\sqrt{\overline{P}'_\kappa(\rho)} \to +\infty$ as $\kappa \to 0$:

$$\begin{cases} \partial_t \rho + u \cdot \nabla \rho + \rho \nabla \cdot u = 0 \\ \partial_t u + u \cdot \nabla u + \frac{1}{\rho} \nabla \overline{P}_\kappa\left(\rho\right) = 0. \end{cases} \tag{1.3}$$

The two approaches coincide if the one parameter family of pressure laws $\overline{P}_\kappa\left(\rho\right)$ satisfies

$$\overline{P}'_\kappa\left(\rho\right) = \frac{1}{\kappa^2} \overline{P}'\left(\rho\right), \tag{1.4}$$

where $\overline{P}$ is the fixed pressure law as in (1.2).

In the incompressible limit, the density is constant in time and space so that the functional dependence of the pressure on the density is lost. Therefore, it is convenient to use the pressure instead of the density as unknown variable. Since $\overline{P}'_\kappa\left(\rho\right) > 0$, we can take the inverse function $R_\kappa\left(p\right) = \left(\overline{P}_\kappa\right)^{-1}\left(p\right)$ and rewrite (1.3) using the pressure p as unknown:

$$\begin{cases} \frac{R'_\kappa(p)}{R_\kappa(p)} \left[\partial_t p + u \cdot \nabla p\right] + \nabla \cdot u = 0 \\ \partial_t u + u \cdot \nabla u + \frac{1}{R_\kappa(p)} \nabla p = 0\,. \end{cases}$$

As $\kappa \to 0$, $\overline{P}'_\kappa\left(\rho\right) \to +\infty$, therefore $R'_\kappa\left(p\right) \to 0$, and $R_\kappa\left(p\right) \to \bar{\rho}$, where $\bar{\rho}$ is the constant density at the incompressible limit. Formally, we get the incompressible equations

$$\begin{cases} \nabla \cdot u = 0 \\ \partial_t u + u \cdot \nabla u + \frac{1}{\bar{\rho}} \nabla p = 0\,. \end{cases}$$

In [12, 13] this limit is proved to hold for smooth solutions and small times. The heart of the matter is finding energy estimates, uniform in the small parameter κ.

Here, we obtain similar convergence results, in a 1D setting, for *all times* and within the framework of merely **BV** *weak entropy solutions*, following the papers [6, 7, 8, 11].

The next section describes the physical setting. Section 3 presents the key estimates and the main convergence results. All technical details are deferred to Section 4.

2 Two Immiscible Fluids

In a 1D setting, an incompressible fluid behaves like a solid and its speed is constant in space. Therefore, we consider two compressible immiscible fluids and let one of the two become incompressible, yielding a singular limit for a free boundary problem. Below, we consider a volume of a compressible inviscid fluid, say the *liquid*, that fills the segment $[a(t), b(t)]$ and is surrounded by another compressible fluid, say the *gas*, filling the rest of the real line (see Figure 1). We assume that the gas obeys a fixed pressure law $\overline{P}_g\left(\rho\right)$, while for the liquid we assume a one parameter

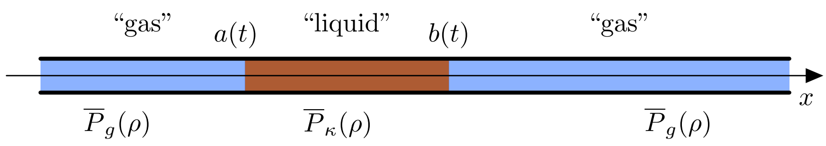

Figure 1: The two immiscible fluids: the liquid is in the middle, while the gas fills the two sides.

family of pressure laws $\overline{P}_\kappa(\rho)$ such that $\overline{P}'_\kappa(\rho) \to +\infty$ as $\kappa \to 0$. The total mass of the liquid is fixed: $\int_{a(t)}^{b(t)} \rho(t,x)\ \mathrm{d}x = m$. Since the two fluids are immiscible, the introduction of the Lagrangian coordinate z and of the specific volume τ is a natural choice [17]:

$$z(t,x) = \int_{a(t)}^{x} \rho(t,\xi)\ \mathrm{d}\xi\,, \qquad \tau = \frac{1}{\rho}, \qquad P_g(\tau) = \overline{P}_g\left(\frac{1}{\tau}\right), \qquad P_\kappa(\tau) = \overline{P}_\kappa\left(\frac{1}{\tau}\right). \tag{2.1}$$

In these coordinates, the liquid and gas phases become the fixed sets (see Figure 2)

$$\mathcal{L} =]0,m[\qquad \text{and} \qquad \mathcal{G} = \mathbb{R} \setminus]0,m[\ .$$

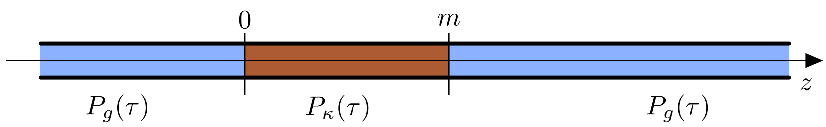

Figure 2: In Lagrangian coordinates, the boundaries separating the two fluids are fixed.

On $P_g(\tau)$ and $P_\kappa(\tau)$, we require the usual hypotheses and the incompressible limit assumption:

$$P_g(\tau)\,,\ P_\kappa(\tau) > 0;\ P'_g(\tau)\,,\ P'_\kappa(\tau) < 0;\ P''_g(\tau)\,,\ P''_\kappa(\tau) > 0;\ P'_\kappa(\tau) \xrightarrow{\kappa\to 0} -\infty\,. \tag{2.2}$$

In the isentropic approximation, the dynamics of the two fluids is described by the p-system [9]

$$\begin{cases} \partial_t \tau - \partial_z v = 0 \\ \partial_t v + \partial_z P_\kappa(z,\tau) = 0, \end{cases} \qquad \text{where} \quad P_\kappa(z,\tau) = \begin{cases} P_\kappa(\tau) & \text{for } z \in \mathcal{L} \\ P_g(\tau) & \text{for } z \in \mathcal{G}, \end{cases} \tag{2.3}$$

$v(t,z)$ being the speed of the fluids at time t and at the Lagrangian coordinate z.

The Rankine–Hugoniot conditions for (2.3), applied at $z = 0$ and $z = m$, imply the following interface conditions (conservation of mass and momentum) for a.e. $t \geq 0$:

$$\begin{cases} v(t,0-) = v(t,0+) \\ P_g\big(\tau(t,0-)\big) = P_\kappa\big(\tau(t,0+)\big)\,, \end{cases} \qquad \begin{cases} v(t,m-) = v(t,m+) \\ P_\kappa\big(\tau(t,m-)\big) = P_g\big(\tau(t,m+)\big)\,. \end{cases}$$

In other words, the pressure and the velocity have to be continuous across the interfaces. Hence, the pressure is a natural choice as unknown, rather than the specific volume. Therefore, we introduce the inverse functions of the pressure laws

$$\mathcal{T}_g(p) = P_g^{-1}(p)\,, \quad \mathcal{T}_\kappa(p) = P_\kappa^{-1}(p)\,, \quad \mathcal{T}'_\kappa(p) \xrightarrow{\kappa\to 0} 0\,, \tag{2.4}$$

the last limit being a consequence of (2.2). Rewrite system (2.3) with (p,v) as unknowns

$$\begin{cases} \partial_t \mathcal{T}_\kappa(z,p) - \partial_z v = 0 \\ \partial_t v + \partial_z p = 0\,, \end{cases} \qquad \text{where} \quad \mathcal{T}_\kappa(z,p) = \begin{cases} \mathcal{T}_\kappa(p) & \text{for } z \in \mathcal{L} \\ \mathcal{T}_g(p) & \text{for } z \in \mathcal{G}\,. \end{cases} \tag{2.5}$$

The conditions at the interfaces become continuity requirements on the unknown functions:

$$\begin{cases} v\,(t,0-) = v\,(t,0+) \\ p\,(t,0-) = p\,(t,0+) \end{cases} \qquad \begin{cases} v\,(t,m-) = v\,(t,m+) \\ p\,(t,m-) = p\,(t,m+) \end{cases} \qquad \text{for a.e. } t \geq 0\,. \tag{2.6}$$

The choice of these unknowns significantly simplifies the study of the Riemann problem at the interfaces.

Particular care is necessary to select the one parameter family of pressure laws, the main constraint being the validity of (1.4) for all κ. Indeed, (1.4) ensures that we recover the same equations obtained through scaling and studied in [12, 13]. The family $\overline{P}_\kappa\,(\rho) = \frac{1}{\kappa^2}\overline{P}\,(\rho)$ chosen in [12] diverges to $+\infty$ as $\kappa \to 0$. This is not a problem when studying only one fluid as in [12] because the pressure enters the equations only through its gradient. In our case, the value of the pressure is very relevant, since it enters the interface conditions (2.6). Therefore, we cannot allow the pressure to grow nonphysically to $+\infty$. We fix the density $\bar\rho$ of the incompressible fluid in the limit and impose that the pressure at that particular density $\bar\rho$ is a constant, independent of κ:

$$\overline{P}_\kappa(\bar\rho) = \bar p, \qquad \text{for all } \kappa \in \,]0,1[\ . \tag{2.7}$$

Now, choose a fixed pressure law $\overline{P} = \overline{P}(\rho)$ (for instance, an admissible choice is the usual γ–law $\overline{P}\,(\rho) = k\,\rho^\gamma$ with $\gamma \geq 1$) and apply conditions (1.4) and (2.7) to get the following expression for $\overline{P}_\kappa\,(\rho)$, with $\overline{P}\,(\bar\rho) = \bar p$:

$$\overline{P}_\kappa\,(\rho) = \bar p + \frac{1}{\kappa^2}\left[\overline{P}\,(\rho) - \bar p\right]\,, \tag{2.8}$$

which, with the substitution $\rho = \frac{1}{\tau}$, becomes:

$$P_\kappa\,(\tau) = \bar p + \frac{1}{\kappa^2}\left[P\,(\tau) - \bar p\right]. \tag{2.9}$$

Finally, in term of the inverse functions $\mathcal{T}_\kappa = P_\kappa^{-1}$ and $\mathcal{T} = P^{-1}$, we have

$$\mathcal{T}_\kappa\,(p) = \mathcal{T}\left(\bar p + \kappa^2\,(p - \bar p)\right)\,, \quad \lim_{\kappa\to 0}\mathcal{T}_\kappa\,(p) = \mathcal{T}\,(\bar p) = \frac{1}{\bar\rho} \doteq \bar\tau\,. \tag{2.10}$$

In [7, 11], (2.10) is approximated linearly:

$$\mathcal{T}\left(\bar p + \kappa^2\,(p-\bar p)\right) \approx \mathcal{T}\,(\bar p) + \kappa^2\mathcal{T}'\,(\bar p)\,(p - \bar p) = \bar\tau + \kappa^2\mathcal{T}'\,(\bar p)\,(p-\bar p)\,, \tag{2.11}$$

so that the liquid phase turns out to be governed by a linear system. This approximation makes all the estimates simpler. Here we study the Cauchy problem in the fully non linear case

$$\begin{cases} \partial_t\mathcal{T}_\kappa\,(z,p) - \partial_z v = 0 \\ \partial_t v + \partial_z p = 0, \end{cases} \qquad \text{where} \quad \mathcal{T}_\kappa\,(z,p) = \begin{cases} \mathcal{T}\left(\bar p + \kappa^2\,(p-\bar p)\right) & \text{for } z \in \mathcal{L} \\ \mathcal{T}_g\,(p) & \text{for } z \in \mathcal{G}. \end{cases} \tag{2.12}$$

Colombo and Schleper in [8, Theorem 2.5] proved that for any fixed small $\kappa > 0$, there exists a Lipschitz semigroup of solutions to (2.12), but their estimates are not uniform with respect to κ. Therefore, as $\kappa \to 0$ the Lipschitz constant of the semigroup could blow up and its domain could shrink, becoming trivial. Here, we provide a full set of new estimates either uniform in κ, or with the dependence on κ made explicit. To this aim, we substantially improve the wave front tracking construction in [4, 8], devising and exploiting a different parametrization of the Lax curves.

The main result of this paper states the rigorous convergence at the incompressible limit in the liquid phase of the solutions to (2.12) to solutions to

$$\begin{cases} \begin{cases} \partial_t \mathcal{T}_g(p) - \partial_z v = 0 \\ \partial_t v + \partial_z p = 0 \end{cases} & z \in \mathcal{G} & \text{gas;} \\ \dot{v} = \frac{p(t,0-)-p(t,m+)}{m} & & \text{liquid;} \\ \begin{cases} v\,(t,0-) &=& v(t) \\ v\,(t,m+) &=& v(t) \end{cases} & & \text{immiscibility and mass conservation.} \end{cases} \tag{2.13}$$

Note that the liquid speed $v(t)$ is independent of the Lagrangian variable z. Therefore, we choose a constant v in the liquid region at time $t = 0$ before letting $\kappa \to 0$. In this very singular limit, the sound speed in the liquid phase tends to $+\infty$; the density converges to a fixed reference value $\bar{\rho}$; the graph of the pressure law $P_\kappa(\tau)$ becomes vertical and the eigenvectors of the Jacobian of the flow tend to coalesce. Moreover, the pressure in the liquid wildly oscillates but, remarkably, we are able to prove the weak* convergence of the pressure to the linear interpolation of the traces of the pressure at the sides of the liquid region, as is to be expected based on physical considerations. A linear example, where all the components of this singular limit can be explicitly computed, can be found in [6].

Recall that problem (2.12), respectively (2.13), is well posed in $\mathbf{L^1}$ globally in time, see [8, Theorem 2.5], respectively [3, Theorem 3.6].

3 Main Result

Throughout, we require that the pressure law P_g in the gas phase and the one parameter family of pressure laws P_κ in the liquid phase, as defined in (2.9), all satisfy the condition

(P): $P \in \mathbf{C^3}(]0, +\infty[\,;]0, +\infty[)$, $P' < 0$ and $P'' > 0$.

The standard choice $p(\tau) = k/\tau^\gamma$ satisfies this condition for all $k > 0$ and $\gamma \geq 1$.

As a starting point, we provide the rigorous definition of solutions to (2.5), with reference to [4, Chapter 4, Definition 4.3 and Admissibility Condition 2].

Definition 3.1. *Fix $T > 0$ and $\kappa > 0$. By* weak solution *to* (2.5) *we mean a map*

$$(p, v) \in \mathbf{C^0}\left([0, T]; (\mathbf{L^1_{loc}} \cap \mathbf{BV})(\mathbb{R}; \mathbb{R}^+ \times \mathbb{R})\right)$$

such that (2.5) *holds in distributional sense. The weak solution u is a* weak entropy solution *to* (2.5) *if both its restrictions to $\mathcal{L}$ and to $\mathcal{G}$ are weak entropy solutions in the sense of [4, Definition 4.3].*

Introduce the mathematical entropy flow $q = pv$ of (2.5), the equalities $(p, v)(t, 0-) = (p, v)(t, 0+)$ and $(p, v)(t, m-) = (p, v)(t, m+)$ (consequences of the Rankine–Hugoniot conditions) imply that the entropy flow is continuous and hence that the entropy is conserved across both interfaces.

In the case of (2.13), we recall [2, Definition 2.5].

