LIMIT VECTOR VARIATIONAL INEQUALITY PROBLEMS VIA SCALARIZATION

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Abstract. We solve a general vector variational inequality problem in a finite - dimensional setting, where only approximation sequences are known instead of exact values of the cost mapping and feasible set. We establish a new equivalence property, which enables us to replace each vector variational inequality with a scalar set-valued variational inequality. Then, we approximate the scalar set-valued variational inequality with a sequence of penalized problems, and we study the convergence of their solutions to solutions of the original one.
Keywords: Vector variational inequality; non-stationarity; set-valued mappings; approximation sequence; penalty method; coercivity conditions.

1. Introduction

Let D be a nonempty convex set in the real n-dimensional space $\mathbb{R}^n$, and let $G : D \to \mathbb{R}^n$ be a mapping. Then one can define the *variational inequality problem* (VI, for short), which is to find an element $x^* \in D$ such that

$$\text{(VI)} \qquad \langle G(x^*), y - x^* \rangle \geq 0 \quad \forall y \in D.$$

VIs give a suitable common format for various applied problems and are closely related with other general problems in nonlinear analysis, such as fixed point, optimization, complementarity, and equilibrium problems; see, e.g., [4]–[9] and the references therein. Moreover, there exist various extensions of the usual scalar VIs, in particular, vector VIs, which are closely related with vector optimization problems; see [6]–[8] for more details.

We recall that the usual *vector variational inequality problem* (in short, VVI) is to find an element $x^* \in D$ such that

$$\text{(VVI)} \qquad G(x^*)(y - x^*) \notin -\text{int}C \quad \forall y \in D,$$

where G is a single-valued mapping from D into $L(\mathbb{R}^n, \mathbb{R}^m)$, and C is some ordering cone of $\mathbb{R}^m$ with $\text{int}C \neq \emptyset$ (see Section 2). Clearly, VVI is an extension of VI in the case $m = 1$ and C is the non-negative ray. Note that

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each value $G(x)$ is an $m \times n$ matrix. However, exact values of the cost mapping G, ordering cone C, and feasible set D may be unknown for many real problems. This situation is clearly invoked by the usual calculation errors and incompleteness of information about the problem under solution. As a result, one can only deal with problems arising from suitable approximations $\{D_k\}$, $\{G_k\}$ and $\{C_k\}$ of the set D, the mapping G and the cone C, respectively.

Our aim is to investigate convergence properties of the approximated problems, following the lines of [12] and [13], where convergence of some penalty based methods for limit variational inequality problems in finite-dimensional spaces was obtained. These approximations do not require special concordance of parameters, and their convergence will be established under suitable coercivity conditions, not necessarily related to any monotonicity assumptions.

The paper is organized as follows: first, we establish a new equivalence property, which enables us to replace each VVI with a scalar set-valued variational inequality. Then, we approximate the scalar set-valued variational inequality with a sequence of penalized problems, and we study the convergence of their solutions to solutions of the original one.

2. Preliminary results

In this section we collect some preliminary notions, and some known results about VVIs. Furthermore, in view of the approximated problems considered in the sequel, we provide some known facts about generalized mixed variational inequalities. The setting is finite-dimensional; every Euclidean space will be endowed by the usual scalar product $\langle \cdot, \cdot \rangle$ inducing the Euclidean norm $\| \cdot \|$. In particular, $B(a, r)$ will denote the open ball centered at a with radius r, and $\overline{B(a, r)}$ its closure.

Let us recall that a nonempty set $C \subset \mathbb{R}^m$ is called a *convex cone* if $\lambda C \subseteq C$ for all $\lambda > 0$, and $C + C = C$. A cone C is called *pointed* if $C \cap (-C) = \{\mathbf{0}\}$, where $\mathbf{0}$ denotes the zero vector. A set X is called *solid* if its interior, denoted by $\mathrm{int} X$, is nonempty. Also, a nonempty set X is called *proper* if it is contained properly in $\mathbb{R}^m$, i.e. $X \neq \mathbb{R}^m$. If C is a convex and proper cone, then $\mathbf{0} \notin \mathrm{int} C$.

Given a convex, proper and solid cone, we can introduce the order $\succ_C$ in $\mathbb{R}^m$ as follows:

$$x \succ_C y \iff x - y \in \mathrm{int} C.$$

In the case where

$$C = \mathbb{R}^m_+ = \{z \in \mathbb{R}^m : \, z_i \geq 0 \quad i = 1, \ldots, m\},$$

we have the weak Paretian order. Note that the cone C is not supposed to be a closed set, since there are some orderings, such as the lexicographic one, whose cones are not closed.

