Aspects of dualities and symmetry enhancements in three and four dimensions

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Declaration

This dissertation is a result of my own efforts. The work to which it refers is based on my PhD research projects:

1. S. Pasquetti and M. Sacchi, *From 3d dualities to 2d free field correlators and back*, JHEP **11** (2019) 081, [1903.10817].
2. S. Pasquetti and M. Sacchi, *3d dualities from 2d free field correlators: recombination and rank stabilization*, JHEP **01** (2020) 061, [1905.05807].
3. S. Pasquetti, S. S. Razamat, M. Sacchi and G. Zafrir, *Rank Q E-string on a torus with flux*, SciPost Phys. **8** (2020) 014, [1908.03278].
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The author also contributed to the other publications which are not contained in this thesis:

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2. I. Garozzo, N. Mekareeya and M. Sacchi, *Duality walls in the* 4d $\mathcal{N}=2$ $SU(N)$ *gauge theory with* $2N$ *flavours*, JHEP **11** (2019) 053, [1909.02832].
3. I. Garozzo, N. Mekareeya, M. Sacchi and G. Zafrir, *Symmetry enhancement and duality walls in 5d gauge theories*, JHEP **06** (2020) 159, [2003.07373].
4. M. Sacchi, *New 2d* $\mathcal{N}$ *= (0, 2) dualities from four dimensions*, JHEP **12** (2020) 009, [2004.13672].
5. E. Beratto, S. Giacomelli, N. Mekareeya, M. Sacchi, *3d mirrors of the circle reduction of twisted* A_{2N} *theories of class* S, JHEP **09** (2020) 161, [2007.05019].
6. E. Beratto, N. Mekareeya and M. Sacchi, *Marginal operators and supersymmetry enhancement in 3d S-fold SCFTs*, JHEP **12** (2020) 017, [2009.10123].

7. C. Hwang, S. Pasquetti and M. Sacchi, *Flips, dualities and symmetry enhancements*, JHEP **05** (2020) 094, [2010.10446].

8. S. Giacomelli, N. Mekareeya and M. Sacchi, *New aspects of Argyres–Douglas theories and their dimensional reduction*, JHEP **03** (2020) 242, [2012.12852].

9. M. Sacchi, O. Sela and G. Zafrir, *Compactifying 5d superconformal field theories to 3d*, 2105.01497, accepted for publication by JHEP.

I hereby declare that except where specific reference is made to the work of others, the contents of this dissertation are original and have not been submitted in whole or in part for consideration for any other degree or qualification in this, or any other university.

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Abstract

The low energy dynamics of quantum field theories can be characterized by several interesting non-perturbative phenomena. Among these, in this thesis we study infra-red dualities and global symmetry enhancements in three and four dimensions. We focus on supersymmetric theories, for which various exact computational tools are at our disposal. Of particular relevance is supersymmetric localization, which allows us to compute partition functions on various compact manifolds. These turn out to be invariant under the renormalization group flow and are thus powerful to probe dualities and symmetry enhancements. Moreover, one can also effectively study flows across dimensions through these supersymmetric partition functions. Equipped with these tools, in this thesis we investigate different perspectives on the program of finding and organizing dualities and symmetry enhancements in $3d$ and $4d$. The approaches that we will employ combine several different concepts that appear in the study of supersymmetric theories. This first one is that we can flow across dimensions via spacetime compactifications. In this way we can find new dualities and symmetry enhancements, by either dimensionally reducing to lower dimensions or uplifting to higher dimensions some that are already known. Another possibility is to compactify a higher dimensional theory, like a $6d$ theory on a Riemann surface so to get a $4d$ theory, and use this geometric construction to predict and systematize dualities and symmetry enhancements. The last important ingredient that will play a role in our analysis consists of correspondences, in particular gauge/CFT correspondences relating partition functions of certein supersymmetric theories to correlation functions of some CFTs. These kind of correspondences can also be exploited to find new results on the gauge theory side from known results on the CFT side. In this thesis we discuss various examples of applications of these ideas. We first present a relation between $3d$ $\mathcal{N}=2$ dualities and identities for $2d$ CFT free field correlators, and we explain how this can be used to uplift known results about $2d$ free fields to new aspects of $3d$ theories. The second topic is the compactification of a particular $6d$ $\mathcal{N}=(1,0)$ SCFT, known as the rank-N E-string theory, on Riemann surfaces with fluxes so to get $4d$ $\mathcal{N}=1$ theories. This construction allows us to predict dualities and symmetry enhancements from known properties of the $6d$ SCFT and geometric considerations. Finally, we discuss a new type of duality for $4d$ $\mathcal{N}=1$ theories that represents a higher dimensional ancestor of the well-known $3d$ mirror symmetry.

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Chapter 1

Introduction

One of the most astonishing achievements of theoretical physics of the twentieth century is Quantum Field Theory (QFT). It provides the best framework available at the moment for describing several physical systems, ranging from high-energy physics where the Standard Model of particles have obtained an incredible number of experimental validations, to Condensed Matter models for phase transitions. Even String Theory, which is the most famous and studied candidate for a quantum theory of gravity, is written in the QFT language. Despite of its many successes, QFT is still far from being fully understood and several issues have to be solved in order to unleash its full power.

Some of these problems are related to the practical difficulty of performing computations in a QFT. Even if we are able to define a QFT that suitably describes a given physical system, we are often far from being able of actually computing quantities that we can compare with experiments. Indeed, when the theory is weakly coupled we can safely apply perturbative methods to perform computations, but when we are in a strongly coupled regime these are not reliable anymore. This is often the case when we go to low energies following the Renormalization Group (RG) flow, since many QFTs, typically in dimension lower than four, become strongly interacting in the infra-red (IR).

Several interesting phenomena can characterize a QFT when we follow its RG flow to low energies. Among these, the ones that we will study in this thesis are IR *dualities* and *symmetry enhancements*. We talk about IR duality when we have two different theories in the ultra-violet (UV), which flow to the same conformal field theory (CFT) fixed point at long distances. Dualities can also be regarded as a tool to extract information on a QFT, since a quantity that is difficult to compute on one side of the duality may be more easily accessible on the other. One can also have a more general situation in which the two theories flow to different CFTs, but these are connected by an *exactly marginal deformation*, that is a deformation by an operator whose scaling dimension is equal to the spacetime dimension d. The space of exactly marginal deformations is called *conformal manifold* and the statement is that the two theories flow to distinct CFTs living at different points of the same conformal manifold. In this thesis we will mostly consider cases in which the theories do flow to the same fixed point at low energies. The most famous example is the Seiberg duality [1], which applies to supersymmetric quantum field theories (SQFT). More precisely, we have that $4d$ $\mathcal{N}=1$ SQCD with $SU(N_c)$ gauge group and N_f fundamental flavors Q^i and $\tilde{Q}_j$ is dual to the $SU(N_f-N_c)$ gauge theory with N_f flavors q_i and $\tilde{q}^i$, N_f^2 gauge singlets $M^i{}_j$ and superpotential $\mathcal{W}=M^i{}_j\tilde{q}^j q_i$. In this example the two theories always flow to the same superconformal field theory (SCFT). Only for $N_f=2N_c$ we have a non-trivial conformal manifold[1], but the two gauge theories still flow to the same point of it at low energies.

Symmetry enhancements, instead, occur when the manifest UV symmetry of a theory gets enlarged to a bigger group as we flow to the IR, which contains the UV symmetry group as a subgroup. This phenomenon may involve not only the global symmetry of a theory but also supersymmetry, namely a non-supersymmetric theory may become supersymmetric in the IR or a supersymmetric theory may acquire new supercharges as we follow the RG flow. In this thesis we will be concerned only with enhancements of global symmetries. This phenomenon is often entangled with the one of duality, in the sense that the presence of one may hint toward the existence of the other. For example, it might be that in a duality the manifest global symmetry is larger in one frame compared to the other. In such a situation, the frame with less symmetry must enjoy a symmetry enhancement so that the two theories have the same global symmetry in the IR. Consider for example again the $4d$ $\mathcal{N}=1$ SQCD with gauge group $SU(2)$ and $N_f\geq 4$ fundamental flavors. Each flavor consists of a pair of chiral and anti-chiral in complex conjugate representations under the gauge group. One would say that the manifest global symmetry is $SU(N_f)\times SU(N_f)$, where each factor rotates independently chirals and anti-chirals respectively. Nevertheless, for $SU(2)$ the fundamental representation is pseudoreal, so chirals and anti-chirals are indistinguishible and they combine to form the fundamental representation of the larger $SU(2N_f)$ symmetry. If we consider the Seiberg dual theory, this is $SU(N_f-2)$ with N_f flavors, the singlets and the superpotential. Here for $N_f>4$ the gauge group is complex so the manifest global symmetry is $SU(N_f)\times SU(N_f)$, while for $N_f=4$ it is $SU(2)$ but the superpotential involving the singlets still preserves only $SU(N_f)\times SU(N_f)$. The duality then implies that on this frame the global symmetry must get enhanced at long distances from $SU(N_f)\times SU(N_f)$ to $SU(2N_f)$[2].

As we have said, these interesting phenomena occur following the RG flow where the theory typically becomes strongly coupled and is in general not even Lagrangian. This is the reason why one of the greatest challenges in QFT is to develop tools to perform exact,

[1]The conformal manifold is generically spanned by suitable combinations of the components of the tensor obtained from the square of the meson matrix $Q^i\tilde{Q}_j$ that are invariant under the complexified global symmetry group [2]. In some particular cases there are additional marginal operators. For $N_c=2$ one should also consider the tensors constructed from the square of the baryon and from the baryon times the meson, while for $N_c=4$ the baryon is itself marginal.

[2]For $N_f=3$ the dual is a Wess–Zumino (WZ) model which is free, so the full IR global symmetry is even larger.

non-perturbative computations. Over the years several routes have been followed to reach this goal. Among the most successful approaches we have for example lattice computations, conformal bootstrap, correspondences, localization techniques, and so on. Many of these rely on the presence of extra symmetries in the theory, such as conformal and supersymmetry. Symmetries tightly constrain the system and in some cases they even allow us to completely solve part of it. In order to make progress it is thus useful to assume that the theory possesses some of these symmetries, and in particular in this thesis we will mostly work with $3d$ and $4d$ theories with four supercharges.

One should also keep in mind that each of these methods has its pros and cons, to the extent that they should be really considered as complementary. Consider for example localization (see [3] for a review and references therein). This is a technique used to compute partition functions and other protected observables of QFTs with enough supersymmetry defined on certain compact manifolds. It is extremely powerful, since it allows us to compute such quantities exactly, including all possible non-perturbative corrections. Moreover, several of these partition functions turn out to be independent of the gauge coupling and are thus invariant along the RG flow. This means that we can compute them in the UV where we usually have a nice Lagrangian description of our theory, and the result must be the same as for the theory to which we flow in the IR. Hence, they represent an incredibly powerful tool to probe the low energy behaviour of SQFTs and to study dualities and symmetry enhancements that they may enjoy, which is the reason why we will use them a lot in this thesis. Nevertheless, localization has a limitation: it can only be applied to Lagrangian theories. The severity of this limitation can be understood from the fact that by now we know plenty of examples of non-Lagrangian theories, one of the first ones being the Argyres–Douglas theory of [4]. In such cases one has to rely on some other methods, like the conformal bootstrap which instead takes as an input only the symmetries of the system and doesn't require an explicit Lagrangian description. Nevertheless, also the conformal boostrap has its limitations, such as being applicable only to conformal theories. In this sense each of the available non-perturbative methods is equally important and has to be considered as complementary to the others. Moreover, despite of this limitation of localization, one should be careful on whether a theory is truly non-Lagrangian or not. In some cases, one can find a Lagrangian description that flows to the non-Lagrangian theory in the IR by admitting that it possesses less supersymmetry, which should then get enhanced, or the non-Lagrangian theory may possess a Lagrangian description upon compactification to lower dimensions. This will often be the case in this thesis, which is why we will be able to make such an intensive use of localization.

Having understood the non-perturbative methods at our disposal, another important task in QFT is to find an organizing principle underlying the many examples of dualities and symmetry enhancements that have been collected over the years and that keep showing up.

In this thesis we will discuss different perspectives on this problem, which combine many different concepts in supersymmetric gauge theories that we are now going to review.

The first important idea is that we can flow across dimensions via spacetime compactifications. This can be in particular applied to the concept of duality. Suppose that we start from a known duality in d dimensions, compactify both of the dual theories on a $(d-d')$-dimensional manifold and flow to energies much smaller than the compactification scale. In this way we obtain two theories in d' dimensions and we can ask ourselves if they are still dual or not. There are several subtleties that one has to take into account when studying the dimensional reduction of a duality and in general it may just happen that the two lower dimensional theories are not dual. This is due to the fact that two different limits are involved in the dimensional reduction and issues of order of limits are typically involved. The first limit consists of flowing to low energies while keeping the compactification radius r fixed. Here is where the duality holds and we expect the two d-dimensional theories to flow to the same fixed point. The second limit is the strict dimensional reduction limit $r\to 0$ while keeping the energy scale fixed. Taking the two limits in this order would give us the lower dimensional version of the fixed point theory of the original d-dimensional dual theories. If we instead take the limit $r\to 0$ first, we obtain two lower dimensional theories that we can conjecture being two different UV descriptions of the aforementioned d'-dimensional fixed point theory. Thus, we understand that this conjecture is true and that the duality survives the dimensional reduction only if the two limits commute, but this is not always true.

This problem has so far been understood at a different level depending on the set-up considered. Restricting ourselves to supersymmetric theories in $4d$, $3d$ and $2d$, some important dimensional reductions of dualities that have been studied in the literature are from $4d$ $\mathcal{N}=1$ to $3d$ $\mathcal{N}=2$ in [5, 6], from $3d$ $\mathcal{N}=2$ to $2d$ $\mathcal{N}=(2,2)$ in [7, 8] and from $4d$ $\mathcal{N}=1$ to $2d$ $\mathcal{N}=(0,2)$ in [9]. In order to exemplify the possible issues that can arise in the compactification of dualities let us consider the $4d$ $\mathcal{N}=1$ to $3d$ $\mathcal{N}=2$ reduction, which will play a relevant role in this thesis. Here one of the main problems is that four-dimensional theories typically possess anomalous $U(1)$ axial symmetries, which are not anomalous in three dimensions since there are no anomalies for continuous symmetries in $3d$. This additional symmetry typically spoils the duality in $3d$, which instead holds only assuming that the symmetry is broken. It turns out that what is breaking the axial symmetry in $3d$ is a monopole superpotential that is dynamically generated in the compactification. In other words, the role of the anomaly in $4d$ is played by the monopole superpotential in $3d$. Such a monopole superpotential can then be removed by integrating out flavors with suitable real mass deformations.

In some cases, when enough insight is gained about the dimensional reduction limit, one can even push this idea further and try to reverse the logic. Namely, we can start from a duality in lower dimensions and use it to guess a still unknown parent duality in higher dimensions. This represents an orthogonal, bottom-up approach to the one of the dimensional reduction, to which we will refer with the word *uplift*. We stress that this should not be intended as an attempt of reversing the RG flow, but just as a hint for the existence of the higher dimensional duality, which should then be carefully tested.

The dimensional reduction of dualities is not the only example of flow across dimensions that exists. Another important line of reasearch of the last decade has been the construction of four-dimensional theories by compactification of six-dimensional SCFTs on Riemann surfaces. The first step in this direction was made by Gaiotto in [10], who constrcuted a large set of $4d$ $\mathcal{N}=2$ SCFTs that are now known as class $\mathcal{S}$ theories from the compactification of $6d$ $\mathcal{N}=(2,0)$ SCFTs on Riemann surfaces. One of the great advantages of this procedure is that we can recover many instaces of S-duality and also predict infinitely many new ones from geometric considerations only. Specifically, distinct $4d$ theories that correspond to different pants decompositions of the same Riemann surface turn out to be S-dual, and the deformation of the surface that is needed to go from one to the other is interpreted in field theory as moving in the parameter space that connects the two theories. This strategy has been later generalized to a less supersymmetric set-up, in which $6d$ $\mathcal{N}=(1,0)$ SCFTs are compactified to $4d$ $\mathcal{N}=1$ theories. In this case we have an even richer structure, since $6d$ $\mathcal{N}=(1,0)$ SCFTs may possess a global symmetry for which we can turn on fluxes through the surface. This can be exploited to predict non-trivial symmetry enhancements, since the symmetry that the $4d$ theory is expected to have should be the subgroup of the $6d$ symmetry that is preserved by the flux, but this might not be fully manifest in $4d$ and in such a case it should get enhanced in the IR.

There is one last important ingredient at our disposal that we can use to enrich our analysis of dualities and symmetry enhancements. This consists of the so-called *gauge/CFT correspondences*, which relate certain observables such as partition functions in supersymmetric gauge theories to completely different quantities in CFTs, like correlation functions of primary operators. What distinguishes correspondences from dualities is that the two theories are in general intrinsically different, for example because they live in different dimensions, and completely different observables are related between the two sides. The most famous example of a gauge/CFT correspondence is the AGT correspondence [11] which, among other things, relates the $\mathbb{S}^4$ partition functions of $4d$ $\mathcal{N}=2$ theories to correlation functions in $2d$ CFTs, like Liouville or Toda theories. The construction of [10] plays an important role also here, since the $4d$ theories involved in the AGT correspondence belong to the set of the class $\mathcal{S}$ theories and the correspondence can be understood in terms of their geometric construction. What is important for us is that gauge/CFT correspondences can be exploited to obtain new results on the gauge theory side from known results on the CFT side. For example, dualities in gauge theory are usually translated into non-trivial identities for CFT correlators.

In this thesis we will make use of all of these ideas to find new dualities and symmetry enhancements in $4d$ $\mathcal{N}=1$ and $3d$ $\mathcal{N}=2$ theories. Let us now briefly summarize the contents of the thesis. In Chapter 2 we will discuss a connection between $3d$ $\mathcal{N}=2$ supersymmetric gauge theories and correlation functions of $2d$ CFTs in the free field realization. This is an example of combination of two of the concepts that we have described, namely dimensional reductions and correspondences. Indeed, the precise statement is that if we consider the $\mathbb{S}^2 \times \mathbb{S}^1$ partition function of certain three-dimensional theories with four supercharges and their limit when the radius of the $\mathbb{S}^1$ is sent to zero, the result looks like not only the $\mathbb{S}^2$ partition functions of some $2d$ $\mathcal{N} = (2, 2)$ gauge theories, but it can also be recast in the form of complex integrals appearing in the context of $2d$ free field correlators. Combining this observation with the concept of duality in $3d$ we obtain an even richer web of connections. Specifically, we find a relation betwen $3d$ $\mathcal{N} = 2$ dualities and integral identities for $2d$ free field correlators. We will first establish this connection by considering the $2d$ limit of a known $3d$ duality and show that the result coincides with a very well-known formula for the 3-point function of Liouville theory. Building on various examples we will be able to fully understand the dictionary of this kind of correspondence. We will then use the obtained knowledge to try to reverse the logic of this limit and uplift some results that are known in the literature on $2d$ CFT free fields to new dualities and symmetry enhancements in three-dimensions. A central role will be played by a $3d$ $\mathcal{N} = 2$ quiver gauge theory that we name $M[SU(N)]$. This theory turns out to flow, after a suitable deformation, to the more known $T[SU(N)]$ theory of Gaiotto–Witten [12], and we will see that some of the properties of $M[SU(N)]$ that we will understand from the $2d$ free fields perspective reduce to similar known properties of $T[SU(N)]$.

In Chapter 3 we will study the compactification of a specific $6d$ $\mathcal{N} = (1, 0)$ SCFT on Riemann surfaces to $4d$. The six-dimensional theory that we will consider is the rank-N E-string theory and the surfaces on which we will focus will be tubes, tori, caps and spheres. We will also have non-trivial fluxes for the $E_8 \times SU(2)_L$ global symmetry of the $6d$ theory through the surfaces. This will allow us to construct four-dimensional models that enjoy interesting dualities and symmetry enhancements, which we will analyze in details. Interestingly, the results of this chapter partially connect with those of Chapter 2. Namely, we will see that the fundamental building block of our constructions will be the theory obtained from compactification on a tube and that this will involve a $4d$ $\mathcal{N} = 1$ quiver gauge theory that we name $E[USp(2N)]$, which after dimensional reduction and suitable deformations reduces to the aforementioned $M[SU(N)]$ theory. Moreover, $E[USp(2N)]$ enjoys the very same set of properties that also $M[SU(N)]$ and $T[SU(N)]$ possess, so we can talk about a $4d$ dimensional uplift of these.

Among the interesting properties that $T[SU(N)]$ enjoys there is mirror symmetry [13]. This is a well-known type of IR duality that is peculiar of three dimensions. One of the features that we will find for $E[USp(2N)]$ is precisely a four-dimensional avatar of $3d$ mirror symmetry for $T[SU(N)]$. Building on this observation and on the fact that many examples of mirror symmetry in $3d$ are known, in Chapter 4 we will introduce a new class of $4d$ $\mathcal{N} = 1$ quiver gauge theories that are related in pairs by this new type of mirror duality in $4d$.

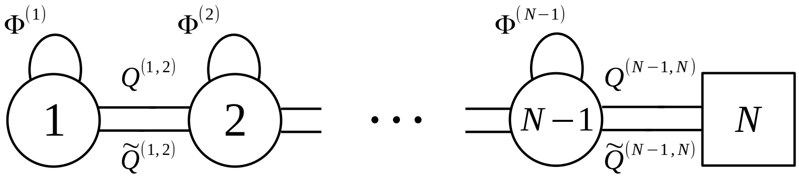

Figure 1.1: Quiver diagram for $T[SU(N)]$ in $\mathcal{N}=2$ notation. Round nodes denote gauge symmetries and square nodes denote global symmetries. Single lines denote chiral fields in representations of the nodes they are connecting. In particular, lines between adjacent nodes denote chiral fields in the bifundamental representations of the two nodes symmetries, while arcs denote chiral fields in the adjoint representation of the corresponding node symmetry.

We will call these theories $E^\sigma_\rho[USp(2N)]$, since they are the four-dimensional counterparts of the three-dimensional $T^\sigma_\rho[SU(N)]$ theories. The latter are basically the class of linear quiver gauge theories with unitary nodes and matter in the fundamental and bifundamental representation only, which are known to be related in pairs by mirror symmetry. This is yet another examples of uplift of a lower dimensional result to higher dimensions.

It is clear that the $T[SU(N)]$ theory plays an important role in the topics of all of the three chapters of this thesis. For this reason, before starting we conclude this introduction we a brief review of this theory.

The $T[SU(N)]$ theory

Given the central role that it will play in this thesis, we give here a brief review of the $T[SU(N)]$ theory and those of its properties that will be important for us. The $T[SU(N)]$ theory admits a Lagrangian description in terms of the quiver in Figure 1.1. The gauge group of the theory is $\prod_{i=1}^{N-1} U(i)$ and each factor is represented by a round node in the quiver. Such a gauge theory can be engineered with a brane setup in Type IIB consisting of N D3-branes stretched between N NS5 and N D5-branes, where the net number of 3-branes ending on each 5-brane is one [14, 12].

The theory has $\mathcal{N}=4$ supersymmetry, as it can be understood in the brane setup where the $SO(4)\cong SU(2)_H\times SU(2)_C$ symmetry that rotates the four directions that are not filled by any brane can be interpreted as the R-symmetry. Nevertheless, for later convenience we will use $\mathcal{N}=2$ notation, where each gauge node carries a vector multiplet and a chiral multiplet $\Phi^{(i)}$ in the adjoint representation of the corresponding gauge symmetry which together form an $\mathcal{N}=4$ vector multiplet. The matter content of the theory consists also of bifundamental chiral fields $Q^{(i,i+1)}_{ab}$ and $\tilde{Q}^{(i,i+1)}_{\tilde{a}\tilde{b}}$ represented in the quiver by lines connecting adjacent nodes, which come from $\mathcal{N}=4$ hypermultiplets[3]. For $i=N-1$ these are actually fundamental fields of the $U(N-1)$ gauge node and they transform under an $SU(N)_X$ global

[3] In our conventions, the bifundamentals $Q^{(i,i+1)}_{ab}$ transform in the representation $\mathbf{i}\otimes\overline{\mathbf{i+1}}$ of $U(i)\times U(i+1)$ and the bifundamental $\tilde{Q}^{(i,i+1)}_{ab}$ transform in the representation $\bar{\mathbf{i}}\otimes\mathbf{i+1}$.

symmetry, which is represented in Figure 1.1 by a square node. In $\mathcal{N}=2$ notation the superpotential of the theory is

$$\mathcal{W}_{T[SU(N)]} = \sum_{i=1}^{N-1} \text{Tr}_i \left[\Phi^{(i)} \left(\text{Tr}_{i+1}\, \mathbb{Q}^{(i,i+1)} - \text{Tr}_{i-1}\, \mathbb{Q}^{(i-1,i)}\right)\right] , \tag{1.1}$$

where we defined the matrix of bifundamentals $\mathbb{Q}^{(i,i+1)} = Q^{(i,i+1)}_{ab}\widetilde{Q}^{(i,i+1)}_{\tilde{a}\tilde{b}}$ connecting the $U(i)$ to the $U(i+1)$ gauge node. On the first node $\mathbb{Q}^{(0,1)} = 0$. The traces Tr_i are taken in the adjoint of the i-th gauge node, except for $i = N$ which corresponds to the trace Tr_X over the global symmetry $SU(N)_X$. The manifest global symmetry of $T[SU(N)]$ is $SU(N)_X \times U(1)^{N-1}$. The $U(1)$ factors corresponding to the topological symmetry of each gauge node are actually enhanced to the second $SU(N)_Y$ symmetry in the IR[4]. For each Cartan in the two $SU(N)$ global symmetries we can turn on real masses. The most suitable parametrization of these masses consists of turning on $2N$ parameters X_i and Y_i with $i = 1, \cdots, N$ subjected to the tracelessness conditions $\sum_{i=1}^N X_i = \sum_{i=1}^N Y_i = 0$.

When we use the $\mathcal{N}=2$ formalism, the $\mathcal{N}=4$ R-symmetry is decomposed as follows. We first consider the Cartans $U(1)_H \subset SU(2)_H$ and $U(1)_c \subset SU(2)_C$. We then consider their diagonal and off-diagonal combinations, $U(1)_{R_0} = U(1)_H + U(1)_C$ and $U(1)_{m_A} = U(1)_C - U(1)_H$. The first one corresponds to the $\mathcal{N}=2$ R-symmetry, while the second one manifests itself as a flavor symmetry in the $\mathcal{N}=2$ theory that we will refer to as *axial symmetry*. Turning on a real mass for the axial symmetry explicitly breaks supersymmetry from $\mathcal{N}=4$ to $\mathcal{N}=2^*$ [15]. From the $\mathcal{N}=2$ perspective, $U(1)_{R_0}$ should be considered as a trial R-symmetry, which can mix with other abelian symmetries in the IR. Since the topological symmetry is non-abelian, $U(1)_{R_0}$ will only mix with $U(1)_{m_A}$. Denoting with r the mixing coefficient and with q_A the charge under $U(1)_{m_A}$, we have that the most general R-charge is

$$R = R_0 + q_A\, r\,. \tag{1.2}$$

Our choice for the parametrization of $U(1)_{m_A}$ and $U(1)_R$ is summarized in Table 1.1. The exact value of r corresponding to the IR superconformal R-symmetry can be fixed by F-extremization [16]. Nevertheless, since the theory is actually $\mathcal{N}=4$ we expect no mixing between $U(1)_{m_A}$ and $U(1)_{R_0}$, namely $r = 0$. As we did for the non-abelian symmetries, we can turn on a real mass $\text{Re}(m_A)$ for the axial symmetry. It is also useful to define the

[4]Enhancements of the topological symmetry in $3d$ $\mathcal{N}=4$ quivers occur every time we have gauge nodes that are *balanced*. For unitary groups, a node is balanced when the number of fundamental hypermultiplets connected to it is twice its rank. In the case of $T[SU(N)]$ this condition is satisfied for every gauge node, so all the topological symmetries of the quiver get enhanced.

	$SU(N)_X$	$SU(N)_Y$	$U(1)_{m_A}$	$U(1)_R$
$Q^{(i-1,i)}$	$\mathbf{1}$	$\mathbf{1}$	1	r
$\tilde{Q}^{(i-1,i)}$	$\mathbf{1}$	$\mathbf{1}$	1	r
$Q^{(N-1,N)}$	$\mathbf{N}$	$\mathbf{1}$	1	r
$\tilde{Q}^{(N-1,N)}$	$\bar{\mathbf{N}}$	$\mathbf{1}$	1	r
$\Phi^{(i)}$	$\mathbf{1}$	$\mathbf{1}$	-2	$2-2r$
$\mathcal{H}$	$\mathbf{N^2-1}$	$\mathbf{1}$	2	$2r$
$\mathcal{C}$	$\mathbf{1}$	$\mathbf{N^2-1}$	-2	$2-2r$

Table 1.1: Charges and representations of the chiral fields and of the chiral ring generators of $T[SU(N)]$ under the global symmetries. In the table $i = 1, \cdots, N-1$ and $Q^{(0,1)} = \tilde{Q}^{(0,1)} = 0$.

following holomorphic combination:

$$m_A = \mathrm{Re}(m_A) + i\frac{Q}{2}r\,. \tag{1.3}$$

Summing up, the complete IR global symmetry of the $\mathcal{N} = 2^*$ version of $T[SU(N)]$ is

$$SU(N)_X \times SU(N)_Y \times U(1)_{m_A}\,. \tag{1.4}$$

The chiral fields of the theory transform under these symmetries according to Table 1.1.

The generators of the chiral ring are the Higgs branch (HB) and the Coulomb branch (CB) moment maps $\mathcal{H}$ and $\mathcal{C}$. The name "moment map" is due to the fact that these multiplets contain the conserved currents for the $SU(N)_X$ and the $SU(N)_Y$ global symmetries that characterize the HB and the CB respectively, which possess a symplectic structure thanks to the high amount of supersymmetry. The HB moment map is

$$\mathcal{H} = \mathcal{Q} - \frac{1}{N}\,\mathrm{Tr}_X\,\mathcal{Q} \tag{1.5}$$

with $\mathcal{Q}$ the $N \times N$ meson matrix

$$\mathcal{Q}_{ij} = \mathrm{Tr}_{N-1}\,\mathbb{Q}^{(N-1,N)}\,. \tag{1.6}$$

As a consequence of the F-term equations deriving from the superpotential (1.1), the HB moment map is nilpotent

$$\mathcal{H}^N = 0\,. \tag{1.7}$$

This implies that the HB is isomorphic to the nilpotent cone of $SU(N)_X$.

The CB branch moment map is instead generated by $\mathrm{Tr}_i\ \Phi^{(i)}$ and monopole operators with magnetic flux vectors $(m_1, \ldots m_{N-1})$, where m_i denotes the unit of flux for the topological $U(1)$ of the i-th node. In particular monopole operators defined with fluxes of the form

$(0^i, (\pm 1)^j, 0^k)$, where 0 and 1 are repeated with integer multiplicities i, j, and k such that $i+j+k=N-1$, have the same R-charge of the adjoint chiral fields and the same charge under $U(1)_{m_A}$. We then collect these $N(N-1)$ monopoles and the traces of the $N-1$ adjoint chirals into a single $N \times N$ traceless matrix[5]. For $N=4$ this matrix reads

$$\mathcal{C} \equiv \begin{pmatrix} 0 & \mathfrak{M}^{(1,0,0)} & \mathfrak{M}^{(1,1,0)} & \mathfrak{M}^{(1,1,1)} \\ \mathfrak{M}^{(\text{-}1,0,0)} & 0 & \mathfrak{M}^{(0,1,0)} & \mathfrak{M}^{(0,1,1)} \\ \mathfrak{M}^{(\text{-}1,\text{-}1,0)} & \mathfrak{M}^{(0,\text{-}1,0)} & 0 & \mathfrak{M}^{(0,0,1)} \\ \mathfrak{M}^{(\text{-}1,\text{-}1,\text{-}1)} & \mathfrak{M}^{(0,\text{-}1,\text{-}1)} & \mathfrak{M}^{(0,0,\text{-}1)} & 0 \end{pmatrix} + \sum_{i=1}^{3} \mathrm{Tr}_i\, \Phi^{(i)} \mathcal{D}_i\,, \tag{1.8}$$

where $\mathcal{D}_i$ are traceless diagonal generators of $SU(N)_Y$. The operator $\mathcal{C}$ constructed in this way transforms in the adjoint representation of $SU(N)_Y$ and thus corresponds to the moment map for this enhanced symmetry.

In Table 1.1 we also report the charges and representations under the global symmetries of the chiral ring generators $\mathcal{H}$ and $\mathcal{C}$ according to our parametrization of $U(1)_{m_A}$ and $U(1)_{R_0}$. Notice that these charges are consistent with the operator map dictated by mirror symmetry which in this case corresponds to a self-duality of the theory, under which the operators of the HB and the CB are exchanged. This in particular implies that also the matrix $\mathcal{C}$ is nilpotent

$$\mathcal{C}^N = 0 \tag{1.9}$$

and the CB is isomorphic to the nilpotent cone of $SU(N)_Y$. The mirror self-duality of $T[SU(N)]$ can be easily understood at the level of the brane setup, where it corresponds to the action of the S element of $SL(2,\mathbb{Z})$ which exchanges NS5 and D5-branes.

One of the main tools we will use to study $T[SU(N)]$ as well as other $3d$ $\mathcal{N}=2$ theories is the supersymmetric partition function on $\mathbb{S}^3_b$ [17, 16, 18, 19] (see Appendix A.3 for our conventions). For $T[SU(N)]$, this will be a function of the parameters in the Cartan of the global symmetry group, which we denoted as X_i, Y_i and m_A. Indeed, the partition function depends only on the holomorphic combination of the real mass for the $U(1)_{m_A}$ abelian symmetry and the mixing coefficient with the trial R-symmetry $U(1)_{R_0}$ [16]. With

[5]The fact that we have as many monopole operators with R-charge 1 as it is necessary to form a $SU(N)_Y$ adjoint representation is a consequence of the gauge nodes being balanced.

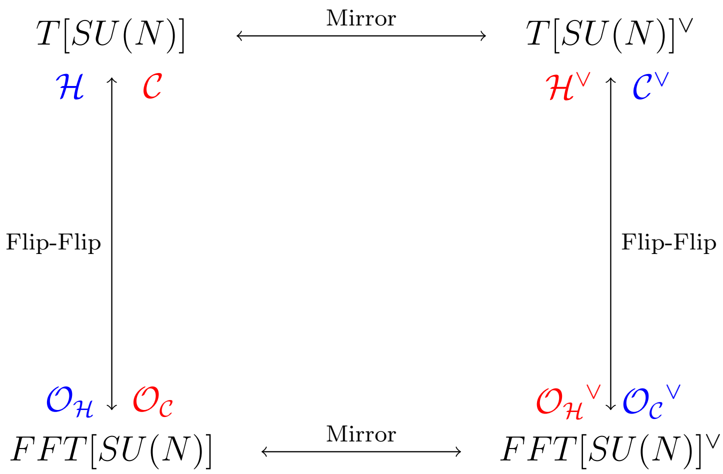

Figure 1.2: Duality web of the $T[SU(N)]$ theory. On the horizontal direction we have the mirror duality, while on the vertical direction we have the flip-flip duality. Operators of the same color are mapped to each other across the dualities.

these conventions, the partition function of $T[SU(N)]$ can be written recursively as

$$\begin{aligned}
\mathcal{Z}_{T[SU(N)]}(\vec{X};\vec{Y};m_A) \;=\; & \int \mathrm{d}\vec{z}^{(N-1)}_{N-1}\, \mathrm{e}^{2\pi i(Y_{N-1}-Y_N)\sum_{a=1}^{N-1} z_a^{(N-1)}} \times \\
& \times \prod_{a,b=1}^{N-1} s_b\left(-i\frac{Q}{2}+\left(z_a^{(N-1)}-z_b^{(N-1)}\right)+2m_A\right)\times \\
& \times \prod_{a=1}^{N-1}\prod_{i=1}^{N} s_b\left(i\frac{Q}{2}\pm(z_a^{(N-1)}-X_i)-m_A\right)\times \\
& \times \;\mathcal{Z}_{T[SU(N-1)]}\left(z_1^{(N-1)},\cdots,z_{N-1}^{(N-1)};Y_1,\cdots,Y_{N-1};m_A\right),
\end{aligned} \tag{1.10}$$

where we defined the measure of integration for the m-th $U(n)$ gauge group on $\mathbb{S}^3_b$ including both the contribution of the $\mathcal{N}=2$ vector multiplet and of the Weyl symmetry factor

$$\mathrm{d}\vec{z}^{(m)}_n = \frac{1}{n!}\frac{\prod_{i=1}^n \mathrm{d}z_i^{(m)}}{\prod_{i<j}^n s_b\left(i\frac{Q}{2}\pm\left(z_i^{(m)}-z_j^{(m)}\right)\right)}\,. \tag{1.11}$$

In [20] it has been observed that $T[SU(N)]$ possesses several duality frames that can be summarized in the commutative diagram of Figure 1.2. One frame is the one obtained applying mirror symmetry, which we denote by $T[SU(N)]^\vee$. As we mentioned before, $T[SU(N)]$ is self-dual under this duality, which acts non-trivially on the chiral ring generators of the theory. In particular, it exchanges the operators charged under $SU(N)_X$ with those charged under $SU(N)_Y$. If we consider the $\mathcal{N}=2^*$ deformation of $T[SU(N)]$, mirror symmetry also acts flipping the sign of the $U(1)_{m_A}$ charges as well as the mixing coefficient of the R-symmetry with this abelian symmetry $r \to 1 - r$. In terms of the mass parameter m_A, we have

$$m_A \to i\frac{Q}{2} - m_A\,. \tag{1.12}$$

In other words, using Table 1.1 we have the following operator map:

$$\begin{aligned} \mathcal{H} \quad &\leftrightarrow \quad \mathcal{C}^\vee \\ \mathcal{C} \quad &\leftrightarrow \quad \mathcal{H}^\vee\,. \end{aligned} \tag{1.13}$$

At the level of the $\mathbb{S}^3_b$ partition function, mirror symmetry for $T[SU(N)]$ translates into the following non-trivial integral identity:

$$\mathcal{Z}_{T[SU(N)]}(\vec{X};\vec{Y};m_A) = \mathcal{Z}_{T[SU(N)]}\left(\vec{Y};\vec{X};i\frac{Q}{2} - m_A\right) = \mathcal{Z}_{T[SU(N)]^\vee}(\vec{X};\vec{Y};m_A)\,. \tag{1.14}$$

This identity can be proven using the fact that $\mathcal{Z}_{T[SU(N)]}$ is an eigenfunction of the trigonometric Ruijsenaars-Schneider model [21].

On top of the mirror dual frame, $T[SU(N)]$ has another interesting dual which was named flip-flip dual $FFT[SU(N)]$ in [20]. This theory is $T[SU(N)]$ with two extra sets of singlet fields $\mathcal{O}_\mathcal{H}$ and $\mathcal{O}_\mathcal{C}$ flipping the HB and CB moment maps

$$\mathcal{W}_{FFT[SU(N)]} = \mathcal{W}_{T[SU(N)]} + \mathrm{Tr}_X\left(\mathcal{O}_\mathcal{H}\,\mathcal{H}^{FF}\right) + \mathrm{Tr}_Y\left(\mathcal{O}_\mathcal{C}\,\mathcal{C}^{FF}\right)\,, \tag{1.15}$$

where $\mathcal{H}_{FF}$ and $\mathcal{C}_{FF}$ denote the operators in the $FFT[SU(N)]$ frame. Flip-flip duality acts trivially on the non-abelian global symmetries of $T[SU(N)]$, while it acts on $U(1)_{m_A}$ and $U(1)_R$ exactly as mirror symmetry (1.12). The operators are accordingly mapped as

$$\begin{aligned} \mathcal{H} \quad &\leftrightarrow \quad \mathcal{O}_\mathcal{H} \\ \mathcal{C} \quad &\leftrightarrow \quad \mathcal{O}_\mathcal{C}\,. \end{aligned} \tag{1.16}$$

This duality implies another non-trivial integral identity satisfied by $\mathcal{Z}_{T[SU(N)]}$

$$\begin{aligned} \mathcal{Z}_{T[SU(N)]}(\vec{X};\vec{Y};m_A) \quad &= \quad \prod_{i,j=1}^N \frac{s_b\left(i\frac{Q}{2} + (X_i - X_j) - 2m_A\right)}{s_b\left(i\frac{Q}{2} + (Y_i - Y_j) - 2m_A\right)} \mathcal{Z}_{T[SU(N)]}\left(\vec{X};\vec{Y};i\frac{Q}{2} - m_A\right) \\ &= \quad \mathcal{Z}_{FFT[SU(N)]}(\vec{X};\vec{Y};m_A)\,, \end{aligned} \tag{1.17}$$

which can also be proven using the trigonometric Ruijsenaars-Schneider model eigenvalue equation [20, 22]. As we will discuss more later in this thesis, the flip-flip duality of $T[SU(N)]$ can be derived by iteratively applying the Aharony duality [23] following a standard procedure.

By combining mirror symmetry and flip-flip duality we can reach a third duality frame $FFT[SU(N)]^\vee$, which again corresponds to $T[SU(N)]$ with two sets of singlet fields $\mathcal{O}_\mathcal{H}^\vee$ and $\mathcal{O}_\mathcal{C}^\vee$ flipping the HB and CB moment maps $\mathcal{H}_{FF}^\vee$ and $\mathcal{C}_{FF}^\vee$

$$\mathcal{W}_{FFT[SU(N)]^\vee} = \mathcal{W}_{T[SU(N)]} + \mathrm{Tr}_Y\left(\mathcal{O}_\mathcal{H}^\vee\,\mathcal{H}^{FF\vee}\right) + \mathrm{Tr}_X\left(\mathcal{O}_\mathcal{C}^\vee\,\mathcal{C}^{FF\vee}\right)\,, \tag{1.18}$$

but in this case the duality acts exchanging $SU(N)_X$ and $SU(N)_Y$, while leaving unchanged $U(1)_{m_A}$ and $U(1)_R$[6]. The operator map between the original $T[SU(N)]$ and $FFT[SU(N)]^\vee$ is

$$\begin{aligned} \mathcal{H} \quad &\leftrightarrow \quad \mathcal{O}_\mathcal{C}^\vee \\ \mathcal{C} \quad &\leftrightarrow \quad \mathcal{O}_\mathcal{H}^\vee\,. \end{aligned} \tag{1.19}$$

[6]In [20] this kind of duality was called spectral duality.

Chapter 2

$3d$ dualities from $2d$ free field correlators

In this chapter we will present a bottom-up approach for finding new dualities for $3d$ $\mathcal{N}=2$ theories by exploiting a connection between their $\mathbb{S}^2 \times \mathbb{S}^1$ partition function and correlators of $2d$ CFTs in the free field realization. The content of this chapter is taken from [24, 25], where in the first reference the correspondence between $3d$ dualities and $2d$ free field correlators was established, while in the second reference it was intensively employed to find new $3d$ $\mathcal{N}=2$ dualities.

2.1 The general idea

As we discussed in the Introduction, in the recent years there has been a lot of new progress in the understanding of IR dualities in various dimensions, especially for supersymmetric theories. This naturally led to the question whether there is a connection between dualities in different dimensions that would allow us to systematically organize them. In the first part of this chapter we will be interested in the reduction of dualities for $3d$ $\mathcal{N}=2$ theories to two dimensions.

This issue was first addressed in [7] in order to related $3d$ mirror symmetry [13] to $2d$ mirror symmetry [26] and later studied in more details and generality in [8][1]. The set-up

[1]Many of the subtleties that appear in this type of dimensional reductions is related to problems in $2d$ IR dualities when the target space is non-compact. In $2d$ the ground state can explore the entire moduli space of the theory because of quantum fluctuations, so that we can't just focus on a single region of it. Moreover, the metric on the target space, which is not protected by supersymmetry, is classically marginal in two dimensions. Consequently, in order to claim for a duality we need a complete knowledge of the target space of the theories at the quantum level, which is in general extremely difficult to achieve. This issue was analyzed in detail in the context of $3d$ dualities reduced to $2d$ in [8, 27]. This problem doesn't appear when the theories have a compact target space or when massive deformations are turned on, since these have the effect of lifting the vacua of the theory, leaving just discrete isolated vacua. For this reason our discussion will not be affected by these subtleties related to the non-compactness of the target space, since all the results we will use include non-vanishing mass deformations.

considered in this type of dimensional reduction is that of a 3d theory with at least $\mathcal{N}=2$ supersymmetry compactified on a circle $\mathbb{S}^1$ of radius R. This preserves the full amount of supersymmetry, so we end up with 2d $\mathcal{N}=(2,2)$ theories. We then have to decide how the parameters of the 3d theory scale with the radius. In 3d we have two types of parameters: real masses and Fayet–Iliopoulos (FI) parameters. In [8] two types of limits were considered. The first one is called *Higgs limit*, since it is characterized by the real masses remaining finite and the FI parameters going to infinity, implying that the matter fields remain light while the 3d Coulomb branch is lifted. The result is typically a 2d $\mathcal{N}=(2,2)$ gauge theory which is usually referred to as gauged linear sigma model (GLSM). The second one is the *Coulomb limit*, which is the opposite limit of FI parameters remaining finite and real masses going to infinity. In this case one usually gets a 2d $\mathcal{N}=(2,2)$ Landau–Ginzburg (LG) model.

We will take a particular point of view on this type of dimensional reduction. The 3d duality implies an identity for the partitions functions of the dual theories which, if the theories are defined on a suitable compact space, can be recast in the form of matrix integrals using localization techniques. Supersymmetric partition functions are indeed a very powerful tool to study dualities, since they are independent of the gauge coupling and this implies that they are invariant under RG flow, which makes it possible to compute them using the UV data and still match them between theories that are dual only in the IR. If we consider, in particular, the partition function on a space of the form $\mathcal{M}\times\mathbb{S}^1$ with $\mathcal{M}$ a compact space we can then explicitly implement the 2d reduction at the level of the matrix integral of the partition function by taking the limit in which the ratio between the radius of the $\mathbb{S}^1$ and the scale of the compact manifold $\mathcal{M}$ goes to zero.

We will be mainly interested in the partition function on $\mathbb{S}^2\times\mathbb{S}^1$, also known as the *supersymmetric index* [28], which was computed using supersymmetric localization in [29, 30]. In Appendix A.2 we give a more detailed review of this partition function, but for the moment we just start mentioning that the corresponding matrix integral takes the following schematic form:

$$\mathcal{Z}^{\mathbb{S}^2\times\mathbb{S}^1}=\sum_{\mathbf{m}}\oint\prod_{j=1}^{\mathrm{rank}\,G}\frac{\mathrm{d}u_j}{2\pi i\,u_j}\mathcal{Z}_{int}\,, \tag{2.1}$$

where we denoted by u_j the fugacities parametrizing the maximal torus of the gauge group whose integration domain is the unit circle in the complex plane and by $\mathbf{m}$ the corresponding GNO magnetic fluxes on $\mathbb{S}^2$. The integrand $\mathcal{Z}_{int}$ can be uniquely determined from the Lagrangian description of the theory and receives contribution from the different multiplets of the theory. Such contributions can be expressed in terms of q-Pochhammer symbols $(a;q)_\infty=\prod_{k=0}^{\infty}(1-aq^k)$, which enjoy the two following properties that will be relevant for us:

$$\lim_{q\to 1}\frac{(q^x;q)_\infty}{(q^y;q)_\infty}=\frac{\Gamma(y)}{\Gamma(x)}(1-q)^{y-x}$$
$$\lim_{q\to 1}\frac{(zq^x;q)_\infty}{(zq^y;q)_\infty}=(1-z)^{y-x}\,. \tag{2.2}$$

The first identity is used when considering the Higgs limit, while the second one is used for the Coulomb limit. Indeed, the Higgs limit typically results in the partition function on $\mathbb{S}^2$ of a GLSM, where the contribution of a $2d$ chiral is written in terms of ordinary Euler gamma functions. Instead, the result of the Coulomb limit with some manipulations can be mapped to the partition function of a Landau-Ginzburg model with logarithmic twisted superpotential. We then want to consider taking these limits on the two sides of an integral identity associated with a $3d$ duality.

There exist two main types of dualities in three dimensions. First, we have dualities that generally go under the name of *mirror symmetry* [13], which are in particular characterized by the fact that the HB of a theory is mapped to the CB of the dual and viceversa (we will have more to say about these dualities in Chapter 4). Then, we have dualities that are more reminiscent of Seiberg duality in four dimensions, under which HB and CB are not swapped. It is clear that if we have a mirror dual pair, if we take the Higgs limit on one side of the duality we are forced to take the Coulomb limit on the opposite side. This is how $3d$ mirror symmetry reduces in $2d$ to the Hori–Vafa duality [7]. Instead, if we have a Seiberg-like duality we can either take the Higgs limit on both sides, which yields a similar Seiberg-like duality between $2d$ gauge theories, or the Coulomb limit on both sides, which leads to a duality between LG models.

The key observation of [24] is that the Coulomb limit of the $\mathbb{S}^2\times\mathbb{S}^1$ partition function of a $3d$ $\mathcal{N}=2$ theory can be rewritten in the form of a complex integral that appears in the context of $2d$ CFT correlators in the free field realization, also known as Dotsenko–Fateev (DF) integral. Specifically, by looking at the second limit in (2.2) one can understand that if the $3d$ theory is non-chiral, that is for each chiral field in a representation $\mathcal{R}$ of the gauge group there exists another chiral field in the complex conjugate representation $\bar{\mathcal{R}}$, then the Coulomb limit of the integrand takes a factorized form into a holomorphic and an anti-holomorphic part. Moreover, calling $q=\mathrm{e}^\beta$ where β is the ratio between the radius of the $\mathbb{S}^1$ and the radius of the $\mathbb{S}^2$, then also the integration measure, as we will show, takes the form of that of a complex integral

$$\sum_{\mathbf{m}}\oint\prod_{j=1}^{\mathrm{rank}\,G}\frac{\mathrm{d}u_j}{2\pi i\,u_j}\underset{\beta\to 0}{\longrightarrow}\int_{\mathbb{C}}\prod_{j=1}^{\mathrm{rank}\,G}\frac{\mathrm{d}^2z_j}{\pi\beta|z_j|^2}\,. \tag{2.3}$$

$\mathbb{S}^2 \times \mathbb{S}^1$ partition function	Free field correlator
Rank of the gauge group	Number of inserted screening operators
Gauge fugacities	Insertion points of the screening operators
Vector symmetry fugacities	Insertion points of the external operators
Axial and topological symmetry fugacities	Momenta of the external operators
Superpotential	Constraints on the momenta

Table 2.1: Main entries of the dictionary between $\mathbb{S}^2 \times \mathbb{S}^1$ partition functions of $3d$ $\mathcal{N} = 2$ gauge theories and free field correlators of $2d$ CFTs. For simplicity we stick to the case in which the gauge group of the $3d$ theory has a single unitary factor.

The Coulomb limit of the identity of the $\mathbb{S}^2 \times \mathbb{S}^1$ partition functions of a pair of Seiberg dual theories can then be understood not only as the identity of the partition functions of two dual LG models, but also as a non-trivial complex integral identity between $2d$ free field correlators[2]. In Table 2.1 we summarize the main entries of the dictionary that we will find for this correspondence.

From the point of view of a field theorist who is fond of $3d$ dualities, the interesting part of this connection is that there exist many results for free field correlators that are known in the $2d$ CFT literature and for which a $3d$ counterpart it not known yet. Indeed, it will be possible not only to work out the dictionary in Table 2.1, but even to map each factor in the integrand of the matrix integral of the $\mathbb{S}^2 \times \mathbb{S}^1$ partition function to each factor in the integrand of the complex integral of the free field correlator. This insight allows us to literally read a $3d$ $\mathcal{N} = 2$ gauge theory out of a free field integral. Equipped with this knowledge, one can then try to reverse the logic and *uplift* some known identities for $2d$ free field integrals to genuine (IR, not mass deformed) dualities between $3d$ gauge theories. This is the bottom-up point of view that was employed in [24, 25]. It is important to clarify that with the word "uplift" we don't refer to some attempt of reversing the RG flow across dimensions, but rather that we will use the $2d$ results as a hint to guess a new $3d$ duality, which should then be tested with all the standard tools.