Definition 3.2. *Fix $T > 0$. By a* solution *to* (2.13) *we mean a pair of maps*

$$\begin{aligned} (p^*, v^*) &\in \mathbf{C^0}\left([0, T]; (\mathbf{L^1_{loc}} \cap \mathbf{BV})(\mathcal{G}; \mathbb{R}^+ \times \mathbb{R})\right) \\ v_l &\in \mathbf{W^{1,\infty}}([0, T]; \mathbb{R}) \end{aligned}$$

such that:

1. *(p^*, v^*) is a weak entropy solution to* $\begin{cases} \partial_t \mathcal{T}_g(p) - \partial_z v = 0 \\ \partial_t v + \partial_z p = 0 \end{cases}$ *in $[0, T] \times \mathcal{G}$;*

2. *for a.e.* $t \in [0,T]$, $\dot v_l(t) = \frac{1}{m}\left(p^*(t,0-) - p^*(t,m+)\right)$;

3. *for a.e.* $t \in [0,T]$, $v^*(t,0-) = v^*(t,m+) = v_l(t)$.

The existence of solutions to (2.12) follows from the next theorem, that also provides the basic estimates for the subsequent compressible to incompressible limit. In this context, a natural requirement is the smallness of the total variation of the initial datum. Aiming at the incompressible limit, it is natural to introduce the weighted total variation

$$\mathrm{WTV}_\kappa\,(p,v) = \mathrm{TV}\,(p,\mathbb{R}) + \mathrm{TV}\,(v,\mathcal{G}) + \frac{1}{\kappa}\ \mathrm{TV}\,(v,\mathcal{L}) \tag{3.1}$$

whose boundedness requires that the initial total variation of the particles speed in the liquid vanishes with κ.

Theorem 3.3. *Fix the total mass of the liquid $m>0$ and a pressure $p_o>0$. Let P, P_g satisfy **(P)**, define $\mathcal{T}_g$ as in* (2.4) *and $\mathcal{T}_\kappa$ as in* (2.10)*. Then, there exist positive δ, Δ, L, κ_* with $\kappa_*<1$ such that for any $\kappa \in \,]0,\kappa_*[$, for any initial datum $(\tilde p,\tilde v) \in \mathbf{L}^1_{\mathbf{loc}}(\mathbb{R};\mathbb{R}^+\times\mathbb{R})$, under the assumptions*

$$\mathrm{WTV}_\kappa(\tilde p,\tilde v) \le \delta\,,\quad \|\tilde p - p_o\|_{\mathbf{L}^\infty(\mathbb{R};\mathbb{R})} \le \delta\,,\quad \tilde v(0+) = \tilde v(0) \quad \text{and} \quad \tilde v(m-) = \tilde v(m)\,, \tag{3.2}$$

problem (2.12) *with initial datum $(\tilde p,\tilde v)$ admits a weak entropy solution (p^κ,v^κ) in the sense of Definition 3.1 defined for all $t\in\mathbb{R}^+$. Moreover, since the specific volume is $\tau^\kappa(t,z) = \mathcal{T}_\kappa\left(z,p^\kappa(t,z)\right)$, for any $t,t_1,t_2 \ge 0$*

$$\begin{array}{rll}
\mathrm{WTV}_\kappa\left((p^\kappa,v^\kappa)(t,\cdot)\right) \le & \Delta, & \\
\mathrm{TV}\left(p^\kappa(t,\cdot),\mathcal{L}\right) \le & \Delta, & \int_{\mathcal{L}} \left|p^\kappa(t_2,z) - p^\kappa(t_1,z)\right| \mathrm{d}z \le \frac{1}{\kappa} L\,|t_2-t_1|\,,\\
\mathrm{TV}\left(v^\kappa(t,\cdot),\mathcal{L}\right) \le & \kappa\Delta, & \int_{\mathcal{L}} \left|v^\kappa(t_2,z) - v^\kappa(t_1,z)\right| \mathrm{d}z \le L\,|t_2-t_1|\,,\\
\mathrm{TV}\left(\tau^\kappa(t,\cdot),\mathcal{L}\right) \le & \kappa^2\Delta, & \int_{\mathcal{L}} \left|\tau^\kappa(t_2,z) - \tau^\kappa(t_1,z)\right| \mathrm{d}z \le \kappa L\,|t_2-t_1|\,,\\
\mathrm{TV}\left(p^\kappa(t,\cdot),\mathcal{G}\right) \le & \Delta, & \int_{\mathcal{G}} \left|p^\kappa(t_2,z) - p^\kappa(t_1,z)\right| \mathrm{d}z \le L\,|t_2-t_1|\,,\\
\mathrm{TV}\left(v^\kappa(t,\cdot),\mathcal{G}\right) \le & \Delta, & \int_{\mathcal{G}} \left|v^\kappa(t_2,z) - v^\kappa(t_1,z)\right| \mathrm{d}z \le L\,|t_2-t_1|\,,\\
\mathrm{TV}\left(\tau^\kappa(t,\cdot),\mathcal{G}\right) \le & \Delta, & \int_{\mathcal{G}} \left|\tau^\kappa(t_2,z) - \tau^\kappa(t_1,z)\right| \mathrm{d}z \le L\,|t_2-t_1|\,;
\end{array} \tag{3.3}$$

for any $z,z_1,z_2 \in \mathcal{L}$

$$\begin{array}{rll}
\mathrm{TV}\left(p^\kappa(\cdot,z),\mathbb{R}^+\right) \le & \frac{\Delta}{\kappa}, & \int_{\mathbb{R}^+} \left|p^\kappa(t,z_2) - p^\kappa(t,z_1)\right| \mathrm{d}t \le L\,|z_2-z_1|\,,\\
\mathrm{TV}\left(v^\kappa(\cdot,z),\mathbb{R}^+\right) \le & \Delta, & \int_{\mathbb{R}^+} \left|v^\kappa(t,z_2) - v^\kappa(t,z_1)\right| \mathrm{d}t \le \kappa L\,|z_2-z_1|\,,\\
\mathrm{TV}\left(\tau^\kappa(\cdot,z),\mathbb{R}^+\right) \le & \kappa\Delta, & \int_{\mathbb{R}^+} \left|\tau^\kappa(t,z_2) - \tau^\kappa(t,z_1)\right| \mathrm{d}t \le \kappa^2 L\,|z_2-z_1|\,;
\end{array} \tag{3.4}$$

for any $z,z_1,z_2 \in \mathcal{G}$

$$\begin{array}{rll}
\mathrm{TV}\left(p^\kappa(\cdot,z),\mathbb{R}^+\right) \le \Delta, & \int_{\mathbb{R}^+} \left|p^\kappa(t,z_2) - p^\kappa(t,z_1)\right| \mathrm{d}t \le L\,|z_2-z_1|\,,\\
\mathrm{TV}\left(v^\kappa(\cdot,z),\mathbb{R}^+\right) \le \Delta, & \int_{\mathbb{R}^+} \left|v^\kappa(t,z_2) - v^\kappa(t,z_1)\right| \mathrm{d}t \le L\,|z_2-z_1|\,,\\
\mathrm{TV}\left(\tau^\kappa(\cdot,z),\mathbb{R}^+\right) \le \Delta, & \int_{\mathbb{R}^+} \left|\tau^\kappa(t,z_2) - \tau^\kappa(t,z_1)\right| \mathrm{d}t \le L\,|z_2-z_1|\,;
\end{array} \tag{3.5}$$

for any $z,z_1,z_2 \in \mathbb{R}$

$$\begin{array}{rll}
\mathrm{TV}\left(p^\kappa(\cdot,z),\mathbb{R}^+\right) \le \frac{\Delta}{\kappa}, & \int_{\mathbb{R}^+} \left|p^\kappa(t,z_2) - p^\kappa(t,z_1)\right| \mathrm{d}t \le \frac{L}{\kappa}\,|z_2-z_1|\,,\\
\mathrm{TV}\left(v^\kappa(\cdot,z),\mathbb{R}^+\right) \le \Delta, & \int_{\mathbb{R}^+} \left|v^\kappa(t,z_2) - v^\kappa(t,z_1)\right| \mathrm{d}t \le L\,|z_2-z_1|\,.
\end{array} \tag{3.6}$$

The above existence result can be completed with uniqueness and Lipschitz continuous dependence of the solutions on the data exploiting the results in [8, Theorem 2.5]. Note however that the estimates provided therein, differently from the ones presented here, are not uniform in κ.

We now pass to the key limit $\kappa \to 0$.

Theorem 3.4. *Fix the total mass of the liquid $m > 0$ and a pressure $p_o > 0$. Let P, P_g satisfy **(P)**, define $\mathcal{T}_g$ as in (2.4) and $\mathcal{T}_\kappa$ as in (2.10). Let δ, Δ, L and κ_* be as in Theorem 3.3. For any $v_o \in \mathbb{R}$ and $(\tilde p, \tilde v) \in \mathbf{L^1_{loc}}(\mathbb{R}; \mathbb{R}^+ \times \mathbb{R})$ satisfying*

$$\|\tilde p - p_o\|_{\mathbf{L}^\infty(\mathbb{R};\mathbb{R})} < \delta\,, \qquad \mathrm{TV}(\tilde p) + \mathrm{TV}(\tilde v) \leq \delta \quad \text{and} \quad \tilde v(z) = v_o \quad \forall z \in [0, m]\,, \tag{3.7}$$

the Cauchy problem for (2.12) with initial datum $(\tilde p, \tilde v)$ admits for any $\kappa \in \left]0, \kappa^*\right[$ a weak entropy solution (p^κ, v^κ) satisfying (3.3) – (3.4) – (3.5) – (3.6).

Moreover, there exist functions

$$\begin{array}{rclrcl} p^* & \in & \mathbf{C^0}\left(\mathbb{R}^+; (\mathbf{L^1_{loc}} \cap \mathbf{BV})(\mathcal{G}; \mathbb{R}^+)\right), & p_l & \in & \mathbf{L}^\infty(\mathbb{R}^+ \times \mathcal{L}; \mathbb{R}^+), \\ v^* & \in & \mathbf{C^0}\left(\mathbb{R}^+; (\mathbf{L^1_{loc}} \cap \mathbf{BV})(\mathbb{R}; \mathbb{R}^+)\right), & v_l & \in & \mathbf{W^{1,\infty}}(\mathbb{R}^+; \mathbb{R}), \end{array}$$

such that $(p^*, v^*_{|\mathcal{G}})$ and v_l solve (2.13) with initial datum

$$\begin{array}{rcll} (p^*, v^*)(0, z) & = & (\tilde p, \tilde v)(z) & \text{a.e. } z \in \mathcal{G} \\ v_l(0) & = & v_o & \end{array}$$

in the sense of Definition 3.2. Up to subsequences, as $\kappa \to 0$,

$$\begin{array}{lllll} v^\kappa(t, \cdot) \to v^*(t, \cdot) & \text{in} & \mathbf{L^1_{loc}}(\mathbb{R}; \mathbb{R})\,, & t \geq 0 & \\ v^\kappa(\cdot, z) \to v^*(\cdot, z) & \text{in} & \mathbf{L^1_{loc}}(\mathbb{R}^+; \mathbb{R}), & z \in \mathbb{R} & \\ \mathrm{TV}\left(v^*(t, \cdot), \mathbb{R}\right) \leq \Delta, & \int_{\mathbb{R}} \left|v^*(t_2, z) - v^*(t_1, z)\right| \mathrm{d}z & \leq L\left|t_2 - t_1\right|, & t, t_1, t_2 \geq 0, & \\ \mathrm{TV}\left(v^*(\cdot, z), \mathbb{R}^+\right) \leq \Delta, & \int_{\mathbb{R}^+} \left|v^*(t, z_2) - v^*(t, z_1)\right| \mathrm{d}t & \leq L\left|z_2 - z_1\right|, & z, z_1, z_2 \in \mathbb{R}, & \\ v^*(t, z) = v_l(t), \ \text{a.e. } (t, z) \in \mathbb{R}^+ \times \mathcal{L}. & & & & \end{array} \tag{3.8}$$

$$\begin{array}{lllll} p^\kappa(t, \cdot) \to p^*(t, \cdot) & \text{in} & \mathbf{L^1_{loc}}(\mathcal{G}; \mathbb{R}), & t \geq 0 & \\ p^\kappa(\cdot, z) \to p^*(\cdot, z) & \text{in} & \mathbf{L^1_{loc}}(\mathbb{R}^+; \mathbb{R}), & z \in \mathcal{G} & \\ \mathrm{TV}\left(p^*(t, \cdot), \mathcal{G}\right) \leq \Delta, & \int_{\mathcal{G}} \left|p^*(t_2, z) - p^*(t_1, z)\right| \mathrm{d}z & \leq L\left|t_2 - t_1\right|, & t, t_1, t_2 \geq 0, & \\ \mathrm{TV}\left(p^*(\cdot, z), \mathbb{R}^+\right) \leq \Delta, & \int_{\mathbb{R}^+} \left|p^*(t, z_2) - p^*(t, z_1)\right| \mathrm{d}t & \leq L\left|z_2 - z_1\right|, & z, z_1, z_2 \in \mathcal{G}, & \\ p^\kappa(\cdot, \cdot) \overset{\star}{\rightharpoonup} p_l(\cdot, \cdot), & \text{in} & \mathbf{L}^\infty(\mathcal{L} \times \mathbb{R}^+; \mathbb{R}^+), & & \\ p_l(t, z) = \left(1 - \frac{z}{m}\right) p^*(t, 0-) + \frac{z}{m} p^*(t, m+), & \text{a.e. } (t, z) \in \mathbb{R}^+ \times \mathcal{L}\,, & & & \\ \tau^\kappa(\cdot, \cdot) \to \bar\tau, & \text{uniformly in } \mathcal{L} \times \mathbb{R}^+. & & & \end{array} \tag{3.9}$$

where the specific volume is $\tau^\kappa(t, z) = \mathcal{T}_\kappa\left(z, p^\kappa(t, z)\right)$.

From the Eulerian coordinates' point of view, the locations of the boundaries of the liquid phase can be recovered through a time integration. Let $x = a_o$ be the initial location of the left interface that we keep fixed with respect to κ. Since in Theorem 3.4 the initial pressure is chosen independently of κ, the initial specific volume in the liquid is given by $\tilde\tau^\kappa(z) = \mathcal{T}\left(\bar p + \kappa^2\left(\tilde p(z) - \bar p\right)\right)$, which may depend on κ. The total mass m of the liquid is fixed. Hence, the initial location of the right interface in general depends on κ, say $x = b_o^\kappa$. Since $\tilde\tau^\kappa(z) \to \bar\tau$ as $\kappa \to 0$, we have $b_o^\kappa \to b_o = a_o + m\bar\tau$. Note however that in the particular case of constant initial pressure $\tilde p(z) = \bar p$ in the liquid, also b_o^κ turns out to be independent of κ.

Let $a^\kappa(t)$ and $b^\kappa(t)$ be the locations of the interfaces (in Eulerian coordinates) at time t for positive κ, while $a(t)$ and $b(t)$ be the corresponding limits as $\kappa \to 0$. Then, we have:

$$\begin{array}{rclrcl} a^\kappa(t) & = & a_o + \int_0^t v^\kappa(\xi, 0)\ \mathrm{d}\xi & a(t) & = & a_o + \int_0^t v_l(\xi)\ \mathrm{d}\xi \\ b^\kappa(t) & = & b_o^\kappa + \int_0^t v^\kappa(\xi, m)\ \mathrm{d}\xi & b(t) & = & b_o + \int_0^t v_l(\xi)\ \mathrm{d}\xi\,. \end{array} \tag{3.10}$$

Using Theorem 3.4 we can see that the boundaries of the two phases are Lipschitz continuous functions of t. Moreover, as $\kappa \to 0$, $a^\kappa \to a$ and $b^\kappa \to b$ uniformly on bounded time intervals. An explicit expression for these boundaries and their limit in a linear framework can be found in [6].

4 Technical Details

Throughout, we suppose that P, P_g in theorems 3.3, 3.4 satisfy condition **(P)** and denote by $\mathcal{O}(1)$ a quantity that depends only on P, P_g and on uniform bounds on the initial data.