Given a set K in $\mathbb{R}^m$, define

$$S(K) = K \cap S(\mathbf{0}, 1), \qquad P(K) = \mathrm{conv} S(K),$$

where $S(\mathbf{0}, 1) = \{z \in \mathbb{R}^m : \, \|z\| = 1\}$, and $\mathrm{conv} A$ denotes the convex hull of the set A. Furthermore, the polar (or conjugate) cone of a set K in $\mathbb{R}^m$ is given by

$$K^* = \{q \in \mathbb{R}^m : \langle p, q\rangle \geq 0, \quad \forall p \in K\}.$$

It is clear that K^* is a convex and closed cone. If K is itself a cone in $\mathbb{R}^m$, then $K^{**} = \overline{K}$. Moreover, if the cone K is proper, convex and solid, then K^* has nonzero elements due to the separation theorem.

In the next lemma a characterization of the interior points of closed and convex cones is provided.

Lemma 2.1. (see e.g. [11, Lemma 1]).

i. Let K be a subset of $\mathbb{R}^m$. If $p \in \mathrm{int}K$, then

$$\langle p, q\rangle > 0 \quad \forall q \in K^* \setminus \{\mathbf{0}\}.$$

ii. Let K be a convex and closed cone in $\mathbb{R}^m$. Suppose that

$$\langle p, q\rangle > 0 \quad \forall q \in K^* \setminus \{\mathbf{0}\};$$

then, $p \in \mathrm{int}K$ (i.e. $\mathrm{int}K \neq \emptyset$).

Let us consider the following VVI: find $x^* \in D$ such that

$$G(x^*)(y - x^*) \notin -\mathrm{int}(C), \quad \forall y \in D,$$

where $D \subseteq \mathbb{R}^n$, and $G = [G_1, G_2, \ldots, G_m]^\top$, with $G_j : \mathbb{R}^n \to \mathbb{R}^n$ for every $j = 1, 2, \ldots, m$.

For every element $q \in C^* \setminus \{\mathbf{0}\}$ we define the *scalarized problem* (VI_q, for short): find a point $x_q \in D$ such that

$$(\mathrm{VI}_q) \qquad \langle G(x_q)(y - x_q), q\rangle \geq 0 \quad \forall y \in D,$$

or, equivalently,

$$\langle G^\top(x_q)q, y - x_q\rangle \geq 0 \quad \forall y \in D.$$

Remark 2.2. From Lemma 2.1, it is easy to prove that any solution of VVI is a solution of problem VI_q, for some $q \in C^* \setminus \{\mathbf{0}\}$. As a matter of fact, if x^* is a solution of VVI, from the convexity of D it follows that the set $G(x^*)(D - x^*)$ is convex; moreover, it contains 0, and does not intersect $-\mathrm{int}C$. By the separation theorem, there exists $q \in \mathbb{R}^n \setminus \{0\}$ such that, for all $y \in D$ and all $v \in -\mathrm{int}C$, one has $\langle G(x^*)(y - x^*), q\rangle > \langle v, q\rangle$. It follows that $\langle v, q\rangle < 0$ for all $v \in -\mathrm{int}C$, and $\langle v, q\rangle \leq 0$ for all $v \in -C$. Hence $q \in C^*$, and x^* is a solution of VI_q. In addition, any solution of problem VI_q, with $q \in C^* \setminus \{\mathbf{0}\}$, is a solution of VVI.

In order to replace the original VVI with an equivalent formulation, we define the set-valued mapping $F : \mathbb{R}^n \to \Pi(\mathbb{R}^n)$ as follows:

$$(2.1) \qquad F(x) = \{f \in \mathbb{R}^n : f = G^\top(x)q = \sum_{j=1}^m G_j(x)q_j, \ q \in P(C^*)\}.$$

Here and below $\Pi(A)$ denotes the family of all nonempty subsets of a set A. The set-valued map F is trivially nonempty, compact and convex valued in $\mathbb{R}^n$, and the set $P(C^*)$ is compact. Under the assumption of continuity of the maps G_j, by adapting the proof of Theorem 16.34 in [3], we can easily show that F is upper semicontinuous.

