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Aspects of Quantum Field Theories in Three Dimensions

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Alle mie Nonne.

"Non hai veramente capito qualcosa fino a quando
non sei in grado di spiegarlo a tua nonna."
A. Einstein

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I hope the reader could understand and forgive me if I find more convenient to switch, from the very beginning, to italian as he could easily imagine.
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[1] Si lo so, non uso mai questa espressione, però è pur sempre una tesi di dottorato e le parolacce è meglio evitarle.

[2] E' una torta, per altro ottima da immergere nel cappuccino, maliziosi che non siete altro.

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Abstract

In this thesis we construct the one-dimensional topological sector of $\mathcal{N} = 6$ ABJ(M) theory and study its relation with the mass-deformed partition function on $\mathbb{S}^3$. Supersymmetric localization provides an exact representation of this partition function as a matrix integral and it has been proposed that correlation functions of certain topological operators are computed through derivatives with respect to the masses. We present non-trivial evidence for this relation by computing the three- and four- point functions up to one loop and the two-point function at two loops, successfully matching the matrix model expansion at weak coupling and finite ranks. As a by-product, we obtain the two-loop explicit expression for the central charge c_T of ABJ(M) theory. We then shift our attention to the study of the infrared phases of two-node quiver Chern-Simons theories with minimal supersymmetry in three dimensions. We discuss both the cases of Chern-Simons levels with the same and with opposite signs, where the latter case turn out to be more non-trivial. The determination of their phase diagrams allows us to conjecture certain infrared dualities involving either two quiver theories, or a quiver and adjoint QCD_3.

Publications

The present thesis is based on the following publications by the author:

- Peer-reviewed

  N. Gorini, L. Griguolo, L. Guerrini, S. Penati, D. Seminara, P. Soresina. "The topological line of ABJ(M) theory". In: *JHEP* 06 (2021), p. 091. DOI: 10.1007/JHEP06(2021)091. arXiv: 2012.11613 [hep-th].

- Under peer-review

  V. Bashmakov, N. Gorini. "Phases of $\mathcal{N} = 1$ Quivers in $2 + 1$ Dimensions". In: (Sept. 2021). arXiv: 2109.11862 [hep-th].

Some of the future directions discussed at the end of this thesis will be deepened in a forthcoming paper:

N. Gorini, L. Griguolo, L. Guerrini, S. Penati, D. Seminara, P. Soresina. *To appear.*

Overview

One of the most important achievements of Theoretical Physics is the Standard Model, a theory which allows us to describe with huge precision the properties of the fundamental particles we find in nature. The main framework used for describing such a successful theory is relativistic *Quantum Field Theory* (QFT).

Despite the great success for its reliable predictivity, Quantum Field Theory is as much powerful as difficult to study. Indeed, thanks to perturbation theory, whose physical intuition is given by Feynman diagrams, is possible to obtain many informations about the behavior of quantum fields when interactions are weak and, very rarely, strong.

Many of the open problems in theoretical physics are indeed intrinsically related to strongly coupled systems, first of all, the *confinement* phenomenon [1, 2] occurring in Quantum Chromodynamics (QCD) in the low-energy, or large-distances, limit. For this reason, in the last three decades, many efforts have been dedicated to the study of similar, but much more constrained, theories [3] for which having some grasps of non-perturbative phenomena is possible. In this sense, *supersymmetry* and *conformal symmetry* have played a pivotal role in gaining some intuitions, and often analytic control, on the strongly coupled dynamics [4, 5]. Supersymmetry is in fact a powerful tool which relates bosonic and fermionic particles of a physical model. This highly non-trivial relation among the fundamental degrees of freedom heavily constraints both the perturbative and non-perturbative dynamics of the models and allows for huge simplifications in the computation of observables such as the *correlation functions* or the *partition function* of the theory. Indeed, thanks to supersymmetry, it is sometimes possible to formulate an exact description of the model by reducing the infinite-dimensional path integral computing the partition function, to a finite-dimensional integral, which is usually called matrix integral or *matrix model*. This can be achieved through a very powerful mathematical procedure called *localization* [6, 7] as we will discuss in what follows. In this context, a special role is also played by *BPS operators*, namely operators preserving a certain fraction of supersymmetry. Indeed, their correlation functions constitute the most important set of protected observables of the theory since protected operators enjoy robustness against quantum corrections to their quantum numbers. An analogous role is played by conformal symmetry, whose importance and broad applicability, has lead to many non-trivial results concerning the dynamics of theoretical, statistical and condensed matter physics models. The dynamics of conformal field theories (CFTs) is so constrained that any n-point function can be recursively reduced to a linear combination of $n-1$-point functions, where the coefficients, together with the quantum numbers of the fields, constitute the set of *CFT data*. At the level of the operators this property is known as the *operator product expansion* (OPE). This powerful property, combined with the so called *crossing symmetry* for four-point functions, allow one to rephrase the problem of solving a CFT into the determination of the complete set of CFT data. This is the philosophy behind the *conformal bootstrap* program [8–12].

From a more abstract perspective, conformal field theories manifest themselves when we try to explore the landscape of QFTs, or equivalently, their space of couplings. In particular, when we vary the couplings of a theory by implementing some *relevant* deformations, we

trigger the so called *renormalization group* (RG) flow [13], a complicated path whose (fixed) endpoints, if they exist, are exactly described by conformal invariant theories. In this sense we can see a generic QFT as nothing but a deformation of a certain original CFT, therefore we are safely allowed to think at CFTs as the mother theories of all QFTs. This is one of the main reasons why it is important to study and classify all the conformal field theories. It is possible for different QFTs in the high-energy, or ultraviolet (UV), limit to flow to the same CFT in the low-energy, or infrared (IR), limit. This property, known as *universality*, plays a fundamental role in the formulation of the so called *IR dualities* [14, 15]. Such dualities find their most important application in three spacetime dimensions, by giving some prescriptions to relate, in a very non-trivial way, different quantum (gauge) field theories exhibiting a non-perturbative dynamics in the IR regime.
Many of the concepts presented above are discussed in this thesis, which constitutes an attempt to explore the features of supersymmetric and conformal field theories (SCFTs) in three dimensions when both the high and low-energy limits are considered.

In the first part of this thesis we will focus on a family of three-dimensional $\mathcal{N} = 6$ SCFTs, also known as ABJ(M) theories, which are particularly relevant in the explicit realization of the AdS_4/CFT_3 correspondence [16–18], since they possess M-theory duals on $AdS_4 \times S^7/\mathbb{Z}_k$ or type IIA string theory duals on $AdS_4 \times \mathbb{CP}^3$, depending on the particular range of the coupling constants [19, 20]. We will focus however on purely field-theoretic aspects of ABJ(M) theories by giving a complete and explicit characterization of its one-dimensional topological sector, a protected sector consisting of a completely solvable set of space-time independent correlation functions. Thanks to its peculiar properties, the dynamics will be fully accessible in the perturbative regime and, thanks to supersymmetric localization, also in the non-perturbative regime.
Topological sectors are in general particularly relevant because one can extract useful information regarding the quantum theory, like CFT data, bounds on numerical factors involved in the *bootstrap* technique, coefficients of Witten diagrams in the AdS duals, or computing certain exact quantities interpolating between the weak and strong coupling regimes [21–24]. A prototypical example of the topological sector appears in $\mathcal{N} = 4$ SYM in four dimensions [25–27]. The dynamics of a particular subset of chiral primary operators and Wilson loops, living on the same $\mathbb{S}^2$ embedded in the full space-time, is completely controlled by the zero-instanton sector of the two-dimensional Yang-Mills theory [28]. All the correlation functions do not depend on space-time positions and can be computed in terms of (multi)matrix models [29]. In three dimensions, general properties of the superconformal algebra suggest that SCFTs with $\mathcal{N} \geq 4$ always contain a topological sector [22, 30]. In the $\mathcal{N} = 4$ case, a one-dimensional topological sector has been explicitly constructed in [31] as a family of twisted Higgs branch operators belonging to the cohomology of a BRST-like supercharge. As proved there, the cohomological supercharge can be used to perform supersymmetric localization in a large class of $\mathcal{N} = 4$ theories placed on $\mathbb{S}^3$. The result is a matrix model for a topological quantum mechanics in which correlation functions can be computed in terms of matrix-integrals. The existence of a one-dimensional topological sector finds interesting applications also in the study of $\mathcal{N} = 8$ and $\mathcal{N} = 6$ three-dimensional theories where, due to the enhanced supersymmetry, correlation functions of certain topological operators are related to the stress-energy tensor ones in a particular kinematic configuration. The topological sector has thus played a notable role in performing a precision study of maximally supersymmetric ($\mathcal{N} = 8$) SCFTs through conformal bootstrap, allowing to compute exactly some OPE data and constraining "islands" in the parameter space [22, 23, 32]. Here, we will focus on the topological line of $\mathcal{N} = 6$ $U(N_1)_k \times U(N_2)_{-k}$ ABJ(M) theory and study the relation between correlation functions of dimension-one topological operators and the mass-deformed matrix model of the theory.

In the second part of this thesis we will shift our attention to more theoretical aspects of the low-energy physics of three-dimensional quantum field theories with minimal supersymmetry in the presence of a Chern-Simons term. Three-dimensional quantum field theories with $\mathcal{N} = 1$ supersymmetry indeed constitute a remarkable bridge between theories with $\mathcal{N} = 2$ supersymmetry, which are much more constrained thanks to holomorphy [33–35], and genuine non-supersymmetric theories, whose dynamics is a challenging subject. Given that we do not have neither non-renormalization theorems, nor localization techniques at our disposal, it may appear that $\mathcal{N} = 1$ supersymmetry does not give any advantage compared to cases without supersymmetry. Nevertheless, in the recent years our understanding of these theories has overcome a new twist [36–44] (see also [45–48] for earlier considerations). In particular, new tools for studying phase diagrams of $\mathcal{N} = 1$ theories were introduced and applied to the analysis of IR dynamics of a vector multiplet coupled to matter in the adjoint or fundamental representations [41, 42]. Among various interesting phenomena observed in those examples, we mention the existence of *walls* in the parameter space, where the value of a topological quantity, known as the *Witten index*, can jump, as well as the presence of second-order phase transition points. Both features seem to be ubiquitous for known $\mathcal{N} = 1$ theories and stem from the fact that the theories possess real parameters. This property is also shared by $\mathcal{N} = 2$ theories in three dimensions but, differently from the minimal supersymmetric case, the Witten index is not allowed to jump when a wall is present [33]. For this purpose we initiate the study of the phase diagrams of three-dimensional $\mathcal{N} = 1$ *quiver* gauge theories, i.e. theories whose gauge group is given by a product of several non-Abelian factors, coupled to bi-fundamental matter. In particular, we restrict ourselves to $SU(2) \times SU(2)$ and $SU(2) \times U(2)$ two-nodes quivers with a single bi-fundamental matter multiplet. This simple setup allows for a detailed treatment and is still rich enough to accommodate a variety of interesting phenomena. In particular, we reveal a rather non-trivial phase structure of these theories and identify a collection of $\mathcal{N} = 1$ superconformal field theories (SCFT) describing their low-energy dynamics. Basing on our understanding of phase diagrams, we are able to conjecture certain IR dualities, some of which can be understood as the dualization of a node of the quiver, and some others as the confinement of a node. Finally, we comment the time-reversal invariance property of the theory when the gauge groups are the same, and the Chern-Simons levels are opposite. This symmetry gives rise to certain non-renormalization theorems [39] which put strong restrictions on the form of the effective superpotential.

We conclude the thesis with a brief summary and a discussion oriented to the possible future directions.

Chapter 1

Introduction

In this chapter we review some fundamental notions regarding the language which will be used throughout this thesis, the one of Superconformal Field Theories (SCFTs), and we will introduce a particular case of such theories, namely ABJ(M) theories.
In section 1.1 we review some general aspects of conformal field theories, in section 1.2 we add susy to the previous discussion ending up with superconformal field theories. In section 1.3 we review some basic aspects of Chern-Simons and Chern-Simons-matter theories and discuss some of their relevant supersymmetric versions. Finally, in section 1.4 we introduce and discuss the main aspects of the ABJ(M) theory.

1.1 Conformal Field Theories

Conformal Field Theories are particular kinds of Quantum Field Theories enjoying conformal invariance. This additional requirement can be encoded in a bosonic extension of the usual Poincaré group to the so called *conformal group*. In this thesis we will focus on Euclidean (Super-)Conformal Field Theories in $d = 3$, thus we will now review the main aspects of conformal symmetry when $d \geq 3$.[1] For a complete review on this topics see e.g. [49–53].

1.1.1 Conformal Algebra

The conformal group is the group generated by those transformations for which, under a generic coordinate transformation $x' = x'(x)$, the metric tensor transforms as

$$g'_{\mu\nu}(x') = \frac{\partial x'^{\rho}}{\partial x^{\mu}} \frac{\partial x'^{\sigma}}{\partial x^{\nu}} g_{\rho\sigma}(x) = \Omega(x)^2 g_{\rho\sigma}(x). \tag{1.1}$$

We notice that when $\Omega(x) = 1$ and $g_{\mu\nu} = \eta_{\mu\nu}$, we get back to the Poincaré group which thus constitute a subgroup of the conformal group. Since the theories analyzed in this thesis are defined on $\mathbb{R}^3$, we will restrict from now on to the flat space case, i.e. $g_{\mu\nu} = \eta_{\mu\nu}$.
At the infinitesimal level, by considering a transformation of the form $x'^{\mu} = x^{\mu} + \epsilon^{\mu}(x)$, equation (1.1) can be recasted in the following differential condition

$$\partial_{\mu}\epsilon_{\nu} + \partial_{\nu}\epsilon_{\mu} = \frac{2}{d}(\partial \cdot \epsilon)\eta_{\mu\nu}. \tag{1.2}$$

By acting on the above equation twice with respectively ∂_{ν} and ∂^{ν} and combining the result with (1.1), we get the following expression

$$\left(\eta_{\mu\nu}\Box + (d-2)\partial_{\mu}\partial_{\nu}\right)(\partial \cdot \epsilon) = 0, \tag{1.3}$$

[1] The $d = 2$ case it is a well-known very special case that deserves a separate discussion which we are not going to make here.

which, after contracting both sides with $\eta^{\mu\nu}$, becomes

$$(d-1)\Box(\partial\cdot\epsilon)=0. \tag{1.4}$$

From the equations in (1.3) and (1.4) it is now easy to obtain the constraints

$$\Box(\partial\cdot\epsilon)=0, \qquad \partial_\mu\partial_\nu(\partial\cdot\epsilon)=0, \tag{1.5}$$

which fix the general structure of $\epsilon^\mu(x)$, namely a polynomial at most quadratic in x^μ. The solution can be then finally expressed in the usual familiar structure as

$$\epsilon_\mu(x)=a_\mu+\omega_{\mu\nu}x^\nu+\lambda x^\mu-2(x\cdot b)x^\mu+x^2b^\mu, \tag{1.6}$$

which represents the most general form of the so called *conformal Killing vector*.
From (1.6) we can immediately read the infinitesimal transformation for translations (generated by P_μ and parametrized by a^μ), rotations (generated by $M_{\mu\nu}$ and parametrized by $\omega_{\mu\nu}$), dilatations (generated by D and parametrized by λ) and the so called *special conformal transformations* (generated by K_μ and parametrized by b^μ).
We notice that the number of independent components of the parameters introduced above are respectively $d+\frac{d(d-1)}{2}+1+d=\frac{(d+2)(d+1)}{2}$ which is the dimension of an $\mathfrak{so}(2+d)$-like algebra. The commutation relations satisfied by the generators are indeed

$$\begin{gathered}[D,P_\mu]=iP_\mu, \qquad [D,K_\mu]=-iK_\mu, \qquad [K_\mu,P_\nu]=2i(\eta_{\mu\nu}D-M_{\mu\nu})\\ [M_{\mu\nu},P_\rho]=-i(\eta_{\mu\rho}P_\nu-\eta_{\nu\rho}P_\mu), \qquad [M_{\mu\nu},K_\rho]=-i(\eta_{\mu\rho}K_\nu-\eta_{\nu\rho}K_\mu)\\ [M_{\mu\nu},M_{\rho\sigma}]=i(\eta_{\mu\sigma}M_{\nu\rho}+\eta_{\nu\rho}M_{\mu\sigma}-\eta_{\mu\rho}M_{\nu\sigma}-\eta_{\mu\sigma}M_{\nu\rho})\end{gathered} \tag{1.7}$$

which actually describe an $\mathfrak{so}(1,d+1)$ algebra. An elegant way to achieve this is to define the following set of generators

$$J_{\mu\nu}=M_{\mu\nu}, \qquad J_{\mu,0}=\frac{P_\mu-K_\mu}{2}, \qquad J_{d+1,\mu}=\frac{P_\mu+K_\mu}{2}, \qquad J_{d+1,0}=D \tag{1.8}$$

which satisfy the commutation relation

$$[J_{AB},J_{CD}]=i(\eta_{AD}J_{BC}+\eta_{BC}J_{AD}-\eta_{AC}J_{BD}-\eta_{BD}J_{AC}), \qquad J_{AB}=-J_{BA} \tag{1.9}$$

where $A,B,C,D=0,\mu,d+1$ with $\mu=1,\dots,d$ and $\eta_{AB}=\mathrm{diag}(-1,1,\dots,1)$ is the metric of the extended physical space $\mathbb{R}^{1,d+1}$. Is it now clear from (1.9) that the algebra generated by J_{AB} is $\mathfrak{so}(1,d+1)$. In the same spirit one can show that in general, for $\mathbb{R}^{p,q}$ with $d=p+q$, the conformal algebra is $\mathfrak{so}(p+1,q+1)$.

1.1.2 Correlators

At this point, it is reasonable to ask what is the spectrum of a CFT and what are the observables that can be computed. For this purpose, let us introduce operators as the irreducible representations of the above conformal algebra.
It is possible to notice from the relations in (1.7), that the generator of dilatations D defines a real Cartan subalgebra $\mathbb{R}\simeq\mathfrak{so}(1,1)$ of $\mathfrak{so}(1,d+1)$. This means that any other generator belonging to the latter has some weight under its action. We call this weight *scaling dimension* and indicate it with Δ. It is immediate to see that $\Delta_P=1$, $\Delta_K=-1$ and $\Delta_M=0$, thus P_μ and K_μ play the role of raising and lowering operator respectively for Δ.

In the usual radial quantization scheme[2] for CFTs, the dilatation operator D plays the role of the Hamiltonian of our theory. Since every physical state must have a bounded

[2]Since the cylinder $\mathbb{S}^{d-1}\times\mathbb{R}$ and the Euclidean space $\mathbb{R}^d$ are conformally equivalent, we can quantize a CFT either on the former or the latter. This implies that a CFT quantized on a d-dimensional cylinder on equal time slices is perfectly equivalent to a CFT quantized on d-dimensional flat space on equal radius slices.

energy spectrum from below, there always exists one state which has the lowest possible "energy", in other words there must exists a lowest weight state for Δ, namely a state annihilated by K_μ. Such states, which we call (*quasi-*)*primary states*, sit also in irreducible representations[3] of the Lorentz algebra $\mathfrak{so}(d)$. We shall indicate them as $|[L]_\Delta\rangle$, where $[L]$ label the $\mathfrak{so}(d)$ representation. At this point we define a so called *conformal primary state* as a state satisfying the following conditions

$$\begin{aligned} M_{\mu\nu}\,|[L]_\Delta\rangle_s &= (\Sigma_{\mu\nu})_r{}^s\,|[L]_\Delta\rangle_s\,, \\ D\,|[L]_\Delta\rangle &= -i\Delta\,|[L]_\Delta\rangle\,, \\ K_\mu\,|[L]_\Delta\rangle &= 0, \end{aligned} \tag{1.10}$$

where $\Sigma_{\mu\nu}$ is the spin matrix and r, s are indices in a suitable Lorentz representation.
As we saw above, K_μ and P_μ act as ladder operators for Δ, therefore, it is possible to construct irreducible representations of the conformal algebra by acting with P_μ on primary states. The generic structure of such states will be then

$$P_{\mu_1}\dots P_{\mu_n}\,|[L]_\Delta\rangle\,. \tag{1.11}$$

States obtained in this way are called *descendants* and the set of primary states together with their descendants form the so called *conformal family*.
From the discussion we made above, since we want to work in a theory of fields, we need now to associate states to fields. This can be achieved thanks to the celebrated *state-operator correspondence* which, when we consider the radial quantization scheme, is realized by associating states living on the sphere to operators defined in the interior of it. Operators inserted at the origin $x^\mu = 0$ are called *primary operators* $\Phi_{L,\Delta}$ and are explicitly associated to primary states as follows

$$\Phi_{L,\Delta}(0)\,|0\rangle \equiv |[L]_\Delta\rangle\,. \tag{1.12}$$

At this point it is easy to show that the relations in (1.10) are correctly recovered if primary operators satisfy the following conditions

$$\begin{aligned} [M_{\mu\nu}, \Phi_{L,\Delta}(0)] &= \Sigma_{\mu\nu}\Phi_{L,\Delta}(0), \\ [D, \Phi_{L,\Delta}(0)] &= -i\Delta\Phi_{L,\Delta}(0), \\ [K, \Phi_{L,\Delta}(0)] &= 0. \end{aligned} \tag{1.13}$$

It would be tempting now to say that operators inserted away from the origin can be associated to descendant states and, in fact, this is not completely true. Indeed states associated to such operators turn out to be an infinite linear combination of descendant states

$$|\Psi\rangle \equiv \Phi_{L,\Delta}(x)\,|0\rangle = e^{ix\cdot P}\Phi_{L,\Delta}(0)e^{-ix\cdot P}\,|0\rangle = e^{ix\cdot P}\,|[L]_\Delta\rangle = \sum_n \frac{1}{n!}(ix\cdot P)^n\,|[L]_\Delta\rangle\,, \tag{1.14}$$

whereas descendant states are associated to the derivatives of operators computed at the origin, for example

$$-i\partial_\mu\Phi_{L,\Delta}(x)|_{x=0}\,|0\rangle = [P_\mu, \Phi_{L,\Delta}(0)]\,|0\rangle = P_\mu\,|[L]_\Delta\rangle\,. \tag{1.15}$$

[3]Note that a primary state is unique only if we require that it is also an highest weight state of the Lorentz group.

We are now ready to come back to our main discussion and introduce how operators transform under the conformal group. Primary fields transform under the conformal group as

$$\Phi_i'(x') = \Omega(x)^{-\Delta}\mathcal{D}_i^{\ j}(\mathcal{R}(x))\Phi_j(x) \tag{1.16}$$

where $\mathcal{D}(\mathcal{R})$ is the proper representation of the Lorentz group and Ω is the conformal factor we introduced in (1.1).
All the fields in a unitary CFT can only be primaries or descendants and, being descendants obtained from primaries by acting with the momentum generator, all the relevant (local) informations about the theory are captured by the dynamics of primary fields. For this reason let us introduce correlators of primary fields as the main, but not unique, observables of our theory.

In a conformal invariant theory, correlators are heavily constrained by conformal invariance. As an example, by considering the two-point function of scalar primaries we get

$$\langle\phi_1(x)\phi_2(y)\rangle = f(|x-y|), \tag{1.17}$$

thanks to rotation and translation invariance. From (1.16), is then easy to implement invariance under dilatations, namely $x' = \lambda x$, and get

$$\langle\phi_1(x)\phi_2(y)\rangle = \frac{C_{12}}{|x-y|^{\Delta_1+\Delta_2}}. \tag{1.18}$$

where C_{12} is some constant.
Recalling now that the finite form for a special conformal transformation is $x'^{\mu} = \frac{x^{\mu}+x^2b^{\mu}}{1+2(b\cdot x)+b^2x^2}$ and using (1.16) again, we can implement this transformation to constraint even more the result above. At the end of the story, the two-point function of scalar primaries becomes

$$\langle\phi_1(x)\phi_2(y)\rangle = \begin{cases} \frac{C_{12}}{|x-y|^{2\Delta}} & \text{if } \Delta_1=\Delta_2=\Delta \\ 0 & \text{if } \Delta_1\neq\Delta_2 \end{cases} \tag{1.19}$$

Notice that, after a proper normalization of the fields, we can safely set $C_{12} = 1$.
The same game can be played with the next non-trivial correlator involving scalar primaries, the three-point function. Again conformal invariance constraints its structure which turns out to be

$$\langle\phi_1(x)\phi_2(y)\phi_3(z)\rangle = \frac{C_{123}}{|x-y|^{\Delta_1+\Delta_2-\Delta_3}|y-z|^{\Delta_2+\Delta_3-\Delta_1}|z-x|^{\Delta_3+\Delta_1-\Delta_2}} \tag{1.20}$$

where now C_{123} is a constant that cannot be fixed by any conformal symmetry. The fact that the three-point function can be fixed up to a constant tells us that such a constant represents something which is physically relevant for the theory. This is actually the case, since all the higher n-point functions can be in principle fully determined once we know all the three-point function constants for all the primaries of the theory and their scaling dimensions[4]. Such quantities $\{C_{ijk}, \Delta_l\}$ are usually called *conformal data*.
If we proceed with the four-point function of scalar primaries, we notice that conformal invariance can fix its structure up to a function of particular conformal invariant combinations of the coordinates called *cross-ratios*. Explicitly we have

$$\langle\phi_1(x_1)\phi_2(x_2)\phi_3(x_3)\phi_4(x_4)\rangle = f(u,v)\prod_{i<j}^{4} x_{ij}^{\Delta/3-\Delta_i-\Delta_j}, \qquad u = \frac{x_{12}x_{34}}{x_{13}x_{24}},\ v = \frac{x_{12}x_{34}}{x_{23}x_{14}} \tag{1.21}$$

[4]We need to know also the representation of the Lorentz group under which they transform.

where we used for shorthand notation that $x_{ij} = x_i - x_j$, $\Delta = \sum_{i=1}^4 \Delta_i$ and u, v are the two inequivalent cross-ratios we can build out of four space-time coordinates.[5]
As we previously anticipated, the function $f(u,v)$ can be fully determined once all the conformal data are known. It general those data are not known at all and some other constraints must be found in order to determine $f(u,v)$.
By looking for example at (1.21) and imposing that all the scalar primaries appearing there are the same, we get

$$\langle \phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4)\rangle = \frac{g(u,v)}{x_{12}^{2\Delta}x_{34}^{2\Delta}}. \tag{1.22}$$

The correlator is now manifestly invariant under permutations of x_i and this fact leads to the following consistency conditions

$$g(u,v) = g\left(\frac{u}{v}, \frac{1}{v}\right), \tag{1.23}$$

$$g(u,v) = \left(\frac{u}{v}\right)^{\Delta} g(v,u). \tag{1.24}$$

The equations above derive from a more general result known as *crossing symmetry equation*, which states nothing but that the value of a n-point correlation function must be independent of the channel we choose for computing it. This is the general idea on which the *conformal bootstrap* [4, 12, 52] is based on.

1.1.3 Unitarity Bounds

The power of conformal invariance goes beyond the constraints we found above about observables. Indeed, for a unitary CFT, it also allows to constraint the scaling dimension Δ of primary operators by simply requiring the non-negativity of the norm of every state appearing in a conformal multiplet. [6]
In Euclidean space, hermitian conjugation maps $t \to -t$, which, in the radial quantization scheme, is equivalent to the action of the inversion element $\mathcal{I}$ of the conformal group[7].
At the level of conformal generators, hermitian conjugation should act by the same operator and what results are the following transformations

$$\begin{aligned} P_\mu^\dagger &= \mathcal{I}P_\mu\mathcal{I}^{-1} = K_\mu, & K_\mu^\dagger &= P_\mu, \\ M_{\mu\nu}^\dagger &= M_{\mu\nu}, & D^\dagger &= -D. \end{aligned} \tag{1.25}$$

It is easy to realize that the transformations above leave the algebra in (1.7) invariant.
In order to obtain the so called *unitarity bounds*, we can now compute the norms of certain descendants and impose the non-negativity of them. Let us start from the level-one[8] descendants of scalar primaries. Assuming to normalize the norm of primary states to one, we have

$$\begin{aligned} 0 \le ||a^\mu P_\mu \,|[\phi]_\Delta\rangle\,||^2 &= a^\mu a^\nu \,\langle [\phi]_\Delta|\, K_\mu P_\nu \,|[\phi]_\Delta\rangle \\ &= a^\mu a^\nu \,\langle [\phi]_\Delta|\, [K_\mu, P_\nu] \,|[\phi]_\Delta\rangle \\ &= a^\mu a^\nu \,\langle [\phi]_\Delta|\, 2i(\eta_{\mu\nu}D - M_{\mu\nu}) \,|[\phi]_\Delta\rangle \\ &= |a|^2 2i(-i\Delta) = 2|a|^2\Delta \end{aligned} \tag{1.26}$$

[5]Note that for a generic n-point function they are exactly $\frac{n(n-3)}{2}$.
[6]For Euclidean theories this property is actually called *reflection positivity*.
[7]Formally, the inversion element does not belong to the connected component of the conformal group being an $O(1, d+1)$ transformation. This means that its finite action cannot be obtained by exponentiating any generator in the $\mathfrak{so}(1, d+1)$ algebra.
[8]The "level" refers to the number of times we apply the momentum generator to our primary field.

where a^μ is a rank-one tensor parameter and in the third inequality we used the conformal algebra in (1.7). From the result obtained above, we get the expected constraint $\Delta \geq 0$.
One can also proceed in computing the norm of level-two descendants and find

$$\begin{aligned} 0 \leq ||a^{\mu\nu} P_\mu P_\nu \,|[\phi]_\Delta\rangle\, ||^2 &= a^{\mu\nu} a^{\rho\sigma} \langle [\phi]_\Delta |\, K_\sigma K_\rho P_\mu P_\nu \,|[\phi]_\Delta\rangle \\ &= \dots \\ &= 4\Delta(2(\Delta+1) a^{\mu\nu} a_{\mu\nu} - a^\mu{}_\mu a^\nu{}_\nu) \\ &= 4\Delta(2(\Delta+1)d - d^2). \end{aligned} \tag{1.27}$$

where in the last equality we have chosen[9] $a^{\mu\nu} = \delta^{\mu\nu}$. By putting together the results found in (1.26) and (1.27), the final result for the unitarity bound of scalars primaries becomes

$$\Delta = 0 \qquad \text{or} \qquad \Delta \geq \frac{d}{2} - 1. \tag{1.28}$$

One could in principle compute norms at higher level but it turns out that the constraint found in (1.28) is actually exact regardless the level we consider.
For generic representations $|[h_a]_\Delta\rangle$, one can compute the only non-trivial term in the third line of (1.26), namely

$$\langle [h^a]_\Delta | M_{\mu\nu} | [h^b]_\Delta \rangle. \tag{1.29}$$

The expression above can be evaluated by exploiting the following trick

$$(M_{\mu\nu})^{ab} = \frac{1}{2} (V^{\rho\sigma}_{\mu\nu})(M_{\rho\sigma})^{ab} = (V \cdot M)^{ab}_{\mu\nu} \tag{1.30}$$

where V is the generator of rotations in the vector representation of the Lorentz algebra and we defined the inner product for generic representations as $A \cdot B \equiv \frac{1}{2} A_{ab} B_{ab}$.
The inner product above can be recasted in a useful form when acting on the states in (1.29), namely

$$\begin{aligned} V \cdot M &= \frac{1}{2} \left((V+M)^2 - V^2 - M^2 \right) \\ &= \frac{1}{2} \left(-\mathcal{C}(V \otimes \mathcal{R}_h) + \mathcal{C}(V) + \mathcal{C}(\mathcal{R}_h) \right) \end{aligned} \tag{1.31}$$

where $\mathcal{R}_h$ is the $\mathfrak{so}(d)$ representation of $|[h^a]_\Delta\rangle$ and $\mathcal{C}$ stands for the quadratic Casimir. The bound we get for Δ thus becomes

$$\Delta \geq \frac{1}{2} \left(\mathcal{C}(V) + \mathcal{C}(\mathcal{R}_h) - \mathcal{C}(V \otimes \mathcal{R}_h) \right). \tag{1.32}$$

The most restrictive bound we can get for Δ is the one for which $\mathcal{C}(V \otimes \mathcal{R}_h)$ assumes its minimal value, therefore

$$\Delta \geq \frac{1}{2} \left(\mathcal{C}(V) + \mathcal{C}(\mathcal{R}_h) - \mathcal{C}(\mathcal{R}_{\min}) \right). \tag{1.33}$$

with $\mathcal{R}_{\min} \in V \otimes \mathcal{R}_h$. Notice that the result found for scalars in (1.28) is the only more restrictive case than the general rule of (1.33).
Primary states for which the inequality in (1.33) is strict give rise to the so called *long multiplets*, if instead the inequality becomes an equality then they give rise to *short multiplets*.

One important consequence of (1.33) is that, when the bound is saturated, we find the so called *null states*. Primaries admitting this kind of states must saturate the bound and, more importantly, their scaling dimension cannot acquire any correction at quantum level. In other words, they represent some *protected* quantities. Some trivial examples are

[9] The starting point of the relations above is that the matrix of inner products should be non-negative. Here we consider just the trace part, but the same result follows by taking the symmetric traceless part.

- Massless scalar: $P_\mu P^\mu \phi = \partial_\mu \partial^\mu \phi = 0 \quad \Longrightarrow \quad \Delta = \frac{d-2}{2}$,
- Massless Dirac fermion: $\gamma^\mu P_\mu \psi = \partial\!\!\!/ \psi = 0 \quad \Longrightarrow \quad \Delta = \frac{d-1}{2}$,
- Traceless stress-energy tensor: $P_\mu T^{\mu\nu} = \partial_\mu T^{\mu\nu} = 0 \quad \Longrightarrow \quad \Delta = d$,
- Conserved currents: $P_\mu J^\mu = \partial_\mu J^\mu = 0 \quad \Longrightarrow \quad \Delta = d - 1$.

We can in general imagine the structure of a conformal multiplet as a function of the scaling dimension of its primary state. The scaling dimension depends itself on the parameters of the theory and, when they are varied, it will in general change. When Δ satisfies the strict inequality of (1.33) the multiplet is long, but, when the bound is hit, a null state will appear at a certain level l causing the multiplet to become short. This very important phenomenon is usually called *Recombination*.
What happens in general is that short multiplets (or long multiplets with at least one short multiplet) may merge and generate a long multiplet. Schematically we have

$$[h^a]_\Delta + [h']_{\Delta+l} \longrightarrow [h^a]_{\Delta+\epsilon}, \tag{1.34}$$

with Δ saturating the unitarity bound and h' being the Lorentz representation of the primary state filling the gap left by the first null state appearing in the short multiplet.
An instructive example is the following

$$[V^\mu]_{\Delta=d-1} \oplus [\phi]_{\Delta=d} \longrightarrow [V^\mu]_{\Delta=d-1+\epsilon}, \tag{1.35}$$

which, on the field theoretic side, can be interpreted as

$$\partial_\mu J^\mu = 0 \ \oplus \ \mathbb{O} \longrightarrow \partial_\mu J^\mu = g\mathbb{O}. \tag{1.36}$$

The relation above corresponds to a deformation of the starting CFT implemented by turning on a coupling g and adding a relevant (marginal) operator $\mathbb{O}$ causing the current to be no more conserved and, therefore, no more protected.

At this point we are ready to add more ingredients to the conformal field theory environment, such as supersymmetry, and discuss some important aspects of superconformal field theories.

1.2 Superconformal Field Theories

In this section we want to briefly introduce supersymmetry and promote conformal field theories to superconformal field theories. Nice reviews covering many aspects of superconformal field theories are [54–58]. Many aspects of conformal field theories will hold also in this context but some new important features will appear and enrich the set of tools at our disposal for obtaining constrained, and sometimes also exact, results. Detailed reviews on supersymmetry and its field-theoretical implications are [59, 60].

1.2.1 Supersymmetry

Supersymmetry is a fermionic symmetry which, roughly speaking, associates fermionic partners to bosonic particles and viceversa. Such symmetry is generated by $\mathcal{N}$ spin$-\frac{1}{2}$ fermionic generators Q^I_α, where $I = 1, \dots, \mathcal{N}$, $\alpha = 1, 2$, called *supercharges*[10].

[10] In four dimensions we have Q but also $\bar{Q}$, which are respectively $(\frac{1}{2}, 0)$ and $(0, \frac{1}{2})$ Lorentz spinors.

From a more theoretical point of view, supersymmetry is the only known consistent extension to the well-known Coleman-Mandula theorem [61], which states that the most general continuous symmetry group preserving the S-matrix[11] has to have the following form

$$G_{CM} = \text{Poincaré} \times \text{Internal Symmetries}. \tag{1.37}$$

The Coleman-Mandula theorem assumes that the algebra $\mathcal{G}$ involves only commutator and, in fact, the only way to avoid this "no-go" theorem is indeed to introduce fermionic generators, as was shown by Haag, Lopuszanski and Sohnius [62]. For supersymmetric quantum field theories therefore we have

$$G_{HLS} = \text{superPoincaré} \times \text{Internal Symmetries}. \tag{1.38}$$

By focusing on the Poincaré algebra, its supersymmetric extension becomes

$$\begin{aligned} &[M_{\mu\nu}, M_{\rho\sigma}] = i(\eta_{\mu\sigma}M_{\nu\rho} + \eta_{\nu\rho}M_{\mu\sigma} - \eta_{\mu\rho}M_{\nu\sigma} - \eta_{\mu\sigma}M_{\nu\rho}) \\ &[M_{\mu\nu}, P_\rho] = -i(g_{\mu\rho}P_\nu - g_{\nu\rho}P_\mu) \qquad \{Q^I_\alpha, Q^J_\beta\} = \varepsilon_{\alpha\beta}Z^{IJ} \\ &[M_{\mu\nu}, Q^I_\alpha] = \frac{1}{2}(\sigma^{\mu\nu})_\alpha{}^\beta Q^I_\beta \qquad \{Q^I_\alpha, \bar{Q}^{J\beta}\} = 2(\sigma^\mu)_\alpha{}^\beta P_\mu \delta^{IJ} \end{aligned} \tag{1.39}$$

where $Z^{IJ} = -Z^{JI}$ is the *central charges matrix*. For a generic number $\mathcal{N}$ of supersymmetries, the algebra in (1.39) admits an (outer) automorphism given by rotations of the supercharges, which is called *R-symmetry*. In general the maximal R-symmetry group is $U(\mathcal{N})$ but it may happen that additional conditions reduce it to some of its subgroups.[12]

At this point, it is easy to realize that $P_\mu P^\mu$ is still a Casimir for the algebra in (1.39) while $W_\mu W^\mu$ is not[13]. This means that only the mass is a good quantum number for the fundamental objects living in a supersymmetric theory and not the spin. Such fundamental objects are called *supermultiplets*. Supermultiplets can be constructed by observing that (suitable combinations of) Q and $\bar{Q}$ play the role of ladder operators for the spin, just like P and K do in the conformal case for the scaling dimension. Again, multiplets admitting null descendant states are called *short* or *BPS*, otherwise they are called *long*.
With this notions at hand, let us introduce the supersymmetrization of the conformal algebra we saw in (1.7) and discuss more in detail the structure of (super)conformal multiplets.

1.2.2 Superconformal Algebra

After having introduced the supersymmetric extension of the Poincaré algebra, one could then ask what happens if we extend the same construction to the conformal algebra of (1.7). The generalization is not as straigthforward as it seems, indeed there are some peculiarities that should be mentioned. Firstly, the supersymmetry algebra can be extended to the superconformal one only for $d \leq 6$ space-time dimensions [63, 64]. Secondly, spinor representations of the conformal group can be explicitly represented as two combined Lorentz spinors[14]. This fact induces us to introduce a new set of fermionic generators S, called *superconformal charges*[15], which allow us to explicitly write the superconformal algebra in

[11] Assuming the usual axioms of quantum field theories like locality, causality, unitarity etc... .

[12] This is the case for example of $\mathcal{N} = 4$ SYM theory, for which the $U(1)$ inside $U(4)$ is central. Moreover, the scalars can be packed into a real six-dimensional antisymmetric representation which is allowed only for $SU(4) \simeq SO(6)$ and not $U(4)$. An alternative explanation is that, when compactifying the ten-dimensional SYM theory [3] on $\mathbb{T}^6$, we get the breaking pattern $SO(1,9) \to SO(1,3) \times SO(6)$, which reproduces the correct R- and space-time symmetries of the 4d theory.

[13] Where $W_\mu = \frac{1}{2}\epsilon^{\mu\nu\rho\sigma} p_\nu M_{\rho\sigma}$ is the usual Pauli-Lubanski pseudo-vector.

[14] In Minkowskian signature this happens because $\mathrm{Cl}(1, d-1) \otimes_{\mathbb{R}} \mathrm{Cl}(1, d-1) \simeq \mathrm{Cl}(2, d)$.

[15] To avoid confusion, from now on, we will refer to S charges as superconformal charges and to Q charges as Poincaré supercharges.

a more confortable way (though less compact), namely in terms of bosonic generators and Lorentz spinors only. Last but not least, in third instance, the R-symmetry is no more simply an automorphism of the algebra but is in fact part of the algebra. This will clearly have an impact on the classification of irreducible representations of the full superconformal algebra.