In order to establish the correspondence we will start by considering, as a prototypical example, the duality between the $U(N)$ gauge theory with one adjoint chiral and one flavor with no superpotential and the WZ model of $3N$ chirals with a cubic superpotential that was

[2] We should mention that this connection between $\mathbb{S}^2 \times \mathbb{S}^1$ partition functions of $3d$ $\mathcal{N} = 2$ theories and $2d$ free field correlators is not totally unexpected. Indeed, in recent years the connection between $3d$ $\mathcal{N} = 2$ gauge theories and free field correlators in Toda CFT has been considered from various perspectives. In particular, in [31, 32] a dictionary to map $3d$ quiver theories to q-deformed conformal blocks in the free field realization was proposed (this map was further explored on various compact spaces in [33] and from a different perspective in [34]). The limit that we are considering here consists in the $q \to 1$ limit in which we recover the standard conformal correlators. One difference between our work and theirs is that they relate the partition function on $\mathbb{D}_2 \times \mathbb{S}^1$ to the conformal block, while we relate the partition function on $\mathbb{S}^2 \times \mathbb{S}^1$ to the full conformal correlator. The connection between the two set-ups, which is clear from the CFT perspective, becomes also clear from the gauge theory perspective thanks to the factorization property of $3d$ partition functions discussed in [35, 36]. The other important difference is in the application of this connection to $3d$ dualities and integral identities for free field correlators that we are considering.

first proposed in [37]. We will show that the Coulomb limit of the $\mathbb{S}^2 \times \mathbb{S}^1$ partition function of the gauge theory yields a complex integral that coincides with the free field representation of the 3-point function of Liouville theory, which was studied in [38]. The Coulomb limit of the WZ theory will then give us an evaluation formula for such free field integral, which, as we are going to review, can be related to the famous result of Dorn, Otto, Zamolodchikov and Zamolodchikov [39, 40] for the 3-point function of Liouville theory known as the *DOZZ formula*.

We will then be able to uplift an interesting result of [38] for this 3-point function to the $3d$ duality. More precisely, in [38] the aforementioned evaluation formula of the free field integral was proven by iterating a more fundamental identity for some special complex integrals. We will show that such fundamental identity can also be obtained from a $3d$ duality, which is an analogue of Aharony duality [23] but for theories with monopole superpotentials that was proposed in [41]. We will then be able to mimic the manipulations of [38] performed in CFT to the free field integral so to give a derivation of the confining duality for $U(N)$ with one adjoint and one flavor by iterating some fundamental monopole duality.

There is also another result in CFT that can be uplifted to the $3d$ duality. Namely, in order to relate the evaluation formula of the free field integral of [38] to the DOZZ formula, one needs to perform an analytic continuation in the number of screening charges. As summarized in Table 2.1, this is related to the rank N of the gauge group. We will show that such analytic continuation can also be performed at the level of the partition functions (either on $\mathbb{S}^3$ or $\mathbb{S}^2 \times \mathbb{S}^1$) of the dual $3d$ theory. More precisely, the partition function of the WZ theory can be recast in a form suitable for analytic continuation in N, which can then be interpreted as the partition function (either on $\mathbb{S}^5$ or $\mathbb{S}^4 \times \mathbb{S}^1$) of four free hypers in $5d$. This theory of four free hypers is also known as the $5d$ version of the T_2 theory [10, 42]. The $U(N)$ gauge theory is instead understood as arising when one of the Kähler parameters in the (p, q)-web of the T_2 theory is quantized, in which case the theory undergoes geometric transition.

After having worked out the dictionary for the relation between $3d$ dualities and $2d$ free field correlators, we will try to invert the direction and uplift some of the results of [38] for the free field correlators of Liouville theory to new $3d$ dualities.

We will first introduce a new $3d$ $\mathcal{N} = 2$ quiver gauge theory called $M[SU(N)]$ theory. The name is due to the fact that it can be understood as an $\mathcal{N} = 2$ generalization with monopole superpotential of the famous $T[SU(N)]$ theory [12], to which it flows after a suitable real mass deformation. The $M[SU(N)]$ theory enjoys many properties that reduce after this deformation to similar properties that are known for $T[SU(N)]$, such as an enhancement of the global symmetry and a web of self-dualities under mirror symmetry and flip-flip duality [20]. This $M[SU(N)]$ theory has a $2d$ analogue in terms of a complex integral and all of its properties translate into similar properties for such integral that have been used in [38].

We will then use $M[SU(N)]$ as a building block to construct more complicated theories that we will show enjoy some interesting dualities. Such dualities can be guessed by again uplifting some of the results of [38] for the free field correlators of Liouville theory, but we will also test them using more standard methods, such as mapping the gauge invariant operators and matching the supersymmetric partition functions.

2.2 Free field correlators in Liouville theory

In this subsection we briefly summarize some results on Liouville theory and the free field integral representation of their correlators that we will need later.

Liouville theory is the CFT of a complex scalar field ϕ with Lagrangian density

$$\mathcal{L} = \frac{1}{4\pi}(\partial_a \phi)^2 + \mu \mathrm{e}^{2b\phi} \,. \tag{2.4}$$

This theory has a holomorphic stress-energy tensor

$$T(z) = -(\partial_z \phi)^2 + Q\partial_z^2 \phi \,, \tag{2.5}$$

which generates a Virasoro algebra with central charge

$$c_L = 1 + 6Q^2, \qquad Q = b + b^{-1} \,. \tag{2.6}$$

The basic operators in this theory are

$$V_\alpha(z, \bar{z}) = \mathrm{e}^{2\alpha\phi} \,. \tag{2.7}$$

These are Virasoro primaries with conformal dimension $\Delta_L = \alpha(Q - \alpha)$, where α is a continuous parameter that we will call "momentum" of the operator. It was shown in [43] that the correlation function of k of such operators exhibits poles when the momenta satisfy the screening quantization condition

$$\alpha \equiv \alpha_1 + \cdots + \alpha_k = Q - Nb, \qquad N \in \mathbb{N} \,, \tag{2.8}$$

and that its residue at these poles takes the form of a complex integral known as *free field* or *Dotsenko–Fateev* (DF) correlator

$$\begin{aligned} \underset{\alpha=Q-Nb}{\mathrm{res}} \langle V_{\alpha_1}(z_1) V_{\alpha_2}(z_2) \cdots V_{\alpha_k}(z_k) \rangle \;&=\; (-\pi\mu)^N \prod_{i<j}^{k} |z_i - z_j|^{-4\alpha_i\alpha_j} \times \\ &\times \int \mathrm{d}^2 \vec{x}_N \prod_{i<j}^{N} |x_i - x_j|^{-4b^2} \prod_{i=1}^{N} \prod_{j=1}^{k} |x_i - z_j|^{-4b\alpha_k} \,. \end{aligned} \tag{2.9}$$

where we defined the integration measure

$$\mathrm{d}^2\vec{x}_N = \frac{1}{\pi^N N!}\prod_{j=1}^{N}\mathrm{d}^2x_j\,, \qquad \mathrm{d}^2x_j = \mathrm{dRe}(x_j)\,\mathrm{dIm}(x_j)\,. \tag{2.10}$$

The reason for the name "free field correlator" is that, as it was shown in [43], by integrating over the Liouville zero modes one can re-express the original correlator in terms of the same correlator in the free theory, but with the insertion of N *screening charge operators* $\mathrm{e}^{2b\phi}$.

Let us start considering the case of the 3-point function, which gives the structure constant $C(\alpha_1,\alpha_2,\alpha_3)$. In this case we have

$$\underset{\alpha=Q-Nb}{\mathrm{res}}\, C(\alpha_1,\alpha_2,\alpha_3) = (-\pi\mu)^N I_N(\alpha_1,\alpha_2,\alpha_3)\,, \tag{2.11}$$

where I_N is the free field integral[3]

$$I_N(\alpha_1,\alpha_2,\alpha_3) = \int \mathrm{d}^2\vec{u}_N \prod_{a<b}^{N}|u_a-u_b|^{-4b^2}\prod_{a=1}^{N}|u_a|^{-4b\alpha_1}|u_a-1|^{-4b\alpha_2} \tag{2.12}$$

The integral (2.12) was calculated exactly in [44] and then used to guess the form of the 3-point function via analytic continuation, as we will review. However, a different derivation of the evaluation formula of (2.12) was provided in [38], which was based on the following fundamental identity for complex integrals [45]:

$$\begin{aligned}\int \mathrm{d}^2\vec{u}_{N_c}\prod_{i<j}^{N_c}|u_i-u_j|^2\prod_{i=1}^{N_c}\prod_{a=1}^{N_f}|u_i-\tau_a|^{2p_a} &= \frac{\prod_{a=1}^{N_f}\gamma(1+p_a)}{\gamma(1+N_c+\sum_a p_a)}\prod_{a<b}^{N_f}|\tau_a-\tau_b|^{2(1+p_a+p_b)}\times\\ &\times\int \mathrm{d}^2 v_{N_f-N_c-1}\prod_{i<j}^{N_f-N_c-1}|v_i-v_j|^2\prod_{i=1}^{N_f-N_c-1}\prod_{a=1}^{N_f}|v_i-\tau_a|^{-2(1+p_a)}\,.\end{aligned} \tag{2.13}$$

As we will show, this identity can be obtained as the $2d$ Coulomb limit of the identity for the $\mathbb{S}^2\times\mathbb{S}^1$ partition functions associated with the duality with one monopole in the superpotential proposed in [41]. By iterating this basic identity, one can rewrite the original integral (2.12) in a similar form, but with the dimension of the integral decreased by one unit, with a shift of the parameters and with a prefactor

$$I_N(\alpha_1,\alpha_2,\alpha_3) = \frac{\gamma(-Nb^2)}{\gamma(-b^2)}\frac{1}{\gamma(2b\alpha_1)\gamma(2b\alpha_2)\gamma(2b\alpha_3+(N-1)b^2)}\, I_{N-1}\left(\alpha_1+\frac{b}{2},\alpha_2+\frac{b}{2},\alpha_3\right)\,, \tag{2.14}$$

[3] Compared to (2.9) we have set the insertion points of the three operators at $z_1=0$, $z_2=1$ and $z_3=\infty$.

where $\gamma(x) = \Gamma(x)/\Gamma(1-x)$. This recursive relation is extremely powerful, since it can be used to completely evaluate the integral. Iterating it N times, we indeed obtain

$$I_N(\alpha_1, \alpha_2, \alpha_3) = \prod_{j=1}^{N} \left(\frac{\gamma(-jb^2)}{\gamma(-b^2)} \right) \prod_{k=0}^{N-1} \frac{1}{\gamma(2b\alpha_1 + kb^2)\gamma(2b\alpha_2 + kb^2)\gamma(2b\alpha_3 + kb^2)} . \tag{2.15}$$

This evaluation formula is still not enough. Indeed, we have to remember that what we are computing is only the residue of the full correlator when the screening condition (2.8) is satisfied. Hence, we would like to find an expression which depends parametrically on N, so that we can analytically continue it to non-integer values lifting the condition (2.8) and reconstructing the structure constant $C(\alpha_1, \alpha_2, \alpha_3)$ for generic values of the momenta. This was done by [39, 40] (see also [46])

$$C(\alpha_1, \alpha_2, \alpha_3) = \left[\pi\mu\gamma(b^2) b^{2-2b^2} \right]^{\frac{(Q-\alpha)}{b}} \frac{\Upsilon'(0) \prod_{j=1}^{3} \Upsilon(2\alpha_j)}{\Upsilon(\alpha - Q) \prod_{j=1}^{3} \Upsilon(\alpha - 2\alpha_j)} . \tag{2.16}$$

What allowed us to rewrite $C(\alpha_1, \alpha_2, \alpha_3)$ in such a form is the following periodicity property of the function $\Upsilon(x)$

$$\begin{aligned} \Upsilon(x+b) &= \gamma(bx) b^{1-2bx} \Upsilon(x) \\ \Upsilon(x+b^{-1}) &= \gamma(b^{-1}x) b^{2b^{-1}x-1} \Upsilon(x) , \end{aligned} \tag{2.17}$$

which can be used to trade a product of N γ-function in (2.15) for a ratio of Υ-functions. Moreover, $\Upsilon(x)$ is the unique function having the correct set of zero points

$$x = \begin{cases} -mb - nb^{-1}, \\ Q + mb + nb^{-1}, \end{cases} \qquad m, n = 0, 1, 2, \dots , \tag{2.18}$$

which means that the analytic continuation (2.16) of (2.15) truly is the full 3-point function.

The next case that will be our interest is the $(3+m)$-point correlation function where 3 primary operators are generic while the remaining m are degenerate, namely their momentum is set to $-\frac{b}{2}$. The free field integral associated to this correlator was also studied in [38], where it was rewritten in a form that is again suitable for analytic continuation in N

$$\begin{aligned} \langle V_{-\frac{b}{2}}(z_1) \dots V_{-\frac{b}{2}}(z_m) V_{\alpha_1}(0) V_{\alpha_2}(1) V_{\alpha_3}(\infty) \rangle &= \Omega_m^N(\alpha_1, \alpha_2, \alpha_3) \prod_{a=1}^{m} |z_a|^{2b\alpha_1} |z_a - 1|^{2b\alpha_2} \times \\ \times \prod_{a<b}^{m} |z_a - z_b|^{-b^2} \int \mathrm{d}^2 \vec{x}_k \prod_{a<b}^{m} |x_a - x_b|^{-4b^2} &\prod_{a=1}^{m} |x_a|^{2A} |x_a - 1|^{2B} K_m^C(x_1, .., x_m | z_1, .., z_m) , \end{aligned} \tag{2.19}$$

where

$$
\begin{aligned}
A &= b\left(\alpha - 2\alpha_1 - Q + k\frac{b}{2}\right), \\
B &= b\left(\alpha - 2\alpha_2 - Q + k\frac{b}{2}\right), \\
C &= b\left(Q + \frac{(2-k)b}{2} - \alpha\right).
\end{aligned} \tag{2.20}
$$

The function $K^C_m(x_1,..,x_m|z_1,..,z_m)$, which was referred to as *kernel function* in [38], also admits a representation in terms of a complex rank $k(k-1)/2$ integral. Specifically, it admits the following recursive definition:

$$
\begin{aligned}
K^\Delta_N(m_1,..,m_N|t_1,..,t_N) = \frac{\gamma(-Nb^2)}{\gamma(-b^2)^N} \prod_{a<b}^N |m_a - m_b|^{2+4b^2} \prod_{a=1}^N |m_a - t_N|^{2\Delta} \times \\
\times \int \mathrm{d}^2\vec{u}_{N-1} \prod_{i<j}^{N-1} |u_i - u_j|^2 K^{\Delta+b^2}_{N-1}(u_1,..,u_{N-1}|t_1,..,t_{N-1}) \times \\
\times \prod_{i=1}^{N-1} |u_i - t_N|^{-2\Delta+2b^2} \prod_{a=1}^{N} |u_i - m_a|^{-2-2b^2}.
\end{aligned} \tag{2.21}
$$

This function possesses a remarkable symmetry property that is hidden is this integral definition

$$
K^\Delta_N(m_1,..,m_N|t_1,..,t_N) = K^\Delta_N(t_1,..,t_N|m_1,..,m_N). \tag{2.22}
$$

In particular, this implies that it is invariant not only under permutations of the parameters m_a, which is a symmetry that is manifest in the definition (2.21), but also of the parameters t_a.

The integral on the r.h.s of eq. (2.19) is suitable for analytic continuation in N, since it only enters as a parameter in the sum of the momenta α, fixed by the screening condition (2.8), appearing in A, B, C. The prefactor $\Omega^N_k(\alpha_1,\alpha_2,\alpha_3)$ instead is the product of $4N-3k$ factors of the function $\gamma(x) = \Gamma(x)/\Gamma(1-x)$. However , by using again the periodicity property (2.17) of the Υ-function, we can re-express the contribution of $N-k$ γ-functions in terms of a single Υ-function moving the dependence on N inside the argument of the Υ, so that also $\Omega^N_k(\alpha_1,\alpha_2,\alpha_3)$ depends parametrically on N. This equivalent form of the free field correlator is then suitable for analytic continuation in N.

This concludes our review of free field correlators in Liouville theory. As we will see, we will find a $3d$ $\mathcal{N}=2$ gauge theory avatar of the main free field integrals we discussed (2.9)-(2.12)-(2.13)-(2.19)-(2.21) and we will be able to reinterpret the identities (2.13)-(2.15)-(2.19)-(2.22) that they satisfy as non-trivial dualities for the $3d$ gauge theories.

2.3 Free field identities from $3d$ dualities

In this section we will establish the connection between the $\mathbb{S}^2 \times \mathbb{S}^1$ partition function of $3d$ $\mathcal{N} = 2$ gauge theories and free field complex integrals. We will take as a prototypical example the duality between the $U(N)$ gauge theory with one adjoint chiral and one flavor with no superpotential and the WZ model of $3N$ chirals with a cubic superpotential discussed in [37]. We will show how the associated identity of the $\mathbb{S}^2 \times \mathbb{S}^1$ partition functions reduces in the $2d$ Coulomb limit to the evaluation formula (2.15) of the free field correlator of the 3-point function of Liouville theory. We will then be able to uplift to three dimensions its derivation by first identifying the necessary identity (2.13) as coming from the $3d$ duality for the $U(N_c)$ gauge theory with N_f flavors and one of the fundamental monopoles turned on in the superpotential discussed in [41]. Finally, we will be able to interpret in gauge theory the analytic continuation in N (2.16) that we saw from the CFT point of view. This will be related to the $5d$ T_2 theory.

2.3.1 Confining duality for $U(1)$ with 1 adjoint and 1 flavor and its $2d$ limit

We start considering the following $3d$ $\mathcal{N} = 2$ duality [37]:

Theory A: $U(N)$ gauge theory with one adjoint chiral Φ, one fundamental flavor P, $\tilde{P}$, N chiral singlets b_j and superpotential

$$\mathcal{W} = \sum_{j=1}^{N} b_j \operatorname{Tr} \Phi^j \,. \tag{2.23}$$

Theory B: WZ model with $3N$ chiral fields α_j, $T_j^{\pm}$ and cubic superpotential

$$\hat{\mathcal{W}} = \sum_{i,j,l=1}^{N} \alpha_i T_j^{+} T_{N-l+1}^{-} \delta_{i+j+l,2N+1} \,. \tag{2.24}$$

This duality can be understood as a non-abelian generalization of the duality between SQED with one flavor and the XYZ model[4] [47]. A key role is played by the b-fields, whose equations of motion have the effect of setting to zero all the Casimir operators $\operatorname{Tr}\Phi^j$ in the chiral ring. Indeed, such Casimir operators are expected to violate the unitarity bound, which means that they become a decoupled free sector of the theory in the IR. In [48] it was shown that the correct way to deal with such operators is to flip them by introducing some additional gauge singlet chiral fields that couple to them in the superpotential. The b-fields have the interesting property of vanishing in the chiral ring because of some quantum effects. One argument for this was used in [49], following [50]. If any of these gauge singlets, say b_k,

[4]For $N = 1$ Φ is a gauge singlet and the superpotential $\mathcal{W} = b_1\Phi$ implies that both b_1 and Φ are massive. Integrating them out we get a $U(1)$ gauge theory with one chiral of charge $+1$ and one chiral of charge -1.

	$U(1)_s$	$U(1)_p$	$U(1)_\omega$	$U(1)_R$
P	0	1	0	r
$\tilde{P}$	0	1	0	r
Φ	2	0	0	$2(1-R)$
b_j	$-2j$	0	0	$2-2j(1-R)$
$\mathfrak{M}^{\pm}_{\Phi^s}$	$-2(N-s-1)$	-1	± 1	$1-r-2(N-s-1)(1-R)$
$\mathrm{Tr}\left(\tilde{P}\Phi^s P\right)$	$2s$	2	0	$2r+2s(1-R)$
α_i	$2(i-1)$	2	0	$2r+2(i-1)(1-R)$
T^+_j	$-2(N-j)$	-1	1	$1-r-2(N-j)(1-R)$
T^-_{N-l+1}	$-2(N-l)$	-1	-1	$1-r-2(N-l)(1-R)$

Table 2.2: Charges under the global symmetries and R-charges of all the chiral fields and of the main gauge invariant chiral operators of the dual theories.

acquires a non-vanishing VEV, then a superpotential of the form $\mathcal{W} = \mathrm{Tr}\,\Phi^k$ is generated, but the theory with such a superpotential has no stable supersymmetric vacua because of the very low number of flavors.

The global symmetry group of Theory A consists of three non-R abelian global symmetries: two of them are flavor symmetries that rotate independently the adjoint chiral and the fundamental flavor, while the third one is the topological symmetry

$$U(1)_s \times U(1)_p \times U(1)_\omega\,. \tag{2.25}$$

The labels for each $U(1)$ factor are related to the letter we will use to denote the corresponding parameters in the supersymmetric partition functions. The two abelian flavor symmetries can mix with the R-symmetry in the IR [16], so to each of them we also associate a mixing coefficient. We parametrize the mixing with $U(1)_s$ by $1-R$ and the one with $U(1)_p$ by r, namely

$$R = R_0 + q_s(1-R) + q_p r\,, \tag{2.26}$$

where R is the charge of a chiral field under a generic R-symmetry, R_0 is the charge under a reference R-symmetry compatible with the superpotential and q_s, q_p are the charges under $U(1)_s$, $U(1)_p$ respectively.

The one in (2.25) is also the global symmetry group of Theory B because of the cubic superpotential. In Table 2.2 we summarize the charges of all the chiral fields of the two theories under the global symmetries and we also specify our parametrization of the R-charges in terms of r and R.

As a first check of the duality, we can map the gauge invariant operators in the chiral rings of the two theories. This is basically a non-abelian generalization of the map for the SQED/XYZ duality, where the two monopole operators and the meson of the electric theory are mapped into the three gauge singlets of the magnetic theory. In our non-abelian case, we can dress these fundamental operators with powers of the adjoint chiral Φ [51]. In total, we get $3N$ independent operators on the side of Theory A

$$\mathfrak{M}^{\pm}_{\Phi^s}\,, \qquad \mathrm{Tr}\left(\tilde{P}\Phi^s P\right)\,, \qquad s=0,\cdots,N-1\,. \tag{2.27}$$

Their charges under the global symmetries are listed in Table 2.2. These operators directly map under the duality into the $3N$ singlets of the WZ theory

$$\begin{aligned} \mathfrak{M}^{+}_{\Phi^s} \quad &\leftrightarrow \quad T^{+}_{s+1} \\ \mathfrak{M}^{-}_{\Phi^s} \quad &\leftrightarrow \quad T^{-}_{N-s} \\ \mathrm{Tr}\left(\tilde{P}\Phi^s P\right) \quad &\leftrightarrow \quad \alpha_{s+1}\,, \qquad s=0,\cdots,N-1\,. \end{aligned} \tag{2.28}$$

Let us now consider the identity of the $\mathbb{S}^2\times\mathbb{S}^1$ partition functions, a.k.a. supersymmetric index, of the dual theories. These can be refined with fugacities for the global symmetries (2.25). The supersymmetric index of Theory A then takes the form (see Appendix A.2 for more details on our conventions for the $\mathbb{S}^2\times\mathbb{S}^1$ partition function)

$$\begin{aligned} \mathcal{I}_{\mathcal{T}_A} \;=\;& \prod_{j=1}^{N}\frac{\left(s^{2j}x^{2j(1-R)};x^2\right)_\infty}{\left(s^{-2j}x^{2-2j(1-R)};x^2\right)_\infty}\sum_{\vec{m}\in\mathbb{Z}^N}\frac{\prod_{i=1}^N\omega^{m_i}}{N!}\oint\prod_{i=1}^N\frac{\mathrm{d}z_i}{2\pi i\, z_i}s^{-2\sum_{i<j}^N|m_i-m_j|}\times \\ &\times\; p^{-\sum_{i=1}^N|m_i|}x^{2(R-1)\sum_{i<j}^N|m_i-m_j|-(r-1)\sum_{i=1}^N|m_i|}\prod_{i<j}^N\left(1-\left(\frac{z_i}{z_j}\right)^{\pm1}x^{|m_i-m_j|}\right)\times \\ &\times\;\prod_{i,j=1}^N\frac{\left(\frac{z_i}{z_j}s^{-2}x^{2R+|m_i-m_j|};x^2\right)_\infty}{\left(\frac{z_j}{z_i}s^2x^{2(1-R)+|m_i-m_j|};x^2\right)_\infty}\prod_{i=1}^N\frac{z_i^{\pm1}p^{-1}x^{2-r+|m_i|}}{z_i^{\mp1}px^{r+|m_i|}}\,, \end{aligned} \tag{2.29}$$

while the index of Theory B is

$$\begin{aligned} \mathcal{I}_{\mathcal{T}_B} \;=\;& \prod_{j=1}^N\frac{\left(s^{-2(j-1)}p^{-2}x^{2-2(j-1)(1-R)-2r};x^2\right)_\infty}{\left(s^{2(j-1)}p^2x^{2(j-1)(1-R)+2r};x^2\right)_\infty}\frac{\left(s^{2(N-j)}p\,\omega^{-1}x^{1+2(N-j)(1-R)+r};x^2\right)_\infty}{\left(s^{-2(N-j)}p^{-1}\omega\, x^{1-2(N-j)(1-R)-r};x^2\right)_\infty}\times \\ &\times\;\frac{\left(s^{2(j-1)}p\,\omega\, x^{1+2(j-1)(1-R)+r};x^2\right)_\infty}{\left(s^{-2(j-1)}p^{-1}\omega^{-1}x^{1-2(j-1)(1-R-r)};x^2\right)_\infty}\,. \end{aligned} \tag{2.30}$$

As an additional test of the duality, this identity can be checked perturbatively for low values of the rank N by expanding both sides in the variable x, see Section 2.3.3 of [24] for more details.

We would like to show that this identity reduces, in a suitable 2*d* limit, to the evaluation formula (2.15) for the free field integral of the 3-point function of Liouville theory. Let us start discussing the general strategy for getting a free field integral identity as a limit of an identity between $\mathbb{S}^2 \times \mathbb{S}^1$ partition functions and then apply it explicitly to our duality.

First of all, we use the identity [52]

$$(-x)^{\frac{|m|-m}{2}} \zeta^{-\frac{|m|-m}{2}} \frac{\left(\zeta^{-1} x^{2+r+|m|}; x^2\right)_\infty}{\left(\zeta\, x^{r+|m|}; x^2\right)_\infty} = \frac{\left(\zeta^{-1} x^{2+r+m}; x^2\right)_\infty}{\left(\zeta\, x^{r+m}; x^2\right)_\infty}\,, \tag{2.31}$$

to rewrite the index of our theory in a form which does not contain absolute values of the magnetic fluxes. For example, the contribution to the index of a chiral field (A.33) becomes

$$Z_{chi} = \prod_{\rho\in\mathcal{R}} \prod_{\sigma\in\mathcal{R}_G} \left((-z)^\rho t^\sigma x^{r-1}\right)^{-\frac{\rho(m)+\sigma(n)}{2}} \frac{\left(z^{-\rho} t^{-\sigma} x^{2-r+\rho(m)+\sigma(n)}; x^2\right)_\infty}{\left(z^\rho t^\sigma x^{r+\rho(m)+\sigma(n)}; x^2\right)_\infty}\,, \tag{2.32}$$

where we also turned on magnetic fluxes n for the global symmetry corresponding to the fugacity t [30]. We then define the complex variables

$$u_i = z_i x^{-m_i} \qquad \bar{u}_i = z_i^{-1} x^{-m_i}\,. \tag{2.33}$$

In this way, the sum over the magnetic fluxes and the contour integral over the gauge fugacities transform into an integral over the entire complex plane. To be precise, with this change of variables we are interpreting $u = \mathrm{e}^{i\theta}$ as the phase and $r = \mathrm{e}^{-\frac{\beta}{2}m}$ as the radius of the complex coordinate (recall that $x^2 = \mathrm{e}^{-\beta}$, where β is the radius of the $\mathbb{S}^1$), with m taking discrete values. Hence, for finite values of β we are only integrating over a discrete set of concentric circles, but we will recover the integral over the whole complex plane after taking the limit $\beta \to 0$. The new integration measure will be

$$\sum_{m\in\mathbb{Z}} \oint \frac{\mathrm{d}z}{2\pi i\, z} = \sum_{r\in\mathrm{e}^{-\frac{\beta}{2}\mathbb{Z}}} \int_0^{2\pi} \frac{\mathrm{d}\theta}{2\pi} \underset{\beta\to 0}{\longrightarrow} \int_{\mathbb{C}} \frac{\mathrm{d}^2 u}{\pi\beta|u|^2}\,, \tag{2.34}$$

where $z = r\mathrm{e}^{i\theta}$ and, after the $2d$ limit, we are using the same conventions of [38], namely $\mathrm{d}^2 z = \mathrm{d}x\,\mathrm{d}y$ with $z = x + iy$. Intuitively, since the integrand depends on the combination βm with $m \in \mathbb{Z}$, in the limit $\beta \to 0$ this becomes a continuous variable and the concentric circles fill the entire complex plane.

To be more precise on this continuum limit of the sum over the magnetic fluxes, we can think that the limit of the index that we are considering is of the form

$$\mathcal{I} = \lim_{\beta\to 0} \sum_{m\in\mathbb{Z}} f(\beta m) = \lim_{\beta\to 0} \lim_{M\to\infty} \sum_{m=-M}^{+M} f(\beta m)\,. \tag{2.35}$$

We can use the Euler–Maclaurin formula to approximate a finite sum with an integral

$$\sum_{j=m}^{n} f(j) = \int_m^n f(x)\,\mathrm{d}x + \frac{f(n)+f(m)}{2} + R\left(f^{(k)}\right) , \tag{2.36}$$

where R is the error that we are doing in the approximation and depends polynomially on the derivatives of the function f. Since our integrand is actually a function of $j = \beta m$, we have that these corrections are of order $\mathcal{O}(\beta)$ and can thus be neglected in the $\beta \to 0$ limit. Hence, we can write

$$\begin{aligned} \mathcal{I} &= \lim_{\beta\to 0}\lim_{M\to\infty}\left(\int_{-M}^{M} f(\beta x)\,\mathrm{d}x + \frac{f(M)+f(-M)}{2}\right) \\ &= \lim_{\beta\to 0}\lim_{M\to\infty}\left(\int_{-\beta M}^{\beta M} f(y)\frac{\mathrm{d}y}{\beta} + \frac{f(M)+f(-M)}{2}\right) \\ &= \lim_{\beta\to 0}\int_{-\infty}^{\infty} f(y)\frac{\mathrm{d}y}{\beta} . \end{aligned} \tag{2.37}$$

This justifies why in the 2*d* limit we can replace the summation over the magnetic fluxes with an additional integration. The expression for the integration measure (2.34) is then obtained from (2.37) upon the change of variable $r = \mathrm{e}^{-y/2}$.

Now that we have two real integrations for each element of the Cartan of the gauge group, we can re-arrange them into a single complex integral. Indeed, after the change of variables (2.33) the integrand can be factorized into a holomorphic and an anti-holomorphic part. For the contribution of a vector multiplet this is immediately done by just using the change of variables (2.33)

$$\begin{aligned} Z_{vec} &= \prod_{i<j}^{N_c} x^{-(m_i-m_j)}\left(1-\left(\frac{z_i}{z_j}\right)^{\pm 1} x^{m_i-m_j}\right) = \\ &= \prod_{i<j}^{N_c}\left|\frac{u_i}{u_j}\right|\left|1-\frac{u_j}{u_i}\right|^2 = \prod_{i=1}^{N_c}|u_i|^{-N_c+1}\prod_{i<j}^{N_c}|u_i-u_j|^2 \end{aligned} \tag{2.38}$$

For the contribution of a chiral multiplet, instead, this can be achieved with some manipulations done by means of the identity

$$(-\zeta)^{-m}\frac{(\zeta^{-1}x^{2+m};x^2)_\infty}{(\zeta\,x^m;x^2)_\infty} = x^{-m}\frac{(\zeta^{-1}x^{2-m};x^2)_\infty}{(\zeta\,x^{-m};x^2)_\infty} , \tag{2.39}$$

The holomorphic and the anti-holomorphic parts will then combine in the 2*d* limit so to give a complex integral.

Finally, the 2*d* limit we will consider will be obtained by properly scaling the parameters of the matrix integral of the partition function with the radius β. The scaling that we will be interested in is associated with the $2d$ Coulomb limit studied in [8]. This limit will be implemented using the following asymptotic properties of the q-Pochhammer symbol:

$$\begin{aligned}\lim_{x\to1}\frac{\left(x^{2a};x^2\right)_\infty}{\left(x^{2b};x^2\right)_\infty}&=\frac{\Gamma(b)}{\Gamma(a)}(1-x^2)^{b-a}\,,\\ \lim_{x\to1}\frac{\left(zx^{2a};x^2\right)_\infty}{\left(zx^{2b};x^2\right)_\infty}&=(1-z)^{b-a}\,.\end{aligned}\tag{2.40}$$

We will now apply this strategy to the identity between the indices (2.29) and (2.30) associated with our $3d$ duality. First of all, we notice that in both of the expressions for the indices we can reabsorb the dependence on the mixing coefficients with the R-symmetry inside the fugacities s and p by rescaling

$$s\to s\,x^{R-1}\,,\qquad p\to p\,x^{-r}\,.\tag{2.41}$$

Moreover, it is convenient to consider the version of the indices where we remove the absolute values by means of (2.31). After these manipulations, the indices of the dual theories become

$$\begin{aligned}\mathcal{I}_{\mathcal{T}_A}&=\prod_{j=1}^N\frac{\left(s^{-2}x^2;x^2\right)_\infty}{\left(s^2;x^2\right)_\infty}\frac{\left(s^{2j};x^2\right)_\infty}{\left(s^{-2j}x^2;x^2\right)_\infty}\sum_{\vec{m}\in\mathbb{Z}^N}\frac{\prod_{a=1}^N\omega^{m_a}}{N!}\oint\prod_{a=1}^N\frac{\mathrm{d}z_a}{2\pi i z_a}\times\\ &\times\prod_{a<b}^N\left(\frac{s}{x}\right)^{2(m_a-m_b)}\left(1-\left(\frac{z_a}{z_b}\right)^{\pm1}x^{m_a-m_b}\right)\frac{\left(\left(\frac{z_a}{z_b}\right)^{\mp1}s^{-2}x^{2-(m_a-m_b)};x^2\right)_\infty}{\left(\left(\frac{z_a}{z_b}\right)^{\pm1}s^2x^{-(m_a-m_b)};x^2\right)_\infty}\times\\ &\times\prod_{a=1}^N\left(\frac{p}{x}\right)^{m_a}\frac{\left(z_a^{\mp1}p^{-1}x^{2-m_a};x^2\right)_\infty}{\left(z_a^{\pm1}p\,x^{-m_a};x^2\right)_\infty}\\ \mathcal{I}_{\mathcal{T}_B}&=\prod_{j=1}^N\frac{\left(s^{-2(j-1)}p^{-2}x^2;x^2\right)_\infty}{\left(s^{2(j-1)}p^2;x^2\right)_\infty}\frac{\left(s^{2(N-j)}p\,\omega^{-1}x;x^2\right)_\infty}{\left(s^{-2(N-j)}p^{-1}\omega\,x;x^2\right)_\infty}\frac{\left(s^{2(j-1)}p\,\omega\,x;x^2\right)_\infty}{\left(s^{-2(j-1)}p^{-1}\omega^{-1}x;x^2\right)_\infty}\,,\end{aligned}\tag{2.42}$$

where we already used (2.39) to manipulate the contributions of the chiral fields.

At this point, we have to decide how the parameters of the theory scale with the radius β of the $\mathbb{S}^1$ as it goes to zero. In this choice is encoded the physics of the $2d$ limit [8]. One possibility is the so-called *Higgs limit*, in which we rescale both the vector masses and the axial masses with the radius, while we don't rescale the FI parameters. In this way, all the matter fields remain light and the Higgs branch is preserved, while the Coulomb branch is lifted. Moreover, we also need to rescale the integration variables, which means that we are looking at the theory close to the origin of the moduli space. This limit yields an integral identity corresponding to a (massive) duality between $2d$ GLSM's.

Instead, we are interested in the opposite limit, that is called *Coulomb limit*. In this limit the axial masses and the FI parameter remain small, while the vector masses become very large. In the specific case of the 3*d* duality we are considering there are no vector masses, so this limit is simply achieved by rescaling

$$s = x^{2\phi_1}, \qquad p = x^{2\phi_2}, \qquad \omega = x^{2\phi_3}\,. \tag{2.43}$$

We also define

$$u_a = z_a x^{-m_a} \qquad \bar{u}_a = z_a^{-1} x^{-m_a}\,. \tag{2.44}$$

Taking this into account, we can rewrite the two indices as

$$\begin{aligned}
\mathcal{I}_{\mathcal{T}_A} &= \prod_{j=1}^{N} \frac{\left(x^{2(1-2\phi_1)};x^2\right)_\infty}{(x^{4\phi_1};x^2)_\infty} \frac{\left(x^{4j\phi_1};x^2\right)_\infty}{(x^{2(1-2j\phi_1)};x^2)_\infty} \frac{1}{N!}\int \prod_{a=1}^{N} \frac{\mathrm{d}^2 u_a}{\pi\beta} \times \\
&\quad \times \prod_{a=1}^{N} |u_a|^{-2\phi_3-2\phi_2-1} \frac{\left(u_a x^{2-2\phi_2};x^2\right)_\infty \left(\bar{u}_a x^{2-2\phi_2};x^2\right)_\infty}{(u_a x^{2\phi_2};x^2)_\infty (\bar{u}_a x^{2\phi_2};x^2)_\infty} \times \\
&\quad \times \prod_{a<b}^{N} \left|\frac{u_b}{u_a}\right|^{4\phi_1} \left|1-\frac{u_a}{u_b}\right|^2 \frac{\left(\frac{u_a}{u_b}x^{2-4\phi_1};x^2\right)_\infty \left(\frac{\bar{u}_a}{\bar{u}_b}x^{2-4\phi_1};x^2\right)_\infty}{\left(\frac{u_a}{u_b}x^{4\phi_1};x^2\right)_\infty \left(\frac{\bar{u}_a}{\bar{u}_b}x^{4\phi_1};x^2\right)_\infty}\,, \\
\mathcal{I}_{\mathcal{T}_B} &= \prod_{j=1}^{N} \frac{\left(x^{2(1-2(j-1)\phi_1-2\phi_2)};x^2\right)_\infty}{(x^{4((j-1)\phi_1+\phi_2)};x^2)_\infty} \frac{\left(x^{1+4(N-j)\phi_1+2\phi_2-2\phi_3};x^2\right)_\infty}{(x^{1-4(N-j)\phi_1-2\phi_2+2\phi_3};x^2)_\infty} \frac{\left(x^{1+4(j-1)\phi_1+2\phi_2+2\phi_3};x^2\right)_\infty}{(x^{1-4(j-1)\phi_1-2\phi_2-2\phi_3};x^2)_\infty}\,.
\end{aligned} \tag{2.45}$$

At this point, we can take the 2*d* limit by sending $\beta \to 0$ and using the identities (2.40). Implementing this limit on the side of Theory A, we get

$$\begin{aligned}
\lim_{\beta\to 0} \mathcal{I}_{\mathcal{T}_A} &= \prod_{j=1}^{N} (1-x^2)^{4\phi_1(1-j)} \frac{\gamma(2\phi_1)}{\gamma(2j\phi_1)} \times \\
&\quad \times \frac{1}{N!}\int \prod_{a=1}^{N} \frac{\mathrm{d}^2 u_a}{\pi\beta} |u_a|^{-2\phi_3-2\phi_2-1} |1-u_a|^{2(2\phi_2-1)} \prod_{a<b}^{N} \left|\frac{u_b}{u_a}\right|^{4\phi_1} \left|1-\frac{u_a}{u_b}\right|^{8\phi_1}\,,
\end{aligned} \tag{2.46}$$

Notice that the result seems to be divergent because of the negative powers of β. Actually, also in the reduction of the index of Theory B we get a similar prefactor, so that considering the $2d$ limit of (2.42) the result is eventually finite

$$\lim_{\beta\to 0}\mathcal{I}_{\mathcal{T}_B} = \prod_{j=1}^{N}(1-x^2)^{-4(N-j)\phi_1-1}\gamma(2(j-1)\phi_1+2\phi_2)\gamma\left(\frac{1}{2}-2(N-j)\phi_1-\phi_2+\phi_3\right)\times$$
$$\times\quad \gamma\left(\frac{1}{2}-2(j-1)\phi_1-\phi_2-\phi_3\right)\,. \tag{2.47}$$

Equating the limit of the two indices and using that $1-x^2\approx\beta$ for β small, we find that (2.42) reduces to

$$\int\prod_{a=1}^{N}\mathrm{d}^2\vec{u}_N\,|u_a|^{-2\phi_3-2\phi_2-4(N-1)\phi_1-1}|1-u_a|^{2(2\phi_2-1)}\prod_{a<b}^{N}|u_a-u_b|^{8\phi_1}=$$
$$=\prod_{j=1}^{N}\frac{\gamma(2j\phi_1)}{\gamma(2\phi_1)}\gamma(2(j-1)\phi_1+2\phi_2)\gamma\left(\frac{1}{2}-2(N-j)\phi_1-\phi_2+\phi_3\right)\times$$
$$\times\gamma\left(\frac{1}{2}-2(j-1)\phi_1-\phi_2-\phi_3\right)\,. \tag{2.48}$$

Thus, we precisely recover (2.15), where the parameters are identified as

$$\begin{cases} b\alpha_1=\frac{1}{4}+(N-1)\phi_1+\frac{\phi_2}{2}+\frac{\phi_3}{2}\\ b\alpha_2=\frac{1}{2}-\phi_2\\ b\alpha_3=\frac{1}{4}+(N-1)\phi_1+\frac{\phi_2}{2}-\frac{\phi_3}{2}\\ b^2=-2\phi_1\end{cases}\,, \tag{2.49}$$

which satisfy the screening condition (2.8)

$$b(\alpha_1+\alpha_2+\alpha_3)=1-(N-1)b^2\,. \tag{2.50}$$

It is useful to go back to the dictionary we gave in Table 2.1 after having worked out this example. First of all, we immediately understand that the rank N of the gauge group determines the dimensionality of the integral and of the summation over magnetic fluxes of the matrix model for the $\mathbb{S}^2\times\mathbb{S}^1$ partition function and, consequently, of the complex integral obtained in the $2d$ Coulomb limit. For this reason, the rank is directly mapped to the number of insertions of the screening charge operator in the free field correlator. Regarding the parameters of the matrix integral, the fugacities s and p for the axial symmetries and ω for the topological symmetry have been mapped in (2.49) to the momenta of the three primary operators. Finally, in the $3d$ gauge theory there was no vector symmetry since the one acting on the fundamental flavor can be reabsorbed in the center of the $U(N)$ gauge group. This is compatible with the fact that vector masses are mapped to the insertion points of the operators in the CFT correlator, since for the case at hand of the 3-point function we can use conformal invariance to set them to 0, 1 and ∞, so to remove any dependence of the correlation function on them.

2.3.2 Dualities with monopole superpotentials and their $2d$ limit

We will now consider another example of a free field integral identity that can be obtained from the $2d$ Coulomb limit of a $3d$ duality. This example will also turn out to be very useful in the next subsection where we will present a derivation of the confining duality for the $U(N)$ gauge theory with one adjoint and one flavor, which mimics the evaluation formula (2.15) for the 3-point function of Liouville theory to which we just saw that the $3d$ duality reduces.

The free field integral identity we are interested in appeared in [38]

$$\int \mathrm{d}^2\vec{u}_{N_c} \prod_{i<j}^{N_c} |u_i - u_j|^2 \prod_{i=1}^{N_c}\prod_{a=1}^{N_f} |u_i - \tau_a|^{2p_a} = \prod_{a=1}^{N_f} \gamma(1+p_a) \prod_{a<b}^{N_f} |\tau_a - \tau_b|^{2(1+p_a+p_b)} \times \\ \times \int \mathrm{d}^2\vec{v}_{N_f-N_c-2} \prod_{i<j}^{N_f-N_c-2} |v_i - v_j|^2 \prod_{i=1}^{N_f-N_c-2} \prod_{a=1}^{N_f} |v_i - \tau_a|^{-2(1+p_a)}\,, \tag{2.51}$$

where the momenta have to satisfy the on-shell condition

$$\sum_{a=1}^{N_f} p_a = -N_c - 1\,. \tag{2.52}$$

We claim that this identity can be obtained as the $2d$ Coulomb limit of the $3d$ duality for the $U(N_c)$ gauge theory with N_f flavors and both of the fundamental monopoles turned on in the superpotential proposed in [41]. Notice that the identity (2.51) is more general than the one (2.13) we used in Section 2.2 to evaluate the free field integral (2.12) of the 3-point function of Liouville theory. The partially specialized identity (2.13) is instead the $2d$ Coulomb limit of the duality with only one monopole in the superpotential, which can be obtained from the one with two monopoles via a real mass deformation as it was shown in [41]. The one-monopole duality also has a further real mass deformation that leads to the Aharony duality [23], which from the free field point of view corresponds to a further specialization of (2.13) that is also used in the derivation of (2.15). Because of this, all of these $3d$ dualities will play an important role in the derivation of the confining duality for the $U(N)$ gauge theory with one adjoint and one flavor that we saw in the previous subsection in the $2d$ CFT context.

To be more precise, the $3d$ duality that reduces to (2.51) is the following[5] [41]:

[5]For $Nc = N$ and $N_f = 2N + 2$ the duality becomes a self-duality. This is related to the fact that in such a case the theory was identified in [53] to be the theory living on the S-duality domain wall of the $4d$ $\mathcal{N} = 2$

Theory 1: $U(N_c)$ with N_f fundamental flavors and superpotential

$$\mathcal{W} = \mathfrak{M}^+ + \mathfrak{M}^- \,. \tag{2.53}$$

Theory 2: $U(N_f - N_c - 2)$ with N_f fundamental flavors, N_f^2 singlets (collected in a matrix ${M^i}_j$) and superpotential

$$\hat{\mathcal{W}} = \sum_{i,j=1}^{N_f} {M^i}_j \tilde{q}_i q^j + \hat{\mathfrak{M}}^+ + \hat{\mathfrak{M}}^- \,. \tag{2.54}$$

The monopole superpotential completely breaks both the axial and the topological symmetry, so that the global symmetry group of the two theories is $SU(N_f) \times SU(N_f)$. It also uniquely fixes the R-charges of the chirals to be $\frac{N_f - N_c - 1}{2Nf}$. The fundamental gauge invariant operators of Theory 1 are the mesons, which are mapped to the singlets M_{ij} in the dual, as standard in Seiberg-like dualities.

We are interested in the integral identity between the $\mathbb{S}^2 \times \mathbb{S}^1$ partition functions of the dual theories. In doing this, we turn on fugacities t_a for the diagonal combination of the non-abelian symmetries $SU(N_f) \times SU(N_f)$ and s_a for their anti-diagonal combination. Moreover, we turn on background magnetic fluxes n_a for the diagonal combination of the non-abelian symmetries $SU(N_f) \times SU(N_f)$ [30]. This is needed in order to get in the $2d$ limit the identity (2.51) with the insertion points taking value on the entire complex plane rather than just on the unit circle, since these will be identified with $\tau_a = t_a x^{-n_a}$ where t_a is the fugacity while n_a is the flux. From the point of view of the complex integrals, this is fundamental since in the derivations presented in [38] the identity (2.51) is applied with the insertion points actually being integration variables. From the $3d$ gauge theory perspective, in the derivation we will present in the next subsection the basic dualities are applied inside a quiver and the diagonal combination of the non-abelian symmetries $SU(N_f) \times SU(N_f)$ is gauged. To consistently implement the gauging at the level of the supersymmetric index, we have to consider its refined version with background magnetic fluxes turned on at least for the global symmetry we want to gauge.

As we did in the previous example, we write the supersymmetric indices in a form in which there is no explicit dependence on the R-charges of the chirals by performing the shift of the fugacities

$$s_a \to s_a \, x^{-\frac{N_f - N_c - 1}{N_f}} \,. \tag{2.55}$$

$SU(N+1)$ gauge theory with $2N+2$ flavors (see also [54, 55] for the $N=2$ case and [56] for the cases of multiple domain walls).

The flavor fugacities then have to satisfy the balancing condition

$$\prod_{a=1}^{N_f} t_a = 1, \qquad \prod_{a=1}^{N_f} s_a = x^{N_f-N_c-1}\,. \tag{2.56}$$

In particular, the second constraint can also be understood as the fact that the monopole superpotential breaks the axial symmetry.

We also use the identity (2.31) to remove absolute values, exactly as we did in the example of the previous subsection. The supersymmetric indices of the dual theories then read

$$\begin{aligned}
\mathcal{I}_{\mathcal{T}_1} &= \frac{1}{N_c!}\sum_{\vec{m}\in\mathbb{Z}^{N_c}}\oint\prod_{i=1}^{N_c}\frac{\mathrm{d}z_i}{2\pi i\, z_i}\prod_{i<j}^{N_c} x^{-(m_i-m_j)}\left(1-\left(\frac{z_i}{z_j}\right)^{\pm1}x^{m_i-m_j}\right)\times\\
&\quad\times\prod_{i=1}^{N_c}\prod_{a=1}^{N_f}((-z_i)t_a)^{-(m_i+n_a)}\frac{\left(z_i^{\mp1}t_a^{\mp1}s_a^{-1}x^{2\pm(m_i+n_a)};x^2\right)_\infty}{\left(z_i^{\pm1}t_a^{\pm1}s_a x^{\pm(m_i+n_a)};x^2\right)_\infty}\\
\mathcal{I}_{\mathcal{T}_2} &= \frac{1}{(N_f-N_c-2)!}\prod_{a,b=1}^{N_f}\left(\frac{t_a}{t_b}s_a s_b x^{-1}\right)^{-\frac{n_a-n_b}{2}}\frac{\left(\frac{t_b}{t_a}s_a^{-1}s_b^{-1}x^{2+(n_a-n_b)};x^2\right)_\infty}{\left(\frac{t_a}{t_b}s_a s_b x^{n_a-n_b};x^2\right)_\infty}\times\\
&\quad\times\sum_{\vec{m}\in\mathbb{Z}^{N_f-N_c-2}}\oint\prod_{i=1}^{N_f-N_c-2}\frac{\mathrm{d}z_i}{2\pi i\, z_i}\prod_{i<j}^{N_f-N_c-2}x^{-(m_i-m_j)}\left(1-\left(\frac{z_i}{z_j}\right)^{\pm1}x^{m_i-m_j}\right)\times\\
&\quad\times\prod_{i=1}^{N_f-N_c-2}\prod_{a=1}^{N_f}\left((-z_i)t_a^{-1}\right)^{-(m_i-n_a)}\frac{\left(z_i^{\mp1}t_a^{\pm1}s_a x^{1\pm(m_i-n_a)};x^2\right)_\infty}{\left(z_i^{\pm1}t_a^{\mp1}s_a^{-1}x^{1\pm(m_i-n_a)};x^2\right)_\infty}\,.
\end{aligned} \tag{2.57}$$

We now want to consider the $2d$ Coulomb limit of this identity. In this case there is no topological symmetry, so the limit consists of keeping the axial masses small as the radius goes to zero, which is achieved in the index by the rescaling

$$s_a = x^{p_a+1}\,, \tag{2.58}$$

while the vector masses become large, which is achieved by not rescaling them. Moreover, we introduce the complex variables

$$u_i = z_i x^{-m_i} \qquad \bar{u}_i = z_i^{-1}x^{-m_i}\,, \qquad \tau_a = t_a x^{-n_a}, \qquad \bar{\tau}_a = t_a^{-1}x^{-n_a}\,. \tag{2.59}$$

Notice that after the rescaling (2.58), the balancing condition (2.56) precisely becomes the on-shell condition (2.52), which suggests that we are on the right track.