Let us now consider the following *generalized variational inequality problem* (GVI, for short): find an element $x^* \in D$ and $f^* \in F(x^*)$ such that

$$\text{(GVI)} \qquad \langle f^*, y - x^* \rangle \geq 0 \quad \forall y \in D.$$

The problem GVI turns out to be equivalent to the problem VVI, as the next proposition shows:

Proposition 2.3. *VVI is equivalent to GVI, where the mapping F is defined in (2.1).*

Proof: If x^* is a solution of VVI, then it solves problem VI_q for some $q \in C^* \setminus \{\mathbf{0}\}$, as a consequence of Lemma 2.1. Hence we can take $q' = (1/\|q\|)q \in S(C^*)$. This means that x^* is a solution of GVI, with $f^* = G^\top(x^*)q'$.

Conversely, let x^* be a solution of GVI. Then it solves VI_q for some $q \in P(C^*)$. By definition, there exist elements $q^i \in S(C^*)$ and numbers $\alpha_i > 0$ such that

$$q = \sum_{i \in I} \alpha_i q^i, \ \sum_{i \in I} \alpha_i = 1,$$

where I is a finite set of indices. Fix any point $y \in D$. Then, there exists an index $l \in I$ such that

$$\langle G(x^*)(y - x^*), q^l \rangle \geq 0,$$

hence, by Lemma 2.1,

$$G(x^*)(y - x^*) \notin -\mathrm{int} C.$$

Therefore x^* solves VVI. □

By the proposition above, existence results for VVI can be obtained by investigating the equivalent GVI. This is a special case of the more general *generalized mixed variational inequality* (GMVI, for short; see e.g. [13]), which is to find an element $x^* \in D$ and $f^* \in F(x^*)$ such that

$$\text{(GMVI)} \qquad \langle f^*, y - x^* \rangle + h(y) - h(x^*) \geq 0 \quad \forall y \in D,$$

where $h : D \to \mathbb{R}$, and $F : D \to \Pi(\mathbb{R}^n)$.

In the following we will consider the problem GMVI under the following basic assumptions:

(A) *D is a nonempty, closed and convex set, $h : D \to \mathbb{R}$ is a lower semicontinuous and convex function, $F : D \to \Pi(\mathbb{R}^n)$ is upper semicontinuous, with nonempty, convex, and compact values on D.*

Let us recall some existence results for GMVI.

Proposition 2.4. (see [13, Proposition 2]). If **A** holds and D is bounded, then GMVI has a solution.

In case the set D is unbounded, some proper coercivity conditions are required. To this purpose, let us recall that a function $\mu : \mathbb{R}^n \to \mathbb{R}$ is said to be *weakly coercive* with respect to a set $\mathcal{D}$ if there exists $\rho \in \mathbb{R}$ such that the set

$$\mathcal{D}(\mu, \rho) := \{x \in \mathcal{D} : \ \mu(x) \leq \rho\}$$

is nonempty and bounded. For every $f \in F(x)$, set

$$\Delta(f, h, x, y) = \langle f, y - x \rangle + h(y) - h(x).$$

We take the following coercivity condition:

(C) *There exist a convex function $\mu : \mathbb{R}^n \to \mathbb{R}$, which is weakly coercive with respect to the set D, and a number r such that, for any point $\bar{x} \in D \setminus D(\mu, r)$ and any $\bar{f} \in F(\bar{x})$ with*

$$\inf_{x \in D(\mu,r)} \Delta(\bar{f}, h, \bar{x}, x) \geq 0, \tag{2.2}$$

there is a point $z \in D$ such that

$$\begin{array}{c} \min\{\Delta(\bar{f}, h, \bar{x}, z), \mu(z) - \mu(\bar{x})\} < 0, \\ \text{and} \\ \max\{\Delta(\bar{f}, h, \bar{x}, z), \mu(z) - \mu(\bar{x})\} \leq 0. \end{array} \tag{2.3}$$

The following existence result holds :

Proposition 2.5. [13, Theorem 1] *If* **A** *and* **C** *are fulfilled, then GMVI has a solution.*

We can somewhat strengthen the above assertion by specializing (2.3), in order to be able to localize the solutions by considering the following coercivity condition:

(C′) *There exist a convex function $\mu : \mathbb{R}^n \to \mathbb{R}$, which is weakly coercive with respect to the set D, and a number r such that for any point $\bar{x} \in D \setminus D(\mu, r)$ and any $\bar{f} \in F(\bar{x})$ satisfying (2.2) there is a point $z \in D$ such that $\mu(z) \leq \mu(\bar{x})$ and*

$$\Delta(\bar{f}, h, \bar{x}, z) < 0. \tag{2.4}$$

Corollary 2.6. *If* **A** *and* **C′** *are fulfilled, then GMVI has a solution, and all the solutions are contained in $D(\mu, r)$.*

Proof: It is enough to note that (2.4) implies (2.3), hence existence of solutions of GMVI follows from Proposition 2.5. Due to (2.4), all these solutions belong to $D(\mu, r)$. □

3. Convergence of penalized approximation problems

Converting VVIs into GVIs enables us to apply the penalty approach. Following some ideas in [12], [13], we intend to establish existence results for the 'limit' problem VVI by investigating the cluster points of the solutions of penalized approximation problems that are defined in terms of more regular data.