For simplicity, we prefer focusing on the superconformal algebra in $d = 2+1$ dimensions with $\mathcal{N}$ supersymmetries, since this will be the main setup for the thesis.
The Lie superalgebra corresponding to the above situation is $\mathfrak{osp}(\mathcal{N}|4)$ whose maximal bosonic subalgebra is $\mathfrak{sp}(4,\mathbb{R}) \oplus \mathfrak{so}(\mathcal{N}) \simeq \mathfrak{so}(2,3) \oplus \mathfrak{so}(\mathcal{N})$. The first bosonic group is nothing but the space-time symmetry algebra and $\mathfrak{so}(\mathcal{N})$ is now the R-symmetry algebra.
By switching to the three-dimensional spinor basis, the algebraic relations involving bosonic generators become

$$
\begin{aligned}
&[M_\alpha{}^\beta, P_{\gamma\delta}] = \delta_\gamma{}^\beta P_{\alpha\delta} + \delta_\delta{}^\beta P_{\alpha\gamma} - \delta_\alpha{}^\beta P_{\gamma\delta} \ , && [D, P_{\alpha\beta}] = P_{\alpha\beta} \ , \\
&[M_\alpha{}^\beta, K^{\gamma\delta}] = -\delta_\alpha{}^\gamma K^{\beta\delta} - \delta_\alpha{}^\delta K^{\beta\gamma} + \delta_\alpha{}^\beta K^{\gamma\delta} \ , && [D, K^{\alpha\beta}] = -K^{\alpha\beta} \ , \\
&[M_\alpha{}^\beta, M_\gamma{}^\delta] = \delta_\gamma{}^\beta M_\alpha{}^\delta - \delta_\alpha{}^\delta M_\gamma{}^\beta \ , && [K^{\alpha\beta}, P_{\gamma\delta}] = 4\delta_{(\gamma}^{(\alpha} M_{\delta)}{}^{\beta)} + 4\delta_{(\gamma}^{\alpha}\delta_{\delta)}^{\beta} D \ , \\
&\qquad [R^{rs}, R^{tu}] = i(\delta^{rt}R^{su} - \delta^{st}R^{ru} - \delta^{ru}R^{st} + \delta^{su}R^{rt}) \ ,
\end{aligned}
\tag{1.40}
$$

whereas the relations involving fermionic generators are

$$
\begin{aligned}
&\{Q_\alpha^r, Q_\beta^s\} = 2\delta^{rs}P_{\alpha\beta} \ , && \{S^{\alpha r}, S^{\beta s}\} = -2\delta^{rs}K^{\alpha\beta} \ , \\
&[K^{\alpha\beta}, Q_\gamma^r] = -i(\delta_\gamma{}^\alpha S^{\beta r} - \delta_\gamma{}^\beta S^{\alpha r}) \ , && [P_{\alpha\beta}, S^{\gamma r}] = -i(\delta_\alpha{}^\gamma Q_\beta^r - \delta_\beta{}^\gamma Q_\alpha^r) \ , \\
&[M_\alpha{}^\beta, Q_\gamma^r] = \delta_\gamma{}^\beta Q_\alpha^r - \frac{1}{2}\delta_\alpha{}^\beta Q_\gamma^r \ , && [M_\alpha{}^\beta, S^{\gamma r}] = -\delta_\alpha{}^\gamma S^{\beta r} + \frac{1}{2}\delta_\alpha{}^\beta S^{\gamma r} \ , \\
&[D, Q_\alpha^r] = \frac{1}{2}Q_\alpha^r \ , && [D, S^{\alpha r}] = -\frac{1}{2}S^{\alpha r} \ , \\
&[R^{rs}, Q_\alpha^t] = i(\delta^{rt}Q_\alpha^s - \delta^{st}Q_\alpha^r) \ , && [R^{rs}, S^{\alpha t}] = i(\delta^{rt}S^{\alpha s} - \delta^{st}S^{\alpha r}) \ , \\
&\qquad \{Q_\alpha^r, S^{\beta s}\} = 2i(M_\alpha{}^\beta + \delta_\alpha{}^\beta D)\delta^{rs} + 2\delta_\alpha{}^\beta R^{rs} \ ,
\end{aligned}
\tag{1.41}
$$

where $r, s, t, u = 1, \ldots, \mathcal{N}$ indices for the vector representation of $\mathfrak{so}(\mathcal{N})$. For all the conventions on spinor indices see for example [22, 65].

1.2.3 Multiplets and Unitary Bounds

Having the full superconformal algebra at our disposal, we are now allowed to study its irreducible representations. Analogously to what we did for the conformal case, let us focus in particular on its highest weight states, namely the so called *superconformal primaries*.
Irreducible representations are univocally identified by the set of Dynkin labels associated to the maximal bosonic subgroup of the superconformal algebra $\mathfrak{so}(2,3) \oplus \mathfrak{so}(\mathcal{N})$.
A generic irreducible representation will be hence identified by $|\Delta, j, r\rangle$ where Δ is the usual scaling dimension, j is the spin and $r = (r_1, \ldots, r_{\frac{\mathcal{N}}{2}})$ are the R-symmetry Dynkin labels (for more details see [56, 65]).
Like the conformal case, the generator K is still a lowering operator for D since again $\Delta_K = -1$. However, in the superconformal case, there are additional generators whose scaling dimension is negative, indeed is easy to observe from (1.41) that $\Delta_S = -\frac{1}{2}$. This means that a superconformal primary state must be annihilated by both K and S generators.
By constructing the usual Chevalley basis for the $\mathfrak{sl}(2)$ algebra associated to each Cartan element of the R-symmetry algebra, namely $\{H_i, E_i^+, E_i^-\}$, the highest weight state condition becomes

$$
\{K^{\alpha\beta}, S^{\alpha r}, J^+, E_i^+\}\, |\Delta, j, r\rangle^{\text{h.w.s.}} = 0 \tag{1.42}
$$

where J^+ and E^+ are raising operators for respectively the spin and the i−th R-symmetry quantum number, with $i = 1, \dots, \frac{\mathcal{N}}{2}$.
Once the h.w.s. is known, descendants states can be built by acting again with the momentum generator P and now also with the supercharges Q, since $\Delta_Q = \frac{1}{2}$. Also in the superconformal case, multiplets whose h.w.s. is annihilated by one or more supercharges Q are called short or BPS-multiplets[16].
Analogously to what we saw in the conformal case, the structure of a multiplet depends on the property of its superconformal primary, in particular on its quantum numbers. By requiring again the non-negativity of the norms of descendant states, one can obtain the superconformal versions of the unitarity bounds discussed in Subsection 1.1.3. For this purpose, it is necessary to introduce the hermiticity properties of superconformal generators which leave (1.40) and (1.41) invariant. These are precisely

$$\begin{gathered}(P_{\alpha\beta})^\dagger = K^{\alpha\beta}, \qquad (K^{\alpha\beta})^\dagger = P_{\alpha\beta}, \qquad (M_\alpha{}^\beta)^\dagger = M_\beta{}^\alpha, \qquad D^\dagger = D \\ (Q^r_\alpha)^\dagger = -iS^{\alpha r}, \qquad (S^{\alpha r})^\dagger = -iQ^r_\alpha, \qquad (R^{rs})^\dagger = R^{rs}.\end{gathered} \tag{1.43}$$

Working out unitarity bounds for superconformal primaries for generic theories is tipically quite involved, nevertheless we can still make some general comments on the results. Since there are more superconformal descendant than conformal ones, one expects that unitarity bounds will be more restricting than in the conformal case. This is indeed obvious since all P-descendants are always Q-descendants but not viceversa.
The general structure of superconformal bounds is the following

$$\Delta \geq f(j, r) \tag{1.44}$$

where now the function f represents a generic linear combination of Lorentz and R-symmetry quantum numbers. The most important thing to notice is that the function $f(j, r)$ can be written as a combination of other linear functions $f(j, r) = f_1(j) + f_2(r) + \delta_{(j,r)}$, where δ is a numerical constant depending on the representation of the considered primary.
This particular structure shows that there exists a much richer classification of superconformal multiplets than of the conformal ones. Indeed, for the former case[17], *isolated short multiplets* may occur for specific values of $f_1(j), f_2(r)$ or the space-time dimension d.
Those particular kinds of superconformal multiplets are indeed specified by a certain value of Δ which cannot be continuously connected to other values for which the bound is strictly satisfied or saturated at the treshold. At the end of the day the possibilities are [56]

- Long Multiplets (L-type): Δ strictly satisfies (1.44).
- Short Multiplets at the threshold (A-type): Δ saturates the inequality (1.44).
- Short Isolated Multiplets (B,C,D-type): Δ assumes a specific value which falls out from the bound but for which the multiplet is still unitary.

The above families are visually summarized in Figure 1.1.
The classification we reviewed above gives some useful hints on the phenomenon of recombination for superconformal multiplets. Indeed we have that an L-type multiplet always fragments into at least an A-type multiplet with the same Lorentz and R-symmetry quantum numbers when it hits the unitarity bound ($\Delta = \Delta_A$). Moreover, it may happen that isolated short multiplets, which in principle can recombine, become *absolutely protected.*

[16] Usually the word "BPS" is accompanied by a fraction which represents the number of supercharges preserved by the primary state w.r.t. to the total number of supercharges of the theory.

[17] In the conformal case is easy to see that the only isolated short multiplet that can arise is for a scalar with $\Delta = 0$, namely the trivial multiplet generated by the identity operator $\mathbb{1}$.

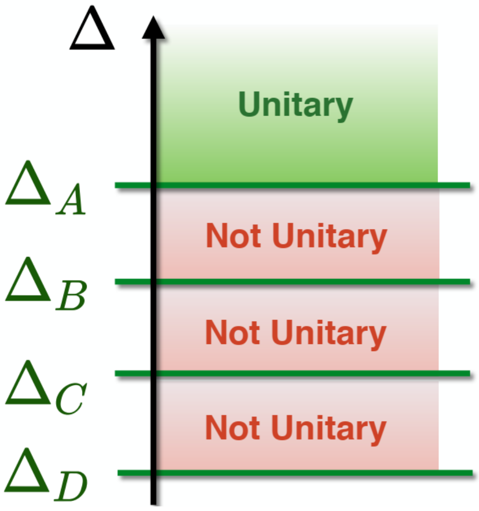

Figure 1.1: Schematic representation of unitary superconformal multiplets (Figure taken from [56]). When $\Delta > \Delta_A$ we have L-type multiplets, when instead $\Delta = \Delta_A$ we have an A-type multiplet. For $\Delta = \Delta_{B,C,D}$ we get the corresponding B,C,D-type multiplets.

When this happens, such multiplets remain always isolated for any exactly marginal deformation[18] of the SCFT and can never participate in any recombination rule. This fact thus heavily constraints the physics of SCFTs when moving on the so called *conformal manifold*.

1.3 Chern-Simons Theories

Let us now shift our attention to another special kind of quantum field theories which will be relevant for the arguments that will be discussed in what follows: Chern-Simons theories. Some nice notes regarding Abelian and non-Abelian Chern-Simons theories can be found in [66–69].
Chern-Simons theories [70] are theories which play a fundamental role both in Physics, being a non-trivial gauge-invariant extension of Yang-Mills theories in odd dimensions, and Mathematics, being for example one of the most powerful tool to compute particular topological invariants [71], e.g. *knot invariants*, for any three-dimensional oriented manifold $\mathcal{M}_3$.

1.3.1 Abelian Theories

Suppose to consider a $U(1)$ Chern-Simons theory. Its action is given by

$$S_{CS} = \frac{k}{4\pi} \int_{\mathcal{M}_3} d^3x \; \varepsilon^{\mu\nu\rho} A_\mu \partial_\nu A_\rho, \tag{1.45}$$

where the coupling $k \in \mathbb{Z}$ is called Chern-Simons *level* and $\mathcal{M}_3$ is a generic three-dimensional oriented manifold. We notice that, a priori, there is no restriction on the possible values that the level k can assume. The integrality condition will follow from requiring the gauge invariance of the partition function as we will see in a moment.
The first thing we want to point out is that (1.45) actually describes a topological field theory, namely the physics depends only on the topology of $\mathcal{M}_3$ and not on the choice of

[18] Exactly marginal deformations are deformations preserving the conformal invariance of the theory.

metric. The action can indeed be written as an integral of a three-form over $\mathcal{M}_3$

$$S_{CS} = \frac{k}{4\pi}\int_{\mathcal{M}_3} A\wedge dA, \tag{1.46}$$

for which there is no need to specify a metric.[19]
Another interesting property of (1.45) is that it is evidently Lorentz-invariant but less evidently gauge-invariant. Indeed, under a gauge transformation $A_\mu \to A_\mu + \partial_\mu\alpha$, the action transforms as

$$\delta S_{CS} = \frac{k}{4\pi}\int_{\mathcal{M}_3} d^3x\ \varepsilon^{\mu\nu\rho}\partial_\mu(\alpha\ \partial_\nu A_\rho), \tag{1.47}$$

which vanishes for suitable boundary conditions. Nevertheless it is very important to understand what happens for a generic function α in a non-trivial topological setup.
By choosing indeed for example $\mathcal{M}_3 = \mathbb{S}^1\times\mathbb{S}^2$, the expression in (1.47) becomes $\delta S_{CS} = 2\pi nk$, where we have $\frac{1}{2\pi}\int_{\mathbb{S}^2} F_{12} = n\in\mathbb{Z}$ for $U(1)$ gauge group. We see that, in general, gauge invariance of the partition function $e^{iS_{CS}}$ is achieved only for $k\in\mathbb{Z}$.

At this point, Chern-Simons theory looks poor of physical content, in particular there no local observables, being both the field-strength $F_{\mu\nu}$ and the stress-energy tensor $T_{\mu\nu}$ vanishing. Despite these discouraging results, one can actually look for non-trivial observables which satisfy the strict conditions imposed by gauge invariance and topologicity of the theory. Such observables are correlators of non-local gauge invariant objects called *Wilson loops*.
Since the theory is topological, it is meaningful to define and compute observables on non-topologically trivial space-time manifolds. We can consider the generic setup $\mathcal{M}_3 = \mathbb{R}\times\Sigma_g$ where Σ_g is a two-dimensional Riemann surface with genus[20] g. The easiest example we can consider is $\Sigma_1 = \mathbb{T}^2$, for which Wilson Loops are defined as follows

$$W_i = \exp\left(i\oint_{\gamma_i} A_j dx^j\right), \tag{1.48}$$

where γ_i, with $i,j = 1,2$, are the two inequivalent non-contractible cycles[21] of the torus. The quantization of the CS-theory on $\mathbb{R}\times\mathbb{T}^2$ allow us to find the following algebraic relation

$$W_1W_2 = e^{\frac{2\pi i}{k}}W_2W_1, \tag{1.49}$$

from which we can infer two important aspects of the theory: The first one is that Wilson Loops are operators satisfying a fractional statistics, usually called *braid* statistics, namely they behave as *anyonic* particles[22]. The second one is that (1.49) is consistent if the theory admits k degenerate ground states. For generic Σ_g the degeneracy must be k^g, meaning that the number of ground states heavily depends on the topology of the space-time.

[19] Another way to see this is the following: By introducing the curved-space Levi-Civita (pseudo-)tensor and volume form, as $\frac{1}{\sqrt{-g}}\epsilon^{\mu\nu\rho}$ and $d^3x\sqrt{-g}$ respectively, it is straightforward to see that the metric dependence always cancels out, leaving the action as in the flat space case.

[20] It is well-known that the topology of two-dimensional Riemann surfaces is completely determined by its genus g or, equivalently, its Euler characteristic $\chi(g) = 2-2g$.

[21] This is because $\pi_1(\mathbb{T}^2) = \mathbb{Z}\oplus\mathbb{Z}$.

[22] We can indeed couple the theory to a point-charge current $J^\mu = \sum_a J^\mu_a$, where

$$J^\mu_a = \oint_{\gamma_a} dx^\mu_a\delta^{(3)}(x-x_a), \quad a = 1,2, \tag{1.50}$$

by inserting the usual interaction term $S_{\text{int}} = \int d^3x A_\mu J^\mu$. At the path-integral level, the deformation S_{int} corresponds to the insertion of $e^{iS_{\text{int}}}$, which is exactly (1.48). The complete partition function of the theory is then nothing but $Z[J] = \frac{1}{Z[0]}\langle W_1W_2\rangle$.

Chern-Simons theory can be thought as a starting point for analyzing different field theoretic aspects that arise when we deform the action in (1.45). The most simple but non-trivial deformation is the insertion of a Maxwell term in the action

$$S_{CSM} = \int d^3x \left(-\frac{1}{4g^2} F_{\mu\nu} F^{\mu\nu} + \frac{k}{4\pi} \varepsilon^{\mu\nu\rho} A_\mu \partial_\nu A_\rho \right). \tag{1.51}$$

The classical equation of motion for the gauge field is

$$\partial_\mu F^{\mu\nu} + \frac{kg^2}{4\pi} \varepsilon^{\nu\rho\sigma} F_{\rho\sigma} = 0, \tag{1.52}$$

which, in terms of the dual field-strength $F^\mu = \frac{1}{2}\varepsilon^{\mu\nu\rho} F_{\nu\rho}$, can be rewritten as

$$(\Box - m^2) F^\mu = 0, \tag{1.53}$$

where we defined $m = \frac{kg^2}{2\pi}$. The above equation makes clear that the fundamental excitations of the theory are massive, meaning that the theory we are considering is now *non-trivially gapped*[23]. We notice that we can recover the dynamics of the pure CS-theory when the low energy limit is taken. This consists in taking $m = \frac{kg^2}{2\pi} \to \infty$, for which all the massive modes decouple and only low-energy probes, i.e. Wilson Loops, are left.

Adding Matter

We can now ask what happens if we couple fermions to the Maxwell-Chern-Simons theory discussed above and we play the same game of looking at the low-energy theory. For this reason let us briefly discuss discrete symmetries for fermions in three dimensions.
We can introduce the three-dimensional Clifford algebra spanned by the usual gamma matrices in the Majorana basis[24]

$$\gamma^0 = \sigma^2, \quad \gamma^1 = i\sigma^1, \quad \gamma^2 = i\sigma^3, \tag{1.54}$$

satisfying the relation

$$\gamma^\mu \gamma^\nu = \eta^{\mu\nu} - i\varepsilon^{\mu\nu\rho}\gamma_\rho, \qquad \eta^{\mu\nu} = \text{diag}(-1, 1, 1). \tag{1.55}$$

The smallest irreducible representation has dimension $2^{[\frac{d}{2}]} = 2$ and is given by a two-component (real) Majorana[25] spinor χ^α. With the prescription $\bar{\chi} = \chi^T \gamma^0$, we can write the lagrangian for a Majorana fermion as

$$\mathcal{L}_\chi = \frac{1}{2} i\bar{\chi}\partial\!\!\!/\chi + \frac{1}{2} m\bar{\chi}\chi. \tag{1.56}$$

By defining *parity* and *time-reversal* symmetries[26] as

$$\mathcal{P}: \; x^1 \to -x^1, \qquad \chi \to i\gamma^1\chi, \tag{1.57}$$
$$\mathcal{T}: \; x^0 \to -x^0, \qquad \chi \to i\gamma^0\chi, \tag{1.58}$$

we immediately see that the kinetic term in (1.56) is invariant whereas the mass term is not. By thinking of the above theory as a point in the space of couplings, we see that turning

[23] The words "non-trivial" mean that non-trivial topological observables can be computed at arbitrary low energy scales. The word "gapped" instead means that the difference between the energy of the vacuum and of the first massive excitation is non-zero, in other words, the mass spectrum is not continuous.

[24] This means that we choose purely imaginary gamma matrices.

[25] A spinor ψ is Majorana if it satisfies the Majorana condition, namely that $\psi^c = \psi$. The corresponsing conventions are taken from Appendix B of [72].

[26] These are both conventionally labelled as *parity* symmetries.

on a mass term automatically breaks parity. The same happens for a Dirac fermion, where, again, only the mass term in the lagrangian breaks parity.
It is obvious to notice that, since the corresponding action of $\mathcal{P}$ on the gauge vector is $\mathcal{P} : A^1 \to -A^1$, which amounts to sending $k \to -k$, the Chern-Simons term is odd under a parity transformation, exactly as the fermionic mass term. As we are going to discuss in what follows, this is not a coincidence, indeed there is actually a deep relation between the Chern-Simons level k and the fermionic mass m. For the moment, let us discuss the so called *parity anomaly* [73].

Let us consider for simplicity a single massless Dirac fermion coupled to a background $U(1)$ gauge field [74]. The action is

$$S = \int d^3x \; i\bar{\psi}\not{D}\psi. \tag{1.59}$$

This theory is classically $\mathcal{P}$ invariant but such symmetry is broken in a very subtle way at quantum level. Indeed, the partition function of this theory is

$$Z[A] = \text{sign}(\det(i\not{D})) \cdot |\det(i\not{D})|, \tag{1.60}$$

where the sign of the Dirac operator is the difference between negative and positive eigenvalues, which must be suitably regularized.
A mathematically rigorous regularization comes from the Atiyah-Patodi-Singer theorem, which allows to write

$$\text{sign}(\det(i\not{D})) = e^{-i\pi\frac{\eta[A]}{2}}, \tag{1.61}$$

where the functional $\eta[A]$ is called the APS *eta-invariant*. This particular functional can be suprisingly represented as a level-one CS-action, namely

$$\pi\eta[A] = \frac{1}{4\pi}\int d^3x \; \varepsilon^{\mu\nu\rho}A_\mu\partial_\nu A_\rho + \dots \; . \tag{1.62}$$

where "..." stand for zero-mode contributions of the connection which can be neglected for the purpose of this subsection. The appearance of a Chern-Simons term at quantum level thus signals the breaking of parity invariance and therefore the anomaly previously mentioned.
In conclusion, if we promote the background field A_μ to a dynamical field by adding a CS-term with level k_0 in the action, parity anomaly causes a highly non-trivial shift in the CS-level as

$$k_0 \to k = k_0 - \frac{N_f}{2}, \tag{1.63}$$

where N_f is the number of fermions of the theory. Notice that k can be integrally or half-integrally quantized whereas k_0 must be integrally quantized because of gauge-invariance.

We can now deepen our discussion by studying what happens when a massive Dirac fermion is coupled to a $U(1)$ CS-theory and what is the resulting theory in the low-energy limit. When we integrate out the fermion, we get a modification of the gauge action with the following effective contribution

$$\begin{aligned} S_{\text{eff}} &= -i\log\det\left(i\not{D} - m\right) \\ &= -i\text{Tr}\log\left(i\not{D} - m\right) \\ &= -i\text{Tr}\left(i\not{\partial} - m\right) - i\text{Tr}\left(\frac{1}{i\not{\partial} - m}\gamma^\mu A_\mu\right) - \frac{i}{2}\text{Tr}\left(\frac{1}{i\not{\partial} - m}\gamma^\mu A_\mu \frac{1}{i\not{\partial} - m}\gamma^\nu A_\nu\right) + \dots, \end{aligned} \tag{1.64}$$

where we have expanded around the classical solution $A_\mu = 0$.
The first term in (1.64) is an overall constant, the second term vanishes, the third term is

the one that contributes to the low-energy physics of the theory and, finally, all the other contributions are zero because of the Coleman-Hill *non-renormalization theorem* [75].
The third term in the expansion above can be interpreted into a one-loop correction of the gauge propagator therefore it can be evaluated in momentum space in the low-energy limit and then brought back in coordinate space. What results is

$$S_{\text{eff}} = \frac{\text{sign}(m)}{2} \cdot \frac{1}{4\pi} \int d^3x \; \varepsilon^{\mu\nu\rho} A_\mu \partial_\nu A_\rho. \tag{1.65}$$

We see that, analogously to what happens for the parity anomaly, the Chern-Simons level get shifted, for a generic number N_f of Dirac fermions, as follows

$$k \to k_{IR} = k + \frac{\text{sign}(m)}{2} N_f, \tag{1.66}$$

where k_{IR} is the CS-level for the low-energy (IR) theory. We notice that, again, gauge-invariance forces k_{IR} to integrally quantized whereas k can be still integrally or half-integrally quantized depending on the number of fermions we integrated out.

1.3.2 Non-Abelian Theories

In this subsection we would like to extend the discussion made in the previous subsections to generic non-Abelian gauge groups G. The action of a non-Abelian Chern-Simons theory reads

$$\begin{aligned} S_{CS} &= \frac{k}{4\pi} \int_{\mathcal{M}_3} d^3x \; \varepsilon^{\mu\nu\rho} \; \text{Tr} \left(A_\mu \partial_\nu A_\rho - \frac{2i}{3} A_\mu A_\nu A_\rho \right) \\ &= \frac{k}{4\pi} \int_{\mathcal{M}_3} \text{Tr} \left(A \wedge dA - \frac{2i}{3} A \wedge A \wedge A \right), \end{aligned} \tag{1.67}$$

where again $k \in \mathbb{Z}$ and $\mathcal{M}_3$ is a generic three-dimensional oriented manifold. Exactly like the Abelian case, also the non-Abelian theory describes a topological field theory, as one can easily see from the second line of the above expression.
The integrality condition for k is still valid also for the non-Abelian case but the proof is slightly more elaborated. Under a gauge transformation $g \in G$ we have

$$A_\mu \;\; \to \;\; g^{-1} A_\mu g + i g^{-1} \partial_\mu g, \tag{1.68}$$

to which corresponds the following variation of the action

$$\begin{aligned} \delta S_{CS} &= \delta S_{CS}^{(ab)} + \frac{k}{4\pi} \int_{\mathcal{M}_3} d^3x \; \frac{1}{3} \varepsilon^{\mu\nu\rho} \; \text{Tr} \left((g^{-1}\partial_\mu g)(g^{-1}\partial_\nu g)(g^{-1}\partial_\rho g) \right) \\ &= \delta S_{CS}^{(ab)} + 2\pi k w(g), \end{aligned} \tag{1.69}$$

where the first term is nothing but the variation of the abelian CS-action (1.47), the second term instead is the so called *Pontryagin index*, or simply *winding number*, of g. For an exhaustive discussion on this topic see [76]. This quantity counts the number of times the map $g : \mathcal{M}_3 \to G$ non-trivially wraps around G. The maps g not continuously connected to the identity element[27] are divided in equivalence classes called *homotopy classes*, precisely labelled by $w(g)$. By taking $\mathcal{M}_3 = \mathbb{S}^3$, all the homotopy classes for the so called *third homotopy group* $\pi_3(G)$ which becomes $\pi_3(G) = \mathbb{Z}$ for any compact simple group G. This fact clearly implies that $w(g) \in \mathbb{Z}$, therefore, by requiring the gauge invariance of the partition function $e^{iS_{CS}}$, it follows that $k \in \mathbb{Z}$.

[27] These gauge transformations are usually called *large gauge transformations*.

The physical properties of non-Abelian CS-theory are, exactly like the Abelian case, encoded in the correlators of Wilson Loops, which constitute the relevant topological observables of the theory. For the non-Abelian case, a Wilson Loop is defined as follows

$$W_R[\gamma] = \mathrm{Tr}_R \left[\mathcal{P} \exp \left(i \oint_\gamma dx^\mu A_\mu \right) \right], \tag{1.70}$$

where $\mathcal{P}$ stands for path-ordering, R is an irreducible representation of G and γ is a closed curve in $\mathcal{M}_3$. In general one can take a disjoint union of r paths γ_i to form what is called a *link* L_r. We can then associate to each γ_i an irreducible representation R_i and define a generic Wilson Loop $W_{R_i}[\gamma_i]$ which can be used to compute observables like

$$Z[\mathcal{M}_3, L_r] = \frac{1}{Z[\mathcal{M}_3, 0]} \int \mathcal{D}A \; e^{iS_{CS}} \prod_{i=1}^{r} W_{R_i}[\gamma_i]. \tag{1.71}$$

Witten argued in [71] that this kind of observables must carry all the relevant topological informations about the link L_r defined on $\mathcal{M}_3$. As a particular case he considered $\mathcal{M}_3 = \mathbb{S}^3$ and $G = SU(2)$, and computed the exact value of (1.71) obtaining what was already understood by mathematicians as the *Jones polynomial* for the link L_r. The formula in (1.71) thus constitutes a powerful generalization for computing topological invariants, also known as *knot invariants*, for any link L_r in $\mathcal{M}_3$.

Adding Matter

As we did in the previous subsection, we can now add matter to this already rich environment and look again at the low-energy theory after integrating out massive degrees of freedom. Since also the non-Abelian CS-action is odd under the parity symmetry discussed previously, coupling the theory to matter gives exactly the same results we discussed in the Abelian case.

The first fact is that parity is still anomalous for f massless fermions coupled to a non-Abelian background gauge field. By promoting it to a dynamical field indeed, we get a shift which now depends on some additional group theoretic structures as follows

$$k_0 \to k = k_0 - \frac{1}{2} \sum_f T(R_f), \tag{1.72}$$

where $T(R_f)$ is the Dynkin index[28] of the real representation R_f in which the fermions sits. In case the representation is complex or pseudo-real, the $\frac{1}{2}$ factor must be dropped from the above relation.

By coupling now f massive fermions to the non-Abelian CS-theory, we again find a formula analogous to (1.66) which reads

$$k \to k_{IR} = k + \frac{1}{2} \mathrm{sign}(m) \sum_f T(R_f), \tag{1.73}$$

where again the same prescription for the $\frac{1}{2}$ factor holds.

In what follows, we will apply the common practice of labeling CS-theories with the parity-translated level k instead of the bare k_0 level (usually called k_{UV}). Parity (or time) symmetry hence will act on the level as simply $k \to -k$.

[28] For the $SU(N)$ case we have $T(F) = \frac{1}{2}$ and $T(A) = \frac{N}{2}$ for respectively the fundamental and the adjoint representations.

1.3.3 $\mathcal{N}=1$ Chern-Simons Theory

In this subsection we introduce the minimal supersymmetric version of the non-Abelian Chern-Simons theory we discussed above. For doing this, let us introduce the so called *superspace* formalism which is the most natural framework for defining and studying supersymmetric theories and that we will employ in the second part of this thesis. A detailed introduction to the three-dimensional superspace formalism can be found in [45].

Superspace is the supersymmetric extension of ordinary space-time, where the standard bosonic coordinates x^μ to which is associated the generator of translations P_μ, are accompanied by Grassmannian coordinates θ^α to which are associated the fermionic generators of supertranslations Q^α.
In three dimensions fermions transform in the fundamental representation of $\mathfrak{so}(1,2)\simeq\mathfrak{sl}(2)$ which is real. They will be thus represented by a real (Majorana) two-component spinor ψ^α, $\alpha=1,2$. In spinor notation, vectors are represented as symmetric matrices $v^{\alpha\beta}$ or traceless matrices $v_\alpha{}^\beta$. In this notation, superspace will be parametrized with $(x^{\alpha\beta},\theta^\gamma)$ where α,β,γ denote spinor indices.
Spinor indices are raised and lowered by

$$C_{\alpha\beta}=-C^{\alpha\beta}=\begin{pmatrix}0 & -i\\ i & 0\end{pmatrix}, \tag{1.74}$$

with the conventions

$$\psi^\alpha\equiv C^{\alpha\beta}\psi_\beta,\quad \psi_\alpha\equiv\psi^\beta C_{\beta\alpha},\quad \psi^2\equiv\frac{1}{2}\psi^\alpha\psi_\alpha. \tag{1.75}$$

Differentiations in superspace is defined as

$$\partial_\alpha\equiv\frac{\partial}{\partial x^\alpha},\qquad \partial_{\alpha\beta}\equiv\frac{\partial}{\partial x^{\alpha\beta}} \tag{1.76}$$

from which we get the following (anti-)commutation relations

$$\partial_\alpha\theta^\beta\equiv\{\partial_\alpha,\theta^\beta\}=\delta_\alpha^\beta,\qquad \partial_{\alpha\beta}x^{\gamma\varepsilon}\equiv[\partial_{\alpha\beta},x^{\gamma\varepsilon}]=\frac{1}{2}\delta^\gamma_{(\alpha}\delta^\varepsilon_{\beta)}. \tag{1.77}$$

In the same spirit we define Grassmannian integration as follows

$$\int d\theta_\alpha=0,\quad \int d\theta_\alpha\theta^\beta=\delta_\alpha^\beta. \tag{1.78}$$

Notice that, by taking a function depending on a single Grassmannian coordinate γ, it can be Taylor expanded as $f(\gamma)=f(0)+\gamma f'(0)$, since $\gamma^2=0$.
If we then integrate such function over γ we get $\int d\gamma f(\gamma)=f'(0)$ meaning that integration and differentiation are equivalent for Grasmmannian coordinates.
This result therefore implies the following fundamental relation

$$\int d\theta_\alpha=\partial_\alpha. \tag{1.79}$$

Since we want introduce a supersymmetric quantum field theory, we need to introduce the concept of field in superspace, usually known as *superfield*. Superfields are represented as functions $Y(x,\theta)$, which, as we saw above, can be Taylor expanded up to second order in the θ coordinate as a set of function exclusively depending on space-time coordinates $f(x)$.
The superspace symmetry group is the superPoincaré group and its associated superalgebra acts on superfields as

$$P_{\alpha\beta}=i\partial_{\alpha\beta},\qquad Q_\alpha=i(\partial_\alpha-i\theta^\mu\partial_{\mu\alpha}) \tag{1.80}$$

Superfields are objects transforming covariantly under supersymmetry transformations. It is easy to show that their spinor derivatives $\partial_\alpha Y(x,\theta)$ do not.
For this reason, it is indeed necessary to introduce spinor superderivatives

$$D_\alpha \equiv \partial_\alpha + i\theta^\mu \partial_{\mu\alpha}, \tag{1.81}$$

such that $D_\alpha Y(x,\theta)$ is still a superfield.
The most general action for a supersymmetric theory can be therefore define as

$$S = \int d^3x d^2\theta \; f(Y, D_\alpha Y, \dots) \tag{1.82}$$

where f is an arbitrary function of superfields.
For setting up a minimally supersymmetric CS-matter theory, we need to introduce the scalar and the vector superfields. The scalar superfield is represented as

$$\Phi(x,\theta) = \phi(x) + \theta\psi(x) - \theta^2 F(x), \tag{1.83}$$

where ϕ and F are real one-component scalars and ψ is a real two-component fermion. Since its components can be selected by using the rules

$$\begin{aligned} \phi(x) &= \Phi(x,0), \\ \psi_\alpha(x) &= D_\alpha \Phi(x,0), \\ F(x) &= D^2\Phi(x,0), \end{aligned} \tag{1.84}$$

the canonical kinetic term can be written in components as

$$\begin{aligned} S_{kin} &= -\frac{1}{2}\int d^3x d^2\theta \; D^\alpha \Phi D_\alpha \Phi \\ &= \frac{1}{2}\int d^3x \left(-\phi\Box\phi + \psi^\alpha i\partial_\alpha{}^\beta \psi_\beta + F^2 \right). \end{aligned} \tag{1.85}$$

Notice that the field F does not have a kinetic term, indeed it is an auxiliary field which can be eliminated from the theory by imposing its equation of motion, i.e. $F = 0$.
Mass terms and interactions can be written by introducing the so called *superpotential* $\mathcal{W}(\Phi)$ which is in general a polynomial in the superfield Φ. The interaction term takes the form

$$S_{int} = \int d^3x d^2\theta \; \mathcal{W}(\Phi) = \int d^3x \left(\mathcal{W}''(\Phi)\psi\psi + \mathcal{W}'(\Phi) F \right). \tag{1.86}$$

A key ingredient for building up a supersymmetric CS-theory is clearly the gauge superfield $\Gamma_\alpha(x,\theta)$ which we now introduce. For simplicity we can define its component from a set of rules analogous to the ones in (1.84)

$$\begin{aligned} \chi_\alpha &= \Gamma_\alpha \big|, & B &= \frac{1}{2} D^\alpha \Gamma_\alpha \big|, \\ A_{\alpha\beta} &= -\frac{i}{2} D_{(\alpha}\Gamma_{\beta)} \big|, & \lambda_\alpha &= \frac{1}{2} D^\beta D_\alpha \Gamma_\beta \big|, \end{aligned} \tag{1.87}$$

where we indicated with $|$ that the terms must be evaluated in $(x, 0)$. Exactly like the usual gauge field, a gauge fixing can be implemented also in this case for eliminating unphysical degrees of freedom which, in this case, are the χ_α and B fields. We are left hence with the physical ones, the gauge vector $A_{\alpha\beta}$ and its superpartner, the *gaugino* λ_α.
Bu defining also the superfield strength $W_\alpha = \frac{1}{2} D^\beta D_\alpha \Gamma_\beta$, satisfying the Bianchi identity $D^\alpha W_\alpha = 0$, one can write the Maxwell and the Chern-Simons terms respectively as

$$S_M = \frac{1}{g^2}\int d^3x d^2\theta \; W^\alpha W_\alpha, \tag{1.88}$$

$$S_{CS} = \frac{k}{8\pi}\int d^3x d^2\theta \; \Gamma^\alpha W_\alpha. \tag{1.89}$$

The generalization to the non-Abelian case follows immediately. The only non-trivial term is the Chern-Simons one which reads

$$S_{CS} = \frac{k}{8\pi}\int d^3x d^2\theta \left(\Gamma^\alpha W_\alpha + \frac{i}{6}\{\Gamma^\alpha,\Gamma^\beta\}D_\beta\Gamma_\alpha + \frac{1}{12}\{\Gamma^\alpha,\Gamma^\beta\}\{\Gamma_\alpha,\Gamma_\beta\}\right). \tag{1.90}$$

What we need to do now is to couple the gauge sector with the matter sector. The standard way to do is to covariantize the superderivative by defining $\nabla_\alpha \equiv D_\alpha + i\Gamma_\alpha$ and thus generalizing the matter kinetic term as

$$S_{kin} = -\frac{1}{2}\int d^3x d^2\theta \; \nabla^\alpha\Phi\nabla_\alpha\Phi. \tag{1.91}$$

Let us now apply some of the arguments we presented in this chapter to discuss the relevant properties, which will be useful for the second part of this thesis, of the pure $\mathcal{N}=1$ $SU(N)_k$ CS-theory. We will discuss other relevant aspects of $\mathcal{N}=1$ CS-matter theories in Section 3.1 and additional details can be found in Appendix B.2.

1.3.4 $\mathcal{N}=1$ $SU(N)_k$ Theory

The $\mathcal{N}=1$ $SU(N)_k$ Yang-Mills-Chern-Simons (YMCS) theory, discussed in detail in [46], contains a vector multiplet only, consisting of a gauge field A and its superpartner λ, the so called *gaugino*, namely a Majorana fermion sitting in the adjoint representation of the gauge group.
The Lagrangian of the theory hence reads

$$\mathcal{L}^{CS}_{\mathcal{N}=1} = -\frac{1}{4g^2}F_{\mu\nu}F^{\mu\nu} + \frac{k}{4\pi}\varepsilon^{\mu\nu\rho}\,\mathrm{Tr}\left(A_\mu\partial_\nu A_\rho - \frac{2i}{3}A_\mu A_\nu A_\rho\right) + i\,\mathrm{Tr}\left(\lambda \not{D}\lambda\right) + m\lambda\lambda. \tag{1.92}$$

As we observed in Subsections 1.3.1 and 1.2.1, the propagating modes of such theory are massive and, the gauge field A and the gaugino λ, since they belong to the same supermultiplet, have the same mass[29] $m = -\frac{kg^2}{2\pi}$.
As we did for the cases discussed above, we can try to study the dynamics of the theory in the low-energy limit, namely when the large k limit is considered. It is immediate to see that, in this limit, fermionic massive modes decouple from the theory and can be safely integrated out. Since we are integrating out a charged fermion, we get a shift of the level k according to (1.73) with $T(adj) = N$ and $m_g < 0$. In this limit the resulting gauge group of the theory is still $SU(N)$ but the CS-level is now shifted to $k - \frac{N}{2}$. The resulting YMCS-theory, being a non-trivially gapped theory [77, 78], still admits bosonic massive propagating modes which, in the deep low-energy limit, are again decoupled. The only degrees of freedom surviving are thus topological and what results is an $SU(N)_{k-\frac{N}{2}}$ CS-theory, which is in fact a topological theory.

It is in general highly non-trivial to understand the properties and the structure of the vacua of a theory when different energy regimes are considered, in particular, whether supersymmetry is preserved or not. A powerful tool which can be defined in any supersymmetric theory for achieving this is the *Witten index* [79].
The Witten index is a topological quantity obtained from the usual definition of the partition function by inserting a certain term which weights in a different way the contributions coming from fermionic and bosonic states. Explicitly it reads

$$I_W = \mathrm{Tr}_{\mathcal{H}}(-1)^F e^{-\beta H}, \tag{1.93}$$

[29]Remember that only k_{bare} appear in the Lagrangian, and the parity-shifted level is used to label the theory only.

where $\mathcal{H}$ is the Hilbert space of physical states, $\beta \in \mathbb{R}^+$ is a parameter F is the fermion number operator, namely an operator for which $F\,|bos\rangle = 0$ and $F\,|ferm\rangle = |ferm\rangle$. The definition given above is well-defined and expicitly computable only when we put a theory on a compact space, since the Hamiltonian acquires a discrete spectrum. It is then easy to show that, because of supersymmetry, only zero energy states contribute to the index, namely

$$I_W = \mathrm{Tr}_{\mathcal{H}}(-1)^F e^{-\beta H} = \mathrm{Tr}_{\mathcal{H}_0}(-1)^F = n_B - n_F, \tag{1.94}$$

where $n_{B,F}$ is the number of bosonic and fermionic ground states respectively. It is also clear from the above expression that I_W is independent of β and, more in general, of any deformation of the couplings of the theory in the parameter space. This means that the Witten index is actually topological invariant and it can be safely computed in any suitable region of the parameter space.

An important consequence of the Witten index is that, if it is different from zero, supersymmetry is preserved. This means that the theory always admit supersymmetric ground states. Viceversa, if the index vanishes,i.e. the number of bosonic and fermionic ground states is the same, we cannot conclude if supersymmetry is preserved or not[30] and other more sofisticated techniques are required.

In [46], it was shown that the Witten index (up to an overall sign[31]) for the $\mathcal{N} = 1$ $SU(N)_{k-\frac{N}{2}}$ vector multiplet theory is

$$\begin{aligned} I &= \frac{1}{(N-1)!}\left(k - \frac{N}{2} + 1\right)\left(k - \frac{N}{2} + 2\right) \ldots \left(k + \frac{N}{2} - 1\right) \\ &= \frac{(k + \frac{N}{2} - 1)!}{(N-1)!(k - \frac{N}{2})!}, \end{aligned} \tag{1.95}$$

which is just the partition function of the low-energy TQFT discussed above on the torus.[32] Since in [46] it was shown that in fact the expression (1.95) holds for all values of k, being the index non-vanishing when $k \geq N/2$, it is natural to conclude that the $\mathcal{N} = 1$ $SU(N)_k$ theory always flows to the (supersymmetric) $SU(N)_{k-\frac{N}{2}}$ topological CS-theory when $k \geq N/2$. Things are much more complicated when instead $0 \leq k < N/2$, for which the Witten index vanishes. In the same work it was indeed conjectured that, when $0 \leq k < N/2$, supersymmetry is spontaneously broken and thus a massless Goldstino emerges in the IR limit. This conjecture found significant support in the results of [36], where it was claimed that, in addition to the Goldstino G_α, there must be topological degrees of freedom described by a $U\left(\frac{N}{2} - k\right)_{\frac{N}{2}+k,N}$ Chern-Simons theory[33].