We define $\mathcal{T}, \mathcal{T}_\kappa, \mathcal{T}_g$ as in (2.4), (2.10) and collect below a few facts about the p-system in Lagrangian coordinates using the (p,v) plane. Consider first the gas phase, where

$$\left\{\begin{array}{l} \partial_t \mathcal{T}_g(p) - \partial_z v = 0 \\ \partial_t v + \partial_z p = 0\,, \end{array}\right. \qquad \text{with eigenvalues} \qquad \begin{array}{rcl} \lambda_1^g(p,v) & = & -\sqrt{-\dfrac{1}{\mathcal{T}_g'(p)}} \\ \lambda_2^g(p,v) & = & \sqrt{-\dfrac{1}{\mathcal{T}_g'(p)}} \end{array} \tag{4.1}$$

so that the Lax shock and rarefaction curves are, see also [7],

$$\begin{array}{rcl} V_1^g(p;p_o,v_o) & = & \left\{\begin{array}{ll} v_o - \displaystyle\int_{p_o}^{p} \sqrt{-\mathcal{T}_g'(\xi)}\,\mathrm{d}\xi & p < p_o \\ v_o - \sqrt{-\left(\mathcal{T}_g(p) - \mathcal{T}_g(p_o)\right)(p-p_o)} & p \geq p_o \end{array}\right. \\ V_2^g(p;p_o,v_o) & = & \left\{\begin{array}{ll} v_o - \sqrt{-\left(\mathcal{T}_g(p) - \mathcal{T}_g(p_o)\right)(p-p_o)} & p < p_o \\ v_o + \displaystyle\int_{p_o}^{p} \sqrt{-\mathcal{T}_g'(\xi)}\,\mathrm{d}\xi & p \geq p_o\,. \end{array}\right. \end{array} \tag{4.2}$$

Similarly, in the liquid phase we have

$$\left\{\begin{array}{l} \partial_t \mathcal{T}_\kappa(p) - \partial_z v = 0 \\ \partial_t v + \partial_z p = 0\,. \end{array}\right. \qquad \text{with eigenvalues} \qquad \begin{array}{rcl} \lambda_1^\kappa(p,v) & = & -\dfrac{1}{\kappa}\sqrt{-\dfrac{1}{\mathcal{T}'\left(\bar p + \kappa^2(p-\bar p)\right)}} \\ \lambda_2^\kappa(p,v) & = & \dfrac{1}{\kappa}\sqrt{-\dfrac{1}{\mathcal{T}'\left(\bar p + \kappa^2(p-\bar p)\right)}} \end{array} \tag{4.3}$$

and the Lax curves are

$$\begin{array}{rcl} V_1^\kappa(p;p_o,v_o) & = & \left\{\begin{array}{ll} v_o - \displaystyle\int_{p_o}^{p} \sqrt{-\mathcal{T}_\kappa'(\xi)}\,\mathrm{d}\xi & p < p_o \\ v_o - \sqrt{-\left(\mathcal{T}_\kappa(p) - \mathcal{T}_\kappa(p_o)\right)(p-p_o)} & p \geq p_o \end{array}\right. \\ V_2^\kappa(p;p_o,v_o) & = & \left\{\begin{array}{ll} v_o - \sqrt{-\left(\mathcal{T}_\kappa(p) - \mathcal{T}_\kappa(p_o)\right)(p-p_o)} & p < p_o \\ v_o + \displaystyle\int_{p_o}^{p} \sqrt{-\mathcal{T}_\kappa'(\xi)}\,\mathrm{d}\xi & p \geq p_o\,. \end{array}\right. \end{array} \tag{4.4}$$

Below we systematically use the parameterizations

$$\sigma_i \to V_i^g(p_o + \sigma_i; p_o, v_o) \qquad \text{and} \qquad \sigma_i \to V_i^\kappa(p_o+\sigma_i;p_o,v_o) \tag{4.5}$$

of the i–Lax curve, σ_i being a pressure difference. Therefore, differently from the usual habit, we have that

Lemma 4.1. *Fix L, l with $L > l > 0$ and $\kappa \in \,]0,1]$. The Lax curves* (4.4) *admit the representation*

$$\begin{array}{rcl} V_1^\kappa(p;p_o,v_o) & = & v_o - \kappa\,(p-p_o)\,F\left(\Pi_\kappa(p), \Pi_\kappa(p_o)\right) \\ V_2^\kappa(p;p_o,v_o) & = & v_o + \kappa\,(p-p_o)\,F\left(\Pi_\kappa(p_o), \Pi_\kappa(p)\right) \end{array} \tag{4.6}$$

$i=1$	$i=2$
$\sigma_1 < 0 \Rightarrow$ rarefaction	$\sigma_2 < 0 \Rightarrow$ shock
$\sigma_1 > 0 \Rightarrow$ shock	$\sigma_2 > 0 \Rightarrow$ rarefaction

Table 1: Types of waves and the signs of the corresponding parameters as in (4.2), (4.4).

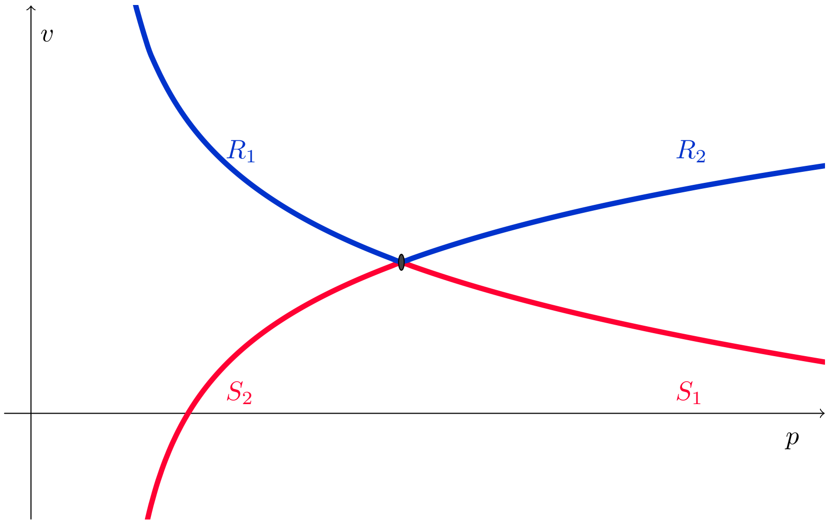

Figure 3: Lax curves (4.4) in the (p, v)–plane.

where

$$\Pi_\kappa(p) = \bar{p} + \kappa^2 (p - \bar{p}),$$

$$F(x,y) = \begin{cases} \int_0^1 \sqrt{-\mathcal{T}'\left(\vartheta x + (1-\vartheta)y\right)}\, \mathrm{d}\vartheta & x < y, \\ \sqrt{-\mathcal{T}'(x)} & x = y, \\ \sqrt{\int_0^1 -\mathcal{T}'\left(\vartheta x + (1-\vartheta)y\right) \mathrm{d}\vartheta} & x > y. \end{cases} \tag{4.7}$$

Moreover,

1. *the function F is of class $\mathbf{C}^{\mathbf{1,1}}([l,L]^2;\mathbb{R})$;*
2. *both restrictions $F_{|x\leq y}$ and $F_{|x\geq y}$ are of class $\mathbf{C}^{\mathbf{2}}([l,L]^2;\mathbb{R})$;*
3. *for $x, y \in [l, L]$, $F(x,y) \in \left[\sqrt{-\mathcal{T}'(L)}, \sqrt{-\mathcal{T}'(l)}\right]$.*

The proof follows from standard computations. A property that plays a key role in the sequel is that the function F above is independent of κ.

Call F_g the function obtained Replacing $\mathcal{T}$ with $\mathcal{T}_g$ in (4.7). Then, Lemma 4.1 in the case $\kappa = 1$, yields a representation for the Lax curve (4.2) in the gas phase.

Riemann Solvers. The wave front tracking algorithm below is, as usual, based on the (possibly, approximate) solutions to Riemann problems.

Throughout, we fix a reference pressure $p_o > 0$. By Galileian invariance, in the statements below only speed differences will be relevant.

Lemma 4.2. *There exists a positive* $\bar\delta$ *such that for all* $\kappa \in \left]0,1\right]$ *and for any couple of states* (p^l, v^l), (p^r, v^r) *with* $\left|p^l - p_o\right| + |p^r - p_o| < \bar\delta$ *and* $\left|v^r - v^l\right| < \kappa\,\bar\delta$, *there exists a unique state* (p^m, v^m) *satisfying*

$$V_1^\kappa(p^m; p^l, v^l) = v^m \qquad \text{and} \qquad V_2^\kappa(p^r; p^m, v^m) = v^r\,.$$

Moreover,

$$\left|p^l - p^m\right| + |p^m - p^r| \le \mathcal{O}(1)\left(\left|p^l - p^r\right| + \frac{\left|v^l - v^r\right|}{\kappa}\right). \tag{4.8}$$

A qualitative justification of (4.8) is provided in Figure 4. In the liquid region, the Lax curves have a slope of order κ (see Lemma 4.1), hence a jump Δv in the velocity generates waves of order $\Delta v/\kappa$.

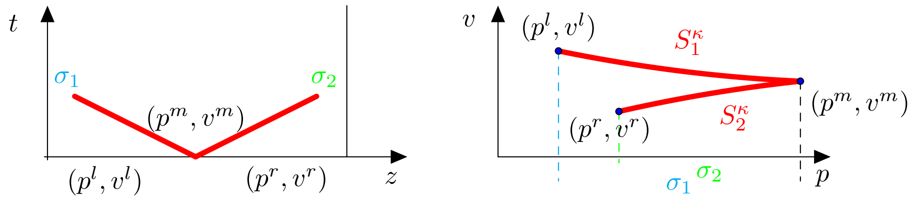

Figure 4: Riemann problem in the liquid. Left, in the (t, z) plane and, right, in the (p, v) plane: $|\sigma_1| + |\sigma_2| = \mathcal{O}(1)\left(|p^l - p^r| + |v^l - v^r|/\kappa\right)$.

Proof. Let $\xi = (v^r - v^l)/\kappa$. We apply the Implicit Function Theorem to $G(p^m, p^l, p^r, \xi) = 0$ where

$$G(p^m, p^l, p^r, \xi) = \xi + (p^m - p^l)\, F\left(\Pi_\kappa(p^m), \Pi_\kappa(p^l)\right) - (p^r - p^m)\, F\left(\Pi_\kappa(p^m), \Pi_\kappa(p^r)\right)$$

to find p^m as a function of (p^l, p^r, ξ), which is possible since the derivative $\partial_{p^m} G$ evaluated at $p^m = p^l = p^r = p_o$ and $\xi = 0$ is

$$\begin{aligned} \partial_{p^m} G(p_o, p_o, p_o, 0) &= 2F\left(\Pi_\kappa(p_o), \Pi_\kappa(p_o)\right) \\ &= 2\sqrt{-\mathcal{T}'\left(\bar p + \kappa^2 (p_o - \bar p)\right)} \ge 2\sqrt{-\mathcal{T}'\left(\max\{\bar p, p_o\}\right)} > 0 \end{aligned}$$

Note also that, in a neighborhood of $(p_o, p_o, p_o, 0)$, all second derivatives of G are bounded uniformly in κ, hence the domain of the implicit function contains a neighborhood of $(p_o, p_o, p_o, 0)$ independent of κ. Finally, v^m can be computed as $v^m = v^l - \kappa\,(p^m - p^l)\, F\left(\Pi_\kappa(p^m), \Pi_\kappa(p^l)\right)$, by Lemma 4.1.

Finally, (4.8) follows from $G\left(p^l, p^l, p^l, 0\right) = G\left(p^r, p^r, p^r, 0\right) = 0$ and the Lipschitz continuity of the implicit function. □

Note that Lemma 4.2 in the case $\kappa = 1$ covers the case of Riemann problems in the gas phase and slightly improves [4, Chapter 5].

The next Lemma refers to the Riemann problem between the gas, on the left, and the liquid, on the right. The symmetric situation is entirely similar.

Lemma 4.3. *There exits a positive* $\bar\delta$ *such that for all* $\kappa \in \left]0,1\right]$ *and for any couple of states* (p^l, v^l), (p^r, v^r) *with* $\left|p^l - p_o\right| + |p^r - p_o| < \bar\delta$ *and* $\left|v^r - v^l\right| < \bar\delta$, *there exists a unique state* (p^m, v^m) *satisfying*

$$V_1^g(p^m; p^l, v^l) = v^m \qquad \text{and} \qquad V_2^\kappa(p^r; p^m, v^m) = v^r\,.$$

Moreover,

$$\begin{aligned}\left|p^m - p^l\right| &= \mathcal{O}(1)\left(\kappa\left|p^r - p^l\right| + \left|v^l - v^r\right|\right) &(4.9)\\ \left|p^m - p^r\right| &= \mathcal{O}(1)\left(\left|p^r - p^l\right| + \left|v^l - v^r\right|\right) &(4.10)\\ \frac{1}{\kappa}\left|v^m - v^r\right| &= \mathcal{O}(1)\left(\left|p^r - p^l\right| + \left|v^l - v^r\right|\right) &(4.11)\end{aligned}$$

Proof. Let $\xi = v^r - v^l$. We apply the Implicit Function Theorem to $G(p^m, p^l, p^r, \xi) = 0$ where

$$G(p^m, p^l, p^r, \xi) = \xi + (p^m - p^l)\, F_g(p^m, p^l) - \kappa\,(p^r - p^m)\, F\left(\Pi_\kappa(p^m), \Pi_\kappa(p^r)\right)$$

to find p^m as a function of (p^l, p^r, ξ), which is possible since the derivative $\partial_{p^m} G$ evaluated at $p^m = p^l = p^r = p_o$ and $\xi = 0$ is

$$\partial_{p^m} G(p_o, p_o, p_o, 0) = \sqrt{-\mathcal{T}_g'(p_o)} + \kappa\, F\left(\Pi_\kappa\,(p_o)\,, \Pi_\kappa\,(p_o)\right) \geq \sqrt{-\mathcal{T}_g'(p_o)} > 0\,.$$

Note also that, in a neighborhood of $(p_o, p_o, p_o, 0)$, all second derivatives of G are bounded uniformly in κ, hence the domain of the implicit function contains a neighborhood of $(p_o, p_o, p_o, 0)$ independent of κ. Moreover, v^m can be computed as $v^m = v^l - (p^m - p^l)\, F_g(p^m, p^l)$, by Lemma 4.1. Concerning the latter estimates, use $G(p^m, p^l, p^r, v^r - v^l) = 0$ to obtain

$$p^m - p^l = \frac{v^l - v^r + \kappa\,(p^r - p^l) F\left(\Pi_\kappa(p^m), \Pi_\kappa(p^r)\right)}{F_g(p^m, p^l) + \kappa\, F\left(\Pi_\kappa(p^m), \Pi_\kappa(p^r)\right)}$$

which implies (4.9) and, together with the simple inequality $\left|p^m - p^r\right| \leq \left|p^m - p^l\right| + \left|p^l - p^r\right|$ also proves (4.10). Finally, the equality $v^r - v^m = \kappa\,(p^r - p^m)\, F\left(\Pi_\kappa(p^m), \Pi_\kappa(p^r)\right)$, together with (4.10), proves (4.11). □

Definition of the Algorithm. We modify the standard construction of the wave front tracking algorithm, see for instance [4, Chapter 4].

First, we identify the state u by means of the pair (p, v). Indeed, we choose to parametrize the Lax curves as in (4.2)–(4.4) and, hence, the waves' sizes are measured through the pressure difference σ between the two states on the sides of the wave.