The last step before considering the $\beta\to0$ limit consists of rewriting the integrand as the product of a holomorphic and an anti-holomorphic part. This is done using (2.38) for the contribution of the vector multiplets and manipulating that of the chiral multiplets by means of (2.39). By doing so, the two supersymmetric indices can be written in the form

$$\begin{aligned}
\mathcal{I}_{\mathcal{T}_1} &= \frac{1}{N_c!}\int\prod_{i=1}^{N_c}\frac{\mathrm{d}^2u_i}{\pi\beta|u_i|^2}\prod_{i=1}^{N_c}|u_i|^{-N_c+1}\prod_{i<j}^{N_c}|u_i-u_j|^2\times\\
&\times\prod_{i=1}^{N_c}\prod_{a=1}^{N_f}|u_i\tau_a|^{-p_a}\frac{(u_i\tau_a x^{1-p_a};x^2)_\infty\,(\bar{u}_i\bar{\tau}_a x^{1-p_a};x^2)_\infty}{(u_i\tau_a x^{1+p_a};x^2)_\infty\,(\bar{u}_i\bar{\tau}_a x^{1+p_a};x^2)_\infty}\\
\mathcal{I}_{\mathcal{T}_2} &= \prod_{a=1}^{N_f}\frac{(x^{-2p_a};x^2)_\infty}{(x^{2(1+p_a)};x^2)_\infty}\prod_{a<b}^{N_f}\left|\frac{\tau_b}{\tau_a}\right|^{1+p_a+p_b}\frac{\left(\frac{\tau_a}{\tau_b}x^{-p_a-p_b};x^2\right)_\infty\left(\frac{\bar{\tau}_a}{\bar{\tau}_b}x^{-p_a-p_b};x^2\right)_\infty}{\left(\frac{\tau_a}{\tau_b}x^{2+p_a+p_b};x^2\right)_\infty\left(\frac{\bar{\tau}_a}{\bar{\tau}_b}x^{2+p_a+p_b};x^2\right)_\infty}\times\\
&\times\frac{1}{(N_f-N_c-2)!}\int\prod_{i=1}^{N_f-N_c-2}\frac{\mathrm{d}^2u_i}{\pi\beta|u_i|^2}\prod_{i=1}^{N_f-N_c-2}|u_i|^{-N_f+N_c+3}\prod_{i<j}^{N_f-N_c-2}|u_i-u_j|^2\times\\
&\times\prod_{i=1}^{N_f-N_c-2}\prod_{a=1}^{N_f}\left|\frac{u_i}{\tau_a}\right|^{p_a+1}\frac{\left(\frac{u_i}{\tau_a}x^{2+p_a};x^2\right)_\infty\left(\frac{\bar{u}_i}{\bar{\tau}_a}x^{2+p_a};x^2\right)_\infty}{\left(\frac{u_i}{\tau_a}x^{-p_a};x^2\right)_\infty\left(\frac{\bar{u}_i}{\bar{\tau}_a}x^{-p_a};x^2\right)_\infty}\,.
\end{aligned}\tag{2.60}$$

We can finally take the $2d$ limit using (2.40). For Theory 1 we have

$$\lim_{\beta\to0}\mathcal{I}_{\mathcal{T}_1}=\frac{\prod_{a=1}^{N_f}|\tau_a|^{-N_cp_a}}{\beta^{N_c}}\int\mathrm{d}^2\vec{u}_{N_c}\prod_{i=1}^{N_c}|u_i|^{-N_c-1-\sum_a p_a}\prod_{i<j}^{N_c}|u_i-u_j|^2\prod_{i=1}^{N_c}\prod_{a=1}^{N_f}|1-u_i\tau_a|^{2p_a}\,.\tag{2.61}$$

Using the on-shell condition (2.52) we can see that the power of $|u_i|$ is actually equal to zero. If we now perform the change of variables $u_i\to u_i^{-1}$, we get

$$\lim_{\beta\to0}\mathcal{I}_{\mathcal{T}_1}=\frac{\prod_{a=1}^{N_f}|\tau_a|^{-N_cp_a}}{\beta^{N_c}}\int\mathrm{d}^2\vec{u}_{N_c}\prod_{i<j}^{N_c}|u_i-u_j|^2\prod_{i=1}^{N_c}\prod_{a=1}^{N_f}|u_i-\tau_a|^{2p_a}\,,\tag{2.62}$$

where again we used the constraint (2.52) to remove a factor of $|u_i|$. Instead, for Theory 2 we have

$$\begin{aligned}
\lim_{\beta\to0}\mathcal{I}_{\mathcal{T}_2} &= \frac{1}{\beta^{N_f-N_c-2}}\prod_{a=1}^{N_f}(1-x^2)^{1+2p_a}\gamma(1+p_a)\prod_{a<b}^{N_f}\left|\frac{\tau_b}{\tau_a}\right|^{1+p_a+p_b}\left|1-\frac{\tau_b}{\tau_a}\right|^{2(1+p_a+p_b)}\times\\
&\times\int\mathrm{d}^2\vec{u}_{N_f-N_c-2}\prod_{i<j}^{N_f-N_c-2}|u_i-u_j|^2\prod_{i=1}^{N_f-N_c-2}\prod_{a=1}^{N_f}\left|1-\frac{u_i}{\tau_a}\right|^{-2(1+p_a)}\,,
\end{aligned}\tag{2.63}$$

which can be rewritten as

$$\lim_{\beta\to 0}\mathcal{I}_{\mathcal{T}_2}=\frac{(1-x^2)^{N_f-2N_c-2}}{\beta^{N_f-N_c-2}}\prod_{a=1}^{N_f}|\tau_a|^{-N_cp_a}\prod_{a=1}^{N_f}\gamma(1+p_a)\prod_{a<b}^{N_f}|\tau_a-\tau_b|^{2(1+p_a+p_b)}\times$$
$$\times\int \mathrm{d}^2\vec{u}_{N_f-N_c-2}\prod_{i<j}^{N_f-N_c-2}|u_i-u_j|^2\prod_{i=1}^{N_f-N_c-2}\prod_{a=1}^{N_f}|u_i-\tau_a|^{-2(1+p_a)}\,, \tag{2.64}$$

Notice that in both of the expressions for the $2d$ limit of the two indices we have a divergent prefactor. Using that for small β we can expand $1-x^2\approx\beta$, these prefactors precisely cancel when we equate them. Also the overall power of $|\tau_a|$ matches and we finally recover the duality between complex free field integrals (2.51).

We are now in position to complete the dictionary of Table 2.1. Specifically, the fugacities t_a for the vector symmetry corresponding to the diagonal combination of $SU(N_f)\times SU(N_f)$ have been mapped in (2.59) to the variables τ_a in the complex integral (2.51), which appear in the same way as the insertion points of the operators in the generic free field correlator (2.9).

2.3.3 Derivation of the confining duality via sequential deconfinement

After having understood the connection between $3d$ dualities and $2d$ free field correlators identities we can try to use it to uplift some known result about $2d$ CFTs to new results in $3d$ gauge theories. For example, we saw in Section 2.2 that the evaluation formula (2.15) for the free field integral representation of the 3-point function of Liouville theory (2.12) can be proven by iterating the fundamental identity (2.13). From the insight that we gained in the two previous subsections, we expect this to be uplifted to $3d$ to a derivation for the confining duality of $U(N)$ with one adjoint and one flavor by iteration of the more fundamental dualities with monopole superpotentials of [41] and the Aharony duality [23].

Let us start reviewing such fundamental dualities. The first one is the analogue of the one we saw in the previous subsection, but this time with only one of the two fundamental monopoles of $U(N)$ in the superpotential[6]:

Theory 1: $U(N_c)$ with N_f fundamental flavors and superpotential

$$\mathcal{W}=\mathfrak{M}^-\,. \tag{2.65}$$

[6] This duality was derived in [41] from the two-monopole duality via a real mass deformation

Theory 2: $U(N_f - N_c - 1)$ with N_f fundamental flavors, N_f^2 singlets (collected in a matrix $M^i{}_j$), an extra singlet S^+ and superpotential

$$\hat{\mathcal{W}} = \sum_{i,j=1}^{N_f} M^i{}_j \tilde{q}_i q^j + \hat{\mathfrak{M}}^+ + S^+ \hat{\mathfrak{M}}^- \,. \tag{2.66}$$

The global symmetry of the theories is $SU(N_f) \times SU(N_f) \times U(1)$. Compared to the two-monopole duality we have an additional abelian global symmetry corresponding to the combination of the axial symmetry and of the topological symmetry that is now preserved since we have only one of the two fundamental monopoles in the superpotential. This $U(1)$ can now mix with the R-symmetry in the IR, so in this case the R-charge of the chirals can't be fixed with the superpotential alone and F-extremization is needed [16]. The operator map is the same of the two-monopole duality, with the addition of the monopole $\mathfrak{M}^+$ which is not in the superpotential being mapped to the singlet S^+.

For simplicity, we are going to present our derivation at the level of the $\mathbb{S}_b^3$ partition function [17, 16, 18, 19] (see also Appendix A.3 for our conventions). For this, we will need the identity of the $\mathbb{S}_b^3$ partition functions associated to this one-monopole duality, which was derived in [41][7]

$$\begin{aligned}
\mathcal{Z}_{\mathcal{T}_1} &= \frac{1}{N_c!}\int \prod_{i=1}^{N_c} \mathrm{d}x_i\, \mathrm{e}^{i\pi(\eta - iQ)\sum_{i=1}^{N_c} x_i} \frac{\prod_{i=1}^{N_c}\prod_{a=1}^{N_f} s_b\left(i\frac{Q}{2} \pm (x_i + M_a) - \mu_a\right)}{\prod_{i<j}^{N_c} s_b\left(i\frac{Q}{2} \pm (x_j - x_i)\right)} = \\
&= \frac{1}{(N_f - N_c - 1)!}\mathrm{e}^{-i\pi\left(2\sum_{a=1}^{N_f} M_a\mu_a + (\eta - iQ)\sum_{a=1}^{N_f} M_a\right)} s_b\left(i\frac{Q}{2} - \eta\right) \times \\
&\quad \times \prod_{a,b=1}^{N_f} s_b\left(i\frac{Q}{2} - (\mu_a + \mu_b - M_a + M_b)\right) \times \\
&\quad \times \int \prod_{i=1}^{N_f - N_c - 1} \mathrm{d}x_i\, \mathrm{e}^{i\pi\eta\sum_{i=1}^{N_c} x_i} \frac{\prod_{i=1}^{N_f - N_c - 1}\prod_{a=1}^{N_f} s_b\left(\pm(x_i - M_a) + \mu_a\right)}{\prod_{i<j}^{N_f - N_c - 1} s_b\left(i\frac{Q}{2} \pm (x_j - x_i)\right)} = \mathcal{Z}_{\mathcal{T}_2}\,.
\end{aligned} \tag{2.67}$$

Here M_a are real masses for the diagonal combination of the $SU(N_f) \times SU(N_f)$ symmetries that rotate the two sets of chirals independently, μ_a are real masses for the anti-diagonal combination of $SU(N_f) \times SU(N_f)$ and η is the parameter associated to the diagonal com-

[7]Since we are going to apply this duality inside a quiver where part of the flavor symmetry is actually gauged as a $U(N_f)$ symmetry rather than $SU(N_f)$, it is important to keep the phases in the prefactor involving the real masses for the flavor symmetries, which are non-trivial if we don't imposes the tracelessness condition $\sum_a M_a = 0$. These factors have been neglected in [41], but they were worked out more in details in [24], where it was also discussed their effect on monopole operators. Moreover, these phases are related to contact terms in the two-point functions of the corresponding conserved currents as shown in [57, 58], where it was also discussed their role in dualities, in particular their importance in order for the matching of partition functions on $\mathbb{S}_b^3$ or $\mathbb{S}^2 \times \mathbb{S}^1$ to work.

bination of the axial and of the topological symmetry that is preserved by the monopole superpotential. These parameters have to satisfy the following balancing condition:

$$\eta + 2\sum_{a=1}^{N_f} \mu_a = iQ(N_f - N_c)\,, \tag{2.68}$$

which is a consequence of the monopole superpotential.

The second fundamental duality we will need is the Aharony duality [23] which, as it was shown in [41], can be obtained as a further real mass deformation of the one-monopole duality:

Theory 1: $U(N_c)$ with N_f flavors and superpotential $\mathcal{W} = 0$.

Theory 2: $U(N_f - N_c)$ with N_f flavors, N_f^2 singlets (collected in a matrix $M^i{}_j$), two extra singlets $S^\pm$ and superpotential $\hat{\mathcal{W}} = \sum_{i,j=1}^{N_f} M^i{}_j \tilde{q}_i q_j + S^- \hat{\mathfrak{M}}^+ + S^+ \hat{\mathfrak{M}}^-$.

The global symmetry of the theories is $SU(N_f) \times SU(N_f) \times U(1)^2$, where now both the axial symmetry and the topological symmetry are preserved since we have no monopole superpotential anymore. Moreover, the operator map now includes also the mapping of the monopole $\mathfrak{M}^-$ to the singlet S^-.

At the level of partition functions, the result of the real mass deformation is

$$\begin{aligned}
\mathcal{Z}_{\mathcal{T}_1} &= \frac{1}{N_c!}\int \prod_{i=1}^{N_c} \mathrm{d}x_i\, \mathrm{e}^{i\pi\xi\sum_{i=1}^{N_c} x_i} \frac{\prod_{i=1}^{N_c}\prod_{a=1}^{N_f} s_b\left(i\frac{Q}{2} \pm (x_i + M_a) - \mu_a\right)}{\prod_{i<j}^{N_c} s_b\left(i\frac{Q}{2} \pm (x_j - x_i)\right)} = \\
&= \mathrm{e}^{-i\pi\xi\sum_{a=1}^{N_f} M_a} s_b\left(i\frac{Q}{2} - \frac{iQ(N_f - N_c + 1) - 2\sum_{a=1}^{N_f}\mu_a \pm \xi}{2}\right) \times \\
&\times \prod_{a,b=1}^{N_f} s_b\left(i\frac{Q}{2} - (\mu_a + \mu_b - M_a + M_b)\right) \times \\
&\times \frac{1}{(N_f - N_c)!}\int \prod_{i=1}^{N_f - N_c} \mathrm{d}x_i\, \mathrm{e}^{i\pi\xi\sum_{i=1}^{N_c} x_i} \frac{\prod_{i=1}^{N_f - N_c}\prod_{a=1}^{N_f} s_b\left(\pm(x_i - M_a) + \mu_a\right)}{\prod_{i<j}^{N_f - N_c} s_b\left(i\frac{Q}{2} \pm (x_j - x_i)\right)} = \mathcal{Z}_{\mathcal{T}_2}\,,
\end{aligned} \tag{2.69}$$

where now we have the additional FI parameter ξ for the topological symmetry and there is no balancing condition anymore.

We will now sketch the general strategy for deriving the confining duality for $U(N)$ with one adjoint and one flavor we saw in Subsection 2.3.1 using these fundamental dualities and then apply it explicitly at the level of the $\mathbb{S}_b^3$ partition functions.

The idea, sketched in Figure 2.1, is to combine the one-monopole and the Aharony duality to find a dual frame for Theory A with lower rank and some extra singlets:

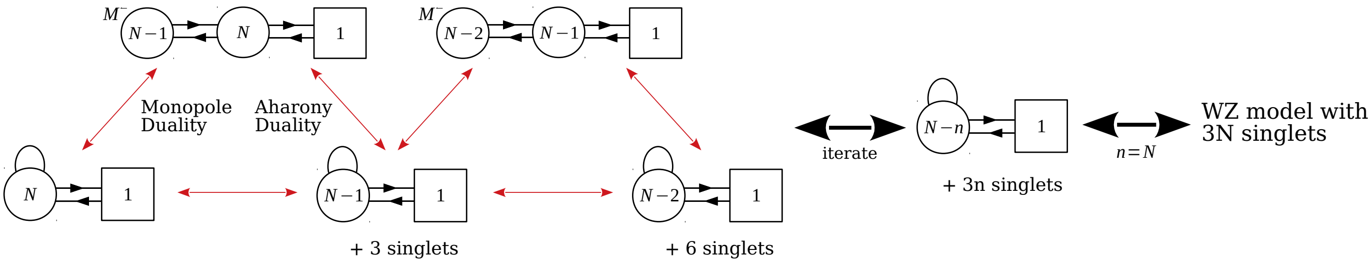

Figure 2.1: Diagrammatic representation of the manipulations we perform in the derivation of the duality.

1) The first step consists in viewing Theory A as the result of the application of the one-monopole duality to an auxiliary $U(N-1)\times U(N)$ quiver theory. The quiver has a flavor attached to the $U(N)$ node and $N-1$ singlets β_k flipping the traces of the $(k-1)$-th powers of the meson constructed with the bifundamental chirals Q, $\tilde{Q}$ connecting the two gauge nodes

$$\mathrm{Tr}_N\, \mathbb{M}^{k-1} \quad k=1,\cdots,N\,, \tag{2.70}$$

where Tr_N denotes the trace over the $U(N)$ color indices and

$$\mathbb{M}=\mathrm{Tr}_{N-1}\,\tilde{Q}Q \tag{2.71}$$

transforms in the adjoint representation of $U(N)$. In the auxiliary quiver theory there is also the negative fundamental monopole of the $U(N-1)$ node turned on in the superpotential and a BF coupling between the axial symmetry $U(1)_\tau$ and the gauge symmetry of the $U(N)$ node[8]. This BF coupling compensates a similar BF coupling which is generated when we apply the one-monopole duality to the $U(N-1)$ node, which confines since the number of flavors connected to it is N, yielding a $U(N)$ theory with one flavor. Moreover, the matrix of gauge singlets M appearing in the magnetic dual of the one-monopole duality reconstructs exactly the adjoint chiral for the $U(N)$ node, while the singlet S^+ is identified with the β_N singlet field. So we recovered Theory A.

2) The second step consists in starting from the auxiliary quiver theory and applying the Aharony duality to the $U(N)$ node, which confines since the number of flavors connected to it is N. Hence, we obtain a $U(N-1)$ theory with one flavor. The matrix of gauge singlets M in the magnetic dual of the Aharony duality then reconstructs the adjoint chiral for the $U(N-1)$ node, the fundamental flavor and a singlet, while two more singlets come from $S^\pm$. Moreover, the Aharony duality produces some contact terms that become

[8]Because of this BF coupling, the fundamental monopoles at the $U(N)$ node with opposite magnetic charge have different charge under $U(1)_\tau$, which implies that charge conjugation is broken in the auxiliary theory.

BF couplings for the $U(N-1)$ node. These BF couplings have the effect of changing the quantum numbers of the monopole operator, which is removed from the superpotential (see Sec. 2.3.2 of [24] for more details). So we obtain a dual frame for Theory A which is actually the same theory but with rank decreased by one unit and three extra singlets. These three singlets map to the highest dressed monopoles and mesons of the theory with $U(N)$ gauge group. Indeed, in the $U(N-1)$ frame we can only construct $3(N-1)$ dressed monopoles and mesons, which map to the same operators in the original $U(N)$ theory.

We thus see that the sequential application of the one-monopole and the Aharony duality only decreases the rank of Theory A (besides producing extra singlets). For this reason, we say that Theory A is *stable* under the sequential application of these two basic dualities. If we iterate this procedure N times, we completely confine the original gauge node and end up with a WZ model with $3N$ gauge singlets, which is the claimed dual theory.

We can repeat the steps we just described in field theory but at the level of the $\mathbb{S}_b^3$ partition functions, thus providing a new analytical proof of the following identity associated with the confining duality for the $U(N)$ gauge theory with one adjoint and one flavor:

$$\begin{aligned} \mathcal{Z}_{\mathcal{T}_A} = \prod_{j=1}^{N} s_b\left(-i\frac{Q}{2}+2j\tau\right) \int \mathrm{d}\vec{x}_N\, \mathrm{e}^{2\pi i\zeta\sum_\alpha x_\alpha} \frac{\prod_{\alpha,\beta=1}^{N} s_b\left(i\frac{Q}{2}+(x_\alpha-x_\beta)-2\tau\right)}{\prod_{\alpha<\beta}^{N} s_b\left(i\frac{Q}{2}\pm(x_\alpha-x_\beta)\right)} \times \\ \times \prod_{\alpha=1}^{N} s_b\left(i\frac{Q}{2}\pm x_\alpha-\mu\right) = \\ = \prod_{j=1}^{N} s_b\left(i\frac{Q}{2}-2\mu-2(j-1)\tau\right) s_b\left(-\zeta+\mu+2(N-j)\tau\right) s_b\left(\zeta+\mu+2(j-1)\tau\right) = \mathcal{Z}_{\mathcal{T}_B}\,, \end{aligned} \tag{2.72}$$

where the integration measure is now defined including the Weyl symmetry factor of the gauge group

$$\mathrm{d}\vec{x}_N = \frac{1}{N!}\prod_{i=1}^{N}\mathrm{d}x_i\,. \tag{2.73}$$

In this identity, τ is the real mass associated to the $U(1)_s$ symmetry, μ is the real mass associated to the $U(1)_p$ symmetry and ζ is the FI parameter associated to the $U(1)_\omega$ topological symmetry. More precisely, τ and μ have been defined as the following holomorphic combinations of the real masses with the corresponding mixing coefficients with the R-symmetry [16]:

$$\tau = \mathrm{Re}(\tau)+i\frac{Q}{2}(1-R)\,,\qquad \mu = \mathrm{Re}(\mu)+i\frac{Q}{2}r\,. \tag{2.74}$$

We start considering the partition function of Theory A

$$\begin{aligned}\mathcal{Z}_N(\tau,\zeta,\mu)\equiv\mathcal{Z}_{\mathcal{T}_A} &= \prod_{j=1}^{N} s_b\left(-i\frac{Q}{2}+2j\tau\right)\frac{1}{N!}\int\prod_{\alpha=1}^{N}\mathrm{d}x_\alpha\,\mathrm{e}^{2\pi i\zeta\sum_\alpha x_\alpha}\times\\ &\times\frac{\prod_{\alpha,\beta=1}^{N}s_b\left(i\frac{Q}{2}+(x_\alpha-x_\beta)-2\tau\right)}{\prod_{\alpha<\beta}^{N}s_b\left(i\frac{Q}{2}\pm(x_\alpha-x_\beta)\right)}\prod_{\alpha=1}^{N}s_b\left(i\frac{Q}{2}\pm x_\alpha-\mu\right).\end{aligned}\tag{2.75}$$

The first step of the derivation consists of replacing the contribution of the adjoint chiral with an auxiliary $U(N-1)$ integral using (2.67), where we identify Φ with the matrix M. In this way, we get the partition function of the auxiliary theory

$$\begin{aligned}\mathcal{Z}_N(\tau,\zeta,\mu) &= \prod_{j=1}^{N-1} s_b\left(-i\frac{Q}{2}+2j\tau\right)\frac{1}{(N-1)!}\int\prod_{\alpha'=1}^{N-1}\mathrm{d}y_{\alpha'}\frac{\mathrm{e}^{-2\pi iN\tau\sum_{\alpha'}y_{\alpha'}}}{\prod_{\alpha'<\beta'}^{N-1}s_b\left(i\frac{Q}{2}\pm(y_{\alpha'}-y_{\beta'})\right)}\times\\ &\times\frac{1}{N!}\int\prod_{\alpha=1}^{N}\mathrm{d}x_\alpha\frac{\mathrm{e}^{2\pi i(\zeta-(N-1)\tau)\sum_\alpha x_\alpha}}{\prod_{\alpha<\beta}^{N}s_b\left(i\frac{Q}{2}\pm(x_\alpha-x_\beta)\right)}\prod_{\alpha=1}^{N}s_b\left(i\frac{Q}{2}\pm x_\alpha-\mu\right)\times\\ &\times\prod_{\alpha'=1}^{N-1}s_b\left(i\frac{Q}{2}\pm(x_\alpha+y'_\alpha)-\tau\right).\end{aligned}\tag{2.76}$$

Notice the shift in the FI parameter of the $U(N)$ node by an amount proportional to the axial mass τ, which represents the BF coupling between the $U(1)_\tau$ symmetry and the gauge symmetry we mentioned above.

Now we can remove the original integral by means of the Aharony duality (2.69), which in the case $N_f=N_c=N$ becomes an evaluation formula

$$\begin{aligned}\mathcal{Z}_N(\tau,\zeta,\mu) &= \prod_{j=1}^{N-1} s_b\left(-i\frac{Q}{2}+2j\tau\right)s_b\left(i\frac{Q}{2}-2\mu\right)s_b\left(\zeta+\mu\right)\times\\ &\times\ s_b\left(-\zeta+\mu+2(N-1)\tau\right)\frac{1}{(N-1)!}\int\prod_{\alpha=1}^{N-1}\mathrm{d}y_\alpha\,\mathrm{e}^{-2\pi i(\zeta+\tau)\sum_\alpha y_\alpha}\times\\ &\times\frac{\prod_{\alpha,\beta=1}^{N-1}s_b\left(i\frac{Q}{2}\pm(y_\alpha-y_\beta)-2\tau\right)}{\prod_{\alpha<\beta}^{N-1}s_b\left(i\frac{Q}{2}\pm(y_\alpha-y_\beta)\right)}\prod_{\alpha=1}^{N-1}s_b\left(i\frac{Q}{2}\pm y_\alpha-\mu-\tau\right).\end{aligned}\tag{2.77}$$

At this point, we notice that we have reconstructed the same structure of the original partition function (up to the change of variables $y_i\to -y_i$), but with a lower dimensional integral, a shift of the parameters and three singlets:

$$\mathcal{Z}_N(\tau,\zeta,\mu) = s_b\left(i\frac{Q}{2}-2\mu\right)s_b\left(\zeta+\mu\right)s_b\left(-\zeta+\mu+2(N-1)\tau\right)\mathcal{Z}_{N-1}(\tau,\zeta+\tau,\mu+\tau)\,. \tag{2.78}$$

We refer to this property of the original partition function saying that it is *stabilized* with respect to the two moves we performed.

Notice that when we applied the Aharony duality to arrive at (2.77), the contact term from (2.69) given by $\mathrm{e}^{-2\pi i(\zeta-(N-1)\tau)\sum_\alpha y_\alpha}$ becomes a BF coupling for the remaining $U(N-1)$ node. More precisely, this contains a mixed CS term between the gauge and the topological symmetry $U(1)_\omega$. Thus, we conclude that the topological symmetry $U(1)_\omega$ is not broken and hence the monopole can't be turned on in the superpotential anymore. We also see that ζ is shifted by τ and we can interpret this as the fact that the monopoles with positive and negative magnetic charge have different charge under the axial symmetry $U(1)_s$, which means that at this stage charge conjugation is broken.

Finally we can use the stabilization property to highly simplify our expression by reducing the dimension of the integral, which from the field theory perspective is the rank of the gauge group. Indeed, iterating n times the two steps that we performed, we get

$$\begin{aligned}\mathcal{Z}_N(\tau,\zeta,\mu) &= \prod_{j=1}^{n}s_b\left(i\frac{Q}{2}-2\mu-2(j-1)\tau\right)s_b\left(-\zeta+\mu+2(N-j)\tau\right)\times\\ &\times\quad s_b\left(\zeta+\mu+2(j-1)\tau\right)\mathcal{Z}_{N-n}(\tau,\zeta+n\tau,\mu+n\tau)\,.\end{aligned} \tag{2.79}$$

If we set $n=N$, we completely confine the original gauge node and obtain the partition function of the WZ dual theory

$$\begin{aligned}\mathcal{Z}_{\mathcal{T}_A} &= \mathcal{Z}_N(\tau,\zeta,\mu) = \prod_{j=1}^{N}s_b\left(i\frac{Q}{2}-2\mu-2(j-1)\tau\right)\times\\ &\times s_b\left(-\zeta+\mu+2(N-j)\tau\right)s_b\left(\zeta+\mu+2(j-1)\tau\right) = \mathcal{Z}_{\mathcal{T}_B}\,.\end{aligned} \tag{2.80}$$

This procedure that we uplifted from $2d$ free field correlators of replacing a rank-2 chiral field with an auxiliary gauge node by means of a confining duality has become quite common in gauge theory and goes under the name of *deconfinement*. This first appeared in the context of $4d$ $\mathcal{N}=1$ theories in [59–61], in $3d$ $\mathcal{N}=2$ in [24] for the case of the duality we considered here and later in [62, 63] for more cases and in $2d$ $\mathcal{N}=(0,2)$ in [64] for a two-dimensional relative of the duality we considered here.

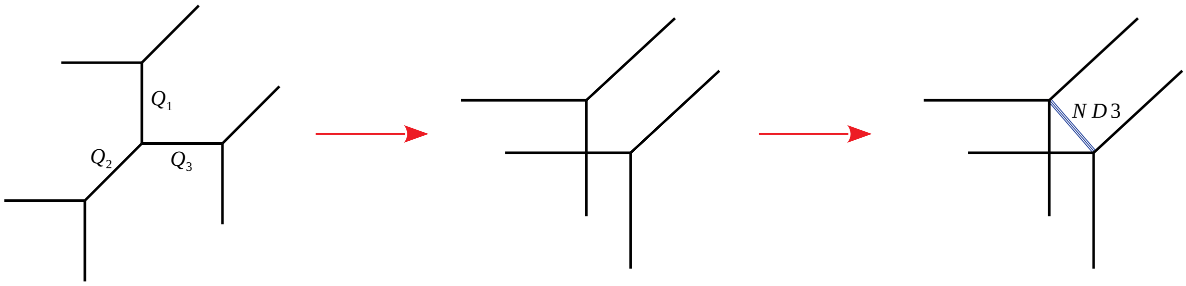

Figure 2.2: Schematic representation of the geometric transition from the $5d$ T_2 theory to the $3d$ $U(N)$ theory. At the first step we un-resolve the singularity of the quantized $\mathbb{P}^1$. At the second step, we move apart the two sets of intersecting five-branes, between which can then stretch N D3 branes. On these D3 branes lives the $U(N)$ theory.

2.3.4 Geometric transition and the $5d$ T_2 theory

We can also uplift the analytic continuation that we saw is needed in CFT to recover the DOZZ formula (2.16) from the evaluation formula (2.15) for the free field integral (2.12) of the 3-point function of Liouville theory.

At a purely mathematical level, we can take the localized partition function of the WZ model that is dual to the $U(N)$ gauge theory with one adjoint and one flavor on $\mathbb{S}^3_b$ or on $\mathbb{S}^2 \times \mathbb{S}^1$ and try to re-express the contribution of the $3d$ chiral fields in a form which depends only parametrically on N using the periodicity property of some special function.

For example, if we work on the three-sphere with squashing parameters $\omega_1 = b$ and $\omega_2 = b^{-1}$, the partition function of the WZ model reads:

$$\begin{aligned}\mathcal{Z}^{\mathbb{S}^3_b}_{WZ} &= \prod_{j=1}^{N} S_2\left(Q+2ij\tau\right) S_2\left(Q+2i\mu+2i(j-1)\tau\right)\times\\ &\times S_2\left(\frac{Q}{2}+i\zeta-i\mu-2i(N-j)\tau\right) S_2\left(\frac{Q}{2}-i\zeta-i\mu-2i(j-1)\tau\right),\end{aligned} \tag{2.81}$$

where $s_b\left(x\right) = S_2\left(\frac{Q}{2}-ix|b,b^{-1}\right) \equiv S_2\left(\frac{Q}{2}-ix\right)$ and to simplify the computation we moved to this side of the duality the contribution of the b-fields. Using the periodicity property

$$S_3(z+\omega_3|\omega_1,\omega_2,\omega_3) = \frac{S_3(z|\omega_1,\omega_2,\omega_3)}{S_2(z|\omega_1,\omega_2)}\,. \tag{2.82}$$

we can rewrite (2.81) in terms of the triple-sine function with $\omega_1 = b, \omega_2 = b^{-1}, \omega_3 = 2i\tau$ as:

$$\mathcal{Z}^{\mathbb{S}^3_b}_{WZ} = \operatorname*{Res}_{N\in\mathbb{N}} \frac{S_3'\left(0\right) S_3\left(-2i\mu+2i\tau\right) S_3\left(\frac{Q}{2}\pm i\zeta-i\mu-2i(N-1)\tau\right)}{S_3\left(-2iN\tau\right) S_3\left(-2i\mu-2i(N-1)\tau\right) S_3\left(\frac{Q}{2}\pm i\zeta-i\mu+2i\tau\right)} = \operatorname*{Res}_{N\in\mathbb{N}} \mathcal{Z}^{\mathbb{S}^5}_{T_2}\,, \tag{2.83}$$

where again for brevity we defined a compact version of the triple-sine function $S_3(z) \equiv S_3(z|b, b^{-1}, 2i\tau)$ in which the dependence on the (specialized) $\omega_{1,2,3}$ parameters is understood. The definition of the $S_2(z)$ and $S_3(z)$ functions as well as some of their properties are collected in the Appendix A.1.1 (more details can be found in [65]).

Therefore, in (2.83) we succeeded in trading our dependence on the number of fields N in the 3d WZ model for a parametric dependence on N inside the triple-sine functions, which is suitable for analytic continuation. But what is the physical interpretation of our result? The triple-sine function appears in the localized partition function of $\mathcal{N}=1$ theories on the five-sphere with squashing parameters $\omega_1, \omega_2, \omega_3$, [66–68]. We claim that the expression we found (2.83) is the five-sphere partition function of the 5d version of the T_2 theory [10], with one of its parameters taking a quantized value, as we will shortly explain. We already noticed that $2i\tau$ is identified with one of the squashing parameter of the five-sphere. The parameters μ, ζ and $2N\tau$ correspond instead to the fugacities for the Cartan subalgebra of the global $SU(2)^3 \subset USp(8)$ symmetry of the T_2 theory. Analytical continuation in N lifts the quantization condition on the fugacity $2iN\tau$ rendering it a free parameter.

The 5d T_N theory can be realized on a (p,q)-web of intersecting five-branes [69], the N-junction, consisting of N $(0,1)$-branes, N $(1,0)$-branes and N $(1,1)$-branes [42]. Equivalently, we can geometrically engineer this theory by M-theory compactified on the toric Calabi-Yau three-fold $\frac{\mathbb{C}^3}{\mathbb{Z}_N\times\mathbb{Z}_N}$, whose toric diagram coincides with the (p,q)-web. One can then use the refined topological vertex to calculate the partition function of the T_N theory, see for example [70]. The toric diagram for the case of T_2 is depicted in Figure 2.2. Each of the three internal lines corresponds to a resolved conifold geometry with Kähler parameter Q_i. The partition function of T_2 on the background $\mathbb{C}^2\times\mathbb{S}^1$ can then be computed using the refined vertex [71] as the topological string partition function associated to the diagram in Figure 2.2. The details can be found in [72] and [70]:

$$\mathcal{Z}_{top}[T_2] = \frac{\left(Q_1Q_2Q_3q^{1/2}t^{1/2};q,t\right)\prod_{i=1}^{3}\left(Q_iq^{1/2}t^{1/2};q,t\right)}{(Q_1Q_2t;q,t)\,(Q_1Q_3q;q,t)\,(Q_2Q_3t;q,t)}\,. \tag{2.84}$$

Finally, the five-sphere partition function of the T_2 theory can be obtained by *gluing* the contribution of three copies of the $\mathbb{C}^2\times\mathbb{S}^1$ partition function which we calculate with $Z_{top}[T_2]$ [67] (see also [73]). Indeed, by using the factorization property of the triple-sine function

$$\begin{aligned} S_3(x|\omega_1,\omega_2,\omega_3) &= \mathrm{e}^{-i\frac{\pi}{3!}B_{33}(x)}\left(\mathrm{e}^{\frac{2\pi i}{e_3}x};q^{-1},t\right)_1\left(\mathrm{e}^{\frac{2\pi i}{e_3}x};q^{-1},t\right)_2\left(\mathrm{e}^{\frac{2\pi i}{e_3}x};q^{-1},t\right)_3 \\ &\equiv \mathrm{e}^{-i\frac{\pi}{3!}B_{33}(x)}\left|\left|\left(\mathrm{e}^{\frac{2\pi i}{e_3}x};q^{-1},t\right)\right|\right|_S^3\,, \end{aligned} \tag{2.85}$$

where $q=\mathrm{e}^{-2\pi i\frac{e_1}{e_3}}$ and $t=\mathrm{e}^{2\pi i\frac{e_2}{e_3}}$ and the parameters e_i are chosen in each sector as in Table 2.3, we see that our expression (2.83) contains three copies of (2.84)[9]:

Sector	e_1	e_2	e_3
1	ω_3	ω_2	ω_1
2	ω_1	ω_3	ω_2
3	ω_1	ω_2	ω_3

Table 2.3: Squashing parameters and equivariant parameters in each sector.

$$\mathcal{Z}_{T_2}^{S^5} \sim ||Z_{top}[T_2]||_S^3 . \tag{2.86}$$

For example, in the first sector we have the following identification of the WZ parameters with the Kähler parameters:

$$Q_1 = \mathrm{e}^{i\pi} q^{-1/2} \mathrm{e}^{\frac{2\pi}{b}(\mu-\zeta)} \quad Q_2 = q^N q^{1/2} t^{-1/2}, \quad Q_3 = \mathrm{e}^{i\pi} q^{-1/2} \mathrm{e}^{\frac{2\pi}{b}(\mu+\zeta)} \tag{2.87}$$

and

$$q = \mathrm{e}^{\frac{4\pi\tau}{b}}, \qquad t = \mathrm{e}^{2\pi i b^{-2}} . \tag{2.88}$$

In particular we observe that the Kähler parameter Q_2 is quantized. The quantization condition of the Kähler parameter $Q_2 = q^N q^{1/2} t^{-1/2}$ signals that the theory can undergo geometric transition, as sketched in Figure 2.2. We shrink the volume of the $\mathbb{P}^1$ corresponding to this leg of the toric diagram returning to the singular conifold point and then we deform the singularity. In terms of the (p,q)-web, we arrive at a configuration of N D3 branes stretched between two 1-junctions of five-branes. The theory living on the N D3 branes is our $3d$ $U(N)$ theory with one adjoint and one flavor.

In the second sector we find the same map of parameters, but $q \leftrightarrow t^{-1}$ and $b \leftrightarrow b^{-1}$. Again Q_2 is quantized and the theory undergoes geometric transition. Instead, in the third sector we find:

$$Q_1 = \mathrm{e}^{\frac{i\pi}{\tau}(\mu-\zeta)} \quad Q_2 = q^{1/2} t^{-1/2}, \quad Q_3 = \mathrm{e}^{\frac{i\pi}{\tau}(\mu+\zeta)} . \tag{2.89}$$

Hence, the third sector actually gives a trivial contribution

$$\frac{(Q_1 Q_3 q; q, t)(Q_1 q^{1/2} t^{1/2}; q, t)(q; q, t)(Q_3 q^{1/2} t^{1/2}; q, t)}{(Q_1 q^{1/2} t^{1/2}; q, t)(Q_1 Q_3 q; q, t)(Q_3 q^{1/2} t^{1/2}; q, t)} = (q; q, t) .$$

So for $N \in \mathbb{N}$ we have only two sectors surviving which are precisely glued to reconstruct the $\mathbb{S}^3$ partition function which can be interpreted as the codimension-two defect theory inside $\mathbb{S}^5$.

Therefore, we managed to interpret the $3d$ duality relating the WZ model to the $U(N)$ theory with one adjoint and one flavor as two descriptions of the same defect theory: as

[9] We are omitting some classical contributions which are not captured by Z_{top}.

the $5d$ T_2 theory with specialized parameters or, after the geometric transition, as the $3d$ $U(N)$ theory on the stretched D3 branes[10]. In particular, the geometric transition is the counterpart of the analytic continuation in the number of screening charges on the CFT side. This interpretation was first put forward in [31, 32] in the context of the Gauge-Liouville triality and here we can see a very neat realization of this idea.

We also notice that the $\mathbb{S}^5$ partition function of the T_2 theory, after analytic continuation, can be identified with the 3-point function for the q-deformed Liouville theory with S-pairing [75]

$$\mathcal{Z}_{T_2}^{\mathbb{S}^5} = \frac{\prod_{i=1}^{3} S_3(2\alpha_i)}{S_3(\sum_{i=1}^{3}\alpha_i - (\omega_1+\omega_2+\omega_3)) \prod_{j=1}^{3} S_3(\sum_{i=1}^{3}\alpha_i - 2\alpha_j)} = C_S(\alpha_1,\alpha_2,\alpha_3)\,. \tag{2.90}$$

In order to see this, we simply need to manipulate the T_2 partition function (2.83) using the property (A.6) of the triple-sine function and use the following dictionary:

$$\begin{cases} \alpha_1 = \frac{Q}{2} + i\mu \\ \alpha_2 = \frac{Q}{4} + i\frac{\zeta}{2} - i\frac{\mu}{2} - i(N-1)\tau \\ \alpha_3 = \frac{Q}{4} - i\frac{\zeta}{2} - i\frac{\mu}{2} - i(N-1)\tau\,. \end{cases} \tag{2.91}$$

We can repeat the discussion above by working with the $\mathbb{S}^2\times\mathbb{S}^1$ partition function. In this case, we can use the periodicity property of the Υ_β function (see Appendix A.1.2)

$$\Upsilon_\beta(x+\epsilon_1|\epsilon_1,\epsilon_2) = \left(\frac{1-\mathrm{e}^{\beta}}{1-\mathrm{e}^{\beta\epsilon}}\right)^{1-\epsilon_2^{-1}x} \gamma_{\beta\epsilon_2}(x\epsilon_2^{-1})\Upsilon_\beta(x|\epsilon_1,\epsilon_2)\,, \tag{2.92}$$

where

$$\gamma_\beta(x) = (1-\mathrm{e}^{\beta})^{1-2x}\frac{\left(\mathrm{e}^{1-\beta x};\mathrm{e}^{\beta}\right)_\infty}{\left(\mathrm{e}^{\beta x};\mathrm{e}^{\beta}\right)_\infty}\,, \tag{2.93}$$

to re-express the contribution of the $3d$ chiral fields to the $\mathbb{S}^2\times\mathbb{S}^1$ partition function in terms of $5d$ hypers on $\mathbb{S}^4\times\mathbb{S}^1$ [76, 77], which can indeed be written using the Υ_β function, with β being the $\mathbb{S}^1$ radius. Hence, in this case we regard the $\mathbb{S}^2\times\mathbb{S}^1$ theory as a codimension-two defect theory inside $\mathbb{S}^4\times\mathbb{S}^1$, with the $3d$ partition function of the WZ model coinciding with the residue of the T_2 theory on $\mathbb{S}^4\times\mathbb{S}^1$.

The $\mathbb{S}^4\times\mathbb{S}^1$ partition function can in turn be obtained by *gluing* two copies of $Z_{top}[T_2]$:

$$\mathcal{Z}_{T_2}^{\mathbb{S}^4\times\mathbb{S}^1} \sim \left|\left|Z_{top}[T_2]\right|\right|_{id}^2 \tag{2.94}$$

[10] A related $5d$ interpretation of this $3d$ duality has been proposed in [74].

as it can be seen from the factorization property of the Υ_β function

$$\Upsilon_\beta(x|\epsilon_1,\epsilon_2) = (1-\mathrm{e}^{\beta})^{-\frac{1}{\epsilon_1\epsilon_2}\left(x-\frac{\epsilon_1+\epsilon_2}{2}\right)^2} \left|\left| \frac{\left(\mathrm{e}^{-\beta x};q,t\right)}{\left(\sqrt{\frac{t}{q}};q,t\right)} \right|\right|_{id}^2 , \tag{2.95}$$

where the id-norm is defined as

$$||(z;q,t)||_{id}^2 \equiv (z;q,t)\left(z^{-1};q^{-1},t^{-1}\right) , \tag{2.96}$$

and

$$q=\mathrm{e}^{-\beta\epsilon_1}, \qquad t=\mathrm{e}^{\beta\epsilon_2} . \tag{2.97}$$

Also in this case working out the dictionary between the WZ parameters and Kähler parameters we discover that Q_2 is quantized and correspondingly the theory undergoes geometric transition.

Finally, with the dictionary (2.91) we can also map $\mathcal{Z}_{T_2}^{\mathbb{S}^4\times\mathbb{S}^1}$ to the 3-point function for q-deformed Liouville theory with id-pairing [75, 78]

$$\mathcal{Z}_{T_2}^{\mathbb{S}^4\times\mathbb{S}^1} = \frac{\Upsilon_\beta'(0)\prod_{i=1}^3\Upsilon_\beta(2\alpha_i)}{\Upsilon_\beta(\sum_{i=1}^3\alpha_i-(\epsilon_1+\epsilon_2))\prod_{j=1}^3\Upsilon_\beta(\sum_{i=1}^3\alpha_i-2\alpha_j)} = C_{id}(\alpha_1,\alpha_2,\alpha_3) . \tag{2.98}$$

The 3-point function $C_{id}(\alpha_1,\alpha_2,\alpha_3)$ is the q-deformed version of the DOZZ formula (2.16) for the 3-point function in Liouville field theory [44, 40], to which it reduces in the limit $\beta\to 0$ thanks to the relation

$$\Upsilon_\beta(x|\epsilon_1,\epsilon_2) \underset{\beta\to 0}{\longrightarrow} \Upsilon(x|\epsilon_1,\epsilon_2) . \tag{2.99}$$

From the field theory point of view, the $\beta\to 0$ limit corresponds to shrinking the $\mathbb{S}^1$ radius, going from $\mathbb{S}^4\times\mathbb{S}^1$ to $\mathbb{S}^4$. This reproduces the familiar AGT map [11] between the partition function of the T_2 theory on $\mathbb{S}^4$ and the 3-point function in Liouville field theory.

2.4 Uplifting free field identities to new $3d$ dualities

In this section we will reverse the logic that we have followed so far and uplift the results about $2d$ free field correlators in Liouville theory that we reviewed in Section 2.2 to new dualities for $3d$ $\mathcal{N}=2$ gauge theories. We start introducing the gauge theory avatar of the kernel function (2.21), which we baptize $M[SU(N)]$, and discuss some of its properties, such as the uplift of the symmetry property (2.22). We then discuss two dualities involving

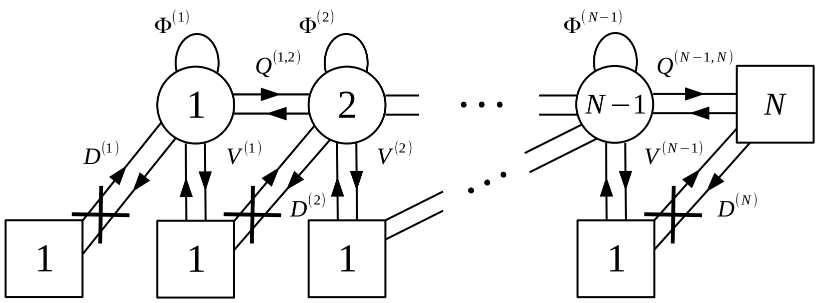

Figure 2.3: Quiver diagram of the $M[SU(N)]$ theory. Round nodes denote gauge symmetries and square nodes denote global symmetries. For this chapter the convention is that all the nodes are associated with unitary groups whose ranks are given by the numbers inside of them. Double-lines connecting two nodes represent pairs of bifundamental chirals in conjugate representations with respect to the corresponding symmetries. Lines that start and end on the same node correspond to chirals in the adjoint representation. The crosses of the diagonal lines represent the singlets β_i that flip the corresponding mesons.

theories that are constructed by gauging part of the global symmetry of $M[SU(N)]$. In particular, we will se that one of these theories is associated with a generalization of the confining duality for $U(N)$ with one adjoint and one flavor to a higher number of flavors.

2.4.1 The $M[SU(N)]$ theory

Lagrangian description and symmetry enhancement

The $3d$ $\mathcal{N}=2$ gauge theory version of the kernel function (2.21) is a quiver theory that we call $M[SU(N)]$[11] and which is represented in Figure 2.3. More precisely, the chiral fields of this theory are:

- $V^{(i)}$, $\tilde{V}^{(i)}$: fundamental flavors connecting the $U(i)$ gauge node with a $U(1)$ flavor node vertically;
- $D^{(i)}$, $\tilde{D}^{(i)}$: fundamental flavors connecting the $U(i)$ node with a $U(1)$ flavor node diagonally;
- $Q^{(i,i+1)}$, $\tilde{Q}^{(i,i+1)}$: bifundamental flavors connecting the i-th node with the $(i+1)$-th one[12]. For $i=N-1$ it connects the last $U(N-1)$ gauge node with the $SU(N)$ flavor symmetry on the very right;

[11] More precisely, what reduces to the kernel function is a variant of the $M[SU(N)]$ theory where we remove the flipping fields β_i and we add a set of singlets $\mathtt{O_H}$ that flip the meson $\mathrm{Tr}_{N-1}\, Q^{(N-1,N)}\tilde{Q}^{(N-1,N)}$ constructed with the last flavor of the saw.

[12] In our conventions, $Q^{(i,i+1)}$ transforms in the representation $\mathbf{i}\otimes\overline{\mathbf{i+1}}$ of $U(i)\times U(i+1)$, while $\tilde{Q}^{(i,i+1)}$ transforms in $\mathbf{i+1}\otimes\bar{\mathbf{i}}$ of $U(i+1)\times U(i)$, so some color indices are understood. For example, for $i=2$ we have $Q^{(i,i+1)}_{na}$ and $\tilde{Q}^{(i,i+1)}_{an}$, with $n=1,2$ and $a=1,2,3$.

- $\Phi^{(i)}$: adjoint chirals corresponding to the i-th gauge node;
- β_i: gauge singlet chiral fields flipping the mesons $D^{(i)}\tilde{D}^{(i)}$ constructed with the i-th diagonal chirals.

In order to write the superpotential of the theory in a compact form, we introduce the following notation. From the bifundamentals $Q^{(i,i+1)}_{na}$ and $\tilde{Q}^{(i,i+1)}_{bm}$ we construct a tensor that represents a chiral field in the representation $(\mathbf{i}\otimes\bar{\mathbf{i}})\otimes(\mathbf{i+1}\otimes\overline{\mathbf{i+1}})$ of $U(i)\times U(i+1)$:

$$\mathbb{Q}^{(i,i+1)}_{nmab}\equiv Q^{(i,i+1)}_{na}\tilde{Q}^{(i,i+1)}_{bm}, \qquad n,m=1,\cdots,i, \quad a,b=1,\cdots,i+1\,. \tag{2.100}$$

Moreover, we denote with Tr_i the trace over the color indices of the i-th gauge group $U(i)$. The superpotential of $M[SU(N)]$ contains the standard $\mathcal{N}=4$ cubic superpotential coupling bifundamentals and adjoints, a linear monopole superpotential involving the two fundamental monopoles of each node, a cubic interaction term coupling the fields in the *saw* to the bifundamentals and the flips of the diagonal mesons

$$\mathcal{W}_{M[SU(N)]}=\mathcal{W}_{mono}+\mathcal{W}_{T[SU(N)]}+\mathcal{W}_{cub}+\mathcal{W}_{flip}\,. \tag{2.101}$$

The first term is a linear monopole superpotential containing monopoles with magnetic flux ±1 with respect to only one of the factors in the gauge group[13]

$$\begin{aligned}\mathcal{W}_{mono} &= \mathfrak{M}^{(1,0,\cdots,0)}+\mathfrak{M}^{(-1,0,\cdots,0)}+\mathfrak{M}^{(0,1,0,\cdots,0)}+\\ &+ \mathfrak{M}^{(0,-1,0,\cdots,0)}+\dots\mathfrak{M}^{(0,\cdots,0,1)}+\mathfrak{M}^{(0,\cdots,0,-1)}\,.\end{aligned} \tag{2.102}$$

The second term is the superpotential of the $T[SU(N)]$ theory (see the Introduction for a brief review of this theory)[14] [12]

$$\mathcal{W}_{T[SU(N)]}=\sum_{i=1}^{N-1}\mathrm{Tr}_i\left[\Phi^{(i)}\left(\mathrm{Tr}_{i+1}\mathbb{Q}^{(i,i+1)}-\mathrm{Tr}_{i-1}\mathbb{Q}^{(i-1,i)}\right)\right]\,, \tag{2.103}$$

where we define $\mathbb{Q}^{(0,1)}=0$. The third term is given by

$$\mathcal{W}_{cub}=\sum_{i=1}^{N-1}\sum_{j=1}^{k}\sum_{l=1}^{i+1}\left(D^{(i+1)}_l\tilde{Q}^{(i,i+1)}_{lj}V^{(i)}_j+\tilde{V}^{(i)}_jQ^{(i,i+1)}_{jl}\tilde{D}^{(i+1)}_a\right)\,. \tag{2.104}$$

[13]Such a monopole superpotential can be understood from the 4d origin of the theory that we will discuss in Section 3.3. Indeed, the monopole superpotential (2.102) is dynamically generated in the dimensional reduction and the requirement in 4d that $U(1)_R$ is non-anomalous translates in 3d in the constraint on the R-charges due to the marginality of the monopoles. See [5, 6] for a general discussion of how monopole superpotentials are dynamically generated when reducing a 4d $\mathcal{N}=1$ theory to 3d with a circle compactification.

[14]We recall that the $T[SU(N)]$ theory admits a quiver representation which is very similar to that of $M[SU(N)]$, but without the fundamental flavors that form the structure of the saw. Moreover, the superpotential consists only of $\mathcal{W}_{T[SU(N)]}$.

	$U(1)_{Y_i}$	$U(1)_{Y_N}$	$SU(N)_X$	$U(1)_{m_A}$	$U(1)_\Delta$	$U(1)_R$
$Q^{(i-1,i)}$	0	0	$\mathbf{1}$	-1	0	$1-R_A$
$\tilde{Q}^{(i-1,i)}$	0	0	$\mathbf{1}$	-1	0	$1-R_A$
$Q^{(N-1,N)}$	0	0	$\bar{\mathbf{N}}$	-1	0	$1-R_A$
$\tilde{Q}^{(N-1,N)}$	0	0	$\mathbf{N}$	-1	0	$1-R_A$
$V^{(i)}$	1	0	$\mathbf{1}$	$i-N+2$	-1	$2+(N-i-2)(1-R_A)-R_\Delta$
$\tilde{V}^{(i)}$	-1	0	$\mathbf{1}$	$i-N+2$	-1	$2+(N-i-2)(1-R_A)-R_\Delta$
$V^{(N-1)}$	0	1	$\mathbf{1}$	1	-1	$1+R_A-R_\Delta$
$\tilde{V}^{(N-1)}$	0	-1	$\mathbf{1}$	1	-1	$1+R_A-R_\Delta$
$D^{(i)}$	-1	0	$\mathbf{1}$	$N-i$	1	$(i-N)(1-R_A)+R_\Delta$
$\tilde{D}^{(i)}$	1	0	$\mathbf{1}$	$N-i$	1	$(i-N)(1-R_A)+R_\Delta$
$D^{(N)}$	0	-1	$\mathbf{N}$	0	1	R_Δ
$\tilde{D}^{(N)}$	0	1	$\bar{\mathbf{N}}$	0	1	R_Δ
$\Phi^{(i)}$	0	0	$\mathbf{1}$	2	0	$2R_A$
β_i	0	0	$\mathbf{1}$	$-2(N-i)$	-2	$2+2(i-N)(R_A-1)-2R_\Delta$

Table 2.4: Representations and charges under the global symmetries of all the chiral fields of the $M[SU(N)]$ theory. In the table, i runs from 1 to $N-1$. By definition, $Q^{(0,1)}=\tilde{Q}^{(0,1)}=0$ and $V^{(0)}=\tilde{V}^{(0)}=0$.