1.4 ABJ(M) Theory

In this last section we introduce the theory that will be discussed in the first part of the thesis and that enjoys all the properties we previously discussed: ABJ(M) theory.

ABJ(M) theory is a three-dimensional superconformal Chern-Simons-matter theory with $\mathcal{N} = 6$ supersymmetries which realizes the AdS/CFT duality [16–18] when the near-horizon

[30] Supersymmetry is broken when there are no zero energy ground states, namely $n_B = n_F = 0$.

[31] As a general comment, the sign depends on the number of Majorana fermions with negative mass in the theory, which we can call n_-. The overall sign is then given by $(-1)^{n_-}$.

[32] Equivalently, it can be computed by counting the number of inequivalent Wilson lines of the theory.

[33] The fact that the Goldstino alone is not enough to describe the IR physics follows from the following observations: at any value of the level k there is a 1-form symmetry in the UV moreover, for $k = 0$, the UV theory enjoys the time reversal symmetry. There are 't Hooft anomalies associated with both symmetries that cannot be matched by the Goldstino only and some other d.o.f. are required.

limit of N coincident M2-branes[34] probing a $\mathbb{C}^4/\mathbb{Z}_k$ singularity is taken [19]. The resulting geometry in this limit becomes $AdS_4 \times \mathbb{S}^7/\mathbb{Z}_k$, thus ABJ(M) is conjectured to play a fundamental role in the understanding of the AdS_4/CFT_3 correspondence, by describing its quantum field theory side at the conformal point. An equivalent gravity description is conjectured to hold for Type IIA Superstring theory on an $AdS_4 \times \mathbb{CP}^3$ background. Since in this thesis we will focus on its purely field-theoretic aspects, let us now briefly review the local and non-local content of the theory and the main exact results that have been obtained so far.

1.4.1 Main Aspects

As we said above, ABJ(M) theory is a superconformal CS-matter theory with $\mathcal{N} = 6$ supersymmetries. The theory thus preserves 12 Poincaré supercharges and 12 superconformal charges giving a total number of 24 preserved fermionic charges. As we reviewed in Subsection 1.2.2, the superconformal algebra for such theories is given by the $\mathfrak{osp}(\mathcal{N}|4)$ superalgebra. For ABJ(M) thus we have $\mathfrak{osp}(6|4)$ whose maximally bosonic subalgebra[35] is thus $\mathfrak{sp}(4) \oplus \mathfrak{so}(6) \simeq \mathfrak{so}(2,3) \oplus \mathfrak{su}(4)$. The theory is moreover a $U(N_1)_k \times U(N_2)_{-k}$ quiver gauge theory, or more specifically a class of theories, depending on the choice of the rank of the gauge groups $N_{1,2}$ and the CS-level[36] k. In general it is usually referred to ABJ theories [20] when $N_1 \neq N_2$ and ABJM theories when $N_1 = N_2 = N$. We will mostly work in the ABJ setup but for simplicity we will refer to it as ABJ(M).

The field content of ABJ(M) theory is given by two gauge vectors $(A_\mu)_i^{\ j}$ and $(\hat{A}_\mu)_{\hat{i}}^{\ \hat{j}}$ transforming in the adjoint representation of the first and second gauge group respectively, four scalar fields $C_I, \bar{C}^I$ and their corresponding fermionic partners $\psi_I^\alpha, \bar{\psi}^{I,\alpha}$ transforming in the (anti-)fundamental representation of the $SU(4)$ R-symmetry group. The couple $(C, \bar{\psi})$ transforms in the bifundamental representation $(N_1, \bar{N}_2)$ of the gauge group, whereas the conjugate fields $(\bar{C}, \psi)$ transform in the anti-bifundamental representation $(\bar{N}_1, N_2)$.
The dynamics of ABJ(M) theory can be studied by taking various limits of the coupling k and the ranks of the gauge groups $N_{1,2}$. The perturbative regime is obtained by taking the $k \gg 1$ limit and keeping $N_{1,2}$ fixed or also with $N_{1,2} \to \infty$. The latter perturbative limit is usually called *planar limit* and is particularly useful in the context of the AdS/CFT correspondence for comparing the value of certain observables on both sides of the duality [24, 80–83]. For this reason it is usually preferred to define the so called *'t Hooft coupling*, $\lambda \equiv \frac{N}{k}$, for which the $\lambda, N \to \infty$ limit describes the duality between ABJ(M) and *eleven-dimensional Supergravity*, i.e. the low-energy limit of M-Theory.

The action of the theory can be schematically represented as

$$S = S_{CS} + S_{matter} + S_{pot}, \qquad \text{with} \quad S_{pot} = S_{pot}^{bos} + S_{pot}^{ferm}, \tag{1.96}$$

where the Chern-Simons and matter terms are, in Euclidean signature,

$$\begin{aligned} S_{CS} &= -i\frac{k}{4\pi}\int d^3x\, \varepsilon^{\mu\nu\rho}\left[\mathrm{Tr}\left(A_\mu\partial_\nu A_\rho + \frac{2i}{3}A_\mu A_\nu A_\rho\right) - \mathrm{Tr}\left(\hat{A}_\mu\partial_\nu\hat{A}_\rho + \frac{2i}{3}\hat{A}_\mu\hat{A}_\nu\hat{A}_\rho\right)\right] \\ S_{mat} &= \int d^3x\ \mathrm{Tr}\left(D_\mu C_I D^\mu \bar{C}^I - i\bar{\psi}^I\gamma^\mu D_\mu\psi_I\right) \end{aligned} \tag{1.97}$$

[34] Such M2-branes are the fundamental dynamical objects, together with M5-branes, of M-Theory.
[35] Notice that this perfectly matches the isometries of the gravity backgrounds for both M-Theory and Type IIA String theory.
[36] By taking for example $k = 1, 2$ and $N = 2$, one can recover the already known BLG model, an $\mathcal{N} = 8$ three-dimensional superconformal CS-matter theory describing the worldvolume theory of two coincident M2-branes manifestly preserving 16 supercharges.

The full action with all the necessary conventions and supersymmetry variations are collected in Appendices A.1 and A.2.
We can now notice from the explicit expression of S_{CS} in the relations above, that ABJM theory, differently from the case of CS-theory with matter discussed in Subsection 1.3.2, does not suffer from parity anomaly. The reason is because, when $N_1 = N_2$, we can apply the same parity symmetry together with a simultaneous exchange of the two gauge fields leaving the action invariant. Nevertheless ABJM theory still suffers from another subtle anomaly, called *framing anomaly*, which is a reminescent effect of the fact that, in pure CS-theory, general covariance is necessarily broken at quantum level. This is because the Faddeev-Popov gauge-fixing procedure forces us to introduce Grassmannian ghost fields whose contributions to the 1-loop partition function are analogous to the ones in (1.60) and (1.61). Being $\eta[A]$ not a topological invariant if zero-modes are present, a *gravitational* CS-counterterm can be introduced in order to regularize such zero-modes effect and restore general covariance at quantum level. By putting together all the contributions, one ends up with an overall phase in the partition function which is now completely topological but still ambiguous[37]. Although ABJ(M) theory is not topological even at classical level because of the presence of matter, such phase contributions are still present in the perturbative evaluation of certain observables which we are now going to introduce.

Wilson Operators

The main observables we can compute in ABJ(M) are correlators of all the local fields we introduced above. However, being it actually a Chern-Simons theory, even in the presence of matter, we can still define Wilson lines and loops and compute their correlators as highly non-trivial observables of the theory. Moreover, since ABJ(M) is a supersymmetric theory, it is natural to ask if such non-local objects preserve all the supersymmetries of the theory or at least a fraction of them. It is immediate to see that the Wilson loop defined in (1.70) does not preserve any supersymmetry and thus cannot represent a protected quantity of the theory. In order to avoid this conclusion, we need to introduce some generalized connection involving the fields of the theory so that the number of preserved supersymmetries get enhanced to some fraction. The general structure of a Wilson operator hence is

$$W_R[\gamma] = \mathrm{Tr}_R \left[\mathcal{P} \exp \left(i \int_\gamma d\tau \mathcal{L}(\tau) \right) \right], \tag{1.98}$$

where $\mathcal{L}(\tau)$ is the generalized connection mentioned above and γ can be either a straight line or a circular loop embedded in a three-dimensional locally conformally flat manifold $\mathcal{M}_3$. By choosing the following purely bosonic connections

$$L_B = A_\mu \dot{x}^\mu - \frac{2\pi i}{k} |\dot{x}| M_I{}^J C_J \bar{C}^I, \tag{1.99a}$$

$$\hat{L}_B = \hat{A}_\mu \dot{x}^\mu - \frac{2\pi i}{k} |\dot{x}| M_I{}^J \bar{C}^I C_J, \tag{1.99b}$$

where $M_I{}^J = \mathrm{diag}(-1,-1,1,1)$ is the constant matrix coupling for scalars, and choosing the contour γ to be a circle or a straigth line, one can define two Wilson operators $W_B[\gamma]$

[37] The gravitational CS-term is still ill defined because, in three-dimensions, there is no canonical choice of a *spin structure* and thus a Levi-Civita spin connection. Different choices are related by a shift of the gravitational CS-term by $2\pi\nu$ where $\nu \in \mathbb{Z}$ is the so called *framing number*. At the level of the partition function we have [71]

$$Z \to e^{\frac{2\pi i \nu d}{24}} Z$$

where d is the dimension of the gauge group G. A more "perturbative" interpretation is given by the fact that the regularization needed for evaluating observables, like vacuum expectation values of Wilson operators, impose the introduction of a new support for the Wilson operator γ_ϵ which may wrap around the original path γ a non-trivial number of times. This number can be found to be exactly ν.

and $\hat{W}_B[\gamma]$ which preserve 4 supercharges out of 24 each, namely are 1/6 BPS [80, 82, 84]. The situation can be improved by placing the connections seen above into a single supermatrix belonging to a $U(N_1|N_2)$ gauge supergroup[38], and compute its holonomy [85]. The explicit expression of this superconnection is

$$\mathcal{L}_F = \begin{pmatrix} L_B & -i\sqrt{\frac{2\pi}{k}}|\dot{x}|\eta_I\bar{\psi}^I \\ -i\sqrt{\frac{2\pi}{k}}|\dot{x}|\psi_I\bar{\eta}^I & \hat{L}_B \end{pmatrix}, \tag{1.100}$$

where $\eta_I, \bar{\eta}^I$ are commuting spinors which allow for the insertion of fermion fields in the superconnection. By setting $M_I{}^J = \text{diag}(-1,1,1,1)$, choosing suitable expressions for $\eta_I, \bar{\eta}^I$ and the contour γ to be a circle or a straigth line, the corresponding Wilson operator is shown to preserve half of the supercharges of the theory, i.e. is 1/2 BPS. We label this operator as $W_{\frac{1}{2}}[\gamma]$.
The bosonic and fermionic Wilson operators defined above seem to be completely independent object giving rise to independent vacuum expectation values. Actually, it was shown in [85] that by merging the two bosonic Wilson operators into a single bosonic operator, which we label as $W_{\frac{1}{6}}[\gamma]$, equipped with the following superconnection

$$\mathcal{L}_B = \begin{pmatrix} L_B & 0 \\ 0 & \hat{L}_B \end{pmatrix}, \tag{1.101}$$

the bosonic and fermionic operators satisfy an highly non-trivial relation[39]

$$W_{\frac{1}{2}}[\gamma] - W_{\frac{1}{6}}[\gamma] = QV, \tag{1.102}$$

where V is a complicated function of the ABJ(M) fields. This relation tells us that the two operators are Q-cohomologically equivalent, namely sit in the same Q-cohomology class. This implies that their properly normalized vacuum expectation value is exactly the same. As we discussed above the beginning of this subsection, some observables in ABJ(M) can still receive additional contributions due to the presence of the intrisic overall phase originated by the framing anomaly. This is exactly the case for the Wilson operators we introduced above, whose vacuum expectation value can be generically expressed as follows

$$\langle W_{\frac{1}{2},\frac{1}{6}}[\gamma]\rangle_\nu = e^{i\pi\Phi(N_1,N_2,\nu)}\ \langle W_{\frac{1}{2},\frac{1}{6}}[\gamma]\rangle_0. \tag{1.103}$$

As noticed in [7, 85], the framing factor can be always exponentiated and factorized so that it is always possible to compare the results obtained with different prescriptions.

It is interesting to notice that both 1/2 and 1/6 BPS Wilson operators can be suitably extended by considering respectively bosonic and fermionic one-parameter families [87] in which γ is taken to be a generic circular path on $\mathbb{S}^2$ inside $\mathbb{R}^3$ whose location is parametrized by an effective angular parameter $\nu_0 \in [0,1]$, called *latitude*. The bosonic operators in this setup are found to be generically 1/12 BPS whereas the fermionic ones 1/6 BPS and their maximally supersymmetric enhanced version can be recovered in the $\nu_0 \to 1$ limit. By computing the vev of such operators at two-loop order, the authors of [86] identified the ν_0 contributions to the result to be exactly the same as the contributions of the same operators computed at a generic framing ν. This result allowed them to conjecture the equivalence between the effective latitude parameter ν_0 and the framing number ν, making manifest the surprising fact that ν have actually a non-integer nature.

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[38] Notice that this procedure do not enhance the gauge group of the theory which has still to be $U(N_1) \times U(N_2)$.

[39] To be precise, this relation holds for $\nu = 1$ framing number. The cohomological equivalence at generic framing number ν can be found in [86].

1.4.2 Exacts Results

In general, all the observables of a theory can be evaluated in some perturbative limit by just Taylor expanding the action in the path-integral to arbitrary loop levels and evaluating all the correlators that contribute to the result. Of course this is a time-consuming process and, moreover, it can be performed only when the perturbative series converges, which is not generically true. We thus understand that some more efficient tools should be implemented in order to obtain reliable results beyond perturbation theory only. Luckily, for supersymmetric theories, one of such methods does exist and is the celebrated *supersymmetric localization* procedure. Some useful reviews on this topic are [88–91].

The main advantage of supersymmetric theories is indeed that they enjoy a fermionic symmetry, i.e. supersymmetry itself, which can be used to heavily constraint certain observables, like the partition function of the theory, such that their computation boils down to a finite-dimensional integral from an infinite-dimensional one. This is the philosophy behind the localization technique.

More practically, suppose to consider a non-anomalous Grassmannian symmetry of the theory δ such that $\delta^2 = \delta_B$, where δ_B can be either zero or a bosonic symmetry of the theory. The Euclidean partition function of the theory can be thus deformed as follows

$$Z = \int_{\mathcal{M}} \mathcal{D}\Phi \; e^{-S[\Phi]} \;\rightarrow\; Z_t = \int_{\mathcal{M}} \mathcal{D}\Phi \; e^{-S[\Phi]-t\delta V[\Phi]}, \tag{1.104}$$

where t is a real parameter and $V[\Phi]$ is a non-trivial functional of the fields of the theory such that $\delta_B V = 0$. By considering the derivative with respect to the parameter t we surprisingly get, given all the assumptions above, that

$$\frac{\partial Z_t}{\partial t} = -\int_{\mathcal{M}} \mathcal{D}\Phi \; (\delta V) \; e^{-S[\Phi]-t\delta V[\Phi]} = -\int_{\mathcal{M}} \mathcal{D}\Phi \; \delta(V \; e^{-S[\Phi]-t\delta V[\Phi]}) = 0. \tag{1.105}$$

This can be achieved by thinking the variation δ to act as a translation along a fermionic coordinate in the (super)space of fields and hence, by interpreting it as a derivative along such a fermionic direction, the term above must vanish under suitable assumptions on the regularity of fields at the boundary of the field space[40].

The result obtained above tells us that the deformed partition function Z_t is actually independent of the parameter t, therefore it is possible to compute it at $t = 0$, i.e. the undeformed partition function, in any limit of t.

The most useful limit we can consider is the $t \to \infty$ limit, in which the dominant contributions to the partition function come from the field configurations for which the $-t\delta V$ term is maximized. Assuming that the bosonic part of the functional V satisfies the inequality $\delta V \geq 0$, the dominant contributions will be given by the field configurations Φ_0 satisfying the equation

$$\delta V[\Phi_0]\big|_{bos} = 0. \tag{1.106}$$

A canonical choice for the expression of the functional V is

$$V = \sum_{\psi} (\delta\psi)^\dagger \psi, \tag{1.107}$$

whose variation essentially tells us that the bosonic zeros are nothing but $\Phi_0 = \delta\psi = 0$, i.e. the solutions of the fermionic *BPS equations*. The submanifold of field configurations satisfying those relations is called *localization locus* and, if it is finite-dimensional, the original path-integral becomes a finite-dimensional integral which can be much easier evaluated.

[40] If boundary contributions are present then the $e^{-t\delta V}$ factor should decrease rapidly enough to guarantee the vanishing of the result. In some specific cases, non-perturbative effects can play a role in determine the exact expression of the partition function as a matrix model. One example is [92].

Such reduced integral is usually called *matrix model*.
It is important to mention that the evaluation of the path-integral can be done through the usual saddle-point approximation by expanding the fields around the localization locus as[41] $\Phi = \Phi_0 + \frac{1}{\sqrt{t}}\tilde{\Phi}$, where $\tilde{\Phi}$ are the fields fluctuations. The astounding result of this procedure is that it actually gives an *exact result*, and not just an approximated one, for the original undeformed partition function of the theory but also of the expectation value of any observable satisfying $\delta\mathbb{O} = 0$.

Supersymmetric localization has been implemented in numerous situations and the partition function of $\mathcal{N} \geq 2$ theories on $\mathbb{S}^3$ [7] and $\mathcal{N} = 2, 4$ theories on $\mathbb{S}^4$ [6] has been successfully reduced to a finite-dimensional integral, i.e. a matrix model. The same procedure has been implemented to obtain the explicit expression of the matrix model for other observables, in particular Wilson Loops, for both the three- and four-dimensional cases.
Let us focalize on the three-dimensional case which will be the relevant case for the next Chapter. In [7] the matrix model for ABJ(M) theory has been explicitly determined by using the localization technique described above. The integral expression for the $N_1 \neq N_2$ case reads [83, 85]

$$\begin{aligned} Z_{ABJ(M)}[\mathbb{S}^3] = C(N_1, N_2) \int \prod_{i=1}^{N_1} \prod_{r=1}^{N_2} d\lambda_i d\hat{\lambda}_r \; e^{-ik\pi(\lambda_i^2 - \hat{\lambda}_r^2)} \\ \times \frac{\prod_{i<j}^{N_1} \sinh^2 \pi(\lambda_i - \lambda_j) \prod_{r<s}^{N_2} \sinh^2 \pi(\hat{\lambda}_r - \hat{\lambda}_s)}{\prod_{i=1}^{N_1} \prod_{r=1}^{N_2} \cosh^2 \pi(\lambda_i - \hat{\lambda}_s)} \end{aligned} \tag{1.108}$$

where $\lambda_i, \hat{\lambda}_r$ are the set of eigenvalues of the Cartan matrices corresponding to the two gauge group factors and $C(N_1, N_2)$ is the overall normalization coefficient whose slightly modified version can be found in [83]. Notably in [93], it was put in evidence that the expression for the ABJM matrix model can be alternatively interpreted as the canonical partition function of a system of N non-interacting fermions with an ad hoc non-trivial one-particle density matrix. This interpretation is usually referred to as the *Fermi gas* approach.
The authors of [7] also showed that by inserting the following quantity

$$W_{\frac{1}{6}} = \frac{1}{N_1} \sum_{i=1}^{N_1} e^{2\pi\lambda_i} \tag{1.109}$$

in (1.108), one obtains the matrix model computing the vacuum expectation value of the 1/6 BPS Wilson loop[42] whose connection have been introduced in (1.99a). Analogously, by substituting $\lambda \to \hat{\lambda}, N_1 \to N_2$ one obtains the same result for (1.99b). In the same fashion, they proposed that also the vev of the 1/2 BPS Wilson loop can be computed from (1.108) by inserting the following expression

$$W_{\frac{1}{2}} = \frac{1}{N_1 + N_2} \left(\sum_{i=1}^{N_1} e^{2\pi\lambda_i} + \sum_{r=1}^{N_2} e^{2\pi\hat{\lambda}_r} \right). \tag{1.110}$$

Thanks to the previous results, the authors of [83] were able to obtain the strong coupling planar and non-planar expressions of the ABJM ($N_1 = N_2$) *free energy* on $\mathbb{S}^3$ and the vevs for both 1/2 and 1/6 BPS Wilson loops, finding perfect agreement with weak coupling holographic calculations. These results constitute a set of fundamental interpolating functions

[41] The normalization of the fluctuations is such that the kinetic term is always canonically normalized, namely no parameter t should appear.

[42] This can be computed if the supercharge used for localizing the path-integral is also preserved by the loop. This turns out to be true only if the Wilson loop constitutes an *Hopf fiber* of the sphere [7]. The Hopf fibration of $\mathbb{S}^3$ is a particular non-trivial fiber bundle for which the base space is $\mathbb{S}^2$, the fiber is $\mathbb{S}^1$ and the total space is $\mathbb{S}^3$. This is perfectly consistent with the fact that $\mathbb{S}^3 = \mathbb{S}^1 \times \mathbb{S}^2$ locally but not globally.

between the weak and strong coupling regime for the ABJM theory.
All the matrix model results for the Wilson loop insertions described above are necessarily computed at framing $\nu = 1$ (see Footnote 42). A very interesting proposal of a matrix model describing the partition function of ABJ(M) and 1/2, 1/6 BPS Wilson loops insertions at generic framing ν has been conjectured in [94]. This highly non-trivial conjecture was accompanied by strong evidences like the three-loops computation of the vev of $\langle W_{\frac{1}{6}} \rangle_\nu$, matching the already known expression for $\nu = 1$, and the planar strong coupling evaluation of the same loop, finding perfect agreement with the leading and next-to-leading order holographic predictions.

Chapter 2

The Topological Line of ABJ(M) Theory

As we saw in the previous chapter, ABJ(M) is a particular theory which enjoys many nice properties like conformal, supersymmetry and parity invariance but also inherits some issues coming from its Chern-Simons theory origin like framing anomaly and, by adding matter, the lost of topologicity at classical and quantum level. In this chapter we address the problem of recovering the topological properties for at least a subsector of the theory, by suitably projecting some selected operators, such that their n-point correlators do not depend on space-time coordinates. The projection procedure will be guided by the implementation of the so called *topological twist* procedure, a technique that allows to consistently combine space-time and global symmetries such that their correlators become invariant under a suitably generalized translation called *twisted translation*. The set of operators satisfying the previous condition populate to all effects a protected but also *solvable* sector, thanks to which many exact data on the whole ABJ(M) theory can be efficiently extracted, like the three-point function constants C_{ijk} introduced in (1.20), or some interpolating functions like the central charge of the theory c_T.

The first implementation of a topological twist for constructing topological quantum field theories (TQFT) goes back to the groundbreaking work of Witten [95], in which it was shown how the application of the procedure to an ad hoc Euclidean four-dimensional $\mathcal{N} = 2$ SYM theory, allows one to render its stress-energy tensor a Q-exact object, which automatically implies the quantum topologicity of the theory[1]. In the same work it was also shown how this technique can be extended to any non-trivial element of the cohomology of Q and thus any observable constructed out of it. Witten's procedure was then adapted to other supersymmetric theories for constructing topological subsectors hosting protected operators. Some examples are the well-known two-dimensional *chiral algebra* sector introduced in [96] existing in any four-dimensional $\mathcal{N} \geq 2$ SYM theory and the one-dimensional topological sector existing in any three-dimensional $\mathcal{N} \geq 4$ theory [22], but explicitly probed only in the $\mathcal{N} = 4, 8$ cases. Precisely motivated by the latter case, we explicitly construct the one-dimensional topological sector for the ABJ(M) theory, we study the dynamics of certain chiral operators living in such sector and prove their topological nature at classical and quantum level. In particular: in section 2.1 we review the formulation for all three-dimensional $\mathcal{N} \geq 4$ theories, in section 2.2 we perform the topological twist and find the explicit field realization for the chiral operators we mentioned above. In section 2.3 we compute the main correlators of such operators up to two-loops for the two-point function and in section 2.2.2 we compare the perturbative results with the exact results obtained from the ABJ(M) mass-deformed matrix model.

[1] Assuming that the symmetry associated to the supercharge Q is non-anomalous and preserves the vacuum.

2.1 A Conjecture for 3D $\mathcal{N} \geq 4$ Theories

In this section we review the main aspects of the conjecture explored in [22–24, 31] regarding the construction of the one-dimensional topological sector for $\mathcal{N} = 4$ SCFTs, its relation with the mass-deformed matrix model computed on $\mathbb{S}^3$ and its generalization to the whole $\mathcal{N} \geq 4$ class. We then discuss the well-motivated speculations regarding ABJ(M) theory, its topological sector and its relation with the corresponding mass-deformed matrix model on $\mathbb{S}^3$.

The construction of topological sectors follows the original idea of Witten [95], in which he gave an explicit recipe for constructing topological observables out of representatives of certain Q-cohomology classes. The argument for which the partition function preserves general covariance at quantum level can be easily imported to correlators as follows. By picking up a Q-exact translation operators $P \sim \{Q, \bar{Q}\}$ and non-trivial operators belonging to inequivalent Q-cohomology classes, what roughly happens is that

$$\begin{aligned}\partial_{x_1} \langle O(x_1) \dots O(x_n) \rangle &= \langle [P, O(x_1)] \dots O(x_n) \rangle \\ &= \langle \{Q, [\bar{Q}, O(x_1)]\} \dots O(x_n) \rangle \\ &= -\sum_i \langle [\bar{Q}, O(x_1)] \dots [Q, O(x_i)] \dots O(x_n) \rangle = 0\end{aligned} \tag{2.1}$$

where in the second line we applied the super-Jacobi identity and in the third line we assumed a vacuum preserving Q. The result obtained above tells us that a generic n-point function of non-trivial Q-closed operators must be independent of x_1. It is straightforward to see that the argument above can be extended to any space-time coordinate x_i when they are all restricted to lie on a one-dimensional submanifold $\mathcal{M}_1$ of $\mathcal{M}_3$. This automatically implies that the correlator must be a function of the couplings of the theory only.
Despite the nice result obtained, the operators we considered above are in general charged under R-symmetry, therefore the only consistent result for the correlator is actually zero.
A non vanishing result can be however recovered if we are able to appropriately compensate the R-symmetry charge of the operators with the introduction of a generalized symmetry under which all the operators are now neutral. The problem of finding neutral non-trivial elements in the Q-cohomology is thus shifted to the determination of non-trivial neutral elements in the cohomology of a so called *twist* or *cohomological supercharge* $\mathcal{Q}$. This is exactly what can be achieved by performing the topological twist we previously mentioned.
For simplicity, the operators satisfying the generalized neutrality condition will be called from now on *topological operators*.

For three-dimensional $\mathcal{N} = 4$ SCFTs such procedure has been successfully implemented in [22] and a one-dimensional topological sector hosting non-trivial topological operators, has been obtained. In the same work it was proven that these operators are always superconformal primaries of certain short multiplets[2] of the form $\mathcal{O}_{a_1 \dots a_n}(\vec{0})$, whose scaling dimension and R-symmetry quantum numbers are $\Delta = j = n/2$ and transform in the $(n+1, 1)$ of the $SO(4) \sim SU(2)_H \times SU(2)_C$ R-symmetry group.
The most interesting aspect regarding the above protected sector is that it is deeply connected with the localization procedure of the partition function of any three-dimensional $\mathcal{N} \geq 4$ SCFT when $\mathcal{M}_3 = \mathbb{S}^3$. Indeed, the localization procedure for such class of theories can be in general performed by using the same fermionic charge as in [7], namely a supercharge which belongs to a three-dimensional $\mathcal{N} = 2$ sub-superalgebra of the corresponding $\mathcal{N} = 4$. However, in [31], it is shown how to equivalently carry out the same localization procedure with the cohomological supercharge $\mathcal{Q}$, which cannot lie in any $\mathcal{N} = 2$ sub-superalgebra, for determining a different but equivalent matrix model as in [7].

[2] In [22, 30] it was shown that the topological sector captures the half-BPS spectrum of the three-dimensional theory. Topological operators indeed form a so called *chiral ring* of half-BPS operators.

Since $\mathcal{Q}$ preserves both the partition function and the topological operators, as we reviewed in Subsection 1.4.2, the localization procedure is allowed for determining the exact value of the correlators of such operators. In [31] it is indeed shown that this can be achieved by coupling the matrix model of the theory to a three-dimensional Gaussian model which, once evaluated on $\mathcal{Q}$-invariant configurations, can be further localized to a one-dimensional Gaussian model placed on the great circle $\mathbb{S}^1$ inside $\mathbb{S}^3$.
All the local topological observables can be therefore represented as follows

$$\langle O(x_1)\dots O(x_n)\rangle = \frac{1}{|\mathcal{W}|}\int_{\text{Cartan}} d\sigma \det_{\text{Adj},\sigma\neq 0}(2\sinh(\pi\sigma)) \int \mathcal{D}O\, e^{-S_\sigma[O]}\langle O(x_1)\dots O(x_n)\rangle_\sigma \tag{2.2}$$

where $|\mathcal{W}|$ is the order of the Weil group, S_σ is the partition function of the one-dimensional model[3] and O are the topological operators. Notice that when there are no insertions in the one-dimensional partition function, the expression above can be reduced to the matrix model computed in [7] without the CS-contribution to the partition function.
The powerful prescription given above for rigorously obtaining all the correlation functions describing the one-dimensional topological sector, can be improved even more. What was indeed noticed in [23, 31] is that the original $\mathcal{N}=4$ SCFT can be deformed by a full supersymmetry-preserving adjoint-valued mass parameter m^a coupled to part of the current supermultiplet of the theory, such that localization is still accessible. The highly non-trivial fact of this modification is that the mass deformation in the original three-dimensional theory can be understood as a mass-deformation of the one-dimensional Gaussian theory given by

$$-4\pi r^2 m^a \int_{-\pi}^{\pi} d\tau\, O^a(\tau). \tag{2.3}$$

The direct implication of the above arguments is that the mass-deformed partition function of the three-dimensional theory $Z[m]$ actually computes the partition function of the one-dimensional theory deformed by (2.3). This means that, by taking the derivatives of the mass-deformed matrix model representing $Z[m]$ with respect to the mass parameters m^a, what we get are *integrated* correlation functions of topological operators living on the great circle $\mathbb{S}^1 \subset \mathbb{S}^3$. From the expression in (2.3), the exact prescription [23] will be given by

$$\left\langle \int_{-\pi}^{\pi} d\tau_1 \dots \int_{-\pi}^{\pi} d\tau_n\, O^{a_1}(\tau_1)\dots O^{a_n}(\tau_n)\right\rangle = \frac{1}{(4\pi r^2)^n}\frac{1}{Z}\frac{\partial^n}{\partial m^{a_1}\dots\partial m^{a_n}} Z[\mathbb{S}^3, m^a]\Big|_{m^a=0}, \tag{2.4}$$

where r is the radius of the sphere $\mathbb{S}^3$. Since the topological correlators are position independent, the integrals on the l.h.s. can be trivially performed leading to the following final recipe which will be used from now on

$$\langle O^{a_1}(\tau_1)\dots O^{a_n}(\tau_n)\rangle = \frac{1}{(8\pi^2 r^2)^n}\frac{1}{Z}\frac{\partial^n}{\partial m^{a_1}\dots\partial m^{a_n}} Z[\mathbb{S}^3, m^a]\Big|_{m^a=0}. \tag{2.5}$$

The most intriguing interpretation of the equation above is that, in principle, if we knew exactly all the local observables involving topological operators, namely we solved the one-dimensional theory describing the topological sector, we would be allowed to reconstruct the exact mass-deformed partition function of the three-dimensional theory on $\mathbb{S}^3$. This interpretation could be seen as the three-dimensional analogous of the *chiral algebra program* [96–98] for four-dimensional $\mathcal{N} \geq 2$ superconformal field theories.

The cohomological construction of the topological sector, together with the prescription in (2.5), are valid also for $\mathcal{N}=8$ SCFTs [23]. We can indeed decompose the $\mathfrak{so}(8)$ R-symmetry algebra into $\mathfrak{so}(4)_R \oplus \mathfrak{so}(4)_F$ R-symmetry and flavour symmetry algebras, such

[3] The detailed discussion and all the relevant informations can be found in [31].

that they can be seen as a subclass of $\mathcal{N}=4$ theories enjoying an $\mathfrak{so}(4)$ flavor symmetry. Moreover, by carefully decomposing representations of the $\mathcal{N}=8$ superconformal algebra in terms of the ones of the $\mathcal{N}=4$ algebra, it is possible to find the corresponding one-dimensional topological sector [22, 23]. In this case the relevant $\mathcal{N}=4$ supercurrent multiplet belong to the $\mathcal{N}=8$ stress-energy tensor multiplet, consequently, topological operators are intimately related to the stress-tensor of the $\mathcal{N}=8$ theory. Superconformal Ward identities hence relate the two-point function $\langle O^a(\tau)O^b(0)\rangle$ to the two-point function of the stress-energy tensor $T_{\mu\nu}$, whose general structure is

$$\langle T_{\mu\nu}(\vec{x})T_{\rho\sigma}(0)\rangle = \frac{c_T}{64}(P_{\mu\rho}P_{\nu\sigma}+P_{\nu\rho}P_{\mu\sigma}-P_{\mu\nu}P_{\rho\sigma})\frac{1}{16\pi^2\vec{x}^{\,2}}, \tag{2.6}$$

where $P_{\mu\nu}=\eta_{\mu\nu}\nabla^2-\partial_\mu\partial_\nu$ and c_T is the central charge of the three-dimensional theory [4]. In particular, one obtains that the two-point function of topological operators is proportional to the central charge c_T of the $\mathcal{N}=8$ theory therefore making manifest the deep relation between an interpolating function of the bulk theory from a protected sector.
Finally, the passages described above are shown to lead to the following formula for computing the central charge of any mass-deformed $\mathcal{N}\geq 4$ SCFT on $\mathbb{S}^3$ [23, 24, 99]

$$c_T = -\frac{64}{\pi^2}\frac{d^2}{dm^2}\log\mathcal{Z}[S^3,m]\Big|_{m=0}. \tag{2.7}$$

The above formula is perfectly consistent being nothing but a particular case of a more general independent formula, which has been proven in [100], relating the *free energy* of any suitably deformed[5] $\mathcal{N}\geq 2$ SCFTs on $\mathbb{S}^3$ to the coefficient of the two-point function of the flavour currents of the theory. In the case at hand, topological operators are the one-dimensional reduction of such currents.
The consistency of the two independent results for c_T thus represents an alternative way to prove the validity of (2.5), at least for $n=2$. For the $\mathcal{N}=8$ theories this has been discussed in detail in [23].

2.2 The ABJ(M) Theory Case

As we mentioned at the beginning of the chapter, we are interested in investigating the previous results for ABJ(M) theory, which is exactly the three-dimensional superconformal field theory lacking of such an explicit construction. Although we should expect things to work similarly, a rigorous proof of the validity of identity (2.5) is still lacking due to the absence of an off-shell formulation of the Chern-Simons sector[6].
Some parallel additional steps have been undertaken in [24], where, assuming the validity of the prescription (2.5) for ABJ(M) theory, some coefficients appearing in the *Mellin amplitudes* of the dual eleven-dimensional supergravity and Type IIA supergravity theories have been fixed by computing four-point functions of topological operators at strong coupling.

The starting point for approaching the ABJ(M) case is to introduce its field content in the $\mathcal{N}=2$ language and turn on a mass deformation in the matrix model[7] corresponding to a (real) mass deformation, with mass spectrum $(m_+,-m_+,m_-,-m_-)$ for the bifundamental chiral multiplets $(\mathcal{W}_1,\bar{\mathcal{Z}}_1,\mathcal{W}_2,\bar{\mathcal{Z}}_2)\equiv W_I$, $I=1,\ldots,4$, in the ABJ(M) action. It follows that, from the formula in (2.5) suitably adapted to the ABJ(M) case, derivatives of the

[4] We conventionally set $c_T=1$ both for a real scalar field and a Majorana fermion.

[5] This recipe is valid also when we place the theory on the squashed sphere S^3_b, where b is called *squashing parameter*. In this case the central charge is given by the second derivative of the free energy w.r.t. to b.

[6] A proof of the validity of (2.5) in the ABJ(M) case will appear soon [101].

[7] The matrix model describing the mass-deformed partition function of ABJ(M) can be evaluated exactly in the large N limit as it was shown in [102, 103].

matrix model with respect to $m_\pm$ should provide integrated correlation functions of certain topological operators constructed out of the superconformal primaries sitting in the stress-energy tensor multiplet (for simplicity we set fermions to zero and consider only the bosonic operators). Their explicit structure is determined to be

$$\mathcal{O}_I{}^J(\vec{x}) = \mathrm{Tr}\big(C_I(\vec{x})\bar{C}^J(\vec{x})\big) - \frac{1}{4}\delta_I{}^J\ \mathrm{Tr}\big(C_K(\vec{x})\bar{C}^K(\vec{x})\big) \tag{2.8}$$

where C_I is the scalar component of W_I. Notice that the above operator is nothing but the symmetric traceless part in the decomposition of the product of the two scalar fields sitting respectively in the $\mathbf{4}$ and $\bar{\mathbf{4}}$ representation of $SU(4)$ R-symmetry, i.e. lives in the $\mathbf{15}_s$ irrep of $SU(4)$.
Superconformal Ward identities, like in the $\mathcal{N}=8$ case, then relate the two-point functions of the operators in (2.8) to correlator (2.6) of the stress-energy tensor giving the following expression

$$\langle \mathcal{O}_I{}^J(\vec{x})\, \mathcal{O}_K^L(\vec{0})\rangle = \frac{c_T}{16}\left(\delta_I^L\delta_K^J - \frac{1}{4}\delta_I^J\delta_K^L\right)\frac{1}{16\pi^2\vec{x}^2} \tag{2.9}$$

Assuming that a one-dimensional topological sector, with the same characteristics as the $\mathcal{N}=4$ case, does exist also in the ABJ(M) case, the operators in (2.8) are related to the topological ones $\mathcal{O}(\tau)$, localized on the great circle $\mathbb{S}^1 \subset \mathbb{S}^3$, by a suitable projection given by the topological twist procedure previously mentioned. Therefore, by exploiting the twisted version of the expression in (2.9), one can simply compute c_T from perturbation theory explicitly. In parallel, the equation in (2.7) is valid also for the ABJ(M) theory as follows

$$c_T = -\,\frac{64}{\pi^2}\frac{\partial^2}{\partial m_\pm^2}\log\mathcal{Z}[S^3, m_\pm]\Big|_{m_\pm=0}, \tag{2.10}$$

and provides an alternative way to compute the same central charge. The matching of the two independent expressions for the central charge allows us to conclude that (2.5) is valid also in the ABJ(M) case.
In what follows we will thus explicitly construct the topological sector for the ABJ(M) theory, determine the structure of topological line operators $\mathcal{O}$ and check the validity of the identity (2.5) for ABJ(M) which reads

$$\langle \mathcal{O}(\tau_1)\mathcal{O}(\tau_2)\rangle = \frac{1}{4\pi^4}\frac{\partial^2}{\partial m_\pm^2}\log\mathcal{Z}[S^3, m^\pm]\Big|_{m_\pm=0} \tag{2.11}$$

This is achieved by matching the weak coupling expansion of the derivatives of the mass deformed ABJ(M) Matrix Model with a two-loop evaluation of the two-point correlator $\langle\mathcal{O}(\tau_1)\mathcal{O}(\tau_2)\rangle$. Notice that this result will provide us the two-loop approximation of the expression of the central charge of ABJ(M) in the weak coupling limit.