In the sequel, we will suppose that D is a set of the form

$$D = V \cap W, \tag{3.1}$$

where V, W are convex and closed sets in the space $\mathbb{R}^n$. In general, V represents geometric constraints, whereas W corresponds to "functional" ones. To identify the points of this latter set, a suitable penalty function will be used, that is a function $P : \mathbb{R}^n \to \mathbb{R}$ such that

$$P(w) \begin{cases} = 0 & w \in W, \\ > 0 & w \notin W. \end{cases}$$

We now define approximations of the data D, G and C. Let $\{V_k\}$ be a sequence of closed, convex subsets of $\mathbb{R}^n$ such that

(A1) $\operatorname{Ls} V_k \subseteq V$, where Ls denotes the topological limit superior, i.e.,

$$\operatorname{Ls} V_k = \{x \in \mathbb{R}^n : x_{n_k} \to x \text{ with } x_{n_k} \in V_{n_k}\}.$$

The set W will be approximated via perturbed penalty functions well behaved with respect to $\{V_k\}$, i.e., a sequence of nonnegative functions $P_k : \mathbb{R}^n \to \mathbb{R}$ such that

(A2)
i. P_k is lower semicontinuous and convex;
ii. if $x_k \in V_k$, $x_k \to \overline{x}$, and $\liminf P_k(x_k) = 0$, then $P(\overline{x}) = 0$, i.e., $\overline{x} \in W$;
iii. for every $w \in V \cap W$, there exists $\{v_k\}$, with $v_k \in V_k$ and $v_k \to w$ such that $P_k(v_k) = 0$ for k large enough.

This means that the set W is approximated implicitly with a sequence $\{W_k\}$. In addition, condition **A2**-iii admits, for some elements of the sequence $\{W_k\}$, to be empty.
The convex, proper and solid cone C in $\mathbb{R}^m$ will be approximated by a sequence of convex, proper and solid cones $\{C_k\}$ satisfying

(A3) $\operatorname{Ls} C_k^* \subseteq C^*$.

Denote by G^k the mapping

$$G^k = [G_1^k, G_2^k, \dots, G_m^k]^\top : \mathbb{R}^n \to L(\mathbb{R}^n, \mathbb{R}^m),$$

and by F^k the set-valued map $F^k : \mathbb{R}^n \to \Pi(\mathbb{R}^n)$, defined as follows:

$$F^k(x) = \{f \in \mathbb{R}^n : f = (G^k)^\top(x)q = \sum_{j=1}^{m} G_j^k(x)q_j, \ q \in P(C_k^*)\}.$$

Let us consider the following assumptions concerning the mappings G_j^k :

(A4)
i. G_j^k is continuous, for every j, k;
ii. for every sequence $\{x_k\}$, with $x_k \in V_k$ and $x_k \to \overline{x}$, the set

$$\{G_j^k(x_k), \ j = 1, 2, \dots, m, \text{ and } k \in \mathbb{N}\}$$

is bounded;
iii. if $x_k \in V_k$, $x_k \to \overline{x}$, $f_k \in F^k(x_k)$ and $f_k \to \overline{f}$, then $\overline{f} \in F(\overline{x})$.

Remark 3.1. Property **A4**-ii implies that the sets $F^k(x_k)$ are uniformly bounded, for every convergent sequence $\{x_k\}$ with $x_k \in V_k$.

For each k, let us now consider the problem GMVI$_k$: find $x_k^* \in V_k$ and $f_k^* \in F^k(x_k^*)$ such that

$$\langle f_k^*, y - x_k^* \rangle + \tau_k(P_k(y) - P_k(x_k^*)) \geq 0, \quad \forall y \in V_k, \tag{3.2}$$

where τ_k is a penalty parameter such that $\tau_k \to +\infty$.
First of all, we need the following technical result:

Lemma 3.2. *Suppose* **A3** *holds. Then each limit point of any sequence* $\{q^k\}$, $q^k \in P(C_k^*)$, *belongs to* $P(C^*)$.