Second, we introduce two strips around the two interfaces $z = 0$ and $z = m$, where all 1-waves have speed -1 and all 2-waves have speed 1. This, together with [1, Lemma 2.5], allows to avoid the introduction of non-physical waves, significantly simplifying the whole procedure.

We consider a representative of the initial datum $\tilde{u} \in \left(\mathbf{BV} \cap \mathbf{L^1}\right)\left(\mathbb{R}, \mathbb{R}^+ \times \mathbb{R}\right)$ such that $\tilde{u}(0+) = \tilde{u}(0)$, $\tilde{u}(m-) = \tilde{u}(m)$. Fix $\varepsilon > 0$. We approximate the initial datum $\tilde{u}$ by a sequence $\tilde{u}^\varepsilon$ of piecewise constant initial data with a finite number of discontinuities such that:

$$\begin{array}{rclrcll} \mathrm{TV}(\tilde{p}^\varepsilon) & \leq & \mathrm{TV}(\tilde{p})\,, & \left\|\tilde{u}^\varepsilon - \tilde{u}\right\|_{\mathbf{L^1}} & \leq & \varepsilon\,, & \\ \mathrm{TV}(\tilde{v}^\varepsilon; \mathcal{G}) & \leq & \mathrm{TV}(\tilde{v}; \mathcal{G})\,, & \tilde{u}^\varepsilon(z) & = & \tilde{u}(0) & \text{for all } z \in [-2\varepsilon^2, 2\varepsilon^2]\,, \\ \mathrm{TV}(\tilde{v}^\varepsilon; \mathcal{L}) & \leq & \mathrm{TV}(\tilde{v}; \mathcal{L})\,, & u^o_\varepsilon(z) & = & u^o(m) & \text{for all } z \in [m - 2\varepsilon^2, m + 2\varepsilon^2]\,. \end{array} \quad (4.12)$$

Observe that a possible jump at the interfaces $z = 0$ and $z = m$ is assigned to the gas region. At each point of jump in the approximate initial datum, we solve the corresponding Riemann problem. As usual, see [4, Chapter 4], we approximate each rarefaction wave by a rarefaction fan consisting of ε-wavelets, each with strength less than ε and traveling with the characteristic speed of the state to its left. On the other hand, each shock wave is assigned its exact Rankine-Hugoniot speed. Similarly to what happens in the usual case, there exists a constant $\delta_o > 0$ such that each of the above Riemann problems has an approximate solution as long as $\mathrm{TV}(\tilde{u}) < \delta_o$. We introduce two strips around the two interfaces $z = 0$ and $z = m$, where all 1-waves have speed -1 and all 2-waves have speed $+1$:

$$\mathcal{I}_\varepsilon^- = [-\varepsilon^2, \varepsilon^2] \times \mathbb{R}^+ \quad \text{and} \quad \mathcal{I}_\varepsilon^+ = [m - \varepsilon^2, m + \varepsilon^2] \times \mathbb{R}^+ .$$

Hence, assign to all 1-waves entering $\mathcal{I}_\varepsilon^- \cup \mathcal{I}_\varepsilon^+$ speed -1, while all 2-waves entering $\mathcal{I}_\varepsilon^- \cup \mathcal{I}_\varepsilon^+$ are given speed $+1$, see Figure 5.

Remark that the actual values attained by the approximate solution are not changed, only the wave speeds are modified. When exiting these strips, every wave is given back its correct speed. By this trick, no interaction among waves of the same family may take place in either of the two strips. This construction can be extended up to the first time t_1 at which two waves interact, or a wave hits one of the interfaces. At time t_1, the so constructed approximate solution is piecewise constant with a finite number of discontinuities. Any such interaction gives rise to a new Riemann problem solved as at time $t = 0$, if the interaction is in the interior of the two phases, or as described in Lemma 4.3, whenever the interaction is along an interface.

Any rarefaction wave, once arisen, is not further split even if its strength exceeds the threshold ε after subsequent interactions, with other waves or with the phase boundaries. The new rarefaction waves that may arise at the interfaces are split, if their strength exceeds ε, when they exit the strips $\mathcal{I}_\varepsilon^\pm$, since inside the strips they all travel with the same speed. We can thus iterate the previous construction at any subsequent interaction, provided suitable upper bounds on the total variation of the approximate solutions are available. As it is usual in this context, see [4, Chapter 7], we may assume that no more than 2 waves interact at any interaction point, or that no interaction happens at the boundaries of the two strips, thanks to a small modification of the speed of waves outside the strips, where necessary.

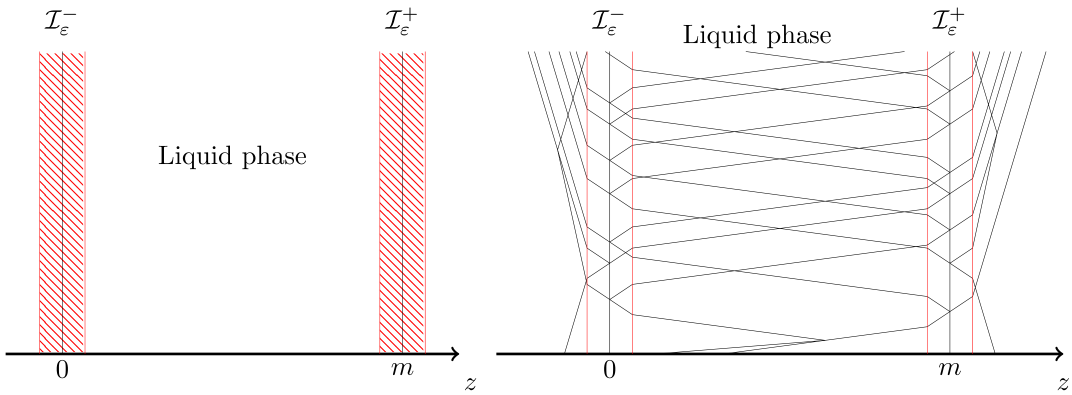

Figure 5: Left, the dashed regions are the strips $\mathcal{I}_\varepsilon^\pm$ surrounding the liquid phase. Right, modification to the usual wave front tracking algorithm: waves in the strips $\mathcal{I}_\varepsilon^-$ and $\mathcal{I}_\varepsilon^+$ are assign speed 1, if belonging to the first family, and -1, if of the second family.

Interaction Estimates. We recall the classical Glimm interaction estimates, see [4, Chapter 7, formulæ (7.31)–(7.32)], which hold for any smooth parametrization of the Lax curves:

$$\begin{array}{rcll} \left|\sigma_1^+ - \sigma_1^-\right| + \left|\sigma_2^+ - \sigma_2^-\right| & \leq & \mathcal{O}(1)\left|\sigma_1^-\sigma_2^-\right| & \text{(Figure 6, left)}, \\ \left|\sigma_1^+\right| + \left|\sigma_2^+ - (\sigma' + \sigma'')\right| & \leq & \mathcal{O}(1)\left|\sigma'\sigma''\right| & \text{(Figure 6, middle)}, \\ \left|\sigma_1^+ - (\sigma' + \sigma'')\right| + \left|\sigma_2^+\right| & \leq & \mathcal{O}(1)\left|\sigma'\sigma''\right| & \text{(Figure 6, right)}, \end{array} \tag{4.13}$$

where we used the notation described in Figure 6.

Aiming at the convergence result, we need more careful interaction estimates in the liquid phase. More precisely, we seek bounds on the constant $\mathcal{O}(1)$ above that allow to control its

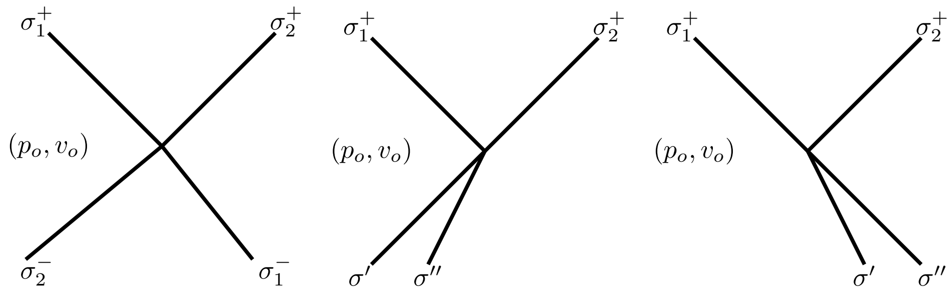

Figure 6: Left, an interaction between waves of different families. Center, an interaction between waves of the second family. Right, an interaction between waves of the first family.

dependence on κ. Remark that the choice of parametrizing Lax curves by means of pressure differences plays a key role in this improvement.

Lemma 4.4. *There exists a $\bar{\delta} > 0$ such that if the interacting waves in Figure 6 hit each other in $\mathcal{L}$ and all have sizes less than $\bar{\delta}$, then, the following estimates hold:*

$$\begin{array}{rcl} \left|\sigma_1^+ - \sigma_1^-\right| + \left|\sigma_2^+ - \sigma_2^-\right| & \leq & \mathcal{O}(1)\, \kappa^2 \left|\sigma_1^- \sigma_2^-\right| \\ \left|\sigma_1^+\right| + \left|\sigma_2^+ - (\sigma' + \sigma'')\right| & \leq & \mathcal{O}(1)\, \kappa^2 \left|\sigma' \sigma''\right| \\ \left|\sigma_1^+ - (\sigma' + \sigma'')\right| + \left|\sigma_2^+\right| & \leq & \mathcal{O}(1)\, \kappa^2 \left|\sigma' \sigma''\right| \end{array} \tag{4.14}$$

Proof. Consider first the case of interacting waves of different families, see Figure 6, left. Then, with straightforward computations, Lemma 4.1 leads to

$$G(\sigma_1^+, \sigma_2^+, \sigma_1^-, \sigma_2^-) = 0 \tag{4.15}$$

where

$$\begin{array}{rcl} G_1(\sigma_1^+, \sigma_2^+, \sigma_1^-, \sigma_2^-) & = & \sigma_1^+ + \sigma_2^+ - \sigma_1^- - \sigma_2^- \\ G_2(\sigma_1^+, \sigma_2^+, \sigma_1^-, \sigma_2^-) & = & \sigma_1^+ F\left(\Pi_\kappa(p_o + \sigma_1^+), \Pi_\kappa(p_o)\right) - \sigma_2^+ F\left(\Pi_\kappa(p_o + \sigma_1^+), \Pi_\kappa(p_o + \sigma_1^+ + \sigma_2^+)\right) \\ & & -\sigma_1^- F\left(\Pi_\kappa(p_o + \sigma_1^- + \sigma_2^-), \Pi_\kappa(p_o + \sigma_2^-)\right) + \sigma_2^- F\left(\Pi_\kappa(p_o), \Pi_\kappa(p_o + \sigma_2^-)\right) \end{array}$$

Note that by 1. and 2. in Lemma 4.1, the function G is of class $\mathbf{C^2}$ and since $\Pi'_\kappa(p) = \kappa^2$ one can compute

$$\left\| D^2 G(\sigma_1^+, \sigma_2^+, \sigma_1^-, \sigma_2^-) \right\|_{\mathbf{L}^\infty} = \mathcal{O}(1)\, \kappa^2 \,. \tag{4.16}$$

Moreover, $G(0,0,0,0) = (0,0)$ and by direct computations, the Jacobian Matrix of G with respect to σ_1^+ and σ_2^+ computed at $(0,0,0,0)$ is

$$\partial_{(\sigma_1^+, \sigma_2^+)} G(0,0,0,0) = \left[\begin{array}{cc} 1 & 1 \\ \sqrt{-\mathcal{T}'\left(\Pi_\kappa(p_o)\right)} & -\sqrt{-\mathcal{T}'\left(\Pi_\kappa(p_o)\right)} \end{array} \right].$$

$$\left| \det \partial_{(\sigma_1^+, \sigma_2^+)} G(0,0,0,0) \right| = 2\sqrt{-\mathcal{T}'\left(\bar{p} + \kappa^2 (p_o - \bar{p})\right)} \geq 2\sqrt{-\mathcal{T}'\left(\max\{\bar{p}, p_o\}\right)} > 0$$

Hence, the Implicit Function Theorem ensures that (4.15) uniquely defines a map Σ^κ of class $\mathbf{C^2}$ such that (4.15) is equivalent to

$$(\sigma_1^+, \sigma_2^+) = \Sigma^\kappa(\sigma_1^-, \sigma_2^-)$$

for all (σ_1^-, σ_2^-) in a neighborhood of $(0,0)$ which can be chosen independently of κ. Moreover, by (4.16),

$$\left\| D^2 \Sigma^\kappa(\sigma_1^-, \sigma_2^-) \right\|_{\mathbf{L}^\infty} \leq \mathcal{O}(1)\, \kappa^2 \,. \tag{4.17}$$

By construction, the following equalities are immediate:

$$\Sigma^\kappa(\sigma_1, 0) = (\sigma_1, 0)\,, \qquad \Sigma^\kappa(0, \sigma_2) = (0, \sigma_2)\,.$$

Using (4.17), compute now

$$\begin{aligned}
\left|\Sigma_1^\kappa(\sigma_1^-, \sigma_2^-) - \sigma_1^-\right| &= \left|\int_0^1 \partial_{\sigma_2^-} \Sigma_1^\kappa(\sigma_1^-, \vartheta\sigma_2^-)\, \mathrm{d}\vartheta\right| \left|\sigma_2^-\right| \\
&= \left|\int_0^1 \left(\partial_{\sigma_2^-} \Sigma_1^\kappa(\sigma_1^-, \vartheta\sigma_2^-) - \partial_{\sigma_2^-} \Sigma_1^\kappa(0, \vartheta\sigma_2^-)\right) \mathrm{d}\vartheta\right| \left|\sigma_2^-\right| \\
&= \left|\int_0^1 \int_0^1 \partial^2_{\sigma_1^-\sigma_2^-} \Sigma_1^\kappa(\vartheta'\sigma_1^-, \vartheta\sigma_2^-)\, \mathrm{d}\vartheta'\, \mathrm{d}\vartheta\right| \left|\sigma_1^-\, \sigma_2^-\right| \\
&= \mathcal{O}(1)\, \kappa^2 \left|\sigma_1^-\, \sigma_2^-\right| .
\end{aligned}$$

We now consider the second estimate in (4.14), corresponding to the case of interacting waves both belonging to the second family. With the notation in Figure 6, middle, we have

$$G(\sigma_1^+, \sigma_2^+, \sigma', \sigma'') = 0 \tag{4.18}$$

where now

$$\begin{aligned}
G_1(\sigma_1^+, \sigma_2^+, \sigma', \sigma'') &= \sigma_1^+ + \sigma_2^+ - \sigma' - \sigma''\,, \\
G_2(\sigma_1^+, \sigma_2^+, \sigma', \sigma'') &= \sigma_1^+\, F\left(\Pi_\kappa(p_o + \sigma_1^+), \Pi_\kappa(p_o)\right) - \sigma_2^+\, F\left(\Pi_\kappa(p_o + \sigma_1^+), \Pi_\kappa(p_o + \sigma_1^+ + \sigma_2^+)\right) \\
&\quad + \sigma' F\left(\Pi_\kappa(p_o), \Pi_\kappa(p_o + \sigma')\right) + \sigma'' F\left(\Pi_\kappa(p_o + \sigma'), \Pi_\kappa(p_o + \sigma' + \sigma'')\right) .
\end{aligned}$$

Note that by 1, 2 in Lemma 4.1, the function G is of class $\mathbf{C^2}$ and since $\Pi'_\kappa(p) = \kappa^2$ one can compute again

$$\left\| D^2 G(\sigma_1^+, \sigma_2^+, \sigma', \sigma'') \right\|_{\mathbf{L}^\infty} = \mathcal{O}(1)\, \kappa^2 \,. \tag{4.19}$$