Finally, the last term involves the singlets β_i, which flip the diagonal mesons

$$\mathcal{W}_{flip}=\sum_{i=1}^{N}\beta_i D^{(i)}\tilde{D}^{(i)}\,. \tag{2.105}$$

The $M[SU(N)]$ theory shares many properties with the more known $T[SU(N)]$ theory (we reviewed some of the properties of $T[SU(N)]$ in the Introduction). The first one consist of the fact that the manifest global symmetry of $M[SU(N)]$[15]

$$SU(N)_X\times\frac{\prod_{i=1}^{N}U(1)_{Y_i}}{U(1)}\times U(1)_{m_A}\times U(1)_\Delta\,, \tag{2.106}$$

gets enhanced in the IR to

$$SU(N)_X\times SU(N)_Y\times U(1)_{m_A}\times U(1)_\Delta\,. \tag{2.107}$$

This symmetry enhancement is analogous to the enhancement of the topological symmetry in $T[SU(N)]$. Our main argument to support this claim is a self-duality which we discuss in the following subsubsection, that swaps the $SU(N)_X$ and the $SU(N)_Y$ symmetries. Another evidence of the symmetry enhancement comes from the fact that the operators in the chiral ring with the same charges under the other global symmetries, included the R-symmetry, re-organize into representations of the full $SU(N)_Y$ symmetry, as we will show below.

[15] The overall $U(1)$ by which we quotient can be re-absorbed in the gauge group.

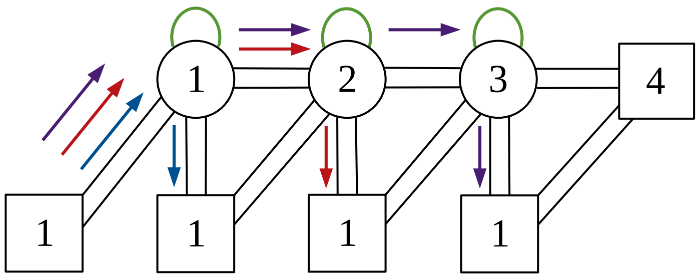

Figure 2.4: Diagrammatic representation of the operators in the first row of the matrix $\mathfrak{C}$. Arrows of the same color represent chiral fields that we assemble to construct an element of the matrix. In order to have a gauge invariant operator, we have to consider sequences of arrows that start and end on a squared node. In this case, this is achieved by starting with one diagonal flavor, going along the tail with an arbitrary number of bifundamentals and ending on a vertical flavor.

Hence, at low energies we only have two abelian global symmetries $U(1)_{m_A}$ and $U(1)_\Delta$ that can mix with the R-symmetry. We denote with R_A and R_Δ respectively the parameters that quantify this mixing. The R-charges of the fields will then be parametrized by these two coefficients as follows. We assign R-charge R_Δ to the last diagonal flavor $D^{(N)}$, $\tilde{D}^{(N)}$ and $1-R_A$ to the last bifundamental $Q^{(N-1,N)}$, $\tilde{Q}^{(N-1,N)}$. Because of the superpotential terms $\mathcal{W}_{T[SU(N)]}$ also all the other bifundamentals will have R-charge $1-R_A$, while the adjoint chirals $\Phi^{(i)}$ will have R-charge R_A. The cubic superpotential $\mathcal{W}_{cub}$ then fixes the R-charge of the last vertical flavor to be $R[V^{(N)},\tilde{V}^{(N)}]=2-(1-R_A)-R_\Delta=1+R_A-R_\Delta$. Then, we have to take into account the monopole superpotential. Requiring that the fundamental monopole operators of the $U(N-1)$ node are exactly marginal, we find that the next diagonal flavor must have R-charge $R[D^{(N-1)},\tilde{D}^{(N-1)}]=-1+R_A+R_\Delta$. Following this procedure along the whole tail, we can fix the R-charges of all the chiral fields in terms of the parameters R_A and R_Δ only. In Table 2.4 we summarize the charges of the chiral fields under all the global symmetries and we specify their R-charges.

The theory possesses several gauge invariant operators that are non-trivial in the chiral ring and in the following we are going to introduce those that will play an important role later. First of all, we have the meson constructed with the last flavors of the tail

$$\mathtt{H}=\mathrm{Tr}_{N-1}\,Q^{(N-1,N)}\tilde{Q}^{(N-1,N)}\,, \tag{2.108}$$

which transforms in the adjoint representation of $SU(N)_X$. By construction, this operator has charge -2 under $U(1)_{m_A}$, is uncharged under $U(1)_\Delta$ and has R-charge $2(1-R_A)$ under the trial R-symmetry $U(1)_R$.

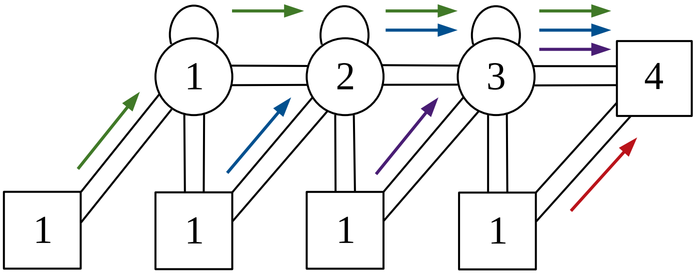

Figure 2.5: Diagrammatic representation of the operator Π. In this case, gauge invariant operators are obtained starting with one diagonal flavor, going along the tail with all the remaining bifundamentals and ending on the bifundamental connected to the last flavor node.

Then, we can construct an operator which transform in the adjoint representation of $SU(N)_Y$ combining the traces of the adjoints at each gauge node on the diagonal and some mixed mesons on the off-diagonal elements. These mesons are built starting from one of the diagonal chirals, moving along the tail with the bifundamentals and ending on a vertical chiral (see Figure 2.4). Explicitly, for $N = 3$ it takes the form

$$\mathtt{C} = \begin{pmatrix} 0 & V^{(1)}D^{(1)} & V_i^{(2)}\tilde{Q}_i^{(1,2)}D^{(1)} \\ \tilde{D}^{(1)}\tilde{V}^{(1)} & 0 & V_i^{(2)}D_i^{(2)} \\ \tilde{D}^{(1)}Q_i^{(1,2)}\tilde{V}_i^{(2)} & \tilde{D}_i^{(2)}\tilde{V}_i^{(2)} & 0 \end{pmatrix} + \sum_{i=1}^{2} \mathrm{Tr}_i\, \Phi^{(i)}\mathcal{D}_i\,, \tag{2.109}$$

where $\mathcal{D}_i$ are traceless diagonal generators of $SU(N)_X$. By construction, this operator has charge $+2$ under $U(1)_{m_A}$, is uncharged under $U(1)_\Delta$ and has R-charge $2R_A$ under the trial R-symmetry $U(1)_R$.

There are two other gauge invariant mixed mesons that one can construct from the chiral fields of the theory. In this case, we still start with a diagonal flavor and move along the tail, but we have to include all the bifundamentals and end with $Q^{(N-1,N)}$ (see Figure 2.5). Such operators can be collected in two vectors that we denote with Π and $\tilde{\Pi}$. Explicitly, for $N = 3$ these operators take the form

$$\Pi = \begin{pmatrix} \tilde{Q}_{i,a}^{(2,3)}\tilde{Q}_i^{(1,2)}D^{(1)} \\ \tilde{Q}_{i,a}^{(2,3)}D_i^{(2)} \\ D_a^{(3)} \end{pmatrix}, \qquad \tilde{\Pi} = \begin{pmatrix} \tilde{D}^{(1)}Q_i^{(1,2)}Q_{i,a}^{(2,3)} \\ \tilde{D}_i^{(2)}Q_{i,a}^{(2,3)} \\ \tilde{D}_a^{(3)} \end{pmatrix}. \tag{2.110}$$

They are uncharged under the axial symmetry $U(1)_{m_A}$, have charge $+1$ under the other abelian symmetry $U(1)_\Delta$, have R-charge R_Δ and transform respectively in the bifundamental $(\mathbf{N}, \overline{\mathbf{N}})$ and anti-bifundamental $(\overline{\mathbf{N}}, \mathbf{N})$ representation of the flavor symmetries $SU(N)_X \times SU(N)_Y$.

Finally, we have some gauge invariant operators that are singlets under the non-abelian global symmetries and are only charged under $U(1)_{m_A}$ and $U(1)_\Delta$. Those that will be important for us are the chiral singlets β_i and the mesons constructed with the vertical chirals and dressed with powers of the adjoints. We can collectively denote these operators by

$$\mathtt{B}_{ij} = \begin{cases} \beta_i & i = 1, \cdots, N, \quad j = 1 \\ \mathrm{Tr}_{i-1}\left[\left(\Phi^{(i-1)}\right)^{j-2} V^{(i-1)} V^{(i-1)}\right] & i = 2, \cdots, N, \quad j = 2, \cdots, i \end{cases} . \tag{2.111}$$

These operators have charge $2(i+j-N-1)$ under $U(1)_{m_A}$, charge -2 under $U(1)_\Delta$ and R-charge $2(1+N-j)+2(i+j-N-1)R_A - 2R_\Delta$ under the trial R-symmetry $U(1)_R$.

The list of the gauge invariant chiral operators with the corresponding charges under the global symmetries is

	$SU(N)_X$	$SU(N)_Y$	$U(1)_{m_A}$	$U(1)_\Delta$	$U(1)_R$
$\mathtt{H}$	$\mathbf{N^1-1}$	$\mathbf{1}$	-2	0	$2(1-R_A)$
$\mathtt{C}$	$\mathbf{1}$	$\mathbf{N^1-1}$	2	0	$2R_A$
Π	$\mathbf{N}$	$\bar{\mathbf{N}}$	0	1	R_Δ
$\tilde{\Pi}$	$\bar{\mathbf{N}}$	$\mathbf{N}$	0	1	R_Δ
$\mathtt{B}_{ij}$	$\mathbf{1}$	$\mathbf{1}$	$2(i+j-N-1)$	-2	$2(1+N-j)+2(i+j-N-1)R_A-2R_\Delta$

Finally, we can write down the partition function of the theory on the squashed three-sphere $\mathbb{S}^3_b$. We turn on real masses in the Cartan of all the factors in the global symmetry group (2.107), that we denote respectively with X_i, Y_i, $\mathrm{Re}(m_A)$ and $\mathrm{Re}(\Delta)$. The parameters for the two $U(1)$ symmetries are defined as holomorphic combinations of the corresponding real masses with the R-symmetry mixing parameters R_A and R_Δ

$$m_A = \mathrm{Re}(m_A) + i\frac{Q}{2}R_A, \qquad \Delta = \mathrm{Re}(\Delta) + i\frac{Q}{2}R_\Delta\,, \tag{2.112}$$

Then, the partition function can be written iteratively as

$$\begin{aligned}
\mathcal{Z}_{M[U(N)]}(\vec{X};\vec{Y};\Delta;m_A) = {} & \underbrace{s_b\left(-i\frac{Q}{2}+2\Delta\right)}_{\beta_N} \underbrace{\prod_{i=1}^{N} s_b\left(i\frac{Q}{2} \pm (X_i - Y_N) - \Delta\right)}_{D^{(N)},\tilde{D}^{(N)}} \times \\
& \times \int \frac{d\vec{z}_{N-1}}{\prod_{a<b}^{N-1} s_b\left(i\frac{Q}{2} \pm (z_a^{(N-1)} - z_b^{(N-1)})\right)} \underbrace{\prod_{a,b=1}^{N-1} s_b\left(i\frac{Q}{2} \pm (z_a^{(N-1)} - z_b^{(N-1)}) - 2m_A\right)}_{\Phi^{(N-1)}} \times \\
& \times \underbrace{\prod_{a=1}^{N-1} s_b\left(\pm(z_a^{(N-1)} - Y_N) + \Delta - m_A\right)}_{V^{(N-1)},\tilde{V}^{(N-1)}} \underbrace{\prod_{i=1}^{N} s_b\left(\pm(z_a^{(N-1)} - X_i) + m_A\right)}_{Q^{(N-1,N)},\tilde{Q}^{(N-1,N)}} \times \\
& \times \mathcal{Z}_{M[U(N-1)]}\left(z_1^{(N-1)}, \cdots, z_{N-1}^{(N-1)}; Y_1, \cdots, Y_{N-1}; \Delta + m_A - i\frac{Q}{2}; m_A\right),
\end{aligned} \tag{2.113}$$

where where we recall that the integration measure is defined including the Weyl symmetry factor of the gauge group

$$\mathrm{d}\vec{x}_k = \frac{1}{k!}\prod_{i=1}^{k}\mathrm{d}x_i^{(k)}\,. \tag{2.114}$$

In order to make sense of the recursive definition we also specify

$$\mathcal{Z}_{M[U(1)]}(X;Y;\Delta) = s_b\left(-i\frac{Q}{2}+2\Delta\right)s_b\left(i\frac{Q}{2}\pm(X-Y)-\Delta\right)\,. \tag{2.115}$$

The $\mathcal{Z}_{M[SU(N)]}(\vec{X};\vec{Y};\Delta;m_A)$ partition function is simply $\mathcal{Z}_{M[U(N)]}(\vec{X};\vec{Y};\Delta;m_A)$ with the tracelessness condition enforced for the fugacities of the $SU(N)_X$ and $SU(N)_Y$ symmetries

$$\sum_{i=1}^{N}X_i = \sum_{i=1}^{N}Y_i = 1\,. \tag{2.116}$$

Self-dualities

Another similarity between $M[SU(N)]$ ad $T[SU(N)]$ is a web of self-dualities that they both enjoy. We discussed in details the duality web of $T[SU(N)]$ in the Introduction, while here we will present the one of $M[SU(N)]$ which is schematically summarized in Figure 2.6. We will only state the self-dualities and their properties here, without discussing any test (some of which can be found in [25]). This is because, as we will explain in Section 3.3, the $M[SU(N)]$ theory can be obtained as a limit of a $4d$ $\mathcal{N}=1$ theory that enjoys exactly the same duality web. Hence, the validity of the dualities that we will discuss in this subsubsection is a direct consequence of the $4d$ dualities we will discuss in Section 3.3.

The first self-duality of $M[SU(N)]$ is a *mirror-like* duality, since it is reminiscent of the self-duality of $T[SU(N)]$ under mirror symmetry. This is depicted as the top horizontal line in Figure 2.6, where the dual theory is labelled $M[SU(N)]^\vee$. It is a self-duality in the sense that the theory is dual to itself, but with a non-trivial map of the gauge invariant operators. Specifically, the duality acts on the global symmetries of $M[SU(N)]$ as follows:

$$SU(N)_X \leftrightarrow SU(N)_Y\,,\qquad U(1)_{m_A}\leftrightarrow -U(1)_{m_A}\,. \tag{2.117}$$

Moreover, it also acts on the R-symmetry by mapping the mixing coefficient R_A as follows:

$$R_A \leftrightarrow 1-R_A\,. \tag{2.118}$$

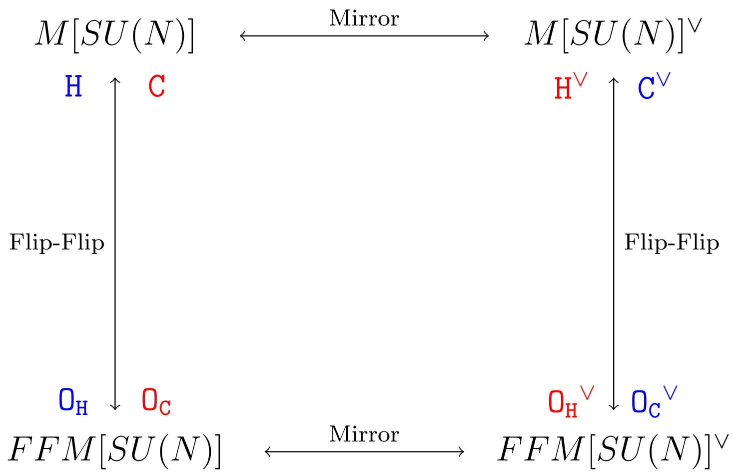

Figure 2.6: Duality web of the $M[SU(N)]$ theory. On the horizontal direction we have the mirror-like duality, while on the vertical direction we have the flip-flip duality. Operators of the same color are mapped to each other across the dualities.

The gauge invariant operators are accordingly mapped as

$$\begin{array}{rcl} \mathtt{H} & \leftrightarrow & \mathtt{C}^\vee \\ \mathtt{C} & \leftrightarrow & \mathtt{H}^\vee \\ \Pi & \leftrightarrow & \tilde{\Pi}^\vee \\ \tilde{\Pi} & \leftrightarrow & \Pi^\vee \\ \mathtt{B}_{ij} & \leftrightarrow & \mathtt{B}^\vee_{N-j+1,N-i+1}\,, \end{array} \tag{2.119}$$

where the label $^\vee$ denotes the operators in the mirror frame $M[SU(N)]^\vee$. As we reviewed in the Introduction, this action is similar to the action of mirror symmetry on $T[SU(N)]$, which swaps Higgs and Coulomb branch. Notice that this duality implies that the flavor symmetry $\prod_{i=1}^{N-1} U(1)_{Y_i}$ on the teeth of the saw is enhanced in the IR to the full non-abelian $SU(N)_Y$. At the level of the $\mathbb{S}^3_b$ partition function, the duality implies the following non-trivial integral identity:

$$\mathcal{Z}_{M[SU(N)]}(\vec{X};\vec{Y};\Delta;m_A) = \mathcal{Z}_{M[SU(N)]}(\vec{Y};\vec{X};\Delta;i\frac{Q}{2}-m_A) = \mathcal{Z}_{M[SU(N)]^\vee}(\vec{X};\vec{Y};\Delta;m_A) \tag{2.120}$$

This identity will derived in Subsection 3.3.3 as a limit of a similar identity for the $\mathbb{S}^3\times\mathbb{S}^1$ partition function of a $4d$ $\mathcal{N}=1$ duality that was proven in [79].

The second self-duality of $M[SU(N)]$ is called *flip-flip* duality, since it is reminiscent of the flip-flip of $T[SU(N)]$ discussed in [20]. This is depicted as the right vertical line in Figure 2.6, where the dual theory is labelled $FFM[SU(N)]$. In this case the theory is dual to itself up to not only a non-trivial map of the symmetries, but also the addition of some gauge singlet chiral fields that flip the operators $\mathtt{H}$ and $\mathtt{C}$ of $M[SU(N)]$. Specifically, the $FFM[SU(N)]$ theory is defined as the $M[SU(N)]$ theory with the addition of two sets of singlets $\mathtt{O}_H$ and $\mathtt{O}_\mathtt{C}$ in the adjoint representation of $SU(N)_X$ and $SU(N)_Y$ respectively, with superpotential

$$\mathcal{W}_{FFM[SU(N)]} = \mathcal{W}_{M[SU(N)]} + \mathrm{Tr}_X\left(\mathtt{O}_\mathtt{H}\mathtt{H}^{FF}\right) + \mathrm{Tr}_Y\left(\mathtt{O}_\mathtt{C}\mathtt{C}^{FF}\right)\,, \tag{2.121}$$

where $\mathrm{Tr}_{X/Y}$ denote the traces over the $SU(N)_{X/Y}$ flavor indices and $\mathtt{H}_{FF}$, $\mathtt{C}_{FF}$ denote the operators $\mathtt{H}$, $\mathtt{C}$ in the flip-flip dual frame. In this case, the non-abelian symmetries are not swapped under the duality, but the $U(1)_{m_A}$ is still inverted as for the mirror duality

$$SU(N)_X \leftrightarrow SU(N)_X\,, \qquad SU(N)_Y \leftrightarrow SU(N)_Y\,, \qquad U(1)_{m_A} \leftrightarrow -U(1)_{m_A}\,. \tag{2.122}$$

The mixing coefficient R_A is also inverted

$$R_A \leftrightarrow 1 - R_A\,. \tag{2.123}$$

The gauge invariant operators are accordingly mapped as

$$\begin{aligned}
\mathtt{H} &\leftrightarrow \mathtt{O}_\mathtt{H}\\
\mathtt{C} &\leftrightarrow \mathtt{O}_\mathtt{C}\\
\Pi &\leftrightarrow \Pi^{FF}\\
\tilde{\Pi} &\leftrightarrow \tilde{\Pi}^{FF}\\
\mathtt{B}_{ij} &\leftrightarrow \mathtt{B}^{FF}_{N-j+1,N-i+1}\,,
\end{aligned} \tag{2.124}$$

where the label ^{FF} denotes the operators in the flip-flip frame $FFM[SU(N)]$. At the level of the $\mathbb{S}^3_b$ partition function, the duality implies the following non-trivial integral identity:

$$\begin{aligned}
\mathcal{Z}_{M[SU(N)]}(\vec{X};\vec{Y};\Delta;m_A) &= \prod_{i,j=1}^{N}\frac{s_b\left(i\frac{Q}{2}\pm(Y_i-Y_j)-2m_A\right)}{s_b\left(i\frac{Q}{2}\pm(X_i-X_j)-2m_A\right)}\mathcal{Z}_{M[SU(N)]}(\vec{X};\vec{Y};\Delta;i\frac{Q}{2}-m_A) = \\
&= \mathcal{Z}_{FFM[SU(N)]}(\vec{X};\vec{Y};\Delta;m_A)\,.
\end{aligned} \tag{2.125}$$

The flip-flip duality is not a fundamental duality, in the sense that it can be derived by sequentially applying a more fundamental duality, which is the two-monopole duality we saw in Subsection 2.3.2[16]. We will discuss a similar derivation of the flip-flip duality for the 4*d* $\mathcal{N}=1$ ancestor of $M[SU(N)]$ by means of the Intriligator–Pouliot duality [81]. The derivation of flip-flip for $M[SU(N)]$ is completely analogous. This derivation also allows us to

[16]It is actually possible to show that also mirror symmetry for $M[SU(N)]$ can be derived by iteratively applying the two-monopole duality, but the derivation is much more involved. See [80] for the derivation in 4*d*, from which the statement for $M[SU(N)]$ descends.

derive the equality (2.125) by just iteratively applying the one for the two-monopole duality. Alternatively, exactly as for the identity of the mirror duality, (2.125) can be obtained as a limit of a similar identity for the $\mathbb{S}^3\times\mathbb{S}^1$ partition function of the $4d$ $\mathcal{N}=1$ duality that was proven in [79].

Mirror and flip-flip are the two fundamental dualities that constitute the diagram of Figure 2.6. Indeed, the fourth duality frame, labelled by $FFM[SU(N)]^\vee$, can just be obtained by applying mirror first and then flip-flip, or viceversa. The $FFM[SU(N)]^\vee$ theory is defined by the superpotential

$$\mathcal{W}_{FFM[SU(N)]^\vee}=\mathcal{W}_{M[SU(N)]}+\mathrm{Tr}_X\left(\mathtt{O}_{\mathtt{H}}^\vee\mathtt{H}^{FF\vee}\right)+\mathrm{Tr}_Y\left(\mathtt{O}_{\mathtt{C}}^\vee\mathtt{C}^{FF\vee}\right)\,, \tag{2.126}$$

and the duality relating $M[SU(N)]$ and $FFM[SU(N)]^\vee$ acts on the global symmetries by exchanging $SU(N)_X$ and $SU(N)_Y$, while leaving the abelian symmetries unchanged

$$SU(N)_X\leftrightarrow SU(N)_Y\,,\qquad U(1)_{m_A}\leftrightarrow U(1)_{m_A}\,. \tag{2.127}$$

The R-symmetry is left unchanged as well. The gauge invariant operators are accordingly mapped as

$$\begin{array}{ccc}\mathtt{H}&\leftrightarrow&\mathtt{O}_{\mathtt{C}}^\vee\\ \mathtt{C}&\leftrightarrow&\mathtt{O}_{\mathtt{H}}^\vee\\ \Pi&\leftrightarrow&\tilde{\Pi}^{FF\vee}\\ \tilde{\Pi}&\leftrightarrow&\Pi^{FF\vee}\\ \mathtt{B}_{ij}&\leftrightarrow&\mathtt{B}_{ij}^{FF\vee}\,.\end{array} \tag{2.128}$$

The equality of the $\mathbb{S}_b^3$ partition functions associated to this duality

$$\begin{aligned}\mathcal{Z}_{M[SU(N)]}(\vec{X};\vec{Y};\Delta;m_A)&=\prod_{i,j=1}^N\frac{s_b\left(i\frac{Q}{2}\pm(Y_i-Y_j)-2m_A\right)}{s_b\left(i\frac{Q}{2}\pm(X_i-X_j)-2m_A\right)}\mathcal{Z}_{M[SU(N)]}(\vec{Y};\vec{X};\Delta;m_A)=\\&=\mathcal{Z}_{FFM[SU(N)]^\vee}(\vec{X};\vec{Y};\Delta;m_A)\,,\end{aligned} \tag{2.129}$$

can be obtained by just applying sequentially (2.120) and (2.125).

This last duality is the natural $3d$ $\mathcal{N}=2$ uplift of the symmetry property of the kernel function (2.22). More precisely, the duality between $M[SU(N)]$ and $FFM[SU(N)]$ implies a non-trivial integral identity between their $\mathbb{S}^2\times\mathbb{S}^1$ partition functions which in the $2d$ Coulomb limit we discussed in the previous section reduces to (2.22). It would be interesting to understand the implications in CFT of the mirror and of the flip-flip duality. Indeed, in [24] only this last duality was discussed and proposed as an uplift of the result for CFT free fields, while the other dualities of the diagram in Figure 2.6 were discovered only later.

Real mass deformation to $T[SU(N)]$

The similarities we have highlighted so far between $M[SU(N)]$ and $T[SU(N)]$ are not an accident. In this section we show that, by taking a real mass deformation associated to the $U(1)_\Delta$ symmetry, the $M[SU(N)]$ theory reduces to the $T[SU(N)]$ theory. When this deformation is turned on, the chirals $D^{(k)}$, $\tilde{D}^{(k)}$ and $V^{(k)}$, $\tilde{V}^{(k)}$ that form the saw of the quiver and are charged under $U(1)_\Delta$ become massive. Integrating out these fields, mixed CS-like couplings between the gauge symmetry and the $\prod_{i=1}^{N-1} U(1)_{Y_i}$ symmetry are generated, so that this is now identified with the restored topological symmetry. This in turns implies that the monopole operators are no longer turned on in the superpotential and that they are part of the chiral ring.

All of the properties we have seen for $M[SU(N)]$ then reduce to similar properties for $T[SU(N)]$, since the real mass deformation doesn't affect them. For example, the symmetry enhancement $\prod_{i=1}^{N-1} U(1)_{Y_i} \to SU(N)_Y$ implies the well-known enhancement of the topological symmetry of $T[SU(N)]$. The role of the operator $\mathtt{C}$ of $M[SU(N)]$ is replaced by that of the monopole matrix $\mathcal{C}$ of $T[SU(N)]$, which is the moment map for the enhanced topological symmetry. Notice in particular that at the end of the flow triggered by the real mass deformation, supersymmetry is enhanced from $\mathcal{N}=2$ to $\mathcal{N}=4$. The operators Π, $\tilde{\Pi}$ are integrated out, since they are charged under the $U(1)_\Delta$ symmetry. Finally, the operator $\mathtt{H}$ is replaced by the operator $\mathcal{H}$, which is the moment map for the flavor symmetry of $T[SU(N)]$. Moreover, taking this limit on each frame of the duality web of $M[SU(N)]$ depicted in Figure 2.6 we precisely recover the duality web of $T[SU(N)]$ depicted in Figure 1.2.

We can also look at the effect of the real mass deformation at the level of the sphere partition function, where it is implemented by taking the limit $\Delta \to \infty$. This limit gives:

$$\lim_{\Delta\to+\infty} \mathcal{Z}_{M[U(N)]}(\vec{X};\vec{Y};\Delta;m_A) = C_N(\Delta,m_A,Q)\mathrm{e}^{-i\pi\sum_{i=1}^N (X_i^2+Y_i^2)}\mathcal{Z}_{T[U(N)]}(\vec{X};\vec{Y};m_A)\,, \tag{2.130}$$

where the prefactor

$$C_N(\Delta,m_A,Q) \;=\; \exp\left\{i\pi\left[\frac{1}{12}N\left(-12\Delta^2-8m_A^2(N-2)(N-1)+\left(2N^2+1\right)Q^2+\right.\right.\right.$$
$$\left.\left.\left.+4im_A(N-1)((2N-1)Q+6i\Delta)+12i\Delta NQ\right)\right]\right\} \tag{2.131}$$

is independent from the flavor fugacities X_i, Y_i and diverges for $\Delta \to \infty$, while the partition function of $T[U(N)]$ is defined iteratively as

$$\begin{aligned}
\mathcal{Z}_{T[U(N)]}(\vec{X};\vec{Y};m_A) = e^{2\pi i Y_N \sum_{i=1}^N X_i} \int \frac{d\vec{z}_{N-1}\, e^{2\pi i (Y_{N-1}-Y_N)\sum_{a=1}^{N-1} z_a^{(N-1)}}}{\prod_{a<b}^N s_b\left(i\frac{Q}{2} \pm (z_a^{(N-1)} - z_b^{(N-1)})\right)} \times \\
\times \prod_{a,b=1}^{N-1} s_b\left(i\frac{Q}{2} \pm (z_a^{(N-1)} - z_b^{(N-1)}) - 2m_A\right) \prod_{a=1}^{N-1}\prod_{i=1}^{N} s_b\left(\pm(z_a^{(N-1)} - X_i) + m_A\right) \times \\
\times \mathcal{Z}_{T[U(N-1)]}(z_1^{(N-1)}, \cdots, z_{N-1}^{(N-1)}; Y_1, \cdots, Y_{N-1} : m_A)\,,
\end{aligned} \tag{2.132}$$

with the case $N = 1$ defined as

$$\mathcal{Z}_{T[U(1)]}(X;Y) = e^{2\pi i XY}\,. \tag{2.133}$$

The proof of (2.130) proceeds by induction. We prove it first for $M[U(2)]$, whose partition function we recall being

$$\begin{aligned}
\mathcal{Z}_2 \equiv \mathcal{Z}_{M[U(2)]}(X_1, X_2; Y_1, Y_2; \Delta; m_A) = \prod_{i=1}^{2} s_b\left(i\frac{Q}{2} \pm (X_i - Y_2) - \Delta\right) \times \\
\times \prod_{i=1}^{2} s_b\left(-i\frac{Q}{2} + 2\Delta + 2(i-2)\left(i\frac{Q}{2} - m_A\right)\right) s_b\left(i\frac{Q}{2} - 2m_A\right) \times \\
\times \int dz\, s_b\left(iQ \pm (z - Y_1) - \Delta - m_A\right) s_b\left(\pm(z - Y_2) + \Delta - m_A\right) \prod_{i=1}^{2} s_b\left(\pm(z - X_i) + m_A\right)\,.
\end{aligned} \tag{2.134}$$

We focus on the limit of the following block of double-sine functions depending on Δ:

$$\begin{aligned}
\mathcal{B}_2 \;=\; & \prod_{i=1}^{2} s_b\left(-i\frac{Q}{2} + 2\Delta + 2(i-2)\left(i\frac{Q}{2} - m_A\right)\right) s_b\left(i\frac{Q}{2} \pm (X_i - Y_2) - \Delta\right) \times \\
\times\; & s_b\left(iQ \pm (z - Y_1) - \Delta - m_A\right) s_b\left(\pm(z - Y_2) + \Delta - m_A\right)\,.
\end{aligned} \tag{2.135}$$

Using the asymptotic behaviour of the double-sine function

$$\lim_{x \to \pm\infty} s_b(x) = e^{\pm i \frac{\pi}{2} x^2}\,, \tag{2.136}$$

we find

$$\begin{aligned}
\lim_{\Delta \to +\infty} \mathcal{B}_2 \;=\; & \exp\left[i\pi\left(\frac{Q^2}{4} - iQm_A + 2m_A^2 + 2\Delta^2 + \right.\right. \\
& \left.\left. - \sum_{i=1}^{2}(X_i^2 + Y_i^2) + 2z(Y_1 - Y_2) + 2Y_2\sum_{i=1}^{2} X_i\right)\right]\,.
\end{aligned} \tag{2.137}$$

The rest of the partition function is independent from Δ, so we find

$$\begin{aligned}\lim_{\Delta\to+\infty}\mathcal{Z}_2 &= C_2(m_A,\Delta)\mathrm{e}^{-i\pi\sum_{i=1}^2(X_i^2+Y_i^2)}\mathrm{e}^{2\pi i Y_2\sum_{i=1}^2 X_i}\times\\ &\times s_b\left(i\frac{Q}{2}-2m_A\right)\int \mathrm{d}z\,\mathrm{e}^{2\pi i(Y_1-Y_2)z}\prod_{i=1}^2 s_b\left(\pm(z-X_i)+m_A\right)=\\ &= C_2(m_A,\Delta)\mathrm{e}^{-i\pi\sum_{i=1}^2(X_i^2+Y_i^2)}\mathcal{Z}_{T[U(2)]}(X_1,X_2;Y_1,Y_2;m_A)\,,\end{aligned}\tag{2.138}$$

where the prefactor $C_2(m_A,\Delta)$ can be read from the first line of (2.137).

Now we consider the recursive definition of the partition function of $M[U(N+1)]$

$$\begin{aligned}\mathcal{Z}_{N+1}\equiv\mathcal{Z}_{M[U(N+1)]}(\vec{X};\vec{Y};\Delta;m_A) &= s_b\left(-i\frac{Q}{2}+2\Delta\right)\prod_{i=1}^{N+1}s_b\left(i\frac{Q}{2}\pm(X_i-Y_{N+1})-\Delta\right)\\ &\times\int\frac{\mathrm{d}\vec{z}_N}{\prod_{a<b}^N s_b\left(i\frac{Q}{2}\pm(z_a-z_b)\right)}\prod_{a,b=1}^N s_b\left(i\frac{Q}{2}\pm(z_a-z_b)-2m_A\right)\times\\ &\times\prod_{a=1}^N s_b\left(\pm(z_a-Y_{N+1})+\Delta-m_A\right)\prod_{i=1}^{N+1}s_b\left(\pm(z_a-X_i)+m_A\right)\times\\ &\times\mathcal{Z}_{M[U(N)]}(z_1,\cdots,z_N;Y_1,\cdots,Y_N;\Delta+m_A-i\frac{Q}{2};m_A)\,.\end{aligned}\tag{2.139}$$

Only two pieces of this partition function are affected by the $\Delta\to\infty$ limit. The first one is the partition function of the $M[U(N)]$ subquiver, whose limit is given by the inductive hypothesis (2.130). The second one is the block of double-sine functions representing the last flipping field β_N and the last flavors of the saw $D^{(N+1)}$, $\tilde{D}^{(N+1)}$ and $V^{(N)}$, $\tilde{V}^{(N)}$

$$\begin{aligned}\mathcal{B}_{N+1} &= s_b\left(-i\frac{Q}{2}+2\Delta\right)\prod_{i=1}^{N+1}s_b\left(i\frac{Q}{2}\pm(X_i-Y_{N+1})-\Delta\right)\prod_{a=1}^N s_b\left(\pm(z_a-Y_{N+1})+\Delta-m_A\right)\\ &\to\exp\left[i\pi\left(Nm_A^2+\frac{2N+1}{8}Q^2-2Nm_A\Delta+iNQ\Delta+\Delta^2+\right.\right.\\ &\left.\left.-\sum_{i=1}^{N+1}X_i^2-Y_{N+1}^2+2Y_{N+1}\sum_{i=1}^{N+1}X_i-2Y_{N+1}\sum_{a=1}^N z_a+\sum_{a=1}^N z_a^2\right)\right]\,.\end{aligned}\tag{2.140}$$

Notice that we have a quadratic term in the integration variable, which represents a CS coupling for the gauge field of the last node of the quiver. This precisely cancels with the corresponding term in (2.130). Hence, combining (2.130) and (2.140) we get

$$\begin{aligned}\lim_{\Delta\to+\infty}\mathcal{Z}_{N+1} &= C_{N+1}(\Delta,m_A)\mathrm{e}^{-i\pi\sum_{i=1}^{N+1}(X_i^2+Y_i^2)}\mathrm{e}^{2\pi iY_{N+1}\sum_{i=1}^{N+1}X_i}\int\mathrm{d}\vec{z}_N\,\mathrm{e}^{-2\pi iY_{N+1}\sum_{a=1}^N z_a}\times\\ &\times\prod_{a=1}^N\prod_{i=1}^{N+1}s_b\left(\pm(z_a-X_i)+m_A\right)\mathcal{Z}_{T[U(N)]}(z_1,\cdots,z_N;Y_1,\cdots,Y_N;m_A)=\\ &= C_{N+1}(\Delta,m_A)\mathrm{e}^{-i\pi\sum_{i=1}^{N+1}(X_i^2+Y_i^2)}\mathcal{Z}_{T[U(N+1)]}(\vec{X};\vec{Y};m_A)\,,\end{aligned}\tag{2.141}$$

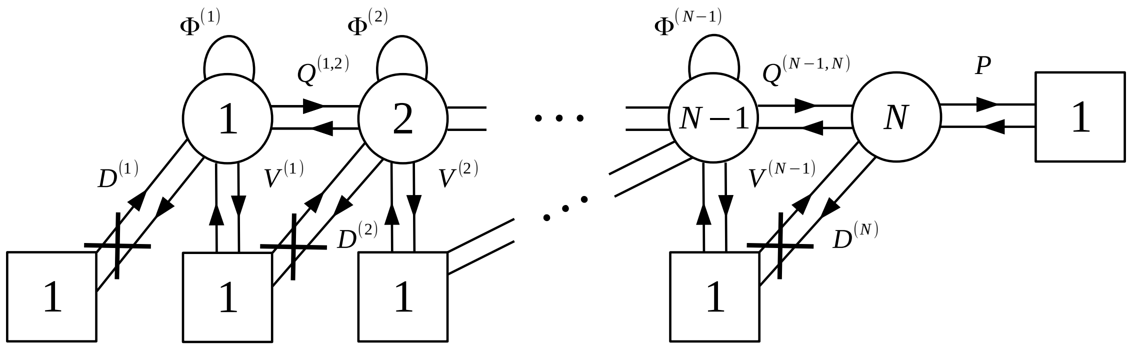

Figure 2.7: Quiver diagram of the $G[U(N)]$ theory.

where in the last step we used the recursive definition (2.132) of the $T[U(N)]$ partition function. This concludes the proof of (2.130) for arbitrary N.

If we take the real mass deformation on the two sides of any of the self-duality identities (2.120)-(2.125)-(2.129), the divergent prefactor $C_N(\Delta, m_A)$ and the background CS terms $\mathrm{e}^{-i\pi\sum_{i=1}^{N}(X_i^2+Y_i^2)}$ cancel out since they are symmetric under both $X_i \leftrightarrow Y_i$ and $m_A \leftrightarrow i\frac{Q}{2} - m_A$, so we obtain similar identities but for the duality web of $T[SU(N)]$. Notice also that the $U(1)$ symmetries of the saw of $M[SU(N)]$ reduce to the topological symmetries of $T[SU(N)]$, so that the symmetry enhancement enjoyed by the latter ones can be understood as a consequence of that of the former ones.

2.4.2 Recombination duality

The $M[SU(N)]$ and the $M[U(N)]$ theories can be used as building blocks to construct more complicated models, where one or both of the non-abelian global symmetries are gauged. In this section and in the following one we discuss some dualities involving such constructions. The first duality we present, which we call *recombination duality*, relates roughly speaking a $M[U(N)]$ with one $U(N)$ symmetry gauged with one flavor to a certain gluing of two smaller tails $M[U(N-k)$ and $M[U(k)]$, where $k = 0, \cdots, N$. This can be understood as the $3d$ uplift of a similar property for the free field integral representation of the kernel function (2.21) that can be found in eq. (B.3) of [38].

The $G[U(N)]$ theory

For simplicity we give the name $G[U(N)]$ to one of the theories involved in the recombination duality. This is depicted in Figure 2.7 and it is obtained from $M[U(N)]$ by gauging the last flavor node[17] and adding one fundamental flavor P, $\tilde{P}$. The superpotential is

$$\mathcal{W}_{G[U(N)]} = \mathcal{W}_{M[U(N)]} = \mathcal{W}_{mono} + \mathcal{W}_{T[U(N)]} + \mathcal{W}_{cub} + \mathcal{W}_{flip}\,. \tag{2.142}$$

Since the extra flavor doesn't interact with any other fields, we have an additional $U(1)_\mu$ flavor symmetry. Moreover, we have no monopole superpotential associated to the $U(N)$ node, which means that its topological symmetry $U(1)_\zeta$ is not broken. Hence, the complete global symmetry group of $G[U(N)]$ is[18]

$$U(N)_z \times U(1)_{m_A} \times U(1)_\Delta \times U(1)_\mu \times U(1)_\zeta\,, \tag{2.143}$$

where the $U(N)_z$ symmetry is not manifest in the UV, but it is enhanced in the IR from the $U(1)$ symmetries of the saw. This can be understood from the fact that the chiral ring generators of $G[U(N)]$ re-organize into representations of $U(N)_z$, as we will show below, but it will become evident also in Section 2.4.3 where we will discuss a dual frame for $G[U(N)]$ in which the full $U(N)_z$ symmetry is manifest.

Since $U(1)_{m_A}$, $U(1)_\Delta$ and $U(1)_\mu$ are abelian symmetries that can mix with the R-symmetry, the corresponding parameters are actually defined as the holomorphic combinations

$$m_A = \mathrm{Re}(m_A) + i\frac{Q}{2}R_A\,, \qquad \Delta = \mathrm{Re}(\Delta) + i\frac{Q}{2}R_\Delta\,, \qquad \mu = \mathrm{Re}(\mu) + i\frac{Q}{2}r \tag{2.144}$$

where R_A, R_Δ, r are the mixing coefficients. In Table 2.5 we summarize the charges under these symmetries of all the chiral fields of the theory.

Some of the chiral ring generators of $G[U(N)]$ are similar to those of the $M[U(N)]$ theory. Firstly, we have the operator $\mathtt{C}$ in the traceless adjoint representation of the enhanced $U(N)_z$ symmetry, which is constructed exactly as for $M[U(N)]$. We then have the operators Ω, $\tilde{\Omega}$ which are constructed by attaching the new chiral fields P, $\tilde{P}$ to the Π, $\tilde{\Pi}$ operators of $M[U(N)]$ so to have gauge invariant objects. For example, for $N=3$ we have

$$\Omega = \begin{pmatrix} P_a\tilde{Q}^{(2,3)}_{i,a}\tilde{Q}^{(1,2)}_i D^{(1)} \\ P_a\tilde{Q}^{(2,3)}_{i,a} D^{(2)}_i \\ P_a D^{(3)}_a \end{pmatrix}\,, \qquad \tilde{\Omega} = \begin{pmatrix} \tilde{D}^{(1)} Q^{(1,2)}_i Q^{(2,3)}_{i,a}\tilde{P}_a \\ \tilde{D}^{(2)}_i Q^{(2,3)}_{i,a}\tilde{P}_a \\ \tilde{D}^{(3)}_a\tilde{P}_a \end{pmatrix}\,. \tag{2.145}$$

Then we have the dressed mesons and the dressed monopoles [51]

$$\mathrm{Tr}_N\left(\tilde{P}\mathbb{M}^s P\right)\,, \qquad \mathfrak{M}^{\pm}_{\mathbb{M}^s}\,, \qquad s = 0,\cdots,N-1\,, \tag{2.146}$$

[17] One can equivalently think of gauging the symmetry that is not manifest in the Lagrangian description of $M[U(N)]$ by just using its mirror self-duality. This would result in a non-Lagrangian gauging, though, so we prefer to stick to this definition of the $G[U(N)]$ theory.

[18] Since we used the freedom due to the gauge symmetry to fix the baryonic symmetry of the flavor P, $\tilde{P}$, the flavor symmetry associated to the saw is now the full $U(N)_z$ group.

	$U(1)_{z_i}$	$U(1)_{m_A}$	$U(1)_\Delta$	$U(1)_\mu$	$U(1)_R$
$Q^{(i-1,i)}$	0	-1	0	0	$1-R_A$
$\tilde{Q}^{(i-1,i)}$	0	-1	0	0	$1-R_A$
P	0	0	0	1	r
$\tilde{P}$	0	0	0	1	r
$V^{(i-1)}$	1	$a-N+1$	-1	0	$2+(N-a-1)(1-R_A)-R_\Delta$
$\tilde{V}^{(i-1)}$	-1	$a-N+1$	-1	0	$2+(N-a-1)(1-R_A)-R_\Delta$
$D^{(i)}$	-1	$N-a$	1	0	$(a-N)(1-R_A)+R_\Delta$
$\tilde{D}^{(i)}$	1	$N-a$	1	0	$(a-N)(1-R_A)+R_\Delta$
$\Phi^{(i)}$	0	2	0	0	$2R_A$
β_i	0	$-2(N-i)$	-2	0	$2+2(i-N)(R_A-1)-2R_\Delta$

Table 2.5: In the table, i runs from 1 to N. By definition, $Q^{(0,1)}=\tilde{Q}^{(0,1)}=0$, $V^{(0)}=\tilde{V}^{(0)}=0$ and $\Phi^{(N)}=0$. We don't report here the topological symmetry $U(1)_\zeta$ since the only operators charged under it come from the non-perturbative monopole sector.

where $\mathfrak{M}^\pm$ are the fundamental monopoles associated to the $U(N)$ gauge node, which are not turned on in the superpotential. The dressing is performed with the meson matrix constructed from the last bifundamental of the $M[U(N)]$ sub-tail

$$\mathbb{M}=\mathrm{Tr}_{N-1}\, Q^{(N-1,N)}\tilde{Q}^{(N-1,N)}\,, \tag{2.147}$$

which transforms in the adjoint representation of $U(N)$. Finally, remember that in the definition of $M[U(N)]$ we have N gauge singlets β_i that flip the diagonal mesons $\tilde{D}^{(i)}D^{(i)}$ for $i=1,\cdots,N$. The transformation rules under the enhanced global symmetry of the gauge invariant chiral operators are

	$U(N)_z$	$U(1)_{m_A}$	$U(1)_\Delta$	$U(1)_\mu$	$U(1)_\zeta$	$U(1)_R$
C	$\mathbf{N^2-1}$	2	0	0	0	$2R_A$
Ω	$\bar{\mathbf{N}}$	0	1	1	0	$R_\Delta+r$
$\tilde{\Omega}$	$\mathbf{N}$	0	1	1	0	$R_\Delta+r$
$\mathfrak{M}^\pm_{\mathbb{M}^s}$	$\mathbf{1}$	$N-2s-1$	-1	-1	±1	$2-(N-2s-1)(1-R_A)-R_\Delta-r$
$\mathrm{Tr}_N\left(\tilde{P}\mathbb{M}^sP\right)$	$\mathbf{1}$	$-2s$	0	2	0	$2s(1-R_A)+2r$
β_i	$\mathbf{1}$	$-2(N-i)$	-2	0	0	$2+2(i-N)(R_A-1)-2R_\Delta$

Recombination dual

We propose a recombination property of $G[U(N)]$, which actually provides a set of several duality frames for the theory. These dual theories are obtained from a $G[U(N-k)]$ and a $G[U(k)]$ tail, where $k\leq N$, glued together with a bifundamental flavor q_{LR}. The fundamental flavors p_L, $\tilde{p}_L$ and p_R, $\tilde{p}_R$ attached to the ends of the two tails transform under the same symmetry $U(1)_\mu$. Moreover, all the $U(1)_{z_{N-i+1}}$ nodes, for $i=1,\cdots,k$, are connected to

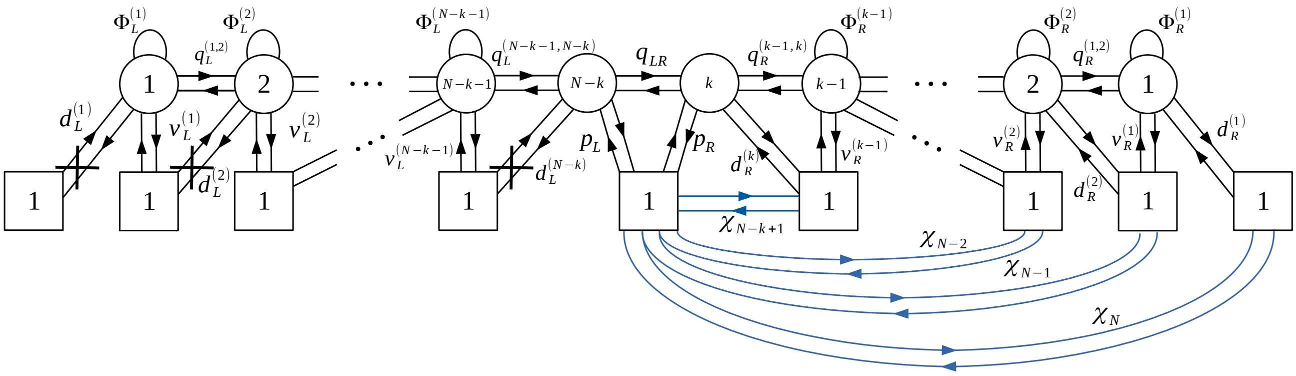

Figure 2.8: Quiver diagram of the recombination dual theory. The blue lines represent gauge singlets χ_i $\tilde{\chi}_i$ that transform under the flavor symmetries of the nodes they connect.

the $U(1)_\mu$ node by some gauge singlets χ_i, $\tilde{\chi}_i$[19]. The β-fields of the right $G[U(k)]$ tail are removed, which can be achieved without modifying the definition of the $G[U(k)]$ building block by adding k additional gauge singlets b_i that flip them. The complete structure of the theory is represented in the quiver of Figure 2.8. Finally, on top of this, we also have $3k$ gauge singlets that we denote by $S_i^\pm$ and α_i.

The full superpotential of the dual theory is

$$\mathcal{W}_{recomb} \;=\; \mathcal{W}_{G[U(N-k)]} + \mathcal{W}_{G[U(k)]} + \mathcal{W}_{mid} + \mathcal{W}_{flips}\,. \tag{2.148}$$

The first two terms are the usual superpotential (2.142) for the two tails $G[U(N-k)]$ and $G[U(k)]$. The third term contains some cubic and quartic couplings and a monopole superpotential that relate the tails

$$\begin{aligned}\mathcal{W}_{mid} \;=\;& \mathrm{Tr}_{N-k}\left(\mathrm{Tr}_k\, q_{LR}\tilde{q}_{LR}\right)\left(\mathrm{Tr}_{N-k-1}\, q_R^{(N-k-1,N-k)}\tilde{q}_R^{(N-k-1,N-k)}\right) + \\ -\;& \mathrm{Tr}_k\left(\mathrm{Tr}_{N-k}\, q_{LR}\tilde{q}_{LR}\right)\left(\mathrm{Tr}_{k-1}\, q_L^{(k-1,k)}\tilde{q}_L^{(k-1,k)}\right) + \\ +\;& \mathrm{Tr}_k\left(p_R\,\mathrm{Tr}_{N-k}\left(q_{LR}\tilde{p}_L\right)\right) + \mathrm{Tr}_{N-k}\left(p_L\,\mathrm{Tr}_k\left(\tilde{q}_{LR}\tilde{p}_R\right)\right) + \\ +\;& \mathfrak{M}^{(0,\cdots,0,1,1,0,\cdots,0)} + \mathfrak{M}^{(0,\cdots,0,-1,-1,0,\cdots,0)}\,.\end{aligned} \tag{2.149}$$

The last term involves the monopoles with non-vanishing magnetic fluxes corresponding to the $U(N-k)$ and $U(k)$ gauge nodes only. This has the effect of breaking the two topological symmetries of these nodes to their anti-diagonal combination, which is mapped to the $U(1)_\zeta$

[19]Notice that the $U(1)_{z_a}$ symmetries of the $G[U(N-k)]$ tail are ordered in the usual way, that is $U(1)_{z_1}$ corresponds to the leftmost square node of the $G[U(N-k)]$ subquiver of Figure 2.8 and $U(1)_{z_{N-k}}$ to the rightmost one, while the $U(1)_{z_{N-i+1}}$ symmetries of the $G[U(k)]$ tail are ordered in the opposite way, that is $U(1)_{z_N}$ corresponds rightmost square node of the $G[U(k)]$ subquiver (which appears reversed in Figure 2.8) and $U(1)_{z_{N-k+1}}$ to the leftmost one. Nevertheless, since all of these symmetries are expected to get enhanced in the IR to $U(N)_z$ the ordering is not really important.