Proof: Note that by **A3** each limit point of the sequence $\{q^k\}$ belongs to C^*. Without loss of generality suppose that $q^k \to \bar{q}$. Then for each k there exist elements $q^{k,i} \in S(C_k^*)$ and numbers $\alpha_i^k \geq 0$, $i = 1, \ldots, m+1$ such that

$$q^k = \sum_{i=1}^{m+1} \alpha_i^k q^{k,i}, \ \sum_{i=1}^{m+1} \alpha_i^k = 1. \tag{3.3}$$

For each fixed i we have the bounded sequence $\{q^{k,i}\}$. Besides, the sequence $\{\alpha^k\}$ with $\alpha^k = (\alpha_1^k, \ldots, \alpha_{m+1}^k)^\top$ is also bounded. Hence, taking $m+2$ times proper subsequences, if necessary, we can suppose that $q^{k,i} \to \bar{q}^i$ for $i = 1, \ldots, m+1$ and $\alpha^k \to \bar{\alpha}$, where, again by **A3**, $\bar{q}^i \in S(C^*)$ for $i = 1, \ldots, m+1$, and

$$\sum_{i}^{m+1} \bar{\alpha}_i = 1, \ \bar{\alpha}_i \geq 0, \ i = 1, \ldots, m+1.$$

By (3.3),

$$\bar{q} = \sum_{i=1}^{m+1} \bar{\alpha}_i \bar{q}^i,$$

hence $\bar{q} \in P(C^*)$. □

In order to provide existence results for (3.2), in the sequel we will assume the following coercivity condition (see, for instance, [12]):

- **(C1)** for each $k = 1, 2, \ldots$, there exist a convex function $\mu_k : \mathbb{R}^n \to \mathbb{R}$, which is weakly coercive with respect to the set V_k, and a number σ_k such that for any point $u \in V_k \setminus V_k(\mu_k, \sigma_k)$ there is a point $v \in V_k$, $\mu_k(v) \leq \mu_k(u)$ such that $P_k(v) \leq P_k(u)$ and $G^k(u)(v-u) \in -\mathrm{int}C_k$.

Remark 3.3. i. Condition **C1** implies condition **C′**, by taking $D = V_k$, $r = \sigma_k$, $\bar{x} = u$, $z = v$, $h = \tau_k P_k$ and $\bar{f} \in F_k(u)$. Indeed, take any point $u \in V_k \setminus V_k(\mu_k, \sigma_k)$; then, by **C1**, there is $v \in V_k$ with $\mu_k(v) \leq \mu_k(u)$ such that $P_k(v) \leq P_k(u)$ and $G^k(u)(v-u) \in -\mathrm{int}C_k$. Then, for any $q^k \in P(C_k^*)$, we have $\langle q^k, G^k(u)(v-u)\rangle < 0$ due to Lemma 2.1. It follows that

$$\langle q^k, G^k(u)(v-u)\rangle + \tau_k(P_k(v) - P_k(u)) < 0.$$

C′ follows by setting $\bar{f} = \sum_{j=1}^m G_j^k(u) q_j^k$.

ii. In case the sets V_k are bounded, condition **C1** can be trivially satisfied by taking, for instance, $\mu_k(x) = 0$ and $\sigma_k \geq 0$, for every k.

The following result holds:

Theorem 3.4. *Let $\{V_k\}$ and $\{P_k\}$ satisfy **A1-4**. Suppose that **C1** is fulfilled. Then*

i. the problem GMVI$_k$ has at least a solution x_k^*, and all the solutions belong to $V_k(\mu_k, \sigma_k)$;

ii. any cluster point x^* of a sequence of solutions $\{x_k^*\}$ is a solution of the problem VVI.