2.2.1 The Topological Twist

We choose for simplicity to perform our construction by selecting a straight-line parallel to the x^3-direction and parametrized as $x^\mu(s) = (0, 0, s)$, with $s \in (-\infty, +\infty)$ being its proper time. All the fields of the original theory hence will decompose into irreducible representation of the superconformal algebra preserving such sector. This is given by an $\mathfrak{su}(1,1|3) \oplus \mathfrak{u}(1)_b$ superalgebra inside the original $\mathfrak{osp}(6|4)$ one. The detailed classification can be found in Appendix A.4.2, but let us present some preliminary facts regarding the structure of the fields living in the one-dimensional sector.
The restriction described above causes all the fields to reorganize accordingly to the maximal bosonic subgroups of $\mathfrak{su}(1,1|3)$ which is $\mathfrak{sl}(2) \oplus \mathfrak{su}(3) \oplus \mathfrak{u}(1)$. The scalars $C_I, \bar{C}^I$ and the

fermions $\psi_I, \bar{\psi}^I$, $I = 1, 2, 3, 4$, will in particular decompose into irreducible representations of the $SU(3)$ residual R-symmetry group, as

$$C_I = (Z, Y_a), \qquad \bar{C}^I = (\bar{Z}, \bar{Y}^a), \qquad \psi_I = (\psi, \chi_a), \qquad \bar{\psi}^I = (\bar{\psi}, \bar{\chi}^a), \tag{2.12}$$

where $Y_a(\bar{Y}^a), \chi_a(\bar{\chi}^a)$, $a = 1, 2, 3$, belong to the $\mathbf{3}(\bar{\mathbf{3}})$ of $SU(3)$, whereas $Z, \bar{Z}, \psi, \bar{\psi}$ are $SU(3)$-singlets. The gauge fields and covariant derivatives (see their definition in (A.12)), split according to the new space-time symmetry as

$$A_\mu = (A \equiv A_1 - iA_2,\ \bar{A} \equiv A_1 + iA_2,\ A_3), \tag{2.13a}$$
$$\hat{A}_\mu = (\hat{A} \equiv \hat{A}_1 - i\hat{A}_2,\ \bar{\hat{A}} \equiv \hat{A}_1 + i\hat{A}_2,\ \hat{A}_3), \tag{2.13b}$$
$$D_\mu = (D \equiv D_1 - iD_2,\ \bar{D} \equiv D_1 + iD_2,\ D_3), \tag{2.13c}$$

We are now ready to implement the twisting procedure for the line considered above and identify the explicit structure of the topological operators we are looking for. We start for simplicity by complexifying the $\mathfrak{su}(1,1|3)$ superalgebra so that we are allowed to take complex combinations of the original generators and such that they can act faithfully on the reduced operators we presented above. Its commutation relations are given in eqs. (A.26, A.28, A.33, A.34).
In order now to perform the topological twist, we need to select an $\mathfrak{su}(1,1)(\simeq \mathfrak{sl}(2))$ subalgebra inside the complexification of $\mathfrak{su}(3)$, which we will take to be generated by

$$\mathfrak{su}(1,1) \simeq \left\langle iR_3{}^1,\ iR_1{}^3,\ \frac{R_1{}^1 - R_3{}^3}{2} \right\rangle \equiv \langle \mathcal{R}_+, \mathcal{R}_-, \mathcal{R}_0 \rangle, \tag{2.14}$$

obeying the following commutation relations

$$[\mathcal{R}_0, \mathcal{R}_\pm] = \pm \mathcal{R}_\pm, \qquad [\mathcal{R}_+, \mathcal{R}_-] = -2\mathcal{R}_0. \tag{2.15}$$

This is important for defining the twisted symmetry algebra which will allow for a non-vanishing result of the correlators in (2.5). We can also define a $\mathfrak{u}(1)$ factor generated by

$$\tilde{\mathcal{R}} \equiv \frac{R_1{}^1 + R_3{}^3}{2}, \tag{2.16}$$

commuting with the algebra in (2.15). In practice, we have broken the complexification of the original $\mathfrak{su}(3)$ into $\mathfrak{su}(1,1) \oplus \mathfrak{u}(1)$.
With respect to this subalgebra, the supercharges split into two doublets (Q^1, Q^3) and (S^1, S^3), and their hermitian conjugates $(\bar{Q}_1, \bar{Q}_3)$, $(\bar{S}_1, \bar{S}_3)$, which transform in the fundamental of $\mathfrak{su}(1,1)$ and have $\mathfrak{u}(1)$ charges $1/6$ and $-1/6$, respectively. The remaining supercharges Q^2, S^2 $(\bar{Q}_2, \bar{S}_2)$ are instead singlets with $U(1)$ charges $-1/3$ $(1/3)$.
The topological twist can now be performed by taking the diagonal sum of the one-dimensional conformal algebra defined in (A.26) with the $\mathfrak{su}(1,1)$ algebra given in (2.14).
The resulting *twisted* generators are

$$\hat{L}_+ = P + \mathcal{R}_+, \qquad \hat{L}_- = K + \mathcal{R}_-, \qquad \hat{L}_0 = D + \mathcal{R}_0, \tag{2.17}$$

which satisfy the following commutation relations

$$[\hat{L}_0, \hat{L}_\pm] = \pm \hat{L}_\pm, \qquad [\hat{L}_+, \hat{L}_-] = -2\hat{L}_0. \tag{2.18}$$

We shall denote this twisted conformal algebra on the line with $\widehat{\mathfrak{su}}(1,1)$.
Under the new spin assignments induced by $\widehat{\mathfrak{su}}(1,1)$, the supercharges Q^3, S^1 and their hermitian conjugates are now scalars. In particular, the linear combinations

$$\mathcal{Q}_1 = Q^3 + iS^1, \qquad \mathcal{Q}_2 = \bar{S}_3 + i\bar{Q}_1 \tag{2.19}$$

define two independent nihilpotent supercharges, $\mathcal{Q}_1^2 = \mathcal{Q}_2^2 = 0$. Remarkably, the generators of $\widehat{\mathfrak{su}}(1,1)$ are $\mathcal{Q}$-exact with respect to both charges. In fact, it is easy to check that

$$\hat{L}_+ = \{\mathcal{Q}_1, \bar{Q}_3\} = -\,i\,\{\mathcal{Q}_2, Q^1\} \qquad \hat{L}_- = -i\,\{\mathcal{Q}_1, \bar{S}_1\} = \{\mathcal{Q}_2, S^3\}$$
$$\hat{L}_0 = \frac{1}{2}\left\{\mathcal{Q}_1, \mathcal{Q}_1^\dagger\right\} = \frac{1}{2}\left\{\mathcal{Q}_2, \mathcal{Q}_2^\dagger\right\} \tag{2.20}$$

The twisted generators $\hat{L}_\pm, \hat{L}_0$ and the charges $\mathcal{Q}_1$ and $\mathcal{Q}_2$ span a superalgebra, which possesses a central extension given by

$$\mathcal{Z} = \frac{1}{4}\{\mathcal{Q}_1, \mathcal{Q}_2\} = \frac{1}{3}M - \tilde{\mathcal{R}} \tag{2.21}$$

where M is the $\mathfrak{u}(1)$ generator defined in (A.29).

2.2.2 $\mathcal{Q}$-Cohomology and Topological States

As we reviewed in Section 2.1, the topological sector of the ABJ(M) theory should contain all the local[8], gauge-invariant operators belonging to the cohomology of a nilpotent charge $\mathcal{Q}$ for which the twisted translations are $\mathcal{Q}$-exact. This is the key ingredient for obtaining non-vanishing constant correlators when we consider topological operators. Since both $\mathcal{Q}_1$ and $\mathcal{Q}_2$ satisfies this property, we can choose either one of them or a suitable linear combination, in any case the results will be independent of which charge we select.[9] For this reason we label any generic combination of cohomological charges as $\mathcal{Q}$ with no loss of generalities.
The defining conditions for an operator $\mathcal{O}(s)$ living in the cohomology of $\mathcal{Q}$ are

$$[\mathcal{Q}, \mathcal{O}(s)\} = 0, \qquad \mathcal{O}(s) \neq [\mathcal{Q}, \mathcal{O}'(s)\}, \tag{2.23}$$

where either commutators or anticommutators appear depending on the spin of $\mathcal{O}$. The above relations are nothing but the requirement that $\mathcal{O}$ must be a non-trivial operator in the $\mathcal{Q}$-cohomology, i.e. $\mathcal{Q}$-closed but not $\mathcal{Q}$-exact.
Such elements have to be selected among the irreducible representations of the one-dimensional superconformal algebra $\mathfrak{su}(1,1|3)$ which are, as we saw in Subsection 1.1.2 classified at the origin of the line. Therefore, the operators we are looking for have to satisfy the relations in (2.23) when $s = 0$. In fact, an operator located at a generic point s can always be obtained from the one evaluated at the origin by applying a twisted translation $\hat{L}_+$ as follows[10]

$$\mathcal{O}(s) \equiv e^{-s\hat{L}_+}\,\mathcal{O}(0)\,e^{s\hat{L}_+}. \tag{2.24}$$

Being the $\hat{L}_+$ generator $\mathcal{Q}$-exact, topological operators $\mathcal{O}(0)$ and $\mathcal{O}(s)$ automatically belong to the same $\mathcal{Q}$-cohomology class.
As briefly reviewed in appendix A.4.2, the operators in an irreducible representation are classified in terms of the conformal weight Δ, the $\mathfrak{u}(1)$ charge m and the two R-symmetry quantum numbers (j_1, j_2) corresponding to the two $\mathfrak{su}(3)$ Cartan generators defined in (A.37). We symbolically write $|\Delta, m, j_1, j_2\rangle$ to denote the corresponding state.

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[8] For our purposes it is sufficient to consider *local* operators.

[9] A useful basis for performing explicit computations is

$$\mathcal{Q}_\pm = \frac{1}{\sqrt{2}}(\mathcal{Q}_1 \pm \mathcal{Q}_2), \qquad \tilde{Q}_\pm = \frac{1}{\sqrt{2}}(\bar{Q}_3 \mp iQ^1), \qquad \tilde{S}_\pm = \frac{1}{\sqrt{2}}(-i\bar{S}_1 \pm S^3), \tag{2.22}$$

for which the two equivalent twisted algebras completely split and the supercharges square to a $\mathcal{Q}$-exact term, i.e. the $\mathcal{Z}$ element of (2.21). Notice that $\{\hat{L}_+, \hat{L}_-, \hat{L}_0, \mathcal{Z}, \mathcal{Q}_+, \mathcal{Q}_+^\dagger, \tilde{Q}_+, \tilde{S}_+\}$ span an $\mathfrak{su}(1,1|1)$ superalgebra.

[10] Notice that the exponentiation of L_+ is performed without the usual i coefficient because, in our convention, the differential representation of the one-dimensional momentum operator in (2.17) is $P = -\partial_s$.

Topological operators can be now easily identified by noting that $\widehat{L}_0$ and $\mathcal{Z}$, being $\mathcal{Q}$-exact, act trivially within each $\mathcal{Q}$-cohomology class (their action on cohomological representatives is always $\mathcal{Q}$-exact). Therefore, operators obeying the condition (2.23) belong necessarily to the zero eigenspaces of $\hat{L}_0$ and $\mathcal{Z}$ [22, 96]. In particular, in a unitary representation, any element of the kernel of $\hat{L}_0$ must be annihilated by $\mathcal{Q}_1$ and $\mathcal{Q}_2$, thanks to the last equation in (2.20). The problem is then reduced to determining the intersection $\mathcal{N} = \mathrm{Ker}(L_0) \cap \mathrm{Ker}(\mathcal{Z})$. To this end, using the $\mathfrak{su}(3)$ Cartan generators defined in (A.37), we rewrite $\hat{L}_0$ and $\mathcal{Z}$ given in eqs. (2.17, 2.21) as $\hat{L}_0 = D - (J_2 + J_1)$ and $\mathcal{Z} = \frac{1}{3}\big(M - (J_2 - J_1)\big)$. Therefore, a state $|\Delta, m, j_1, j_2\rangle$ in a given irreducible unitary representation is an eigenvector of $\hat{L}_0$ and $\mathcal{Z}$ with eigenvalues $\hat{l}_0 = \Delta - \frac{j_2+j_1}{2}$, $3z = m - \frac{j_2-j_1}{2}$.
This state will belong to $\mathcal{N}$ and define a topological operator if and only if

$$\Delta = \frac{j_2 + j_1}{2}\,, \qquad\qquad m = \frac{j_2 - j_1}{2}. \tag{2.25}$$

To identify operators whose quantum numbers satisfy the relations above, let us briefly scan the content of long and shorts $\mathfrak{su}(1,1|3)$ multiplets. The detailed analysis can be found in Appendix A.4.2. We will label multiplets from now on by using the notation of [55] but a useful conversion scheme from this notation to the one in [56] can be found in the latter. The first type of multiplets on which we focus are long multiplets, which we will denote with $\mathcal{A}^{\Delta}_{m;j_1,j_2}$. Those multiplets are characterized by unitarity constraints (A.42). The first of the two constraints on the m quantum number is always incompatible with (2.25), whereas the second one satisfies (2.25) at the threshold. Therefore, the superconformal primaries of the $\mathcal{A}$ multiplets at the threshold certainly belong to the cohomology of $\mathcal{Q}$.
Since topological operators are primary states of long multiplets for which the bound is saturated, more attention is required for a detailed classification. As we reviewed in Subsections 1.1.3 and 1.2.3 indeed, when the threshold is reached, the recombination phenomenon causes the $\mathcal{A}$ multiplets to split into short multiplets according to the decomposition rule in (A.45). From (A.45) we can identify topological operators as superconformal primaries of the short multiplets $\mathcal{B}^{\frac{1}{6},\frac{1}{6}}_{\frac{j_2-j_1}{2};j_1,j_2}$.
Referring to their shortening conditions (A.47, A.51) we immediately see that eqs. (2.25) are always satisfied by the superprimaries of $\mathcal{B}^{\frac{1}{6},0}_{\frac{j_2-j_1}{2};j_1,j_2}$ and $\mathcal{B}^{0,\frac{1}{6}}_{\frac{j_2-j_1}{2};j_1,j_2}$, for generic values of j_1 and j_2. The general result we have found is that topological operators are superconformal primaries of the following three most general supermultiplets

$$\mathcal{B}^{\frac{1}{6},\frac{1}{6}}_{\frac{j_2-j_1}{2};j_1,j_2}, \qquad \mathcal{B}^{\frac{1}{6},0}_{\frac{j_2-j_1}{2};j_1,j_2}, \qquad \mathcal{B}^{0,\frac{1}{6}}_{\frac{j_2-j_1}{2};j_1,j_2}. \tag{2.26}$$

We observe moreover that no other contributions can arise from any descendant state of $\mathcal{B}$ since the application of any supercharge preserving the topological sector would produce either a trivial state or a state violating the constraints in (2.25). Furthermore we can notice that, when j_1, j_2 or both vanish, these multiplets are even shorter and enhance their supersymmetry becoming 1/2 BPS multiplets. With these results at hand, which are in perfect agreement with the general findings of [22, 30, 104], let us now explicitly realize the topological states we identified in terms of the local fields of the theory.

Local Realizations

The primaries of the multiplets determined in the previous subsection can be explicitly realized in terms of fundamental matter fields. In fact, by looking at tables A.2 and A.3, we immediately realise that Y_1 and $\bar{Y}^3$ provide two superconformal primaries of respectively $\mathcal{B}^{\frac{1}{6},\frac{1}{3}}_{\frac{-j_1}{2};j_1,0}$ and $\mathcal{B}^{\frac{1}{3},\frac{1}{6}}_{\frac{j_2}{2};0,j_2}$.
Using these two fundamental fields, the simplest gauge-invariant topological operator on the line can be constructed as the following composite operator

$$\mathcal{O}(s) \equiv e^{-s\hat{L}_+}\mathcal{O}(0)e^{s\hat{L}_+} \qquad \text{with} \qquad \mathcal{O}(0) = \text{Tr}\big(Y_1(0)\bar{Y}^3(0)\big), \tag{2.27}$$

clearly obeying the conditions in (2.25) having $[\Delta, m, j_1, j_2] = [1, 0, 1, 1]$.
Evaluating the twisted translation explicitly, this operator can be written, at a generic point s on the line, as

$$\mathcal{O}(s) = \text{Tr}\Big(Y_a(s)\bar{Y}^b(s)\Big)\; \bar{u}^a(s)\, v_b(s), \tag{2.28}$$

where $\bar{u}^a(s) = (1, 0, is)$ and $v_a(s) = (-is, 0, 1)$, are the so called *polarization vectors*.
These vectors are exactly the projectors we need in order to explicitly realize the twisting procedure at the level of the local untwisted fields and construct coordinate-independent observables, as we discussed around equation (2.9). Notice that the two vectors are actually orthogonal, namely $\bar{u}(s) \cdot v(s) = \bar{u}^a(s)v_b(s)\delta^b_a = 0$.
The contraction with the two polarization vectors leads to a linear combination of single trace operators with coefficients that depend on the insertion points

$$\mathcal{O}(s) = \text{Tr}\big(Y_1\bar{Y}^3\big) - is\,\text{Tr}\big(Y_1\bar{Y}^1\big) + is\,\text{Tr}\big(Y_3\bar{Y}^3\big) + s^2\,\text{Tr}\big(Y_3\bar{Y}^1\big), \tag{2.29}$$

analogously to what happens to certain superprotected operators in the four-dimensional $\mathcal{N} = 4$ SYM theory [85].
An immediate generalization of (2.27) is the following

$$O_n(0) = \text{Tr}\left(\underbrace{Y_1(0)\bar{Y}^3(0)\cdots Y_1(0)\bar{Y}^3(0)}_{n-\text{times}}\right) = \text{Tr}\big(Y_1(0)\bar{Y}^3(0)\big)^n, \tag{2.30}$$

whose quantum numbers are $[\Delta, m, j_1, j_2] = [n, 0, n, n]$.
These operators exhaust the spectrum of topological, gauge invariant local operators suitable for insertions on the topological line.
What makes $\mathcal{O}(0)$ in (2.27) special within the class of operators (2.30) is that it coincides with the scalar chiral superprimary ${\mathcal{O}_2}^4(0)$ we introduced in (2.8), appearing in the supermultiplet of the stress-energy tensor. As discussed at the beginning of this section, its two-point function (2.9) is thus deeply related to one of the stress-energy tensor itself and can be used to perturbatively evaluate the central charge c_T of the theory.

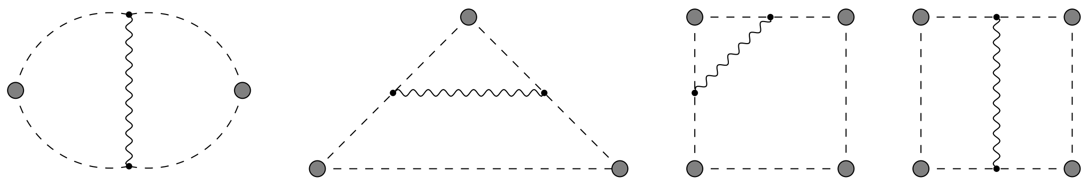


Figure 2.1: Topologies of one-loop diagrams contributing to the correlators.

2.3 Topological Correlators: Perturbative Results

A crucial check of the position independence of the correlators described above comes from their explicit perturbative evaluation. In particular, whether the topological nature is preserved at the quantum level is one of the main questions that can be addressed within this approach. In fact, if the quantum operator is topological, the evaluation of a generic n-point correlator will result in a function whose non-trivial dependence is at most on the coupling constants of the theory. In this section we study the connected two-, three- and four-point functions of the topological operators introduced in (2.28).
While three- and four-point correlators are evaluated up to one loop, we push the calculation for the two-point function up to two loops to provide a check of (2.11) at a non-trivial perturbative order. Correlators are computed on the straight line and later mapped to the great circle $\mathbb{S}^1$ of $\mathbb{S}^3$, in order to allow for a comparison with the localization results of section 2.4.

2.3.1 Correlators on $\mathbb{R}$

The perturbative evaluation of n-point correlation functions relies on the expansion of the Euclidean path integral in powers of the coupling constants N_1/k and N_2/k. All the details regarding the ABJ(M) Euclidean action, Feynman rules and all the related conventions, are collected in Appendices A.1 and A.2.

Using the scalar propagator in (A.13), it is easy to obtain the tree-level results for connected correlators

$$\langle \mathcal{O}(s)\mathcal{O}(0)\rangle^{(0)} = \bar{u}^a(s)\, v_b(s)\, \langle \mathrm{Tr}\left(Y_a\bar{Y}^b\right)\mathrm{Tr}\left(Y_1\bar{Y}^3\right)\rangle^{(0)} = -\frac{N_1N_2}{(4\pi)^2} \tag{2.31}$$

$$\langle \mathcal{O}(t)\mathcal{O}(s)\mathcal{O}(0)\rangle^{(0)} = \bar{u}^a(t)v_b(t)\bar{u}^c(s)v_d(s)\langle \mathrm{Tr}\left(Y_a\bar{Y}^b\right)\mathrm{Tr}\left(Y_c\bar{Y}^d\right)\mathrm{Tr}\left(Y_1\bar{Y}^3\right)\rangle^{(0)} = 0 \tag{2.32}$$

$$\begin{aligned}\langle \mathcal{O}(z)\mathcal{O}(t)\mathcal{O}(s)\mathcal{O}(0)\rangle^{(0)} &= \bar{u}^a(z)v_b(z)\bar{u}^c(t)v_d(t)\bar{u}^e(s)v_f(s)\,\times\\ &\times\langle \mathrm{Tr}\left(Y_a\bar{Y}^b\right)\mathrm{Tr}\left(Y_c\bar{Y}^d\right)\mathrm{Tr}\left(Y_e\bar{Y}^f\right)\mathrm{Tr}\left(Y_1\bar{Y}^3\right)\rangle^{(0)} = 2\frac{N_1N_2}{(4\pi)^4}\end{aligned} \tag{2.33}$$

In the non-vanishing cases, the space-time dependence at the denominator is, as expected, precisely canceled by the contribution coming from the contraction of the untwisted operators with the polarization vectors.
One-loop diagrams contributing to the two-, three- and four-point functions are drawn in figure 2.1. It is easy to realize that they all vanish due to geometrical reasons. Indeed, the combination of the Levi-Civita tensor coming from the gauge propagator (A.15) and the peculiar structure of polarization vectors causes the final result always to vanish.
The first non-trivial information comes therefore at two loops. We restrict the evaluation to the two-point function, whose diagrams at this order are drawn in figures 2.2(a)-2.2(j).

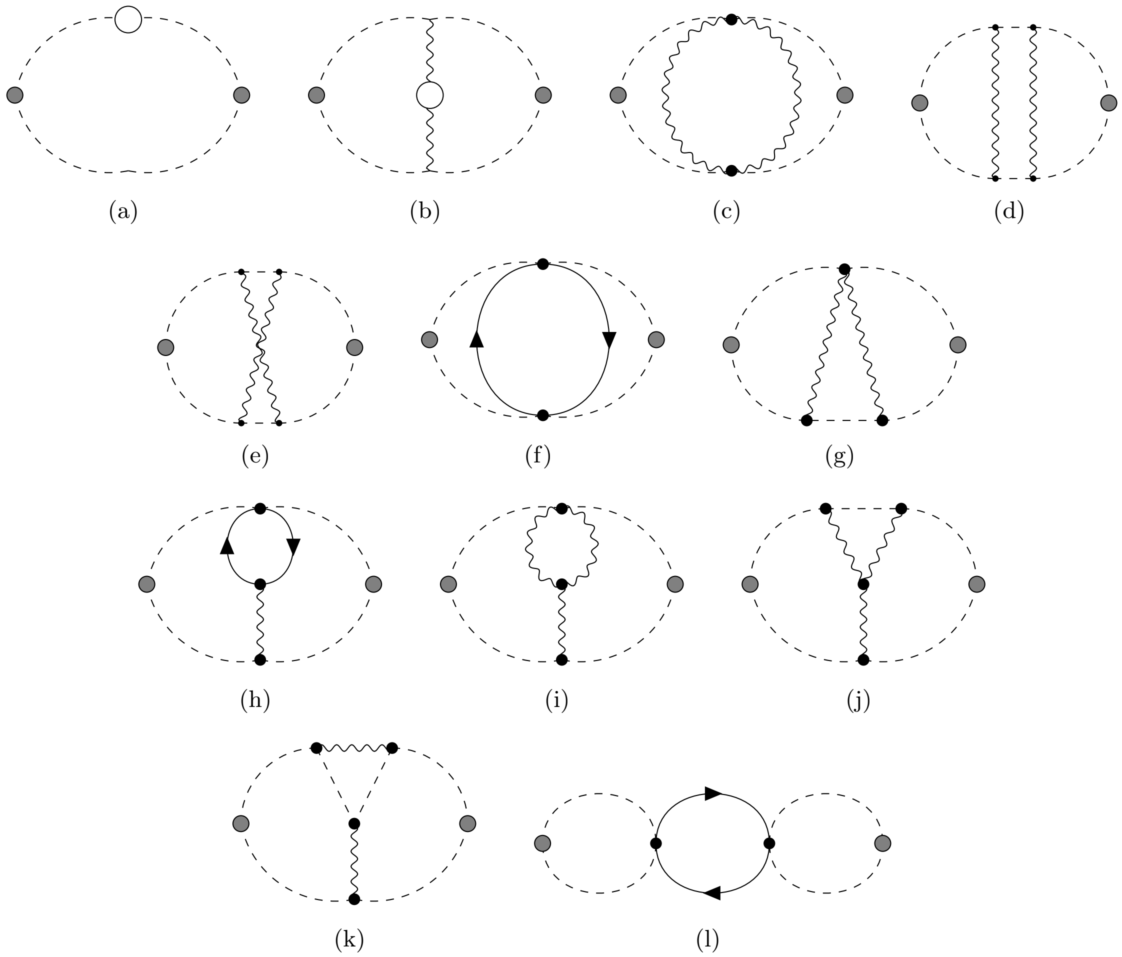


Figure 2.2: Two-loop diagrams for the two-point function. In (a) the white circle is the two-loop correction to the scalar propagator, while in (b) the circle is the one-loop correction to the gauge field propagator. Diagrams (h), (i), (j) and (k) sum up to provide the vertex correction.

The corresponding algebraic expressions, including the combinatorial and color factors are listed in appendix A.6. We evaluate the corresponding integrals by Fourier transforming to momentum space. Potential UV divergences are regularized within the DRED scheme [105, 106]. This amounts to first perform the tensor algebra strictly in three dimensions to reduce the integrals to a linear combination of scalar integrals and then analytically continue the resulting integrals to $d = 3 - 2\epsilon$ dimensions. As usual, we also introduce a dimensionful parameter μ to correct the scale dimensions of the couplings when they are promoted to d dimensions.

Applying *Mathematica* routines[11] based on the uniqueness method the momentum integrals can be analytically evaluated, leading to the results listed in Appendix A.6. Summing all the contributions, UV divergences cancel exactly [12] and we can then safely take the $\epsilon \to 0$ limit. The final result for the two-point function at two loops reads

$$\langle \mathcal{O}(s)\mathcal{O}(0)\rangle^{(2)} = -\frac{N_1 N_2}{(4\pi)^2}\left(1 - \frac{\pi^2}{6k^2}(N_1^2 + N_2^2 - 2)\right) \tag{2.34}$$

[11] We are grateful to Marco Bianchi for sharing with us his routines.

[12] We note that, for dimensional reasons, all the diagramatic contributions have a dipendence on the position of the form $|\mu s|^{8\epsilon}$. In principle, by expanding $|\mu s|^{8\epsilon} \sim (1 + 8\epsilon \log|\mu s| + \cdots)$, we could have produced dangerous finite $\log|\mu s|$ terms that would have spoiled the topological nature of the operators at quantum level.

The highly non-trivial result obtained above gives us strong evidences for the quantum topological nature of the operators (2.28).

The Central Charge c_T

As we have already observed in Subsection 2.2.2, the scalar primary operators appearing in (2.8) are nothing but the untwisted version of the topological operators we explicitly realized in (2.28). This means that, by contracting both sides of the expression (2.9) with suitably constructed polarization vectors, we can relate the perturbative expression of central charge c_T to the two-point function of topological operators we evaluated in (2.34).
For this purpose we write

$$\mathcal{O}(s) = \mathcal{O}_I{}^J(0,0,s)\bar{U}^I(s)V_J(s), \tag{2.35}$$

where we defined, in the $SU(4)$ notation, the following polarization vectors

$$\bar{U}^I(s) = (0,1,0,is)\,, \qquad V_J(s) = (0,-is,0,1). \tag{2.36}$$

At this point it is easy to show that the expression in (2.9) can be rearranged as

$$c_T = -64\,(2\pi)^2\,\langle \mathcal{O}(s)\,\mathcal{O}(0)\rangle \tag{2.37}$$

Inserting in (2.37) the perturbative result (2.34) for the two-point function, we obtain the expansion of the ABJ(M) central charge at second order in the couplings and at generic finite values of the ranks of the gauge group

$$c_T^{(2)} = 16N_1N_2\left(1 - \frac{\pi^2}{6k^2}(N_1^2+N_2^2-2)\right) \tag{2.38}$$

We note that, in the $k \to \infty$ limit, it correctly reproduces the central for a free theory of $4(N_1 \times N_2)$ chiral multiplets, in agreement with our conventions (see footnote 4), while for k fixed and $N_1 = N_2 = 2$, we correctly recover the two-loop approximation of c_T in equation (5.20) of [21].

2.3.2 Correlators on $\mathbb{S}^1$

If we assume that there are no conformal anomalies at quantum level, correlators of twisted operators computed on a line embedded in $\mathbb{R}^3$ and on the great circle $\mathbb{S}^1 \subset \mathbb{S}^3$ should be exactly the same[13] [31]. In other words, it is reasonable to assume that (setting $s = \tan\frac{\tau}{2}$)

$$\langle \mathcal{O}(s_1)\dots\mathcal{O}(s_k)\rangle_{\mathbb{R}^3} = \langle \mathcal{O}(\tau_1)\dots\mathcal{O}(\tau_k)\rangle_{\mathbb{S}^3} \tag{2.39}$$

where $\mathcal{O}(s)$ is the operator in (2.28) on the line, parametrized by the proper time s, and the operator $\mathcal{O}(\tau)$ is its counterpart on the circle, parametrized by the proper time τ. Topological operators on the circle are obtained by contracting the $\mathbb{S}^3$ operator localized on $\mathbb{S}^1$ with polarization vectors $\bar{u}^a(\tau), v_a(\tau)$ on the great circle.
From the background independence of the topological correlators stated in (2.39), it is easy to infer how the polarization vectors get mapped from the line to the great circle. In fact, by taking into account that the ABJ(M) scalar fields transform under a conformal transformation as $Y_1(s) = \Lambda^{\frac{1}{2}} Y_1(\tau)$, $\bar{Y}^3(s) = \Lambda^{\frac{1}{2}}\bar{Y}^3(\tau)$, with $\Lambda = \cos^2\frac{\tau}{2}$ the conformal

[13] Notice that this is still an open problem when we are in presence of an extended defect. A well-known example are the results, in the large N limit, of the vev of 1/2 BPS Wilson line and Wilson loop in $\mathcal{N} = 4$ SYM.

factor associated with the conformal transformation mapping the line into a circle (and viceversa), and the cohomological identification $\mathcal{O}(s) = \mathcal{O}(\tau)$, we obtain

$$\bar{u}^a(\tau) = \Lambda^{\frac{1}{2}} \bar{u}^a(s(\tau)) = \left(\cos\frac{\tau}{2}, 0, \sin\frac{\tau}{2}\right) \tag{2.40}$$

$$v_a(\tau) = \Lambda^{\frac{1}{2}} v_a(s(\tau)) = \left(-\sin\frac{\tau}{2}, 0, \cos\frac{\tau}{2}\right) \tag{2.41}$$

Thanks to the discussion above, we can safely extend the formula for the central charge in (2.37) to the following expression

$$c_T = -64\,(2\pi)^2\, \langle \mathcal{O}(\tau)\, \mathcal{O}(0)\rangle_{\mathbb{S}^1}, \tag{2.42}$$

which will be our starting point for proving the validity of the identity (2.11) for ABJ(M) theory up to two loops.

2.4 Topological Correlators: Matrix Model Results

In this section we explicitly reproduce the result for the central charge in (2.38) by considering the weak coupling expansion of the mass-deformed matrix model of ABJ(M) theory on $\mathbb{S}^3$ [107, 108] and its second derivatives with respect to the masses as prescribed in (2.10). This will be in turn a non-trivial two-loop check of the validity of (2.11).
We start by considering the mass-deformed version of the matrix model we introduced in (1.108) for ABJ(M) theory [24, 99], which reads

$$Z = \int d\lambda\, d\mu\, \frac{e^{i\pi k \sum_i (\lambda_i^2 - \mu_i^2)} \prod_{i<j} 16 \sinh^2\left[\pi\left(\lambda_i - \lambda_j\right)\right] \sinh^2\left[\pi\left(\mu_i - \mu_j\right)\right]}{\prod_{i,j} 4 \cosh\left[\pi(\lambda_i - \mu_j) + \frac{\pi m_+}{2}\right] \cosh\left[\pi(\lambda_i - \mu_j) + \frac{\pi m_-}{2}\right]} \tag{2.43}$$

where we dropped the irrelevant overall factor and the specific mass spectrum is the one introduced in Section 2.2. Taking derivatives respect to m_- (or equivalently with respect to m_+ since the matrix model is invariant under the $m_+ \leftrightarrow m_-$ exchange) we immediately find

$$\frac{\partial^2}{\partial m_-^2} \log Z[S^3, m_\pm]\bigg|_{m_\pm = 0} = \frac{Z''}{Z} - \left(\frac{Z'}{Z}\right)^2 \tag{2.44}$$

where Z is the undeformed matrix model and its derivatives are given by

$$Z' = -\int d\lambda\, d\mu\, e^{i\pi k \sum_i (\lambda_i^2 - \mu_i^2)}\, Z_{1-\mathrm{loop}}(\lambda_i,\, \mu_j) \sum_{i,j} \tanh \pi(\lambda_i - \mu_j) \tag{2.45}$$

$$\begin{aligned} Z'' = &\int d\lambda\, d\mu\, e^{i\pi k \sum_i (\lambda_i^2 - \mu_i^2)}\, Z_{1-\mathrm{loop}}(\lambda_i,\, \mu_j) \\ &\times \frac{\pi^2}{4}\left(\left(\sum_{i,j} \tanh(\pi(\lambda_i - \mu_j))\right)^2 - \sum_{i,j} \frac{1}{\cosh^2(\pi\left(\lambda_i - \mu_j\right))}\right) \end{aligned} \tag{2.46}$$

with

$$Z_{1-\mathrm{loop}}(\lambda_i,\, \mu_j) = \frac{\prod_{i<j} 16 \sinh^2\left[\pi\left(\lambda_i - \lambda_j\right)\right] \sinh^2\left[\pi\left(\mu_i - \mu_j\right)\right]}{\prod_{i,j} 4 \cosh(\pi\left(\lambda_i - \mu_j\right)) \cosh(\pi\left(\lambda_i - \mu_j\right))} \tag{2.47}$$

Since the integrand in Z' is odd under $\lambda \leftrightarrow \mu$ exchange, it vanishes once integrated. Thus we only need to compute contribution (2.46). Performing the following change of variables

$$x_i = \pi\sqrt{k}\lambda_i\,, \qquad y_j = \pi\sqrt{k}\mu_j\,, \qquad g_s = \frac{1}{\sqrt{k}} \tag{2.48}$$

the relevant quantities become

$$Z = \int dX\, dY e^{\frac{i}{\pi}\sum_i (x_i^2 - y_i^2)} f(x,y) \tag{2.49a}$$

$$Z'' = \int dX\, dY e^{\frac{i}{\pi}\sum_i (x_i^2 - y_i^2)} f(x,y)\ \frac{\pi^2}{4}\left(\left(\sum_{i,j}\tanh(g_s(x_i-y_j))\right)^2 - \sum_{i,j}\frac{1}{\cosh^2(g_s\,(x_i-y_j))}\right) \tag{2.49b}$$

where dX, dY are the Haar measures and

$$f(x,y) = \prod_{i<j}\frac{\sinh^2(g_s(x_i-x_j))}{g_s^2(x_i-x_j)^2}\frac{\sinh^2(g_s(y_i-y_j))}{g_s^2(y_i-y_j)^2}\frac{1}{\prod_{i,j}\cosh^2(g_s(x_i-y_j))} \tag{2.50}$$

In order to compute Z and Z'', we find it convenient to canonically normalize them as $Z'' \to Z''/Z_0 \equiv \mathcal{Z}''$, $Z \to Z/Z_0 \equiv \mathcal{Z}$ where

$$Z_0 \equiv \int dX dY e^{\frac{i}{\pi}\sum_i (x_i^2 - y_i^2)} \tag{2.51}$$

is the free partition function. By perturbatively expanding the integrands in (2.49) up to $g_s^4 \sim \frac{1}{k^2}$, i.e. at two loops, and evaluating the normalized gaussian matrix integrals, we obtain

$$\begin{aligned}
\mathcal{Z}'' = -\frac{\pi^2}{4}N_1N_2\Big[1 &+ g_s^2\,\frac{i\pi}{6}\,(N_2-N_1)\,\left(1-(N_2-N_1)^2\right)\\
&- g_s^4\,\frac{\pi^2}{72}\Big(-24+16N_2^2-12N_1(N_2-N_1)+N_2^4+6N_2^2N_1^2+2N_2N_1^3-N_1^4\\
&\qquad + (N_2-N_1)^6\Big) + O(g_s^6)\Big]
\end{aligned}$$

$$\begin{aligned}
\frac{1}{\mathcal{Z}} = 1 &- g_s^2\,\frac{i\pi}{6}\,(N_2-N_1)\left(1-(N_2-N_1)^2\right)\\
&- g_s^4\,\frac{\pi^2}{72}\Big(-2(N_2^2-N_1^2)+8N_2N_1-5N_2^4+2N_2N_1(N_2-N_1)(8N_2-7N_1)-3N_1^4\\
&\qquad + (N_2-N_1)^6\Big) + O(g_s^6)
\end{aligned} \tag{2.52}$$

If we now substitute back $g_s \to \frac{1}{\sqrt{k}}$, the final result reads

$$\frac{1}{\pi^2}\frac{\partial^2}{\partial m_-^2}\log Z[S^3, m_\pm]\Big|_{m_\pm=0} = \frac{1}{\pi^2}\frac{\mathcal{Z}''}{\mathcal{Z}} = -\frac{N_1N_2}{4}\left(1-\frac{\pi^2}{6k^2}(N_1^2+N_2^2-2)\ + O\left(\frac{1}{k^3}\right)\right) \tag{2.53}$$

We can now substitute the result obtained above and find

$$c_T = -\frac{64}{\pi^2}\frac{\partial^2}{\partial m_-^2}\log Z[S^3, m_\pm]\Big|_{m_\pm=0} = 16N_1N_2\left(1-\frac{\pi^2}{6k^2}(N_1^2+N_2^2-2)+O\left(\frac{1}{k^3}\right)\right) \tag{2.54}$$

which beautifully matches with the result of (2.38).
In the next Subsection we present the two-loop result for the four-point function obtained with the same technology as above.

2.4.1 The Four-Point Function at Two Loops

From the general structure of the partition function in (2.43) it is easy to see that all the odd-order mass derivatives evaluated at $m_\pm = 0$ vanish identically due to symmetry reasons, therefore, $(2n+1)$-point functions of topological operators are expected to vanish at any loop order. Notice that this is in agreement with our result for the three-point function in section 2.3.
Even number of derivatives can be instead used to obtain predictions for $(2n)$-point functions of topological operators at weak coupling.
The simplest case we consider beyond the two-point function is the four-point function. By using the prescription (2.5), what we can write is

$$\langle \mathcal{O}(\tau_1)\mathcal{O}(\tau_2)\mathcal{O}(\tau_3)\mathcal{O}(0)\rangle = \frac{1}{(2\pi^2)^4}\frac{\partial^4 \log Z}{\partial m_-^4}\bigg|_{m_\pm=0} = \frac{1}{(2\pi^2)^4}\left(\frac{\mathcal{Z}''''}{\mathcal{Z}} - 3\left(\frac{\mathcal{Z}''}{\mathcal{Z}}\right)^2\right) \tag{2.55}$$

Evaluating explicitly $\mathcal{Z}''''$ at order g_s^4 and using (2.52), we obtain a two-loop prediction for the four-point topological correlator

$$\langle \mathcal{O}(\tau_1)\mathcal{O}(\tau_2)\mathcal{O}(\tau_3)\mathcal{O}(0)\rangle = 2\frac{N_1N_2}{(4\pi)^4} - \frac{N_1N_2(N_1^2+N_2^2-2)}{192\pi^2\, k^2} + O\left(\frac{1}{k^3}\right) \tag{2.56}$$

We note that up to one loop it agrees with our perturbative result (2.33), whereas the $\frac{1}{k^2}$ term constitues a non-trivial prediction which should be checked perturbatively.
We conclude by expressing how all calculations we performed throughout the present chapter point towards the validity of the conjectured relation (2.11) and, more in general, the ABJ(M) version of the formula in (2.5).

Chapter 3

$\mathcal{N} = 1$ Quivers

In this chapter we shift our attention to the study of minimally supersymmetric models in three-dimensions, which interpolates physics from the highly constrained supersymmetric world to the non-supersymmetric one. Three-dimensional theories with $\mathcal{N} = 1$ supersymmetry are not so distant from the non-supersymmetric ones since some exact approaches like localization, non-renormalization theorems or the construction of protected sectors, are not allowed. This fact increases the difficulties in exploring their non-perturbative regime, nevertheless, a careful analysis of the minima of their superpotential allows for a rigorous study of the structure of the phase diagrams. In some cases such phase diagrams show identical phase transitions which uncover certain deep similarities among different theories encoded in what are generally called *infrared dualities* (IR).
Infrared dualities are one of the most powerful tool which can be used for relating completely different, not necessarily supersymmetric, ultraviolet (UV) quantum field theories in the low-energy limit. In particular, what happens is that, for some special submanifolds of the phase diagram where a *phase transition* occurs, we are allowed to describe the same *critical* phenomena with different physical prescriptions. Three-dimensional well-known examples of IR dualities, which are part of an infinite duality web, are the *Particle-Vortex duality*, between the *XY-model* and the Abelian-Higgs model[1], or the so called *Bosonization duality*, between the $U(1)_1$ Abelian-Higgs model and a massive Dirac fermion coupled to an abelian Chern-Simons background gauge field. Since three-dimensional gauge theories are strongly coupled in the low-energy limit, it is generally hard to have an analytic description of such phenomena. Relying on dual physical interpretations on one side may thus allow for a much simpler effective description of the physics on the other side.

In what follows, we will study in detail the phase diagram of $\mathcal{N} = 1$ Chern-Simons quivers gauge theories coupled to bifundamental matter and propose new dualities arising from the matching of certain second (or higher order) phase transitions. This chapter will be organized as follows: In Section 3.1 we briefly review some specific aspects of $\mathcal{N} = 1$ gauge theories with different matter content, in Section 3.2 we compute the one-loop effective superpotential for the $SU(2) \times SU(2)$ case in some physically relevant limits and in Section 3.3 we study the related phase diagrams. In Section 3.4 we try to extend the previous analysis to the $SU(2) \times U(2)$ case and in Section 3.5 we propose some dualities involving quiver theories, as well as the adjoint SQCD. Finally, in Section 3.6 we make some comments regarding theories enjoying parity (time-reversal) symmetry.

[1] Notice that both these theories are described at the critical point by a CFT which is usually called *Wilson-Fisher model*. The fact that a single "quantum" theory describes the physics at the critical point for very different "classical" theories, it is usually known as *universality*.

3.1 $\mathcal{N}=1$ Vector Multiplet Coupled to Matter

In this section we review some known facts regarding the $\mathcal{N}=1$ $SU(N)_k$ theory introduced in Subsection 1.3.4, and $U(N)$ variants thereof, coupled to different matter contents.
Since we are going to deal with $U(N)$ CS-theories, we find useful to introduce the relative standard notation for them.[2]
The $U(N)$ gauge group is equipped with two different, but related, CS-levels and is commonly defined as

$$U(N)_{k_1,k_2} = \frac{SU(N)_{k_1} \times U(1)_{Nk_2}}{\mathbb{Z}_N}, \tag{3.1}$$

for which consistency requires that $k_2 = k_1 \mod N$. We note that while in general time-reversal transformation maps a TQFT to a different TQFT, there are some of which turn out to be invariant, with an example given by $U(N)_{N,2N}$ theories, or the already mentioned ABJ(M) theory. This very important consequence is thus that different Chern-Simons theories may actually describe the same TQFT. An important example of this phenomenon, which will be used in what follows, is given by the *Level-Rank duality* (see [110] for a modern discussion). The dual pair which will be relevant for us is $U(N)_{k,k} \leftrightarrow SU(k)_{-N}$.
Let us now briefly review what happens when we couple a matter multiplet transforming respectively in the adjoint and in the fundamental representation of the gauge group.