Moreover, $G(0,0,0,0) = (0,0)$ and by direct computations, the Jacobian Matrix of G with respect to σ_1^+ and σ_2^+ computed at $(0,0,0,0)$ is, as before,

$$D_{(\sigma_1^+, \sigma_2^+)} G(0,0,0,0) = \left[\begin{array}{cc} 1 & 1 \\ \sqrt{-\mathcal{T}'\left(\Pi_\kappa(p_o)\right)} & -\sqrt{-\mathcal{T}'\left(\Pi_\kappa(p_o)\right)} \end{array}\right].$$

Hence, as before, the Implicit Function Theorem ensures that (4.18) uniquely defines a map Σ^κ of class $\mathbf{C^2}$ such that (4.18) is equivalent to

$$(\sigma_1^+, \sigma_2^+) = \Sigma^\kappa(\sigma', \sigma'')$$

for all (σ', σ'') in a neighborhood of $(0,0)$ which can be chosen independently of κ. Moreover.

$$\left\| D^2 \Sigma^\kappa(\sigma', \sigma'') \right\|_{\mathbf{L}^\infty} \leq \mathcal{O}(1)\, \kappa^2 \,. \tag{4.20}$$

By construction, the following equalities are immediate:

$$\Sigma^\kappa(\sigma', 0) = (0, \sigma')\,, \qquad \Sigma^\kappa(0, \sigma'') = (0, \sigma'')\,.$$

so that, using (4.20)

$$
\begin{aligned}
& \left\|\Sigma^\kappa(\sigma',\sigma'') - (0,\sigma'+\sigma'')\right\| \\
= & \left\| \left(\Sigma^\kappa(\sigma',\sigma'') - (0,\sigma'+\sigma'')\right) - \left(\Sigma^\kappa(\sigma',0) - (0,\sigma')\right)\right\| \\
= & \left\| \sigma'' \int_0^1 \left(\partial_{\sigma''}\Sigma^\kappa(\sigma',\vartheta\sigma'') - (0,1)\right) \mathrm{d}\vartheta \right\| \\
= & \left\| \sigma'' \int_0^1 \left(\left(\partial_{\sigma''}\Sigma^\kappa(\sigma',\vartheta\sigma'') - (0,1)\right) - \left(\partial_{\sigma''}\Sigma^\kappa(0,\vartheta\sigma'') - (0,1)\right)\right) \mathrm{d}\vartheta \right\| \\
= & \left|\sigma'\sigma''\right| \left\| \int_0^1 \int_0^1 \partial^2_{\sigma'\sigma''}\Sigma^\kappa(\vartheta'\sigma',\vartheta\sigma'')\,\mathrm{d}\vartheta\,\mathrm{d}\vartheta' \right\| \\
= & \mathcal{O}(1)\ \kappa^2 \left|\sigma'\,\sigma''\right| ,
\end{aligned}
$$

completing the proof of the second estimate in (4.14). The case of two interacting waves both belonging to the first family in Figure 6, right, is entirely similar. □

The estimates on the waves' sizes in the case of interactions involving the interfaces are as follows.

Lemma 4.5. *There exist positive $\bar\delta$, c and $\kappa_* < 1$ such that, if all the interacting waves in Figure 7 have strength less than $\bar\delta$, then the following estimates hold:*

$$
\begin{array}{rcl}
\left|\sigma_1^+\right| & \leq & \mathcal{O}(1)\ \kappa \left|\sigma_1^-\right| + \left(1+\mathcal{O}(1)\,(\kappa+\bar\delta)\right)\left|\sigma_2^-\right| \\
\left|\sigma_2^+\right| & \leq & (1-c\,\kappa)\left|\sigma_1^-\right| + \qquad \left(2+\mathcal{O}(1)\,\bar\delta\right)\left|\sigma_2^-\right|
\end{array}
\tag{4.21}
$$

uniformly for all $\kappa \in \left]0,\kappa_*\right[$. Moreover:

$$
\sigma_1^+ + \sigma_2^+ = \sigma_1^- + \sigma_2^- \qquad \text{and} \qquad
\left\{
\begin{array}{l}
\sigma_1^- = 0 \quad \Rightarrow \quad \left\{ \begin{array}{l} \left|\sigma_2^+\right| - \left|\sigma_1^+\right| = \left|\sigma_2^-\right|, \\ \sigma_2^-\,\sigma_2^+ \geq 0 \text{ and } \sigma_2^-\,\sigma_1^+ \leq 0 . \end{array}\right. \\[3ex]
\sigma_2^- = 0 \quad \Rightarrow \quad \left\{ \begin{array}{l} \left|\sigma_1^+\right| + \left|\sigma_2^+\right| = \left|\sigma_1^-\right|, \\ \sigma_1^-\,\sigma_1^+ \geq 0 \text{ and } \sigma_1^-\,\sigma_2^+ \geq 0 . \end{array}\right.
\end{array}
\right.
\tag{4.22}
$$

Proof. In the present case, we have

$$
G(\sigma_1^+,\sigma_2^+,\sigma_1^-,\sigma_2^-) = 0 \tag{4.23}
$$

where

$$
\begin{aligned}
G_1(\sigma_1^+,\sigma_2^+,\sigma_1^-,\sigma_2^-) &= \sigma_1^+ + \sigma_2^+ - \sigma_1^- - \sigma_2^- \\
G_2(\sigma_1^+,\sigma_2^+,\sigma_1^-,\sigma_2^-) &= \sigma_2^-\,F_g(p_o,p_o+\sigma_2^-) - \kappa\,\sigma_1^-\,F\left(\Pi_\kappa(p_o+\sigma_1^-+\sigma_2^-),\Pi_\kappa(p_o+\sigma_2^-)\right) \\
& \quad +\sigma_1^+\,F_g(p_o+\sigma_1^+,p_o) - \kappa\,\sigma_2^+\,F\left(\Pi_\kappa(p_o+\sigma_1^+),\Pi_\kappa(p_o+\sigma_1^++\sigma_2^+)\right)
\end{aligned}
$$

with $G(0,0,0,0) = 0$ and Jacobian matrix

$$
\partial_{(\sigma_1^+,\sigma_2^+)}G(0,0,0,0) = \left[\begin{array}{cc} 1 & 1 \\ \sqrt{-\mathcal{T}_g'(p_o)} & -\kappa\sqrt{-\mathcal{T}'\left(\Pi_\kappa(p_o)\right)} \end{array}\right]
$$

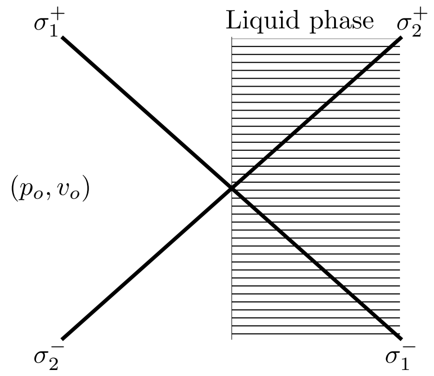

Figure 7: Notation for the proof of Lemma 4.5: σ_1^-, coming from the liquid phase, and σ_2^-, coming from the gas phase, hit against the phase boundary generating σ_2^+ in the liquid phase and σ_1^+ in the gas phase.

$$\left|\det \partial_{(\sigma_1^+,\sigma_2^+)} G(0,0,0,0)\right| = \sqrt{-\mathcal{T}_g'(p_o)} + \kappa\sqrt{-\mathcal{T}'\left(\Pi_\kappa(p_o)\right)} \geq \sqrt{-\mathcal{T}_g'(p_o)} > 0$$

and moreover

$$D^2 G(\sigma_1^+, \sigma_2^+, \sigma_1^-, \sigma_2^-) = \mathcal{O}(1)$$

uniformly in κ, which allows to apply the Implicit Function Theorem in the same neighborhood of radius $\bar{\delta}$ for all small κ, yielding a map $\Sigma^\kappa(\sigma_1^-, \sigma_2^-) = \left(\sigma_1^+, \sigma_2^+\right)$ such that

$$D^2\Sigma^\kappa(\sigma_1^-, \sigma_2^-) = \mathcal{O}(1) \tag{4.24}$$

locally in (σ_1^-, σ_2^-) and uniformly in κ. Moreover,

$$\begin{aligned}
& D\Sigma^\kappa(0,0) \\
= & -\left[D_{(\sigma_1^+,\sigma_2^+)} G(0,0,0,0)\right]^{-1} D_{(\sigma_1^-,\sigma_2^-)} G(0,0,0,0) \\
= & \frac{1}{\sqrt{-\mathcal{T}_g'(p_o)}+\kappa\sqrt{-\mathcal{T}'\left(\Pi_\kappa(p_o)\right)}} \begin{bmatrix} \kappa\sqrt{-\mathcal{T}'\left(\Pi_\kappa(p_o)\right)} & 1 \\ \sqrt{-\mathcal{T}_g'(p_o)} & -1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ \kappa\sqrt{-\mathcal{T}'\left(\Pi_\kappa(p_o)\right)} & -\sqrt{-\mathcal{T}_g'(p_o)} \end{bmatrix} \\
= & \frac{1}{\sqrt{-\mathcal{T}_g'(p_o)}+\kappa\sqrt{-\mathcal{T}'\left(\Pi_\kappa(p_o)\right)}} \begin{bmatrix} 2\,\kappa\sqrt{-\mathcal{T}'\left(\Pi_\kappa(p_o)\right)} & -\sqrt{-\mathcal{T}_g'(p_o)} + \kappa\sqrt{-\mathcal{T}'\left(\Pi_\kappa(p_o)\right)} \\ \sqrt{-\mathcal{T}_g'(p_o)} - \kappa\sqrt{-\mathcal{T}'\left(\Pi_\kappa(p_o)\right)} & 2\sqrt{-\mathcal{T}_g'(p_o)} \end{bmatrix}
\end{aligned}$$

which shows that the following bound $D\Sigma^\kappa(0,0) = \mathcal{O}(1)$ hold uniformly in κ. This, together with (4.24), implies

$$D\Sigma^\kappa(\sigma_1^-, \sigma_2^-) = \mathcal{O}(1)\ ,$$

so that

$$\Sigma^\kappa(\sigma_1^-, \sigma_2^-) = \mathcal{O}(1)\left(\left|\sigma_1^-\right| + \left|\sigma_2^-\right|\right) \tag{4.25}$$

since $\Sigma^\kappa(0,0) = 0$. Solve now $G_2(\sigma_1^+, \sigma_2^+, \sigma_1^-, \sigma_2^-) = 0$ for σ_1^+, use the bound (4.25) and the estimates $\left|\sigma_1^-\right|, \left|\sigma_2^-\right| < \bar{\delta}$ to obtain:

$$\sigma_1^+ \quad = \quad -\frac{F_g(p_o, p_o+\sigma_2^-)}{F_g(p_o+\sigma_1^+, p_o)}\sigma_2^- + \frac{F\left(\Pi_\kappa(p_o+\sigma_1^-+\sigma_2^-), \Pi_\kappa(p_o+\sigma_2^-)\right)}{F_g(p_o+\sigma_1^+, p_o)}\kappa\,\sigma_1^- \tag{4.26}$$

$$
+\frac{F\left(\Pi_\kappa(p_o+\sigma_1^+),\Pi_\kappa(p_o+\sigma_1^++\sigma_2^+)\right)}{F_g(p_o+\sigma_1^+,p_o)}\,\kappa\,\sigma_2^+ \tag{4.27}
$$

$$
\begin{aligned}
&\leq \left(1+\mathcal{O}(1)\,\bar\delta\right)\left|\sigma_2^-\right| + \mathcal{O}(1)\ \kappa\left|\sigma_1^-\right| + \mathcal{O}(1)\ \kappa\left(\left|\sigma_1^-\right|+\left|\sigma_2^-\right|\right)\\
&= \mathcal{O}(1)\ \kappa\left|\sigma_1^-\right| + \left(1+\mathcal{O}(1)\,(\kappa+\bar\delta)\right)\left|\sigma_2^-\right|
\end{aligned}
$$

which gives the first estimate in (4.21). To obtain the second one, use $G_1(\sigma_1^+,\sigma_2^+,\sigma_1^-,\sigma_2^-)=0$ and (4.26)–(4.27):

$$
\left(1+\kappa\,\frac{F\left(\Pi_\kappa(p_o+\sigma_1^+),\Pi_\kappa(p_o+\sigma_1^++\sigma_2^+)\right)}{F_g(p_o+\sigma_1^+,p_o)}\right)\sigma_2^+ \tag{4.28}
$$

$$
= \left(1-\kappa\,\frac{F\left(\Pi_\kappa(p_o+\sigma_1^-+\sigma_2^-),\Pi_\kappa(p_o+\sigma_1^-)\right)}{F_g(p_o+\sigma_1^+,p_o)}\right)\sigma_1^- + \left(1+\frac{F_g(p_o,p_o+\sigma_2^-)}{F_g(p_o+\sigma_1^+,p_o)}\right)\sigma_2^- \tag{4.29}
$$

which implies the second in (4.21), since for a suitable $c>0$,

$$
\begin{aligned}
\frac{F\left(\Pi_\kappa(p_o+\sigma_1^+),\Pi_\kappa(p_o+\sigma_1^++\sigma_2^+)\right)}{F_g(p_o+\sigma_1^+,p_o)} &\geq c\,,\\
\frac{F\left(\Pi_\kappa(p_o+\sigma_1^-+\sigma_2^-),\Pi_\kappa(p_o+\sigma_1^-)\right)}{F_g(p_o+\sigma_1^+,p_o)} &\geq c\,,\\
\frac{F_g(p_o,p_o+\sigma_2^-)}{F_g(p_o+\sigma_1^+,p_o)} &\leq 1+\mathcal{O}(1)\ \bar\delta\,.
\end{aligned}
$$

To prove (4.22), note that in the case $\sigma_1^-=0$, (4.28)–(4.29) imply that σ_2^+ and σ_2^- have the same sign. On the other hand, by (4.26)–(4.27)

$$
\sigma_1^+ = \left(-\frac{F(p_o,p_o+\sigma_2^-)}{F(p_o+\sigma_1^+,p_o)}+\mathcal{O}(1)\ \kappa\right)\sigma_2^-
$$

so that σ_1^+ and σ_2^- have different signs whenever κ is sufficiently small, proving the first equality on the right in (4.22).

Assume now that $\sigma_2^-=0$, so that $\sigma_1^++\sigma_2^+=\sigma_1^-$. By (4.28)–(4.29), σ_1^- and σ_2^+ have the same sign for κ small. The second inequality in (4.21) then ensures that $\left|\sigma_2^+\right|<\left|\sigma_1^-\right|$ and hence also σ_1^+ has the same sign of σ_1^- and σ_2^+. □

Remark that a wave refracted at the phase boundary remains of the same type, whereas the reflected wave changes type when it comes from the liquid and remains of the same type when it comes from the gas, see Table 1 and (4.22).