Proof: Assertion i. follows directly from Corollary 2.6 and Remark 3.3-i. Concerning ii., by **A1**, we get that $x^* \in V$. Let us show that $x^* \in W$. From (3.2), there exists $f_k^* \in F^k(x_k^*)$ such that

$$0 \leq P_k(x_k^*) \leq \tau_k^{-1}\langle f_k^*, y - x_k^*\rangle + P_k(y), \quad \forall y \in V_k. \tag{3.4}$$

Take any $w \in V \cap W$; from **A2**-iii, there exists $\{x_k\}$, with $x_k \in V_k$ such that $x_k \to w$ and $P_k(x_k) = 0$ for k large enough. Therefore, by choosing $y = x_k$ in (3.4), we get

$$0 \leq P_k(x_k^*) \leq \tau_k^{-1}\langle f_k^*, x_k - x_k^* \rangle.$$

Denote by $x_{n_k}^*$ a subsequence converging to x^*. From assumption **A4**-ii we get that

$$0 \leq \liminf P_k(x_{n_k}^*) \leq \lim \tau_{n_k}^{-1}\langle f_{n_k}^*, x_{n_k} - x_{n_k}^* \rangle = 0,$$

i.e., $\liminf P_k(x_{n_k}^*) \to 0$. From **A2**-ii, it follows that $P(x^*) = 0$, i.e. $x^* \in W$. Let us prove that x^* is indeed a solution of VVI. First of all, we have already shown that $x^* \in D$. Let $z \in D$; from **A2**-iii there exists $z_k \in V_k$ such that $z_k \to z$, and $P_k(z_k) = 0$. From (3.2) and the assumptions on P_k we get

$$\langle f_k^*, z_k - x_k^* \rangle \geq \tau_k P_k(x_k^*) \geq 0.$$

From **A4**-ii, there exists a subsequence $\{f_{m_k}^*\}$ such that $f_{m_k}^* \to f^*$. It is easy to show, from **A3** and **A4**-iii, that $f^* \in F(x^*)$. Taking $k \to \infty$ in the inequality above, the assertion easily follows. □

4. Existence results

In the previous section we focused on conditions entailing existence of solutions of the approximating problems. Moreover, we found out that any cluster points of a sequence of solutions provides a solution of the 'limit' problem. In order to apply Theorem 3.4, we are interested in sufficient conditions on the data leading to the existence of cluster points for any sequence of solutions of the problems $\{\text{GMVI}_k\}$.

Following some literature on this subject (see, for instance, [12]), the first result takes into account some additional assumptions involving the weakly coercive functions μ_k and the scalars σ_k associated to condition **C1**.

Proposition 4.1. *Let **C1** be satisfied for every k, and assume that the next conditions are fulfilled:*

- **(C2)** *If $v_k \in V_k$ and $\|v_k\| \to +\infty$, then $\liminf_{k\to\infty} \mu_k(v_k) \geq \sigma''$;*
- **(C3)** $\limsup_{k\to\infty} \sigma_k \leq \sigma' < \sigma''$.

Then, any sequence of solutions $\{x_k^*\}$ has a cluster point.

Proof: From Theorem 3.4, every solution x_k^* belongs to $V_k(\mu_k, \sigma_k)$, therefore $\mu_k(x_k^*) \leq \sigma_k$. Assumptions **C2** and **C3** gives the assertion. □

Let us now denote by $d_H(A, B)$ the Hausdorff distance between two nonempty sets $A, B \subset \mathbb{R}^m$, defined as follows:

$$d_H(A, B) := \max\{\sup_{a\in A} d(a, B), \sup_{b\in B} d(b, A)\},$$

where d denotes the distance endowed by the Euclidean norm. In particular, if both sets are bounded, then $d_H(A, B) \in \mathbb{R}$.

Proposition 4.2. *Suppose that the following assumptions hold:*

(B) *i. there exists $\tilde{x} \in \cap_k V_k$ such that $P_k(\tilde{x}) = 0$;*

ii. for any $\{z_k\}$, *with* $z_k \subset V_k$ *and* $\|z_k\| \to +\infty$,

$$\min_{q \in P(C^*)} \frac{\langle G^\top(z_k)q, z_k - \tilde{x}\rangle}{\|z_k - \tilde{x}\|} \to +\infty;$$

iii. set $d_k(x) = d_H(F_k(x), F(x))$; *if* $z_k \in V_k$ *and* $\|z_k\| \to +\infty$, *then* $\limsup_k d_k(z_k) \in \mathbb{R}$.