3.1.1 Adjoined Matter Multiplet

An adjoined Matter multiplet consists of a real scalar and a Majorana fermion both transforming in the adjoint representation of the gauge group. The Lagrangian we introduced in (1.92) thus get modified by the following additional pieces

$$\mathcal{L}^{\phi}_{kin} = \mathrm{Tr}\left(D_\mu \phi D^\mu \phi\right) \tag{3.2}$$

$$\mathcal{L}^{\psi}_{kin} = i\,\mathrm{Tr}\left(\psi \not{D} \psi\right) \tag{3.3}$$

$$\mathcal{L}_{Yuk} = ig\sqrt{2}\,\mathrm{Tr}\left([\lambda, \phi]\psi\right) \tag{3.4}$$

$$\mathcal{L}_m = \mathrm{Tr}\left(\frac{1}{2}m^2\phi^2 + m\psi\psi\right) \tag{3.5}$$

where the last term is nothing but the (real) matter superpotential term introduced in (1.86) written in components[3]. In what follows we will be interested in the tree-level superpotential given by the mass term only.

The dynamics of an $\mathcal{N}=1$ vector multiplet coupled to a matter multiplet in the adjoint representation, as a function of the matter mass parameter, was described in details in [37]. Here we review the $SU(2)$ gauge group case, which will be relevant in the next sections of this chapter.
For $k \geq 2$ and $m \gg 0$, we can integrate out the matter multiplet and get the pure $SU(2)_{k+1}$ vector multiplet. In the low-energy limit, this theory further flows to a supersymmetric vacuum hosting the $SU(2)_k$ CS-theory where the gaugino, which has negative mass, has been integrated out.
When we consider the $m \ll 0$ case instead, after integrating out the matter multiplet again, we get the pure $SU(2)_{k-1}$ vector multiplet, which flows to a supersymmetric vacuum hosting now a $SU(2)_{k-2}$ CS-theory. Thus, by computing the Witten index as prescribed in (1.95), we immediately notice that there are two large mass asymptotic phases whose Witten index is different. Since the Witten index is invariant under deformation of the parameters of

[2] Useful details can be found in Appendix C of [109] and [14].

[3] Notice that when $m = -\frac{kg^2}{2\pi}$, we get the Lagrangian of a pure $\mathcal{N}=2$ vector multiplet.

the theory, different asymptotic values indicate that non-trivial phenomena must occur in between them.
The transition between these two phases happens in two stages: At $m=0$ the effective potential develops a flat direction, namely a *moduli space* of vacua emerges, and for small and positive mass a new supersymmetric vacuum supporting a $U(1)_{2k}$ CS-theory appears. Physically, this vacuum can be thought as coming in from the infinity of the field space along the flat direction.
The Witten index of this theory is exactly the one that compensates the jump of its two asymptotic values, restoring the matching of the Witten index on both sides of the phase transition. This process can be physically interpreted as follows: At some finite mass value m^* these two vacua merge through a second order phase transition and produce a single vacuum which is the one visible at large positive masses. As a final remark we notice that the jump of the Witten index may happen when a flat direction is developed by the superpotential, since it is ill-defined for a continuous set of vacua as we discussed in Subsection 1.3.4, whereas when a phase transition occurs it consistently stays constant.

For $k=1$, the $m\gg 0$ scenario is not much different from the previous case, indeed we get a single supersymmetric vacuum supporting the $SU(2)_1$ TQFT. On the contrary, when $m\ll 0$, by integrating out matter, we get the $SU(2)_0$ vector multiplet which breaks supersymmetry [46]. The theory in the IR limit is then given by[4]

$$G_\alpha + U(1)_2. \tag{3.6}$$

where G_α is the goldstino. Since the origin of this IR theory is subtle, let us discuss in detail what happens when we try to chart its phase diagram.
At $m=0$ the effective potential develops a flat direction, and for small positive mass we find a new supersymmetric vacuum hosting a $U(1)_2$ TQFT, which came in from infinity of the field space. Using the fact that $U(1)_2$ is time-reversal invariant [36] and applying the Level-Rank duality (cf. [110]),

$$U(1)_2 \overset{\text{T-inv}}{\longleftrightarrow} U(1)_{-2} \overset{\text{L-R}}{\longleftrightarrow} SU(2)_1, \tag{3.7}$$

we recognise the same vacuum as the large positive mass one, therefore, we do not have a phase transition in this case. We note that at positive and small mass values, the non-supersymmetric vacuum hosting the $G_\alpha + U(1)_2$ theory coexists with the supersymmetric one hosting the $U(1)_2$ TQFT, meaning that the former is actually *meta-stable*.

Finally, for $k=0$, in the $m=0$ point we get the supersymmetry enhancement to $\mathcal{N}=2$ and the theory shows a runaway behaviour [111]. For non-zero masses the runaway behaviour stabilizes and the superpotential shows a single trivial supersymmetric vacuum for both positive and negative masses.

3.1.2 Fundamental Matter Multiplets

In this section we review the structure of the phase diagram for $U(2)$ and $SU(2)$ vector multiplets theories coupled to F matter multiplets in the fundamental representation, as a function of the matter mass m (we assume that all flavours are given the same mass). This is a particular case of the results in [41].
Let us first consider the case of $U(2)_{k+1,k}$ theories coupled to F fundamental:

- **$F=1$:** At large positive mass, by integrating out the matter, we get a single vacuum with $U(2)_{k+\frac{3}{2},k+\frac{1}{2}}$ vector multiplet. For large and negative mass we get a single vacuum with $U(2)_{k+\frac{1}{2},k-\frac{1}{2}}$ vector multiplet. In the intermediate phase $0<m<m_*$ (where m_*

[4] The origin of the $U(1)_2$ topological theory is the one discussed at the end of Section 1.3.4.

is some finite and positive value) in addition to the vacuum seen at negative masses, a new vacuum with $U(1)_{k+\frac{1}{2}}$ vector multiplet appears allowing for the compensation of the asymptotic values of the Witten index. At $m=0$ the superpotential shows flat directions which, in the parameter space, can be represented as a wall. At $m=m_*$ a phase transition occurs and the two vacua merge reproducing the single asymptotic vacuum.

- **$F \geq 2$:** At large positive masses, we get a single vacuum with $U(2)_{k+\frac{2+F}{2},k+\frac{F}{2}}$ vector multiplet. For large and negative masses we get a single vacuum with $U(2)_{k+\frac{2-F}{2},k-\frac{F}{2}}$ vector multiplet. In the intermediate phase $0<m<m_*$ in addition to the negative masses vacuum, two new vacua appear from infinity [41].
The first vacuum hosts an $\mathcal{N}=1$ non-linear sigma model (NLSM), whose target space[5] is $\frac{U(F)}{U(F-1)\times U(1)}$, coming from spontaneously broken $SU(F)$ global symmetry, together with the decoupled $U(1)_k$ vector multiplet.
The second hosts instead an $\frac{U(F)}{U(F-2)\times U(2)}$ $\mathcal{N}=1$ NLSM. Again, at $m=0$ a wall is developed and at $m=m_*$ we find the phase transition locus.

A peculiarity of this type of models is that, in contrast to expectations, the three vacua living in the intermediate phase merge in just a single phase transition [41] instead of two distinct ones. This phenomenon was dubbed *supercriticality*.

We now briefly discuss $SU(2)_{k+1}$ theories[6] with F fundamentals. For large and positive mass we get a single vacuum with $SU(2)_{k+\frac{F}{2}+1}$ vector multiplet, whereas for negative masses we have the $SU(2)_{k-\frac{F}{2}+1}$ vector multiplet. In the intermediate phase with $0<m<m_*$, in addition to the negative mass vacuum, a new vacuum come in from infinity through the already discussed mechanism, supporting a $\frac{Sp(F)}{Sp(F-1)\times Sp(1)}$ $\mathcal{N}=1$ NLSM.
In [41], a family of dualities between SQCD theories was described. Among the proposed dualities, we note the following[7]

$$U(2)_{2,1}+2\Phi \longleftrightarrow SU(2)_{-2}+2\tilde{\Phi}, \tag{3.8}$$

which is obtained by substituting $k=0, N_f=N=2$ in their equation (6.1). This duality is quite subtle and, if correct, has quite far-reaching consequences, such as the global symmetry enhancement, on the $U(2)$ side, from $SU(2)\times U(1)$ to $Sp(2)$ at the IR fixed point. A puzzling aspect about this duality is the mismatch of the phases when the theories are deformed away from the fixed point by the mass deformation. In particular, while the $U(2)$ theory has three vacua in the intermediate phase (two of them being the NLSMs we described above), the $SU(2)$ theory has just two of them. The correctness of (3.8) thus implies giving up supercriticality, e.g. due to non-perturbative corrections. This means that, for consistency, there exists an intermediate phase where only two vacua merge on both sides.

While it is easy to get the vacua for large mass values, it is more subtle to understand the dynamics in the intermediate phase for which one needs to know the effective superpotential exactly. For this reason we turn to the evaluation of the effective superpotential for the two-node quiver theories as described at the beginning of the chapter.

[5]We find useful to recall that, for NLSMs, the Witten index actually computes the *Euler characteristic* of the target space manifold. By labelling $\mathcal{M}_{F,n} \equiv \frac{U(F)}{U(F-n)\times U(n)}$, one finds that $I_W=\chi(\mathcal{M}_{F,n})=\binom{F}{n}$.

[6]It worths reminding that the global symmetry of the Lagrangian here is not just $U(F)$, but rather $Sp(F)$, because of the pseudo-real nature of the $SU(2)$ fundamental representation.

[7]We thank Adar Sharon for discussions on this duality.

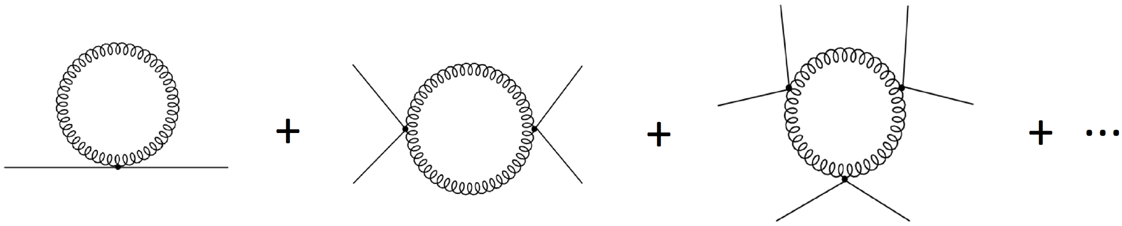


Figure 3.1: Feynman Supergraphs contributing to the 1-loop effective Superpotential.

3.2 Effective Superpotential for $SU(2) \times SU(2)$ Theories

In this section we consider the previously mentioned two-node quiver theories with $SU(2)_{k_1} \times SU(2)_{k_2}$ gauge group with one bi-fundamental matter multiplet. We compute the exact one-loop effective superpotential, the superspace analogue of the Coleman-Weinberg effective potential [76], by using the supergraph formalism[8]. The strategy presented here will closely follow the one of [41]. To keep the discussion as general as possible, we start by considering the $G_1 \times G_2$ case for the gauge groups and then specialize to the $SU(2) \times SU(2)$ case. The generic structure of the effective superpotential is [41]

$$\mathcal{W}_{\text{eff}} = \int \frac{d^3p}{(2\pi)^3} d^2\theta' \, \delta(\theta - \theta')\Sigma(p, \theta')\delta(\theta' - \theta), \tag{3.9}$$

where Σ, at the one-loop order, is given by the sum of the diagrams shown in Figure 3.1. We note that the form of the series expansion is exactly the same as can be found in [76], albeit with Feynman rules replaced by their superspace counterpart.
Let Γ^A_α, $\hat{\Gamma}^M_\alpha$ be vector multiplet superfields of the first and the second gauge groups respectively; here A, M are colour indices and α is the spinor index. The general structure of the YMCS gauge propagator can be found in (B.10). In our case we need the following expressions

$$\langle \Gamma^A_\alpha(p)\Gamma^{B,\beta}(-p)\rangle = \delta^{AB} \, \frac{\delta^\beta_\alpha(\kappa_1 \, D^2 \, + \, p^2) \, + \, (\kappa_1 \, - \, D^2) \, p_\alpha{}^\beta}{p^2 \, (\kappa_1^2 \, + \, p^2)} \equiv \delta^{AB} (\Delta_1)_\alpha{}^\beta, \tag{3.10}$$

$$\langle \hat{\Gamma}^M_\alpha(p)\hat{\Gamma}^{N,\beta}(-p)\rangle = \delta^{MN} \, \frac{\delta^\beta_\alpha(\kappa_2 \, D^2 \, + \, p^2) \, + \, (\kappa_2 \, - \, D^2) \, p_\alpha{}^\beta}{p^2 \, (\kappa_2^2 \, + \, p^2)} \equiv \delta^{MN} (\Delta_2)_\alpha{}^\beta, \tag{3.11}$$

where $\kappa_i = \frac{k g_i^2}{2\pi}$. There are also three types of vertices joining two gauge multiplets and two matter multiplets[9] which partecipate in the computation. The corresponding Feynman rules can be easily obtained from the explicit expressions of the covariant derivatives in (B.12),(B.13) and read

$$\Gamma^{(4)}_{\Gamma\Gamma\bar{\phi}\phi} = \langle \Gamma^A_\alpha(p)\Gamma^{B,\beta}(q)\Phi_i{}^{\hat{j}}(r)\bar{\Phi}_{\hat{k}}{}^l(-p-q-r)\rangle \; = \; -\frac{g_1^2}{2}(T^{(A)})_i{}^n (T^{B)})_n{}^l \, \delta^{\hat{j}}{}_{\hat{k}} \, \delta_\alpha{}^\beta \tag{3.12}$$

$$\Gamma^{(4)}_{\hat{\Gamma}\hat{\Gamma}\bar{\phi}\phi} = \langle \hat{\Gamma}^M_\alpha(p)\hat{\Gamma}^{N,\beta}(q)\Phi_i{}^{\hat{j}}(r)\bar{\Phi}_{\hat{k}}{}^l(-p-q-r)\rangle \; = \; -\frac{g_2^2}{2}(K^{(M)})_{\hat{k}}{}^{\hat{n}} (K^{N)})_{\hat{n}}{}^{\hat{j}} \, \delta_i \, \delta_\alpha{}^\beta, \tag{3.13}$$

$$\Gamma^{(4)}_{\Gamma\hat{\Gamma}\bar{\phi}\phi} = \langle \Gamma^A_\alpha(p)\hat{\Gamma}^{M,\beta}(q)\Phi_i{}^{\hat{j}}(r)\bar{\Phi}_{\hat{k}}{}^l(-p-q-r)\rangle = \frac{1}{2} g_1 g_2 (T^A)_i{}^l (K^M)_{\hat{k}}{}^{\hat{j}} \, \delta_\alpha{}^\beta. \tag{3.14}$$

[8] A detailed discussion of the $\mathcal{N} = 1$ superspace and supergraph formalism can be found in [45].

[9] As usually, there also exist a cubic vertex with two matter legs and one gauge leg but they do not contribute since, in the Landau gauge, the gauge propagators are transverse, namely $(\Delta_{1,2})_\alpha{}^\beta D_\beta = 0$.

Above, unhatted lower case letters $i, j, \ldots$ are fundamental indices and A, B are adjoint indices of G_1, with T^A being its generators. Similarly, hatted lower case letters are fundamental indices and M, N are adjoint indices of G_2, with K^A being its generators.
We can now proceed with the computation of Σ. For this purpose, we find useful to introduce the following matrices

$$M^{AB} = g_1^2 \mathrm{Tr}\, \bar{\Phi} T^{(A} T^{B)} \Phi \tag{3.15a}$$
$$N^{MN} = g_2^2 \mathrm{Tr}\, \Phi K^{(M} K^{N)} \bar{\Phi} \tag{3.15b}$$
$$G^{AM} = -g_1 g_2 \mathrm{Tr}\, \bar{\Phi} T^A \Phi K^M . \tag{3.15c}$$

which allow us a more compact treatment of the computation.
We see indeed that Σ is given, at the leading order, by the sum of the one-loop contributions corresponding to Feynman diagrams appearing in Figure 3.1. With the help of the expressions defined in (3.15), we find

$$\begin{aligned}\Sigma_{\text{1-loop}} = &-\tfrac{1}{2}\,\mathrm{Tr}\, M\, \Delta_1 \,-\, \tfrac{1}{2}\,\mathrm{Tr}\, N\, \Delta_2 + \\ &+ \tfrac{1}{4}\mathrm{Tr}\,(M\, \Delta_1)^2 \,+\, \tfrac{1}{2}\mathrm{Tr}\,(G)^\dagger\, \Delta_1\, G\, \Delta_2 + \tfrac{1}{4}\mathrm{Tr}\,(N\, \Delta_2)^2 - \\ &- \tfrac{1}{6}\mathrm{Tr}\,(M\, \Delta_1)^3 \,-\, \tfrac{1}{2}\mathrm{Tr}\, G^\dagger\, \Delta_1\, M\, \Delta_1\, G\, \Delta_2 \,-\, \tfrac{1}{2}\mathrm{Tr}\, G\, \Delta_2\, N\, \Delta_2\, G^\dagger\, \Delta_1 \\ &- \tfrac{1}{6}\mathrm{Tr}\,(N\, \Delta_2)^3 \,+\, \ldots\,,\end{aligned} \tag{3.16}$$

where the trace is taken over both spinor and colour indices for each gauge group respectively.
To simplify even more the contributions appearing above, we list the following identities involving gauge propagators

$$\begin{aligned}(\Delta_i)_\alpha{}^\alpha &= \frac{2(\kappa_i D^2 + p^2)}{p^2(\kappa_i^2 + p^2)} = \delta_i, \\ (\Delta_1)_\alpha{}^\beta (\Delta_1)_\beta{}^\gamma &= \frac{2(\kappa_1 D^2 + p^2)}{p^2(\kappa_1^2 + p^2)} (\Delta_1)_\alpha{}^\gamma = \delta_1 (\Delta_1)_\alpha{}^\gamma, \\ (\Delta_2)_\alpha{}^\beta (\Delta_2)_\beta{}^\gamma &= \frac{2(\kappa_2 D^2 + p^2)}{p^2(\kappa_2^2 + p^2)} (\Delta_2)_\alpha{}^\gamma = \delta_2 (\Delta_2)_\alpha{}^\gamma, \\ (\Delta_1)_\alpha{}^\beta (\Delta_2)_\beta{}^\gamma &= \delta_1 (\Delta_2)_\alpha{}^\gamma = \delta_2 (\Delta_1)_\alpha{}^\gamma.\end{aligned} \tag{3.17}$$

At this point, by tracing on spinor indices only, we can repackage the expression in (3.16) in terms of the following additional matrices

$$\mathcal{M} = \begin{pmatrix} M^{AB} & G^{AN} \\ (G^\dagger)^{MB} & N^{MN} \end{pmatrix}, \qquad \Delta = \begin{pmatrix} \delta_1 \cdot \mathbf{1}_{N_1\times N_1} & 0 \\ 0 & \delta_2 \cdot \mathbf{1}_{N_2\times N_2} \end{pmatrix}, \tag{3.18}$$

where N_1 and N_2 are the dimensions of the fundamental representations of G_1 and G_2, respectively. With this formalism at hand, we can write the generic one-loop contribution for $2n$ external matter legs as

$$\Sigma^{(n)}_{\text{1-loop}} \equiv \mathrm{Tr}(\mathcal{M}\Delta)^n. \tag{3.19}$$

By summing over all the possible contributions, we get

$$\begin{aligned}\Sigma &= \sum_n \mathcal{S}_n\, \Sigma^{(n)}_{\text{1-loop}} \\ &= -\frac{1}{2}\,\mathrm{Tr}(\mathcal{M}\Delta) + \frac{1}{4}\,\mathrm{Tr}(\mathcal{M}\Delta\mathcal{M}\Delta) - \frac{1}{6}\,\mathrm{Tr}(\mathcal{M}\Delta\mathcal{M}\Delta\mathcal{M}\Delta) + \ldots \\ &= -\frac{1}{2}\,\mathrm{Tr}\log\big(\mathbb{1}_{(N_1+N_2)\times(N_1+N_2)} + \mathcal{M}\Delta\big) \\ &= -\frac{1}{2}\,\log\det\big(\mathbb{1}_{(N_1+N_2)\times(N_1+N_2)} + \mathcal{M}\Delta\big) \\ &= -\frac{1}{2}\,\log\Big(\det\big((\mathbb{1}_{N_1\times N_1} + \delta_1 M) - \delta_1\delta_2 G(\mathbb{1}_{N_2\times N_2} + \delta_2 N)^{-1} G^T\big)\cdot\det\big(\mathbb{1}_{N_2\times N_2} + \delta_2 N\big)\Big)\end{aligned} \tag{3.20}$$

where in the first step we used the following general formula for the symmetry factor associated to each supergraph

$$\mathcal{S}_n = \frac{1}{n!}\left(-\frac{1}{2}\right)^n (2n-2)!! = \frac{(-1)^n}{2n}. \tag{3.21}$$

We now choose to restrict to the $SU(2)\times SU(2)$ case. In this setup we can further simplify the result thanks to few helpful properties which we are now going to present.
We start noticing that, in this case, M, N, and G are all three by three diagonal matrices therefore they all commute. In fact, by considering the expressions in (3.15), we have

$$\begin{aligned} M^{AB} &= \frac{g_1^2 \operatorname{Tr}\bar{\Phi}\Phi}{4}\,\delta^{AB} \\ N^{AB} &= \frac{g_2^2 \operatorname{Tr}\bar{\Phi}\Phi}{4}\,\delta^{AB}. \end{aligned} \tag{3.22}$$

The above expressions thus allow us to simplify the result in (3.20) as

$$\Sigma = -\frac{1}{2}\operatorname{Tr}\log\left[\mathbb{1}_{3\times 3} + \delta_1 M + \delta_2 N + \delta_1\delta_2(MN - G^T G)\right]. \tag{3.23}$$

Taking into account that the eigenvalues of $G^T G$ are given by

$$\left(\frac{g_1^2 g_2^2 \det\Phi\det\bar{\Phi}}{4}, \frac{g_1^2 g_2^2 \det\Phi\det\bar{\Phi}}{4}, \frac{g_1^2 g_2^2 (\operatorname{Tr}\bar{\Phi}\Phi)^2}{16}\right), \tag{3.24}$$

and introducing the notation

$$\rho \equiv \operatorname{Tr}\bar{\Phi}\Phi, \qquad B \equiv 2\det\Phi, \qquad \bar{B} \equiv 2\det\bar{\Phi}, \tag{3.25}$$

we can substitute the above results in (3.23) and obtain

$$\Sigma = -\log\left[1 + \frac{\rho}{4}(g_1^2\delta_1 + g_2^2\delta_2) + \frac{\rho^2 - B\bar{B}}{16} g_1^2 g_2^2 \delta_1\delta_2\right] - \frac{1}{2}\log\left[1 + \frac{\rho}{4}(g_1^2\delta_1 + g_2^2\delta_2)\right]. \tag{3.26}$$

At this point we can substitute everything in (3.9) and get the following complete one-loop expression for the superpotential

$$\begin{aligned} \mathcal{W}_{\text{1-loop}} = &-\int \frac{d^3p}{(2\pi)^3} d^2\theta' \delta(\theta-\theta') \log\left[1 + \frac{\rho}{4}(g_1^2\delta_1 + g_2^2\delta_2) + \frac{\rho^2 - B\bar{B}}{16} g_1^2 g_2^2\delta_1\delta_2\right]\delta^2(\theta'-\theta) \\ &-\frac{1}{2}\int \frac{d^3p}{(2\pi)^3} d^2\theta' \delta(\theta-\theta')\log\left[1 + \frac{\rho}{4}(g_1^2\delta_1 + g_2^2\delta_2)\right]\delta^2(\theta'-\theta). \end{aligned} \tag{3.27}$$

In order to compute the expression above, we first need to deal with the θ integral. The presence of delta functions allow us to compute it with the help of the following identities

$$\begin{aligned} &\delta^2(\theta-\theta')\delta^2(\theta'-\theta) = 0, \\ &\delta^2(\theta-\theta')D^\alpha\delta^2(\theta'-\theta) = 0, \\ &\delta^2(\theta-\theta')D^2\delta^2(\theta'-\theta) = \delta^2(\theta-\theta'), \end{aligned} \tag{3.28}$$

which tell us that only linear expression in D^2 will contribute to the final result.
The effective superpotential thus formally becomes

$$\begin{aligned} \mathcal{W}_{\text{1-loop}} = &-\int \frac{d^3p}{(2\pi)^3}\log\left[1 + \frac{\rho}{4}(g_1^2\delta_1 + g_2^2\delta_2) + \frac{\rho^2 - B\bar{B}}{16} g_1^2 g_2^2 \delta_1\delta_2\right]|_{D^2} \\ &-\frac{1}{2}\int \frac{d^3p}{(2\pi)^3}\log\left[1 + \frac{\rho}{4}(g_1^2\delta_1 + g_2^2\delta_2)\right]|_{D^2}. \end{aligned} \tag{3.29}$$

where $|_{D^2}$ means that we should reduce the functions of D^2 to linear ones, as explained above, and then take the coefficient in front of D^2. The reduction procedure can be performed with the help of the identities in (B.7) and the following non-trivial relation [41]

$$(\kappa D^2+p^2)^n|_{D^2}=\frac{1}{|p|}\mathrm{Im}\left((i\kappa|p|+p^2)^n\right), \tag{3.30}$$

from which we can easily extract all the D^2 coefficients.
Putting together all the recipes explained above and the results obtained in (3.17), the fully reduced expression for the effective superpotential becomes

$$\begin{aligned}\mathcal{W}_{\text{1-loop}} =& -\int\frac{d^3p}{(2\pi)^3}\frac{1}{|p|}\mathrm{Im}\log\left[1+\frac{\rho}{2}\left(\frac{g_1^2}{(p^2-i\kappa_1|p|)}+\frac{g_2^2}{(p^2-i\kappa_2|p|)}\right)\right.\\ &\left.+\frac{g_1^2g_2^2(\rho^2-B\bar{B})}{4(p^2-i\kappa_1|p|)(p^2-i\kappa_2|p|)}\right]\\ &-\frac{1}{2}\int\frac{d^3p}{(2\pi)^3}\frac{1}{|p|}\mathrm{Im}\log\left[1+\frac{\rho}{2}\left(\frac{g_1^2}{(p^2-i\kappa_1|p|)}+\frac{g_2^2}{(p^2-i\kappa_2|p|)}\right)\right].\end{aligned} \tag{3.31}$$

Since we are interested in the classification of the supersymmetric vacua of the theory, which are the minima of the superpotential, the derivatives of $\mathcal{W}_{\text{eff}}$ with respect to ρ, B and $\bar{B}$ are clearly going to play a relevant role. Since we cannot compute the exact closed form of (3.31) in full generality, we focus on the physically relevant limit only, namely the asymptotic behavior at large field values. In this approximation, the leading results can be obtained for the general case $k_1\neq -k_2$ with the help of the *Mathematica* software. The two sub-cases in which we are interested in and that will be discussed in the next sections are

$$\begin{aligned}\partial_\rho\mathcal{W}_{\text{1-loop}} &= \begin{cases}-\frac{F_1+F_4}{\rho^{1/2}}+\mathcal{O}\left(\rho^{-3/2},(\frac{B}{\rho})^2\right) & \text{if } \frac{B}{\rho}\ll 1,\\ -G-\frac{3F_1+F_2}{\rho^{1/2}}+\mathcal{O}\left(\rho^{-3/2}\right) & \text{if } B=\rho\end{cases}\\ \partial_B\mathcal{W}_{\text{1-loop}} &= \begin{cases}\frac{F_3B}{\rho^{3/2}}+B\,\mathcal{O}\left(\rho^{-5/2},\frac{B}{\rho}\right) & \text{if } \frac{B}{\rho}\ll 1,\\ G+\frac{F_2}{\rho^{1/2}},+\mathcal{O}\left(\rho^{-3/2}\right) & \text{if } B=\rho.\end{cases}\end{aligned} \tag{3.32}$$

Here F_1,F_2,F_3,F_4,G_1 are functions of $g_1,g_2,\kappa_1,\kappa_2$:

$$F_1=\frac{\kappa_1g_1^2+\kappa_2g_2^2}{16\sqrt{2}\pi\sqrt{g_1^2+g_2^2}}, \tag{3.33a}$$

$$F_2=\frac{g_1^2g_2^2\left((\kappa_1+\kappa_2)(g_1^2+g_2^2)-3(\kappa_1g_2^2+\kappa_2g_1^2)\right)}{4\sqrt{2}\pi(g_1^2+g_2^2)^{5/2}}, \tag{3.33b}$$

$$F_3=\frac{g_1g_2(\kappa_1g_2+\kappa_2g_1)}{4\sqrt{2}\pi(g_1+g_2)^2}, \tag{3.33c}$$

$$F_4=\frac{g_1\kappa_1+g_2\kappa_2}{8\sqrt{2}\pi} \tag{3.33d}$$

$$G=\frac{g_1^2g_2^2}{4\pi(g_1^2+g_2^2)}. \tag{3.33e}$$

For the case of $k_1=-k_2$ one can go a bit further and get not only the leading asymptotic behaviour but the complete closed form expressions for $\partial_\rho\mathcal{W}_{\text{1-loop}}$ and $\partial_{|B|}\mathcal{W}_{\text{1-loop}}$ in the limits $B\to\rho$ and $B\to 0$. All these results are collected in Appendix B.3.
As is explained in [37], the asymptotic behaviour of the effective superpotential is fully determined by the one-loop contributions and does not receive higher-order corrections. This fact implies that the results in (3.32) and (3.33) are actually exact.

3.3 Phase Diagrams of $SU(2) \times SU(2)$ Models

In this section we classify and study in detail the vacua of $SU(2) \times SU(2)$ quiver theories by starting from the results obtained in the previous section. This will allows us to discuss their IR phases, which will be our starting point for the new dualities we propose at the end of the chapter, and finally chart their complete phase diagrams.
Let us start by adding the one-loop effective superpotential we previously computed to the tree-level mass term appearing in the Lagrangian. The final form of the superpotential we are going to study hence reads

$$\mathcal{W}_{\text{eff}} = m \operatorname{Tr} \bar{\Phi}\Phi + \mathcal{W}_{1-loop}. \tag{3.34}$$

As we have already recalled, the supersymmetric vacua are given by the critical points of the superpotential. They can be thus found by solving the following *F-term* equation

$$\bar{\partial}\mathcal{W} = 0. \tag{3.35}$$

In order to sistematically solve the above equation, we can put in a diagonal form the scalar field in the matter supermultiplet, represented as a two-by-two matrix, by applying two simultaneous gauge transformations. The final structure of a generic vacuum therefore will be simply

$$\Phi = \begin{pmatrix} \phi_{11} & 0 \\ 0 & \phi_{22} \end{pmatrix}, \tag{3.36}$$

where the common phase of ϕ_{11} and ϕ_{22} have been set to zero thanks to the $U(1)$ baryonic symmetry. The equation in (3.35) can be then expanded as

$$\begin{aligned} \phi_{11}\partial_\rho \mathcal{W} + \phi_{22}\partial_{|B|}\mathcal{W} &= 0, \\ \phi_{22}\partial_\rho \mathcal{W} + \phi_{11}\partial_{|B|}\mathcal{W} &= 0. \end{aligned} \tag{3.37}$$

From the equations above one can then easily infer the following possibilities:[10]

1) The vacuum at the origin, $\phi_{11} = \phi_{22} = 0$.

2) $\partial_\rho \mathcal{W} = -\partial_{|B|}\mathcal{W} \neq 0$, $\phi_{11} = \phi_{22}$.

3) $\partial_\rho \mathcal{W} = \partial_{|B|}\mathcal{W} = 0$.

Since the analysis will heavily rely on the relative sign between the two CS-levels $k_{1,2}$, we find convenient to split the discussion into the $\operatorname{sgn}(k_1) = \operatorname{sgn}(k_2)$ and $\operatorname{sgn}(k_1) = -\operatorname{sgn}(k_2)$.

3.3.1 Chern-Simons Levels: Same Sign

In this subsection we deal with the case of CS-levels with the same sign. Without loss of generality we assume that $0 \leq k_1 \leq k_2$[11]whereas we postpone the study of the $k_1 = k_2 = 0$ case, which requires a separate treatment.
When the mass parameter is large and positive, we can integrate the matter out and obtain a single vacuum with the infrared theory given by

$$\mathcal{N} = 1 \qquad SU(2)_{k_1+1} \times SU(2)_{k_2+1}. \tag{3.38}$$

[10]There is also the possibility for which $\partial_\rho \mathcal{W} = \partial_{|B|}\mathcal{W} \neq 0$, $\phi_{11} = -\phi_{22}$, but this is gauge-equivalent to the second one in the list.

[11]The situation of $0 \leq k_2 \leq k_1$ is obtained by exchanging two nodes and the situation of $0 \geq k_2 \geq k_1$ is obtained by applying the time reversal transformation.

This theory preserves supersymmetry and further flows to a topological CS-theory in the IR, as we discussed in Subsection 1.3.4, which reads

$$SU(2)_{k_1} \times SU(2)_{k_2} \quad \text{TQFT}. \tag{3.39}$$

The Witten index of this theory is computed by taking the product of the indices corresponding to the two gauge group factors. The overall sign of the index is determined by the number of fermions with negative mass (see Footnote 31). Therefore, taking into account that there are six negative-mass Majorana gaugini, we get

$$\text{WI}_+ = (k_1+1)(k_2+1). \tag{3.40}$$

If the mass parameter is large and negative, we can again integrate the matter out, and what results is

$$\mathcal{N}=1 \qquad SU(2)_{k_1-1} \times SU(2)_{k_2-1}. \tag{3.41}$$

In order to understand the nature of this vacuum, we need to discuss all the different possibilities depending on the values of k_1 and k_2. The resulting IR theories can then be found accordingly:

- If $k_1, k_2 > 1$, the vacuum preserves supersymmetry and flows to a CS theory.

$$SU(2)_{k_1-2} \times SU(2)_{k_2-2} \quad \text{TQFT}, \tag{3.42}$$

$$\text{WI}_- = (k_1-1)(k_2-1). \tag{3.43}$$

- If $k_1 = 1$, $k_2 > 1$, the vacuum breaks supersymmetry, and we get a Majorana goldstino together with a decoupled CS theory,

$$G_\alpha + U(1)_2 \times SU(2)_{k_2-2} \quad \text{TQFT}. \tag{3.44}$$

- If $k_1 = k_2 = 1$, the vacuum breaks supersymmetry, and the IR theory is

$$G_\alpha + U(1)_2 \times U(1)_2 \quad \text{TQFT}. \tag{3.45}$$

- If $k_1 = 0$, $k_2 > 1$, supersymmetry is preserved, and the IR theory is

$$SU(2)_{k_2-2} \quad \text{TQFT}, \tag{3.46}$$

$$\text{WI}_- = -(k_2-1). \tag{3.47}$$

- If $k_1 = 0$, $k_2 = 1$, supersymmetry is again broken, and we get in the IR

$$G_\alpha + U(1)_2 \quad \text{TQFT}. \tag{3.48}$$

We note in particular that the $m \gg 0$ phase and $m \ll 0$ phase have different Witten indices. In order to understand the transition between the large negative mass phase and the large positive mass phase, it is useful to understand the dynamics near the point $m = 0$, where the asymptotic behaviour of the superpotential changes and the Witten index can jump.
We start by observing that the vacuum of the first kind, namely at the origin of the field space, $\phi_{11} = \phi_{22} = v_1 = 0$, exists for $m = 0$ as well as for m small and positive or small and negative. We identify this vacuum with the semiclassical vacuum we have seen at large and negative mass. (It will be evident in a moment that this choice leads to a consistent phase diagram providing the matching of the Witten index at the phase transition locus). This vacuum either preserves supersymmetry or, if one of the CS levels is equal to one, breaks it non-perturbatively. It is then expected to find new vacua appearing from the infinity of the field space near the line $m = 0$, and whose total Witten index must be different from zero. We thus initiate the search of these vacua, which must be either of the second or of the third type.

Non-Abelian Vacuum

We first turn to the analysis of the vacuum of the second kind, for which $\phi_{11} = \phi_{22} = v_2$ for some real and positive v_2, and $\rho = B = \bar{B} = 2v_2^2$.
The equation $\partial_\rho \mathcal{W} = -\partial_{|B|}\mathcal{W}$ hence turns into

$$-G - \frac{3F_1 + F_2}{\rho^{1/2}} + m = -G - \frac{F_2}{\rho^{1/2}}. \tag{3.49}$$

We immediately notice that, thanks to the positivity of F_1 (see (3.33)), we must necessarily have $m > 0$. Therefore, when we move from the negative-mass region and cross the $m = 0$ line, a new vacuum is found, in precise accordance with our expectations. The expression for the vev can be then easily found and is given by

$$v_2 = \frac{3F_1}{\sqrt{2}m}. \tag{3.50}$$

Next we determine the effective low-energy theory of this vacuum. The vev

$$\Phi = \begin{pmatrix} v_2 & 0 \\ 0 & v_2 \end{pmatrix}, \tag{3.51}$$

breaks the global baryonic symmetry $U(1)_B$, thus we expect to see the corresponding Goldstone boson with its superpartner. The vacuum also breaks the gauge group to the diagonal $SU(2)$ with the induced CS-level equal to $k_1 + k_2$. The CS-level can receive quantum corrections when massive fermions charged under the unbroken $SU(2)$ are integrated out, so we need to understand the fermionic mass spectrum.
The fermionic mass terms originate from the superpotential, from the gaugini-matter coupling terms, and from the gaugini mass term, a supersymmetric counterpart of the CS-term. The quadratic Lagrangian thus reads

$$\begin{aligned}\mathcal{L}_{\psi^2} =& \frac{\partial^2 \mathcal{W}}{\partial \bar{\Phi}_{\hat{i}j} \partial \Phi_{k\hat{l}}} \bar{\Psi}_{\hat{i}j} \Psi_{k\hat{l}} + \frac{1}{2}\left(\frac{\partial^2 \mathcal{W}}{\partial \bar{\Phi}_{\hat{i}j} \partial \bar{\Phi}_{\hat{k}l}} \bar{\Psi}_{\hat{i}j} \Psi^c_{\hat{k}l} + \text{c.c.}\right) + \left(ig_1 \text{Tr}\bar{\Psi}\lambda_1\Phi + ig_2 \text{Tr}\bar{\Psi}^c\lambda_2\bar{\Phi} + \text{c.c.}\right) + \\ & - \kappa_1 \text{Tr}\bar{\lambda}_1\lambda_1 - \kappa_2 \text{Tr}\bar{\lambda}_2\lambda_2,\end{aligned} \tag{3.52}$$

where the indices are put on the same line for convenience, and scalars are assumed to take their vev. While we are given fermions in representations of $SU(2) \times SU(2)$ group, it is natural to decompose them into representations of the preserved diagonal $SU(2)$ group. The decomposition goes as follows,

$$\Psi_i{}^{\hat{j}} = \frac{1}{2}\left[\psi^a_{\text{Re}}(\sigma^a)_i{}^{\hat{j}} + i\psi^a_{\text{Im}}(\sigma^a)_i{}^{\hat{j}} + (\psi_0 + i\psi_G)\delta_i{}^{\hat{j}}\right], \tag{3.53}$$

where we introduced two Majorana modes, ψ_0 and ψ_G, neutral under the diagonal $SU(2)$, and two Majorana multiplets, ψ^a_{Re} and ψ^a_{Im}, transforming in the adjoint representation of $SU(2)$. Two other adjoint multiplets are provided by λ_1 and λ_2.
It follows from (3.52) that the mass of ψ_0 and ψ_G are

$$m_0 = 4v_2^2\left(\partial_\rho^2 \mathcal{W} + 2\partial_\rho\partial_B \mathcal{W} + \partial_B^2 \mathcal{W}\right) = m, \tag{3.54a}$$

$$m_G = 0, \tag{3.54b}$$

thus we can identify the latter with the superpartner of the Goldstone boson associated with the broken $U(1)_B$. The mass of ψ_{Re} instead reads

$$m_{\text{Re}} = 2\partial_\rho \mathcal{W} = -2G - 2m\left(1 + \frac{F_2}{3F_1}\right). \tag{3.55}$$

Notice that, since we must have that $m>0$ from (3.49) and $G>0$ by definition, it follows that $m_{\rm Re}$ must be negative.
Finally, $\psi_{\rm Im}$, λ_1, and λ_2 mix with each other via the mass matrix

$$\begin{pmatrix} 0 & g_1 v_2 & -g_2 v_2 \\ g_1 v_2 & -\kappa_1 & 0 \\ -g_2 v_2 & 0 & -\kappa_2 \end{pmatrix}. \tag{3.56}$$

This mass matrix has one positive eigenvalue and two negative eigenvalues (one of the eigenmodes with a negative eigenvalue can be identified with the gaugino of the unbroken gauge group). Therefore, in total we have three multiplets with negative mass and one multiplet with positive mass transforming in the adjoint representation of the unbroken $SU(2)$. By integrating them out we get a shift of the CS-level equal to -2.
With these results at hand, we are now ready to formulate the infrared dynamics of this vacuum. Unless $k_1=0$ and $k_2=1$, it preserves supersymmetry, and the infrared dynamics is described by

$$\Phi_G \;+\; SU(2)_{k_1+k_2-2} \quad \text{TQFT}, \tag{3.57}$$

where Φ_G is the Goldstone supermultiplet. If, on the contrary, $k_1=0$ and $k_2=1$, supersymmetry is spontaneously broken, and we get in the infrared

$$\phi_G \;+\; G_\alpha \;+\; U(1)_2 \quad \text{TQFT}, \tag{3.58}$$

with ϕ_G being the Goldstone boson.