At any fixed positive time t, the approximate solution is a piecewise constant function $u^\varepsilon(t)=\sum_\alpha u_\alpha\,\chi_{[z_\alpha,z_{\alpha+1}[}$. If t is not an interaction time, we denote by σ_α the size of the wave supported at z_α and introduce the potentials

$$
\begin{aligned}
&V_{\mathcal{G}_{in}} = \sum_{\alpha\in\mathcal{G}_{in}}|\sigma_\alpha| \qquad V_{\mathcal{G}_{out}} = \sum_{\alpha\in\mathcal{G}_{out}}|\sigma_\alpha| \qquad V_{\mathcal{L}} = \sum_{\alpha\in\mathscr{L}}|\sigma_\alpha|\\
&Q_{\mathcal{G}} = \sum_{(\alpha,\beta)\in\mathcal{A}_{\mathcal{G}}}\left|\sigma_\alpha\,\sigma_\beta\right| \qquad Q_{\mathcal{L}} = \sum_{(\alpha,\beta)\in\mathcal{A}_{\mathcal{L}}}\left|\sigma_\alpha\,\sigma_\beta\right|\\
&\Upsilon = K_{in}\,V_{\mathcal{G}_{in}} + V_{\mathcal{G}_{out}} + K_{\mathcal{L}}\,V_{\mathcal{L}} + H_{\mathcal{G}}\,Q_{\mathcal{G}} + \kappa^2\,H_{\mathcal{L}}\,Q_{\mathcal{L}}\,,
\end{aligned} \tag{4.30}
$$

where $K_{in}, K_{\mathcal{L}}, H_{\mathcal{G}}$ and $H_{\mathcal{L}}$ are constants independent of κ to be precisely defined below. Above, we denoted

$\mathcal{G}_{in}$	2-waves supported in $]-\infty,0[$ and 1-waves supported in $]m,+\infty[$.
$\mathcal{G}_{out}$	1-waves supported in $]-\infty,0[$ and 2-waves supported in $]m,+\infty[$.
$\mathscr{L}$	all waves supported in the liquid phase $\mathcal{L}$.
$\mathcal{A}_{\mathcal{G}}$	pairs of approaching waves supported in the gas phase.
$\mathcal{A}_{\mathcal{L}}$	pairs of approaching waves supported in the liquid phase.

Here, we define as *approaching* two waves both supported in the same interval $]-\infty,0[$, $]0,m[$ or $]m,+\infty[$, either of the same family and when one of the two is a shock, or of different families with the one of the first family on the right.

Lemma 4.6. *There exist weights $K_{in}, K_{\mathcal{L}}, H_{\mathcal{G}}$ and $H_{\mathcal{L}}$, all greater than 1, $\kappa_* \in\,]0,1[$ and a positive $\bar\delta$ such that, for all $\kappa \in\,]0,\kappa_*[$ and piecewise constant initial data $\tilde u^\varepsilon$ with the corresponding approximate solution u^ε constructed by the algorithm above satisfying $\Upsilon(u^\varepsilon(0+)) < \bar\delta$, the function $t \to \Upsilon\left(u^\varepsilon(t)\right)$ is non increasing. Moreover, calling $\sigma_\alpha, \sigma_\beta$ the waves interacting at time $\bar t$ and point $\bar z$, with σ_α coming from the left, the following estimates hold:*

$$\begin{array}{lrcl} \bar z \in \mathcal{G} & \Delta\Upsilon & \le & -\left|\sigma_\alpha\,\sigma_\beta\right| \\ \bar z = 0 & \Delta\Upsilon & \le & -\left|\sigma_\alpha\right| - \kappa\left|\sigma_\beta\right| \\ \bar z \in \mathcal{L} & \Delta\Upsilon & \le & -\kappa^2\left|\sigma_\alpha\,\sigma_\beta\right| \\ \bar z = m & \Delta\Upsilon & \le & -\kappa\left|\sigma_\alpha\right| - \left|\sigma_\beta\right|. \end{array} \tag{4.31}$$

Proof. Denote by C, with $C > 1$, a positive constant bounding from above all $\mathcal{O}(1)$ appearing in (4.13), (4.14) and (4.21). Choose $\bar\delta > 0$ such that $\bar\delta < 1/(2C)$, and $\tilde u^\varepsilon$ such that $\Upsilon\left(u^\varepsilon(0+)\right) < \bar\delta$.

Suppose that at time $\bar t$ there is an interaction and that $\Upsilon\left(u^\varepsilon(\bar t-)\right) < \bar\delta$. Consider the different interactions separately. Begin with an interaction in $\mathcal{G}$, as in Figure 6, using (4.13) and definitions (4.30):

$$\begin{array}{rcl} \Delta V_{\mathcal{G}_{in}} & \le & C\left|\sigma_\alpha\,\sigma_\beta\right| \\ \Delta V_{\mathcal{G}_{out}} & \le & C\left|\sigma_\alpha\,\sigma_\beta\right| \\ \Delta V_{\mathcal{L}} & = & 0 \end{array} \qquad \begin{array}{rcl} \Delta Q_{\mathcal{G}} & \le & C\left|\sigma_\alpha\,\sigma_\beta\right|\bar\delta - \left|\sigma_\alpha\,\sigma_\beta\right| \le -\frac12\left|\sigma_\alpha\,\sigma_\beta\right| \\ \Delta Q_{\mathcal{L}} & = & 0 \\ \Delta\Upsilon & \le & (C\,K_{in} + C - \frac12\,H_{\mathcal{G}})\left|\sigma_\alpha\,\sigma_\beta\right|. \end{array}$$

Consider an interaction in the liquid phase, as in Figure 6, using (4.14) and definitions (4.30):

$$\begin{array}{rcl} \Delta V_{\mathcal{G}_{in}} & = & 0 \\ \Delta V_{\mathcal{G}_{out}} & = & 0 \\ \Delta V_{\mathcal{L}} & \le & C\,\kappa^2\left|\sigma_\alpha\,\sigma_\beta\right| \end{array} \qquad \begin{array}{rcl} \Delta Q_{\mathcal{G}} & = & 0 \\ \Delta Q_{\mathcal{L}} & \le & (C\,\kappa^2\,\bar\delta - 1)\left|\sigma_\alpha\,\sigma_\beta\right| \le -\frac12\left|\sigma_\alpha\,\sigma_\beta\right| \\ \Delta\Upsilon & \le & \kappa^2\,(C\,K_{\mathcal{L}} - \frac12\,H_{\mathcal{L}})\left|\sigma_\alpha\,\sigma_\beta\right| \end{array}$$

Consider now the case $\bar z = 0$, the case $\bar z = m$ being entirely analogous. By (4.21), for $\kappa + \bar\delta$ sufficiently small so that $C(\kappa + \bar\delta) < 1$, it follows, using definitions (4.30), that:

$$\begin{array}{rcl} \Delta V_{\mathcal{G}_{in}} & \le & -|\sigma_\alpha| \\ \Delta V_{\mathcal{G}_{out}} & \le & 2|\sigma_\alpha| + C\,\kappa\left|\sigma_\beta\right| \\ \Delta V_{\mathcal{L}} & \le & 3|\sigma_\alpha| - c\,\kappa\left|\sigma_\beta\right| \\ & & \end{array} \qquad \begin{array}{rcl} \Delta Q_{\mathcal{G}} & \le & 2\,\bar\delta\left|\sigma_\alpha\right| + C\,\kappa\,\bar\delta\left|\sigma_\beta\right| \\ \Delta Q_{\mathcal{L}} & \le & 3\,\bar\delta\left|\sigma_\alpha\right| + \bar\delta\left|\sigma_\beta\right| \\ \Delta\Upsilon & \le & \left[2 - K_{in} + 3K_{\mathcal{L}} + (2H_{\mathcal{G}} + 3\kappa^2 H_{\mathcal{L}})\bar\delta\right]\left|\sigma_\alpha\right| \\ & & +\kappa\left[C - c\,K_{\mathcal{L}} + (C\,H_{\mathcal{G}} + \kappa\,H_{\mathcal{L}})\bar\delta\right]\left|\sigma_\beta\right| \end{array}$$

To complete the proof, observe that choosing

1. $K_{\mathcal{L}}$ so that $C - c\,K_{\mathcal{L}} \le -2$;
2. K_{in} so that $2 - K_{in} + 3K_{\mathcal{L}} \le -2$;
3. $H_{\mathcal{G}}$ so that $C(1 + K_{in}) - \frac12\,H_{\mathcal{G}} \le -1$;
4. $H_{\mathcal{L}}$ so that $C\,K_{\mathcal{L}} - \frac12\,H_{\mathcal{L}} \le -1$;
5. $\bar\delta$ so that $(C\,H_{\mathcal{G}} + H_{\mathcal{L}})\bar\delta \le 1$ and $(2H_{\mathcal{G}} + 3H_{\mathcal{L}})\bar\delta \le 1$.

ensures that (4.31) holds. The proof is concluded by induction on the interaction times. □

Lemma 4.7. *With the algorithm defined above, if the piecewise constant initial datum $\tilde{u}^\varepsilon$ is chosen so that $\Upsilon(u^\varepsilon(0+)) < \bar{\delta}$, with $\bar{\delta}$ as in Lemma 4.6, ($u^\varepsilon(t)$ being the approximate solution constructed above) then there exists no cluster point of interaction points.*

Proof. By contradiction, call t_* the first time at which a cluster point (t_*, z_*) of interaction points appears.

First, assume that $z_* \neq 0$ and $z_* \neq m$. Call $\mathcal{U}$ a neighborhood of (t_*, z_*) not intersecting the interfaces $z \in \{0, m\}$. The interactions where there are more than one outgoing waves of the same family are those where

- two waves of the same family hit against each other originating a rarefaction fan of the other family of total size bigger than ε; and
- a wave hits an interface, resulting in a new reflected rarefaction larger than ε which is eventually split as it reaches the boundary of the strip.

Because of the estimates (4.13), (4.14), (4.21) and (4.31), at any of these interactions $\Delta\Upsilon \leq -\frac{\kappa}{C}\varepsilon$. Hence, these interactions may take place only a finite number of times. An application of [1, Lemma 2.5] contradicts the existence of (t_*, z_*).

Assume now $z_* = 0$, the case $z_* = m$ being entirely equivalent. For a small positive η, choose a trapezoid $\mathcal{N}_\eta$ contained in $\mathcal{I}_\varepsilon^-$ of the form

$$\mathcal{N}_\eta = \left\{ (t,z) \in \mathcal{I}^- : t \in \,]t_* - \eta, t_*[\text{ and } \left| \frac{z - z_*}{t - t_* - \eta} \right| \leq 2 \right\}.$$

By construction, finitely many waves cross the lower side of $\mathcal{N}_\eta$ and no wave may enter $\mathcal{N}_\eta$ along the two sides. Inside $\mathcal{N}_\eta$, any wave can generate another wave at most once, when it hits the interface $z = z_*$. Inside $\mathcal{N}_\eta$ waves propagate with speed either 1 or -1 and at interactions between waves with different speeds, no new wave is produced. Hence, the total number of interaction points inside $\mathcal{N}_\eta$ is finite. This contradicts the existence of a cluster point of interaction points. □

To ensure that the value of the functional at $t = 0+$ is sufficiently small in order that all the above interaction estimates hold true, we need some conditions on the total variation of the initial data. The standard estimates on the solution of the Riemann problem (see [4, Chapter 5]) imply that, in the gas, it is sufficient that the initial datum has sufficiently small total variation. On the other hand, in the liquid, the estimates on the Riemann problem depend on the small parameter κ, as shown in (4.8), see also Figure 4. All this justifies the introduction of the weighted total variation (3.1).

Lemma 4.8. *Consider $\bar{\delta}$ as defined in Lemma 4.2 and let $(\tilde{p}^\varepsilon, \tilde{v}^\varepsilon) = \tilde{u}^\varepsilon : \mathbb{R} \to \mathbb{R}^+ \times \mathbb{R}$ be piecewise constant, continuous at $z = 0$ and $z = m$ such that $\|\tilde{p}^\varepsilon - p_o\|_{\mathbf{L}^\infty} < \bar{\delta}$. If u^ε is the approximate solution constructed above, then, there exists a positive C, which can be chosen bounding from above all $\mathcal{O}(1)$ appearing in* (4.13), (4.14) *and* (4.21), *such that*

$$\frac{1}{C} \operatorname{WTV}_\kappa(\tilde{u}^\varepsilon) \leq \Upsilon(u^\varepsilon(0+)) \leq C \operatorname{WTV}_\kappa(\tilde{u}^\varepsilon)$$

with $\operatorname{WTV}_\kappa$ *defined as in* (3.1) *and* Υ *as in* (4.30) *with the weights $K_{in}, K_{\mathcal{L}}, H_{\mathcal{G}}$ and $H_{\mathcal{L}}$ chosen as in Lemma 4.6.*

Proof. Let σ_α be the sizes of the waves in $u^\varepsilon(0+)$ and z_α be their locations. Consider the estimate on the left. The strength of a wave is the absolute value of the pressure difference between the states on its sides, therefore, because of the weights' choice in Lemma 4.6 (they are all greater than 1), we have

$$\operatorname{TV}(\tilde{p}^\varepsilon) \leq \operatorname{TV}(p^\varepsilon(0+)) = \sum_\alpha |\sigma_\alpha| \leq \Upsilon(u^\varepsilon(0+)).$$

The slopes of Lax curves in the gas do not depend on κ (4.2), hence, along a Lax curve, the jump in the speed is uniformly controlled by the jump in the pressure:

$$\mathrm{TV}(\tilde{v}^{\varepsilon};\mathcal{G}) \leq \mathrm{TV}(v^{\varepsilon}(0+);\mathcal{G}) \leq \mathcal{O}(1)\sum_{\alpha}|\sigma_\alpha| \leq \mathcal{O}(1)\,\Upsilon(u^{\varepsilon}(0+))\,.$$

Finally, in the liquid we use (4.6) which shows that along a Lax curve in the liquid, the jump in the speed is controlled by κ times the jump in the pressure:

$$\mathrm{TV}(\tilde{v}^{\varepsilon};\mathcal{L}) \leq \mathrm{TV}(v^{\varepsilon}(0+);\mathcal{L}) \leq \mathcal{O}(1)\,\kappa\sum_{\alpha}|\sigma_\alpha| \leq \mathcal{O}(1)\,\kappa\Upsilon(u^{\varepsilon}(0+))\,.$$

This concludes the proof of the left estimate.