	$U(1)_{z_i}$	$U(1)_{z_{N-n+1}}$	$U(1)_{m_A}$	$U(1)_\Delta$	$U(1)_\mu$	$U(1)_\zeta$	$U(1)_R$
$S_i^\pm$	0	0	$N-2i+1$	-1	-1	±1	$2-(N-2i+1)(1-R_A)-R_\Delta-r$
α_i	0	0	$-2(i-1)$	0	2	0	$2(i-1)(1-R_A)+2r$
χ_{N-i+1}	0	1	0	1	1	0	$R_\Delta+r$
$\tilde{\chi}_{N-i+1}$	0	-1	0	1	1	0	$R_\Delta+r$
$q_L^{(a-1,a)}$	0	0	-1	0	0	0	$1-R_A$
$\tilde{q}_L^{(a-1,a)}$	0	0	-1	0	0	0	$1-R_A$
$q_R^{(i-1,i)}$	0	0	-1	0	0	0	$1-R_A$
$\tilde{q}_R^{(i-1,i)}$	0	0	-1	0	0	0	$1-R_A$
q_{LR}	0	0	1	0	0	0	R_A
$\tilde{q}_{LR}$	0	0	1	0	0	0	R_A
p_L	0	0	$-k$	0	1	0	$k(1-R_A)+r$
$\tilde{p}_L$	0	0	$-k$	0	1	0	$k(1-R_A)+r$
p_R	0	0	$k-1$	0	-1	0	$1-(k-1)(1-R_A)-r$
$\tilde{p}_R$	0	0	$k-1$	0	-1	0	$1-(k-1)(1-R_A)-r$
$v_L^{(a-1)}$	1	0	$a-N+1$	-1	0	0	$2+(N-a-1)(1-R_A)-R_\Delta$
$\tilde{v}_L^{(a-1)}$	-1	0	$a-N+1$	-1	0	0	$2+(N-a-1)(1-R_A)-R_\Delta$
$v_R^{(i-1)}$	0	1	i	1	0	0	$1-i(1-R_A)+R_\Delta$
$\tilde{v}_R^{(i-1)}$	0	-1	i	1	0	0	$1-i(1-R_A)+R_\Delta$
$d_L^{(a)}$	-1	0	$N-a$	1	0	0	$(a-N)(1-R_A)+R_\Delta$
$\tilde{d}_L^{(a)}$	1	0	$N-a$	1	0	0	$(a-N)(1-R_A)+R_\Delta$
$d_R^{(i)}$	0	-1	$1-i$	-1	0	0	$1+(i-1)(1-R_A)-R_\Delta$
$\tilde{d}_R^{(i)}$	0	1	$1-i$	-1	0	0	$1+(i-1)(1-R_A)-R_\Delta$
$\Phi_L^{(b)}$	0	0	2	0	0	0	$2R_A$
$\Phi_R^{(j)}$	0	0	2	0	0	0	$2R_A$
$\beta_{L,a}$	0	0	$2(N-a)$	-2	0	0	$2+2(N-a)(1-R_A)-2R_\Delta$

Table 2.6: Transformation rules of the chiral fields of the recombination dual theory under the global symmetry. In the table, a runs from 1 to $N-k$, b from 1 to $N-k-1$, i from 1 to k and j from 1 to $k-1$.

symmetry of the dual $G[U(N)]$ theory. Finally, we have some flip terms

$$\mathcal{W}_{flips}=\sum_{i=1}^{k}\left(S_i^\pm\mathfrak{M}^\pm_{(k)\mathbb{M}_R^{k-i}}+\alpha_i\,\mathrm{Tr}_k\left(\tilde{p}_R\mathbb{M}_R^{i-1}p_R\right)+\beta_{R,i}b_i+\chi_{N-i+1}\Omega_{R,i}+\tilde{\chi}_{N-i+1}\tilde{\Omega}_{R,i}\right)\,, \tag{2.150}$$

where $\mathfrak{M}^\pm_{(k)}$ denote the fundamental monopoles of the $U(k)$ gauge node, which can be dressed with the meson matrix

$$\mathbb{M}_R=\mathrm{Tr}_{k-1}\,q_R^{(k-1,k)}\tilde{q}_R^{(k-1,k)} \tag{2.151}$$

transforming in the adjoint representation of $U(k)$, and $\Omega_{R,i}$ denotes the i-th component of the vector Ω associated to the right $G[U(k)]$ tail. In Table 2.6 we summarize the charges under the global symmetries of all the chiral fields of the theory.

The chiral ring operators are basically obtained by gluing those of the two $G[U(N)]$ tails. First, we have an operator that we denote $\hat{\mathsf{C}}$ which transforms in the adjoint representation of

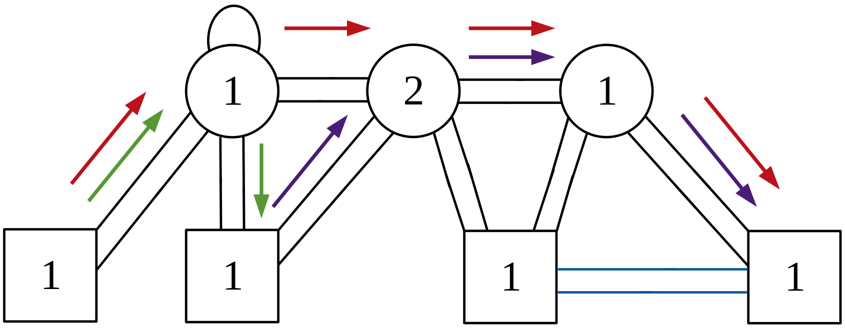

Figure 2.9: Diagrammatic representation of the operator $\hat{\mathtt{C}}$ in the case $N = 3$ and $k = 1$. Arrows of the same color represent chiral fields that we assemble to construct an element of the matrix. In order to have a gauge invariant operator, we have to consider sequences of arrows that start and end on a squared node.

$U(N)_z$. This consists of four blocks. The two on the diagonal are respectively $(N-k)\times(N-k)$ and $k \times k$ matrices that correspond to the usual $\mathtt{C}$ operators of $G[U(N-k)]$ and $G[U(k)]$. Recall that these are constructed starting with one of the diagonal flavors, moving along the tail following the bifundamentals and then ending on one of the vertical flavors. On the diagonal we still have the traces of the adjoint chirals, but since we have only $N-2$ of them one element has to be

$$\mathrm{Tr}\,\mathbb{M}_{LR} = \mathrm{Tr}_{N-k}\,\mathrm{Tr}_k\, q_{LR}\tilde{q}_{LR}\,. \tag{2.152}$$

The off-diagonal blocks are built in a similar way, but going from one tail to the other using the bifundamental q_{LR} as a link and ending on one of the diagonal flavors of the opposite tail rather than a vertical one (see Figure 2.9). For example, for $N = 3$ and $k = 1$ this matrix takes the form

$$\hat{\mathtt{C}} = \begin{pmatrix} 0 & v_L^{(1)} d_L^{(1)} & d_R^{(1)} \tilde{q}_{LR,i} \tilde{q}_i^{(1,2)} d_L^{(1)} \\ \tilde{d}_L^{(1)} \tilde{v}_L^{(1)} & 0 & d_R^{(1)} \tilde{q}_{LR,i} d_{L,i}^{(2)} \\ \tilde{d}_L^{(1)} q_i^{(1,2)} q_{LR,i} \tilde{d}_R^{(1)} & \tilde{d}_{L,i}^{(2)} q_{LR,i} \tilde{d}_R^{(1)} & 0 \end{pmatrix} + \Phi^{(1)}\mathcal{D}_1 + \mathrm{Tr}\,\mathbb{M}_{LR}\mathcal{D}_2\,, \tag{2.153}$$

Then, we have the operators $\hat{\Omega}$, $\hat{\tilde{\Omega}}$. One may think that they are obtained by simply juxtaposing the vectors Ω_L and Ω_R of the two tails, but this is not possible since they have not the same charges under the global symmetries. Moreover, the operators of the right tail are set to zero in the chiral ring by the equations of motion of the flipping fields χ_i. The correct operators are then

$$\hat{\Omega} = \tilde{\Omega}_R \oplus \begin{pmatrix} \chi_{N-k+1} \\ \vdots \\ \chi_N \end{pmatrix}, \qquad \hat{\tilde{\Omega}} = \Omega_R \oplus \begin{pmatrix} \tilde{\chi}_{N-k+1} \\ \vdots \\ \chi_N \end{pmatrix}. \tag{2.154}$$

These transform in the fundamental and anti-fundamental representation respectively of the flavor symmetry $U(N)_z$. Something similar happens for the mesonic operators of the saw. Namely, the mesons constructed with the diagonal chirals of the right tail are now non-trivial operators since we removed the flipping fields $\beta_{R,i}$ and they can be collected with the singlets $\beta_{L,a}$ of the left tail. Hence, the complete tower of N generators of this type is

$$\begin{cases} \beta_{L,a} & a = 1, \cdots, N-k \\ d_R^{(i)} \tilde{d}_R^{(i)} & i = 1, \cdots, k \end{cases}. \tag{2.155}$$

Let's now consider the monopole operators and their dressings. Only those associated to the $U(N-k)$ node are non-trivial, since those at the $U(k)$ node are flipped by the singlets $S_i^\pm$ (recall that the monopoles of the other gauge nodes are turned on in the superpotential and so they are removed from the chiral ring). Hence, we have the following $2N$ generators:

$$\begin{cases} \mathfrak{M}^\pm_{\mathbb{M}_L^s} & s = 0, \cdots, N-k-1 \\ S_i^\pm & i = 1, \cdots, k \end{cases}, \tag{2.156}$$

where $\mathfrak{M}^\pm$ denotes the fundamental monopoles of the $U(N-k)$ node, which are dressed with the field

$$\mathbb{M}_L = \mathrm{Tr}_{N-k-1}\, q_L^{(N-k-1,N-k)} \tilde{q}_L^{(N-k-1,N-k)} \tag{2.157}$$

that transforms in the adjoint representation of $U(N-k)$. Finally, we have the (dressed) mesons associated to the extra flavors of the two tails p_L, $\tilde{p}_L$, p_R, $\tilde{p}_R$, where the dressing is made using the matrices $\mathbb{M}_L$ and $\mathbb{M}_R$. Again, these operators are flipped in the right tail by the gauge singlets α_i. Thus, the last set of N gauge invariant chiral operators is

$$\begin{cases} \mathrm{Tr}_{N-k}\left(\tilde{p}_L \mathbb{M}_L^s p_L\right) & s = 0, \cdots, N-k-1 \\ \alpha_i & i = 1, \cdots, k \end{cases}, \tag{2.158}$$

Summing up, the main gauge invariant operators of the recombination dual theory and their transformation rules under the global symmetries are

	$U(N)_z$	$U(1)_{m_A}$	$U(1)_\Delta$	$U(1)_\mu$	$U(1)_\zeta$	$U(1)_R$
$\hat{\mathtt{C}}$	adj	2	0	0	0	$2R_A$
$\hat{\Omega}$	$\bar{\mathbf{N}}$	0	1	1	0	$R_\Delta + r$
$\hat{\tilde{\Omega}}$	$\mathbf{N}$	0	1	1	0	$R_\Delta + r$
$\beta_{L,a}$	0	$2(N-a)$	-2	0	0	$2+2(N-a)(1-R_A)-2R_\Delta$
$d_R^{(N-i+1)}\tilde{d}_R^{(N-i+1)}$	0	$2(N-i)$	-2	0	0	$2+2(N-i)(1-R_A)-2R_\Delta$
$S_i^\pm$	0	$N-2i+1$	-1	-1	± 1	$2-(N-2i+1)(1-R_A)-R_\Delta-r$
$\mathfrak{M}^\pm_{\mathbb{M}_L^s}$	0	$N-2k-2s-1$	-1	-1	± 1	$2-(N-2k-2s-1)(1-R_A)-R_\Delta-r$
α_i	0	$-2(i-1)$	0	2	0	$2(i-1)(1-R_A)+2r$
$\mathrm{Tr}_{N-k}\left(p_L\mathbb{M}_L^s\tilde{p}_L\right)$	0	$-2(k+s)$	0	2	0	$2(k+s-1)(1-R_A)+2r$

As a first check of the duality we can map the gauge invariant chiral operators of the two theories

$$\begin{array}{ccl}
\mathtt{C} & \leftrightarrow & \hat{\mathtt{C}} \\
\Omega & \leftrightarrow & \hat{\Omega} \\
\tilde{\Omega} & \leftrightarrow & \hat{\tilde{\Omega}} \\
\beta_i & \leftrightarrow & \begin{cases} \beta_{L,i} & i=1,\cdots,N-k \\ d_R^{(N-i+1)}\tilde{d}_R^{(N-i+1)} & i=N-k+1,\cdots,N \end{cases} \\
\mathfrak{M}^\pm_{\mathbb{M}^{s-1}} & \leftrightarrow & \begin{cases} S_s^\pm & s=1,\cdots,k \\ \mathfrak{M}^\pm_{\mathbb{M}_L^{s-k-1}} & s=k+1,\cdots,N \end{cases} \\
\mathrm{Tr}_N\left(P\mathbb{M}^{s-1}\tilde{P}\right) & \leftrightarrow & \begin{cases} \alpha_s & s=1,\cdots,k \\ \mathrm{Tr}_{N-k}\left(p_L\mathbb{M}_L^{s-k-1}\tilde{p}_L\right) & s=k+1,\cdots,N \end{cases}
\end{array} \tag{2.159}$$

At the level of $\mathbb{S}_b^3$ partition functions, the recombination duality is represented by the following integral identity:

$$\begin{aligned}
\mathcal{Z}_{G[U(N)]}(\vec{z};\zeta;\mu;\Delta;m_A) = \int \mathrm{d}\vec{u}_N\, \mathrm{e}^{2\pi i\zeta\sum_{i=1}^N u_i^{(N)}} \frac{\prod_{i=1}^N s_b\left(i\frac{Q}{2}\pm u_i^{(N)}-\mu\right)}{\prod_{i<j}^N s_b\left(i\frac{Q}{2}\pm(u_i^{(N)}-u_j^{(N)})\right)} \times \\
\times \mathcal{Z}_{M[U(N)]}(\vec{u^{(N)}};\vec{z};\Delta;m_A) = \\
= \Lambda_k^N(m_A,\Delta,\zeta,\mu) \prod_{i=N-k+1}^N \mathrm{e}^{2\pi i\zeta z_i} s_b\left(i\frac{Q}{2}\pm z_i-\mu-\Delta\right) \times \\
\times \int \mathrm{d}\vec{u}_{N-k}\, \mathrm{e}^{2\pi i\zeta\sum_{a=1}^{N-k} u_a^{(N-k)}} \frac{\prod_{a=1}^{N-k} s_b\left(i\frac{Q}{2}\pm u_a^{(N-k)}-\mu-k(i\frac{Q}{2}-m_A)\right)}{\prod_{a<b}^{N-k} s_b\left(i\frac{Q}{2}\pm(u_a^{(N-k)}-u_b^{(N-k)})\right)} \times \\
\times \mathcal{Z}_{M[U(N-k)]}\left(u_1^{(N-k)},\cdots,u_{N-k}^{(N-k)};z_1,\cdots,z_{N-k};\Delta-k\left(i\frac{Q}{2}-m_A;m_A\right)\right) \times
\end{aligned}$$

$$\times \int \mathrm{d}\vec{v}_k\, \mathrm{e}^{-2\pi i \zeta \sum_{i=1}^{k} v_i^{(k)}} \frac{\prod_{i=1}^{k} s_b\left(\pm v^{(i)} + \mu + (k-1)\left(i\frac{Q}{2} - m_A\right)\right)}{\prod_{i<j}^{k} s_b\left(i\frac{Q}{2} \pm (v_i^{(k)} - v_j^{(k)})\right)} \times$$
$$\times \prod_{i=1}^{k}\prod_{a=1}^{N-k} s_b\left(i\frac{Q}{2} \pm (v_i^{(k)} - u_a^{(N-k)}) - m_A\right)$$
$$\times \mathcal{Z}_{M[U(k)]}\left(v_1^{(k)}, \cdots, v_k^{(k)}; z_N, \cdots, z_{N-k+1}; m_A - \Delta + k\left(i\frac{Q}{2} - m_A; m_A\right)\right), \tag{2.160}$$

where Λ_k^N is the contribution of the $4k$ flipping singlets $S_i^{\pm}$, α_i and b_i

$$\begin{aligned} \Lambda_k^N(m_A, \Delta, \zeta, \mu) \quad = \quad & \prod_{n=1}^{k} s_b\left(\pm\zeta + \mu + \Delta - m_A + (N-2n)\left(i\frac{Q}{2} - m_A\right)\right) \times \\ \times \quad & s_b\left(i\frac{Q}{2} - 2\mu - 2(n-1)\left(i\frac{Q}{2} - m_A\right)\right) \times \\ \times \quad & s_b\left(-i\frac{Q}{2} + 2\Delta - 2(i-1)\left(i\frac{Q}{2} - m_A\right)\right). \end{aligned} \tag{2.161}$$

Notice in particular that the double-sine functions in the last line, representing the fields b_i, precisely cancel the contribution of the flipping fields $\beta_{R,i}$ from the right $G[U(k)]$ tail. The parameters on which the partition function depends are the real masses z_i for the flavor symmetry $U(N)_z$, the axial masses m_A, Δ, μ for the axial symmetries $U(1)_{m_A} \times U(1)_\Delta \times U(1)_\mu$ and the FI parameter ζ corresponding to the topological symmetry $U(1)_\zeta$.

Sketch of the derivation

The recombination duality we just described can be derived by sequentially applying a more fundamental duality which is the Aharony duality that we saw in Subsection 2.3.3. We are now going to schematically describe the main steps of this derivation. This can be formalized in various ways, either at the field theory or at the partition function level. For example, in Subsection 3.3 of [25] the precise superpotential at each step of the derivation has been worked out. This task is particularly complicated since it requires the knowledge of the operator map of monopole operators at each dualization (see [24, 62, 63] for more details on how to work this out[20]). A more efficient strategy is to implement this derivation at the level of some localized partition function, by sequentially applying the corresponding identities for the Aharony duality. In Appendix B.1 we show this explicitly at the level of the $\mathbb{S}_b^3$ partition function for $N = 3$.

The basic strategy is to apply the Aharony duality locally on a single node of the quiver at a time. In order for this to be possible, we should have no adjoint matter at that node,

[20] See also Appendix A of [82] and the Appendix C of [83].

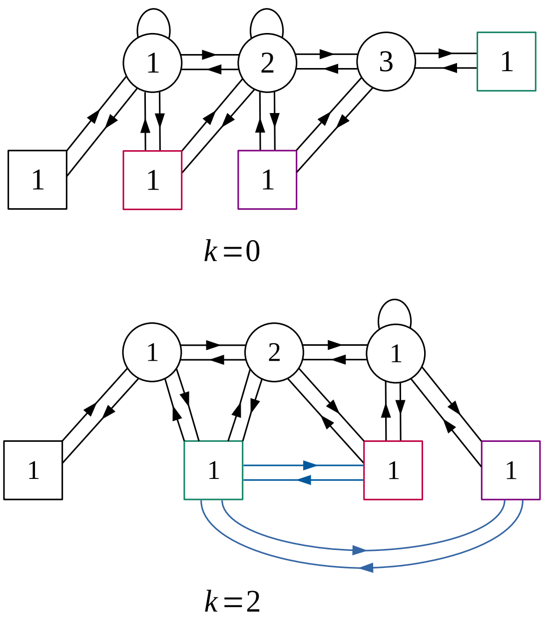

Figure 2.10: Quiver diagrams of the theories at each step of the derivation of the recombination duality. We use different colors for each square node so to keep track of the fields charged under the corresponding global symmetry.

since the Aharony duality contemplates only fundamental matter. In our case, the node of the $G[U(N)]$ quiver that satisfies such requirement is the last $U(N)$ node:

- At the first step, we apply the Aharony duality to the $U(N)$ node. Since this node sees $N+1$ flavors, its rank is decreased to 1. Taking into account all the singlet fields that are produced in the dualization, we find exactly the recombination dual theory for $k=1$. Some of these fields are actually charged under the gauge symmetry of the adjacent nodes now. In particular, some of them give mass to the adjoint chiral of the adjacent $U(N-1)$ node, as expected looking at the quiver in Figure 2.8.

- Since we don't have an adjoint chiral at the $U(N-1)$ node anymore, we can repeat this procedure and apply the Aharony duality to it[21]. Now this node sees $N+1$ flavors, since the rank of the node on its right has been decreased to 1. Hence, when we dualize it, it becomes a $U(2)$ gauge node. Again we produce various flipping fields, some of which give mass to the adjoint chiral of the $U(N-2)$ node on the left, which allows us to iterate this procedure again. Moreover, we don't produce any link between the $U(N-2)$ node on the left and the $U(1)$ node on the right, since these turn out to be massive. The result is precisely the recombination dual frame for $k=2$.

[21]One can also check using the results of [24, 62, 63] that, after the dualization of the $U(N)$ gauge node, the fundamental monopoles of the $U(N-1)$ node are not turned on in the superpotential anymore.

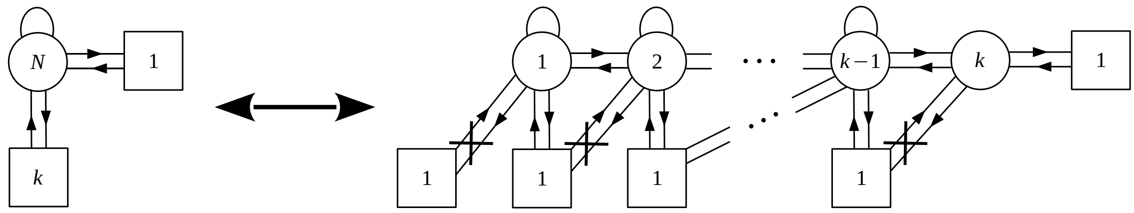

Figure 2.11: Schematic representation of the rank stabilization duality for generic N and k. We don't represent gauge singlet fields to avoid clutter the drawing, except for the β-fields of the $G[U(k)]$ theory which flip the diagonal mesons.

- Iterating this procedure k times we get the recombination dual frame for generic k.

In Figure 2.10 we show the structure of the quivers at each step of this derivation for $N = 3$ and $k = 0, 1, 2, 3$.

From the derivation we just discussed it becomes clear an interesting property of the $G[U(N)]$ theory. As we go along the tail applying the Aharony duality, we initially decrease the rank of the gauge node to which we apply it, until we reach the middle of the tail. From this point, the rank starts to increase back and when we finally arrive at the end of the tail we recover the same original $G[U(N)]$ theory, but reversed. Hence, for a particular number k of iterations of the Aharony duality we reach a configuration in which the dual theory has minimal rank. For even N this happens exactly at $k = N/2$, while for odd N we have two possibilities $k = (N \pm 1)/2$. The rank of the original theory was

$$\text{rank}(\mathcal{T}_{G[U(N)]}) = \sum_{i=1}^{N} i = \frac{N(N+1)}{2} \,. \tag{2.162}$$

Instead, when we use the recombination duality to get to the configuration with minimal rank, we have

$$\text{rank}(\mathcal{T}_{min}) = \begin{cases} \frac{N}{2}\left(\frac{N}{2}+1\right) & N \text{ even} \\ \frac{N-1}{2}\left(\frac{N-1}{2}+1\right) + \frac{N+1}{2} & N \text{ odd} \end{cases} \,. \tag{2.163}$$

2.4.3 Rank stabilization duality

The next duality that we consider is a generalization of the confining duality for $U(N)$ with one adjoint and one fundamental flavor we saw in Subsection 2.3.1 to a higher number of fundamental flavors. This duality can be understood as the $3d$ uplift of the integral identity (2.19) for the free field integral of the $(k+3)$-point correlation function of Liouville theory, where 3 of the primary operators are arbitrary while k are degenerate. The fact that we add k operators implies that in the $3d$ gauge theory we have k additional fundamental flavors, while the fact that they are degenerate implies that the k flavors are involved in some superpotential interaction. Indeed, we recall that "degenerate" means that the momenta of the operators are specified in terms of the coupling constant b of Liouville theory, so that we don't have the corresponding additional parameters in the matrix integral of the free field correlator, and the absence of such parameters in the 3d gauge theory is interpreted as the fact that the axial symmetries associated to the k flavors are broken by some superpotential interaction. As expected from the free field identity (2.19), the dual theory contains the $M[U(k)]$ theory as a building block and it will turn out to be the $G[U(k)]$ theory that we introduced. The duality is schematically represented in Figure 2.11. We also pointed out in Section 2.2 that on the l.h.s. of (2.19) the number of screening charges N doesn't appear anymore as the dimension of the integral, but only as a parameter. The same will be true for the 3d duality, namely the rank N of the $U(N)$ gauge group only appears as a parameter on the dual side. For this reason, we call this *rank stabilization duality*.

Theory A

The first theory involved in the duality is the $U(N)$ gauge theory with $k+1$ fundamental flavors Q_i, $\tilde{Q}^i$ with $i = 1, \cdots, k$ and P, $\tilde{P}$, plus one adjoint chiral Φ and $N-k$ singlets b_j with superpotential

$$\mathcal{W}_A = \mathrm{Tr}_N\left(\Phi\,\mathrm{Tr}_k\, Q\tilde{Q}\right) + \sum_{j=1}^{N-k} b_j\,\mathrm{Tr}_N\,\Phi^j = \sum_{i=1}^{k}\sum_{a,b=1}^{N} Q_i^a {\Phi^b}_a {\tilde{Q}^i}_b + \sum_{j=1}^{N-k} b_j\,\mathrm{Tr}_N\,\Phi^j\,, \quad (2.164)$$

with $k < N$. Recall that in the case $k = 0$ all the Casimir operators are flipped by the b-fields since they are expected to violate the unitarity bound and decouple in the IR [48]. Moreover, the b-fields can't acquire a VEV because of quantum effects [49]. As we increase the number of flavors, the superconformal R-charge of the adjoint chiral Φ is expected to increase and the highest Casimir operators start to go above the unitarity bound. Hence, for a fixed value of k we only need to flip the first $N-k$ Casimir operators.

The global symmetry group of the theory is[22]

$$U(k)_z \times U(1)_\tau \times U(1)_\mu \times U(1)_\zeta\,. \quad (2.165)$$

Since $U(1)_\tau$ and $U(1)_\mu$ are abelian flavor symmetries that can mix with the R-symmetry, the corresponding parameters are actually defined as the holomorphic combinations

$$\tau = \mathrm{Re}(\tau) + i\frac{Q}{2}(1-R), \qquad \mu = \mathrm{Re}(\mu) + i\frac{Q}{2}r\,, \quad (2.166)$$

[22] In our convention, we choose to gauge the baryonic symmetry associated to the flavor P, $\tilde{P}$ that doesn't enter in the superpotential. For this reason, the symmetry associated to the flavors Q_i, $\tilde{Q}^i$ is $U(k)$ rather than $SU(k)$. Moreover, compared to the case $k = 0$ of Subsection 2.3.1 we are re-labelling the abelian factors of the global symmetries as $U(1)_s \to U(1)_\tau$, $U(1)_p \to U(1)_\mu$ and $U(1)_\zeta \to U(1)_\omega$, since we will mostly deal with the $\mathbb{S}^3_b$ partition function rather than the $\mathbb{S}^2 \times \mathbb{S}^1$ one.

where r and R are the mixing coefficients. Notice that the $\mathcal{N}=4$ like superpotential for the flavors Q and $\tilde{Q}$ implies that only the anti-diagonal combination $U(k)_z$ of the symmetries that rotate them independently is preserved, while the diagonal axial symmetry is broken. This is the manifestation in $3d$ of the fact that the momenta of the k additional operators in the correlator of the $2d$ CFT are fixed. The transformation rules of all the chiral fields of the theory under the global symmetries and their R-charges are

	$U(k)_z$	$U(1)_\tau$	$U(1)_\mu$	$U(1)_R$
Q	$\mathbf{k}$	-1	0	R
$\tilde{Q}$	$\mathbf{k}$	-1	0	R
P	0	0	1	r
$\tilde{P}$	0	0	1	r
Φ	0	2	0	$2(1-R)$
b_j	0	$-2j$	0	$2-2j(1-R)$

This theory possesses various types of gauge invariant chiral operators. First of all, we have the Casimirs of the gauge group built from the adjoint chiral Φ. The first $N-k$ of these are actually flipped by the b-fields, so that we only have k operators of this kind

$$\mathrm{Tr}_N\,\Phi^j\,,\qquad j=N-k+1,\cdots,N\,. \tag{2.167}$$

Then, we have the fundamental monopole operators $\mathfrak{M}^\pm$ which can also be dressed with Φ in the adjoint representation of the residual gauge group that survives in the monopole background [51]. In total, we have $2N$ independent operators of this form, which we denote by

$$\mathfrak{M}^\pm_{\Phi^s}\,,\qquad s=0,\cdots,N-1\,. \tag{2.168}$$

The mesonic operators can be of different types, depending on which flavor we use to construct them. We can have mesons built from the P, $\tilde{P}$ flavor, which can also be dressed with the adjoint chiral Φ

$$\mathrm{Tr}_N\left(\tilde{P}\Phi^s P\right),\qquad s=0,\cdots,N-1\,. \tag{2.169}$$

Another possibility is to combine the flavor P, $\tilde{P}$ with one of the flavors Q, $\tilde{Q}$. In this case, we can't have dressed mesons because the F-term equations of Q, $\tilde{Q}$ set them to zero. Hence, we only have $2k$ of them

$$Q_i\tilde{P}\,,\qquad P\tilde{Q}_i\,,\qquad i=1,\cdots,k\,, \tag{2.170}$$

which can be collected in two vectors transforming in the anti-fundamental and fundamental representation respectively of $U(k)_z$. Finally, we have the meson obtained combining Q and

$\tilde{Q}$. Also such a meson can't be dressed because of the equations of motion of Q, $\tilde{Q}$. Hence, we have k^2 of them

$$Q_i\tilde{Q}_j\,, \qquad i,j=1,\cdots,k\,, \tag{2.171}$$

which can be collected into a matrix transforming in the traceless adjoint representation of $U(k)_z$. The trace part is indeed set to zero by the F-term equation of Φ. The charges of these operators under the global symmetries are

	$U(k)_z$	$U(1)_\tau$	$U(1)_\mu$	$U(1)_\zeta$	$U(1)_R$
$\text{Tr}_N\,\Phi^j$	0	$2j$	0	0	$2j(1-R)$
$\mathfrak{M}^{\pm}_{\Phi^s}$	0	$-2N+k+2s+2$	-1	±1	$1-r-(2N-k-2s-2)(1-R)$
$\text{Tr}_N\left(\tilde{P}\Phi^s P\right)$	0	$2s$	2	0	$2r+2s(1-R)$
$Q\tilde{P}$	$\bar{\mathbf{k}}$	-1	1	0	$r+R$
$P\tilde{Q}$	$\mathbf{k}$	-1	1	0	$r+R$
$Q\tilde{Q}$	$\mathbf{k^2-1}$	-2	0	0	$2R$

Theory B

The dual theory is $G[U(k)]$[23] with $3(N-k)$ additional gauge singlets α_i, T^+_j, T^-_{N-l+1} with $i,j,l=1,\cdots,N-k$ and superpotential (recall that we are limiting ourselves to the regime $k<N$)

$$\mathcal{W}_B=\mathcal{W}_{G[U(k)]}+\mathcal{W}_{int}\,, \tag{2.172}$$

where $\mathcal{W}_{int}$ is a cubic superpotential that encodes interactions between the extra singlets α_i, T^+_j, T^-_l and the operators of the $G[U(k)]$ tail

$$\begin{aligned}
\mathcal{W}_{int} \;=\; & \sum_{i,j,l=1}^{N-k}\alpha_i T^+_j T^-_{N-l+1}\delta_{i+j+l,2N-k+1}+\sum_{j,l=1}^{N-k}\sum_{r=0}^{k-1}\text{Tr}_k\left(\tilde{p}\mathbb{M}^r p\right)T^+_j T^-_{N-l+1}\delta_{r+j+l,N}+\\
&+\sum_{i,j=1}^{N-k}\sum_{s=0}^{k-1}\alpha_i\mathfrak{M}^+_{\mathbb{M}^s}T^-_{N-l+1}\delta_{i+s+l,N}+\sum_{i,j=1}^{N-k}\sum_{t=0}^{k-1}\alpha_i T^+_j\mathfrak{M}^-_{\mathbb{M}^t}\delta_{i+j+t,N}+\\
&+\sum_{l=1}^{N-k}\sum_{r,s=0}^{k-1}\text{Tr}_k\left(\tilde{p}\mathbb{M}^r p\right)\mathfrak{M}^+_{\mathbb{M}^s}T^-_{N-l+1}\delta_{r+s+l,k-1}+\sum_{j=1}^{N-k}\sum_{r,t=0}^{k-1}\text{Tr}_k\left(\tilde{p}\mathbb{M}^r p\right)T^+_j\mathfrak{M}^-_{\mathbb{M}^t}\delta_{r+j+t,k-1}+\\
&+\sum_{i=1}^{N-k}\sum_{s,t=0}^{k-1}\alpha_i\mathfrak{M}^+_{\mathbb{M}^s}\mathfrak{M}^-_{\mathbb{M}^t}\delta_{i+s+t,k-1}+\sum_{r,s,t=0}^{k-1}\text{Tr}_k\left(\tilde{p}\mathbb{M}^r p\right)\mathfrak{M}^+_{\mathbb{M}^s}\mathfrak{M}^-_{\mathbb{M}^t}\delta_{r+s+t,2k-N-2}\,.
\end{aligned} \tag{2.173}$$

[23] We denote the fields of the $G[U(k)]$ theory with lower case letters, in contrast to the convention we used in Subsection 2.4.3 to avoid confusion with the fields of Theory A.

	$U(1)_{z_a}$	$U(1)_\tau$	$U(1)_\mu$	$U(1)_\zeta$	$U(1)_R$
α_i	0	$2(i-1)$	2	0	$2r+2(i-1)(1-R)$
T_j^+	0	$-2N+k+2j$	-1	1	$1-r-(2N-k-2j)(1-R)$
T_{N-l+1}^-	0	$-2N+k+2l$	-1	-1	$1-r-(2N-k-2l)(1-R)$
β_a	0	$2(N-k+a)$	0	0	$2(N-k+a)(1-R)$
$q^{(a-1,a)}$	0	1	0	0	$1-R$
$\tilde{q}^{(a-1,a)}$	0	1	0	0	$1-R$
p	0	$N-k$	1	0	$r+(N-k)(1-R)$
$\tilde{p}$	0	$N-k$	1	0	$r+(N-k)(1-R)$
$v^{(a-1)}$	1	$N-a$	0	0	$1+(N-a)(1-R)$
$\tilde{v}^{(a-1)}$	-1	$N-a$	0	0	$1+(N-a)(1-R)$
$d^{(a)}$	-1	$-N+a-1$	0	0	$1-(N-a+1)(1-R)$
$\tilde{d}^{(a)}$	1	$-N+a-1$	0	0	$1-(N-a+1)(1-R)$
$\Phi^{(a)}$	0	-2	0	0	$2R$

Table 2.7: Representations and charges under the global symmetries of all the chiral fields of Theory B. In the table the indices i, j, l run from 1 to $N-k$, while a from 1 to k. By convention, $q^{(0,1)}=\tilde{q}^{(0,1)}=0$, $v^{(0)}=\tilde{v}^{(0)}=0$ and $\Phi^{(k)}=0$.

Both the meson and the monopole operators of $G[U(k)]$ are dressed with the matrix

$$\mathbb{M}=\text{Tr}_{k-1}\, q^{(k-1,k)}\tilde{q}^{(k-1,k)}\,, \tag{2.174}$$

which transforms in the adjoint representation of the $U(k)$ factor of the gauge group. Notice that for $k=0$ the superpotential reduces to that of the WZ dual we saw in Subsection 2.3.1.

The last term in the superpotential (2.173) involves only the operators of the $G[U(k)]$ part of the theory and has the effect of breaking one of the $U(1)$ axial symmetries of $G[U(k)]$ (2.143), so now the global symmetries of Theory A and Theory B match (at least at the level of the Cartan subalgebra). Indeed, in order for such a term to be uncharged under all the global symmetries and have R-charge 2, the axial masses of $G[U(k)]$ (2.144) have to satisfy the constraint

$$\Delta=(N-k+1)m_A-i\frac{Q}{2}(N-k)\,, \tag{2.175}$$

which can be consistently solved in terms of a single parameter τ

$$m_A=i\frac{Q}{2}-\tau,\qquad \Delta=i\frac{Q}{2}-(N-k+1)\tau\,, \tag{2.176}$$

Hence, we see that the two axial symmetries are broken to this particular combination

$$U(1)_{m_A}\times U(1)_\Delta\quad\rightarrow\quad U(1)_\tau \tag{2.177}$$

	$U(k)_z$	$U(1)_\tau$	$U(1)_\mu$	$U(1)_\zeta$	$U(1)_R$
α_i	0	$2(i-1)$	2	0	$2r+2(i-1)(1-R)$
T_j^+	0	$-2N+k+2j$	-1	1	$1-r-(2N-k-2j)(1-R)$
T_{N-l+1}^-	0	$-2N+k+2l$	-1	-1	$1-r-(2N-k-2l)(1-R)$
β_a	0	$2(N-k+a)$	0	0	$2(N-k+a)(1-R)$
$\mathtt{C}$	$\mathbf{k^2-1}$	-2	0	0	$2R$
Ω	$\bar{\mathbf{k}}$	-1	1	0	$r+R$
$\tilde{\Omega}$	$\mathbf{k}$	-1	1	0	$r+R$
$\mathfrak{M}^{\pm}_{\mathbb{M}^s}$	0	$-k+2s+2$	-1	±1	$1-r-(k-2s-2)(1-R)$
$\mathrm{Tr}\,(\tilde{p}\mathbb{M}^s p)$	0	$2(N-k+s)$	2	0	$2r+2(N-k+s)(1-R)$

Table 2.8: The main gauge invariant operators of Theory B and their transformation rules under the (enhanced) global symmetry.

Taking this into account the global symmetry group of Theory B is

$$\prod_{a=1}^{k} U(1)_{z_a} \times U(1)_\tau \times U(1)_\mu \times U(1)_\zeta \,. \tag{2.178}$$

On this side of the duality, the full flavor symmetry $U(k)_z$ is not visible in the UV, but it is enhanced at low energies, so that the global symmetry group coincides with that of Theory A

$$U(k)_z \times U(1)_\tau \times U(1)_\mu \times U(1)_\zeta \,. \tag{2.179}$$

This feature is motivated by the validity of the duality, but also by the fact that the gauge invariant operators of $G[U(k)]$ re-organize into representations of $U(k)_z$, as showed in Subsubsection 2.4.2. We list the transformation rules of the chiral fields under the global symmetries and their R-charges in Table 2.7.

The gauge invariant chiral operators are precisely those of $G[U(k)]$. We summarize them in Table 2.8, where we also specify their charges under the global symmetries and their R-charges. From this, we can find the map between the chiral ring generators of the dual theories, which provides a first non-trivial test of the duality

$$\begin{aligned}
\mathrm{Tr}_N\,\Phi^{N-k+a} \quad &\leftrightarrow \quad \beta_a, \qquad a=1,\cdots,k\\
\mathfrak{M}^+_{\Phi^s} \quad &\leftrightarrow \quad \begin{cases} T^+_{s+1} & s=0,\cdots,N-k-1\\ \mathfrak{M}^+_{\mathbb{M}^{k-N+s}} & s=N-k,\cdots,N \end{cases}\\
\mathfrak{M}^-_{\Phi^s} \quad &\leftrightarrow \quad \begin{cases} T^-_{N-s} & s=0,\cdots,N-k-1\\ \mathfrak{M}^-_{\mathbb{M}^{k-N+s}} & s=N-k,\cdots,N \end{cases}
\end{aligned}$$

$$
\begin{array}{rcl}
\mathrm{Tr}_N\left(\tilde{P}\Phi^s P\right) & \leftrightarrow & \begin{cases}\alpha_{s+1} & s=0,\cdots,N-k-1\\ \mathrm{Tr}_k\left(\tilde{p}\mathbb{M}^{k-N+s}p\right) & s=N-k,\cdots,N-1\end{cases}\\
Q\tilde{P} & \leftrightarrow & \Omega\\
P\tilde{Q} & \leftrightarrow & \tilde{\Omega}\\
Q\tilde{Q} & \leftrightarrow & \mathtt{C}\,.
\end{array} \tag{2.180}
$$

At the level of the three-sphere partition functions the duality is expressed by the identity

$$
\begin{aligned}
\mathcal{Z}_{\mathcal{T}_A} = & \prod_{j=1}^{N-k} \underbrace{s_b\left(-i\frac{Q}{2}+2j\tau\right)}_{\beta_j} \int \mathrm{d}x_N\, \mathrm{e}^{2\pi i\zeta\sum_\alpha x_\alpha} \frac{\prod_{\alpha,\beta=1}^{N} s_b\left(i\frac{Q}{2}+(x_\alpha-x_\beta)-2\tau\right)}{\prod_{\alpha<\beta}^{N} s_b\left(i\frac{Q}{2}\pm(x_\alpha-x_\beta)\right)}\times\\
\times & \prod_{\alpha=1}^{N} s_b\left(i\frac{Q}{2}\pm x_\alpha-\mu\right)\prod_{a=1}^{k} s_b\left(\pm(x_\alpha-z_a)+\tau\right) = \prod_{j=1}^{N-k}\underbrace{s_b\left(i\frac{Q}{2}-2\mu-2(j-1)\tau\right)}_{\alpha_j}\times\\
\times & \underbrace{s_b\left(-\zeta+\mu+(2N-k-2j)\tau\right)}_{T_j^+}\prod_{j=k+1}^{N}\underbrace{s_b\left(\zeta+\mu+(-k+2j-2)\tau\right)}_{T_j^-}\times\\
\times & \mathcal{Z}_{G[U(k)]}\left(\vec{z};\zeta;\mu+(N-k)\tau;i\frac{Q}{2}-(N-k+1)\tau;i\frac{Q}{2}-\tau\right) = \mathcal{Z}_{\mathcal{T}_B}\,.
\end{aligned} \tag{2.181}
$$

In Appendix B.2 we prove this identity for $k=1,2$ and generic N by iteratively applying the one-monopole and the Aharony duality. The derivation is very similar to the one for the confining duality that we have in the $k=0$ case. The idea is to find a dual frame which is then *stable* under a definite sequence of duality moves, in the sense that every time we apply those moves we go back to the same theory but with a lower rank of the gauge group and with some additional singlets. Exploiting this stability of the theory, we are able to find the dual frame of $G[U(k)]$, where now the original rank N only appears as a parameter. This again justifies the name "rank stabilization" that we give to this duality. The duality has been also tested for $k=3$ and $N=4,5$ by means of a perturbative expansion of the supersymmetric index in Appendix C.2.3 of [25].

Finally, it is a tedious but straightforward exercise to show that, starting from the duality identity for the $\mathbb{S}^2\times\mathbb{S}^1$ partition functions and taking the $2d$ Coulomb limit that we explained in Section 2.3, we recover the duality identity for the free-field correlator (2.19) of the $(k+3)$-point correlator of Liouville theory that was proven in [38].

Rank analytic continuation

As we have mentioned, the rank stabilization duality relating the $U(N)$ theory with an adjoint and $k+1$ flavors to the $G[U(k)]$ quiver theory can be considered as the $3d$ uplift of the duality relation (2.19) for the free field representation with N screening charges of the correlator with 3 primaries and k degenerate operators in the Liouville theory. In Section 2.2 we stressed the fact that the identity (2.19) provides a form suitable for analytic continuation in N which allows us to reconstruct the correlator for generic values of the momenta lifting the screening condition (2.8).

Similarly to what we saw for $k = 0$ in Subsection 2.3.4, also the $3d$ partition function enjoys a similar property. Indeed the partition function of Theory B in (2.181) consists of two blocks, the partition function of $G[U(k)]$ and the contribution of the gauge singlets. In the former N enters as a parameter inside the charges of the various fields, while in the latter it also counts the number of singlets

$$\begin{aligned}\mathcal{Z}_{\mathcal{T}_B} = &\prod_{j=1}^{N} S_2\left(Q+2ij\tau\right)\prod_{j=1}^{N-k} S_2\left(Q+2i\mu+2i(j-1)\tau\right)\times\\ &\times S_2\left(\frac{Q}{2}+i\zeta-i\mu-i(2N-k-2j)\tau\right)\prod_{j=k+1}^{N} S_2\left(\frac{Q}{2}-i\zeta-i\mu-i(2j-k-2)\tau\right)\times\\ &\times\mathcal{Z}_{G'[U(k)]}\left(\vec{z};\zeta;\mu+(N-k)\tau;i\frac{Q}{2}-(N-k+1)\tau;i\frac{Q}{2}-\tau\right),\end{aligned} \tag{2.182}$$

Here we have written explicitly the contribution of the β-fields contained in the $G[U(k)]$ theory, so by $G'[U(k)]$ we denote the same quiver gauge theory but without those flipping fields. Moreover, we moved to this side of the duality the contribution of the b-fields that flipped some of the Casimirs on the $USp(2N)$ gauge theory side and we used that $s_b(x) = S_2\left(\frac{Q}{2}-ix|b,b^{-1}\right) \equiv S_2\left(\frac{Q}{2}-ix\right)$. Now we can use the periodicity property of the triple-sine function

$$S_3(z+\omega_3|\omega_1,\omega_2,\omega_3) = \frac{S_3(z|\omega_1,\omega_2,\omega_3)}{S_2(z|\omega_1,\omega_2)} \tag{2.183}$$

to move the dependence on N inside the argument of the triple-sine function, making it appear in a form suitable for analytic continuation since the $3d$ partition function can be expressed as:

$$\begin{aligned}\mathcal{Z}_{\mathcal{T}_B} = &\operatorname*{Res}_{N\in\mathbb{N}}\left\{\frac{S_3'(0)S_3\left(-2i\mu+2i\tau\right)S_3\left(\frac{Q}{2}\pm i\zeta-i\mu-i(2N-k-2)\tau\right)}{S_3\left(-2iN\tau\right)S_3\left(-2i\mu-2i(N-k-1)\tau\right)S_3\left(\frac{Q}{2}\pm i\zeta-i\mu-i(k-2)\tau\right)}\right\}\times\\ &\times\mathcal{Z}_{G[U(k)]}\left(z_a,\zeta,\mu+(N-k)\tau,i\frac{Q}{2}-\tau,i\frac{Q}{2}-(N-k+1)\tau\right),\end{aligned} \tag{2.184}$$

where $S_3(x) \equiv S_3(x|b,b^{-1},2i\tau)$. Inside the brackets we recognize the five-sphere partition function of the $5d$ T_2 theory, which can be realized on the toric CY geometry $\mathbb{C}^3/\mathbb{Z}_2\times\mathbb{Z}_2$ [42], with quantized Kähler parameters. This is the result that we got in the $k = 0$ case in Subsection 2.3.4. The analytic continuation in N is then reinterpreted as geometric transition with the $3d$ theory appearing as a codimension-two defect theory at the point in the moduli space of the $5d$ T_2 theory specialized by the quantized values of the Kähler parameters as proposed in [31, 32].

The $(k+3)$-point correlator corresponds via the AGT map [84, 85] to the T_2 theory (two M5 wrapping the 3-punctured sphere) coupled to k co-dimension-two defects (k M2 branes which are points on the 3-punctured sphere). In our case the $5d$ theory emerging after the geometric transition can be realized as the $5d$ T_2 geometry with the insertion of k toric branes[24] and the contribution of the $G[U(k)]$ theory captures how the defects interact among themselves.

2.5 Outlook

In this chapter we established a connection between a specific observable of $3d$ $\mathcal{N}=2$ gauge theories, namely their partition function on $\mathbb{S}^2\times\mathbb{S}^1$, and complex integrals that correspond to free field realizations of $2d$ CFT correlators. We saw that the latter can be obtained from the former by considering a particular $2d$ limit where the $\mathbb{S}^1$ is shrunk to a point that is called "Coulomb limit". We determined the precise dictionary of this correspondence and then exploited it to find new results about the dynamics of $3d$ $\mathcal{N}=2$ gauge theories by starting from known result about $2d$ free field correlators.

From this point, there are several directions that we may follow. We already mentioned that $3d$ dualities can be obtained as circle reductions of $4d$ dualities [5, 6]. Given the success that we achieved in trying to reverse the similar limit that relates $3d$ dualities to $2d$ dualities, it is then very tempting to try to understand whether we can further uplift our results in $3d$ to new dualities and symmetry enhancements in four dimensions.

In order to understand this, it is again useful to consider a prototypical example in which the more standard top-down approach is well understood. Remember that our starting example was the confining duality in $3d$ relating the $U(N)$ gauge theory with one adjoint and one fundamental flavor to a WZ model of $3N$ chirals with a cubic superpotential. It turns out that this duality has a $4d$ $\mathcal{N}=1$ parent which was first proposed in [86]:

Theory A: $USp(2N)$ gauge theory with one antisymmetric chiral A, six fundamental chirals Q_a and N chiral singlets b_i with superpotential[25]

$$\mathcal{W}=\sum_{i=1}^{N} b_i\,\mathrm{Tr}_N\,A^i\,. \tag{2.186}$$

[24] In [72] the contribution of k toric branes in the length-two strip geometry, which is closely related to the T_2 geometry, was shown to reproduce the $(k+3)$-point conformal blocks.

[25] The $USp(2N)$ indices are contracted using the totally antisymmetric tensor

$$J^{(N)}=\mathbb{I}_N\otimes i\,\sigma_2\,. \tag{2.185}$$

Theory B: WZ model with $15N$ chiral singlets $\mu_{ab;i}$ for $i = 1, \cdots, N$, $a < b = 1, \cdots, 6$ interacting with the cubic superpotential

$$\hat{\mathcal{W}} = \sum_{i,j,k=1}^{N} \sum_{a,b,c,d,e,f=1}^{6} \epsilon_{abcdef}\mu_{ab;i}\mu_{cd;j}\mu_{ef;k}\delta_{i+j+k,2N+1} \, . \tag{2.187}$$

Observe how similar this duality is to the $3d$ one. We can indeed flow from the $4d$ duality to the $3d$ one by first compactifying it on $\mathbb{S}^1$ and then considering a deformation that consists both of a Coulomb branch VEV breaking the gauge group from $USp(2N)$ to $U(N)$ and a real mass deformation that keeps some of the matter fields massless[26]. This is actually how the $3d$ duality was first derived in [37]. Moreover, also the four-dimensional duality can be derived with a deconfinement technique similar to the one we presented in Subsection 2.3.3 for the $3d$ one[27]. This was first shown in [90] at the level of the $\mathbb{S}^3 \times \mathbb{S}^1$ partition function to prove the associated integral identity that was first conjectured in [91] (see also [92, 93] for a field theory point of view).

The upshot of this discussion is that the $4d$ duality seems to be the more fundamental one, from which we can derive the $3d$ duality as a limit and the evaluation formula for the 3-point function of Liouville theory as a further limit. This suggests that also the other $3d$ dualities that we discussed in this chapter and that were obtained as uplifts of known results for $2d$ free field correlators can be further uplifted to $4d$. This seems even more plausible if we believe that all the dualities in low dimensions can be obtained as limits of a restricted set of dualities in higher dimensions.

In the next chapter we will indeed discuss many results about the dynamics of $4d$ $\mathcal{N} = 1$ theories, but we will take a completely different perspective which is in a sense orthogonal to the point of view we adopted so far. We will indeed follow a top-down approach instead of a bottom-up one, by constructing $4d$ theories as compactifications of $6d$ SCFTs on Riemann surfaces. We will see how the $4d$ analogue of the $M[SU(N)]$ theory appears naturally in this framework. We will also recover the confining duality for $USp(2N)$ with one antisymmetric and six fundamental chirals. We will not be able to find the $4d$ versions of the recombination and the rankstabilization duality, which will be discussed in an upcoming paper [92]. It would be very interesting to obtain them also from the $6d$ construction we will discuss in the next chapter.

It is also important to stress the fact that here we just focused on a very specific $2d$ CFT as a starting point for obtaining new $3d$ dualities, namely Liouville theory, but there are many other results in the CFT literature that we can try to uplift. For example, free field

[26] In the $\mathbb{S}^1$ reduction a monopole superpotential is generated, but this gets lifted after the real mass deformation.

[27] It is also known a $2d$ reduction of this $4d$ duality. The reduction can be performed on a two-sphere $\mathbb{S}^2$ with a topological twist that preserves half of the supersymmetry [87–89], so to get a duality between $\mathcal{N} = (0, 2)$ theories [9]. This has been discussed in [64], where it has also been derived with a deconfinement technique similar to the one for the $3d$ and $4d$ analogues of the duality.

correlators for Toda A_n theory have been studied in [94, 95]. The difference between Toda and Liouville is that now we don't have just one scalar field, but a set of n scalar fields. The free field integrals for correlation functions in Toda theory then contain the insertion of n distinct sets of screening charge operators, which will have dimensions $N_1, \cdots, N_n$. The corresponding $3d$ gauge theory will then be a quiver theory with n unitary nodes of ranks $N_1, \cdots, N_n$ and the free field integral identities studied in [94, 95] will uplift to dualities for such quiver gauge theories. In an upcoming paper [92] we will study some of the dualities that can be obtained in this way, but uplifting them directly to four dimensions.

Finally, we conclude mentioning the fact that it is of course possible to exploit the correspondence between $3d$ dualities and $2d$ free field correlators in the standard top-down direction. Indeed over the years tons of dualities for $3d$ $\mathcal{N} = 2$ theories have been discovered and it would be very interesting to understand if they can lead to new insights into the realm of $2d$ CFTs.

Chapter 3

$4d$ compactifications of the $6d$ E-string theory

In this chapter we will present an approach for finding new dualities and symmetry enhancements in four dimensions that is in a sense orthogonal to the one we used in the first chapter, since it is intrinsically top-down. We will indeed construct $4d$ $\mathcal{N}=1$ theories as compactifications of $6d$ $\mathcal{N}=(1,0)$ SCFTs on Riemann surfaces, focusing in particular on the $6d$ E-string theory. We will discuss how dualities and symmetry enhancements for the resulting models can be understood geometrically thanks to this construction. The content of this chapter is based on [96, 97], where in the first reference the compactification of the E-string theory on tubes and tori has been discussed, while in the second one cap and sphere compactifications have been considered.