Abelian Vacuum

Next we consider the vacuum of the third kind appearing from infinity in the field space. The condition $\partial_B \mathcal{W}=0$ requires that $B=0$[12]. The condition $\partial_\rho \mathcal{W}$ then reads as

$$m-\frac{F_1+F_4}{\rho^{1/2}}=0. \tag{3.59}$$

Since $F_1+F_4>0$, as can be seen from (3.33), there are no solutions for $m\le 0$, whereas for $m>0$ we find

$$\Phi=\begin{pmatrix} v_3 & 0 \\ 0 & 0\end{pmatrix}, \qquad v_3=\frac{F_1+F_4}{m}. \tag{3.60}$$

The next step is to study the IR dynamics of this vacuum. The $U(1)_B$ is preserved here, and is generated by [13]

$$\frac{i}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1\end{pmatrix} \oplus \frac{i}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1\end{pmatrix}_2, \tag{3.61}$$

and the gauge symmetry is broken to $U(1)$, generated by

$$\frac{i}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1\end{pmatrix}_1 \oplus \frac{i}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1\end{pmatrix}_2. \tag{3.62}$$

where the subscripts $1,2$ indicate respectively the first or the second gauge group factor. This Abelian gauge field inherits the CS-level $2(k_1+k_2)$[14].

[12] One can first observe that $B\le\rho$, and from (3.32) we see that for $B=\rho\to\infty$, $\partial_B\mathcal{W}>0$, while for $B=0$, $\partial_B\mathcal{W}=0$, as desired. It can be found then numerically that there are no more solutions for $0<B<\rho$.

[13] We note that this choice of the preserved $U(1)_B$ is not unique, and is defined up to an action of the gauge transformation.

[14] The factor of two in front of the CS-levels is due to the different normalization between $SU(2)$ generators and the corresponding $U(1)$ subgroup generator.

Mode	η	χ	ψ_0	ψ_+	ψ_-	$\lambda_{1,0}$	$\lambda_{1,+}$	$\lambda_{2,0}$	$\lambda_{2,+}$
Type	**M**	**M**	**D**	**D**	**D**	**M**	**D**	**M**	**D**
$U(1)$	0	0	0	1	-1	0	1	0	1
$U(1)_B$	0	0	1	1	0	0	0	0	1

Table 3.1: Fermion modes

We can now classify fermions according to their charges with respect to the unbroken $U(1)\times U(1)_B$. The matter multiplet fermions and two types of gaugini can be decomposed as

$$\Psi=\begin{pmatrix}\frac{\eta+i\chi}{\sqrt{2}} & \psi_+\\ \psi_- & \psi_0\end{pmatrix},\qquad \lambda_1=\frac{1}{2}\begin{pmatrix}\lambda_{1,0} & \sqrt{2}\lambda_{1,+}\\ \sqrt{2}\lambda^c_{1,+} & -\lambda_{1,0}\end{pmatrix},\qquad \lambda_2=\frac{1}{2}\begin{pmatrix}\lambda_{2,0} & \sqrt{2}\lambda_{2,+}\\ \sqrt{2}\lambda^c_{2,+} & -\lambda_{2,0}\end{pmatrix}. \tag{3.63}$$

All the types of the modes (Majorana or Dirac) as well as their charges are summarized in Table 3.1. The masses of fermions neutral under the $U(1)$ gauge group are determined as follows. By starting again from (3.52), we get that η and ψ_0 masses come from the superpotential,

$$\begin{aligned} m_\eta &= 2v_3^2\frac{\partial^2\mathcal{W}}{\partial\rho^2}=m,\\ m_{\psi_0} &= v_3^2\left(\frac{\partial^2\mathcal{W}}{\partial B^2}+\frac{1}{B}\frac{\partial\mathcal{W}}{\partial B}\right)=2v_3^2\frac{\partial^2\mathcal{W}}{\partial B^2}=2m\frac{F_3}{F_1+F_4}, \end{aligned} \tag{3.64}$$

whereas χ, $\lambda_{1,0}$, $\lambda_{2,0}$ mix with each other through the mass matrix

$$\begin{pmatrix} 0 & \frac{1}{\sqrt{2}}g_1v_3 & -\frac{1}{\sqrt{2}}g_2v_3\\ \frac{1}{\sqrt{2}}g_1v_3 & -\kappa_1 & 0\\ -\frac{1}{\sqrt{2}}g_2v_3 & 0 & -\kappa_2\end{pmatrix}. \tag{3.65}$$

There are also mixing modes charged under the $U(1)$ gauge group, in particular, ψ_- mixes with $\lambda^c_{1,+}$ via

$$\begin{pmatrix} 0 & i\frac{1}{\sqrt{2}}g_1v_3\\ -i\frac{1}{\sqrt{2}}g_1v_3 & -\kappa_1\end{pmatrix}, \tag{3.66}$$

and ψ_+ mixes with $\lambda_{2,+}$ via

$$\begin{pmatrix} 0 & -i\frac{1}{\sqrt{2}}g_2v_3\\ i\frac{1}{\sqrt{2}}g_2v_3 & -\kappa_2\end{pmatrix}. \tag{3.67}$$

Both matrices have one positive eigenvalue and one negative eigenvalue which implies that the $U(1)$ CS-level does not get renormalized when these massive modes are integrated out. We thus conclude that at low energies we get a pure Abelian CS-theory

$$U(1)_{2(k_1+k_2)}\quad \text{TQFT}. \tag{3.68}$$

Let us now comment the results obtained so far. First, we were able to follow the appearance of two new vacua as far as the line $m=0$ is crossed. This process is controlled just by the leading asymptotic of the effective superpotential, which in turn is determined by the one-loop contribution only [37]. We thus conclude that we have rigorously derived the existence of these vacua. Second, there might be supersymmetric vacua emerging for some values of the parameters g_1, g_2, m, which don't come from infinity, but rather appear at finite field values. These vacua should have vanishing total Witten index, and their dynamics is a priori governed by perturbation theory at all orders, and not just at one-loop level. We do not have reliable tools to study them and, moreover, there are no consistency requirements (i.e. Witten index matching) that would necessitate their existence.

Phase Diagrams

We are now able to formulate the phase diagram of the theory under consideration. We start with a generic case of $k_1 > 1$ and $k_2 > 1$, and the relevant phase diagram is schematically depicted in Figure 3.2, where we attempt to reflect only its topology[15].
At large and negative masses, up to the $m = 0$ line, there is the supersymmetric semiclassical vacuum described by

$$SU(2)_{k_1-2} \times SU(2)_{k_2-2} \quad \text{TQFT}, \tag{3.69}$$

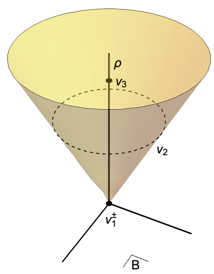
as we saw in (3.42). This vacuum is denoted on the figure as v_1^-, and this phase corresponds to the purple region. The Witten index of this vacuum is

$$\text{WI}_1 = (k_1 - 1)(k_2 - 1). \tag{3.70}$$

When we cross the wall at $m = 0$, two new vacua come in from infinity,

$$\begin{aligned} &\Phi_G + SU(2)_{k_1+k_2-2} \quad \text{TQFT}, \quad \text{WI}_2 = 0, \\ &U(1)_{2(k_1+k_2)} \quad \text{TQFT}, \quad \text{WI}_3 = 2(k_1 + k_2), \end{aligned} \tag{3.71}$$

giving us a phase with three vacua (the new vacua are v_2 and v_3, and the corresponding region in Figure 3.2 is the light blue one). These three vacua must undergo, generically, two second-order phase transitions (lines m_* and m_{**}, which are actually functions of g_1, g_2), merging into a single vacuum seen at large positive masses which we indicate with v_1^+. These phase transitions are supposed to happen somewhere around the origin of the field space, where the physics is strongly coupled, and we do not have much control over it. The structure of vevs in each vacuum suggests (see the figure on the right) that at the first phase transition either v_2 merges with v_1^-, or v_3 merges with v_1^-: we conjecture, basing on a duality proposed in Section 3.5, that the first option is realized. In the intermediate phase we then still get the Abelian vacuum v_3 and some other vacuum, v_q, which is guessed to support again the $SU(2)_{k_1-2} \times SU(2)_{k_2-2}$ CS-theory (the yellow region in Figure 3.2). At the second phase transition two vacua merge and produce the large positive mass vacuum v_1^+.



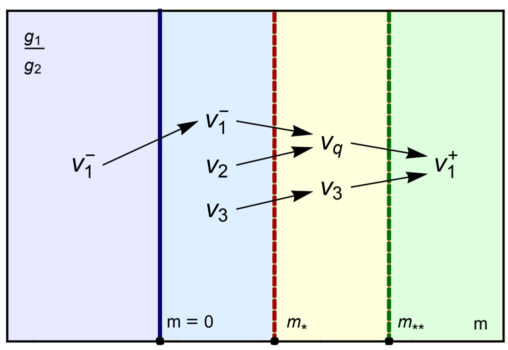


Figure 3.2: Structure of the phase diagrams of the $SU(2)_{k_1} \times SU(2)_{k_2}$ quivers and, as will be clearer later, the $SU(2)_{k_1} \times U(2)_{k_2,k_3}$ quivers with either $k_1, k_2 > 1$, or $k_1 = 0$, $k_2 > 1$. Dashed lines correspond to the second order phase transitions, while the solid line is the wall. Supersymmetric vacua at each phase are indicated.

[15] In particular, various straight lines appearing on the figure should in practice be curved.

Let us now discuss some special cases of the CS-levels

- If $k_1 = 1$, $k_2 \geq 1$, the negative mass vacuum breaks supersymmetry. As soon as the wall $m = 0$ is crossed, two supersymmetric vacua appear,

$$\begin{aligned} &\Phi_G + SU(2)_{k_2-1} \quad \text{TQFT}, \quad \text{WI}_2 = 0, \\ &U(1)_{2(k_2+1)} \quad \text{TQFT}, \quad \text{WI}_3 = 2(k_2+1), \end{aligned} \tag{3.72}$$

at some value of the mass parameter they merge and give rise to the large mass vacuum,

$$SU(2)_1 \times SU(2)_{k_2} \quad \text{TQFT}. \tag{3.73}$$

The picture is illustrated in Figure 3.3.

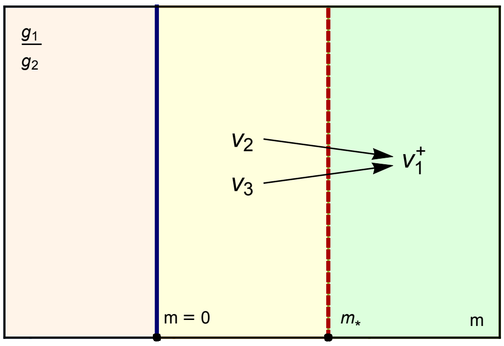


Figure 3.3: Phase diagram for $k_1 = 1$, $k_2 \geq 1$. Dashed line correspond to the second order phase transition, while the solid line is the wall. Supersymmetric vacua at each phase are indicated.

- If $k_1 = 0$ and $k_2 > 1$, the negative mass vacuum is supersymmetric and the theory flows to

$$SU(2)_{k_2-2} \quad \text{TQFT}, \quad \text{WI}_0 = -(k_2 - 1). \tag{3.74}$$

When we cross the line $m = 0$, two supersymmetric vacua appear

$$\begin{aligned} &\Phi_G + SU(2)_{k_2-2} \quad \text{TQFT}, \quad \text{WI}_2 = 0, \\ &U(1)_{2k_2} \quad \text{TQFT}, \quad \text{WI}_3 = 2k_2. \end{aligned} \tag{3.75}$$

When m is increased, the resulting three vacua undergo two phase transition and produce a supersymmetric vacuum with

$$SU(2)_{k_2} \quad \text{TQFT}. \tag{3.76}$$

- If $k_1 = 0$ and $k_2 = 1$, the negative mass vacuum is not supersymmetric. When the wall is crossed, there appears one supersymmetry-breaking vacuum (see (3.58)) and one supersymmetric vacuum with

$$U(1)_2 \quad \text{TQFT} \tag{3.77}$$

in the IR. Using the chain of dualities for TQFTs, we observe that

$$U(1)_2 \longleftrightarrow U(1)_{-2} \longleftrightarrow SU(2)_1, \tag{3.78}$$

and so the new supersymmetric vacuum came in from infinity exactly reproduces the semiclassical large mass vacuum. Therefore, in this case the theory does not undergo any phase transition. (See Figure 3.4)

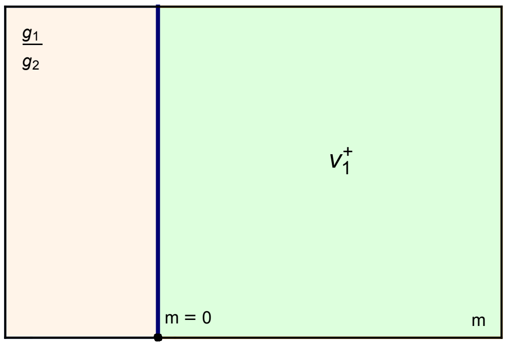


Figure 3.4: Phase diagram for $k_1 = 0$, $k_2 = 1$. The solid line correspond to the wall. The supersymmetric vacum is indicated.

An important check of the picture that we are suggesting here is the matching of the Witten indices at each phase transition locus. As an example, we can consider the case of $k_1, k_2 > 1$, for which we have

$$\mathrm{WI}_1 + \mathrm{WI}_2 + \mathrm{WI}_3 = (k_1 - 1)(k_2 - 1) + 0 + 2(k_1 + k_2) = (k_1 + 1)(k_2 + 1). \tag{3.79}$$

The right-hand side is nothing but the Witten index of the large mass vacuum WI_+ we saw in (3.40).

3.3.2 Chern-Simons Levels: Opposite Sign

Having understood the phase diagram for the case of CS-levels having the same sign, we now turn to the study of the case where the CS-levels have opposite sign. Without loss of generality we consider the first CS-level to be positive and the second to be negative. The theory we are considering thus in fact is

$$SU(2)_{k_1} \times SU(2)_{-k_2}, \tag{3.80}$$

coupled as before to a bi-fundamental matter multiplet Φ, with $k_1, k_2 > 0$. We will also assumefor simplicity that $k_1 > k_2$. The results of the following discussion are summarized in Figure 3.5.

As before, it is useful to start the analysis by considering the large mass semiclassical phases. For large and negative masses we find a supersymmetric vacuum with

$$SU(2)_{k_1-2} \times SU(2)_{-k_2} \quad \mathrm{TQFT} \tag{3.81}$$

in the IR (v_1^- in Figure 3.5). In the large positive mass phase we see the following picture:

- When $k_2 > 1$, we get a supersymmetric vacuum (v_1^+ on Figure (3.5)) hosting a CS theory,
$$SU(2)_{k_1} \times SU(2)_{-k_2+2} \quad \mathrm{TQFT}. \tag{3.82}$$
- When $k_2 = 1$, SUSY gets broken, and the IR theory is given by
$$G_\alpha + SU(2)_{k_1} \times U(1)_2 \quad \mathrm{TQFT}. \tag{3.83}$$

We first discuss in details the case of $k_2 > 1$, and then comment on the changes in the picture when $k_2 = 1$.

As soon as the large mass phases are understood, the next step is to study the behaviour near the wall, at $m = 0$. Again, we see that the $\phi_{11} = \phi_{22} = 0$ vacuum exists on both sides of the wall. It will again be natural to identify this vacuum with one of the large mass vacua, however this time it is less obvious to decide which of the two should be chosen. We also remark that, while moving along the line $m = 0$, three special points can be distinguished. These are the points for which the asymptotic behaviour of the effective superpotential (3.32) changes in a certain way, and are given by

$$F_1 = 0, \quad \frac{g_1}{g_2} = \left(\frac{k_2}{k_1}\right)^{\frac{1}{4}}, \tag{3.84a}$$

$$F_1 + F_4 = 0, \quad \frac{g_1}{g_2} = \alpha, \tag{3.84b}$$

$$F_3 = 0, \quad \frac{g_1}{g_2} = \frac{k_2}{k_1}. \tag{3.84c}$$

Here α is the single positive root of the equation $x^4 - \frac{k_2}{k_1} + 2(x^3 - \frac{k_2}{k_1})\sqrt{x^2+1} = 0$, and we note that for $k_1 > k_2$ we have

$$\frac{k_2}{k_1} < \alpha < \left(\frac{k_2}{k_1}\right)^{\frac{1}{4}} \tag{3.85}$$

As we did in the previous subsection, we now give a detailed discussion of the vacuuum structure near the wall.

Non-Abelian Vacuum

We first search for the vacuum of the second type. The equation $\rho = B = \bar{B} = 2v_2^2$ again reduces to

$$-\frac{3F_1}{\rho^{1/2}} + m = 0, \tag{3.86}$$

but now F_1 changes the sign when g_1, g_2 are varied, indeed what happens is that

$$F_1 \gtrless 0, \quad \frac{g_1}{g_2} \gtrless \left(\frac{k_2}{k_2}\right)^{1/4}. \tag{3.87}$$

We thus conclude that a vacuum of the second type is still given by

$$v_2 = \frac{3F_1}{\sqrt{2}m}, \tag{3.88}$$

and exists when either $m > 0$, $\frac{g_1}{g_2} > \left(\frac{k_2}{k_2}\right)^{1/4}$ or $m < 0$, $\frac{g_1}{g_2} < \left(\frac{k_2}{k_2}\right)^{1/4}$ (v_2 in the orange, light blue, grey and brown regions of Figure 3.5). At the point $\frac{g_1}{g_2} = \left(\frac{k_2}{k_2}\right)^{1/4}$ the quantum potential develops an asymptotic direction with zero energy. This is the first special point mentioned above. The gauge and global symmetry breaking pattern in this vacuum is the same as for the CS levels of the same sign, $U(1)_B$ is spontaneously broken, and the unbroken gauge group is $SU(2)_{k_1-k_2}$. We can also apply the previously obtained results for the fermionic mass spectrum, which does not undergo any changes.

The resulting low-energy theory again depends on the values of the levels:

- When $k_1 > k_2 + 1$, supersymmetry is preserved, and at low energies we get the Goldstone multiplet and a CS-theory,

$$\Phi_G + SU(2)_{k_1-k_2-2} \quad \text{TQFT}. \tag{3.89}$$

- When $k_1 = k_2 + 1$, supersymmetry is broken, and we get in the IR

$$\phi_G + G_\alpha + U(1)_2 \quad \text{TQFT}, \tag{3.90}$$

where ϕ_G is the Goldstone boson.

Abelian Vacuum

Finally, we look for Abelian vacua. There is still a solution given by

$$\Phi = \begin{pmatrix} v_3 & 0 \\ 0 & 0 \end{pmatrix}, \quad v_3 = \frac{F_1 + F_4}{m}. \tag{3.91}$$

Introducing the critical value $\frac{g_1}{g_2} = \alpha$ such that $F_1(\alpha) + F_4(\alpha) = 0$, we see that the solution exists either for $m > 0$, $\frac{g_1}{g_2} > \alpha$, or for $m < 0$, $\frac{g_1}{g_2} < \alpha$. At the point $\frac{g_1}{g_2} = \alpha$ the quantum superpotential again developes an asymptotic direction with zero energy.

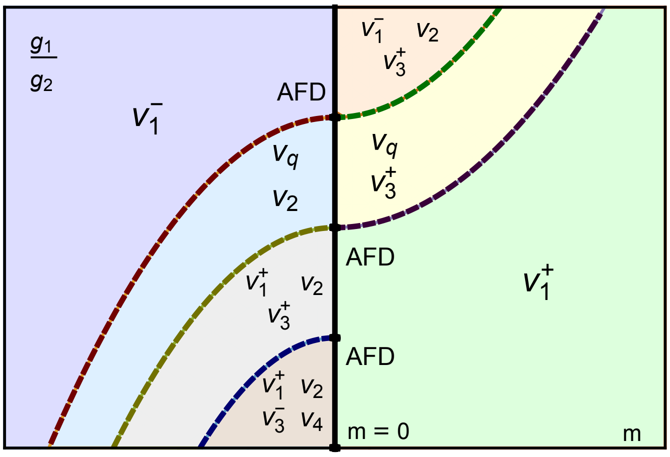


Figure 3.5: Structure of the phase diagrams of the $SU(2)_{k_1} \times SU(2)_{-k_2}$ quivers and, as will be clear later, the $SU(2)_{k_1} \times U(2)_{-k_2,-k_3}$ quivers with $k_1, k_2 > 0$ and $k_1 > k_2$. Dashed lines correspond to the second order phase transitions, while the solid line is the wall. Supersymmetric vacua at each phase are indicated.

We also note that, as follows from the definition of F_4, $\alpha < \left(\frac{k_2}{k_1}\right)^{1/4}$. This means that when we gradually move along the $m = 0$ line from the region with $g_1 \gg g_2$ to the region with $g_1 \ll g_2$, we first see the flipping point for the Non-Abelian vacuum (where it moves from the positive mass region to the negative mass region), and then the flipping point for the Abelian vacuum.

To determine the IR physics of this vacuum, one has to reexamine the fermionic mass spectrum. It follows from (3.64)-(3.67) that upon passing by the point $\frac{g_1}{g_2} = \alpha$, the charged modes do not change mass signs (but one neutral Dirac mode does), thus the IR description is given by

$$U(1)_{2(k_1-k_2)} \quad \text{TQFT}. \tag{3.92}$$

However, this is not the end of the story. Indeed, when the point $\frac{g_1}{g_2} = \frac{k_2}{k_1}$ is passed by, a new solution for the equations $\partial_{|B|}\mathcal{W} = \partial_\rho \mathcal{W} = 0$ is found. This can be seen in the following way. We note that $0 \le |B| \le \rho$, and

$$\partial_{|B|}\mathcal{W}\|_{B|=\rho} > 0,$$
$$\partial_{|B|}\mathcal{W}\|_{B|=0} = 0,$$

for any values of the parameters. But $\partial^2_{|B|}\mathcal{W} \propto F_3$ changes the sign exactly at the point $\frac{g_1}{g_2} = \frac{k_2}{k_1}$. In fact, when $\frac{g_1}{g_2} > \frac{k_2}{k_1}$ then we have that $\partial^2_{|B|}\mathcal{W}\Big|_{|B|=0} > 0$, and so it is possible that $B = 0$ is the only zero of $\partial_{|B|}\mathcal{W} = 0$. This reasonable claim is confirmed by the numerical study of the superpotential. On the other hand, when $\frac{g_1}{g_2} < \frac{k_2}{k_1}$, $\partial^2_{|B|}\mathcal{W}\Big|_{|B|=0} < 0$, and so there is at least one more solution with $B \neq 0$: the numerical study confirms that there is indeed only one such solution. This new vacuum still breaks the gauge group down to $U(1)$, but it also breaks the global $U(1)_B$. The IR physics is thus represented by

$$\Phi_G + U(1)_{2(k_1-k_2)} \quad \text{TQFT.} \tag{3.94}$$

Phase Diagrams

We now summarize the picture we suggest for the phase diagram, starting from the case $k_1, k_2 > 1$ and keeping in mind the relation in (3.85). We recall that we started by determining the large mass phases, depicted by the purple and the green regions of Figure 3.5. The next step was to understand the near-the-wall behaviour. When $g_1 \gg g_2$, there is just one vacuum on the left from the wall, but two new vacua, the Non-Abelian and the Abelian ones, appear on the right from the wall (orange regions of Figure 3.5). While moving down along the wall, we encounter the first special point, after which the Non-Abelian vacuum is found on the left from the wall, while the Abelian vacuum is still on the right: this corresponds to the light blue and the yellow regions. In both these phases there is also a vacuum at the origin. We do not have a weak coupling limit that would allow the direct study of this vacuum, but we propose that its IR description is identical to the large negative mass vacuum, v_1^-, since it provides the correct Witten index, and automatically matches the UV 1-form symmetry 't Hooft anomaly.

When we decrease $\frac{g_1}{g_2}$ even further, the second special point is found. While passing it, we find that there is just one vacuum on the right from the wall (the green phase in Figure 3.5), and three vacua on the left: a vacuum at the origin together with the Non-Abelian and the Abelian vacua discussed above (the grey phase in Figure 3.5). The vacuum at the origin is now identified with the large mass vacuum.

If we go even further down the wall, the Abelian vacuum splits into two Abelian vacua (v_3^- and v_4 in the brown region of Figure 3.5). The v_4 vacuum was described above (3.94), and v_3^- does not differ much from v_3^+: in fact, only the counter-terms for background fields associated to the global symmetry (e.g. the metric) are going to be different.

The special case of $k_1 > 1$ and $k_2 = 1$, is pretty much similar, and the resulting phase diagram is depicted in Figure 3.6. We note though that in this case the large positive mass phase (pink region) does not have any supersymmetric vacua, consequently there are just two vacua in the grey phase and three vacua in the brown phase. It also implies that at the transition line between the yellow region and the grey region two supersymmetric vacua collide and, instead of producing a new supersymmetric vacuum, get lifted. In the other special case, when $k_2 = k_1 + 1$, in the v_2 vacuum supersymmetry is broken and so the phase transition between the purple phase and the light blue phase is absent.

We conclude this section with the following observation. While sitting exactly at the wall and moving along it, we notice that at the flipping point for v_3 (the second special point) the Witten index jumps (in fact, for $k_2 = 1$ it vanishes below the flipping point, while being non-zero above). This is consistent with the fact that an asymptotically flat direction opens up at this point. Indeed, the transition can be arranged in the following way: at the special point one vacuum goes away to infinity, while another vacuum comes in from there. We thus conclude that exactly at the second flipping point there are no supersymmetric vacua, and the model exhibits the runaway behaviour (however, there might be meta-stable supersymmetry breaking vacua). The first flipping point does not show any special behaviour for the $SU(2) \times SU(2)$ quivers, but, as we will see below, there is an analogous phase transition at the first flipping point for the $SU(2) \times U(2)$ quivers. We thus expect the following picture: when we move down along the wall, at the first flipping point the supersymmetric vacuum goes away to infinity, and after this point the identical vacuum comes in from infinity. Again, at the special point the runaway behaviour takes place.

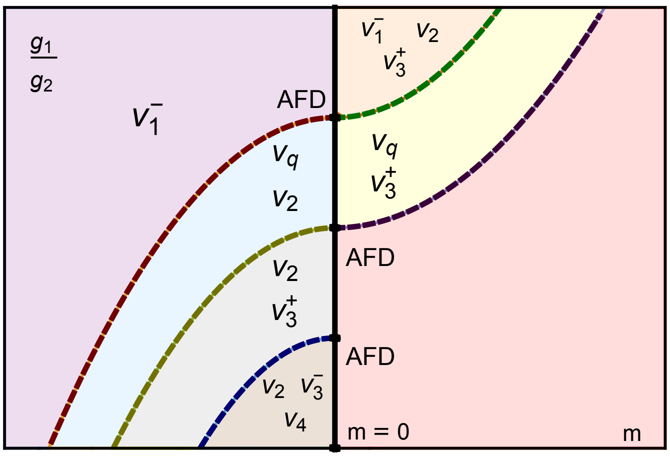


Figure 3.6: Phase diagram of the $SU(2)_{k_1} \times SU(2)_{-1}$ quivers with $k_1 > 1$. Dashed lines correspond to the second order phase transitions, while the solid line is the wall. The supersymmetric vacua in each phase are indicated.

3.4 Phase Diagrams of $SU(2) \times U(2)$ Models

In this section we discuss the phases of $SU(2) \times U(2)$ quiver theories, again with one bi-fundamental matter multiplet. Even though in principle one should recompute the effective superpotential for this case, we will appeal to a shortcut, and just assume that the vacuum structure (and in particular the symmetry breaking patterns) are the same as we have seen before. The main motivation for this assumption is that the Abelian factor inside U(2), for large CS-level, does not modify the behaviour of the vacua.

3.4.1 Chern-Simons Levels: Same Sign

We start by considering the models of the form

$$\mathcal{N} = 1 \quad SU(2)_{k_1} \times U(2)_{k_2,k_3} \tag{3.95}$$

with the coupling to bi-fundamental matter, and we will restrict ourselves with the case of $k_1 \geq 0$, $k_2 > 1$.
As before, we can readily understand the large mass phases. When the mass is large and positive, we get the IR theory

$$SU(2)_{k_1} \times U(2)_{k_2,k_3+1} \quad \text{TQFT} \tag{3.96}$$

with the index[16]

$$\text{WI}_+ = -\frac{(k_1+1)(k_2+1)|k_3+1|}{2}. \tag{3.97}$$

When the mass is large and negative, few different cases can be discussed:

- When $k_1 > 1$ and $k_2 > 1$, there is one supersymmetric vacuum whose IR theory is given by
$$SU(2)_{k_1-2} \times U(2)_{k_2-2,k_3-1} \quad \text{TQFT}. \tag{3.98}$$

[16] In the case of $U(N)_{k_2,k_3}$ the Witten index reads

$$\text{WI} = \frac{(k_2+N-1)!\ k_3}{N!\ k_2!}$$

The Witten index is

$$\mathrm{WI}_- = -\frac{(k_1-1)(k_2-1)|k_3-1|}{2}. \tag{3.99}$$

- When $k_1 = 1$, supersymmetry is spontaneously broken, and the IR theory is given by

$$G_\alpha + U(1)_2 \times U(2)_{k_2-2,k_3-1} \quad \text{TQFT}. \tag{3.100}$$

- When $k_1 = 0$, we again see a supersymmetric vacuum hosting a CS theory,

$$U(2)_{k_2-2,k_3-1} \quad \text{TQFT}, \tag{3.101}$$

and the index is

$$\mathrm{WI}_- = \frac{(k_2-1)|k_3-1|}{2}. \tag{3.102}$$

Following the familiar strategy, it is then useful to understand the dynamics near the wall, $m = 0$, which we do now.

Non-Abelian Vacuum

By assumption, there again exists a vacuum of the form

$$\Phi = \begin{pmatrix} v_2 & 0 \\ 0 & v_2 \end{pmatrix}. \tag{3.103}$$

The gauge group is still broken to $SU(2)$ with the induced CS level $k_1 + k_2$, but since the baryonic symmetry is now gauged, there are no Goldstone modes in the IR. In fact, the would-be Goldstone boson superpartner ψ_G gets mixed with the $U(1)$ gaugino via the mass matrix

$$\begin{pmatrix} 0 & -g_2 v_2 \\ -g_2 v_2 & -\kappa_3 \end{pmatrix}. \tag{3.104}$$

We assume that the rest of the fermionic spectrum is qualitatively the same, and thus the IR theory is given by

$$SU(2)_{k_1+k_2-2} \quad \text{TQFT}. \tag{3.105}$$

There are ten negative-mass Majorana modes, so the Witten index is

$$\mathrm{WI}_2 = k_1 + k_2 - 1. \tag{3.106}$$

Abelian Vacuum

In the same way we expect to find a vacuum of the form

$$\Phi = \begin{pmatrix} v_3 & 0 \\ 0 & 0 \end{pmatrix}.$$

It breaks the gauge group to $U(1)\times U(1)$, and the induced CS levels are given by the matrix

$$K = \begin{pmatrix} 2(k_1+k_2) & -k_2 \\ -k_2 & \frac{1}{2}(k_2+k_3) \end{pmatrix}. \tag{3.107}$$

We can now use the fermionic charges and masses computed in Section 3.3.1 to obtain the quantum corrections to the level matrix induced upon the integration out of the fermions:

$$K_{IR} = \begin{pmatrix} 2(k_1+k_2) & -k_2 \\ -k_2 & \frac{1}{2}(k_2+k_3)+\frac{1}{2} \end{pmatrix}. \tag{3.108}$$

The Witten index of this vacuum is given (up to a sign) by the number of lines in the corresponding Abelian CS theory,

$$\mathrm{WI}_3 = \det K_{IR} = -|(k_1+k_2)(k_2+k_3)+(k_1+k_2)-k_2^2|. \tag{3.109}$$

The overall structure of the phase diagram is identical to the one depicted on Figures (3.2),(3.3). We conjecture (following the pattern discussed in Section 3.3.1) that for $k_3 \neq 0$ at the intermediate (yellow) phase there is still the Abelian vacuum, as well as some other vacuum, resulting from the merging of the Non-Abelian vacuum and the vacuum at the origin. This *quantum* vacuum is expected to support a TQFT or/and a non-linear sigma model with the Witten index fixed by the matching condition. When $k_3 = 1$, the vacuum structure becomes quite different, and will be discussed in Section 3.5.

3.4.2 Chern-Simons Levels: Opposite Signs

Next we discuss the models of the form

$$\mathcal{N}=1 \quad SU(2)_{k_1}\times U(2)_{-k_2,-k_3} \tag{3.110}$$

with, $k_1 > k_2 > 1$.
When the matter mass is large and positive, we obtain

$$SU(2)_{k_1}\times U(2)_{-k_2+2,-k_3+1} \tag{3.111}$$

in the IR. There are three or four negative mass Majorana modes, depending on whether k_3 is positive or negative, so the index is

$$\mathrm{WI}_1 = -\frac{(k_1+1)(k_2-1)(k_3-1)}{2}. \tag{3.112}$$

When instead the mass is large and negative, we find a supersymmetric vacuum with

$$SU(2)_{k_1-2}\times U(2)_{-k_2,-k_3-1} \quad \text{TQFT}, \tag{3.113}$$

$$\mathrm{WI}_1 = \mathrm{sgn}(k_3)\frac{(k_1-1)(k_2+1)|k_3+1|}{2}. \tag{3.114}$$

Non-Abelian Vacuum

Similarly to the $SU(2)\times SU(2)$ case, the non-Abelian vacuum is expected to exist on the right from the wall for $g_1 \gg g_2$, and on the left from the wall for $g_1 \ll g_2$, with a flipping point for some value of $\frac{g_1}{g_2}$. The gauge group is broken to $SU(2)_{k_1-k_2}$. The masses of fermions transforming in the adjoint representation don't change upon the crossing of the flipping point, there are always one of them with a positive mass and three with negative masses. On the contrary, one of the neutral Majorana fermions change the sign of its mass, such that there are seven negative-mass Majorana modes when $m > 0$ and eight negative-mass Majorana modes when $m < 0$. We therefore get in the IR

$$SU(2)_{k_1-k_2-2} \quad \text{TQFT}, \quad \mathrm{WI} = -\mathrm{sgn}(m)(k_1-k_2+1). \tag{3.115}$$

Abelian Vacuum

Finally, we suppose that there is an Abelian vacuum supporting the $U(1)\times U(1)$ CS theory. This vacuum is also expected to flip from one side of the wall to another at some value of

$\frac{g_1}{g_2}$ (the second special point), and fermions charged under the unbroken gauge group do not flip the signs of their masses, and so the level matrix is given by

$$K_{IR}^{+} = \begin{pmatrix} 2(k_1 - k_2) & k_2 \\ k_2 & -\frac{1}{2}(k_2 + k_3) + \frac{1}{2} \end{pmatrix}. \tag{3.116}$$

on both sides of the wall. Some neutral modes though flip their masses, so that the Witten index is negative for $m > 0$ and positive for $m < 0$.

Decreasing the ratio $\frac{g_1}{g_2}$ even further, we expect to face the third special point where a new Abelian vacuum with $B \neq 0$ appears (v_4 in the brown region of Figure 3.5). This vacuum preserves just one Abelian factor, and supports

$$U(1)_{2(k_1-k_2)} \quad \text{TQFT} \tag{3.117}$$

in the IR. The Abelian vacuum discussed above also undergoes some changes when the third special point is passed. Namely, one of the fermions charged under the second $U(1)$ gets a negative mass, which leads to the corrected level matrix:

$$K_{IR}^{-} = \begin{pmatrix} 2(k_1 - k_2) & k_2 \\ k_2 & -\frac{1}{2}(k_2 + k_3) - \frac{1}{2} \end{pmatrix}. \tag{3.118}$$

The Abelian vacuum with the $U(1) \times U(1)$ gauge group and the level matrix given above is denoted by v_3^- in Figure 3.5.
The overall phase diagram looks quite similar to what is seen in Figures 3.5 and 3.6, despite it is now harder to guess the vacuum at the origin in yellow and purple phases. We conjecture that for $k_2 = 2, k_3 = 1$ it is given by the Abelian CS theory

$$U(1)_{2k_1}. \tag{3.119}$$

This conjecture is motivated by a duality discussed in the next section. Slightly more generally, for $k_3 = 1$ it is natural to expect that this vacuum supports the same TQFT in the IR, as does the Abelian vacuum also existing in this phase, namely

$$[U(1) \times U(1)]_K \quad \text{TQFT}, \tag{3.120}$$

$$K = \begin{pmatrix} 2(k_1 - k_2) & k_2 \\ k_2 & -\frac{1}{2}k_2 \end{pmatrix}. \tag{3.121}$$

The two proposal are consistent, since, as can be easily verified, for $k_2 = 2$ $[U(1) \times U(1)]_K$ is dual to $U(1)_{2k_1}$

The last comment concerns the dynamics at the special points. Similarly to the discussion at the end of Section 3.3.2, we observe two phase transitions, at the first and at the second special points. As before, they are organized by first sending a supersymmetric vacuum to infinity, and then receiving a new supersymmetric vacuum, generically with a different TQFT and Witten index, from infinity, with the runaway behaviour at the transition point.

3.5 Dualities

The discussion of the previous two sections demonstrated that a generic three-dimensional $\mathcal{N} = 1$ quiver gauge theory has multiple second-order phase transitions with associated IR fixed points. In this section we will provide few conjectures stating that certain CFTs that appear as IR limits of different quiver theories may in fact be the same.
Some of such dualities were already used above to guess certain aspects of the phase diagrams (namely, the vacuum structures in the intermediate phases).

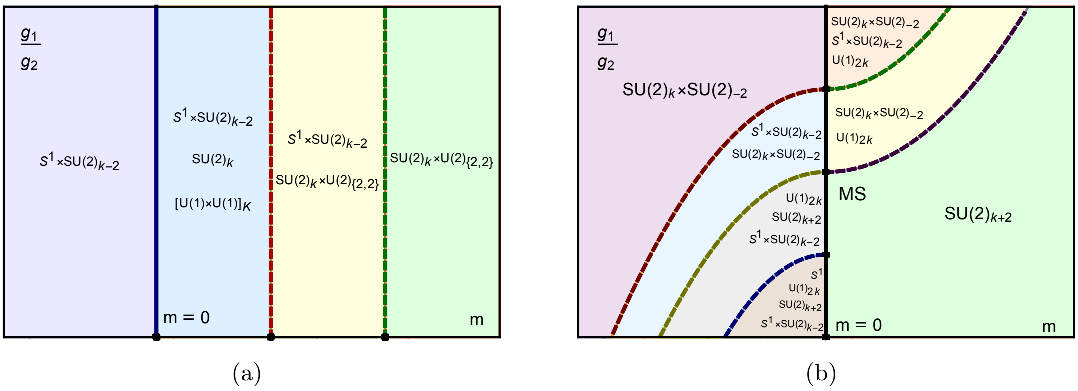


Figure 3.7: Phase diagram for the $SU(2)_k \times U(2)_{2,1}$ quiver (a) and for the $SU(2)_{k+2} \times SU(2)_{-2}$ quiver (b).

3.5.1 Dualities between $SU(2) \times SU(2)$ and $SU(2) \times U(2)$

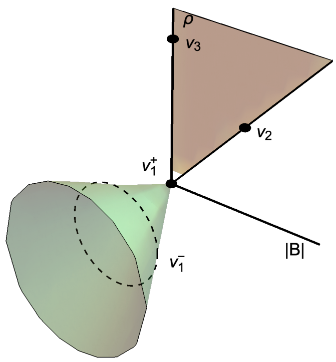
The first pair of theories we consider is $SU(2)_k \times U(2)_{2,1}$ and $SU(2)_{k+2} \times SU(2)_{-2}$ quivers with $k > 0$; the corresponding phase diagrams are shown in Figure 3.7, and the yellow phase of 3.7(a) as well as the yellow and light blue phases of 3.7(b) are conjectures. We observe, using the level-rank duality

$$SU(2)_{-2} \longleftrightarrow U(2)_{2,2}, \tag{3.122}$$

that the transition in Figure 3.7(a) between the yellow and the green phases is identical to the transition in Figure 3.7(b) between the light blue phase and the purple phase, with the phases on both sides of the transition given by

$$\mathbb{S}^1 \times SU(2)_{k-2} + SU(2)_k \times SU(2)_{-2} \longrightarrow SU(2)_k \times SU(2)_{-2}. \tag{3.123}$$

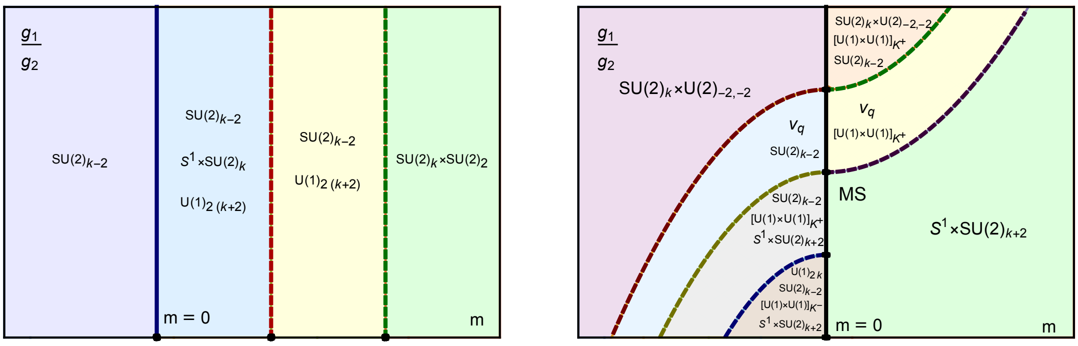
While making the conjecture about the vacua in the yellow phase of Figure 3.7(a), we assumed that at the transition point $m = m_*$ the non-Abelian vacuum merges with the Abelian one, while the vacuum at the origin stays apart. This is in contrast with what was assumed in Sections 3.3.1 and 3.4.1. The difference comes from the fact that here a new "branch" of vacua, parametrized by the dual photon, emerges. The space of possible vacua can be then visualized as the two-dimensional space parametrized by the scalar vevs, together with a cone attached at the origin (see the Figure on the right). The angular direction of the cone is the dual photon, and the radial direction gives the radius of the circle (which is not a dynamical field, but rather a function of the parameters). It is then possible that first the non-Abelian and the Abelian vacua meet at the origin, and then the dual photon radius (as a function of m) shrinks to zero, and the second phase transition happens.