Passing to the right inequality, recall the usual bound $\Upsilon(u^\varepsilon(0+)) \leq \mathcal{O}(1)\sum_\alpha|\sigma_\alpha|$ which clearly holds also for Υ as defined in (4.30). Proceed using the classical estimate for the solutions to the Riemann problems in the gas and (4.8) in the liquid:

$$\begin{aligned}
\Upsilon(u^{\varepsilon}(0+)) &= \mathcal{O}(1)\sum_{\alpha}|\sigma_\alpha| \\
&= \mathcal{O}(1)\left(\sum_{z_\alpha\in\mathring{\mathcal{G}}}|\sigma_\alpha| + \sum_{z_\alpha\in\mathcal{L}}|\sigma_\alpha|\right) \\
&= \mathcal{O}(1)\left(\sum_{z_\alpha\in\mathring{\mathcal{G}}}\Big(\left|\tilde{p}^{\varepsilon}(z_\alpha+)-\tilde{p}^{\varepsilon}(z_\alpha-)\right| + \left|\tilde{v}^{\varepsilon}(z_\alpha+)-\tilde{v}^{\varepsilon}(z_\alpha-)\right|\Big)\right. \\
&\qquad\left. + \sum_{z_\alpha\in\mathcal{L}}\Big(\left|\tilde{p}^{\varepsilon}(z_\alpha+)-\tilde{p}^{\varepsilon}(z_\alpha-)\right| + \frac{1}{\kappa}\left|\tilde{v}^{\varepsilon}(z_\alpha+)-\tilde{v}^{\varepsilon}(z_\alpha-)\right|\Big)\right) \\
&= \mathcal{O}(1)\ \mathrm{WTV}_\kappa(\tilde{u}^{\varepsilon})\,,
\end{aligned}$$

completing the proof. □

Proposition 4.9. *Fix a positive pressure p_o and let P^g, P satisfy **(P)**. There exist constants $\delta,\Delta,L,\kappa_* > 0$, with $\kappa_* < 1$, such that, for any $\kappa\in\,]0,\kappa_*[$, for any piecewise constant initial datum $\tilde{u} = (\tilde{p},\tilde{v})$, continuous at the points $z=0$, $z=m$, satisfying $\mathrm{WTV}_\kappa(\tilde{u}) \leq \delta$ and $\|\tilde{p}-p_o\|_{\mathbf{L}^\infty} \leq \delta$, the wave front tracking approximate solution $u^{\kappa,\varepsilon} = (p^{\kappa,\varepsilon}, v^{\kappa,\varepsilon})$ to the Cauchy problem for (2.12) can be constructed for all times $t\geq 0$. Moreover, given the specific volume as $\tau^{\kappa,\varepsilon}(t,z) = \mathcal{T}_\kappa\left(z, p^{\kappa,\varepsilon}(t,z)\right)$, the following estimates hold.*

For any $t, t_1, t_2 \geq 0$

$$\begin{array}{lll}
\mathrm{WTV}_\kappa\left(u^{\kappa,\varepsilon}(t,\cdot)\right) \leq & \Delta, & \\
\mathrm{TV}\left(p^{\kappa,\varepsilon}(t,\cdot),\mathcal{L}\right) \leq & \Delta, & \int_{\mathcal{L}}\left|p^{\kappa,\varepsilon}(t_2,z)-p^{\kappa,\varepsilon}(t_1,z)\right|\mathrm{d}z \leq \frac{1}{\kappa}L\left|t_2-t_1\right|, \\
\mathrm{TV}\left(v^{\kappa,\varepsilon}(t,\cdot),\mathcal{L}\right) \leq & \kappa\Delta, & \int_{\mathcal{L}}\left|v^{\kappa,\varepsilon}(t_2,z)-v^{\kappa,\varepsilon}(t_1,z)\right|\mathrm{d}z \leq L\left|t_2-t_1\right|, \\
\mathrm{TV}\left(\tau^{\kappa,\varepsilon}(t,\cdot),\mathcal{L}\right) \leq & \kappa^2\Delta, & \int_{\mathcal{L}}\left|\tau^{\kappa,\varepsilon}(t_2,z)-\tau^{\kappa,\varepsilon}(t_1,z)\right|\mathrm{d}z \leq \kappa L\left|t_2-t_1\right|, \\
\mathrm{TV}\left(p^{\kappa,\varepsilon}(t,\cdot),\mathcal{G}\right) \leq & \Delta, & \int_{\mathcal{G}}\left|p^{\kappa,\varepsilon}(t_2,z)-p^{\kappa,\varepsilon}(t_1,z)\right|\mathrm{d}z \leq L\left|t_2-t_1\right|, \\
\mathrm{TV}\left(v^{\kappa,\varepsilon}(t,\cdot),\mathcal{G}\right) \leq & \Delta, & \int_{\mathcal{G}}\left|v^{\kappa,\varepsilon}(t_2,z)-v^{\kappa,\varepsilon}(t_1,z)\right|\mathrm{d}z \leq L\left|t_2-t_1\right|, \\
\mathrm{TV}\left(\tau^{\kappa,\varepsilon}(t,\cdot),\mathcal{G}\right) \leq & \Delta, & \int_{\mathcal{G}}\left|\tau^{\kappa,\varepsilon}(t_2,z)-\tau^{\kappa,\varepsilon}(t_1,z)\right|\mathrm{d}z \leq L\left|t_2-t_1\right|.
\end{array} \tag{4.32}$$

For any $z \in \mathcal{L}$, $z_1, z_2 \in \mathcal{L} \setminus ([-\varepsilon^2, \varepsilon^2] \cup [m-\varepsilon^2, m+\varepsilon^2])$

$$\begin{aligned}
\mathrm{TV}\left(p^{\kappa,\varepsilon}(\cdot,z), \mathbb{R}^+\right) &\leq \frac{\Delta}{\kappa}, & \int_{\mathbb{R}^+} \left|p^{\kappa,\varepsilon}(t,z_2) - p^{\kappa,\varepsilon}(t,z_1)\right| \mathrm{d}t &\leq L\left|z_2 - z_1\right|, \\
\mathrm{TV}\left(v^{\kappa,\varepsilon}(\cdot,z), \mathbb{R}^+\right) &\leq \Delta, & \int_{\mathbb{R}^+} \left|v^{\kappa,\varepsilon}(t,z_2) - v^{\kappa,\varepsilon}(t,z_1)\right| \mathrm{d}t &\leq \kappa L\left|z_2 - z_1\right|, \\
\mathrm{TV}\left(\tau^{\kappa,\varepsilon}(\cdot,z), \mathbb{R}^+\right) &\leq \kappa\Delta, & \int_{\mathbb{R}^+} \left|\tau^{\kappa,\varepsilon}(t,z_2) - \tau^{\kappa,\varepsilon}(t,z_1)\right| \mathrm{d}t &\leq \kappa^2 L\left|z_2 - z_1\right|.
\end{aligned} \tag{4.33}$$

For any $z, z_1, z_2 \in \mathcal{G}$

$$\begin{aligned}
\mathrm{TV}\left(p^{\kappa,\varepsilon}(\cdot,z), \mathbb{R}^+\right) &\leq \Delta, & \int_{\mathbb{R}^+} \left|p^{\kappa,\varepsilon}(t,z_2) - p^{\kappa,\varepsilon}(t,z_1)\right| \mathrm{d}t &\leq L\left|z_2 - z_1\right|, \\
\mathrm{TV}\left(v^{\kappa,\varepsilon}(\cdot,z), \mathbb{R}^+\right) &\leq \Delta, & \int_{\mathbb{R}^+} \left|v^{\kappa,\varepsilon}(t,z_2) - v^{\kappa,\varepsilon}(t,z_1)\right| \mathrm{d}t &\leq L\left|z_2 - z_1\right|, \\
\mathrm{TV}\left(\tau^{\kappa,\varepsilon}(\cdot,z), \mathbb{R}^+\right) &\leq \Delta, & \int_{\mathbb{R}^+} \left|\tau^{\kappa,\varepsilon}(t,z_2) - \tau^{\kappa,\varepsilon}(t,z_1)\right| \mathrm{d}t &\leq L\left|z_2 - z_1\right|.
\end{aligned} \tag{4.34}$$

For any $z, z_1, z_2 \in \mathbb{R}$

$$\begin{aligned}
\mathrm{TV}\left(p^{\kappa,\varepsilon}(\cdot,z), \mathbb{R}^+\right) &\leq \frac{\Delta}{\kappa}, & \int_{\mathbb{R}^+} \left|p^{\kappa,\varepsilon}(t,z_2) - p^{\kappa,\varepsilon}(t,z_1)\right| \mathrm{d}t &\leq \frac{L}{\kappa}\left|z_2 - z_1\right|, \\
\mathrm{TV}\left(v^{\kappa,\varepsilon}(\cdot,z), \mathbb{R}^+\right) &\leq \Delta, & \int_{\mathbb{R}^+} \left|v^{\kappa,\varepsilon}(t,z_2) - v^{\kappa,\varepsilon}(t,z_1)\right| \mathrm{d}t &\leq L\left|z_2 - z_1\right|.
\end{aligned} \tag{4.35}$$

Moreover, the maximal size of rarefaction waves is uniformly bounded by a constant, independent of κ, times ε.

Proof. Choose $\bar\delta$ as in Lemma 4.6 and κ_* as in Lemma 4.5. Define $\delta = \bar\delta / C$, with C as in Lemma 4.8. Using the piecewise constant initial data $\tilde u$, we use the previously described algorithm and call $u^{\kappa,\varepsilon}$ the piecewise constant approximate solution so obtained. By Lemma 4.8, we have $\Upsilon\left(u^{\kappa,\varepsilon}(0+)\right) \leq C \cdot \mathrm{WTV}_\kappa\left(\tilde u\right) < \bar\delta$. By Lemma 4.6, the map $t \to \Upsilon\left(u^{\kappa,\varepsilon}(t)\right)$ is not increasing so that $\Upsilon\left(u^{\kappa,\varepsilon}(t)\right) < \bar\delta$ for all positive times. Lemma 4.7 ensures that $u^{\kappa,\varepsilon}$ can be constructed for all times $t \geq 0$. Again, Lemma 4.8 implies the estimate

$$\mathrm{WTV}_\kappa\left(u^{\kappa,\varepsilon}(t)\right) \leq C\,\Upsilon\left(u^{\kappa,\varepsilon}(t)\right) \leq C\,\Upsilon\left(u^{\kappa,\varepsilon}(0+)\right) \leq C^2\,\mathrm{WTV}_\kappa\left(\tilde u\right) \leq C^2\,\delta = C\bar\delta.$$

The estimates on the total variation of $p^{\kappa,\varepsilon}$ and $v^{\kappa,\varepsilon}$ in (4.32) immediately follow. To obtain the bounds on the total variation of the specific volume in the liquid, use (2.10). The Lipschitz continuity estimates in (4.32) are now a standard consequence, see e.g. [4, Section 7.4], since the wave propagation speed in the gas is uniformly bounded independently of κ and in the liquid (also in $\mathcal{I}^\pm_\varepsilon$) is bounded by $\mathcal{O}(1)/\kappa$.

Pass to (4.33). Observe first that from the proof of Lemma 4.6 it follows that

$$6\,C^3\bar\delta \leq 1 \qquad \text{and} \qquad C \geq 1\,. \tag{4.36}$$

As usual, we call σ_α the size of the wave supported at z_α. For $z \in \mathcal{L}$, define

$$\begin{aligned}
\Upsilon_z(t) &= W_z(t) + V_z(t) + \frac{3C}{\kappa}\,\Upsilon(t) \\
W_z(t) &= \sum_{\tau\in[0,t]} \left|\Delta p(\tau,z)\right| \\
V_z(t) &= \sum_{\alpha\in I_z(t)} \left|\sigma_\alpha\right| \\
I_z(t) &= \left\{\alpha\colon z_\alpha \in \mathcal{L} \text{ and the wave at } z_\alpha \text{ is of the } \left\{\begin{array}{l}\text{first family and } z_\alpha > z\\ \text{second family and } z_\alpha < z\end{array}\right\}\right. \\
\Upsilon(t) &= \Upsilon\left(u^{\kappa,\varepsilon}(t)\right)
\end{aligned}$$

Note that the sum defining W_z is actually a finite sum, since the total number of waves is finite by Lemma 4.7. We claim that $t \to \Upsilon_z(t)$ is non increasing. Indeed, $\Upsilon_z(t)$ may change its value at a time t when:

1. A wave with size $\sigma_{\bar\alpha}$ crosses z and no other interaction occurs. Then, $\Delta W_z(t) = \left|\Delta p(t,z)\right| = |\sigma_{\bar\alpha}|$, $\Delta V_z(t) = -|\sigma_{\bar\alpha}|$ and $\Delta\Upsilon(t) = 0$. Hence, $\Delta\Upsilon_z(t) = 0$.
2. An interaction in $\mathcal{G}$ occurs and no wave crosses z. Then, $\Delta W_z(t) = 0$, $\Delta V_z(t) = 0$ and $\Delta\Upsilon(t) \leq 0$. Hence, $\Delta\Upsilon_z(t) \leq 0$.
3. An interaction in $\mathcal{L}$ occurs and no wave crosses z. Then, $\Delta W_z(t) = 0$; calling σ_α, σ_β the sizes of the interacting waves, $\Delta V_z(t) \leq C\kappa^2\left|\sigma_\alpha\,\sigma_\beta\right|$ by (4.14) and $\Delta\Upsilon(t) \leq -\kappa^2\left|\sigma_\alpha\sigma_\beta\right|$ by (4.31). Hence, $\Delta\Upsilon_z(t) \leq C\left(1-\frac{3}{\kappa}\right)\kappa^2\left|\sigma_\alpha\sigma_\beta\right| \leq 0$.
4. A 2-wave with size σ_α, coming from $\mathcal{G}$, and a 1-wave with size σ_β, coming from $\mathcal{L}$, interact at $\bar z = 0$. Then, $\Delta W_z(t) = 0$; by Lemma 4.5, $\Delta V_z(t) \leq (1-c\kappa)\left|\sigma_\beta\right| + (2+C\bar\delta)|\sigma_\alpha|$; by Lemma 4.6 $\Delta\Upsilon(t) \leq -|\sigma_\alpha| - \kappa\left|\sigma_\beta\right|$. Hence, $\Delta\Upsilon_z(t) \leq (1-c\kappa-3C)\left|\sigma_\beta\right| + \left(2+C\bar\delta-\frac{3C}{\kappa}\right)|\sigma_\alpha| \leq 0$.
5. Two waves interact at $z = m$: the same procedure as above applies.

The remaining times where Υ_z may change value consist in the superposition of two or more of the cases considered above and can be dealt superimposing the corresponding inequalities. Therefore,

$$\begin{array}{rclclcl}
\mathrm{TV}\left(p^{\kappa,\varepsilon}(\cdot,z)\right) & = & \sup\limits_{T>0}\mathrm{TV}\left(p^{\kappa,\varepsilon}(\cdot,z);[0,T]\right) & = & \sup\limits_{T>0} W_z(T) & \leq & \sup\limits_{T>0}\Upsilon_z(T)\\
& \leq & \Upsilon_z(0+) & = & V_z(0+) + \frac{3C}{\kappa}\Upsilon(0+) & \leq & \frac{\Delta}{\kappa}
\end{array}$$

provided $\Delta > 2\bar\delta$, completing the proof of the first estimate on the total variation in (4.33). The remaining total variation bounds in (4.33) follow from the estimates

$$\left|\Delta v(t,z)\right| \leq \mathcal{O}(1)\,\kappa\left|\Delta p(t,z)\right| \quad \text{and} \quad \left|\Delta\tau(t,z)\right| \leq \mathcal{O}(1)\,\kappa^2\left|\Delta p(t,z)\right|$$

which hold along Lax curves by Lemma 4.1 and (2.10). The Lipschitz continuity estimates in (4.33) are now a standard consequence, see e.g. [4, Section 7.4], since the wave propagation speed in $\mathcal{L}\setminus\left([-\varepsilon^2,\varepsilon^2]\cup[m-\varepsilon^2,m+\varepsilon^2]\right)$ is of order $1/\kappa$.

The proof of the estimates (4.34) is obtained from that of (4.33) completed above, formally setting $\kappa = 1$ and with obvious modifications to the definition of Υ_z.

The estimates on all the real line (4.35) are obtained choosing a common upper bound on the total variation and a common lower bound on the wave speeds in the liquid, in the gas and in the two strips $\mathcal{I}_\varepsilon^\pm$ and observing that $z \mapsto u^{\kappa,\varepsilon}(t,z)$ is continuous at $z = 0$, $z = m$ for every $t \geq 0$ in which no wave interacts with the interfaces. Observe that a similar Lipschitz estimate does not hold for the specific volume τ, since at $z = 0$ and $z = m$ it is not continuous.

Finally, the estimate on the maximal size of rarefaction waves follows the lines in [4, Section 7.3, Step 5]. Indeed, call $\bar\sigma(t)$ the size at time t of a rarefaction wave in the wave front tracking approximation. We claim that, if in the interval $[t_o,\tau]$ the wave does not leave the phase in which it is found at time t_o and does not disappear due to possible interactions with shocks of the same family, then $\left|\bar\sigma(\tau)\right| \leq 6\left|\bar\sigma(t_o)\right|$.