Then, any sequence of solutions $\{x_k^*\}$ *admits a cluster point.*

Proof: Let us argue by contradiction, by assuming that $\|x_k^* - \tilde{x}\| \to +\infty$. Since x_k^* is a solution for GMVI$_k$, there exists $f_k^* \in F_k(x_k^*)$ such that

$$\langle f_k^*, y - x_k^*\rangle + \tau_k(P_k(y) - P_k(x_k^*)) \geq 0, \quad \forall y \in V_k.$$

Set $y = \tilde{x}$; then, from **B**-i, we get, for every k,

$$\langle f_k^*, \tilde{x} - x_k^*\rangle \geq \tau_k P_k(x_k^*)) \geq 0.$$

From the definition of d_k in **B**-iii, and the closedness of the set $F(x)$ for all $x \in D$, there exists $\phi_k \in F(x_k^*)$ such that

$$\|\phi_k - f_k^*\| \leq d_k(x_k^*), \tag{4.1}$$

and

$$\langle \phi_k, \tilde{x} - x_k^*\rangle + \langle f_k^* - \phi_k, \tilde{x} - x_k^*\rangle \geq 0,$$

implying that

$$\frac{\langle \phi_k, x_k^* - \tilde{x}\rangle}{\|x_k^* - \tilde{x}\|} \leq \frac{\langle f_k^* - \phi_k, \tilde{x} - x_k^*\rangle}{\|x_k^* - \tilde{x}\|}. \tag{4.2}$$

Note that

$$\phi_k = G(x_k^*)^\top q^k, \quad \text{for some } q^k \in P(C^*);$$

in particular, from **B**-ii, we have that

$$\frac{\langle \phi_k, x_k^* - \tilde{x}\rangle}{\|x_k^* - \tilde{x}\|} \to +\infty.$$

As a matter of fact, from (4.1) we have that

$$\frac{\langle f_k^* - \phi_k, \tilde{x} - x_k^*\rangle}{\|x_k^* - \tilde{x}\|} \leq \|f_k^* - \phi_k\| \leq d_k(x_k^*).$$

Therefore, from (4.2), we get

$$+\infty = \limsup_k d_k(x_k^*),$$

contradicting the assumption **B**-iii. □

Remark 4.3. In case $C = C^* = \mathbb{R}^m_+$, condition **B**-ii reduces to the usual coercivity condition for the components G_j, $j = 1, 2, \ldots, m$:

$$\frac{\langle G_j(z_k), z_k - \tilde{x}\rangle}{\|z_k - \tilde{x}\|} \to +\infty, \quad \forall j = 1, 2, \ldots, m.$$

If $C = C_{\text{lex}}$, where C_{lex} denotes the cone associated to the lexicographic order, the C^* is given by

$$C^* = \{q \in \mathbb{R}^m : q = (t, 0, \ldots, 0), \, t \geq 0\},$$

and $P(C^*) = \{\mathbf{e}_1\}$. Thus, condition **B**-ii reduces to the usual coercivity condition for the first component G_1 of G only:

$$\frac{\langle G_1(z_k), z_k - \tilde{x}\rangle}{\|z_k - \tilde{x}\|} \to +\infty.$$

Under different conditions, another existence result can be stated in the framework of C-monotone maps G. Let us first recall that $G : \mathbb{R}^n \subseteq \mathbb{R}^n \to L(\mathbb{R}^n, \mathbb{R}^m)$ is said to be C-monotone on $D \subseteq \mathbb{R}^n$ if

$$(G(x') - G(x))(x' - x) \in C, \quad \forall x', x \in D.$$

This is equivalent to say that the map $x \mapsto G^\top(x)q$ is monotone on D, for every $q \in P(C^*)$.

In the sequel, we will suppose that the set D has nonempty interior. Denote by $\overline{x}$ a point in $\operatorname{int}(D)$. From Lemma 2 in [2] (see also [1]), for any $q \in P(C^*)$, there exist positive numbers $r_q = r_q(\overline{x})$ and $c_q = c_q(\overline{x})$ such that

$$\langle G^\top(x)q, x - \overline{x}\rangle \geq r_q\|G^\top(x)q\| - c_q(\|x - \overline{x}\| + r_q), \quad \forall x \in D; \tag{4.3}$$

in particular, $c_q = \sup_{x \in \overline{B}(\overline{x}, r_q)} \|G^\top(x)q\| < +\infty$.

Proposition 4.4. *Let $\overline{x} \in \operatorname{int}(D)$, and set*

$$r' = \inf_{q \in P(C^*)} r_q, \quad r'' = \sup_{q \in P(C^*)} r_q, \quad c' = \sup_{q \in P(C^*)} c_q.$$

*Suppose that **A2**-iii holds, and the following assumptions are satisfied:*

(B′) *i. the map $G : \mathbb{R}^n \to L(\mathbb{R}^n, \mathbb{R}^m)$ is C-monotone on D;*
ii. for every $\beta \in \mathbb{R}$ the set

$$L_\beta(G) := \{x \in \mathbb{R}^n : \sup_{q \in P(C^*)} \|G^\top(x)q\| \leq \beta\|x\|\}$$

is bounded;
iii. set $d'_k(x) = \sup_{a \in F_k(x), b \in F(x)} d(a, b)$; if $z_k \in V_k$ and $\|z_k\| \to +\infty$, then $\limsup_k d'_k(z_k) \in \mathbb{R}$;
iv. $r' > 0$, $r'', c' \in \mathbb{R}$.