3.1 The general idea

In recent years there has been a lot of work on trying to construct four-dimensional theories, especially with supersymmetry, as the compactification of some higher dimensional theories. One possibility it to start with a $6d$ SCFT and compactify it on a Riemann surface, so to get an effective $4d$ theory at low energies. In the process one can also preserve part of the supersymmetry of the $6d$ theory by performing a topological twist, that is turning on a background gauge field for a subgroup of the six-dimensional R-symmetry in such a way that it cancels the contribution of the spin connection of the Riemann surface in the supersymmetry variations of the fermions.

This approach has been pioneered in [10, 98], where $4d$ $\mathcal{N}=2$ SCFTs have been constructed as compactifications of $6d$ $\mathcal{N}=(2,0)$ SCFTs. One of the key features of this procedure is that the $4d$ theory only depends on the topological properties of the 2-surface $\Sigma_{g,n}$, namely the genus g and the number of punctures n, while it doesn't depend on the metric on it. This allows us to understand dualities between the $4d$ theories in a geometric

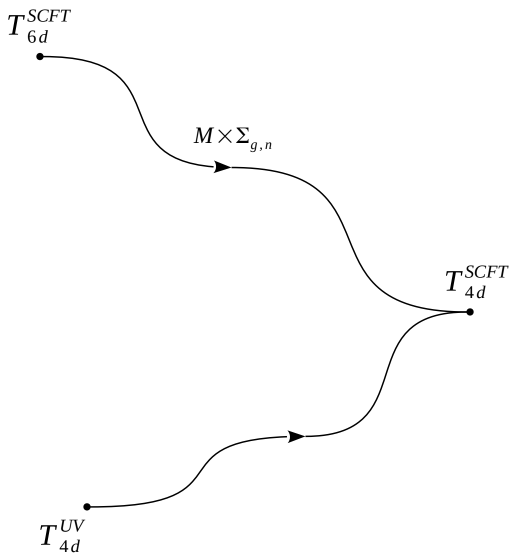

Figure 3.1: Schematic representation of the different RG flows that we can use to reach the same $4d$ SCFT.

way. The idea is to first figure out what are the theories related to the fundamental building block, which is the sphere with 3 punctures. These theories are usually called *trinions* and gluing them together we can construct models associated to arbitrary Riemann surfaces. Indeed, the process of gluing is understood field theoretically as gauging mutual global symmetries of the trinion theories. Starting from a Riemann surface, we can consider different degeneration limits corresponding to different pants decomposition, namely different gluings of 3-punctured spheres. Since the gluing is different for different pants decompositions, one would associate to each of them a distinct theory. Nevertheless, the $4d$ theory is expected to depend only on the topology of the surface, so all the pants decompositions are equivalent and this implies a duality between the different looking $4d$ theories. In some instances one can also have symmetry enhancements, when the surface is built from trinions that individually preserve less symmetry than that expected from the full surface.

This strategy has proved to be incredibly successful for the program of the classification of $4d$ $\mathcal{N}=2$ SCFTs. The only flaw is that most of the times we are not able to get a Lagrangian description of the resulting theories from this perspective. In particular, the trinion theories are typically non-Lagrangian. Nevertheless, it may be possible to find a Lagrangian that is not conformal in the UV, but that flows to the desired SCFT in the IR, provided that we relax the assumption on the amount of supersymmetry. Indeed, in the recent years many $4d$ $\mathcal{N}=1$ Lagrangians for $\mathcal{N}=2$ SCFTs have been found [99–108][1]. In other words, we can try to look for a $4d$ $\mathcal{N}=1$ Lagrangian theory that flows in the IR to the same SCFT that is the

[1]See also [109] for an $\mathcal{N}=1$ Lagrangian of an $\mathcal{N}=3$ SCFT.

result of a compactification of a $6d$ theory[2], as depicted in Figure 3.1. This phenomenon is sometimes referred to as *across dimensional duality*. When this occurs, we can predict IR dualities and symmetry enhancements for such $4d$ theories using the geometric picture, as described above.

Given that we are now allowing for lower supersymmetry, we can also start from a $6d$ $\mathcal{N}=(1,0)$ SCFT, compactify it on a Riemann surface so to get a $4d$ $\mathcal{N}=1$ SCFT and then try to find a UV Lagrangian that flows to it in the IR. Again this is done performing a topological twist that involves the Cartan of the $6d$ $SU(2)_R$ R-symmetry, but this time the $6d$ theory may also possess a global symmetry and we can turn on fluxes for this symmetry through the Riemann surface, still preserving $\mathcal{N}=1$ supersymmetry in $4d$. This allows us to get a richer set of examples of symmetry enhancements. Indeed, when we compactify on a surface without punctures the expected global symmetry will be the one preserved by the flux, but this symmetry may not be fully manifest in the UV Lagrangian description that we found. This means that it must get enhanced at low energies. We can also predict IR dualities using the structure of the fluxes. More precisely, it may happen that different fluxes, to which we would associate distinct UV Lagrangians in $4d$, are related by an element of the Weyl group of the $6d$ global symmetry and are thus equivalent. This means that the SCFT to which the two theories are flowing is the same, which is equivalent to saying that they are IR dual. This will be particularly useful for us, since we will mostly consider compactifications on surfaces that don't have more than one pant decomposition, so it will allow us to get interesting structures in $4d$ despite of this.

The realm of $6d$ $\mathcal{N}=(1,0)$ SCFTs is very wide and the compactification of such theories to four dimensions has been intensively studied in the recent years (see for example [110–123, 96, 124–126, 97])[3]. In this thesis we will focus on the particular case of compactifications of the rank-N E-string theory [127, 128]. Compactifications of this $6d$ SCFT have been considered in [116] for the rank-1 case and then in [96, 97] for higher rank N. Here we will present the results of the latter two papers.

We will start by reviewing the basic properties of this $6d$ SCFT and explaining how we can use them to make predictions about the $4d$ theories that result from the compactification. Checking that these expectations are indeed satisfied by our models we will give strong evidence of our claim that they are UV Larangians for the SCFTs obtained from the compactification of the E-string theory. In particular, we can use the knowledge of the 8-form anomaly polynomial of the $6d$ E-string theory to predict what should be the anomalies of the four-dimensional theories. Moreover, we can check that our models possess the global

[2]More generally, it may be that the $4d$ SCFT theory obtained from the $6d$ compactification sits on a point of a non-trivial conformal manifold $\mathcal{M}_c$ and that one is able to find a $4d$ UV Lagrangian that doesn't flow exactly to the same SCFT, but to another one which still lives on a different corner of $\mathcal{M}_c$.

[3]Very recently this strategy has been also extended in [83] to the compactification of $5d$ $\mathcal{N}=1$ SCFTs, so to find $3d$ $\mathcal{N}=2$ models enjoying dualities and symmetry enhancements.

symmetry preserved by the flux by means of the supersymmetric index, also known as the $\mathbb{S}^3 \times \mathbb{S}^1$ partition function.

We will then discuss what will be the fundamental building block of our constructions, namely the theory obtained by compactification of the E-string theory on a tube, that is a 2-punctured sphere. We will make a very specific choice of flux that will give us a sort of fundamental tube, from which we can construct more tubes with different choices of flux. The $4d$ theory associated to the fundamental tube will involve a quiver gauge theory called $E[USp(2N)]$, which is a four-dimensional ancestor of the three-dimensional $M[SU(N)]$ theory we saw in Section 2.4.1. We will discuss many of its properties, including a duality web that it enjoys and which is very reminiscent of the one of $M[SU(N)]$. We will then explicitly show how to flow from $E[USp(2N)]$ to $M[SU(N)]$ at the level of supersymmetric partition functions, which will allows us to understand the duality web of $M[SU(N)]$ as a limit of that of $E[USp(2N)]$.

From the tube theory we will then construct tori with various values of the flux and verify that they pass all the necessary tests. In particular, we will check that they enjoy the symmetry enhancements that are expected from the six-dimensional construction by means of the supersymmetric index. We will also find a duality between models corresponding to fluxes that are equivalent up to an element of the Weyl group of the $6d$ global symmetry. This will be explained in terms of a *braid duality* that the $E[USp(2N)]$ theory enjoys and which generalizes the confining Seiberg duality for the $SU(2)$ gauge theory with 3 flavors [1]. Hence, our construction will give us a geometric derivation of such duality.

We will then consider E-string compactifications on a cap, that is a sphere with one puncture. The cap can be obtained from the tube by *closing* one of the two punctures, which turns out to be a well-defined operation in field theory. This corresponds indeed to giving a vacuum expectation value (VEV) to a gauge invariant operator of the tube theory that completely breaks the global symmetry carried by the puncture. From the cap model we will then construct models corresponding to spheres. Here an interesting new feature will arise. Namely, we will see that the $SO(3)_{\text{ISO}} \cong SU(2)_{\text{ISO}}$ isometry of the two-sphere will show up in the $4d$ theory as a flavor symmetry. The $U(1)$ Cartan of such symmetry can be traced back to one of the symmetries of the tube model. Such a symmetry is typically anomalous when we glue tubes to build tori, so that the torus models don't possess it. Instead, it is preserved by the VEV that is used to obtain the cap and gets enhanced to $SU(2)_{\text{ISO}}$ when we construct the sphere models. We will check that this symmetry can be correctly identified with the isometry of the two-sphere by comparing its anomalies with those computed from the Bott–Cattaneo formula [129] (see also [130] for an application in physics).

Finally, also in the case of the sphere compactifications we will encounter two models that are associated to fluxes which are equivalent up to an element of the Weyl group of the $6d$ global symmetry. This again implies an IR duality between the two theories, which will turn out to be nothing but the confining duality for $USp(2N)$ with one antisymmetric and 6 fundamental chirals that we mentioned at the end of the previous chapter.

3.2 Properties of the compactification of the E-string theory

The $6d$ SCFT that we are going to compactify is the rank N E-string theory, and we shall begin our discussion by listing several properties of this SCFT that will be useful later. The rank N E-string SCFT can be engineered in string theory as the theory living on N M5-branes probing an M9-plane. In addition to the $6d$ superconformal symmetry, it has an $SU(2)_L \times E_8$ global symmetry. In the brane construction the E_8 comes from the gauge symmetry on the M9-plane, while the $SU(2)_L$ comes from the $SO(4) = SU(2)_L \times SU(2)_R$ symmetry acting on the directions of the M9-plane orthogonal to the M5-branes, where the other $SU(2)_R$ is the R-symmetry. The matter spectrum consists of N tensor multiplets. While it has no known Lagrangian description in $6d$, its compactification to lower dimensions leads to more approachable theories and we shall consider these.

It is known that when compactified on a finite radius circle to $5d$ with a proper holonomy inside E_8,[4] it flows to a $5d$ gauge theory with a $USp(2N)$ gauge group, an antisymmetric hypermultiplet and eight fundamental hypermultiplets [132]. We can also consider the compactification without the holonomy in the zero radius limit, where the theory flows to a $5d$ SCFT with $SU(2)_L \times E_8$ global symmetry, which was originally found in [131]. We can consider turning the holonomy back on, which is mapped to a mass deformation that causes the $5d$ SCFT to flow to a $5d$ gauge theory with gauge group $USp(2N)$ and matter being an antisymmetric and seven fundamental hypermultiplets. We note that one can continue with circle compactification to get other interesting theories with E_8 global symmetry in lower dimensions. For instance the compactification on a torus leads [132] to the rank N Minahan-Nemeschansky E_8 strongly interacting SCFTs [133].

In the following we are going to discuss some properties for the $4d$ $\mathcal{N} = 1$ theories obtained from the compactification of the E-string theory on various Riemann surfaces with fluxes that we can predict from our six-dimensional construction. Verifying that these properties are enjoyed by the $4d$ Lagrangian descriptions that we will propose for the compactified theories will be a very strong check that our conjecture is correct.

3.2.1 Anomalies

One of the tools that we will use consists of anomalies. In this subsection we are going to review how to compute the anomalies for the $4d$ theories by knowing those of the original $6d$

[4] That an holonomy is necessary can be seen from the fact that the E_8 symmetry is broken in the low-energy gauge theory. More specifically, to get the low-energy $5d$ gauge theory the holonomy must be tuned with the radius, see [131].

theory. We will then test in the following sections that the anomalies of our models match with this 6*d* prediction.

We are interested in the compactification of the E-string SCFTs on Riemann surfaces with fluxes in their E_8 global symmetry[5]. The majority of the discussion in this section was already worked out in [116, 96, 97] and we shall merely summarize the main parts here. As previously mentioned we are interested in compactifications that have a non-trivial flux in the $U(1)$ subgroups of the E_8 global symmetry. To enumerate the fluxes it is convenient to introduce a flux basis. We will mostly use two bases of fluxes associated to the E_8 factor of the 6*d* global symmetry. One corresponds to the $U(1)^8$ Cartan of the $SU(8) \times U(1)_c$ subgroup of the E_8 global symmetry. In this basis the fluxes are parametrized by a vector $(n^{U(1)_c}; n_1^{SU(8)}, \cdots, n_8^{SU(8)})$ with the constraint that $\sum_{a=1}^{8} n_a^{SU(8)} = 0$. Because of this, we will sometimes refer to it as the *overcomplete basis*. Another possible basis, which we will instead call *complete basis*, is associated to the $SO(2)^8$ Cartan of the $SO(16) \subset E_8$ subgroup. In this basis the fluxes are parametrized by a vector $(n_1^{SO(16)}, \cdots, n_8^{SO(16)})$. The change of basis is given by

$$n_a^{SO(16)} = n^{U(1)_c} + 2n_a^{SU(8)}, \qquad a = 1, \cdots, 8\,. \tag{3.1}$$

For the $SU(2)_L$ part we have only one choice, that is a flux n_t for its $U(1)_t$ Cartan. We refer the reader to [116] and to Appendix A of [96] for more details on the fluxes.

As we have said, one of the major things that will concern us will be the determination of the anomalies of the resulting 4*d* theories from the anomalies of the original 6*d* SCFT. The anomalies generally receive two contributions. One is from the integration of the anomaly polynomial of the 6*d* SCFT on the Riemann surface [110], while the other is the contribution from the degrees of freedom associated with the punctures, if these are present. We shall begin by discussing the first contribution and then move on to discuss the second one.

Barring the issue of punctures, the anomalies of the resulting 4*d* theories can be evaluated by integrating the 8-form anomaly polynomial of the 6*d* SCFT on the Riemann surface. This can be done since the anomaly polynomial of the rank N E-string SCFTs is known [135, 136].

We can first consider the case in which the flux is turned on for only one $U(1)$ inside E_8. These will break $E_8 \to U(1) \times G$, where the $U(1)$ is the part for which we turn on the flux, which we will denote by z. There are 8 choices for such $U(1)$, each of which corresponds to a different node of the E_8 Dynkin diagram (see Figure 3.2). We introduce the coefficient ξ_G parametrizing such choice. Namely, for $\xi_G = 1$ the commutant G of the $U(1)$ inside E_8 is E_7, for $\xi_G = 2$ it is $SO(14)$, for $\xi_G = 3$ it is $E_6 \times SU(2)$, for $\xi_G = 4$ it is $SU(8)$, for $\xi_G = 6$ it is $SU(3) \times SO(10)$, for $\xi_G = 7$ it is $SU(2) \times SU(7)$, for $\xi_G = 10$ it is $SU(4) \times SU(5)$, and for $\xi_G = 15$ it is $SU(2) \times SU(3) \times SU(5)$. The $U(1)_R$ symmetry we use descends from the Cartan of the 6*d* $SU(2)_R$. Its 6*d* origin makes it useful to work with for the purpose

[5] Flux compactifications of 6*d* SCFTs to four dimensions were first discussed in [134].

of anomaly calculations, though it is in general not the superconformal R-symmetry. With these conventions, the anomalies of the $4d$ theory are [116]

$$\begin{aligned}
&\mathrm{Tr}(U(1)_R^3) = (g-1+\frac{s}{2})N(4N^2+6N+3), \quad \mathrm{Tr}(U(1)_R) = -(g-1+\frac{s}{2})N(6N+5), \\
&\mathrm{Tr}(U(1)) = -12Nz\xi_G, \qquad \mathrm{Tr}(U(1)^3) = -12Nz\xi_G^2, \\
&\mathrm{Tr}(U(1)_R U(1)^2) = -2N(N+1)(g-1+\frac{s}{2})\xi_G, \qquad \mathrm{Tr}(U(1)U(1)_R^2) = 2N(N+1)\xi_G z \\
&\mathrm{Tr}(U(1)_R SU(2)_L^2) = -\frac{N(N^2-1)(g-1+\frac{s}{2})}{3}, \qquad \mathrm{Tr}(U(1)SU(2)_L^2) = -\frac{N(N-1)}{2}\xi_G z\,,
\end{aligned} \tag{3.2}$$

where g denotes the genus of the Riemann surface and s the number of punctures.

We can use the above anomalies to write a trial a function and perform a- maximization to obtain candidate values for the superconformal a and c anomalies. This always comes with the caveat of having no accidental abelian symmetries, which is not always satisfied. Nevertheless, if we have matched the symmetries between $4d$ and $6d$ the analogous naive computation should produce the same result and thus we quote it here. The anomalies are [116]

$$a = \frac{\sqrt{2\xi_G}Q(3Q+5)^{\frac{3}{2}}}{16}|z|\,, \qquad c = \frac{\sqrt{2\xi_G(3Q+5)}Q(3Q+7)}{16}|z|\,. \tag{3.3}$$

For the cases of tubes and tori compactifications this will be enough for what concerns the contribution to the anomalies from the bulk of the surface, since we will always deal with the simpler cases where the flux is only for one $U(1)$ inside E_8. When we will close one of the punctures of the tube to get a cap, instead, we will see that the VEV procedure will introduce a non-trivial flux for all the Cartans of E_8, as well as for $SU(2)_L$. We then need to compute the $4d$ anomalies in such a case. This has been done in [97] and we review the computation in Appendix D. The result expressed in the overcomplete basis is

$$\begin{aligned}
&\mathrm{Tr}\left(U(1)_R^3\right) = \left(g+\frac{s}{2}-1\right)N\left(4N^2+6N+3\right), \\
&\mathrm{Tr}\left(U(1)_R\right) = -\left(g+\frac{s}{2}-1\right)N\left(6N+5\right), \\
&\mathrm{Tr}\left(U(1)_c^3\right) = -12Nn_c, \quad \mathrm{Tr}\left(U(1)_c\right) = -12Nn_c\,, \\
&\mathrm{Tr}\left(U(1)_{u_a}^3\right) = -6N\left(n_a-n_8\right), \quad \mathrm{Tr}\left(U(1)_{u_a}\right) = -6N\left(n_a-n_8\right), \\
&\mathrm{Tr}\left(U(1)_R U(1)_c^2\right) = -2\left(g+\frac{s}{2}-1\right)N\left(N+1\right), \quad \mathrm{Tr}\left(U(1)_R^2 U(1)_c\right) = 2N\left(N+1\right)n_c\,, \\
&\mathrm{Tr}\left(U(1)_R U(1)_{u_a}^2\right) = -2\left(g+\frac{s}{2}-1\right)N\left(N+1\right), \\
&\mathrm{Tr}\left(U(1)_R^2 U(1)_{u_a}\right) = N\left(N+1\right)\left(n_a-n_8\right),
\end{aligned}$$

$$\begin{aligned}
&\mathrm{Tr}\left(U(1)_R U(1)_{u_a} U(1)_{u_b}\right) = -\left(g+\frac{s}{2}-1\right) N\left(N+1\right) ,\\
&\mathrm{Tr}\left(U(1)_c U(1)^2_{u_a}\right) = -4Nn_c, \quad \mathrm{Tr}\left(U(1)^2_c U(1)_{u_a}\right) = -2N\left(n_a-n_8\right) ,\\
&\mathrm{Tr}\left(U(1)_{u_a} U(1)^2_{u_b}\right) = -2N\left(n_a+n_b-2n_8\right) , \quad \mathrm{Tr}\left(U(1)_c U(1)_{u_a} U(1)_{u_b}\right) = -2Nn_c ,\\
&\mathrm{Tr}\left(U(1)_{u_a} U(1)_{u_b} U(1)_{u_d}\right) = -N\left(n_a+n_b+n_d-3n_8\right) ,\\
&\mathrm{Tr}\left(U(1)^3_t\right) = -\left(N-1\right)\left(4N^2-2N+1\right) n_t, \quad \mathrm{Tr}\left(U(1)_t\right) = -\left(N-1\right)\left(6N+1\right) n_t ,\\
&\mathrm{Tr}\left(U(1)_R U(1)^2_t\right) = -\frac{4}{3}\left(g+\frac{s}{2}-1\right) N\left(N^2-1\right) ,\\
&\mathrm{Tr}\left(U(1)^2_R U(1)_t\right) = \frac{4}{3} N\left(N^2-1\right) n_t ,\\
&\mathrm{Tr}\left(U(1)_t U(1)^2_{c/u_a}\right) = -2N\left(N-1\right) n_t, \quad \mathrm{Tr}\left(U(1)^2_t U(1)_c\right) = -2N\left(N-1\right) n_c ,\\
&\mathrm{Tr}\left(U(1)^2_t U(1)_{u_a}\right) = -N\left(N-1\right)\left(n_a-n_8\right) ,\\
&\mathrm{Tr}\left(U(1)_t U(1)_{u_a} U(1)_{u_b}\right) = -N\left(N-1\right) n_t ,
\end{aligned} \tag{3.4}$$

with the constraint $n_8 = -\sum_{a=1}^{7} n_a$ and where we are denoting by $U(1)_{u_a}$ the Cartans of $SU(8)$. The rest of the anomalies that don't appear in (3.4) vanish.

This will still be not enough for the case of sphere compactifications without punctures. Indeed, as we mentioned in our general discussion, the sphere has an $SU(2)_{\mathrm{ISO}}$ isometry that will manifest itself as a flavor symmetry in 4d, and we would like to have a prediction for its anomalies. Again this was done in [97] and we review the computation in Appendix D, while here we shall only quote the result. For a sphere with no punctures $g = s = 0$ and the anomalies for are

$$\begin{aligned}
\mathrm{Tr}\left(SU(2)^2_{\mathrm{ISO}} U(1)_R\right) &= \frac{N(N+1)}{12}\left(-8+6n_c^2+4(N-1)n_t^2-4N+3\sum_{a=1}^{8} n_a^2\right)\\
\mathrm{Tr}\left(SU(2)^2_{\mathrm{ISO}} U(1)_t\right) &= -\frac{N-1}{12} n_t\left(-1+2N(3n_c^2-5)+(4N^2-2N+1)n_t^2+\right.\\
&\qquad\left.+N(-4N+3\sum_{a=1}^{8} n_a^2)\right)\\
\mathrm{Tr}\left(SU(2)^2_{\mathrm{ISO}} U(1)_c\right) &= -\frac{N}{2} n_c\left(-3+2n_c^2+(N-1)n_t^2-N+\sum_{a=1}^{8} n_a^2\right)\\
\mathrm{Tr}\left(SU(2)^2_{\mathrm{ISO}} U(1)_{u_a}\right) &= -\frac{N}{4}(n_a-n_8)\left(-3+2n_c^2+(N-1)n_t^2-N+\sum_{a=1}^{8} n_a^2\right) .
\end{aligned} \tag{3.5}$$

We now move on to discussing the contribution of the punctures to the anomalies, where we specifically concentrate on the contribution from the degrees of freedom associated with the punctures rather than the geometric contribution which was previously discussed. The calculation of this contribution of the punctures to the anomalies was set up in [116, 118, 120][6] and applied in particular to the case of the rank-N E-string theory in [96]. The basic idea is to consider the region around a puncture and deform it so as to look like a long thin tube ending at the puncture. We can then compactify the $6d$ SCFT on the circle of the tube and get the reduced $5d$ theory on an interval ending with the puncture. Particularly, we shall assume that the necessary holonomy as been turned on around the tube so that the reduced $5d$ theory is the IR free $USp(2N)$ gauge theory with an antisymmetric hyper and eight fundamental hypers that was introduced previously. The puncture can then be described as a boundary condition of this $5d$ gauge theory.

This leads us to consider boundary conditions of $5d$ gauge theories preserving half of the supersymmetry, that is four supercharges. These can be described as giving Dirichlet or Neumann boundary conditions to various multiplets on the boundary. Specifically, close to the boundary the $5d$ bulk fields approach the $4d$ boundary and can be decomposed in terms of $4d$ $\mathcal{N}=1$ superfields. The boundary conditions can then be described as assigning Dirichlet or Neumann boundary conditions to those superfields.

There are in principal many different possible boundary conditions leading to the many different punctures that exist in these types of construction. Here we shall only consider one type, which is the one considered in [116] for $N=1$ and in [96] for generic N. This type of puncture can be thought of as a generalization of the so called maximal punctures of class $\mathcal{S}$ theories [10]. The boundary conditions associated with this choice are as follows. First we decompose the $5d$ vector multiplet into a $4d$ $\mathcal{N}=1$ vector multiplet and an adjoint chiral on the boundary. We then give Dirichlet boundary conditions to the $\mathcal{N}=1$ vector and Neumann boundary conditions to the adjoint chiral. Note that as the vector multiplet is given Dirichlet boundary conditions, the $5d$ $USp(2N)$ gauge symmetry becomes non-dynamical at the boundary. As a result it becomes a global symmetry associated with the puncture.

We can similarly decompose the hypermultiplets into two chiral fields in conjugate representations and give Dirichlet boundary conditions to one and Neumann boundary conditions to the other. Here we have a choice for which chiral gets which boundary conditions and this leads to slightly different punctures. This difference is usually referred to as the sign of the puncture.

We next want to consider the contribution of the degrees of freedom at the boundary to the anomalies. This is known to be given by half the $4d$ anomalies expected from the matter given Neumann boundary conditions, see [116] for the details. We next evaluate these for the punctures considered here. First we consider the anomalies involving the $U(1)_R$ Cartan of the $SU(2)_R$ symmetry. These only receive contributions from the adjoint chiral as the fermions in the hypermultiplets are $SU(2)_R$ singlets. Specifically the fermion in the adjoint chiral has charge -1 under $U(1)_R$, is in the adjoint of the $USp(2N)$ symmetry associated with the

[6] See [137] for the discussion in the case of the $6d$ $(2,0)$ SCFT.

puncture and is a singlet under the other global symmetries. As a result it contributes to the anomalies as follows:

$$\mathrm{Tr}(U(1)_R^3) = -\frac{N(2N+1)}{2}, \qquad \mathrm{Tr}(U(1)_R) = -\frac{N(2N+1)}{2}, \tag{3.6}$$
$$\mathrm{Tr}(U(1)_R USp(2N)^2) = -\frac{N+1}{2}.$$

Then we consider the anomalies for the $SU(2)_L$ global symmetry. These receive contributions only from the antisymmetric hyper, the two chirals in which form a doublet of this symmetry. As we give different boundary conditions to them, the puncture breaks $SU(2)_L$ to its $U(1)_t$ Cartan and the anomalies expected for this symmetry are

$$\mathrm{Tr}(U(1)_L^3) = q^3\frac{(Q(2Q-1)-1)}{2}, \qquad \mathrm{Tr}(U(1)_L) = q\frac{(Q(2Q-1)-1)}{2}, \tag{3.7}$$
$$\mathrm{Tr}(U(1)_L USp(2Q)^2) = q\frac{(Q-1)}{2}.$$

Here q is the charge under $U(1)_t$ which depends on the normalization and the sign. We will use in what follows a normalization of the charges such that $q = -\frac{1}{2}$[7].

Finally, the anomalies for Cartans of the E_8 global symmetry, for which we are turning on fluxes, receive contributions only from the octet of the $USp(2N)$ fundamental hypers. Denoting again with $U(1)_{u_a}$ for $a = 1, \cdots, 8$ the Cartans of the $SU(8)$ part of the $U(1)_c \times SU(8) \subset E_8$ subgroup, these are given by

$$\begin{aligned}
&\mathrm{Tr}(U(1)_c^3) = N\sum_{a=1}^{8} q_a^3, \quad \mathrm{Tr}(U(1)_c) = N\sum_{a=1}^{8} q_a, \quad \mathrm{Tr}(U(1)_c USp(2Q)^2) = \frac{1}{4}\sum_{a=1}^{8} q_a\\
&\mathrm{Tr}\, U(1)_c U(1)_{u_a}^2 = 2N q_a \quad a = 1,\cdots,7, \qquad \mathrm{Tr}\, U(1)_c U(1)_{u_a} U(1)_{u_b} = N\, q_a \quad a \neq b\\
&\mathrm{Tr}\, U(1)_{u_a} U(1)_{u_b} U(1)_{u_d} = N, \quad a \neq b, d\,.
\end{aligned} \tag{3.8}$$

Here q_a are the charges of each octet hyper under such $U(8)$ subgroup of E_8 that is manifest in the gauge theory description and we shall use a normalization where $q_a = -\frac{1}{2}$.

All the anomalies that were not mentioned here receive no contribution from the punctures.

3.2.2 Symmetries

Another test that we will perform consists of verifying that our 4d Lagrangians possess a global symmetry which is at least the one predicted from the compactification. Indeed, from the 6d perspective we expect that this global symmetry should be the subgroup of the $SU(2)_L \times E_8$ global symmetry of the E-string theory that is preserved by the flux. This

[7] The correct normalization can be fixed by trying to match the anomalies computed from 6d with those computed from the 4d Lagrangian description that we are going to propose.

symmetry may not be fully manifest in our $4d$ Lagrangian. In such cases, it must get enhanced in the IR.

Enhancements of symmetries can be checked by computing the supersymmetric index of the theory. This is a quantity that is invariant along the RG flow, so it can be calculated using the UV Lagrangian description and this would give us the same result as for the IR SCFT. The global symmetry can be understood from the index because, when computed as a power series in some of its fugacities denoted by p and q (see Appendix A.4 for more details), we can see at each order characters of such symmetry. Moreover, if computed with the superconformal R-charge it coincides with the superconformal index [138–140] and, using the representation theory of the superconformal algebra, one can show that at order pq it receives contributions only from marginal operators with positive sign and conserved currents with negative sign [141]. Hence, if we know the marginal operators of our model we can deduce what is its true global symmetry and check whether it is larger than the one manifest in the UV.

It is then crucial to be able to determine what is the global symmetry preserved by the flux. The fluxes are vectors in the root lattice of E_8 and we first need to choose a basis of Cartan generators to use for it. Here we will consider the complete basis we discussed before, which is associated with the $SO(16)$ subgroup of E_8. Under this subgroup, the adjoint representation of E_8 decomposes as

$$\mathbf{248}_{E_8} \to \mathbf{120}_{SO(16)} + \mathbf{128}_{SO(16)}\,, \tag{3.9}$$

where the $\mathbf{120}_{SO(16)}$ is the adjoint of $SO(16)$ and the $\mathbf{128}_{SO(16)}$ is one of its chiral spinors. We choose to represent the non-zero roots of E_8 as follows:

$$\begin{aligned}&(\pm 2, \pm 2, 0, 0, 0, 0, 0, 0) \;+\; \text{permutations}\\ &(\pm 1, \pm 1, \pm 1, \pm 1, \pm 1, \pm 1, \pm 1, \pm 1) \text{ with even number of minus signs}\,,\end{aligned} \tag{3.10}$$

where the first line comes from the adjoint representation of $SO(16)$, while the second line comes from its spinor representation. This is equivalent to saying that we choose to span the Cartan of $SO(16)$ in a basis such that the characters of its vector representation is

$$\mathbf{16}_{SO(16)} = \sum_{a=1}^{8} x_a^2 + x_a^{-2}\,, \tag{3.11}$$

where a_i are the fugacities for the chosen Cartans. With this normalization, the entries of the flux vector should be integrally quantized because of the Dirac quantization condition. Fractional fluxes may also be allowed, but they should be compensated by a flux for the center of the non-abelian symmetry preserved by the flux, with the effect of breaking it to a subgroup of lower rank (see appendix C of [116] for more details).

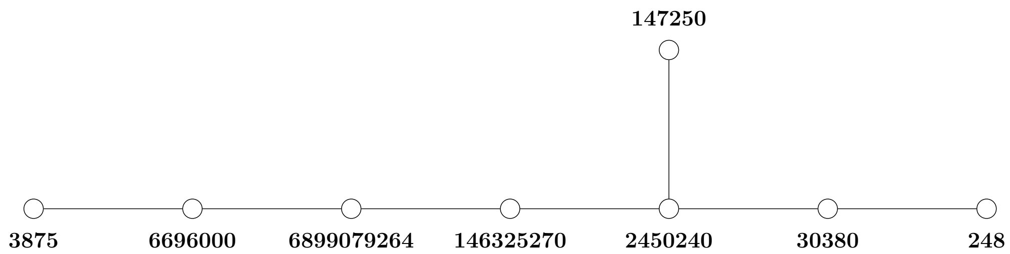

Figure 3.2: The Dynkin diagram of E_8. We also specify which representation is associated to each node.

After having established our conventions for the fluxes, understanding what is the global symmetry that a given flux preserves is conceptually very simple. This is the subgroup of E_8 defined by the subset of the roots (3.10) that are orthogonal to the flux. This is because Weyl groups are generated by reflections in the plane orthogonal to the associated root vector, so the Weyl element associated to a given root will fix the flux vector if and only if the flux vector is orthogonal to the associated root. Therefore, the roots of the preserved symmetry are the subset of all E_8 roots orthogonal to the flux vector.

Despite the simplicity of the problem, this might be a bit tricky to implement in some cases. There is also an equivalent criterion which is instead easier to use. Namely, to each flux vector we can associated a representation of E_8 such that the flux corresponds to its highest weight. Once we determine this representation, we can associate to it some of the nodes of the E_8 Dynkin diagram by just looking at the Dynkin labels of the representation. The preserved symmetry is then the one whose Dynkin diagram is obtained by *chopping* these nodes from the E_8 Dynkin diagram, where to each chopped node we associate a $U(1)$ factor. In Figure 3.2 we represent the Dynkin diagram of E_8 specifying which representation is associated to each node.

Let us consider a simple example to illustrate this. Suppose that the flux vector is

$$\mathcal{F} = (1, 1, 1, 1, 1, 1, 1, 1)\,. \tag{3.12}$$

This is one of the roots of E_8, in particular it is one of the weights of the spinor representation **128** of $SO(16)$. Hence, the E_8 representation associated to it is the adjoint **248**. This representation corresponds to the last node of the E_8 Dynkin diagram (see Figure 3.2). Removing this node we get the Dynkin diagram of E_7. The removed node corresponds to the only $U(1)$ inside E_8 for which we are turning on a flux, which in this case is the minimal flux 1. Hence, the preserved symmetry is $U(1) \times E_7$.

The simplest fluxes that we can consider are those whose associated representation corresponds to a single node in the E_8 Dynkin diagram, as we specify in Figure 3.2. These are the cases which correspond to a flux for a single $U(1)$ inside E_8 and the value of the flux is the minimal one, that is 1. We summarize these cases and the associated preserved

Flux vector	Representation	Preserved symmetry	ξ_G
$(2,2,0,0,0,0,0,0)$ $(1,1,1,1,1,1,1,1)$	**248**	$U(1)\times E_7$	1
$(4,0,0,0,0,0,0,0)$ $(2,2,2,2,0,0,0,0)$	**3875**	$U(1)\times SO(14)$	2
$(2,2,2,2,2,2,0,0)$ $(3,3,1,1,1,1,1,1)$	**30380**	$U(1)\times SU(2)\times E_6$	3
$(4,2,2,2,2,0,0,0)$ $(5,1,1,1,1,1,1,1)$	**147250**	$U(1)\times SU(8)$	4
$(4,4,2,2,2,2,0,0)$ $(4,4,4,0,0,0,0,0)$	**2450240**	$U(1)\times SU(3)\times SO(10)$	6
$(6,2,2,2,2,2,0,0)$	**6696000**	$U(1)\times SU(2)\times SU(7)$	7
$(4,4,4,4,4,0,0,0)$ $(5,5,5,1,1,1,1,1)$	**146325270**	$U(1)\times SU(4)\times SU(5)$	10
$(9,3,3,3,3,1,1)$	**6899079264**	$U(1)\times SU(2)\times SU(3)\times SU(5)$	15

Table 3.1: Vectors associated to fluxes for a single $U(1)$ inside E_8 with the associated representation, preserved symmetry and value of ξ_G. For each line we represent only those choices of the flux vector that are not related by an element of the Weyl group of E_8. These are also the minimal fluxes allowed and higher fluxes can be obtained from integer multiples of them.

symmetry in Table 3.1, where we also specify the value of the parameter ξ_G that we used to write the anomalies (3.2). These are the main fluxes that we will consider for the torus compactifications in Section 3.4. Instead for the sphere compactifications of Section 3.5 we will have to consider more complicated fluxes, so it is useful to keep in mind the strategy we just explained.

There is also another test that we can perform which is related to symmetries. The symmetry expected from $6d$ is usually realized by some operators of the $4d$ theory whose presence can be predicted from the higher dimensional construction as well. Specifically, there are two types of operators in the $6d$ theory that are known and which reduce to operators in $4d$. These are the stress-energy tensor and the conserved current for the global symmetry. When we compactify the theory on the Riemann surface, they lead to various relevant, marginal or irrelevant operators in the $4d$ theory. We are now going to review the expectations for how these operators should look like in the lower dimensional theory, particularly through the supersymmetric index. For a detailed derivation we refer the reader to [142], while here we shall only quote the result (see also [143] and Appendix E of [116]).

Suppose that we compactify our theory on a Riemann surface of genus g with flux F in a $U(1)$ subgroup of its global symmetry G[8], which we shall denote as $U(1)_\alpha$. Additionally, we shall assume that there are no punctures on the surface. The presence of the flux breaks G

[8]For simplicity we assume that there is flux only in one $U(1)$. The generalization to the case of flux in multiple $U(1)$ groups is straightforward.

to the subgroup $U(1)_\alpha \times \tilde{G}$. We can then decompose the character of the adjoint of G as follows:

$$\chi_{adj}(G) = \sum_i \alpha^{q_i} \chi_{R_i}(\tilde{G}), \tag{3.13}$$

where q_i is the charge under $U(1)_\alpha$ of the representation R_i of $\tilde{G}$ appearing in the decomposition of the adjoint representation of G. Here we have used α as the fugacity of $U(1)_\alpha$. While the representations R_i depend on the choice of $U(1)_\alpha$, they always contain the adjoint of $\tilde{G}$ and a singlet corresponding to the adjoint of $U(1)_\alpha$.

The main result of [142] that is important for us is that for a generic Riemann surface without punctures and flux F the index of the lower dimensional theory has a special form when written using the $U(1)_R^{6d}$ symmetry that is the Cartan of the $SU(2)_R$ symmetry, which is the R-symmetry of 6d $\mathcal{N} = (1,0)$. This form is

$$\begin{aligned} \mathcal{I} = \quad 1 \quad &+ \left(\sum_{i|q_i>0} \alpha^{q_i} \chi_{R_i}(\tilde{G})(g-1+q_i F) \right) pq + \Big(3g-3+(1+\chi_{adj}(\tilde{G}))(g-1)\Big)\, pq \\ &+ \left(\sum_{i|q_i<0} \alpha^{q_i} \chi_{R_i}(\tilde{G})(g-1+q_i F) \right) pq + \dots \end{aligned} \tag{3.14}$$

A crucial observation is that the 6d R-symmetry that we are using is not necessarily the superconformal R-symmetry of the 4d model. In particular, the 4d theory will possess the abelian symmetry $U(1)_\alpha$ that can mix with the R-symmetry in the IR [144]. This mixing can be implemented in the index by shifting the fugacity α by some power of pq, namely $\alpha \to \alpha (pq)^{\frac{R_\alpha}{2}}$ where R_α is the mixing coefficient between $U(1)_\alpha$ and $U(1)_R^{6d}$. From (3.14) we then see that the operators coming from the decomposition (3.13) which are charged under $U(1)_\alpha$, corresponding to the first and last terms, will move in the index to an order which is higher or lower than pq depending on the relative sign of their charge q_i and of the mixing coefficient R_α, meaning that they are relevant or irrelevant. The operators uncharged under $U(1)_\alpha$ on the other hand, corresponding to the middle terms of (3.14), will stay at order pq, meaning that they are marginal.

Another comment is about the origin of the various terms in (3.14). The $3g-3$ part of the middle term comes from the stress-energy tensor. Notice that for a torus this is zero, while for a sphere it is -3. Hence, for genus zero it doesn't contribute as a marginal operator, but rather as a current. We interpret this as the current for the flavor symmetry descending from the $SU(2)_{\text{ISO}}$ of the sphere, which transforms in the adjoint representation **3**. All of the other terms, instead, come from the 6d conserved current. Hence, from the conserved current we can get relevant, marginal and irrelevant operators depending on the charge under $U(1)_\alpha$ of the states contained in it, which can be understood from the decomposition (3.13).

Focusing on the states uncharged under $U(1)_\alpha$, which appear in the middle part of (3.14), we can see that these also disappear for $g = 1$, similarly to the contribution of the stress-energy tensor. Nevertheless, we still expect the theory to possess the global symmetry $U(1)_\alpha \times \tilde{G}$, so there should be a conserved current for this symmetry that appears at order pq with negative sign. This means that there has to be a marginal operator in the adjoint of $U(1)_\alpha \times \tilde{G}$ that cancels its contribution to the index. In conclusion, we expect the order pq of the index, when computed with the $6d$ R-charge, to be zero for all torus compactifications for this reason. For the case $g = 0$ of the sphere, instead, we have one copy of these states contributing with a negative sign, meaning that they play the role of the conserved current of $U(1)_\alpha \times \tilde{G}$. This means that there has to be no marginal operator for sphere compactifications, so that the contribution of the conserved current of $U(1)_\alpha \times \tilde{G}$ is not canceled.

3.3 The $E[USp(2N)]$ theory

We are now going to present a $4d$ $\mathcal{N} = 1$ quiver gauge theory that we call $E[USp(2N)]$, which was first introduced in [96] and also studied in [145, 108, 97]. This theory is one of the main characters of this thesis for several reasons. From the point of view of this chapter, this theory is involved in the construction of the models associated to the compactification on a tube of the rank-N E-string theory, from which we will then derive torus, cap and sphere compactifications. Moreover, it is the $4d$ uplift of the three-dimensional $M[SU(N)]$ theory we presented in Section 2.4.1. We will indeed show that it enjoys some properties that are very similar to those of $M[SU(N)]$ that we already encountered. This is no an accident, since we will also be able to explicitly show how to flow from $E[USp(2N)]$ to $M[SU(N)]$. Finally, it will also play an important role in the next chapter when we will discuss the $4d$ uplift of the $3d$ mirror symmetry.

3.3.1 Lagrangian description and symmetry enhancement

The $4d$ $\mathcal{N} = 1$ $E[USp(2N)]$ theory admits a Lagrangian description in terms of the quiver represented in Figure 3.3. The gauge group is $\prod_{i=1}^{N-1} USp(2i)$ and the matter content consists of the following chiral fields in the singlet, fundamental, bifundamental and antisymmetric representation:

- a chiral field $Q^{(i,i+1)}$ in the bifundamental representation of $USp(2i) \times USp(2(i+1))$, with $i = 1, \cdots, N-1$;
- two chiral fields $D_\alpha^{(i)}$ in the fundamental representation of $USp(2i)$, which form a doublet of the i-th $SU(2)$ flavor symmetry of the saw, with $i = 1, \cdots, N$;
- two chiral fields $V_\alpha^{(i)}$ in the fundamental representation of $USp(2i)$, which form a doublet of the $(i+1)$-th $SU(2)$ flavor symmetry of the saw, with $i = 1, \cdots, N-1$;

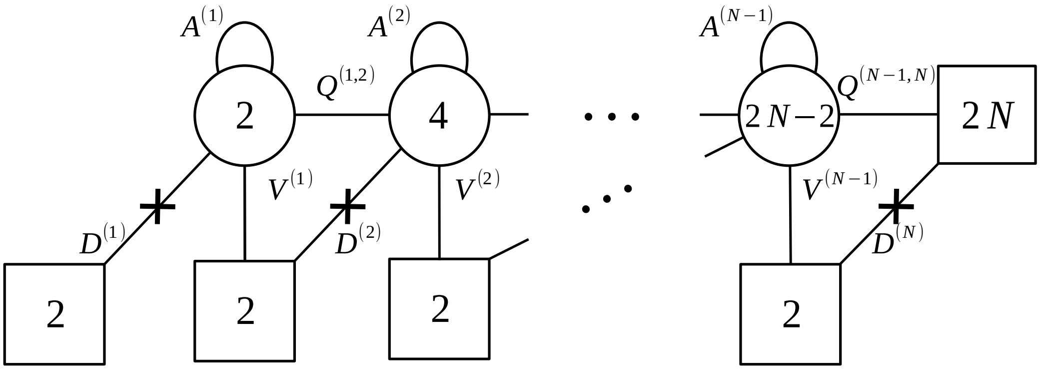

Figure 3.3: Quiver diagram of the $E[USp(2N)]$ theory. Round nodes denote gauge symmetries and square nodes denote global symmetries. In contrast with the convention for the drawings of Chapter 2, all the nodes correspond to symplectic groups whose rank is given by the number inside of them. Single lines denote chiral fields in representations of the nodes they are connecting. In particular, lines between adjacent nodes denote a chiral field in the bifundamental representation of the two nodes symmetries, while arcs denote chiral fields in the antisymmetric representation of the corresponding node symmetry. Crosses represent the singlets β_i that flip the diagonal mesons.

- a chiral field $A^{(i)}$ in the antisymmetric representation of $USp(2i)$, with $i = 1, \cdots, N-1$;
- a gauge singlet β_i that is coupled to the gauge invariant meson built from $D^{(i)}$ through a superpotential which will be discussed momentarily.

In order to write the superpotential in a compact form, we define

$$\mathbb{Q}^{(i,i+1)}_{nmab} = Q^{(i,i+1)}_{na} Q^{(i,i+1)}_{mb} \tag{3.15}$$

The superpotential consists of three main types of interactions: a cubic coupling between the bifundamentals and the antisymmetrics, another cubic coupling between the chirals in each triangle of the quiver and finally the flip terms with the singlets β_n coupled to the diagonal mesons

$$\begin{aligned} \mathcal{W}_{E[USp(2N)]} &= \sum_{i=1}^{N-1} \mathrm{Tr}_i \left[A^{(i)} \left(\mathrm{Tr}_{i+1}\, \mathbb{Q}^{(i,i+1)} - \mathrm{Tr}_{i-1}\, \mathbb{Q}^{(i-1,i)} \right) \right] \\ &+ \sum_{i=1}^{N-1} \mathrm{Tr}_{y_{i+1}}\, \mathrm{Tr}_i\, \mathrm{Tr}_{i+1} \left(V^{(i)} Q^{(i,i+1)} D^{(i+1)} \right) + \sum_{i=1}^{N-1} \beta_n\, \mathrm{Tr}_{y_i}\, \mathrm{Tr}_i \left(D^{(i)} D^{(i)} \right) . \end{aligned} \tag{3.16}$$

The traces are labelled as follows: Tr_i denotes the trace over the color indices of the i-th gauge node, while Tr_{y_i} denotes the trace over the i-th $SU(2)$ flavor symmetry. Notice that for $i = N$ we have the trace over the $USp(2N)_x$ flavor symmetry, which we will also denote

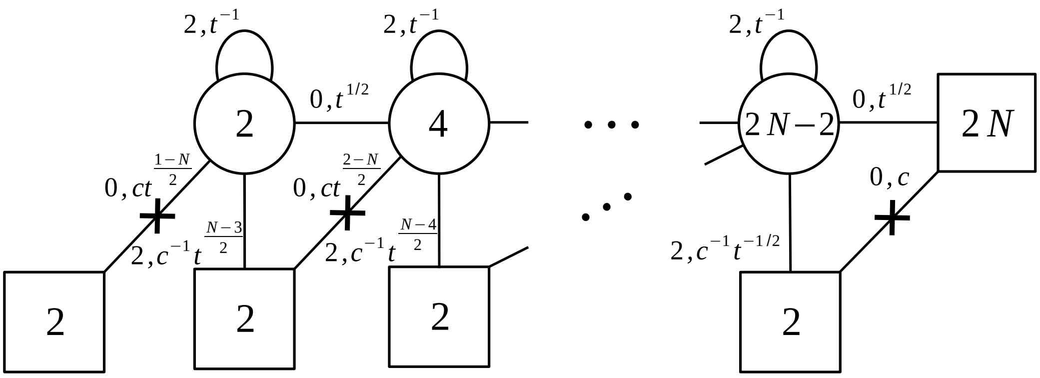

Figure 3.4: Trial R-charges and charges under the abelian symmetries. The power of c is the charge under $U(1)_c$, while the power of t is the charge under $U(1)_t$.

by $\mathrm{Tr}_N = \mathrm{Tr}_x$. All the traces are defined including the antisymmetric tensor J of $USp(2n)$

$$J^{(n)} = \mathbb{I}_n \otimes i\,\sigma_2\,. \tag{3.17}$$

For example, given a $2n \times 2n$ matrix A, we define

$$\mathrm{Tr}\,(A) = J^{(n)}_{ij} A^{ij}\,. \tag{3.18}$$

In this Lagrangian description the following non-anomalous global symmetry is manifest:

$$USp(2N)_x \times \prod_{n=1}^{N} SU(2)_{y_n} \times U(1)_t \times U(1)_c\,. \tag{3.19}$$

This symmetry gets actually enhanced in the IR to

$$USp(2N)_x \times USp(2N)_y \times U(1)_t \times U(1)_c\,. \tag{3.20}$$

We would like to stress the similarity between this symmetry enhancement and the one of $M[SU(N)]$. We will indeed relate the $USp(2N)_y$ symmetry of $E[USp(2N)]$ to the $SU(N)_Y$ symmetry of $M[SU(N)]$ when we will show how to flow from one theory to the other. Using the observation that we made in Subsection 2.4.1 when relating $M[SU(N)]$ and $T[SU(N)]$, we can start understanding that the $SU(2)$ symmetries of the saw of $E[USp(2N)]$ should be considered as a $4d$ avatar of the $U(1)$ topological symmetries of $T[SU(N)]$ in $3d$. In [96] this enhancement of the global symmetry of $E[USp(2N)]$ was argued studying the gauge invariant operators, which re-arrange into representations of the enhanced $USp(2N)_y$ symmetry, and using infra-red dualities. Indeed, as we will review shortly, there exists a dual frame of $E[USp(2N)]$ where $USp(2N)_y$ is manifest, while $USp(2N)_x$ is enhanced.

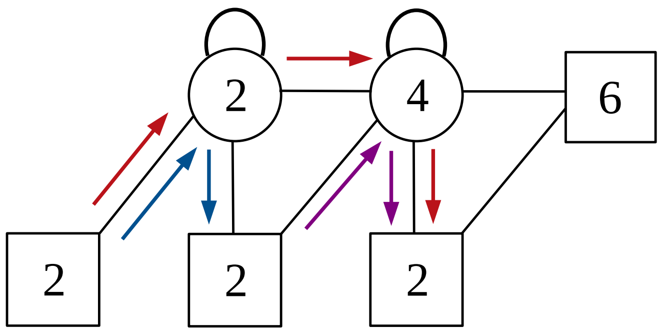

Figure 3.5: $SU(2)\times SU(2)$ bifundamental operators contributing to C in the $N=3$ case.

We define a trial R-symmetry, which we denote as $U(1)_{R_0}$, such that the fields $Q^{(i,i+1)}$ and $D^{(i)}$ have R-charge 0, while the fields β_i, $A^{(i)}$ and $V^{(i)}$ have R-charge 2. This is not the superconformal R-symmetry, but it is anomaly free and consistent with the superpotential (3.16). Moreover, we define the $U(1)_c$ and $U(1)_t$ symmetries by assigning charges 0 and $\frac{1}{2}$ to $Q^{(N-1,N)}$ and 1 and 0 to $D^{(N)}$. The charges of all the other chiral fields are then fixed by the superpotential and by the requirement that $U(1)_R$ is not anomalous at each gauge node, where $U(1)_R$ is defined taking into account the possible mixing of the abelian symmetries with the trial R-symmetry $U(1)_{R_0}$

$$R = R_0 + \mathfrak{c} q_c + \mathfrak{t} q_t \,, \tag{3.21}$$

where q_c and q_t are the charges under the two $U(1)$ symmetries and $\mathfrak{c}$ and $\mathfrak{t}$ are the mixing coefficients. Among this two parameter family of R-charges, we can determine the exact superconformal one by a-maximization [144]. The charges of all the chiral fields under the two $U(1)$ symmetries as well as their trial R-charges in our conventions are summarized in Figure 3.4.