Indeed, consider the liquid phase, let $\bar z(t)$ be the location of the wave at time t and define

$$\begin{array}{rcl}
s(t) & = & \left|\bar\sigma(t)\right|\left[1 + 6\,C^2\,\kappa^2\,V_s(t) + 24\,C^3\,\kappa\,\Upsilon(t)\right]\\
V_s(t) & = & \sum_{\alpha\in I_s(t)}|\sigma_\alpha|\\
I_s(t) & = & \left\{\alpha\colon z_\alpha\in\mathcal{L},\ \text{the wave at } z_\alpha \text{ is approaching the wave at } \bar z\right\}.
\end{array}$$

The function $t \to s(t)$ is non increasing in the interval $[t_o,\tau]$. Indeed, $s(t)$ may change its value at the following times:

1. At time t a wave σ_α interacts with the wave at $\bar z(t)$ and no other interaction occurs. Then, by (4.14) $\Delta\left|\bar\sigma(t)\right| \le C\kappa^2\left|\bar\sigma(t-)\sigma_\alpha\right|$; $\Delta V_s(t) = -\left|\sigma_\alpha\right|$; $\Delta\Upsilon(t) < 0$. Hence, by (4.36),

$$\begin{aligned}\Delta s(t) &= \Delta\left|\bar\sigma(t)\right|\left[1+6C^2\kappa^2 V_s(t+)+24C^3\kappa\Upsilon(t+)\right]+\left|\bar\sigma(t-)\right|\left[6C^2\kappa^2\Delta V_s(t)+24C^3\kappa\Delta\Upsilon(t)\right]\\ &\le C\kappa^2\left|\bar\sigma(t-)\sigma_\alpha\right|\left[1+6C^2\kappa^2\bar\delta+24C^3\kappa\bar\delta\right]-6C^2\kappa^2\left|\bar\sigma(t-)\right|\left|\sigma_\alpha\right|\\ &\le C\kappa^2\left|\bar\sigma(t-)\sigma_\alpha\right|\left[1+6C^2\kappa^2\bar\delta+24C^3\kappa\bar\delta-6C\right]\\ &\le C\kappa^2\left|\bar\sigma(t-)\sigma_\alpha\right|\left[1+1+4-6C\right]\le 0\,.\end{aligned}$$

2. At time t, an interaction in $\mathcal{G}$ occurs and no wave crosses $\bar z(t)$. Then, $\Delta\left|\bar\sigma(t)\right| = 0$, $\Delta V_s(t) = 0$ and $\Delta\Upsilon(t)\le 0$. Hence, $\Delta s(t)\le 0$.

3. At time t, an interaction in $\mathcal{L}$ occurs and no wave crosses $\bar z(t)$. Then, $\Delta\left|\bar\sigma(t)\right| = 0$; calling σ_α, σ_β the sizes of the interacting waves, $\Delta V_s(t)\le C\kappa^2\left|\sigma_\alpha\,\sigma_\beta\right|$ by (4.14) and $\Delta\Upsilon(t)\le-\kappa^2\left|\sigma_\alpha\sigma_\beta\right|$ by (4.31). Hence,

$$\Delta s(t)\le\left|\bar\sigma(t)\right|\left[6C^3\kappa^4\left|\sigma_\alpha\sigma_\beta\right|-24C^3\kappa^3\left|\sigma_\alpha\sigma_\beta\right|\right]\le\left|\bar\sigma(t)\right|\left|\sigma_\alpha\sigma_\beta\right|6C^3\kappa^3(\kappa-4)\le 0\,.$$

4. At time t, an interaction occurs at $z=0$ and no wave crosses $\bar z(t)$. Call σ_α the size of the wave coming from $\mathcal{G}$, and σ_β the size of the wave coming from $\mathcal{L}$. Then, $\Delta\left|\bar\sigma(t)\right| = 0$. By Lemma 4.5, $\Delta V_s(t)\le(1-c\kappa)\left|\sigma_\beta\right|+(2+C\bar\delta)\left|\sigma_\alpha\right|$; by Lemma 4.6 $\Delta\Upsilon(t)\le-\left|\sigma_\alpha\right|-\kappa\left|\sigma_\beta\right|$. Hence,

$$\begin{aligned}\Delta s(t) &\le\left|\bar\sigma(t)\right|\left[6C^2\kappa^2\left((1-c\kappa)\left|\sigma_\beta\right|+(2+C\bar\delta)\left|\sigma_\alpha\right|\right)-24C^3\kappa\left(\left|\sigma_\alpha\right|+\kappa\left|\sigma_\beta\right|\right)\right]\\ &\le 6C^2\kappa\left|\bar\sigma(t)\right|\left[\kappa\left((1-c\kappa)\left|\sigma_\beta\right|+(2+C\bar\delta)\left|\sigma_\alpha\right|\right)-4C\left(\left|\sigma_\alpha\right|+\kappa\left|\sigma_\beta\right|\right)\right]\\ &\le 6C^2\kappa\left|\bar\sigma(t)\right|\left[\kappa\left|\sigma_\beta\right|(1-c\kappa-4C)+\left|\sigma_\alpha\right|\left(2\kappa+C\kappa\bar\delta-4C\right)\right]\le 0\end{aligned}$$

5. Two waves interact at $z=m$: the same procedure as above applies.

The remaining times where $s(t)$ may change value consist in the superposition of two or more of the cases considered above and can be dealt superimposing the corresponding inequalities proved above. Therefore $s(\tau)\le s(t_o)$ which implies

$$\left|\bar\sigma(\tau)\right|\le\left|\bar\sigma(t_o)\right|\frac{1+6C^2\kappa^2V_s(t_o)+24C^3\kappa\Upsilon(t_o)}{1+6C^2\kappa^2V_s(\tau)+24C^3\kappa\Upsilon(\tau)}\le\left[1+6C^2\bar\delta+24C^3\bar\delta\right]\left|\bar\sigma(t_o)\right|\le 6\left|\bar\sigma(t_o)\right|$$

This proves the claim in the liquid. In the case of a wave in the gas, the argument is similar: it is sufficient to set $\kappa=1$ in the definition of $s(t)$ and make the obvious modifications to the map V_s.

Finally, we observe now that when a wave crosses the interfaces, the refracted wave has a strength given by the strength of the incoming wave times a constant bounded uniformly with respect to κ, for instance we can choose $3C$ (see Lemma 4.5). Moreover, when a rarefaction is born, its strength is less than ε and it can cross at most an interface once. Therefore, also the last claim of the Proposition is proved with the constant $1944\,C^2\varepsilon$. □

Proof of Theorem 3.3. Use $\delta,\Delta,L,\kappa_*>0$ as defined in Proposition 4.9 and choose any $\kappa\in\,]0,\kappa^*[$. Fix a suitable sequence ε_ν strictly decreasing to 0. Approximate the initial datum $\tilde u=(\tilde p,\tilde v)$ with an approximate, piecewise constant initial datum $\tilde u^{\varepsilon_\nu}$ satisfying (4.12), so that $\mathrm{WTV}_\kappa\left(\tilde u^{\varepsilon_\nu}\right)\le\mathrm{WTV}_\kappa\left(\tilde u\right)\le\delta$, $\|\tilde p^{\varepsilon_\nu}-p_o\|_{\mathbf{L}^\infty}\le\delta$.

Proposition 4.9 ensures that it is possible to construct a wave front tracking ε_ν–approximate solution $(p^{\kappa,\varepsilon_\nu},v^{\kappa,\varepsilon_\nu})$ that satisfies all properties stated therein.

Using (4.32) and (4.35), a repeated application of Helly Theorem [4, Theorem 2.4], ensures the convergence of a suitable subsequence, which we still denote by $u^{\kappa,\varepsilon_\nu}$, to a function $u^\kappa = (p^\kappa, v^\kappa)$ in the following sense

$$\begin{aligned}
\lim_{\nu\to+\infty} \left\|(p^{\kappa,\varepsilon_\nu}, v^{\kappa,\varepsilon_\nu})(t,\cdot) - (p^\kappa, v^\kappa)(t,\cdot)\right\|_{\mathbf{L^1}([-M,M];\mathbb{R}^+\times\mathbb{R})} &= 0, \text{ for any } t \geq 0,\ M > 0 \\
\lim_{\nu\to+\infty} \left\|(p^{\kappa,\varepsilon_\nu}, v^{\kappa,\varepsilon_\nu})(\cdot,z) - (p^\kappa, v^\kappa)(\cdot,z)\right\|_{\mathbf{L^1}([0,M];\mathbb{R}^+\times\mathbb{R})} &= 0, \text{ for any } z \in \mathbb{R},\ M > 0 \\
(p^\kappa, v^\kappa)(0,\cdot) &= (\tilde{p}^\kappa, \tilde{v}^\kappa)(\cdot).
\end{aligned}$$

Passing to the limit in (4.32), (4.33), (4.34), (4.35), we obtain (3.3), (3.4), (3.5) and (3.6).

Since the bounds on the total variation are uniform in ε and since the strength of rarefactions is uniformly bounded by a constant times ε, standard techniques in wave front tracking [4, Section 7.4] can be used to show that the limit u^κ is a weak entropy solution to (2.12) in the open regions $z < 0$, $0 < z < m$, $z > m$. By (3.6), we have that the map $z \to u^\kappa(\cdot, z)$ is continuous in $\mathbf{L^1}$, in particular it is continuous across $z = 0$ and $z = m$. Therefore, u^κ trivially satisfies there the Rankine-Hugoniot conditions and the entropy (in)equality. Hence, u^κ is a weak entropy solution to (2.12) in all $\mathbb{R}^+ \times \mathbb{R}$. □

Proof of Theorem 3.4. By (3.7), $\mathrm{WTV}_\kappa(\tilde{u}) < \delta$ so that Theorem 3.3 applies, ensuring the existence of a solution $u^\kappa = (p^\kappa, v^\kappa)$ to (2.12) satisfying (3.3), (3.4), (3.5) and (3.6).

Since $\kappa < 1$, from (3.3) and (3.6) we have for v^κ:

$$\begin{array}{rclrcll}
\mathrm{TV}\left(v^\kappa(t,\cdot),\mathbb{R}\right) & \leq & \Delta, & \int_{\mathbb{R}} \left|v^\kappa(t_2,z) - v^\kappa(t_1,z)\right| \mathrm{d}z & \leq & L\left|t_2 - t_1\right|, & t, t_1, t_2 \geq 0, \\
\mathrm{TV}\left(v^\kappa(\cdot,z),\mathbb{R}^+\right) & \leq & \Delta, & \int_{\mathbb{R}^+} \left|v^\kappa(t,z_2) - v^\kappa(t,z_1)\right| \mathrm{d}t & \leq & L\left|z_2 - z_1\right|, & z, z_1, z_2 \in \mathbb{R}.
\end{array} \tag{4.37}$$

Helly Theorem [4, Theorem 2.4] implies the existence of a subsequence (that we call again v^κ) converging to a limit v^* in the sense of (3.8). From the bound in (3.3) on the total variation of v^κ or from the Lipschitz estimate in (3.4) for v^κ in the liquid, it is straightforward to obtain that $v^*(t,z) = v_l(t)$ for all $z \in \mathcal{L}$ and $t \geq 0$, where $v_l(t)$ is a function which depends on time only, completing the proof of (3.8) and of 3. in Definition 3.2.

The same procedure can be carried out for the pressure in the gas region, proving the first four lines in (3.9). Observe that for the pressure, we cannot apply Helly Theorem in the liquid since there the estimates blow up as $\kappa \to 0$. Because of the strong convergence in the gas region of both the velocity and the pressure, the limit $u^* = (p^*, v^*)$ satisfies 1. in Definition 3.2 and the initial condition $u^*(0,z) = \tilde{u}(z)$ a.e. $z \in \mathcal{G}$.

The uniform convergence of τ^κ in the liquid is a straightforward consequence of (2.10) and of the uniform bound on the $\mathbf{L}^\infty$ norm of p^κ.

Since the pressure is uniformly bounded, we have a weak* convergence (possibly passing to further subsequences) $p^\kappa \overset{\star}{\rightharpoonup} p^*$ in $\mathbf{L}^\infty\left(\mathbb{R}^+ \times \mathbb{R}, \mathbb{R}\right)$ [5, Section 4.3 Point C.]. If we define $p_l = p^*_{|\mathbb{R}^+\times\mathcal{L}}$ we get the fifth line in (3.9).

By (3.3) and (3.6), the second equation in (2.12) can be written in integral form in $[t_1, t_2] \times \mathcal{L}$:

$$\int_0^m v^\kappa\left(t_1, z\right)\, \mathrm{d}z - \int_0^m v^\kappa\left(t_2, z\right)\, \mathrm{d}z + \int_{t_1}^{t_2} p^\kappa\left(t, 0\right)\, \mathrm{d}t - \int_{t_1}^{t_2} p^\kappa\left(t, m\right)\, \mathrm{d}t = 0. \tag{4.38}$$

Now, we use the strong convergence of both p^κ and v^κ in the gas region and the fact that in $\mathcal{L}$, v^* is constant to obtain

$$m\left[v_l\left(t_2\right) - v_l\left(t_1\right)\right] = \int_{t_1}^{t_2} p^*\left(t, 0\right)\, \mathrm{d}t - \int_{t_1}^{t_2} p^*\left(t, m\right)\, \mathrm{d}t\,.$$

Setting $t_1 = 0$ and $t_2 = t$ in the last expression above,

$$v_l\left(t\right) = v_l\left(0\right) + \frac{1}{m}\int_0^t \left[p^*\left(s, 0\right) - p^*\left(s, m\right)\right]\, \mathrm{d}s = v_l\left(0\right) + \frac{1}{m}\int_0^t \left[p^*\left(s, 0-\right) - p^*\left(s, m+\right)\right]\, \mathrm{d}s$$

which means that v_l is Lipschitz continuous and satisfies 2. in Definition 3.2.

Observe that the non linear term $\mathcal{T}_\kappa(z, p^\kappa)$ converges strongly to

$$\tau^*(t,z) = \begin{cases} \bar{\tau} & \text{for } z \in \mathcal{L}, \\ \mathcal{T}_g\left(p^*(t,z)\right) & \text{for } z \in \mathcal{G}, \end{cases}$$

hence we can pass to the limit in (2.12) in distributional sense to obtain

$$\begin{cases} \partial_t \tau^* - \partial_z v^* = 0 \\ \partial_t v^* + \partial_z p^* = 0, \end{cases} \quad \text{in } \mathbb{R}^+ \times \mathbb{R}. \tag{4.39}$$

Since in the liquid region $v^*(t,z) = v_l(t)$ with v_l Lipschitz continuous, the second equation in (4.39) becomes

$$\partial_z p^*(t,z) = -\dot{v}_l(t) \text{ in } \mathbb{R}^+ \times \mathcal{L}.$$

Therefore there exists a measurable function $\beta(t)$ such that the function

$$p_l(t,z) = -z\dot{v}_l(t) + \beta(t)$$

can be chosen as a representative of the limit pressure p^* restricted to the liquid. This implies the existence of the two limits

$$\lim_{z\to 0^+} p_l(t,z) = \beta(t), \qquad \lim_{z\to m^-} p_l(t,z) = \beta(t) - z\dot{v}_l(t).$$

The fourth line in (3.9) ensures the existence of the corresponding limits from the gas region:

$$\lim_{z\to 0^-} p^*(t,z) = p^*(t,0), \qquad \lim_{z\to m^+} p^*(t,z) = p^*(t,m), \text{ a.e. } t \in \mathbb{R}^+,$$

hence Rankine-Hugoniot conditions for (4.39) applied along $z = 0$ and $z = m$ imply that the right and the left limit of the pressure must coincide along $z = 0$ and $z = m$ for a.e. $t \geq 0$. Therefore, we have

$$\begin{cases} p^*(t,0) = \beta(t) \\ p^*(t,m) = -m\dot{v}_l(t) + \beta(t) \end{cases} \quad \text{for a.e. } t \geq 0,$$

which implies the remaining equality to be proved in (3.9). □

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