The second pair is $SU(2)_k \times SU(2)_2$ and $SU(2)_{k+2} \times U(2)_{-2,-1}$ quivers with $k > 0$; the corresponding phase diagrams are presented in Figure 3.8, and the yellow phase of Figure 3.8(a) is a conjecture. We propose that the "quantum" vacuum v_q in Figure 3.8(b) is given by

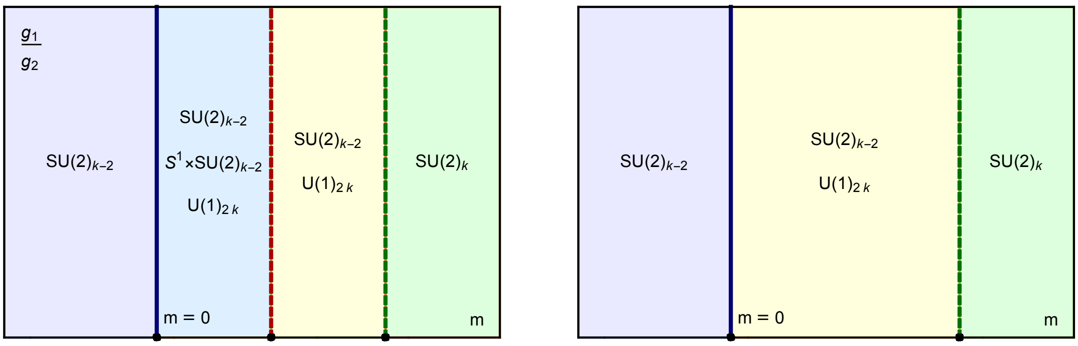
$$U(1)_{2(k+2)} \quad \text{TQFT}. \tag{3.124}$$

Using again the level-rank duality (3.122), we again observe that the transition in Figure 3.8(a) between the yellow and the green phases is identical to the transition in Figure 3.8(b)



(a) (b)

Figure 3.8: Phase diagram for the $SU(2)_k \times SU(2)_0$ quiver (a) and for the $SU(2)_k$ adjoint QCD (b).



(a) (b)

Figure 3.9: Phase diagram for the $SU(2)_k \times SU(2)_0$ quiver (a) and for the $SU(2)_k$ adjoint QCD (b).

between the light blue phase and the purple phase,

$$SU(2)_{k-2} + U(1)_{2(k+2)} \longrightarrow SU(2)_k \times SU(2)_2. \tag{3.125}$$

The two dualities we have just described can be obtained from the duality (3.8) by gauging the flavour $SU(2)$ (sub)groups on both sides.

3.5.2 Duality between $SU(2) \times SU(2)$ quiver and adjoint QCD

The first model considered here is the $SU(2)_k \times SU(2)_0$ quiver, discussed in section 3.3.1. The phase diagram can be found in Figure 3.9(a), where the form of the yellow phase is a conjecture. The phases of the $SU(2)_k$ adjoint QCD were reviewed in Section 3.1, and are depicted on Figure 3.9(b).

Evidently, the phase transitions between the yellow phases and the green phases are identical, and this hints towards the possibility of the duality:

$$SU(2)_k \times SU(2)_0 \quad \text{with a bi-fundamental} \quad \longleftrightarrow \quad SU(2)_k \quad \text{with an adjoint.} \tag{3.126}$$

This duality if correct has a quite clear meaning. Assuming that the $SU(2)_0$ node of the quiver confines, we can describe the low-energy physics in terms of the bilinears

$$X = \Phi\Phi^\dagger, \tag{3.127}$$

which indeed transforms in the adjoint representation of $SU(2)_k$. There is one point in this picture that may seem disturbing. The quiver theory possesses the baryonic symmetry $U(1)_B$, and there are charged operators $B = \det \Phi$. Neither the symmetry, nor would-be dual operators appear on the QCD side. This issue can be resolved in two ways: either the quiver theory baryons happen to be massive, and do not appear in the IR fixed point, or they are actually massless at the CFT point, but decouple from the rest. In the latter case the QCD side should be supplemented by a decoupled free complex multiplet.

3.6 Time reversal invariant models

We have already mentioned that $3d$ $\mathcal{N}=1$ theories with time reversal invariance have a beautiful property: their superpotentials admit only corrections odd under the action of T-transformation [39]. It significantly restricts the possible form of the effective superpotential, and sometimes superpotential even turns out to be fully protected.

Examples of T-invariant theories can be found also among the quiver theories. For example, a two-node quiver with opposite CS levels,

$$\mathcal{N}=1 \quad SU(2)_k \times SU(2)_{-k} + \text{a bi-fundamental}, \tag{3.128}$$

enjoys this property at the point $g_1 = g_2$, $m = 0$[17]. It is easy to see that there are no parity odd terms that could be written in the effective superpotential, implying that we have an example with full protection at hands. In fact, one can check that the 1-loop superpotential computed in Section 3.2 vanishes at this point. It follows that the theory has a moduli space of vacua, which coincides with the classical one,

$$\mathcal{M} = \mathbb{S}^1 \times \mathbb{R}^2 / S_2. \tag{3.129}$$

At the origin of the moduli space we expect to find a SCFT. At a point away from the origin the IR physics is described by three real massless moduli without any topological sector.

We can then deform the theory from the T-invariant point by turning on the mass term or changing the ratio $\frac{g_1}{g_2}$ and study the resulting IR phases. The large mass phases for $k > 1$ are supersymmetric and are given by

$$SU(2)_k \times SU(2)_{-k+2} \quad \text{TQFT} \tag{3.130}$$

for large positive masses and

$$SU(2)_{k-2} \times SU(2)_{-k} \quad \text{TQFT} \tag{3.131}$$

for large negative masses. When $k = 1$, supersymmetry is spontaneously broken, and the IR description is

$$G_\alpha + U(1)_2 \times U(1)_2 \quad \text{TQFT} \tag{3.132}$$

for large positive masses and the same for large positive masses, where the level-rank duality (3.78) has been used.

We can also study behaviour near the wall and find that for $\frac{g_1}{g_2} > 1$ two new vacua appear when $m > 0$, while for $\frac{g_1}{g_2} < 1$ we find them when $m < 0$. These are would-be Abelian and non-Abelian vacua familiar from above, despite the fact that they do not support any topological degrees of freedom. Still, they are not trivial and host S^1 Goldstone bosons: Abelian vacuum breaks spontaneously magnetic symmetry of the preserved $U(1)$ gauge group, while non-Abelian vacuum breaks the baryonic symmetry. The phase diagram is depicted on Figure 3.10.

[17] To be more precise, T-transformation must be augmented by the exchange of two gauge group factors.

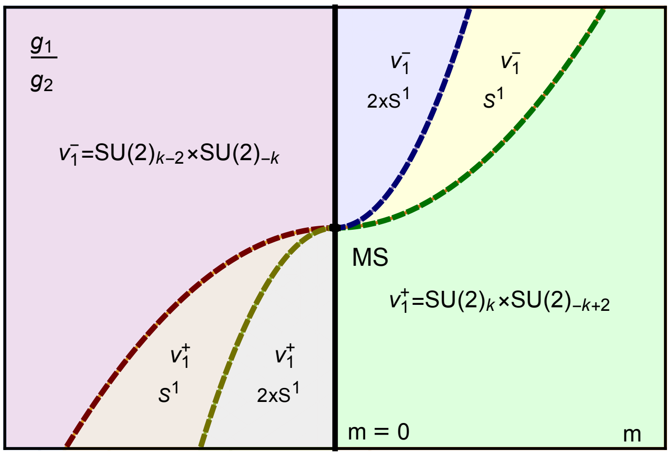


Figure 3.10: Structure of the phase diagram for the $SU(2)_k \times SU(2)_{-k}$ quiver. Dashed lines correspond to the second order phase transitions, while the solid line is the wall. The supersymmetric vacua in each phase are indicated.

Conclusions and Future Directions

In first part of the thesis, we have constructed the one-dimensional topological sector of $\mathcal{N}=6$ ABJ(M) theory, taking a slightly different point of view with respect to previous investigations [24, 112]. We started directly from the superconformal algebra $\mathfrak{su}(1,1|3) \oplus \mathfrak{u}(1)_b$ and obtained the relevant $\mathcal{Q}$-cohomology in the same formalism. The topological operators have been correctly identified as non-trivial neutral elements of such cohomology which consistently turned out to be certain superconformal primaries of the one-dimensional superalgebra. We have constructed gauge-invariant composite operators and computed their correlation functions which have the interesting property of being related to the correlators of the stress-energy tensor [24]. We have computed the two-point function at two loops and the three- and four-point functions at one loop founding perfect agreement with suitable derivatives of the mass-deformed matrix model of the partition function of ABJ(M) in the weak coupling expansion. Our result strongly supports the proposal in [24] for which the mass-deformed partition function of any three-dimensional $\mathcal{N} \geq 4$ SCFT is the generating functional of the (integrated) topological correlation functions. As a by-product we have also obtained the explicit expression for the central charge c_T at two loops, for generic ranks N_1, N_2.

A natural generalization of the results summarized above would concern the construction of topological operators inserted into the 1/2 BPS Wilson line, whose related defect CFT has been examined in [113]. Defect conformal field theories supported on the 1/2 BPS Wilson line have been studied in four-dimensional $\mathcal{N}=4$ SYM [114, 115] and its topological sector has been extensively studied in a series of papers [116–118]. In the ABJ(M) case, since the relevant symmetry of the 1/2 BPS Wilson loop is exactly $\mathfrak{su}(1,1|3) \oplus \mathfrak{u}(1)_b$, we expect that an explicit representation of the topological operators can be constructed, although in terms of supermatrices, as done in [113]. A work in this direction is in progress. Another interesting perspective would be to apply conformal bootstrap techniques in this context. In the $\mathcal{N}=4$ case the OPE data in the relevant topological quantum mechanics can be obtained or constrained imposing the associativity and unitarity of the operator algebra [30, 119]. This procedure is dubbed mini-bootstrap (or micro-bootstrap in four-dimensions [120]) because it concerns a closed subsystem of the full bootstrap equations. The generalization to the $\mathcal{N} \geq 4$ case could give further hints on the underlying structure of the topological quantum mechanics.

In the second part of the thesis, we have initiated the sistematic study of the IR phases of three-dimensional $\mathcal{N}=1$ quiver theories. We have observed how the simplest possible setup of a two-node quiver, namely with $SU(2)$ gauge groups and one bi-fundamental matter multiplet, reveals quite rich and diverse physical pictures. We find the characteristic features of theories with two supercharges observed previously in the literature: walls in the parameter space at which the Witten index jumps, multiple phases with second order phase transitions between them, vacua with spontaneously broken supersymmetry, which can be either stable or meta-stable. Especially interesting phase diagrams are found in theories whose CS-levels have different signs (Sections 3.3.2, 3.4.2). The study of the phase diagrams have allowed us to conjecture two generalized dualities which have been previously conjectured for SQCD-like theories and can be obtained from them by gauging the flavour symmetries on both sides. This tool for generating new dualities is well-known for theories with greater amount of supersymmetry, but our results suggest that it is also applicable in the landscape of minimally supersymmetric three-dimensional theories. The third duality we discuss involves the confinement of a node, and corresponds to the situation for which at low energies the physics can be described in terms of gauge-invariant (with respect to a given node) composite operators in terms of the original matter field.

The results obtained in the present thesis offer directions for many generalizations. The most obvious one is to consider multi-node quivers or quivers with higher-rank gauge groups, as $SU(N)$ or $U(N)$, for which the structure of the phase diagrams is expected to be more complicated reflecting the different structure of the one-loop superpotential. More general dualities of [41] can then be used, together with the node-dualization technique, to conjecture new dualities between quivers. In this case, a more detailed analysis is required in order to establish to which of the multiple SCFTs the duality applies.

Another interesting generalization would be to take several bi-fundamental matter multiplets. The resulting vacua will generically break flavour symmetries therefore the IR description will be given in terms of non-linear sigma models, in addition to the topological sectors which we have observed.

Appendix A

ABJM Conventions

We work in Euclidean space with coordinates $x^\mu = (x^1, x^2, x^3)$ and metric $\delta_{\mu\nu}$. Gamma matrices satisfying the usual Clifford algebra $\{\gamma^\mu, \gamma^\nu\} = 2\delta^{\mu\nu}\mathbb{1}$, are chosen to be the usual Pauli matrices

$$(\gamma^\mu)_\alpha^{\ \beta} \equiv (\sigma^\mu)_\alpha^{\ \beta} \qquad \mu = 1, 2, 3 \tag{A.1}$$

Standard relations which are useful for perturbative calculations are

$$\gamma^\mu\gamma^\nu = \delta^{\mu\nu} + i\varepsilon^{\mu\nu\rho}\gamma_\rho \tag{A.2}$$

$$\gamma^\mu\gamma^\nu\gamma^\rho = \delta^{\mu\nu}\gamma^\rho - \delta^{\mu\rho}\gamma^\nu + \delta^{\nu\rho}\gamma^\mu + i\varepsilon^{\mu\nu\rho} \tag{A.3}$$

Moreover, we define $\gamma^{\mu\nu} \equiv \frac{1}{2}[\gamma^\mu, \gamma^\nu]$.
Spinor indices are raised and lowered according to the following rules

$$\psi^\alpha = \varepsilon^{\alpha\beta}\psi_\beta, \qquad \psi_\alpha = \varepsilon_{\alpha\beta}\psi^\beta$$

with $\varepsilon^{12} = -\varepsilon_{12} = 1$. Consequently, we define $(\gamma^\mu)_{\alpha\beta} \equiv \varepsilon_{\beta\gamma}(\gamma^\mu)_\alpha^{\ \gamma} = (-\sigma^3, i\mathrm{I}, \sigma^1)$ and $(\gamma^\mu)^{\alpha\beta} \equiv \varepsilon^{\alpha\gamma}(\gamma^\mu)_\gamma^{\ \beta} = (\sigma^3, i\mathrm{I}, -\sigma^1)$. They satisfy $(\gamma^\mu)_{\alpha\beta} = (\gamma^\mu)_{\beta\alpha}$ and $(\gamma^\mu)^{\alpha\beta} = (\gamma^\mu)^{\beta\alpha}$.

A.1 ABJ(M) Action

The Euclidean gauge-fixed ABJ(M) theory action has the following structure

$$S = S_{\text{CS}} + S_{\text{gf}} + S_{\text{mat}} + S_{\text{pot}}. \tag{A.4}$$

By using the conventions in [94] with a convenient rescaling of the gauge fields and the corresponding ghosts, $A \to \frac{1}{\sqrt{k}}A$, $\hat{A} \to \frac{1}{\sqrt{k}}\hat{A}$, $c \to \frac{1}{\sqrt{k}}c$, $\hat{c} \to \frac{1}{\sqrt{k}}\hat{c}$, the explicit expression of the action above reads

$$S_{\text{CS}} = -\frac{i}{4\pi}\int d^3x\, \varepsilon^{\mu\nu\rho}\left[\text{Tr}\left(A_\mu\partial_\nu A_\rho + \frac{2i}{3\sqrt{k}}A_\mu A_\nu A_\rho\right) - \text{Tr}\left(\hat{A}_\mu\partial_\nu\hat{A}_\rho + \frac{2i}{3\sqrt{k}}\hat{A}_\mu\hat{A}_\nu\hat{A}_\rho\right)\right] \tag{A.5}$$

$$S_{\text{gf}} = \frac{1}{4\pi}\int d^3x\ \text{Tr}\left[\frac{1}{\xi}(\partial_\mu A^\mu)^2 + \partial_\mu\bar{c}D^\mu c - \frac{1}{\xi}\left(\partial_\mu\hat{A}^\mu\right)^2 - \partial_\mu\bar{\hat{c}}D^\mu\hat{c}\right] \tag{A.6}$$

$$S_{\text{mat}} = \int d^3x\ \text{Tr}\left[D_\mu C_I D^\mu\bar{C}^I - i\bar{\psi}^I\gamma^\mu D_\mu\psi_I\right] \tag{A.7}$$

$$\begin{aligned} &= \int d^3x\ \text{Tr}\Big[\partial_\mu C_I\partial^\mu\bar{C}^I - i\bar{\psi}^I\gamma^\mu\partial_\mu\psi_I + \frac{1}{\sqrt{k}}\left(\bar{\psi}^I\gamma^\mu\hat{A}_\mu\psi_I - \bar{\psi}^I\gamma^\mu\psi_I A_\mu\right) \\ &\quad + \frac{i}{\sqrt{k}}\left(A_\mu C_I\partial^\mu\bar{C}^I - C_I\hat{A}_\mu\partial^\mu\bar{C}^I - \partial_\mu C_I\bar{C}^I A^\mu + \partial_\mu C_I\hat{A}^\mu\bar{C}^I\right) \\ &\quad + \frac{1}{k}\left(A_\mu C_I\bar{C}^I A^\mu - A_\mu C_I\hat{A}^\mu\bar{C}^I - C_I\hat{A}_\mu\bar{C}^I A^\mu + C_I\hat{A}_\mu\hat{A}^\mu\bar{C}^I\right)\Big] \end{aligned} \tag{A.8}$$

$$
\begin{aligned}
S_{\text{pot}} &\equiv S_{6\text{pt}} + S_{4\text{pt}} \\
&= -\frac{4\pi^2}{3k^2}\int d^3x\ \text{Tr}\Big[C_I\bar{C}^I C_J\bar{C}^J C_K\bar{C}^K + \bar{C}^I C_I\bar{C}^J C_J\bar{C}^K C_K \\
&\qquad\qquad +4\,C_I\bar{C}^J C_K\bar{C}^I C_J\bar{C}^K - 6\,C_I\bar{C}^J C_J\bar{C}^I C_K\bar{C}^K\Big] \qquad \text{(A.9)}\\
&\quad -\frac{2\pi i}{k}\int d^3x\ \text{Tr}\Big[\bar{C}^I C_I\Psi_J\bar{\Psi}^J - C_I\bar{C}^I\bar{\Psi}^J\Psi_J + 2\,C_I\bar{C}^J\bar{\Psi}^I\Psi_J \\
&\qquad\qquad -2\,\bar{C}^I C_J\Psi_I\bar{\Psi}^J - \epsilon_{IJKL}\bar{C}^I\bar{\Psi}^J\bar{C}^K\bar{\Psi}^L + \epsilon^{IJKL}C_I\Psi_J C_K\Psi_L\Big] \qquad \text{(A.10)}
\end{aligned}
$$

where $\epsilon_{1234} = \epsilon^{1234} = 1$, for the $\mathfrak{su}(4)$ generators we use the following relations

$$
\text{Tr}\big(T^A T^B\big) = \delta^{AB}\ , \qquad [T^A, T^B] = i f^{AB}{}_C\, T^C, \tag{A.11}
$$

and where the covariant derivatives have been defined as follows

$$
\begin{aligned}
D_\mu C_I &= \partial_\mu C_I + \frac{i}{\sqrt{k}}A_\mu C_I - \frac{i}{\sqrt{k}}C_I\hat{A}_\mu, \qquad & D_\mu\bar{C}^I &= \partial_\mu\bar{C}^I + \frac{i}{\sqrt{k}}\hat{A}_\mu\bar{C}^I - \frac{i}{\sqrt{k}}\bar{C}^I A_\mu \\
D_\mu\bar{\psi}^I &= \partial_\mu\bar{\psi}^I + \frac{i}{\sqrt{k}}A_\mu\bar{\psi}^I - \frac{i}{\sqrt{k}}\bar{\psi}^I\hat{A}_\mu, \qquad & D_\mu\psi_I &= \partial_\mu\psi_I + \frac{i}{\sqrt{k}}\hat{A}_\mu\psi_I - \frac{i}{\sqrt{k}}\psi_I A_\mu
\end{aligned} \tag{A.12}
$$

A.2 ABJ(M) Feynman rules

The corresponding propagators at tree and loop orders, as needed for the two-loop calculations, are:

- Scalar propagator

$$
\langle (C_I)_i{}^{\hat{j}}(x)\ (\bar{C}^J)_{\hat{k}}{}^{l}(y)\rangle^{(0)} = \delta_I^J\delta_i^l\delta_{\hat{k}}^{\hat{j}}\ \frac{\Gamma(\frac{1}{2}-\epsilon)}{4\pi^{\frac{3}{2}-\epsilon}}\frac{1}{|x-y|^{1-2\epsilon}} \tag{A.13}
$$

$$
\langle (C_I)_i{}^{\hat{j}}(x)\ (\bar{C}^J)_{\hat{k}}{}^{l}(y)\rangle^{(1)} = 0 \tag{A.14}
$$

- Vector propagators in Landau gauge

$$
\begin{aligned}
\langle (A_\mu)_i{}^j(x)\ (A_\nu)_k{}^l(y)\rangle^{(0)} &= \delta_i^l\delta_k^j\ i\,\frac{\Gamma(\frac{3}{2}-\epsilon)}{\pi^{\frac{1}{2}-\epsilon}}\ \varepsilon_{\mu\nu\rho}\ \frac{(x-y)^\rho}{|x-y|^{3-2\epsilon}} \\
\langle (\hat{A}_\mu)_{\hat{i}}{}^{\hat{j}}(x)\ (\hat{A}_\nu)_{\hat{k}}{}^{\hat{l}}(y)\rangle^{(0)} &= -\,\delta_{\hat{i}}^{\hat{l}}\delta_{\hat{k}}^{\hat{j}}\ i\,\frac{\Gamma(\frac{3}{2}-\epsilon)}{\pi^{\frac{1}{2}-\epsilon}}\ \varepsilon_{\mu\nu\rho}\ \frac{(x-y)^\rho}{|x-y|^{3-2\epsilon}}
\end{aligned} \tag{A.15}
$$

$$
\begin{aligned}
\langle (A_\mu)_i{}^j(x)\ (A_\nu)_k{}^l(y)\rangle^{(1)} &= \delta_i^l\delta_k^j\ \frac{N_2}{k}\ \frac{\Gamma^2(\frac{1}{2}-\epsilon)}{\pi^{1-2\epsilon}}\left(\frac{\delta_{\mu\nu}}{|x-y|^{2-4\epsilon}} - \partial_\mu\partial_\nu\frac{|x-y|^{4\epsilon}}{4\epsilon(1+2\epsilon)}\right) \\
\langle (\hat{A}_\mu)_{\hat{i}}{}^{\hat{j}}(x)\ (\hat{A}_\nu)_{\hat{k}}{}^{\hat{l}}(y)\rangle^{(1)} &= \delta_i^l\delta_k^j\ \frac{N_1}{k}\ \frac{\Gamma^2(\frac{1}{2}-\epsilon)}{\pi^{1-2\epsilon}}\left(\frac{\delta_{\mu\nu}}{|x-y|^{2-4\epsilon}} - \partial_\mu\partial_\nu\frac{|x-y|^{4\epsilon}}{4\epsilon(1+2\epsilon)}\right)
\end{aligned} \tag{A.16}
$$

- Fermion propagator

$$
\langle (\psi_{\alpha I})_{\hat{i}}{}^{j}(x)\ (\bar{\psi}^{J\beta})^{\hat{l}}{}_{k}(y)\rangle^{(0)} = \delta_I^J\delta_{\hat{i}}^{\hat{l}}\delta_k^j\ i\,\frac{\Gamma(\frac{3}{2}-\epsilon)}{2\pi^{\frac{3}{2}-\epsilon}}\ (\gamma^\mu)_\alpha{}^\beta\ \frac{(x-y)_\mu}{|x-y|^{3-2\epsilon}} \tag{A.17}
$$

$$\langle (\psi_{\alpha I})_{\hat{i}}^{\ j}(x)\ (\bar{\psi}^{J\beta})^{\hat{l}}_{\ k}(y)\rangle^{(1)} = -\delta_I^J \delta_{\hat{i}}^{\hat{l}} \delta_k^j\ \delta_\alpha^{\ \beta} \left(\frac{N_1 - N_2}{k}\right) i\, \frac{\Gamma^2(\frac{1}{2}-\epsilon)}{8\pi^{2-2\epsilon}}\, \frac{1}{|x-y|^{2-4\epsilon}} \tag{A.18}$$

We note that in the ABJ(M) limit, $N_1 = N_2$, the one-loop correction to the fermionic propagator vanishes.

The vertices entering the perturbative calculations of Section 2.3 can be easily read from terms (A.5), (A.8) and (A.10) of the action.

A.3 Euclidean $\mathfrak{osp}(6|4)$ Superalgebra

In Euclidean signature the generators of the bosonic conformal algebra contained in the $\mathfrak{osp}(6|4)$ superalgebra satisfy the following commutation rules

$$\begin{aligned}
&[M^{\mu\nu}, M^{\rho\sigma}] = \delta^{\sigma\mu} M^{\nu\rho} - \delta^{\sigma\nu} M^{\mu\rho} + \delta^{\rho\nu} M^{\mu\sigma} - \delta^{\rho\mu} M^{\nu\sigma} \qquad && [P^\mu, K^\nu] = 2(\delta^{\mu\nu} D + M^{\mu\nu})\\
&[P^\mu, M^{\nu\rho}] = \delta^{\mu\nu} P^\rho - \delta^{\mu\rho} P^\nu && [K^\mu, M^{\nu\rho}] = \delta^{\mu\nu} K^\rho - \delta^{\mu\rho} K^\nu\\
&[D, P^\mu] = P^\mu && [D, K^\mu] = -K^\mu
\end{aligned} \tag{A.19}$$

The $\mathfrak{su}(4) \simeq \mathfrak{so}(6)$ R-symmetry generators $J_I^{\ J}$, with $I, J = 1, \dots, 4$, are traceless matrices that satisfy the relation

$$[J_I^{\ J}, J_K^{\ L}] = \delta_I^L J_K^{\ J} - \delta_K^J J_I^{\ L} \tag{A.20}$$

The fermionic generators Q_α^{IJ}, S_α^{IJ} satisfy the following anticommutation rules

$$\begin{aligned}
&\{Q_\alpha^{IJ}, Q^{KL,\beta}\} = \varepsilon^{IJKL} (\gamma^\mu)_\alpha^{\ \beta} P_\mu \qquad \{S_\alpha^{IJ}, S^{\beta KL}\} = \varepsilon^{IJKL} (\gamma^\mu)_\alpha^{\ \beta} K_\mu\\
&\{Q_\alpha^{IJ}, S^{\beta KL}\} = \varepsilon^{IJKL} \left(\frac{1}{2}(\gamma^{\mu\nu})_\alpha^{\ \beta} M_{\mu\nu} + \delta_\alpha^\beta D\right) + \delta_\alpha^\beta \varepsilon^{KLMN} (\delta_M^J J_N^{\ I} - \delta_M^I J_N^{\ J})
\end{aligned} \tag{A.21}$$

and similarly for $\bar{Q}_{\alpha IJ} = \frac{1}{2}\varepsilon_{IJKL} Q_\alpha^{KL}$ and $\bar{S}_{\alpha IJ} = \frac{1}{2}\varepsilon_{IJKL} S_\alpha^{KL}$.

The full $\mathfrak{osp}(6|4)$ superalgebra is obtained by completing the picture with the mixed commutators

$$\begin{aligned}
&[K^\mu, Q_\alpha^{IJ}] = (\gamma^\mu)_\alpha^{\ \beta} S_\beta^{IJ} && [P^\mu, S_\alpha^{IJ}] = (\gamma^\mu)_\alpha^{\ \beta} Q_\beta^{IJ}\\
&[M^{\mu\nu}, Q_\alpha^{IJ}] = -\frac{1}{2}(\gamma^{\mu\nu})_\alpha^{\ \beta} Q_\beta^{IJ} && [M^{\mu\nu}, S_\alpha^{IJ}] = -\frac{1}{2}(\gamma^{\mu\nu})_\alpha^{\ \beta} S_\beta^{IJ}\\
&[D, Q_\alpha^{IJ}] = \frac{1}{2} Q_\alpha^{IJ} && [D, S^{\alpha IJ}] = -\frac{1}{2} S^{\alpha IJ}\\
&[J_I^{\ J}, Q_\alpha^{KL}] = \delta_I^K Q_\alpha^{JL} + \delta_I^L Q_\alpha^{KJ} - \frac{1}{2}\delta_I^J Q_\alpha^{KL} \quad && [J_I^{\ J}, S^{\alpha KL}] = \delta_I^K S^{\alpha JL} + \delta_I^L S^{\alpha KJ} - \frac{1}{2}\delta_I^J S^{\alpha KL}
\end{aligned} \tag{A.22}$$

The bosonic generators in (A.19), (A.20) are taken to satisfy the following conjugation rules

$$(P^\mu)^\dagger = -K^\mu \qquad (K^\mu)^\dagger = -P^\mu \qquad D^\dagger = D \qquad (M^{\mu\nu})^\dagger = -M^{\mu\nu} \qquad (J_K^{\ L})^\dagger = J_L^{\ K} \tag{A.23}$$

whereas the fermionic ones are subject to the following hermicity conditions

$$(Q_\alpha^{IJ})^\dagger = \frac{1}{2}\varepsilon_{IJKL}\ S^{KL\alpha} = \bar{S}_{IJ}^\alpha \qquad (S_\alpha^{IJ})^\dagger = \frac{1}{2}\varepsilon_{IJKL}\ Q^{KL\alpha} = \bar{Q}_{IJ}^\alpha \tag{A.24}$$

The action of the $\mathfrak{su}(4)$ R-symmetry generators on fields Φ_I ($\bar{\Phi}^I$) in the (anti-)fundamental representation reads

$$[J_I^{\ J}, \Phi_K] = \frac{1}{4}\delta_I^J \Phi_K - \delta_K^J \Phi_I \qquad [J_I^{\ J}, \bar{\Phi}^K] = \delta_I^K \bar{\Phi}^J - \frac{1}{4}\delta_I^J \bar{\Phi}^K \tag{A.25}$$

The full analysis of the relevant multiplets of $\mathfrak{osp}(6|4)$ is discussed in [104].

A.4 The $\mathfrak{su}(1,1|3)$ Superalgebra

In this appendix we describe the immersion of the $\mathfrak{su}(1,1|3)$ superalgebra inside $\mathfrak{osp}(6|4)$ and the classification of its irreducible representations.

A.4.1 Details

The maximal bosonic subalgebra of $\mathfrak{su}(1,1|3)$ is $\mathfrak{sl}(2) \oplus \mathfrak{su}(3) \oplus \mathfrak{u}(1)$, where $\mathfrak{sl}(2) \simeq \mathfrak{su}(1,1)$ is the euclidean conformal algebra in one dimension and $\mathfrak{su}(3) \oplus \mathfrak{u}(1)$ is the R-symmetry algebra. The $\mathfrak{su}(1,1)$ algebra is generated by $\{P \equiv iP_3, K \equiv iK_3, D\}$ satisfying the following commutation relations

$$[D,P] = P, \qquad [D,K] = -K, \qquad [P,K] = -2D. \tag{A.26}$$

The $\mathfrak{su}(3)$ R-symmetry subalgebra is generated by traceless operators ${R_a}^b$, whose explicit form reads

$${R_a}^b = \begin{pmatrix} {J_2}^2 + \frac{1}{3}{J_1}^1 & {J_2}^3 & {J_2}^4 \\ {J_3}^2 & {J_3}^3 + \frac{1}{3}{J_1}^1 & {J_3}^4 \\ {J_4}^2 & {J_4}^3 & -{J_3}^3 - {J_2}^2 - \frac{2}{3}{J_1}^1 \end{pmatrix}. \tag{A.27}$$

These generators satisfy the algebraic relation

$$[{R_a}^b, {R_c}^d] = \delta_a^d {R_c}^b - \delta_c^b {R_a}^d. \tag{A.28}$$

The spectrum of bosonic generators of $\mathfrak{su}(1,1|3)$ is completed by a residual $\mathfrak{u}(1)$ generator M, defined as

$$M \equiv 3iM_{12} - 2{J_1}^1. \tag{A.29}$$

We now move to the fermionic sector of the superalgebra. Since we have placed the line along the x^3-direction, the fermionic generators of the one-dimensional superconformal algebra are identified with the following supercharges

$$Q_1^{12}, Q_1^{13}, Q_1^{14}, Q_2^{23}, Q_2^{24}, Q_2^{34} \qquad \text{and} \qquad S_1^{12}, S_1^{13}, S_1^{14}, S_2^{23}, S_2^{24}, S_2^{34} \tag{A.30}$$

It is useful to rewrite these generators in a more compact way, through the following definitions

$$\begin{aligned} Q^{k-1} &\equiv Q_1^{1k}, \qquad & \bar{Q}_{k-1} &\equiv \frac{i}{2}\,\epsilon_{klm}\,Q_2^{lm}, \\ S^{k-1} &\equiv i\,S_1^{1k}, \qquad & \bar{S}_{k-1} &\equiv \frac{1}{2}\,\epsilon_{klm}\,S_2^{lm}, \qquad k,l,m = 2,3,4 \end{aligned} \tag{A.31}$$

and make the shift $Q^{k-1} \to Q^a$, $\bar{Q}_{k-1} \to \bar{Q}_a$ with $a = 1,2,3$, and similarly for the superconformal charges.

This set of generators inherits the following hermicity conditions

$$\begin{aligned} (Q^a)^\dagger &= \bar{S}_a, \qquad & (\bar{Q}_a)^\dagger &= S^a, \\ (S^a)^\dagger &= \bar{Q}_a, \qquad & (\bar{S}_a)^\dagger &= Q^a, \end{aligned} \qquad a = 1,2,3 \tag{A.32}$$

and the following anti-commutation relations

$$\begin{aligned} \{Q^a, \bar{Q}_b\} &= \delta_b^a\,P & \{S^a, \bar{S}_b\} &= \delta_b^a\,K \\ \{Q^a, \bar{S}_b\} &= \delta_b^a\left(D + \frac{1}{3}M\right) - {R_b}^a \quad & \{\bar{Q}_a, S^b\} &= \delta_a^b\left(D - \frac{1}{3}M\right) + {R_a}^b \end{aligned} \tag{A.33}$$

together with the mixed commutation rules

$$
\begin{array}{llll}
[D,Q^a]=\frac{1}{2}Q^a & [K,Q^a]=S^a & [R_a{}^b,Q^c]=\delta_a^cQ^b-\frac{1}{3}\delta_a^bQ^c & [M,Q^a]=\frac{1}{2}Q^a \\
[D,\bar{Q}_a]=\frac{1}{2}\bar{Q}_a & [K,\bar{Q}_a]=\bar{S}_a & [R_a{}^b,\bar{Q}_c]=-\delta_c^b\bar{Q}_a+\frac{1}{3}\delta_a^b\bar{Q}_c & [M,\bar{Q}_a]=-\frac{1}{2}\bar{Q}_a \\
[D,S^a]=-\frac{1}{2}S^a & [P,S^a]=-Q^a & [R_a{}^b,S^c]=\delta_a^cS^b-\frac{1}{3}\delta_a^bS^c & [M,S^a]=\frac{1}{2}S^a \\
[D,\bar{S}_a]=-\frac{1}{2}\bar{S}_a & [P,\bar{S}_a]=-\bar{Q}_a & [R_a{}^b,\bar{S}_c]=-\delta_c^b\bar{S}_b+\frac{1}{3}\delta_a^b\bar{S}_c & [M,\bar{S}_a]=-\frac{1}{2}\bar{S}_a
\end{array}
\tag{A.34}
$$

From eq. (A.25) and definitions (A.27) it follows that the action of the $SU(3)$ R-symmetry generators on fields in the (anti-)fundamental representation is

$$
[R_a{}^b,\Phi_c]=\frac{1}{3}\delta_a^b\Phi_c-\delta_c^b\Phi_a, \qquad [R_a{}^b,\bar{\Phi}^c]=\delta_a^c\bar{\Phi}^b-\frac{1}{3}\delta_a^b\bar{\Phi}^c. \tag{A.35}
$$

Notice that, as we introduced in the main text, the decomposition of the three-dimensional superalgebra to the one-dimensional one comprehend a decoupled $\mathfrak{u}(1)_b$ factor generated by

$$
B=M_{12}+iJ_1{}^1. \tag{A.36}
$$

A.4.2 Irreducible Representations

In this appendix, we shall briefly review the classification of the multiplet of $\mathfrak{su}(1,1|3)$ presented in [121]. We shall classify the states in terms of the four Dynkin labels $[\Delta,m,j_1,j_2]$ associated to the bosonic subalgebra $\mathfrak{su}(1,1)\oplus\mathfrak{su}(3)\oplus\mathfrak{u}(1)$. Here Δ stands for the conformal weight, m for the $\mathfrak{u}(1)$ charge and (j_1,j_2) are the eigenvalues corresponding to the two $\mathfrak{su}(3)$ Cartan generators J_1 and J_2. We choose

$$
\begin{aligned}
J_1 &\equiv \frac{R_2{}^2-R_1{}^1}{2}=-\frac{2R_1{}^1+R_3{}^3}{2}, \\
J_2 &\equiv \frac{R_3{}^3-R_2{}^2}{2}=\frac{R_1{}^1+2R_3{}^3}{2},
\end{aligned}
\tag{A.37}
$$

where we have exploited the traceless property $R_a{}^a=0$ to remove the dependence on $R_2{}^2$. The commutations rules (A.28) implies that we can associate an $\mathfrak{sl}(2)$ subalgebra with each Cartan generator. In fact, the two sets of operators

$$
\{R_2{}^1,R_1{}^2,J_1\}\equiv\{E_1^-,E_1^+,J_1\} \quad , \quad \{R_3{}^2,R_2{}^3,J_2\}\equiv\{E_2^-,E_2^+,J_2\} \tag{A.38}
$$

satisfy the following algebraic relations

$$
[E_i^+,E_i^-]=2J_i \qquad [J_i,E_i^\pm]=\pm E_i^\pm \qquad i=1,2 \tag{A.39}
$$

and define the raising and lowering operators used to construct the representations of $\mathfrak{su}(3)$. In the main text, we have chosen a different $\mathfrak{sl}(2)$ to define the twisted algebra. We have preferred to use the one generated by $\{R_3{}^1,R_1{}^3,-J_1-J_2\}$, which treats the two Dynkin labels (j_1,j_2) symmetrically. Moreover, the supercharges with this choice of basis possess well-defined Dynkin labels, whose values are displayed in Table A.1.

When localized on the line, the ABJ(M) fundamental fields also have definite quantum numbers with respect to $\mathfrak{su}(1,1)\oplus\mathfrak{su}(3)\oplus\mathfrak{u}(1)$. Their values are listed in Table A.2 for the scalar fields and in Table A.3 for the fermionic ones.

Generators	$[\Delta, m, j_1, j_2]$	
Q^1 $\bar{Q}_1$	$\left[\frac{1}{2}, \frac{1}{2}, -1, 0\right]$	$\left[\frac{1}{2}, -\frac{1}{2}, 1, 0\right]$
Q^2 $\bar{Q}_2$	$\left[\frac{1}{2}, \frac{1}{2}, 1, -1\right]$	$\left[\frac{1}{2}, -\frac{1}{2}, -1, 1\right]$
Q^3 $\bar{Q}_3$	$\left[\frac{1}{2}, \frac{1}{2}, 0, 1\right]$	$\left[\frac{1}{2}, -\frac{1}{2}, 0, -1\right]$
S^1 $\bar{S}_1$	$\left[-\frac{1}{2}, \frac{1}{2}, -1, 0\right]$	$\left[-\frac{1}{2}, -\frac{1}{2}, 1, 0\right]$
S^2 $\bar{S}_2$	$\left[-\frac{1}{2}, \frac{1}{2}, 1, -1\right]$	$\left[-\frac{1}{2}, -\frac{1}{2}, -1, 1\right]$
S^3 $\bar{S}_3$	$\left[-\frac{1}{2}, \frac{1}{2}, 0, 1\right]$	$\left[-\frac{1}{2}, -\frac{1}{2}, 0, -1\right]$

Table A.1: Table of Dynkin labels of fermionic generators. For a generic element v_μ transforming in a weight-μ representation, the Dynkin label corresponding to a generator H_i of the Cartan subalgebra is defined as $j_i(v_\mu) \equiv 2[H_i, v_\mu]$.

Scalar fields	$[\Delta, m, j_1, j_2]$	
Z , $\bar{Z}$	$\left[\frac{1}{2}, \frac{3}{2}, 0, 0\right]$	$\left[\frac{1}{2}, -\frac{3}{2}, 0, 0\right]$
Y_1 , $\bar{Y}^1$	$\left[\frac{1}{2}, -\frac{1}{2}, 1, 0\right]$	$\left[\frac{1}{2}, \frac{1}{2}, -1, 0\right]$
Y_2 , $\bar{Y}^2$	$\left[\frac{1}{2}, -\frac{1}{2}, -1, 1\right]$	$\left[\frac{1}{2}, \frac{1}{2}, 1, -1\right]$
Y_3 , $\bar{Y}^3$	$\left[\frac{1}{2}, -\frac{1}{2}, 0, -1\right]$	$\left[\frac{1}{2}, \frac{1}{2}, 0, 1\right]$

Table A.2: Quantum number assignments to scalar matter fields of the ABJ(M) theory defined in eq. (2.12).

Fermionic fields	$[\Delta, m, j_1, j_2]$	
$(\psi)_1$, $(\psi)_2$	$[1, 3, 0, 0]$	$[1, 0, 0, 0]$
$(\bar{\psi})_1$, $(\bar{\psi})_2$	$[1, 0, 0, 0]$	$[1, -3, 0, 0]$
$(\chi_1)_1$, $(\chi_1)_2$	$[1, 1, 1, 0]$	$[1, -2, 1, 0]$
$(\bar{\chi}^1)_1$, $(\bar{\chi}^1)_2$	$[1, 2, -1, 0]$	$[1, -1, -1, 0]$
$(\chi_2)_1$, $(\chi_2)_2$	$[1, 1, -1, 1]$	$[1, -2, -1, 1]$
$(\bar{\chi}^2)_1$, $(\bar{\chi}^2)_2$	$[1, 2, 1, -1]$	$[1, -1, 1, -1]$
$(\chi_3)_1$, $(\chi_3)_2$	$[1, 1, 0, -1]$	$[1, -2, 0, -1]$
$(\bar{\chi}^3)_1$, $(\bar{\chi}^3)_2$	$[1, 2, 0, 1]$	$[1, -1, 0, 1]$

Table A.3: Quantum number assignments to fermionic matter fields of the ABJ(M) theory defined in eq. (2.12).

Finally we do not consider directly the gauge fields, but their covariant derivatives. Their Dynkin labels are given by

$$D\ [1, 3, 0, 0] \qquad \bar{D}\ [1, -3, 0, 0] \qquad D_3\ [1, 0, 0, 0] \tag{A.40}$$

Therefore their action on an operator that is an eigenstate $|\Delta, m, j_1, j_2\rangle$ of the Cartan generators simply shifts the the first two quantum numbers. Next we summarize the relevant superconformal multiplets constructed in [121].