The gauge invariant operators of $E[USp(2N)]$ that will be important for us are of three main types. First, we have two operators, which we denote by H and C, in the traceless antisymmetric representation of $USp(2N)_x$ and $USp(2N)_y$ respectively. The first one is just the meson matrix

$$\mathsf{H} = \mathrm{Tr}_{N-1}\left[Q^{(N-1,N)}Q^{(N-1,N)} - \frac{1}{N}\,\mathrm{Tr}_X\left(Q^{(N-1,N)}Q^{(N-1,N)}\right)\right] . \tag{3.22}$$

This operator has also $U(1)_c$ and $U(1)_t$ charge 0 and 1 respectively and trial R-charge 0. The operator C is instead constructed collecting different gauge invariant operators, $N-1$ of them are singlets under the non-abelian global symmetries while the others are in the bifundamental representations of all the possible pairs of $SU(2)$ manifest symmetries of the saw. These have indeed the same charges under the abelian symmetries and the same trial R-charge and together they reconstruct the traceless antisymmetric representation of the enhanced

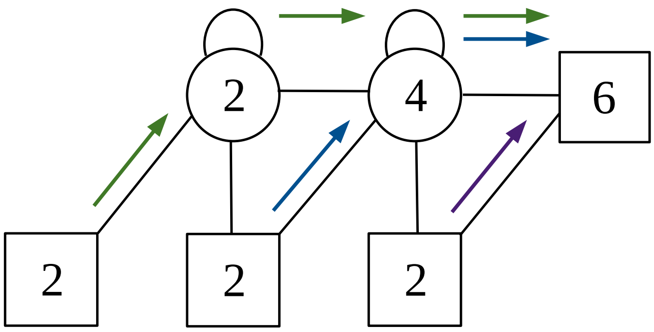

Figure 3.6: Operators contributing to Π in the $N = 3$ case.

$USp(2N)_y$ according to the branching rule under the subgroup $SU(2)^N \subset USp(2N)$

$$\mathbf{N(2N-1)-1} \to (N-1) \times (\mathbf{1}, \cdots, \mathbf{1}) \oplus [(\mathbf{2}, \mathbf{2}, \mathbf{1}, \cdots, \mathbf{1}) \oplus (\text{all possible permutations})] \, . \tag{3.23}$$

The $N-1$ singlets are the traces of the antisymmetric chirals at each gauge node

$$\text{Tr}_i \, A^{(i)}, \qquad i = 1, \cdots, N-1 \, , \tag{3.24}$$

while the bifundamentals are constructed starting from one diagonal flavor, going along the tail with an arbitrary number of bifundamentals $Q^{(i,i+1)}$ and ending on a vertical chiral, with all the needed contractions of color indices (see Figure 3.5). All these operators have $U(1)_c$ and $U(1)_t$ charge 0 and -1 respectively and trial R-charge 2.

There is also an operator Π in the bifundamental representation of $USp(2N)_x \times USp(2N)_y$. This is constructed collecting N operators in the fundamental representation of $USp(2N)_x$ and of each of the $SU(2)$ symmetries according to the branching rule under $SU(2)^N \subset USp(2N)$

$$\mathbf{2N} \to (\mathbf{2}, \mathbf{1}, \cdots, \mathbf{1}) \oplus (\mathbf{1}, \mathbf{2}, \mathbf{1}, \cdots, \mathbf{1}) \oplus \cdots \oplus (\mathbf{1}, \cdots, \mathbf{1}, \mathbf{2}) \, . \tag{3.25}$$

These N operators are obtained starting with one diagonal flavor and going along the tail with all the remaining bifundamentals ending on $Q^{(N-1,N)}$ (see Figure 3.6). All these operators have $U(1)_c$ and $U(1)_t$ charge 1 and 0 respectively and trial R-charge 0.

Finally, we have some gauge invariant operators that are also singlets under the non-abelian global symmetries and are only charged under $U(1)_c$ and $U(1)_t$. Those that will be important for us are the chiral singlets β_i and the mesons constructed with the vertical chirals and dressed with powers of the antisymmetrics. We can collectively denote these

	$USp(2N)_x$	$USp(2N)_y$	$U(1)_t$	$U(1)_c$	$U(1)_{R_0}$
H	$\mathbf{N(2N-1)-1}$	$\mathbf{1}$	1	0	0
C	$\mathbf{1}$	$\mathbf{N(2N-1)-1}$	-1	0	2
Π	$\mathbf{N}$	$\mathbf{N}$	0	$+1$	0
B_{ij}	$\mathbf{1}$	$\mathbf{1}$	$N-i-j+1$	-2	$2j$

Table 3.2: Transformation rules of the main operators of $E[USp(2N)]$.

operators with

$$B_{ij} = \begin{cases} \beta_i & i = 1, \cdots, N, \quad j = 1 \\ \mathrm{Tr}_{i-1}\left[\left(A^{(i-1)}\right)^{j-2} V^{(i-1)} V^{(i-1)}\right] & i = 2, \cdots, N, \quad j = 2, \cdots, i \end{cases} . \tag{3.26}$$

These operators have $U(1)_c$ charge -2, $U(1)_t$ charge $N-i-j+1$ and trial R-charge $2j$. The charges and representations of all these operators under the global symmetry are given in Table 3.2.

As we already mentioned, we will show later that $E[USp(2N)]$ has a limit to $M[SU(N)]$, which we know has a further limit to $T[SU(N)]$. The names H and C for the operators of $E[USp(2N)]$ is motivated from this perspective by the fact that after these limits they reduce to the operators $\mathtt{H}$ and $\mathtt{C}$ of $M[SU(N)]$ and, eventually, to the operators $\mathcal{H}$ and $\mathcal{C}$ of $T[SU(N)]$. Indeed, we can embed $U(1)\times SU(N) \subset USp(2N)$ and the traceless antisymmetric of $USp(2N)$ accordingly decomposes as

$$\mathbf{N(2N-1)-1} \to (\mathbf{N^2-1})^0 \oplus \left(\frac{\mathbf{N(N-1)}}{\mathbf{2}}\right)^2 \oplus \left(\overline{\frac{\mathbf{N(N-1)}}{\mathbf{2}}}\right)^{-2} . \tag{3.27}$$

When flowing from $E[USp(2N)]$ to $M[SU(N)]$ we will consider a Coulomb branch VEV accompanied by a real mass deformation that makes the fields charged under the $U(1)$ part massive and leaves only the adjoint of $SU(N)$ components of H and C massless, which we identify with $\mathtt{H}$ and $\mathtt{C}$. The operators Π and B_{ij} of $E[USp(2N)]$ similarly reduce to the operators $\mathtt{\Pi}$ and $\mathtt{B}_{ij}$ of $M[SU(N)]$, which we saw that become massive after the further flow to $T[SU(N)]$.

One of our main tools for studying $E[USp(2N)]$, its dualities and deformations will be the supersymmetric index [138–140] (see also [146] for a review and Appendix A.4 for our conventions). This will depend on fugacities for the $USp(2N)_x \times USp(2N)_y \times U(1)_c \times U(1)_t$ global symmetries that we accordingly denote by x_i, y_i, c and t. It can be expressed with the following recursive definition:

$$\begin{aligned}\mathcal{I}_{E[USp(2N)]}(\vec{x};\vec{y};c;t) &= \underbrace{\Gamma_e\left(pq\,c^{-2}\right)}_{\beta_N}\underbrace{\prod_{i=1}^{N}\Gamma_e\left(c\,y_N^{\pm1}x_i^{\pm1}\right)}_{D^{(N)}}\\ &\times \oint \mathrm{d}\vec{w}^{(N-1)}_{N-1}\,\underbrace{\Gamma_e\left(pq\,t^{-1}\right)^{N-1}\prod_{a<b}^{N-1}\Gamma_e\left(pq\,t^{-1}w_a^{(N-1)\pm1}w_b^{(N-1)\pm1}\right)}_{A^{(N-1)}}\\ &\times \prod_{a=1}^{N-1}\underbrace{\Gamma_e\left(pq\,t^{-1/2}c^{-1}\,y_N^{\pm1}w_a^{(N-1)\pm1}\right)}_{V^{(N-1)}}\underbrace{\prod_{i=1}^{N}\Gamma_e\left(t^{1/2}w_a^{(N-1)\pm1}x_i^{\pm1}\right)}_{Q^{(N-1,N)}}\\ &\times \mathcal{I}_{E[USp(2(N-1))]}\left(w_1^{(N-1)},\cdots,w_{N-1}^{(N-1)};y_1,\cdots,y_{N-1};t^{-1/2}c;t\right)\,,\end{aligned} \tag{3.28}$$

with the base of the iteration defined as

$$\mathcal{I}_{E[USp(2)]}(x;y;c) = \Gamma_e\left(pq\,c^{-2}\right)\Gamma_e\left(c\,y^{\pm1}x^{\pm1}\right)\,. \tag{3.29}$$

We also defined the integration measure of the m-th $USp(2n)$ gauge node as

$$\mathrm{d}\vec{w}^{(m)}_n = \frac{[(p;p)(q;q)]^n}{2^n n!}\prod_{i=1}^{n}\frac{\mathrm{d}w_i^{(m)}}{2\pi i\,w_i^{(n)}}\frac{1}{\prod_{i<j}^{n}\Gamma_e\left(w_i^{(m)\pm1}w_j^{(m)\pm1}\right)\prod_{i=1}^{n}\Gamma_e\left(w_i^{(m)\pm2}\right)}\,, \tag{3.30}$$

which includes the vector multiplet contribution and the Weyl symmetry factor. This index is defined using the assignment of R-charges as depicted in Figure 3.4. If one wishes to use another non-anomalous assignment of R-charges then the parameters should be redefined as

$$c \to c\,(pq)^{\mathfrak{c}/2}, \qquad t \to t\,(pq)^{\mathfrak{t}/2}\,, \tag{3.31}$$

where $\mathfrak{c}$ and $\mathfrak{t}$ are the mixing coefficients appearing in (3.21). As pointed out in [96], the expression (3.28) coincides with the interpolation kernel $\mathcal{K}_c(x,y)$ studied in [79], where many integral identities for this function were proven which support the dualities of $E[USp(2N)]$ that we are going to present.

3.3.2 Dualities and properties under RG flow

Duality web

$E[USp(2N)]$ enjoys a web of self-dualities that is completely analogous to the one of $T[SU(N)]$ that we described in the Introduction and the one of $M[SU(N)]$ that we saw in Subsubsection

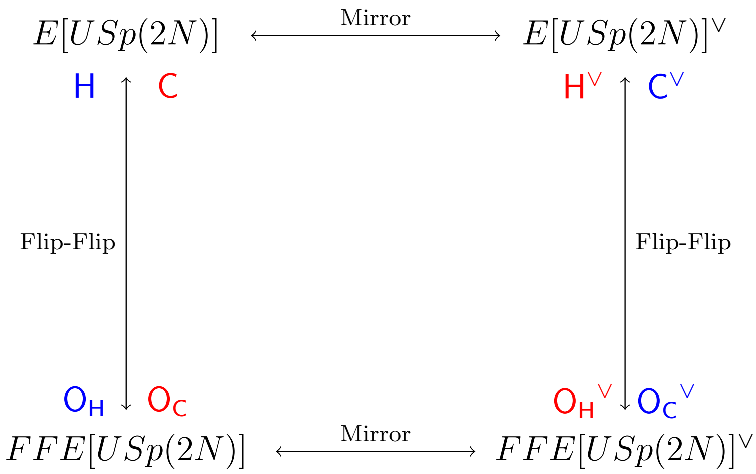

Figure 3.7: Duality web of the $E[USp(2N)]$ theory. On the horizontal direction we have the mirror-like duality, while on the vertical direction we have the flip-flip duality. Operators of the same color are mapped to each other across the dualities.

2.4.1. This is schematically sketched in Figure 3.7. First of all, we have a dual frame we denote by $E[USp(2N)]^\vee$ where the $USp(2N)_x$ and $USp(2N)_y$ symmetries are exchanged and the $U(1)_t$ fugacity is mapped to

$$t \to \frac{pq}{t}\,, \tag{3.32}$$

which means that all the charges under $U(1)_t$ are flipped and that the mixing coefficient is redefined as $\mathfrak{t} \to 2 - \mathfrak{t}$. In other words, $E[USp(2N)]$ is self-dual with a non-trivial map of the gauge invariant operators

$$\begin{array}{ccc} \mathsf{H} & \leftrightarrow & \mathsf{C}^\vee \\ \mathsf{C} & \leftrightarrow & \mathsf{H}^\vee \\ \Pi & \leftrightarrow & \Pi^\vee \\ B_{ij} & \leftrightarrow & B^\vee_{N-j+1,N-i+1}\,. \end{array} \tag{3.33}$$

We will refer to this duality as a $4d$ version of mirror symmetry, since it reduces to the mirror-like self-duality of $M[SU(N)]$, which has a further limit to the self-duality of $T[SU(N)]$ under mirror symmetry. At the level of the index we have the following identity:

$$\begin{aligned} \mathcal{I}_{E[USp(2N)]}(\vec{x};\vec{y};c;t) &= \mathcal{I}_{E[USp(2N)]}(\vec{y};\vec{x};c;pq/t) = \\ &= \mathcal{I}_{E[USp(2N)]^\vee}(\vec{x};\vec{y};c;t)\,, \end{aligned} \tag{3.34}$$

which has been proven in Theorem 3.1 of [79] and which reduces to the identity (2.120) for the mirror self-duality of $M[SU(N)]$ in a suitable limit. This duality strongly supports the enhancement to $USp(2N)_y$, since this symmetry is explicitly manifest in the $E[USp(2N)]^\vee$ dual frame.

On top of the mirror dual frame we have a second frame we denote by $FFE[USp(2N)]$, which is defined as $E[USp(2N)]$ plus two sets of singlets $\mathsf{O_H}$ and $\mathsf{O_C}$ flipping the two operators H^{FF} and C^{FF}

$$\mathcal{W}_{FFE[USp(2N)]} = \mathcal{W}_{E[USp(2N)]} + \mathrm{Tr}_x\left(\mathsf{O_H}\mathsf{H}^{FF}\right) + \mathrm{Tr}_y\left(\mathsf{O_C}\mathsf{C}^{FF}\right) . \tag{3.35}$$

In this case the $USp(2N)_x$ and $USp(2N)_y$ symmetries are left unchanged, while only the $U(1)_t$ fugacity transforms as in (3.32). The operator map is indeed

$$\begin{aligned} \mathsf{H} \quad &\leftrightarrow \quad \mathsf{O_H} \\ \mathsf{C} \quad &\leftrightarrow \quad \mathsf{O_C} \\ \Pi \quad &\leftrightarrow \quad \Pi^{FF} \\ B_{ij} \quad &\leftrightarrow \quad B^F_{N-j+1,N-i+1} \,. \end{aligned} \tag{3.36}$$

We will refer to this duality as a $4d$ version of flip-flip duality, since it reduces to the flip-flip duality of $M[SU(N)]$ and of $T[SU(N)]$.

We mentioned that in the three-dimensional case, both for $M[SU(N)]$ and for $T[SU(N)]$, this flip-flip dual frame can be reached by iteratively applying some more fundamental dualities, which are the two-monopole duality for $M[SU(N)]$ and the Aharony duality for $T[SU(N)]$. A similar statement holds for the $4d$ flip-flip duality of $E[USp(2N)]$. In this case, the fundamental duality that we should iterate is the Intriligator–Pouliot duality [81]. This can be understood as a variant for symplectic gauge groups of the Seiberg duality [1]:

Theory A: $USp(2N_c)$ gauge theory with $2N_f$ fundamental chirals and no superpotential $\mathcal{W} = 0$.

Theory B: $USp(2N_f - 2N_c - 4)$ gauge theory with $2N_f$ fundamental chirals, $N_f(2N_f - 1)$ singlets (collected in an antisymmetric matrix M_{ab}) and superpotential $\hat{\mathcal{W}} = M^{ab}q_a q_b$.

We will now briefly describe how this derivation works[9]:

- At the first iteration we start from the $USp(2)$ node, whose antisymmetric chiral is just a singlet. The Intriligator–Pouliot duality has the effect of making the antisymmetric chiral field of the adjacent $USp(4)$ node massive, so that we can then apply again the Intriligator–Pouliot duality on it. We continue applying iteratively the Intriligator–Pouliot duality until we reach the last $USp(2(N-1))$ node. Notice that since every

[9]The reader can find more details of the derivation of the flip-flip duality of $T[SU(N)]$ in [145] from the point of view of the $\mathbb{S}^3_b$ partition function and in [82] from the field theory point of view. The $4d$ and the $3d$ derivations are completely analogous, if not for the fact that in $4d$ we have no monopole operators which makes the task of mapping the superpotential of the theory at each step easier.

$USp(2n)$ node sees $4n+4$ chirals the ranks do not change when we apply the duality. Moreover, some of the singlet fields expected from the Intriligator–Pouliot duality are massive (as it can be understood from their charge assignments) and no new links between gauge nodes are created.

- At the second iteration we start again from the $USp(2)$ node and proceed along the tail, but this time we stop at the second last node $USp(2(N-2))$. This allows us to restore the antisymmetric at the last $USp(2N)$ node.

- We iterate this procedure for a total of $N-1$ times, meaning that we apply Intriligator–Pouliot duality $N(N-1)/2$ times.

- The singlet fields appearing in the Intriligator–Pouliot duality reconstruct the singlet matrices $\mathsf{O_H}$ and $\mathsf{O_C}$.

At the level of the supersymmetric index, the flip-flip duality is encoded in the following integral identity:

$$\begin{aligned}\mathcal{I}_{E[USp(2N)]}(\vec{x};\vec{y};c;t) &= \prod_{i<j}^{N}\frac{\Gamma_e\left(tx_i^{\pm1}x_j^{\pm1}\right)}{\Gamma_e\left(ty_i^{\pm1}y_j^{\pm1}\right)}\mathcal{I}_{E[USp(2N)]}(\vec{x};\vec{y};c,pq/t) = \\ &= \mathcal{I}_{FFE[USp(2N)]}(\vec{x};\vec{y};c;t)\,, \end{aligned} \tag{3.37}$$

which is proven in Proposition 3.5 of [79] and can be alternatively derived by applying iteratively the integral identity (C.1) for Intriligator–Pouliot duality as explained above. We show this in Appendix C.1 for $N=3$.

Finally, we can combine the two previous dualities to find a third dual frame and complete the duality web of Figure 3.7. We denote this frame by $FFE[USp(2N)]^\vee$ and its superpotential is

$$\mathcal{W}_{FFE[USp(2N)]^\vee} = \mathcal{W}_{E[USp(2N)]} + \text{Tr}_y\left(\mathsf{O}_\mathsf{H}^\vee \mathsf{H}^{FF,\vee}\right) + \text{Tr}_x\left(\mathsf{O}_\mathsf{C}^\vee \mathsf{C}^{FF,\vee}\right)\,. \tag{3.38}$$

Across this duality the $USp(2N)_x$ and $USp(2N)_y$ symmetries are exchanged, while $U(1)_t$ is left unchanged. Accordingly we have the operator map

$$\begin{aligned} \mathsf{H} &\leftrightarrow \mathsf{O}_\mathsf{C}^\vee \\ \mathsf{C} &\leftrightarrow \mathsf{O}_\mathsf{H}^\vee \\ \Pi &\leftrightarrow \Pi^{FF,\vee} \\ B_{ij} &\leftrightarrow B_{ij}^{FF,\vee}\,. \end{aligned} \tag{3.39}$$

The equality of the $\mathbb{S}^3 \times \mathbb{S}^1$ partition functions associated to this duality

$$\begin{aligned}\mathcal{I}_{E[USp(2N)]}(\vec{x};\vec{y};c;t) &= \prod_{i<j}^{N} \frac{\Gamma_e\left(t x_i^{\pm 1} x_j^{\pm 1}\right)}{\Gamma_e\left(t y_i^{\pm 1} y_j^{\pm 1}\right)} \mathcal{I}_{E[USp(2N)]}(\vec{y};\vec{x};c;t) = \\ &= \mathcal{I}_{FFE[USp(2N)]^\vee}(\vec{x};\vec{y};c;t)\,, \end{aligned} \tag{3.40}$$

can be obtained by just applying sequentially (3.34) and (3.37).

Some interesting RG flows

In addition to dualities the $E[USp(2N)]$ theory enjoys interesting properties under RG flows triggered by turning on VEVs for various operators. The first flow we consider makes the $E[USp(2N)]$ quiver theory reduce to a smaller quiver tail, causing the breaking of the non-abelian global symmetry $USp(2N)_x \times USp(2N)_y \to USp(2(N-1))_x \times USp(2(N-1))_y$. More precisely, the deformation in question corresponds to a minimal VEV for the operator Π, i.e.$\langle \Pi_{2N,2N}\rangle \neq 0$. This can be achieved by introducing an additional singlet field that flips this operator and turning on such singlet linearly in the superpotential. The equation of motion of the singlet then implies that the operator acquired a non-vanishing VEV. At the level of the supersymmetric index, this deformation implies the constraint $x_N = c y_N$, for which we have (see Lemma 3.1 of [79])

$$\begin{aligned}&\lim_{x_N \to c\, y_N} \mathcal{I}_{E[USp(2N)]}(\vec{x};\vec{y};c;t) \frac{\Gamma_e\left(c^2\right)}{\Gamma_e\left(c\, x_N^{\pm 1} y_N^{\pm 1}\right)} = \\ &= \prod_{i=1}^{N} \frac{\Gamma_e\left(c\, y_N x_i^{\pm 1}\right) \Gamma_e\left(y_N^{-1} y_i^{\pm 1}\right)}{\Gamma_e\left(t\, c^{-1} y_N^{-1} x_i^{\pm 1}\right) \Gamma_e\left(t\, y_N^{-1} y_i^{\pm 1}\right)} \mathcal{I}_{E[USp(2(N-1))]}(\vec{x};\vec{y};c;t)\,. \end{aligned} \tag{3.41}$$

Another interesting VEV that we can consider can be implemented by adding a linear term $\delta\mathcal{W} = b_{N-1}$ to the superpotential. This induces a VEV for the meson constructed from the second last diagonal flavor $d^{(N-1)}$ which partially Higgses the last gauge node. The $E[USp(2N)]$ theory reduces after this deformation to a bifundamental of $USp(2N)_x \times USp(2N)_y$ plus an antisymmetric for the $USp(2N)_y$ symmetry. It is easy to see how this works in the $N = 2$ case, where the index is given by

$$\begin{aligned}&\mathcal{I}_{E[USp(4)]}(x_1,x_2;y_1,y_2;c;t) = \\ &= \frac{\prod_{i=1}^2 \Gamma_e\left(c\, y_2^{\pm 1} x_i^{\pm 1}\right)}{\Gamma_e\left(c^2\right)\Gamma_e\left(t^{-1}c^2\right)} \oint \mathrm{d}w_1 \frac{\Gamma_e\left(t^{-1/2} c\, w^{\pm 1} y_1^{\pm 1}\right) \prod_{i=1}^2 \Gamma_e\left(t^{1/2} w^{\pm 1} x_i^{\pm 1}\right)}{\Gamma_e(w^{\pm 2})\Gamma_e(t^{1/2} c\, w^{\pm 1} y_2^{\pm 1})}\,. \end{aligned} \tag{3.42}$$

The condition of b_1 entering in the superpotential corresponds to $c \to \sqrt{t}$. In this limit the poles of the integrand at $w = t^{1/2}c^{-1}y_1^\pm$ and $w = t^{-1/2}c\,y_1^\pm$ pinch the integration contour in two points and we can evaluate the index by taking the residue at these two points as in [147]. Both poles give the same contribution to the index and we get

$$\lim_{c\to\sqrt{t}} \mathcal{I}_{E[USp(4)]}(x_1, x_2; y_1, y_2; c, t) = \frac{\prod_{i,j=1}^{2} \Gamma_e(\sqrt{t}x_i^\pm y_j^\pm)}{\Gamma_e(t)^2\, \Gamma_2(ty_1^\pm y_2^\pm)}. \tag{3.43}$$

At higher rank, the condition of b_{N-1} entering in the superpotential still corresponds to $c \to \sqrt{t}$ and the reduction of the index follows by Proposition 3.5 in [79]

$$\lim_{c\to\sqrt{t}} \mathcal{I}_{E[USp(2N)]}(\vec{x}; \vec{y}; c; t) = \frac{\prod_{i,j=1}^{N} \Gamma_e(\sqrt{t}x_i^\pm y_j^\pm)}{\Gamma_e(t)^N \prod_{i<j}^{N} \Gamma_e(ty_i^\pm y_j^\pm)}\,. \tag{3.44}$$

3.3.3 Flowing to $3d$

We highlighted several times the similarities between the $4d$ $E[USp(2N)]$ theory and the $3d$ $M[SU(N)]$ and $T[SU(N)]$ theories. These are not just an accident, since we can obtain the three-dimensional theories by compactifying $E[USp(2N)]$ on a circle and performing various real mass deformations. All the properties of $M[SU(N)]$ that we described in Subsection 2.4.1, in particular the enhancement of the symmetry of the saw and the duality web, can be obtained as a limit of those of $E[USp(2N)]$. We will now explicitly analyze the flow relating the two theories using as a tool their supersymmetric partition functions. It is known indeed that the $\mathbb{S}^3 \times \mathbb{S}^1$ and the $\mathbb{S}_b^3$ partition functions are related by a limit which is interpreted as sending the relative radius of the $\mathbb{S}^1$ and the $\mathbb{S}^3$ to zero [148–151, 5, 6]. We will also give a field theory interpretation of these limits along the way.

Compactification to $E[USp(2N)]_{3d}$

If we compactify the $E[USp(2N)]$ theory on a circle we obtain a $3d$ $\mathcal{N} = 2$ quiver theory we denote as $E[USp(2N)]_{3d}$ which has the same gauge and matter content and superpotential

$$\mathcal{W}_{E[USp(2N)]_{3d}} = \mathcal{W}_{E[USp(2N)]} + \mathcal{W}_{mon}\,, \tag{3.45}$$

where $\mathcal{W}_{mon}$ is the contribution of KK monopoles turned on for each gauge node which are dynamically generated in the reduction as discussed in [5, 6]. This monopole superpotential ensures that $E[USp(2N)]_{3d}$ and $E[USp(2N)]$ have the same global symmetry

$$USp(2N)_X \times USp(2N)_Y \times U(1)_{m_A} \times U(1)_\Delta\,, \tag{3.46}$$

since the condition of marginality of the $USp(2n)$ monopoles in $3d$ is equivalent to the requirement that $U(1)_R$ is non-anomalous in $4d$. The gauge invariant operators of $E[USp(2N)]_{3d}$ can be constructed in the same way as those of $E[USp(2N)]$ since the monopole superpotential also implies that the monopole operators are not in the chiral ring.

We can implement the $3d$ limit on the $\mathbb{S}^3 \times \mathbb{S}^1$ supersymmetric index by rescaling the global and gauge fugacities with the $\mathbb{S}^1$ radius r as

$$x_i \to \mathrm{e}^{2\pi i r X_i}, \quad y_i \to \mathrm{e}^{2\pi i r Y_i}, \quad c \to \mathrm{e}^{2\pi i r \Delta}, \quad t \to \mathrm{e}^{2\pi i r (iQ - 2m_A)}, \quad w_\alpha^{(j)} \to \mathrm{e}^{2\pi i r z_\alpha^{(j)}} \tag{3.47}$$

where $i = 1, \cdots, N$, $j = 1, \cdots, N-1$ and $\alpha = 1, \cdots, j$ and taking the hyperbolic limit of the elliptic Gamma function

$$\lim_{r\to 0} \Gamma_e\left(\mathrm{e}^{2\pi i r x}; \mathrm{e}^{-2\pi r b}, \mathrm{e}^{-2\pi r b^{-1}}\right) = \mathrm{e}^{-\frac{i\pi}{6r}\left(i\frac{Q}{2} - x\right)} s_b\left(i\frac{Q}{2} - x\right), \tag{3.48}$$

where we recall that $Q = b + b^{-1}$. By doing so we find

$$\lim_{r\to 0} \mathcal{I}_{E[USp(2N)]}(\vec{x}; \vec{y}; c; t) \quad = \quad C_N^{3d}(\Delta, Q, r) \mathcal{Z}_{E[USp(2N)]_{3d}}(\vec{X}; \vec{Y}; \Delta; m_A), \tag{3.49}$$

where

$$\begin{aligned} \mathcal{Z}_{E[USp(2N)]_{3d}}(\vec{X}; \vec{Y}; \Delta; m_A) &= s_b\left(-i\frac{Q}{2} + 2\Delta\right) \prod_{i=1}^N s_b\left(i\frac{Q}{2} \pm Y_N \pm X_i - \Delta\right) \times \\ &\times \int \frac{\prod_{a=1}^{N-1} \mathrm{d}z_a^{(N-1)}}{2^{N-1}(N-1)!} s_b\left(i\frac{Q}{2} - 2m_A\right)^{N-1} \prod_{a<b}^{N-1} s_b\left(i\frac{Q}{2} \pm z_a^{(N-1)} \pm z_b^{(N-1)} - 2m_A\right) \times \\ &\times \frac{\prod_{a=1}^{N-1} s_b\left(\pm Y_N \pm z_a^{(N-1)} - m_A + \Delta\right) \prod_{i=1}^N s_b\left(\pm z_a^{(N-1)} \pm X_i + m_A\right)}{\prod_{a<b}^{N-1} s_b\left(i\frac{Q}{2} \pm z_i^{(N-1)} \pm z_j^{(N-1)}\right) \prod_{a=1}^{N-1} s_b\left(i\frac{Q}{2} \pm 2z_i^{(N-1)}\right)} \times \\ &\times \mathcal{Z}_{E[USp(2N)]_{3d}}\left(z_1^{(N-1)}, \cdots, z_{N-1}^{(N-1)}; Y_1, \cdots, Y_{Q-1}; \Delta + m_a - i\frac{Q}{2}; m_A\right), \end{aligned} \tag{3.50}$$

is the partition function of the $E[USp(2N)]_{3d}$ theory on $\mathbb{S}_b^3$ and the prefactor is

$$C_N^{3d}(\Delta, Q, r) = \left[r(\mathrm{e}^{-2\pi r b}; \mathrm{e}^{-2\pi r b})_\infty (\mathrm{e}^{-2\pi r b^{-1}}; \mathrm{e}^{-2\pi r b^{-1}})_\infty\right]^{\frac{N(N-1)}{2}} \mathrm{e}^{-\frac{i\pi}{6r}\left(\frac{iQ}{4}N(5N+1) - 2N\Delta\right)}. \tag{3.51}$$

By taking this limit on each side of the integral identities (3.34)-(3.37)-(3.40) encoding the duality web of $E[USp(2N)]$ we get similar identities but for the $3d$ theory, meaning that also $E[USp(2N)]_{3d}$ enjoys the same dualities. Indeed, notice that the prefactor $C_N^{3d}(\Delta, r)$, which is divergent as $r \to 0$, is independent of the parameters X_i, Y_i and m_A, meaning that it cancels between the two sides of the identities (3.34)-(3.37)-(3.40) when we take the limit[10].

[10]The singlets in (3.37)-(3.40) give a trivial contribution to this divergent prefactor.

Flow to $M[SU(N)]$

We can now perform some deformations to flow to other $3d$ quiver theories. For example we can proceed as in [41] and a perform a combination of a Coulomb branch VEV and of a real mass deformation that makes us flow to the $M[SU(N)]$ theory we encountered in Section 2.4.1. The Coulomb branch VEV has the effect of Higgsing the gauge groups from $USp(2N)$ to $U(n)$. The real mass deformation is then needed to reach a vacuum where part of the chirals remain massless. This flow has also the effect of generating non-perturbative contributions due to the breaking of the gauge groups. These contributions together with the original KK monopoles combine in a contribution to the superpotential consisting of the sum of the two fundamental monopole operators of opposite magnetic charge at each gauge node $\mathfrak{M}^+ + \mathfrak{M}^-$. The theory that we get is then precisely the $M[SU(N)]$ theory. As we already mentioned, the operators of $E[USp(2N)]$ reduce to operators of $M[SU(N)]$ that we labelled with the same letters, but with a different font. In particular, the operators H and C in the antisymmetric representations of $USp(2N)_x$ and $USp(2N)_y$ reduce to the operators $\mathtt{H}$ and $\mathtt{C}$ in the adjoint representations of $SU(N)_X$ and $SU(N)_Y$.

At the level of the sphere partition function this deformation is implemented [151, 5] by taking

$$X_i \to X_i + s, \quad Y_i \to Y_i + s, \quad s \to +\infty \tag{3.52}$$

and by shifting all the integration variables

$$z_\alpha^{(i)} \to z_\alpha^{(i)} + s\,. \tag{3.53}$$

The shift of the gauge variables corresponds to the Coulomb branch VEV and the shift of the real masses corresponds to the real mass deformation that we mentioned. Notice that for each node since the integrands are symmetric we can rewrite the integrals as

$$\int_{-\infty}^{+\infty} \prod_{i=1}^{n} \mathrm{d}z_i \; f(z_i) = 2^n \int_{0}^{+\infty} \prod_{i=1}^{n} \mathrm{d}z_i \; f(z_i) \to 2^n \int_{-s}^{+\infty} \prod_{i=1}^{n} \mathrm{d}z_i \; f(z_i+s)\,. \tag{3.54}$$

This has the effect of canceling the 2^n factor in the $USp(2n)$ measure.

As we explained when studying the limit from $M[SU(N)]$ to $T[SU(N)]$, the real mass deformation is implemented using

$$\lim_{x\to\pm\infty} s_b\left(x\right) = \mathrm{e}^{\pm i\frac{\pi}{2}x^2}\,, \tag{3.55}$$

The result is

$$\lim_{s\to+\infty} \mathcal{Z}_{E[USp(2N)]_{3d}}(\vec{X};\vec{Y};\Delta;m_A) = C_N(\vec{X},\vec{Y},m_A,\Delta,Q,s)\mathcal{Z}_{M[U(N)]}(\vec{X};\vec{Y};\Delta;m_A)\,, \tag{3.56}$$

where we recall that the partition function of $M[U(N)]$ is defined recursively as in (2.113) and that to get the one of $M[SU(N)]$ we just have to impose the tracelessness conditions $\sum_{i=1}^N X_i = \sum_{i=1}^N Y_i = 0$. The prefactor is

$$\begin{aligned} C_N(\vec{X},\vec{Y},m_A,\Delta,Q,s) &= \exp\left\{i\pi\left[iQ\left(N(N+1)+\sum_{i=1}^N(X_i+NY_i)\right)+\right.\right. \\ &- \left.\left. 2\Delta\left(2Ns+\sum_{i=1}^N(X_i+Y_i)\right)+2(N-1)m_A\sum_{i=1}^N(X_i-Y_i)\right]\right\}\,. \end{aligned} \tag{3.57}$$

Again, we can take this limit on each side of the duality identities of $E[USp(2N)]_{3d}$, which we derived as limits of those (3.34)-(3.37)-(3.40) for $E[USp(2N)$, so to recover the identities (2.120)-(2.125)-(2.129) for the duality web of $M[SU(N)]$. One can check that the divergent prefactor perfectly cancels between the two sides of each of these identities. For example, the prefactor (3.57) is invariant under the simultaneous exchange of $X_i \leftrightarrow Y_i$ and $m_A \leftrightarrow i\frac{Q}{2} - m_A$, which implies that the limit of the identity (3.34) is finite and leads to (2.120)[11].

3.3.4 A variant: the $FE[USp(2N)]$ theory

In our study of the compactifications of the rank-N E-string theory we will use a close relative of $E[USp(2N)]$. We already met a variant of $E[USp(2N)]$ that we called $FFE[USp(2N)]$, since it is defined as $E[USp(2N)]$ with the addition of two sets of operators $\mathsf{O_H}$ and $\mathsf{O_C}$ in the antisymmetric representation of $USp(2N)_x$ and $USp(2N)_y$, respectively, which flip the operators H and C. We now define the $FE[USp(2N)]$ theory as the theory obtained by introducing $\mathsf{O_H}$ only and not $\mathsf{O_C}$[12]

$$\mathcal{W}_{FE[USp(2N)]} = \mathcal{W}_{E[USp(2N)]} + \mathrm{Tr}_x\left(\mathsf{O_H}\mathsf{H}\right)\,. \tag{3.58}$$

This theory inherits all the dualities that we saw for $E[USp(2N)]$, since we just have to add the singlets $\mathsf{O_H}$ in each duality frame. Nevertheless, its role is different in each frame, since the operator H is mapped non-trivially under the dualities (see Figure 3.7). For

[11] For this particular limit it is important to consider the contribution of the singlets in (3.37)-(3.40) to get the finite result coinciding with (2.125)-(2.129)

[12] The $3d$ analogues of $FE[USp(2N)]$ are usually denoted as $FM[SU(N)]$ and $FT[SU(N)]$. The $FT[SU(N)]$ theory was first introduced in [20].

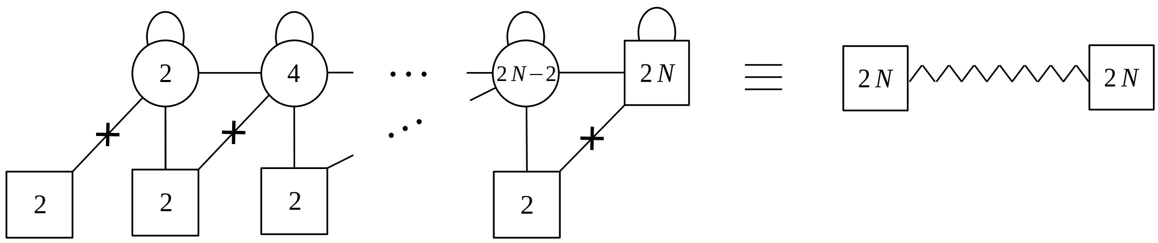

Figure 3.8: Schematic representation of the IR SCFT to which $FE[USp(2N)]$ flows, which explicitly displays both of its $USp(2N)$ global symmetries.

brevity, we summarize the action of the various dualities by giving the identities for the supersymmetric indices. For the mirror-like duality we have

$$\begin{aligned}\mathcal{I}_{FE[USp(2N)]}(\vec{x};\vec{y};c;t) &= \Gamma_e\left(pq\,t^{-1}\right)^{2(N-1)}\prod_{i<j}^{N}\Gamma_e\left(pq\,t^{-1}x_i^{\pm1}x_j^{\pm1}\right)\Gamma_e\left(pq\,t^{-1}y_i^{\pm1}y_j^{\pm1}\right)\times\\ &\times\;\mathcal{I}_{FE[USp(2N)]}(\vec{y};\vec{x};c;pq/t)\,. \end{aligned} \tag{3.59}$$

For the flip-flip duality we have

$$\begin{aligned}\mathcal{I}_{FE[USp(2N)]}(\vec{x};\vec{y};c;t) &= \Gamma_e\left(pq\,t^{-1}\right)^{2(N-1)}\prod_{i<j}^{N}\Gamma_e\left(pq\,t^{-1}x_i^{\pm1}x_j^{\pm1}\right)\Gamma_e\left(pq\,t^{-1}y_i^{\pm1}y_j^{\pm1}\right)\times\\ &\times\;\mathcal{I}_{E[USp(2N)]}(\vec{x};\vec{y};c,pq/t)\,. \end{aligned} \tag{3.60}$$

Finally, combining these two dualities we find a third duality which is extremely simple, since it just exchanges $USp(2N)_x$ and $USp(2N)_y$ and it doesn't add any flipping fields[13]

$$\mathcal{I}_{FE[USp(2N)]}(\vec{x};\vec{y};c;t)=\mathcal{I}_{FE[USp(2N)]}(\vec{y};\vec{x};c;t)\,. \tag{3.61}$$

This tells us that $FE[USp(2N)]$ is symmetric under the exchange of $USp(2N)_x$ and $USp(2N)_y$, and it can be understood as the 4d uplift of the symmetry property (2.22) for the kernel function (2.21) that appeared in our study of 2d free field correlators.

In our constructions of E-string compactifications we will use $FE[USp(2N)]$ as a building block by gauging its two $USp(2N)$ global symmetries. Since one of these two symmetries is not manifest in the quiver description of the theory, it is useful to introduce the notation of Figure 3.8 to represent the IR SCFT to which $FE[USp(2N)]$ flows. In this way we can explicitly depict the gaugings involved in the E-string models, including the non-Lagrangian

[13]The 3d analogue of this duality for the $FT[SU(N)]$ theory was first proposed in [20] where it was called *spectral duality*.

one. Notice that this depiction of the theory is totally symmetric under the exchange of the two $USp(2N)$ symmetries, reflecting the last duality we discussed for $FE[USp(2N)]$.

There is a VEV deformation that we haven't discussed previously for $E[USp(2N)]$ since it is better understood in terms of $FE[USp(2N)]$. Indeed, this corresponds to turning on linearly in the superpotential some of the components of the operator O_{H} as follows:

$$\delta\mathcal{W} = \mathsf{J}_N \mathsf{O}_{\mathsf{H}}\,, \tag{3.62}$$

where

$$\mathsf{J}_N = \frac{i\sigma_2}{2} \otimes \left(J_N + J_N^T\right) \tag{3.63}$$

and J_N is the Jordan matrix of dimension N. This deformation induces a VEV for the operator H

$$\langle \mathsf{O}_{\mathsf{H}} \rangle = \mathsf{J}_N\,. \tag{3.64}$$

This VEV breaks the flavor symmetry $USp(2N)_x$ down to $SU(2)_v$. Moreover, it makes the original $FE[USp(2N)]$ theory flow to a simple WZ model. This can be understood from Corollary 2.8 of [79]

$$\begin{aligned}
\mathcal{I}_{FE[USp(2N)]}(v, t\,v, \cdots, t^{N-1}v; \vec{y}; c, t) &= \prod_{j=2}^{N} \frac{1}{\Gamma_e\left(t^j\right)} \prod_{i=1}^{N} \frac{\Gamma_e\left(v\,c\,y_i^{\pm 1}\right)\Gamma_e\left(v^{-1}c\,t^{1-N}y_i^{\pm 1}\right)}{\Gamma_e\left(t^{1-i}c^2\right)} \\
&\rightarrow \prod_{j=2}^{N} \frac{1}{\Gamma_e\left(t^j\right)} \prod_{i=1}^{N} \frac{\Gamma_e\left(c\,t^{\frac{1-N}{2}}v^{\pm 1}y_i^{\pm 1}\right)}{\Gamma_e\left(c^2 t^{1-i}\right)}\,,
\end{aligned} \tag{3.65}$$

where at the second step we redefined $v \to t^{\frac{1-N}{2}}v$ to make the residual $SU(2)_v$ symmetry manifest. This VEV will be extremely important for us to construct the cap model starting from the tube model in Section 3.5. Moreover, in the next chapter we will consider generalizations of it to block diagonal Jordan matrices, which will lead us to introduce the new class of $E_\rho^\sigma[USp(2N)]$ theories.

3.3.5 Braid relation: generalized Seiberg duality

The $E[USp(2N)]$ and the $FE[USp(2N)]$ theories enjoy another interesting property. This can be equivalently stated using any of the two theories, but for later convenience we prefer to present it in terms of $FE[USp(2N)]$. If we glue two $FE[USp(2N)]$ blocks by gauging a diagonal combination of one of the two $USp(2N)$ symmetries of each tail together with an extra flavor f in the fundamental representation of this gauge symmetry, we re-obtain $FE[USp(2N)]$ plus two extra sets of singlets O_L and O_R, as depicted in Figure 3.9. Notice

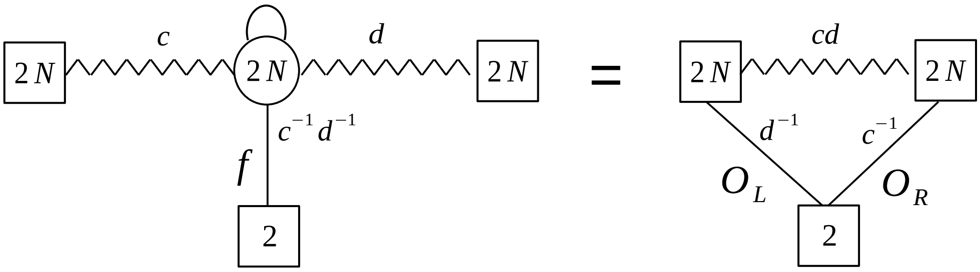

Figure 3.9: Schematic representation of the braid relation.

that for $N = 1$ the braid relation reduces to Seiberg duality for $SU(2)$ with 3 flavors, which is dual to a WZ model.

At the level of the index this duality is encoded in the following braid relation given in Proposition 2.12 of [79]:

$$\oint \mathrm{d}\vec{z}_N \, \Gamma_e\left(t\right)^{N-1} \prod_{i<j}^{N} \Gamma_e\left(t\, z_i^{\pm 1} z_j^{\pm 1}\right) \prod_{i=1}^{N} \Gamma_e\left(u_0 z_i^{\pm 1}\right) \Gamma_e\left(u_1 z_i^{\pm 1}\right) \times$$
$$\times \mathcal{I}_{FE[USp(2N)]}(\vec{z};\vec{x};c;t) \mathcal{I}_{FE[USp(2N)]}(\vec{z};\vec{y};d;t) =$$
$$= \prod_{i=1}^{N} \Gamma_e\left(c\, u_0 x_i^{\pm 1}\right) \Gamma_e\left(c\, u_1 x_i^{\pm 1}\right) \Gamma_e\left(d\, u_0 y_i^{\pm 1}\right) \Gamma_e\left(d\, u_1 y_i^{\pm 1}\right) \mathcal{I}_{FE[USp(2N)]}(\vec{x};\vec{y};cd;t)\,, \tag{3.66}$$

which holds if the following balancing condition is satisfied:

$$u_0 u_1 = \frac{pq}{c^2 d^2}\,. \tag{3.67}$$

Let's discuss in more details the superpotential of the dual theories and their gauge invariant operators. We first consider the l.h.s. of the duality where we glue two $FE[USp(2N)]$ blocks. We name the fields as in Figure 3.10.

In this case the superpotential is the one of the two $FE[USp(2N)]$ tails

$$\mathcal{W} \;=\; \mathcal{W}^{L}_{FE[USp(2N)]} + \mathcal{W}^{R}_{FE[USp(2N)]} + A\left(\mathsf{O}^{L}_{\mathsf{H}} + \mathsf{O}^{R}_{\mathsf{H}}\right)\,. \tag{3.68}$$

Notice that we also added to the two $FE[USp(2N)]$ a chiral field in the antisymmetric representation of the $USp(2N)$ gauge node which couples to the operators O_{H} of the two tails. The superpotential is such that A and one combination of $\mathsf{O}^{L}_{\mathsf{H}}$ and $\mathsf{O}^{R}_{\mathsf{H}}$ are massive. The remaining massless combination is denoted as $A^{(N)}$ in Figure 3.10 and couples to $Q_L^{(N-1,N)}$ and $Q_R^{(N-1,N)}$. Moreover, it has the effect of identifying the $U(1)_t$ symmetries of the two $FE[USp(2N)]$ blocks.

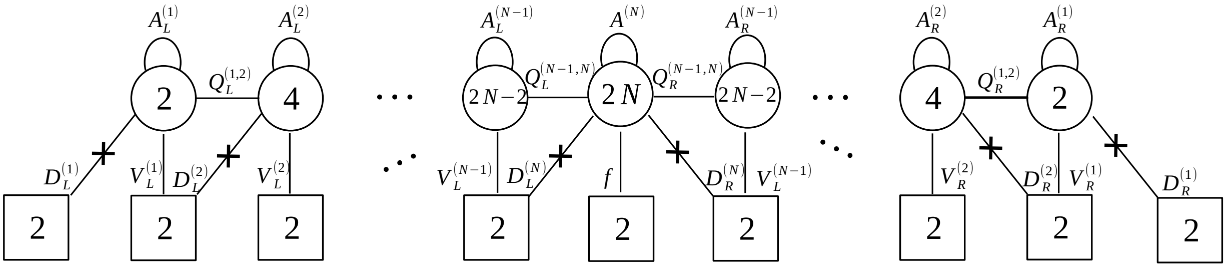

Figure 3.10: Fields appearing in the l.h.s. of the braid relation. Here we are assuming that the two $FE[USp(2N)]$ are glued with the Lagrangian gauging of their manifest $USp(2N)$ symmetries.

The balancing condition here follows by requiring $U(1)_R$ to be non-anomalous at the central $USp(2N)$ node. If we rescale the fugacities for the two chirals f_1, f_2 as

$$u_0 \to u_0 \frac{\sqrt{pq}}{cd}, \qquad u_1 \to u_1 \frac{\sqrt{pq}}{cd} \tag{3.69}$$

we can see that the balancing condition (3.67) becomes the standard tracelessness condition $u_0 u_1 = 1$ for the $SU(2)$ symmetry rotating them. Hence, the global symmetry is

$$USp(2N)_x \times USp(2N)_y \times U(1)_t \times U(1)_c \times U(1)_d \times SU(2)_u\,. \tag{3.70}$$

Observe that the two $USp(2N)$ factors are enhanced at low energies from the $SU(2)$ symmetries of the saw if we take the gauging to be the Lagrangian one as in Figure 3.3.5.

The gauge invariant operators are constructed starting from those of the two $FE[USp(2N)]$ tails:

- operators C_L, C_R constructed as described in Subsection 3.3.1 using the diagonal, vertical and bifundamental chirals of the left and right $FE[USp(2N)]$ blocks respectively;

- operator Ξ constructed starting from one of the diagonals of the left $FE[USp(2N)]$ and terminating on a diagonal of the right $FE[USp(2N)]$ including bifundamentals.

- operators Ω_L, Ω_R constructed by joining the operators Π_L and Π_R in the bifundamental representation of $USp(2N)_x \times USp(2N)_z$ and $USp(2N)_y \times USp(2N)_z$ respectively with the two fundamental chirals f_i

$$\Omega_L = \text{Tr}_N\left(\Pi_L f\right), \qquad \Omega_R = \text{Tr}_N\left(\Pi_R f\right)\,; \tag{3.71}$$

- long mesons constructed with the bifundamentals and the chirals f_i

$$\Theta_i = \mathrm{Tr}_u\, \mathrm{Tr}_{i,L/R} \left(f \prod_{a=i}^{N-1} q_{L/R}^{(a,a+1)} \right)^2 , \tag{3.72}$$

where Tr_u is the trace over the $SU(2)_u$ flavor indices, $\mathrm{Tr}_{i,L/R}$ is the trace over the gauge indices of the i-th $USp(2i)$ gauge node of the left and right tail respectively, and L/R means that we can construct two sets of operators of this form with the bifundamentals of the left or the right tail respectively, but we expect them to actually coincide in pairs in the chiral ring;

- flipping fields β_i^L, β_i^R of the diagonal mesons on the left and right $FE[USp(2N)]$ tails.

The transformation rules of these operators under the global symmetries are summarized in the following table:

	$USp(2N)_x$	$USp(2N)_y$	$U(1)_t$	$U(1)_c$	$U(1)_d$	$SU(2)_u$	$U(1)_{R_0}$
C_L	$\mathbf{N(2N-1)-1}$	$\mathbf{1}$	-1	0	0	$\mathbf{1}$	2
C_R	$\mathbf{1}$	$\mathbf{N(2N-1)-1}$	-1	0	0	$\mathbf{1}$	2
Ξ	$\mathbf{N}$	$\mathbf{N}$	0	1	1	$\mathbf{1}$	0
Ω_L	$\mathbf{N}$	$\mathbf{1}$	0	-1	0	$\mathbf{2}$	1
Ω_R	$\mathbf{1}$	$\mathbf{N}$	0	0	-1	$\mathbf{2}$	1
β_i^L	$\mathbf{1}$	$\mathbf{1}$	$N-i$	-2	0	$\mathbf{1}$	2
β_i^R	$\mathbf{1}$	$\mathbf{1}$	$N-i$	0	-2	$\mathbf{1}$	2
Θ_i	$\mathbf{1}$	$\mathbf{1}$	$N-i$	-2	-2	$\mathbf{1}$	2

On the r.h.s. we have $FE[USp(2N)]$ with two sets of chiral singlets O_L and O_R in the bifundamental representation of the global $USp(2N)_x \times SU(2)_u$ and $SU(2)_u \times USp(2N)_y$ symmetries respectively, which interact with the $FE[USp(2N)]$ block through the superpotential

$$\hat{\mathcal{W}} = \mathcal{W}_{FE[USp(2N)]} + \mathrm{Tr}_u\, \mathrm{Tr}_x\, \mathrm{Tr}_y\, O^L \Pi O^R . \tag{3.73}$$

Because of this superpotential, the global symmetry of the theory precisely matches with (3.70).