Then, any sequence of solutions $\{x_k^*\}$ admits a cluster point.

Proof: Let $x \in D$ and $f \in F(x)$. Then, there exists $q \in P(C^*)$ such that $f = G^\top(x)q$. Therefore,

$$\langle f, x - \overline{x}\rangle \geq r_q\|f\| - c_q(\|x - \overline{x}\| + r_q),$$

i.e.

$$r_q\|f\| \leq \langle f, x - \overline{x}\rangle + c_q(\|x - \overline{x}\| + r_q).$$

From **B′**-iv,

$$r'\|f\| \leq \langle f, x - \overline{x}\rangle + c'(\|x - \overline{x}\| + r''). \tag{4.4}$$

Let now x_k^* be a solution of GMVI$_k$. Then, for some $f_k^* \in F_k(x_k^*)$,

$$\langle f_k^*, y - x_k^*\rangle + \tau_k(P_k(y) - P_k(x_k^*)) \geq 0 \quad \forall y \in V_k.$$

From assumption **A2**-iii, there exist $v_k \in V_k$, $v_k \to \overline{x}$ such that $P_k(v_k) = 0$ for any k. Therefore, from the previous inequality, taking into account the nonnegativity of P_k, we get

$$\langle f_k^*, v_k - x_k^*\rangle \geq 0. \tag{4.5}$$

From (4.5), for any $f_k \in F(x_k^*)$ we have

$$\begin{aligned}\langle f_k, x_k^* - \overline{x}\rangle &= \langle f_k - f_k^*, x_k^* - \overline{x}\rangle + \langle f_k^*, x_k^* - \overline{x}\rangle\\ &\leq \langle f_k - f_k^*, x_k^* - \overline{x}\rangle + \langle f_k^*, x_k^* - v_k\rangle + \langle f_k^*, v_k - \overline{x}\rangle\\ &\leq \langle f_k - f_k^*, x_k^* - \overline{x}\rangle + \langle f_k^*, v_k - \overline{x}\rangle\\ &\leq d_k'(x_k^*)\|x_k^* - \overline{x}\| + \langle f_k^*, v_k - \overline{x}\rangle.\end{aligned}$$

From (4.4), setting $x = x_k^*$, we have

$$r'\|f_k\| \leq \langle f_k, x_k^* - \overline{x}\rangle + c'(\|x_k^* - \overline{x}\| + r''),$$

therefore

$$\begin{aligned}r'\|f_k\| \leq d_k'(x_k^*)\|x_k^* - \overline{x}\| &+ \langle f_k^* - f_k, v_k - \overline{x}\rangle +\\ &+ \langle f_k, v_k - \overline{x}\rangle + c'(\|x_k^* - \overline{x}\| + r'')\\ \leq (c' + d_k'(x_k^*))\|x_k^* - \overline{x}\| &+ d_k'(x_k^*)\|v_k - \overline{x}\| + \|f_k\|\|v_k - \overline{x}\|.\end{aligned}$$

Since $v_k \to \overline{x}$, there exists an integer k' such that $(r' - \|v_k - \overline{x}\|) > r'/2$. Rearranging the terms we have:

$$r'/2\|f_k\| \leq (r' - \|v_k - \overline{x}\|)\|f_k\| \leq (c' + d_k'(x_k^*))\|x_k^* - \overline{x}\| + d_k'(x_k^*)\|v_k - \overline{x}\|$$

for any $k \geq k'$.

Suppose that $\|x_k^*\| \to +\infty$. Then, from $\mathbf{B}'$-iii, the sequence $\{d_k'(x_k^*)\}$ is bounded from above, and therefore there exists a positive β such that

$$\|f_k\| \leq \beta\|x_k^*\|.$$

In particular, $\sup_{q\in P(C^*)} \|G^\top(x_k^*)q\| \leq \beta\|x_k^*\|$, i.e. $x_k^* \in L_\beta(G)$, contradicting the boundedness assumption $\mathbf{B}'$-ii. $\square$

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