The $\mathcal{A}$ Multiplets

We start with the so-called long multiplets, denoted by $\mathcal{A}^{\Delta}_{m;j_1,j_2}$. Their highest weight of the representations, namely the super-conformal primary (SCP), is identified by requiring that

$$S^a\,|\Delta, m, j_1, j_2\rangle^{\text{hw}} = 0 \qquad \bar{S}_a\,|\Delta, m, j_1, j_2\rangle^{\text{hw}} = 0 \qquad E_a^+\,|\Delta, m, j_1, j_2\rangle^{\text{hw}} = 0 \tag{A.41}$$

Then the entire multiplet is built by acting with the supercharges Q^a and $\bar{Q}_a$. For unitary representations, the Dynkin label of the highest weight are constrained by the following inequalities

$$\Delta \geq \begin{cases} \frac{1}{3}(2j_2 + j_1 - m), & m < \frac{j_2 - j_1}{2} \\ \frac{1}{3}(j_2 + 2j_1 + m), & m \geq \frac{j_2 - j_1}{2} \end{cases} \tag{A.42}$$

At the threshold of the unitary region, these multiplets split into shorter ones because of the recombination phenomenon. For $m < \frac{j_2 - j_1}{2}$ the unitarity bound is for $\Delta = \frac{1}{3}(2j_2 + j_1 - m)$ and one can verify that

$$\mathcal{A}^{-\frac{1}{3}m + \frac{1}{3}j_1 + \frac{2}{3}j_2}_{m,j_1,j_2} = \mathcal{B}^{\frac{1}{6},0}_{m,j_1,j_2} \oplus \mathcal{B}^{\frac{1}{6},0}_{m+\frac{1}{2},j_1,j_2+1} \tag{A.43}$$

Equivalently, for $m > \frac{j_2 - j_1}{2}$ one has

$$\mathcal{A}^{\frac{1}{3}m + \frac{2}{3}j_1 + \frac{1}{3}j_2}_{m,j_1,j_2} = \mathcal{B}^{0,\frac{1}{6}}_{m,j_1,j_2} \oplus \mathcal{B}^{0,\frac{1}{6}}_{m-\frac{1}{2},j_1+1,j_2} \tag{A.44}$$

For the particular case $m = \frac{j_2 - j_1}{2}$ we have

$$\mathcal{A}^{\frac{j_2+j_1}{2}}_{\frac{j_2-j_1}{2};j_1,j_2} = \mathcal{B}^{\frac{1}{6},\frac{1}{6}}_{\frac{j_2-j_1}{2};j_1,j_2} \oplus \mathcal{B}^{\frac{1}{6},\frac{1}{6}}_{\frac{j_2-j_1}{2}+\frac{1}{2};j_1,j_2+1} \oplus \mathcal{B}^{\frac{1}{6},\frac{1}{6}}_{\frac{j_2-j_1}{2}-\frac{1}{2};j_1+1,j_2+1} \oplus \mathcal{B}^{\frac{1}{6},\frac{1}{6}}_{\frac{j_2-j_1}{2};j_1+1,j_2+1}. \tag{A.45}$$

Above the symbols $\mathcal{B}^{\frac{1}{N},\frac{1}{M}}_{m;j_1,j_2}$ stand for a type of short multiplets (see below). The two superscripts denote respectively the fraction of Q and $\bar{Q}$ charges with respect to the total number of charges $(Q + \bar{Q})$, which annihilates the super-conformal primary.

The $\mathcal{B}$ Multiplets

Let us now have a closer look to short multiplets. They are obtained by imposing that the highest weight is annihilated by some of the Q and $\bar{Q}$ charges. First we consider the case

$$Q^a\,|\Delta, m, j_1, j_2\rangle^{\text{hw}} = 0 \tag{A.46}$$

from which we get three possible short supermultiplets

$$a = 3 \qquad \Delta = \frac{1}{3}(j_1 + 2j_2 - m) \qquad \mathcal{B}^{\frac{1}{6},0}_{m;j_1,j_2} \tag{A.47}$$

$$a = 3,2 \qquad \Delta = \frac{1}{3}(j_1 - m), \quad j_2 = 0 \qquad \mathcal{B}^{\frac{1}{3},0}_{m;j_1,0} \tag{A.48}$$

$$a = 3,2,1 \qquad \Delta = -\frac{1}{3}m, \quad j_1 = j_2 = 0 \qquad \mathcal{B}^{\frac{1}{2},0}_{m;0,0} \tag{A.49}$$

according to the number of charges obeying the condition (A.46). Obviously we can also consider the conjugate shortening condition

$$\bar{Q}_a |\Delta, m, j_1, j_2\rangle^{\rm hw} = 0 \tag{A.50}$$

which yields short multiplets conjugate to the ones considered above

$$a = 1 \qquad \Delta = \frac{1}{3}(j_2 + 2j_1 + m) \qquad \mathcal{B}^{0,\frac{1}{6}}_{m;j_1,j_2} \tag{A.51}$$

$$a = 1,2 \qquad \Delta = \frac{1}{3}(j_2 + m), \quad j_1 = 0 \qquad \mathcal{B}^{0,\frac{1}{3}}_{m;0,j_2} \tag{A.52}$$

$$a = 1,2,3 \qquad \Delta = \frac{1}{3}m, \quad j_1 = j_2 = 0 \qquad \mathcal{B}^{0,\frac{1}{2}}_{m;0,0} \tag{A.53}$$

Finally we may have mixed multiplets where the highest weight is annihilated both by Q^a and $\bar{Q}_a$. Those include

$$\mathcal{B}^{\frac{1}{6},\frac{1}{6}}_{m;j_1,j_2} \qquad \Delta = \frac{j_2 + j_1}{2} \qquad m = \frac{j_2 - j_1}{2} \tag{A.54}$$

$$\mathcal{B}^{\frac{1}{3},\frac{1}{6}}_{m;j_1,0} \qquad \Delta = \frac{j_1}{2} \qquad m = -\frac{j_1}{2} \qquad j_2 = 0 \tag{A.55}$$

$$\mathcal{B}^{\frac{1}{6},\frac{1}{3}}_{m;0,j_2} \qquad \Delta = \frac{j_2}{2} \qquad m = \frac{j_2}{2} \qquad j_1 = 0 \tag{A.56}$$

A.5 Supersymmetry Transformations

Here we list all the supersymmetry transformations, using both the original $SU(4)$ formalism and the $SU(3)$ one, which are relevant for the construction of the twisted superalgebra, its irreducible representations and multiplets.

A.5.1 $SU(4)$ Notations

The ABJ(M) action in (A.4) is invariant under the following superconformal transformations

$$\begin{aligned}
\delta C_K &= -\bar{\zeta}^{IJ,\alpha}\,\varepsilon_{IJKL}\,\bar{\psi}^L_\alpha \\
\delta \bar{C}^K &= 2\bar{\zeta}^{KL,\alpha}\,\psi_{L,\alpha} \\
\delta \bar{\psi}^{K,\beta} &= 2i\bar{\zeta}^{KL,\alpha}(\gamma^\mu)_\alpha{}^\beta D_\mu C_L + \frac{4\pi i}{k}\bar{\zeta}^{KL,\beta}(C_L\bar{C}^M C_M - C_M\bar{C}^M C_L) + \frac{8\pi i}{k}\bar{\zeta}^{IJ,\beta}C_I\bar{C}^K C_J \\
&\quad + 2i\bar{\epsilon}^{KL,\beta}C_L \\
\delta \psi^\beta_K &= -i\bar{\zeta}^{IJ,\alpha}\varepsilon_{IJKL}(\gamma^\mu)_\alpha{}^\beta D_\mu \bar{C}^L + \frac{2\pi i}{k}\bar{\zeta}^{IJ,\beta}\varepsilon_{IJKL}(\bar{C}^L C_M\bar{C}^M - \bar{C}^M C_M\bar{C}^L) \\
&\quad + \frac{4\pi i}{k}\bar{\zeta}^{IJ,\beta}\varepsilon_{IJML}\bar{C}^M C_K\bar{C}^L - i\bar{\epsilon}^{IJ,\beta}\varepsilon_{IJKL}\bar{C}^L \\
\delta A_\mu &= \frac{4\pi i}{k}\bar{\zeta}^{IJ,\alpha}(\gamma_\mu)_\alpha{}^\beta\left(C_I\psi_{J\beta} - \frac{1}{2}\varepsilon_{IJKL}\bar{\psi}^K_\beta\bar{C}^L\right) \\
\delta \hat{A}_\mu &= \frac{4\pi i}{k}\bar{\zeta}^{IJ,\alpha}(\gamma_\mu)_\alpha{}^\beta\left(\psi_{J\beta}C_I - \frac{1}{2}\varepsilon_{IJKL}\bar{C}^L\bar{\psi}^K_\beta\right)
\end{aligned} \tag{A.57}$$

where the parameters of the transformations are expressed in terms of supersymmetry and superconformal parameters as

$$\bar{\zeta}^{IJ}_\alpha = \bar{\Theta}^{IJ}_\alpha - x^\mu(\gamma_\mu)^\beta_\alpha\bar{\epsilon}^{IJ}_\beta \tag{A.58}$$

We recall that they satisfy $\bar{\zeta}^{IJ} = -\bar{\zeta}^{JI}$, and are subject to the reality conditions $\bar{\zeta}^{IJ} = (\zeta_{IJ})^*$ with $\zeta_{IJ} = \frac{1}{2}\epsilon_{IJKL}\bar{\zeta}^{KL}$.

If we set $\bar{\epsilon}^{IJ} = 0$ in (A.57) we obtain $\mathcal{N} = 6$ supersymmetry transformations. Expressing them as

$$\delta\Phi = [\bar{\Theta}^{IJ}\bar{Q}_{IJ}, \Phi] = [\Theta_{IJ}Q^{IJ}, \Phi] \tag{A.59}$$

for a generic field Φ, it is easy to realize that the Q^{IJ} supercharges (or equivalently $\bar{Q}_{IJ}$) satisfy the $\mathfrak{osp}(6|4)$ algebra (A.21) under the identification $P_\mu = i\partial_\mu$.

A.5.2 $SU(3)$ Notations

The generic supersymmetry transformation defined in (A.59) can be specialized to the $\mathfrak{su}(1,1|3)$ supercharges $(Q^a, \bar{Q}_a)$ defined in (A.31,A.33). For a generic field $\tilde{\Phi}$ in a given representation of the $\mathfrak{su}(3)$ R-symmetry algebra it reduces to

$$\delta\tilde{\Phi} = [\theta_a Q^a + \bar{\theta}^a\bar{Q}_a, \tilde{\Phi}] \tag{A.60}$$

under the parameter identification

$$\theta_a = 2\,\Theta^1_{1(a+1)} \tag{A.61}$$

$$\bar{\theta}^a = -i\;\varepsilon^{a+1,b+1,c+1}\Theta^2_{b+1,c+1} \qquad a,b,c = 1,2,3 \tag{A.62}$$

From the variations in (A.57) we can easily read the supersymmetry transformations of the ABJ(M) fundamental fields reorganized in $\mathfrak{su}(3)$ R-symmetry representations (see eqs. (2.12) and (2.13)). Comparing these transformations with the general variation defined in (A.60) we obtain the action of the supercharges on the fields, which takes the following form

- Scalar fields

$$\begin{aligned} &Q^a Z = -\bar{\chi}^a_1 && \bar{Q}_a Z = 0 && Q^a\bar{Z} = 0 && \bar{Q}_a\bar{Z} = i\chi^1_a \\ &Q^a Y_b = \delta^a_b\bar{\psi}_1 && \bar{Q}_a Y_b = -i\epsilon_{abc}\bar{\chi}^c_2 && Q^a\bar{Y}^b = -\epsilon^{abc}\chi^2_c && \bar{Q}_a\bar{Y}^b = -i\delta^b_a\psi^1 \end{aligned} \tag{A.63}$$

- Fermions

$$\bar{Q}_a\psi^1 = 0 \qquad Q^a\psi^1 = -iD_3\bar{Y}^a - \frac{2\pi i}{k}\left(\bar{Y}^a l_B - \hat{l}_B\bar{Y}^a\right) \tag{A.64a}$$

$$Q^a\psi^2 = -iD\bar{Y}^a \qquad \bar{Q}_a\psi^2 = -\frac{4\pi}{k}\epsilon_{abc}\bar{Y}^b Z\bar{Y}^c \tag{A.64b}$$

$$\bar{Q}_a\chi^1_b = \epsilon_{abc}\,\bar{D}\bar{Y}^c \qquad Q^a\chi^1_b = i\delta^a_b D_3\bar{Z} + \frac{4\pi i}{k}\left(\bar{Z}\Lambda^a_b - \hat{\Lambda}^a_b\bar{Z}\right) \tag{A.64c}$$

$$Q^a\chi^2_b = i\delta^a_b\,D\bar{Z} \qquad \bar{Q}_a\chi^2_b = -\epsilon_{abc}D_3\bar{Y}^c - \frac{2\pi}{k}\epsilon_{acd}\left(\bar{Y}^c\Theta^d_b - \hat{\Theta}^d_b\bar{Y}^c\right) \tag{A.64d}$$

$$Q^a\bar{\psi}_1 = 0 \qquad \bar{Q}_a\bar{\psi}_1 = -D_3 Y_a - \frac{2\pi}{k}\left(Y_a\hat{l}_B - l_B Y_a\right) \tag{A.64e}$$

$$\bar{Q}_a\bar{\psi}_2 = -\bar{D}Y_a \qquad Q^a\bar{\psi}_2 = \frac{4\pi i}{k}\epsilon^{abc}Y_b\bar{Z}Y_c \tag{A.64f}$$

$$Q^a\bar{\chi}^b_1 = -i\epsilon^{abc}\,DY_c \qquad \bar{Q}_a\bar{\chi}^b_1 = \delta^a_b D_3 Z + \frac{4\pi}{k}\left(Z\hat{\Lambda}^a_b - \Lambda^a_b Z\right) \tag{A.64g}$$

$$\bar{Q}_a\bar{\chi}^b_2 = \delta^b_a\,\bar{D}Z \qquad Q^a\bar{\chi}^b_2 = i\epsilon^{abc}D_3 Y_c + \frac{2\pi i}{k}\epsilon^{acd}\left(Y_c\hat{\Theta}^b_d - \Theta^b_d Y_c\right) \tag{A.64h}$$

- Gauge fields

$$
\begin{aligned}
Q^a A_3 &= -\frac{2\pi i}{k}\left(\bar\psi_1 \bar Y^a - \bar\chi_1^a \bar Z + \epsilon^{abc} Y_b \chi_c^2\right) & \bar Q_a A_3 &= \frac{2\pi}{k}\left(Z\chi_a^1 - Y_a\psi^1 - \epsilon_{abc}\bar\chi_2^b \bar Y^c\right)\\
Q^a A &= 0 & \bar Q_a A \ &= -\frac{4\pi}{k}\left(Y_a\psi^2 - Z\chi_a^2 - \epsilon_{abc}\bar\chi_1^b \bar Y^c\right)\\
Q^a \bar A &= -\frac{4\pi i}{k}\left(\bar\psi_2 \bar Y^a - \bar\chi_2^a \bar Z - \epsilon^{abc} Y_b \chi_c^1\right) & \bar Q_a \bar A \ &= 0\\
Q^a \hat A_3 &= -\frac{2\pi i}{k}\left(\bar Y^a \bar\psi_1 - \bar Z \bar\chi_1^a + \epsilon^{abc} \chi_c^2 Y_b\right) & \bar Q_a \hat A_3 &= \frac{2\pi}{k}\left(\chi_a^1 Z - \psi^1 Y_a - \epsilon_{abc}\bar Y^c \bar\chi_2^b\right)\\
Q^a \hat A &= 0 & \bar Q_a \hat A \ &= \frac{4\pi}{k}\left(\psi^2 Y_a - \chi_a^2 Z - \epsilon_{abc}\bar Y^c \bar\chi_1^b\right)\\
Q^a \hat{\bar A} &= -\frac{4\pi i}{k}\left(\bar Y^a \bar\psi_2 - \bar Z \bar\chi_2^a - \epsilon^{abc} \chi_c^1 Y_b\right) & \bar Q_a \hat{\bar A} \ &= 0
\end{aligned}
\tag{A.65}
$$

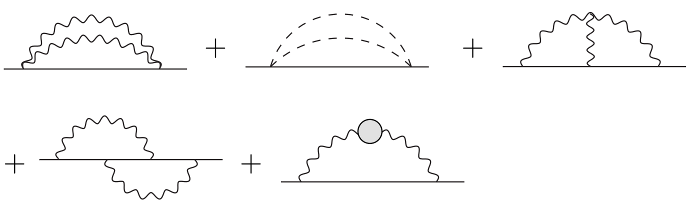
where we have defined the bilinear scalar fields

$$
\begin{aligned}
\begin{pmatrix} \Lambda_a^b & 0 \\ 0 & \hat\Lambda_a^b \end{pmatrix} &= \begin{pmatrix} Y_a \bar Y^b + \frac{1}{2}\delta_a^b l_B & 0 \\ 0 & \bar Y^b Y_a + \frac{1}{2}\delta_a^b \hat l_B \end{pmatrix}\\
\begin{pmatrix} \Theta_a^b & 0 \\ 0 & \hat\Theta_a^b \end{pmatrix} &= \begin{pmatrix} Y_a \bar Y^b - \delta_a^b (Z\bar Z + Y_c \bar Y^c) & 0 \\ 0 & \bar Y^b Y_a - \delta_a^b (\bar Z Z + \bar Y^c Y_c) \end{pmatrix}\\
\begin{pmatrix} l_B & 0 \\ 0 & \hat l_B \end{pmatrix} &= \begin{pmatrix} Z\bar Z - Y_c \bar Y^c & 0 \\ 0 & \bar Z Z - \bar Y^c Y_c \end{pmatrix}
\end{aligned}
\tag{A.66}
$$

A.6 Two-loop Integrals

In this appendix we list the integrals corresponding to the two-loop diagrams in figures 2.2(a)-2.2(l), dressed by their color factors.
Diagram 2.2(a) contains the two-loop correction to the scalar propagator. This has been computed in [94] and reads

$$\mathcal{C}(N_1, N_2) \equiv$$



$$
\begin{aligned}
&= \frac{N_1 N_2}{k^2}\left(N_1^2 + N_2^2 - 4N_1N_2 + 2\right)\left(\frac{\pi}{3\epsilon} + 2\pi + O(\epsilon)\right)\\
&+ \frac{N_1 N_2}{k^2}\left(N_1^2 + N_2^2 - 2\right)\left(-\frac{4\pi}{3\epsilon} + \pi\left(\pi^2 - 8\right) + O(\epsilon)\right)\\
&+ \frac{N_1 N_2}{k^2}\left(N_1N_2 - 1\right)\left(-\frac{8\pi}{3\epsilon} + 4\pi(\pi^2 - 20\pi) + O(\epsilon)\right)
\end{aligned}
\tag{A.67}
$$

To compute the contributions of the other diagrams it is sufficient to rely on Feynman rules listed in appendix A.2, together with the product of polarization vectors. Explicitly, we find

$$
(2.2(\text{b})) = -s^2 \frac{\Gamma^6\left(\frac{1}{2}-\epsilon\right)}{32\,\pi^{7-6\epsilon}} \frac{N_1^2 N_2^2}{k^2} \int d^dx d^dy \frac{x^\mu y^\nu}{(x^2)^{\frac{3}{2}-\epsilon}(y^2)^{\frac{3}{2}-\epsilon}((x-s)^2)^{\frac{1}{2}-\epsilon}((y-s)^2)^{\frac{1}{2}-\epsilon}} \times \left[\frac{\delta_{\mu\nu}}{\left[(x-y)^2\right]^{1-2\epsilon}} - \partial_\mu \partial_\nu \frac{\left[(x-y)^2\right]^{2\epsilon}}{4\epsilon(1+2\epsilon)} \right] \tag{A.68}
$$

$$
(2.2(\text{c})) = s^2 \frac{\Gamma^4\left(\frac{1}{2}-\epsilon\right)\Gamma^2\left(\frac{3}{2}-\epsilon\right)}{128\,\pi^{7-6\epsilon}} \frac{N_1 N_2}{k^2}((N_1-N_2)^2 - 2N_1N_2 + 2) \times \int \frac{d^dx d^dy}{(x^2)^{\frac{1}{2}-\epsilon}(y^2)^{\frac{1}{2}-\epsilon}((x-y)^2)^{2-2\epsilon}((x-s)^2)^{\frac{1}{2}-\epsilon}((y-s)^2)^{\frac{1}{2}-\epsilon}} \tag{A.69}
$$

$$
(2.2(\text{d})) = s^2 \frac{\Gamma^6\left(\frac{1}{2}-\epsilon\right)\Gamma^2\left(\frac{3}{2}-\epsilon\right)}{256\,\pi^{10-8\epsilon}} \frac{N_1 N_2}{k^2} (N_1-N_2)^2 \; \varepsilon_{\mu\nu\eta}\varepsilon_{\rho\sigma\tau} \times \int d^dx d^dy d^dz d^dw \frac{(x-y)^\eta (z-w)^\tau}{((x-y)^2)^{\frac{3}{2}-\epsilon}((z-w)^2)^{\frac{3}{2}-\epsilon}((x-s)^2)^{\frac{1}{2}-\epsilon}((y-s)^2)^{\frac{1}{2}-\epsilon}(z^2)^{\frac{1}{2}-\epsilon}(w^2)^{\frac{1}{2}-\epsilon}} \times \partial^\mu \partial^\rho \frac{1}{((x-z)^2)^{\frac{1}{2}-\epsilon}} \partial^\nu \partial^\sigma \frac{1}{((y-w)^2)^{\frac{1}{2}-\epsilon}} \tag{A.70}
$$

$$
(2.2(\text{e})) = -s^2 \frac{\Gamma^6\left(\frac{1}{2}-\epsilon\right)\Gamma^2\left(\frac{3}{2}-\epsilon\right)}{128\,\pi^{10-8\epsilon}} \frac{N_1 N_2}{k^2} (N_1N_2-1) \; \varepsilon_{\mu\nu\eta}\varepsilon_{\rho\sigma\tau} \times \int d^dx d^dy d^dz d^dw \frac{(x-y)^\eta (z-w)^\tau}{((x-y)^2)^{\frac{3}{2}-\epsilon}((z-w)^2)^{\frac{3}{2}-\epsilon}((x-s)^2)^{\frac{1}{2}-\epsilon}((w-s)^2)^{\frac{1}{2}-\epsilon}(y^2)^{\frac{1}{2}-\epsilon}(z^2)^{\frac{1}{2}-\epsilon}} \times \partial^\mu \partial^\rho \frac{1}{((x-z)^2)^{\frac{1}{2}-\epsilon}} \partial^\nu \partial^\sigma \frac{1}{((y-w)^2)^{\frac{1}{2}-\epsilon}} \tag{A.71}
$$

$$
(2.2(\text{f})) = s^2 \frac{\Gamma^4\left(\frac{1}{2}-\epsilon\right)\Gamma^2\left(\frac{3}{2}-\epsilon\right)}{16\,\pi^{7-6\epsilon}} \frac{N_1 N_2}{k^2}(N_1N_2-1) \times \int \frac{d^dx d^dy}{(x^2)^{\frac{1}{2}-\epsilon}(y^2)^{\frac{1}{2}-\epsilon}((x-y)^2)^{2-2\epsilon}((x-s)^2)^{\frac{1}{2}-\epsilon}((y-s)^2)^{\frac{1}{2}-\epsilon}} \tag{A.72}
$$

We note that in the large N_1, N_2 approximation we obtain (2.2(f)) $= -4$(2.2(c)), in agreement with the results in [122].

$$
(2.2(\text{g})) = -s^2 \frac{\Gamma^5\left(\frac{1}{2}-\epsilon\right)\Gamma^2\left(\frac{3}{2}-\epsilon\right)}{128\,\pi^{\frac{17}{2}-7\epsilon}} \frac{N_1 N_2}{k^2}(N_1^2+N_2^2-4N_1N_2+2)\; \varepsilon_{\mu\rho\sigma}\varepsilon_{\mu\nu\eta} \times \int d^dx d^dy d^dz \frac{(x-z)^\sigma}{((x-z)^2)^{\frac{3}{2}-\epsilon}} \frac{(x-y)^\eta}{((x-y)^2)^{\frac{3}{2}-\epsilon}} \times \partial^\rho \frac{1}{((y-z)^2)^{\frac{1}{2}-\epsilon}} \partial^\nu \frac{1}{((y-s)^2)^{\frac{1}{2}-\epsilon}} \frac{1}{(x^2)^{\frac{1}{2}-\epsilon}(z^2)^{\frac{1}{2}-\epsilon}((x-s)^2)^{\frac{1}{2}-\epsilon}} \tag{A.73}
$$

$$(2.2(h)) = 0 \qquad\qquad (2.2(i)) = 0 \tag{A.74}$$

$$\begin{aligned}(2.2(j)) = s^2\, &\frac{\Gamma^5\left(\frac{1}{2}-\epsilon\right)\Gamma^3\left(\frac{3}{2}-\epsilon\right)}{128\,\pi^{10-8\epsilon}}\,\frac{N_1N_2}{k^2}(N_1^2+N_2^2-2)\,\varepsilon_{\rho\nu\tau}\varepsilon_{\rho\eta\sigma}\varepsilon_{\nu\mu\varphi}\varepsilon_{\tau\chi\xi}\times\\ \int d^dxd^dyd^dzd^dw&\frac{(x-z)^\varphi(y-z)^\xi(w-z)^\sigma}{((x-z)^2)^{\frac{3}{2}-\epsilon}((y-z)^2)^{\frac{3}{2}-\epsilon}((w-z)^2)^{\frac{3}{2}-\epsilon}}\\ &\times\,\partial^\eta\frac{1}{((w-s)^2)^{\frac{1}{2}-\epsilon}}\,\partial^\chi\frac{1}{((x-y)^2)^{\frac{1}{2}-\epsilon}}\,\partial^\mu\frac{1}{((x-s)^2)^{\frac{1}{2}-\epsilon}}\,\frac{1}{(y^2)^{\frac{1}{2}-\epsilon}(w^2)^{\frac{1}{2}-\epsilon}}\end{aligned} \tag{A.75}$$

$$\begin{aligned}(2.2(k)) = s^2\, &\frac{\Gamma^6\left(\frac{1}{2}-\epsilon\right)\Gamma^2\left(\frac{3}{2}-\epsilon\right)}{256\,\pi^{10-8\epsilon}}\,\frac{N_1N_2}{k^2}(N_1N_2-2)\,\varepsilon_{\mu\nu\epsilon}\varepsilon_{\rho\sigma\tau}\times\\ &\int d^dxd^dyd^dzd^dw\frac{(x-y)^\epsilon(z-w)^\tau}{((x-y)^2)^{\frac{3}{2}-\epsilon}((z-w)^2)^{\frac{3}{2}-\epsilon}}\,\frac{1}{((w-s)^2)^{\frac{1}{2}-\epsilon}}\\ &\times\,\partial^\rho\frac{1}{((x-z)^2)^{\frac{1}{2}-\epsilon}}\,\partial^\nu\frac{1}{((y-z)^2)^{\frac{1}{2}-\epsilon}}\partial^\sigma\frac{1}{(w^2)^{\frac{1}{2}-\epsilon}}\\ &\qquad\times\left[\partial^\mu\frac{1}{((x-s)^2)^{\frac{1}{2}-\epsilon}}\,\frac{1}{(y^2)^{\frac{1}{2}-\epsilon}}-\partial^\mu\frac{1}{(x^2)^{\frac{1}{2}-\epsilon}}\,\frac{1}{((y-s)^2)^{\frac{1}{2}-\epsilon}}\right]\end{aligned} \tag{A.76}$$

$$\begin{aligned}(2.2(l)) = -s^2\, &\frac{\Gamma^4\left(\frac{1}{2}-\epsilon\right)\Gamma^2\left(\frac{3}{2}-\epsilon\right)}{32\,\pi^{7-6\epsilon}}\frac{N_1N_2}{k^2}(N_1-N_2)^2\\ &\times\int\frac{d^dxd^dy}{((x-s)^2)^{1-2\epsilon}((x-y)^2)^{2-2\epsilon}(y^2)^{1-2\epsilon}}\end{aligned} \tag{A.77}$$

The explicit results for the expressions above, in the $\epsilon \to 0$ limit, are

$$(2.2(a)) = \frac{N_1 N_2}{k^2} \frac{1}{128\pi^2} \Big[-\left(N_1^2 + N_2^2 + 4N_1N_2 - 6\right) \frac{1}{\epsilon} + (N_1^2 + N_2^2 - 2)\left(\pi^2 - 2(3 + \log 2)\right) + 4(N_1N_2 - 1)\left(\pi^2 - 2(11 + \log 2)\right) \Big] |\mu s|^{8\epsilon} \tag{A.78}$$

$$(2.2(b)) = -\frac{N_1 N_2}{k^2} (N_1N_2 - 1) \frac{\left(\pi^2 - 12\right)}{16\pi^2} |\mu s|^{8\epsilon} \tag{A.79}$$

$$(2.2(c)) = \frac{N_1 N_2}{k^2} \left(N_1^2 + N_2^2 - 4N_1N_2 + 2\right) \left(\frac{1}{128\pi^2} \frac{1}{\epsilon} + \frac{1 + \log 2}{64\pi^2} \right) |\mu s|^{8\epsilon} \tag{A.80}$$

$$(2.2(d)) = 0 \tag{A.81}$$

$$(2.2(e)) = -\frac{N_1 N_2}{k^2} (N_1N_2 - 1) \frac{\left(5\pi^2 - 48\right)}{96\pi^2} |\mu s|^{8\epsilon} \tag{A.82}$$

$$(2.2(f)) = \frac{N_1 N_2}{k^2} (N_1N_2 - 1) \left(\frac{1}{16\pi^2} \frac{1}{\epsilon} + \frac{1 + \log 2}{8\pi^2} \right) |\mu s|^{8\epsilon} \tag{A.83}$$

$$(2.2(g)) = -\frac{N_1 N_2}{k^2} \left(N_1^2 + N_2^2 - 4N_1N_2 + 2\right) \frac{\left(\pi^2 - 12\right)}{128\pi^2} |\mu s|^{8\epsilon} \tag{A.84}$$

$$(2.2(h)) = 0 \tag{A.85}$$

$$(2.2(i)) = 0 \tag{A.86}$$

$$(2.2(j)) = \frac{N_1 N_2}{k^2} (N_1N_2 - 1) \frac{\left(\pi^2 - 12\right)}{48\pi^2} |\mu s|^{8\epsilon} \tag{A.87}$$

$$(2.2(k)) = \frac{N_1 N_2}{k^2} \left(N_1^2 + N_2^2 - 2\right) \frac{\left(\pi^2 - 12\right)}{192\pi^2} |\mu s|^{8\epsilon} \tag{A.88}$$

$$(2.2(l)) = -\frac{N_1 N_2}{k^2} (N_1 - N_2)^2 \frac{1}{64} |\mu s|^{8\epsilon} \tag{A.89}$$

Appendix B

$\mathcal{N} = 1$ Gauge Theories

B.1 Group Theory Conventions

Let us list here some useful group-theoretical conventions which will be used throughout the Chapter 3.
The $\mathfrak{su}(2)$ Lie algebra is defined through the usual relation

$$[\tau^A, \tau^B] = i\epsilon^{ABC}\tau^C \tag{B.1}$$

where A, B, C are adjoint indices. We choose $\tau^A \equiv \frac{1}{2}\sigma^A$, where σ^A are the Pauli matrices, so that the canonical normalization

$$\mathrm{Tr}\left(\tau^A\tau^B\right) = \frac{1}{2}\delta^{AB} \tag{B.2}$$

holds. We make also use of the following notation

$$\tau^{(A}\tau^{B)} \equiv \frac{1}{2}\{\tau^A, \tau^B\}, \qquad \tau^{[A}\tau^{B]} \equiv \frac{1}{2}[\tau^A, \tau^B] \tag{B.3}$$

Since we are considering a quiver gauge theory with the matter sitting in the (anti-)bifundamental representation of $G_1 \times G_2$, namely $(\bar{R}, R)$ and $(R, \bar{R})$ respectively, we recall the action of G on them. The gauge group transformations $U \in G_1$ and $V \in G_2$ will then act in the following way

$$(\Phi')_i{}^{\hat{j}} = (U\Phi V^\dagger)_i{}^{\hat{j}}, \qquad (\bar{\Phi}')_{\hat{i}}{}^{j} = (V\bar{\Phi}U^\dagger)_{\hat{i}}{}^{j} \tag{B.4}$$

and at the level of algebra we have

$$\delta^{(A)}\Phi_i{}^{\hat{j}} = i\left[g_1(T^{(A)})_i{}^k\Phi_k{}^{\hat{j}} - g_2\Phi_i{}^{\hat{k}}(K^{(A)})_{\hat{k}}{}^{\hat{j}}\right], \qquad \delta^{(A)}\bar{\Phi}_{\hat{i}}{}^{j} = i\left[g_2(K^{(A)})_{\hat{i}}{}^{\hat{k}}\bar{\Phi}_{\hat{k}}{}^{j} - g_1\Phi_{\hat{i}}{}^{k}(T^{(A)})_k{}^{j}\right] \tag{B.5}$$

where $g_{1,2}$ are the two gauge couplings for $G_{1,2}$ and $T^{(A)}$, $K^{(A)}$ are generators of the $\mathfrak{g}_1, \mathfrak{g}_2$ Lie algebras respectively.

B.2 Superspace Conventions

The complete set of conventions can be found in [41, 45].
Here we complement the relations we stated in Subsection 1.3.3 with all the necessary ingredient we exploited for obtaining the results discussed in the main text.
The graded commutation relation for derivatives is

$$\{D_\alpha, D_\beta\} = 2i\partial_{\alpha\beta} \tag{B.6}$$

where $\partial_{\alpha\beta}$ is the ordinary spacetime derivative. Derivatives also satisfy the following useful identities

$$\begin{gathered}\partial^{\alpha\gamma}\partial_{\beta\gamma} = \delta^\alpha_\beta\Box, \quad D_\alpha D_\beta = i\partial_{\alpha\beta} + C_{\alpha\beta}D^2 \\ D_\alpha D_\beta D^\alpha = 0, \quad D^2 D_\alpha = -D_\alpha D^2 = i\partial_{\alpha\beta}D^\beta, \quad (D^2)^2 = \Box\end{gathered} \tag{B.7}$$

B.2.1 $\mathcal{N}=1$ Gauge Theories

The most generic kinetic terms for three-dimensional supersymmetric gauge theories are Chern-Simons and Yang-Mills ones. In the $\mathcal{N}=1$ notation, the Lagrangian for such terms reads

$$\mathcal{L}_{\text{CS-YM}} = -\frac{k}{4\pi}\,\text{Tr}\Big(2i\Gamma^\alpha\partial_{\alpha\beta}\Gamma^\beta + \Gamma^\alpha D_\alpha D^\beta \Gamma_\beta\Big) + \frac{1}{2g^2}\,\text{Tr}\Big(\Gamma^\alpha \Box \Gamma_\alpha - i\Gamma^\alpha\partial_{\alpha\beta}D^2\Gamma^\beta\Big) \quad \text{(B.8)}$$

which, with some effort, can be recasted in the following gauge-fixed form

$$\begin{aligned}\mathcal{L}^{\text{gf}}_{\text{CS-YM}} =& -\frac{k}{4\pi}\,\text{Tr}\bigg(2i\Gamma^\alpha\partial_{\alpha\beta}\Gamma^\beta + \left(1-\frac{1}{\beta}\right)\Gamma^\alpha D_\alpha D^\beta\Gamma_\beta\bigg) \\ &+\frac{1}{2g^2}\,\text{Tr}\bigg(\left(1+\frac{1}{\alpha}\right)\Gamma^\alpha\Box\Gamma_\alpha - i\left(1-\frac{1}{\alpha}\right)\Gamma^\alpha\partial_{\alpha\beta}D^2\Gamma^\beta\bigg)\end{aligned} \quad \text{(B.9)}$$

By taking the Landau gauge-fixing limit ($\alpha,\beta\to 0$) one can obtain the final form for the CS-YM gauge propagator, which reads

$$\Delta_\alpha^{\ \beta} = g^2\frac{\delta_\alpha^\beta(\kappa D^2+p^2)+(\kappa-D^2)p_\alpha^{\ \beta}}{p^2(\kappa^2+p^2)}, \qquad \kappa = \frac{kg^2}{2\pi} \quad \text{(B.10)}$$

Matter is coupled through the following action

$$S_{\text{matter}} = -\frac{1}{2}\int d^3x d^2\theta\ (\nabla^\alpha\bar\Phi)(\nabla_\alpha\Phi) \quad \text{(B.11)}$$

where in our setup, the covariant derivatives take the following explicit form

$$\nabla_\alpha\Phi_i^{\ \hat{j}} = D_\alpha\Phi_i^{\ \hat{j}} - ig_1\ \Gamma^A_\alpha(T^A)_i^{\ k}\Phi_k^{\ \hat{j}} + ig_2\ \hat\Gamma^M_\alpha\ \Phi_i^{\ \hat{k}}(K^M)_{\hat{k}}^{\ \hat{j}} \quad \text{(B.12)}$$

$$\nabla_\alpha\bar\Phi_{\hat{j}}^{\ i} = D_\alpha\bar\Phi_{\hat{j}}^{\ i} - ig_2\ \hat\Gamma^M_\alpha(K^M)_{\hat{j}}^{\ \hat{k}}\bar\Phi_{\hat{k}}^{\ i} + ig_1\ \Gamma^A_\alpha\ \bar\Phi_{\hat{j}}^{\ k}(T^A)_k^{\ i} \quad \text{(B.13)}$$

B.3 Effective superpotential at $k_1=-k_2$

When the CS-levels are equal by the absolute value but opposite, one can compute derivatives of the superpotential explicitly. Introducing $k_1=-k_2=k$, $\kappa_1=\frac{kg_1^2}{2\pi}$, $\kappa_2=\frac{kg_1^2}{2\pi}$, we get for $B\to\rho$:

$$\begin{aligned}\partial_\rho\mathcal{W}_{\text{1-loop}} =& -\frac{(g_1^2+g_2^2)(\kappa_1-\kappa_2)\left[(9g_1^4+22g_1^2g_2^2+9g_2^4)\rho+6(g_1^2+g_2^2)(\kappa_1^2+\kappa_1\kappa_2+\kappa_2^2)\right]}{16\pi(2\kappa_1\kappa_2+\rho(g_1^2+g_2^2))\sqrt{(k_1+k_2)^2+2\rho(g_1^2+g_2^2)}} \\ &\overset{\rho\to\infty}{\longrightarrow} -\frac{(\kappa_1-\kappa_2)(9g_1^4+22g_1^2g_2^2+9g_2^4)}{16\pi\sqrt{2}\sqrt{(g_1^2+g_2^2)\rho^{1/2}}},\end{aligned} \quad \text{(B.14a)}$$

$$\begin{aligned}\partial_{|B|}\mathcal{W}_{\text{1-loop}} =& \frac{g_1^2g_2^2(\kappa_1-\kappa_2)\rho}{4\pi(2\kappa_1\kappa_2+\rho(g_1^2+g_2^2))\sqrt{(k_1+k_2)^2+2\rho(g_1^2+g_2^2)}} \\ &\overset{\rho\to\infty}{\longrightarrow} \frac{g_1^2g_2^2(\kappa_1-\kappa_2)}{4\sqrt{2}\pi(g_1^2+g_2^2)^{3/2}\rho^{1/2}},\end{aligned} \quad \text{(B.14b)}$$

and in the limit $B \to 0$ the result is

$$\partial_\rho \mathcal{W}_{\text{1-loop}} = -\frac{\kappa_1 - \kappa_2}{8\pi(g_1^2 - g_2^2)^2 D}\left[\frac{N_1}{\sqrt{\kappa_1^2 + 2g_1^2\rho}} + \frac{N_2}{\sqrt{\kappa_2^2 + 2g_2^2\rho}}\right] -$$
$$- \frac{(g_1^2 + g_2^2)(\kappa_1 - \kappa_2)(2\kappa_1^2 + 2\kappa_1\kappa_2 + 2\kappa_2^2 + 3\rho(g_1^2 + g_2^2))}{16\pi(2\kappa_1\kappa_2 + \rho(g_1^2 + g_2^2))\sqrt{(\kappa_1 + \kappa_2)^2 + 2\rho(g_1^2 + g_2^2)}}$$
$$\xrightarrow{\rho\to\infty} -\frac{g_1\kappa_1^2 - g_2\kappa_2^2}{8\sqrt{2}\pi} - \frac{\kappa_1^2 g_1^2 - \kappa_2^2 g_2^2}{16\sqrt{2}\pi\sqrt{g_1^2 + g_2^2}\rho^{1/2}}, \tag{B.15a}$$

$$\partial_{|B|} \mathcal{W}_{\text{1-loop}} = \frac{g_1^2 g_2^2(\kappa_1 - \kappa_2)B}{4\pi(g_1^2 - g_2^2)D}\left[\frac{g_1^4 + 3g_1^2 g_2^2}{\sqrt{\kappa_1^2 + 2g_1^2\rho}} - \frac{g_2^4 + 3g_1^2 g_2^2}{\sqrt{\kappa_2^2 + 2g_2^2\rho}}\right]$$
$$\xrightarrow{\rho\to\infty} \frac{g_1 g_2(\kappa_1 g_2 - \kappa_2 g_2)}{4\sqrt{2}\pi(g_1 + g_2)^2}. \tag{B.15b}$$

where

$$D = (g_1^2 - g_2^2)^2\rho - 4(\kappa_1^2 g_2^2 + \kappa_2^2 g_1^2). \tag{B.16a}$$
$$N_1 = g_1^4(g_1^2 - g_2^2)^3\rho - 4g_1^8\kappa_2^2 + 4g_1^2 g_2^2\kappa_1^2(-g_1^4 + g_1^2 g_2^2 + g_2^4), \tag{B.16b}$$
$$N_2 = g_2^4(g_2^2 - g_1^2)^3\rho - 4g_2^8\kappa_1^2 + 4g_2^2 g_1^2\kappa_2^2(-g_2^4 + g_1^2 g_2^2 + g_1^4). \tag{B.16c}$$

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