The gauge invariant operators are the same of $FE[USp(2N)]$ with the addition of the two sets of singlets O_L and O_R. Moreover, we can construct some long mesons of the form

$$\Theta_i^R = \mathrm{Tr}_u\, \mathrm{Tr}_i\, \mathrm{Tr}_x \left(O_R \prod_{a=i}^{N-1} q^{(a,a+1)} \right)^2 \tag{3.74}$$

and similar ones Θ_i^L involving O_L which have not a simple expression in terms of fundamental fields since the $USp(2N)_y$ symmetry is not manifest in the quiver description, but whose existence is guaranteed by the self-duality of the $FE[USp(2N)]$ block under the exchange $USp(2N)_x \leftrightarrow USp(2N)_y$. They transform under the global symmetries as follows:

	$USp(2Q)_x$	$USp(2Q)_y$	$U(1)_t$	$U(1)_c$	$U(1)_d$	$SU(2)_u$	$U(1)_{R_0}$
$\mathsf{O_H}$	$\mathbf{N(2N-1)-1}$	$\mathbf{1}$	-1	0	0	$\mathbf{1}$	2
C	$\mathbf{1}$	$\mathbf{N(2N-1)-1}$	-1	0	0	$\mathbf{1}$	2
Π	$\mathbf{N}$	$\mathbf{N}$	0	1	1	$\mathbf{1}$	0
β_i	$\mathbf{1}$	$\mathbf{1}$	$N-i$	-2	-2	$\mathbf{1}$	2
O_L	$\mathbf{N}$	$\mathbf{1}$	0	0	-1	$\mathbf{2}$	1
O_R	$\mathbf{1}$	$\mathbf{N}$	0	-1	0	$\mathbf{2}$	1
Θ_i^R	$\mathbf{1}$	$\mathbf{1}$	$N-i$	-2	0	$\mathbf{1}$	2
Θ_i^L	$\mathbf{1}$	$\mathbf{1}$	$N-i$	0	-2	$\mathbf{1}$	2

The operator map across the duality is then

$$\begin{aligned} \mathsf{C}_L &\leftrightarrow \mathsf{O_H}\,, \\ \mathsf{C}_R &\leftrightarrow \mathsf{C}\,, \\ \Xi &\leftrightarrow \Pi\,, \\ \Omega_L &\leftrightarrow O_L\,, \\ \Omega_R &\leftrightarrow O_R\,, \\ \Theta_i &\leftrightarrow \beta_i\,, \\ \beta_i^R &\leftrightarrow \Theta_i^L\,, \\ \beta_i^L &\leftrightarrow \Theta_i^R\,. \end{aligned} \tag{3.75}$$

3.4 E-string compactifications on tubes and tori

3.4.1 The basic tube model and gluing prescription

Let us start from a geometric definition of the tube model. The tube model we discuss here is a four-dimensional theory corresponding to the compactification of the $6d$ E-string theory on a two punctured sphere with some particular value of flux for the E_8 symmetry. The punctures have one $USp(2N)$ symmetry associated to each of them and they come in different types. This is due to the fact that the punctures are expected to break the E_8 global symmetry to $SO(16)$ and we have some freedom in how the $SO(16)$ is embedded inside the E_8. Specifically, we are free to act with any inner automorphism of E_8, which are just the Weyl transformations, to potentially get different embeddings. Likewise the flux, being a vector in the root lattice of E_8, is also affected by Weyl transformations. Thus, given a tube we can generate an equivalent tube by acting with an E_8 Weyl transformation. Tubes differing in this way are ultimately the same tube, but the gluing of two tubes is affected if these differ by a relative Weyl group action. When we glue two punctures together the

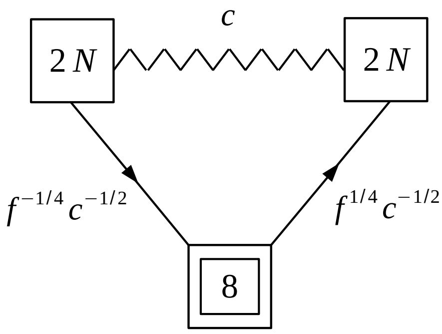

Figure 3.11: The basic tube with E_7 flux $\mathcal{F} = (\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2})$. The squares denote USp groups and double squares SU groups.

associated flux for the combined tube is the sum with appropriate signs, as we shall see, of their fluxes.

Now we need to define the tube and gluing at the level of the physical theories. In a tube theory each puncture comes equipped with an octet of operators M_a with $a = 1, \cdots, 8$ in the fundamental representation of $USp(2N)$, which we will refer to by an abuse of notation as *moment maps*. These moment map operators are charged under the $U(1)$ symmetries comprising the Cartan generators of the E_8. Different types of punctures have moment map operators charged differently under these symmetries. Again, the difference of the charges can be associated to an action of the Weyl group of E_8. Consider fixing a specific $SO(16)$ subgroup of E_8, as we have done when we chose an $SO(16)$ flux basis. Then for simplicity, when we glue two punctures together we can first limit ourselves to only gluing punctures of the same type up to the action of the Weyl group of the chosen $SO(16)$ subgroup of E_8 (a more general gluing will be discussed in Section 3.4.3). The Weyl group of $SO(16)$ is comprised of permutations of eight elements and of flips of any even number of them. In particular it means that the moment map operators M_a and M'_a of the two punctures we are gluing have same charges under the Cartan symmetries of E_8 up to permutations of the indices and flips of signs for even number of components. Let us denote the permutation by σ and the set of indices with flipped signs by $\mathfrak{F}$. The punctures also have operators A in the antisymmetric traceless representation of $USp(2N)$. We glue punctures by gauging a diagonal combination of the two $USp(2N)$ symmetries and by introducing a chiral field, $\hat{A}$, in the traceless antisymmetric representation of $USp(2N)$ and a set of fundamental fields, Φ_a for $a \notin \mathfrak{F}$, which couple to the moment map operators through a superpotential

$$\mathcal{W} = \sum_{a \notin \mathfrak{F}} (M_a - M'_{\sigma(a)})\, \Phi^a + \sum_{a \in \mathfrak{F}} M_a M'_{\sigma(a)} + (A - A')\hat{A}\,. \tag{3.76}$$

The first type of superpotential terms are usually referred to as Φ-gluing and the second ones as S-gluing. The third term only appears for higher rank E-string as for rank one we do not have traceless antisymmetric representations. Physically the restriction to only having even number of flipped charges is related to Witten global anomaly obstruction [152]. We will be gauging the $USp(2N)$ symmetry and the absence of the Witten anomaly implies that the number of chiral fields in the fundamental representation here is even. Finally, if the fluxes of the theories we are gluing are $\mathcal{F}$ and $\mathcal{F}'$, the flux of the combined theory, $\mathcal{F}^{glued}$, will be given by

$$a \notin \mathfrak{F} : \mathcal{F}_a^{glued} = \mathcal{F}_a + \mathcal{F}'_{\sigma(a)}\,, \qquad a \in \mathfrak{F} : \mathcal{F}_a^{glued} = \mathcal{F}_a - \mathcal{F}'_{\sigma(a)}\,. \tag{3.77}$$

Next we need to define at least one tube model from which we can build other tubes and torus theories. The simplest tube, depicted in Figure 3.11, is constructed by coupling the $FE[USp(2N)]$ block to two octets M, M' with superpotential

$$\mathcal{W} = \sum_{a=1}^{8} M^a \Pi M'_a\,. \tag{3.78}$$

In the tube model constructed in such a way, the operators in the antisymmetric representations of the $USp(2N)$ symmetries of the puncture are $\mathsf{O_H}$ and C[14]. This tube model is associated to a flux breaking $E_8 \to U(1)_c \times E_7$ which in the $SO(2)^8 \subset SO(16)$ complete basis corresponds to the vector

$$\mathcal{F} = \left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right)\,. \tag{3.79}$$

The basic tube theory has global symmetry $USp(2N) \times USp(2N) \times U(1)_t \times U(1)_c \times SU(8)_u \times U(1)_f$. The $U(1)_t$ symmetry is hidden inside the block and when we glue tubes into tori it will enhance to the $SU(2)_L$ symmetry of the E-string theory. The $U(1)_f$ symmetry, when we glue tubes into tori, always disappears because of anomalies and superpotential constraints. For this reason we will omit it from our discussion of torus compactifications setting $f = 1$, but it will become crucial in sphere compactifications since it represents the Cartan of the $SU(2)_{\text{ISO}}$ isometry of $\mathbb{S}^2$ as we will see in Section 3.5. We use the $U(1)_c$ fugacity to define charges of the moment maps on the right as $\eta_a = c^{-1/2} u_a$, where u_a are $SU(8)_u$ fugacities satisfying $\prod_{a=1}^{8} u_a = 1$. On the left the fugacities are $\eta'_a = c^{-1/2} u_a^{-1}$. The map between η_a to η'_a consists of charge conjugation for the $SU(8)$, but without acting on $U(1)_c$. This is not a Weyl group element of $SO(16)$, which contains charge conjugation for both groups, but not for each one separately. However, it is an element of the $E_7 \subset E_8$ Weyl symmetry group.

[14] We remark again that thanks to the symmetry property of $FE[USp(2N)]$ under the exchange of $USp(2N)_x$ and $USp(2N)_y$ it doesn't matter which puncture we are identifying with the symmetry that is manifest in the Lagrangian description and which with the one that is enhanced at low energies. The only distinction between the two punctures, as we explained, relies in the octet fields and their charges under the E_8 subgroup.

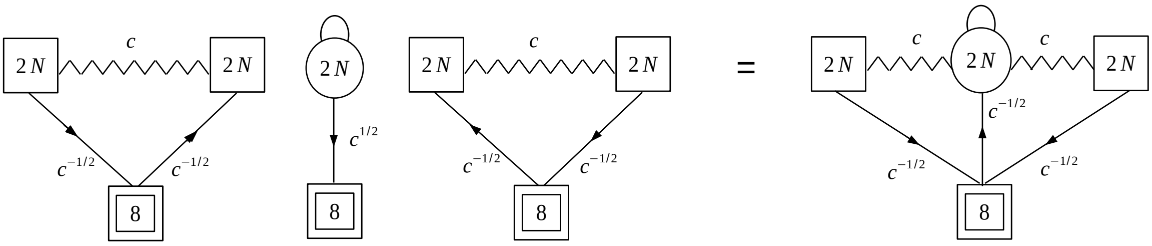

Figure 3.12: Gluing two basic tubes together with a trivial element of the Weyl group we obtain a tube with flux $(1,1,1,1,1,1,1,1)$.

Let us illustrate how we glue two such tubes together with concrete examples. We can glue two basic tubes with a trivial identification of the moment maps as depicted in Figure 3.12. If we assign fugacities η_a to moment maps of one glued puncture and ξ_a to the other glued puncture, we identify the charges with a trivial action of the $SO(16)$ Weyl symmetry group as follows:

$$a = 1\dots 8 \;:\;\; \eta_a = \xi_a\,. \tag{3.80}$$

The two basic tubes then are glued with the superpotential

$$\mathcal{W} = \sum_{a=1}^{8}(M_a - M'_{\sigma(a)})\Phi^a + (A - A')\hat{A}\,, \tag{3.81}$$

where A, A' are either the operators $\mathsf{O_H}$ or C of the $FE[USp(2N)]$ associated to the tubes we are gluing. Integrating out the massive fields, the fields M_a and $M'_{\sigma(a)}$ are identified and we get the quiver on the right of Figure 3.12. The flux of the combined model is obtained summing the fluxes $\mathcal{F}$ and $\mathcal{F}'$ of the two glued theories. Since in this case there are no flips of fugacities the tube model which we obtain has flux

$$\begin{aligned}&(\mathcal{F}_1+\mathcal{F}'_1,\mathcal{F}_2+\mathcal{F}'_2,\mathcal{F}_3+\mathcal{F}'_3,\mathcal{F}_4+\mathcal{F}'_4,\mathcal{F}_5+\mathcal{F}'_5,\mathcal{F}_6+\mathcal{F}'_6,\mathcal{F}_7+\mathcal{F}'_7,\mathcal{F}_8+\mathcal{F}'_8) = \\ &= (1,1,1,1,1,1,1,1)\,,\end{aligned} \tag{3.82}$$

which corresponds to a unit of flux $z = 1$ for the $U(1)$ whose commutant in E_8 is E_7.

We can also glue two basic tubes with a non-trivial identification of the moment maps. Denoting again as η_a and ξ_a the fugacities of the punctures we are gluing, we identify the charges with the following action of the $SO(16)$ Weyl symmetry group:

$$\begin{aligned}a = 1\dots 4 \;&:\;\; \eta_a = \xi_a\,,\\ a = 5\dots 8 \;&:\;\; \eta_a = \frac{1}{\xi_a}\,.\end{aligned} \tag{3.83}$$

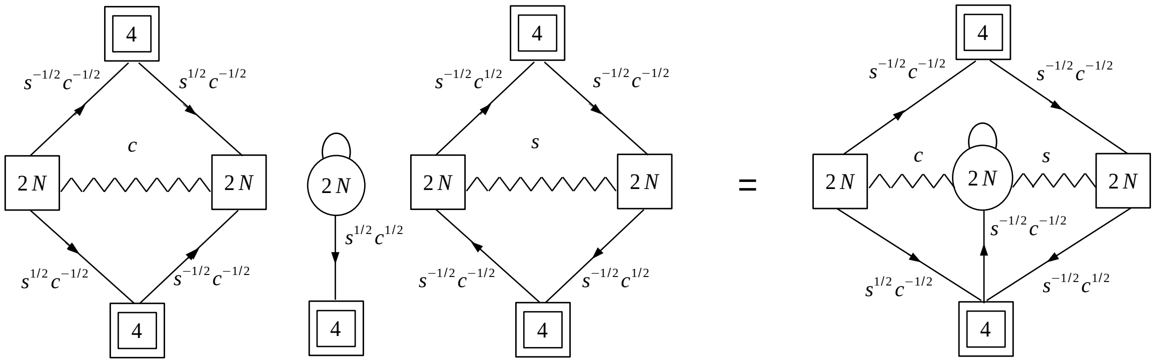

Figure 3.13: Example of gluing two basic tubes together with a non-trivial element of the Weyl group of $SO(16)$. The resulting tube will have flux $(1,1,1,1,0,0,0,0)$.

We decompose the $SU(8)_u \times U(1)_c$ fugacities of our basic tube into $SU(4)_v \times SU(4)_w \times U(1)_s \times U(1)_c$ fugacities taking for the first moment map fugacities $\eta_a = c^{-\frac{1}{2}}s^{-\frac{1}{2}}v_a$ for $a = 1\dots 4$ and $\eta_a = c^{-\frac{1}{2}}s^{\frac{1}{2}}w_a$ for $a = 5\dots 8$, with $\prod_{a=1}^4 v_a = \prod_{a=5}^8 w_a = 1$. Analogously for the second moment map we take $\xi_a = c'^{-\frac{1}{2}}s'^{-\frac{1}{2}}v'_a$ for $a = 1\dots 4$ and $\xi_a = c'^{-\frac{1}{2}}s'^{\frac{1}{2}}w'_a$ for $a = 5\dots 8$. The identification above then sets $c = s'$, $s = c'$, $v_a = v'_a$, and $w_a = 1/w'_a$. The gluing of the two tubes is depicted in Figure 3.13. In this case, the two basic tubes are glued with the superpotential

$$\mathcal{W} = \sum_{a=1}^{4}(M_a - M'_{\sigma(a)})\Phi^a + \sum_{a=5}^{8} M_a M'_{\sigma(a)} + (A - A')\hat{A}\,. \tag{3.84}$$

Now half of the fugacities are flipped and consequently the tube model we obtain has flux

$$\begin{aligned}&(\mathcal{F}_1 + \mathcal{F}'_1, \mathcal{F}_2 + \mathcal{F}'_2, \mathcal{F}_3 + \mathcal{F}'_3, \mathcal{F}_4 + \mathcal{F}'_4, \mathcal{F}_5 - \mathcal{F}'_5, \mathcal{F}_6 - \mathcal{F}'_6, \mathcal{F}_7 - \mathcal{F}'_7, \mathcal{F}_8 - \mathcal{F}'_8)\\ &= (1,1,1,1,0,0,0,0)\,,\end{aligned} \tag{3.85}$$

which corresponds to half a unit of flux $z = \frac{1}{2}$ for the $U(1)$ whose commutant in E_8 is $SO(14)$.

We can further glue these tubes. For example, by gluing two $(1,1,1,1,0,0,0,0)$ tubes with a trivial action of the $SO(16)$ Weyl symmetry group, adding eight Φ_a fundamentals, as shown in Figure 3.14, we obtain a tube with flux $(2,2,2,2,0,0,0,0)$ which corresponds to a unit of flux $z = 1$ for the $U(1)$ whose commutant in E_8 is $SO(14)$.

Using these simple definitions we will now construct a large set of models with interesting properties. Before doing this we will verify that the 't Hooft anomalies under all the

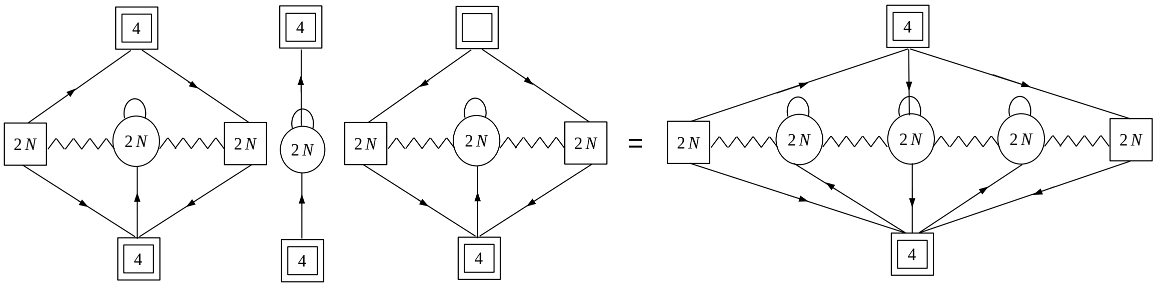

Figure 3.14: Gluing two $(1,1,1,1,0,0,0,0)$ tubes together with a trivial element of the Weyl group of $SO(16)$ we obtain a tube with flux $(2,2,2,2,0,0,0,0)$.

symmetries of the conjectured tube theory match the six dimensional predictions (3.2), (3.6), (3.7), and (3.8).

Let us also note here that the RG flow between $FE[USp(2N)]$ to $FE[USp(2(N-1))]$ that we have discussed in section 3.3.2 has a $6d$ meaning. This flow corresponds to separating one M5 brane from the rest and flowing to a lower rank E-string theory. Note that such a flow keeps the six dimensional symmetry $E_8 \times SU(2)_L$ intact. However as the symmetries corresponding to the punctures in the $FE[USp(2N)]$ and $FE[USp(2(N-1))]$ theories are different, the VEV breaks $USp(2N)$ down to $USp(2(N-1))$. We also note that the flow to WZ model discussed in 3.3.2 corresponding to setting $c = \sqrt{t}$ was considered in the context of rank N E-string compactification in Appendix B of [116] and corresponds to some relevant deformation of the theories obtained in the compactifications.

3.4.2 Anomalies of the basic tube

Let us compute various anomalies of the basic tube theory. We have defined the basic model using a certain R-symmetry and definition of $U(1)_c$ and $U(1)_t$ using which the charges of various fields take the simplest form. Also these are the definitions used by Rains in [79]. However, to compare with the $6d$ computation we need to perform some slight redefinitions. In general, as we mentioned before, different choices of R-symmetry are related by admixture of abelian symmetries

$$R = R_0 + \mathfrak{c} q_c + \mathfrak{t} q_t \,. \tag{3.86}$$

Here we will use the six-dimensional R-symmetry, denoted by $\hat{R}$, which corresponds to taking $\mathfrak{c} = 0$ and $\mathfrak{t} = 1$ so $\hat{R} = R_0 + q_t$. Using this R-symmetry we find that the linear anomalies are

$$\operatorname{Tr} U(1)_{\hat{R}} = -N(1+2N)\,, \qquad \operatorname{Tr} U(1)_t = 1 + N - 2N^2\,, \qquad \operatorname{Tr} U(1)_c = -14N\,. \tag{3.87}$$

Next consider anomalies with puncture symmetries

$$\mathrm{Tr}\, U(1)_{\hat{R}} USp(2N)^2 = -\frac{1+N}{2}\,, \qquad \mathrm{Tr}\, U(1)_c USp(2N)^2 = -1\,,$$
$$\mathrm{Tr}\, U(1)_t USp(2N)^2 = \frac{1-N}{2}\,. \tag{3.88}$$

Then we have cubic anomalies involving a single symmetry

$$\mathrm{Tr}\, U(1)^3_{\hat{R}} = -N(1+2N)\,, \qquad \mathrm{Tr}\, U(1)^3_t = 1+N-2N^2\,, \qquad \mathrm{Tr}\, U(1)^3_c = -8N\,. \tag{3.89}$$

Finally we have cubic anomalies involving several $U(1)$ symmetries

$$\mathrm{Tr}\, U(1)_{\hat{R}} U(1)^2_t = 0\,, \qquad \mathrm{Tr}\, U(1)_{\hat{R}} U(1)^2_c = 0\,, \qquad \mathrm{Tr}\, U(1)_c U(1)^2_t = -N(N-1)\,,$$
$$\mathrm{Tr}\, U(1)_t U(1)^2_c = 0\,, \qquad \mathrm{Tr}\, U(1)^2_{\hat{R}} U(1)_t = 0\,, \qquad \mathrm{Tr}\, U(1)^2_{\hat{R}} U(1)_c = N(N+1)\,. \tag{3.90}$$

To compare with the six-dimensional prediction we have to sum the bulk contribution to the inflow contribution of the two punctures with $z = 1/2$, $\xi_G = 1$, $q = -1/2$ and $q_a = -1/2$ for $a = 1, \cdots 8$. For example

$$\mathrm{Tr}\, U(1)_c = \underbrace{-12\times(1/2)\times 1\, N}_{\text{geometric}} + \underbrace{2\times 8\times(-1/2)\, N}_{\text{inflow}} = -14N$$
$$\mathrm{Tr}\, U(1)^3_c = \underbrace{-12\times(1/2)\times 1^2\, N}_{\text{geometric}} + \underbrace{2\times 8\times(-1/2)^3\, N}_{\text{inflow}} = -8N\,, \tag{3.91}$$

and further, taking contributions from the punctures only,

$$\mathrm{Tr}\, U(1)_t = -\frac{2N^2-N-1}{2}$$
$$\mathrm{Tr}\, U(1)^3_t = -\frac{2N^2-N-1}{8}$$
$$\mathrm{Tr}(U(1)_c SU(2)^2_L) = -\frac{N(N-1)}{4}\,. \tag{3.92}$$

In order to match the anomalies (3.92) with the ones computed in $4d$ we also need to redefine $q_t \to \frac{1}{2} q_t \equiv q_{\hat{t}}$. In this normalization for the $U(1)_{\hat{t}}$ charges the character of the fundamental representation of $SU(2)_L$ is $\hat{t}^{\frac{1}{2}} + \hat{t}^{-\frac{1}{2}}$ and thus $\mathrm{Tr}\, U(1) SU(2)^2_L = \mathrm{Tr}\, U(1) U(1)^2_{\hat{t}}$. In particular,

$$\mathrm{Tr}\, U(1)_{\hat{t}} = \frac{1}{2}\, \mathrm{Tr}\, U(1)_t = -\frac{2N^2-N-1}{2}$$
$$\mathrm{Tr}\, U(1)^3_{\hat{t}} = \frac{1}{8}\, \mathrm{Tr}\, U(1)^3_t = -\frac{2N^2-N-1}{8}$$

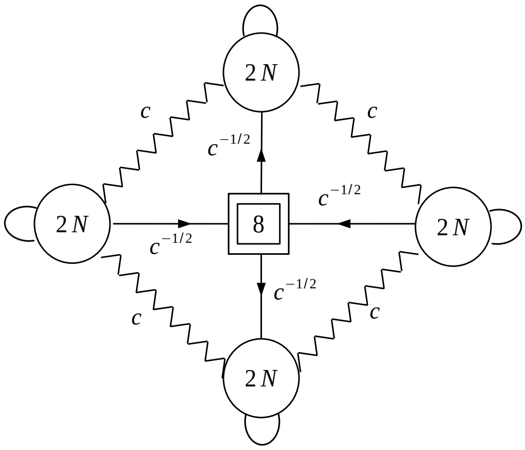

Figure 3.15: Gluing $2n$ $(\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2})$ tubes we obtain a torus with n units of flux preserving $SU(2)_L \times E_7 \times U(1)$. Here we are representing the model for $n = 2$.

$$\mathrm{Tr}\, U(1)_c U(1)_t^2 = \mathrm{Tr}(U(1)_c SU(2)_L^2) = -\frac{N(N-1)}{4}\,. \tag{3.93}$$

The matching of the anomalies that we just performed is a strong test of our claim that the model constructed with $FE[USp(2N)]$ and the two octets of chirals is the result of the compactification of the rank-N E-string theory on a tube with flux $(\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2})$, which is flux $z = \frac{1}{2}$ for the $U(1)$ whose commutant in E_8 is E_7. Now we will perform additional tests which consist of using this basic tube to construct torus models with various values of flux and verifying that they possess the anomalies and the global symmetries expected from $6d$.

3.4.3 Torus compactifications

We now use the basic E_7 tube to construct models corresponding to torus compactifications with various values of fluxes by performing gluings with different $SO(16)$ Weyl elements as we just described.

Tori with $\mathcal{F} = (n,n,n,n,n,n,n,n)$

The simplest torus models we can build are obtained by combining the basic E_7 tubes together with a trivial action of the Weyl group. Taking an even number of such tubes we do not break any of the symmetries. In particular combining $2n$ tubes we obtain the torus compactification of the E-string with n units of flux for the $U(1)$ whose commutant in E_8 is E_7 (see Figure 3.15).

Anomalies:

We compute some of the anomalies of this torus theory. It is convenient to package the anomalies of abelian symmetries into the trial a and c anomalies. Using the trial R-charge $R_0 + \mathfrak{t}q_t + \mathfrak{c}q_c$ we first calculate the trial a and c anomalies of $FE[USp(2N)]$

$$\begin{aligned} a^{FE[USp(2N)]}(\mathfrak{c},\mathfrak{t}) &= \frac{3}{32}N\left(\mathfrak{c}(16-9(N-1)(\mathfrak{t}-2)\mathfrak{t})-(2N-1)\mathfrak{t}(3(\mathfrak{t}-3)\mathfrak{t}+8)-4\right)+ \\ &\quad -\frac{3}{32}\left(12n\mathfrak{c}^3+3(1-\mathfrak{t})^3-(1-\mathfrak{t})\right), \\ c^{FE[USp(2N)]}(\mathfrak{c},\mathfrak{t}) &= \frac{1}{32}N\left(\mathfrak{c}(44-27(N-1)(\mathfrak{t}-2)\mathfrak{t})-(2N-1)\mathfrak{t}(9(\mathfrak{t}-3)\mathfrak{t}+22)-8\right)+ \\ &\quad -\frac{1}{32}\left(36N\mathfrak{c}^3+9(1-\mathfrak{t})^3-5(1-\mathfrak{t})\right). \end{aligned} \tag{3.94}$$

When we glue the tubes to a torus we add an octet of fundamental fields Φ_a, the antisymmetric field $\hat{A}$, and gauge the $USp(2N)$ symmetry. The contribution of the gluing to the anomaly is then,

$$\begin{aligned} a^{glue\;(8,0)}(\mathfrak{c},\mathfrak{t}) &= \frac{3}{32}\left(-6\mathfrak{c}^3N+8\mathfrak{c}N-(N(2N-1)-1)\left(-3(\mathfrak{t}-1)^3+\mathfrak{t}-1\right)+\right. \\ &+ \left. 2(2N+1)N\right) \\ c^{glue\;(8,0)}(\mathfrak{c},\mathfrak{t}) &= \frac{1}{32}\left(2\mathfrak{c}\left(20-9\mathfrak{c}^2\right)N+\left(2N^2-N-1\right)\left(9\mathfrak{t}^3-27\mathfrak{t}^2+22\mathfrak{t}-4\right)+\right. \\ &+ \left. 4(2N+1)N\right). \end{aligned} \tag{3.95}$$

Here the label $(8,0)$ denotes the fact that we glue with an octet of Φ_a as opposed to gluing with less than 8 fields, as we do when we consider a non-trivial identification with the action of the Weyl symmetry group. The total anomaly is given by

$$\begin{aligned} a^{E_7,2n}(\mathfrak{c},\mathfrak{t}) &= 2n(a^{FE[USp(2N)]}(\mathfrak{c},\mathfrak{t})+a^{glue\,(8,0)}(\mathfrak{c},\mathfrak{t})), \\ a^{E_7,2n}(\mathfrak{c},\mathfrak{t}) &= 2n(c^{FE[USp(2N)]}(\mathfrak{c},\mathfrak{t})+c^{glue\,(8,0)}(\mathfrak{c},\mathfrak{t})). \end{aligned} \tag{3.96}$$

We can maximize $a^{E_7,2n}$ with respect to $\mathfrak{c}$ and $\mathfrak{t}$ and obtain

$$\mathfrak{c} = \frac{2\sqrt{2}}{3}\sqrt{3N+5}, \qquad \mathfrak{t}=0, \tag{3.97}$$

for which we get

$$a = \frac{\sqrt{2}N(3N+5)^{\frac{3}{2}}}{16}n, \qquad c = \frac{\sqrt{2(3N+5)}Q(3N+7)}{16}n, \tag{3.98}$$

which matches the six dimensional prediction (3.3) with $z = n$ and $\xi_G = 1$, the value corresponding to the flux preserving $U(1)\times E_7$. Notice that the absence of mixing with $U(1)_t$ is compatible with the fact that this symmetry is enhanced to $SU(2)_L$ for which we have no flux. We can also match the abelian anomalies. For example from the $4d$ theory we extract

$$\mathrm{Tr}\, U(1)_c = -12Nn\,, \qquad \mathrm{Tr}\, U(1)_c^3 = -12Nn \tag{3.99}$$

which perfectly match the $6d$ prediction.

Index:

Next we can compute the index of the torus theory and check whether the expected symmetry makes an appearance. The index for the basic tube theory with flux $\mathcal{F} = \left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right)$ is given by[15]

$$\begin{aligned} \mathcal{I}_{tube}^{(z=\frac{1}{2})}(\vec{x};\vec{y};c;t;\vec{u}) \;&=\; \prod_{i=1}^{N}\prod_{a=1}^{8}\Gamma_e\left((pq)^{\frac{1}{2}}c^{-\frac{1}{2}}u_i a x_i^{\pm1}\right)\Gamma_e\left((pq)^{\frac{1}{2}}c^{-\frac{1}{2}}u_a^{-1}y_i^{\pm1}\right)\times \\ &\times\; \mathcal{I}_{FE[USp(2N)]}(\vec{x};\vec{y};c;t)\,. \end{aligned} \tag{3.100}$$

We also define the contribution of the gluing as

$$\Delta_N(\vec{z};c;t;\vec{u}) = \prod_{i=1}^{N}\prod_{a=1}^{8}\Gamma_e\left((qp)^{\frac{1}{2}}c^{\frac{1}{2}}u_a^{-1}z_i^{\pm1}\right)\Delta_N(\vec{z};t)\,, \tag{3.101}$$

where the contribution of the traceless antisymmetric of $USp(2N)$ is

$$\Delta_N(\vec{z};t) = \Gamma_e\,(t)^{N-1}\prod_{i<j}^{N}\Gamma_e\left(t\,z_i^{\pm1}z_j^{\pm1}\right)\,. \tag{3.102}$$

Then the torus model with $z = n$ units of flux has the following index

$$\mathcal{I}_N^{(z=n)} = \oint \mathrm{d}\vec{z}_N^{(1)} \cdots \oint \mathrm{d}\vec{z}_N^{(2n)} \prod_{i=1}^{2n}\mathcal{I}_{tube}^{(z=\frac{1}{2})}(\vec{z}^{(i)},\vec{z}^{(i+1)};c;t;\vec{u})\Delta_N(\vec{z}^{(i+1)};\vec{u};c;t)\,. \tag{3.103}$$

In order to analyze the symmetries of the theory we expand the index using the $6d$ R-symmetry $\hat{R}$. The case of $N = 1$ was discussed in detail in [116], here we give the result for $N = 2$ and generic value of z[16]. With the $6d$ R-symmetry $\hat{R}$ and with $t = \hat{t}^{\frac{1}{2}}$ we obtain for flux $z > 1$

$$\begin{aligned} \mathcal{I}_{N=2}^{(z=n)} &= 1 + c^{2z} + c^{4z} + \cdots + pq(z\mathbf{56}c^{-1} - z\mathbf{56}c + 2zc^{-2}) + \\ &+pq(p+q)(z\mathbf{56}c^{-1} + 2zc^{-2}) + (pq)^{\frac{3}{2}}(z\mathbf{56}c^{-1} + 2zc^{-2})(\hat{t}^{\frac{1}{2}} + \hat{t}^{-\frac{1}{2}}) + \cdots\,. \end{aligned} \tag{3.104}$$

We see that the representations of $SU(8)_u$ recombine into representations of E_7. In particular, $\overline{\mathbf{28}} + \mathbf{28} \to \mathbf{56}$. The fugacity $\hat{t}^{\frac{1}{2}}$ is the Cartan of $SU(2)_L$. We can easily identify

[15] In the following expressions we turn off the fugacity f.

[16] For low values of flux there can be additional operators with low charges contributing in low orders of the expansion of the index.

some of these operators in the quiver. For example, the operators charged c^{2z} are $\prod_{i=1}^{2n} \Pi_{(i)}$ where $\Pi_{(i)}$ is the Π operator for the i-th $FE[USp(2N)]$ block. The operators in the $\mathbf{28}$ and $\overline{\mathbf{28}}$ are built from the octet fields as $\mathrm{Tr}\, M_a M_a$. Note that as half of M_a are in the fundamental and half in the anti-fundamental of $SU(8)_u$ we get exactly n copies of $\mathbf{56}$s.

We will now compare the $4d$ spectrum that we see from the index with what we expect from the $6d$ construction. As we discussed around eq. (3.14) the lowest BPS operators contributing to the index of the $4d$ theory come from the $6d$ conserved currents and the energy momentum tensor. Since we are considering torus compactifications with flux breaking $E_8 \to U(1)_c \times E_7$ we expect that the contribution to the index of these operators will be in the representations appearing in the branching rule for the decomposition of the adjoint of $E_8 \to U(1)_c \times E_7$

$$\mathbf{248} \to \mathbf{1}^{\pm 2} \oplus \mathbf{1}^0 \oplus \mathbf{133}^0 \oplus \mathbf{56}^{\pm 1}\,, \tag{3.105}$$

where the superscripts indicate the $U(1)_c$ charges. The multiplicities with which these operators contribute depend on the charges under the $U(1)$ for which we turn on the flux and on the flux z. For example, in our normalization[17] an operator with charge $+1$ under $U(1)_c$ will be a fermion and contribute with multiplicity $-z$ to the index, while an operator charged -2 will be a boson and will contribute with multiplicity $+2z$ to the index. These operators will appear at order pq in the expansion of the $4d$ index when the $6d$ R-charge $\hat{R}$ is chosen. This is the expected pattern. Indeed we see that in (3.104) at order pq we have operators in the $\mathbf{56}^{\pm}$ and $\mathbf{1}^2$. However, we are missing the $\mathbf{1}^{-2}$ operator. It is not clear what eliminates it from the $4d$ theory, and it will be interesting to figure this out. One possibility is that it get canceled against defect operators wrapping the torus.

We then can think of $0 = \mathbf{1}^0 + \mathbf{133}^0 + \mathbf{3}^0_{SU(2)_L} - (\mathbf{1}^0 + \mathbf{133}^0 + \mathbf{3}^0_{SU(2)_L})$ as the cancellation of marginal operators and conserved currents. This is expected for torus compactifications as we discussed in Subsection 3.2.2. In particular, we would conclude that the conformal manifold, having dimension 8, is big enough to accommodate the E_7 symmetry enhancement.

Another property that we immediately see is that the index is invariant under exchange of $\hat{t}$ with $\hat{t}^{-1}$. This is consistent with the expectation that $\hat{t}^{\frac{1}{2}}$ is an $SU(2)_L$ fugacity as the operation of flipping $\hat{t}$ is the Weyl operation of the $SU(2)_L$. This conforms to our expectations from $6d$ since this model has no flux for $SU(2)_L$. This property holds for generic rank N and flux z thanks to the flip-flip duality of $FE[USp(2N)]$ and in particular (3.37). Note that, as we always introduce the antisymmetric field $\hat{A}$ for each gluing, this will guarantee that the index is invariant under the Weyl transformation of $SU(2)_L$ for tori with any choice of flux. This is not true for tubes however as the punctures break the $SU(2)_L$ symmetry to $U(1)_{\hat{t}}$.

[17]Specifically, the $U(1)$ for which we have flux z is identified with $-U(1)_c$.

Tori with $\mathcal{F} = \left(\frac{k}{2}, \frac{k}{2}, \frac{k}{2}, \frac{k}{2}, \frac{k}{2}, \frac{k}{2}, \frac{k}{2}, \frac{k}{2}\right)$

If we glue an odd number of basic tubes we obtain theories with half-integer fluxes $\mathcal{F} = \left(\frac{k}{2}, \frac{k}{2}, \frac{k}{2}, \frac{k}{2}, \frac{k}{2}, \frac{k}{2}, \frac{k}{2}, \frac{k}{2}\right)$, with k an odd integer. This flux violates the Dirac quantization condition. Nevertheless, fractional fluxes can in general still be allowed, provided that these are accompanied by a flux in the center of the non-abelian part of the global symmetry that is the commutant of the $U(1)$ inside the full $6d$ global symmetry for which we are turning on a flux. The effect of this flux for the center group is to further break the non-abelian global symmetry to some subgroup. Such a flux for the center group can be generated by turning on two holonomies, one for each cycle of the torus, that are almost commuting, that is they commute up to an element of the center group[18]. The symmetry preserved by the flux corresponds then to the one preserved by the holonomies. For a more in depth analysis we refer the reader to Appendix C of [116].

For a given group, there may be more than one such choices of holonomies, which preserve a different subgroup and which can be continuously connected one to the other. The field theory interpretation of this is that the theories obtained from the compactification will possess a conformal manifold and their marginal deformations are related to the aforementioned holonomies. If we are on a point of the conformal manifold where the global symmetry is the one preserved by one choice of holonomies, we can move on a generic point where the symmetry is broken to its maximal torus $U(1)^r$, with r the rank of the preserved global symmetry group, and then to a special point where these $U(1)$ symmetries reassemble into a different symmetry group associated to a different choice of holonomies. For the case at hand, the fractional flux will in general break the E_7 symmetry to its maximal torus $U(1)^4$, but we may expect this to further enhance at most to F_4 [116].

Tori with half-integer fluxes can be obtained by gluing an odd number of copies of the basic tube. We focus on the case of a single tube self-glued to give a torus with $\mathcal{F} = \left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right)$. Recall that the two punctures of the tube are of types that are not related by an action of the Weyl group of $SO(16)$. Hence, we can't perform a gluing that preserves the $SU(8)$ symmetry, as expected. Instead, we will perform a gauging that explicitly breaks the $SU(8)$ symmetry to $SU(4) \times U(1)$, which actually is enhanced to $SO(8)$ in the Lagrangian since the gauge group is $USp(2N)$ and the chirals transforming under $SU(4) \times U(1)$ are in the (anti-)fundamental representation of $SU(4)$. We will then discuss the possibility for this to further get enhanced to F_4.

More precisely, we start from the tube theory of Figure 3.11 and we split the octets of chiral fields in two groups, as depicted on the left of Figure 3.16. This corresponds to the group decomposition $SU(8)_u \to SU(4)_v \times SU(4)_w \times U(1)_s$. We also rewrite the superpotential

[18] This is sometimes also refereed to as a non-trivial second Stiefel-Whitney class. See [121] and references therein, for some discussion of this aimed at physicists.

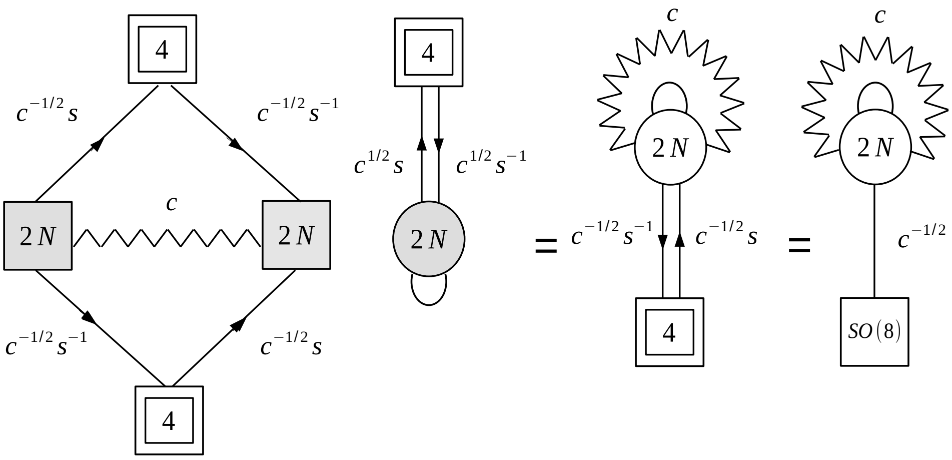

Figure 3.16: Self-gluing of the basic tube yields the torus with flux $\left(\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2}\right)$. The shaded nodes indicate the gauging which identifies the $USp(2N)$ symmetries of the basic tube.

as

$$\mathcal{W}=\sum_{a=1}^{4}M^a\Pi M'_a+\sum_{b=5}^{8}M^b\Pi M'_b\,. \tag{3.106}$$

We perform a gauging that breaks the upper $SU(4)$ symmetry by identifying the two $USp(2N)$ symmetries and adding a pair Φ, Φ' of bifundamental and anti-bifundamental of $SU(4)\times USp(2N)$ and a $USp(2N)$ traceless antisymmetric $\hat{A}$ with superpotential

$$\mathcal{W}=\sum_{a=1}^{4}M^a\Pi M'_a+\sum_{b=5}^{8}M^b\Pi M'_b+\sum_{b=5}^{8}\left(M^b\Phi_b+M'_b\Phi'^b\right)+\hat{A}(\mathsf{O_H}-\mathsf{C})\,. \tag{3.107}$$

Integrating out the massive fields we get the quiver on the right of Figure 3.16 with superpotential

$$\mathcal{W}=\sum_{a=1}^{4}M^a\Pi M'_a\,, \tag{3.108}$$

where the global symmetry is actually $SO(8)$. This is the theory corresponding to a torus with flux $\mathcal{F}=\left(\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2}\right)$.

In order to discuss the possible enhancement of the $SO(8)$ symmetry to F_4, we consider the supersymmetric index of this theory

$$\mathcal{I}_{torus}^{\left(z=\frac{1}{2}\right)}(c;t;\vec{u}) \;=\; \oint \mathrm{d}\vec{z}_N\,\Delta_N(\vec{z};t)\prod_{i=1}^{N}\prod_{a=5}^{8}\Gamma_e\left((pq)^{\frac{1}{2}}c^{\frac{1}{2}}\left(s^{-1}w_a\right)^{\pm1}z_i^{\pm1}\right)\mathcal{I}_{tube}^{\left(z=\frac{1}{2}\right)}(\vec{z};\vec{z};c;t\vec{u})=$$

$$= \oint \mathrm{d}\vec{z}_N \, \Delta_N(\vec{z};t) \prod_{i=1}^{N} \prod_{a=1}^{4} \Gamma_e \left((pq)^{\frac{1}{2}} c^{-\frac{1}{2}} \, (s\, v_a)^{\pm 1} \, z_i^{\pm 1} \right) \mathcal{I}_{FE[USp(2N)]}(\vec{z};\vec{z};c;t) \,, \tag{3.109}$$

where we decomposed the $SU(8)_u$ fugacities into $SU(4)_v \times SU(4)_w \times U(1)_s$ fugacities according to

$$u_a = s\, v_a, \qquad u_{b+4} = s^{-1} w_b \tag{3.110}$$

with the constraints $\prod_{a=1}^{4} v_a = \prod_{b=5}^{8} w_b = 1$. Notice that the index is manifestly $SO(8)$ invariant in the variables $u_a = s\, v_a$. It is also secretly F_4 invariant, since according to Theorem 3.22 of [79] it is invariant under

$$u_a \to \frac{u_a}{\sqrt{u_1 u_2 u_3 u_4}} \,. \tag{3.111}$$

This implies that if we expand the index in powers of p and q, the characters of $SO(8)$ should actually re-arrange into characters of F_4. Indeed, using the $6d$ R-charge $\hat{R}$ and rescaling $t = \hat{t}^{\frac{1}{2}}$ we find that the index of the model for rank $N = 2$ is

$$\begin{aligned} \mathcal{I}_{torus}^{(z=\frac{1}{2})} &= 1 + c + 2c^2 + \cdots + qp \left(\mathbf{28} + 1 + c^{-2} + (\mathbf{28}+1)c^{-1} + c \right) + \\ &+ pq(p+q) \left(\mathbf{28} + 2 + c^{-2} + (\mathbf{28}+1)c^{-1} + (\mathbf{28}+2)c \right) + \\ &+ (pq)^{\frac{3}{2}} \left(c^{-2} + \mathbf{28}c^{-1} - \mathbf{28}c \right) (\hat{t}^{\frac{1}{2}} + \hat{t}^{-\frac{1}{2}}) + \cdots \,, \end{aligned} \tag{3.112}$$

where $\mathbf{28}$ is the representation of $SO(8)$, which can also be thought of as the representations $\mathbf{26} + 1 + 1$ of F_4. From this expression we can see that if we compute the index with the $4d$ superconformal R-charge we get a pq term equal to $(\mathbf{28}+1)pq$, which doesn't contain a conserved current for F_4. If we assume a cancellation of the current due to marginal operators, we find that the conformal manifold is bigger than the one predicted from $6d$. The expansion of the index then can be re-arranged into characters of F_4 and there is no a priori contradiction with the conformal manifold having a locus on which the symmetry is enhanced to F_4. This is to be contrasted with the $N = 1$ case where with minimal flux $z = \frac{1}{2}$ the conformal manifold did not contain an F_4 locus [116].

We can also consider the theory corresponding to higher half-integer flux $z > \frac{1}{2}$ (see Figure 3.17 for the case $z = \frac{3}{2}$)

$$\begin{aligned} \mathcal{I}_{torus}^{(z=\frac{n}{2})} &= 1 + c^{2z} + 2c^{4z} + \cdots + qp \left(2z(\mathbf{28}-1)c^2 - 2z(\mathbf{28}-1)c + 2z\mathbf{28}c^{-1} + 2zc^{-2} \right) + \\ &+ pq(p+q) \left(2z\mathbf{28}c^{-1} + 2zc^{-2} - 2z(\mathbf{28}-1)c \right) + \\ &+ (pq)^{\frac{3}{2}} \left(2z\mathbf{28}c^{-1} + 2zc^{-2} - 2z\mathbf{28}c \right) (\hat{t}^{\frac{1}{2}} + \hat{t}^{-\frac{1}{2}}) + \cdots \,. \end{aligned} \tag{3.113}$$

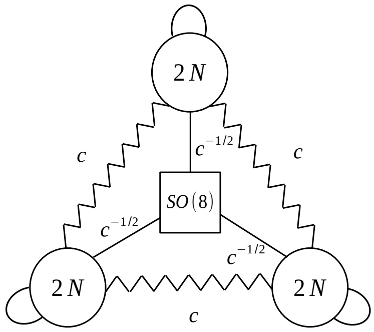

Figure 3.17: Gluing three basic tubes to form a tube with flux $\left(\frac{3}{2},\frac{3}{2},\frac{3}{2},\frac{3}{2},\frac{3}{2},\frac{3}{2},\frac{3}{2},\frac{3}{2}\right)$.

We notice that in this case there is no pq term corresponding to an operator uncharged under c. This means that computing the index with the $4d$ superconformal R-charge the pq term vanishes. Hence, we find no contradiction with the enhancement to F_4 on some point of the conformal manifold. Moreover, given that the mixing coefficient with $U(1)_c$ (3.97) is positive, computing the index with the $4d$ superconformal R-charge there will be no contribution from relevant fermionic operators [141].

Tori with $\mathcal{F} = (2n, 2n, 2n, 2n, 2n, 2n, 0, 0)$

Let us consider gluing two basic tubes with the element of $SO(16)$ Weyl symmetry group which flips two of its fugacities, that is

$$a = 1\dots 6 \ : \quad \eta_a = \xi_a\,, \qquad a = 7,8 \ : \quad \eta_a = \frac{1}{\xi_a}\,. \tag{3.114}$$

We split the fugacities into $SU(6)_u \times SU(2)_v \times U(1)_s \times U(1)_c$ as,

$$a = 1\dots 6 \ : \quad \eta_a = s^{\frac{1}{2}} c^{-\frac{1}{2}} u_a\,, \qquad a = 7,8 \ : \quad \eta_a = s^{-\frac{3}{2}} c^{-\frac{1}{2}} v_a\,, \tag{3.115}$$

$$a = 1\dots 6 \ : \quad \xi_a = s'^{\frac{1}{2}} c'^{-\frac{1}{2}} u'_a\,, \qquad a = 7,8 \ : \quad \xi_a = s'^{-\frac{3}{2}} c'^{-\frac{1}{2}} v'_a\,, \tag{3.116}$$

with $\prod_{a=1}^{6} u_a = \prod_{a=1}^{6} u'_a = \prod_{a=7}^{8} v_a = \prod_{a=7}^{8} v_a = 1$. Then the map between the charges (3.114) implies $c' = s^{-\frac{3}{2}} c^{\frac{1}{2}}$, $s' = s^{-\frac{1}{2}} c^{-\frac{1}{2}}$, $u_a = u'_a$ and $v_a = 1/v'_a$.

Since only two fugacities are flipped the gluing will involve six $USp(2N)$ fundamentals Φ_a, $a = 1, \cdots, 6$ as shown in Figure 3.18 and the flux associated to this tube will be

$$\left(\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2}\right) + \left(\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},-\frac{1}{2},-\frac{1}{2}\right) = (1,1,1,1,1,1,0,0)\,, \tag{3.117}$$

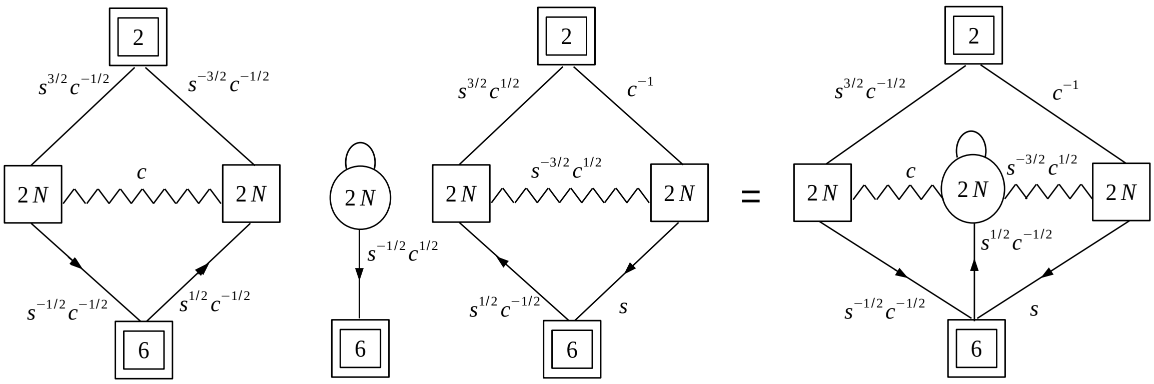

Figure 3.18: Gluing two basic tubes to form a tube with flux $(1,1,1,1,1,1,0,0)$. We avoid drawing arrows for lines connecting $USp(2N)$ nodes to $SU(2)$ nodes.

corresponding to half a unit of flux $z=\frac{1}{2}$ for the $U(1)$ whose commutant in E_8 is $E_6\times SU(2)$.

If we now glue $2n$ such tubes with a trivial element of the $SO(16)$ Weyl group we obtain the theory corresponding to the compactification on a torus with $z=n$ units of flux in this $U(1)$ and we expect the symmetry on some locus of the conformal manifold to be $SU(2)_L\times E_6\times SU(2)\times U(1)$. The torus theory is depicted in Figure 3.19. We proceed to perform some checks of the proposal.

Anomalies:

We first calculate the conformal anomalies for this torus theory and obtain:

$$a=\frac{1}{8}\sqrt{\frac{3}{2}}nN(3N+5)^{3/2}\,,\qquad c=\frac{1}{8}\sqrt{\frac{3}{2}}nN\sqrt{3N+5}(3N+7)\,. \tag{3.118}$$

This matches the six-dimensional prediction (3.3) for $SU(2)\times E_6$ preserving n units of flux, that is with $\xi_G=3$ and $z=n$.

Index:

We can compute the index. Again we will consider the case with rank $N=2$ and arbitrary flux $z=n$ for simplicity (for the case $N=1$ see [116]). Using the six-dimensional R-charge $\hat{R}$ and rescaling $t=\hat{t}^{\frac{1}{2}}$ we find

$$\begin{aligned}\mathcal{I}^{(z=n)}_{N=2}=&\,1+\cdots+pq\Big(3z(\mathbf{2},\mathbf{1})m^{-3}+2z(\mathbf{1},\mathbf{27})m^{-2}+z(\mathbf{2},\overline{\mathbf{27}})m^{-1}+\\&-2z(\mathbf{1},\overline{\mathbf{27}})m^{2}-z(\mathbf{2},\mathbf{27})m\Big)+pq(p+q)(3z(\mathbf{2},\mathbf{1})m^{-3}+2z(\mathbf{1},\mathbf{27})m^{-2}+\\&+z(\mathbf{2},\overline{\mathbf{27}})m^{-1}-z(\mathbf{2},\mathbf{27})m)+(pq)^{\frac{3}{2}}(3z(\mathbf{2},\mathbf{1})m^{-3}+2z(\mathbf{1},\mathbf{27})m^{-2}+\\&+z(\mathbf{2},\overline{\mathbf{27}})m^{-1}-z(\mathbf{2},\mathbf{27})m)(\hat{t}^{\frac{1}{2}}+\hat{t}^{-\frac{1}{2}})+\cdots\end{aligned} \tag{3.119}$$

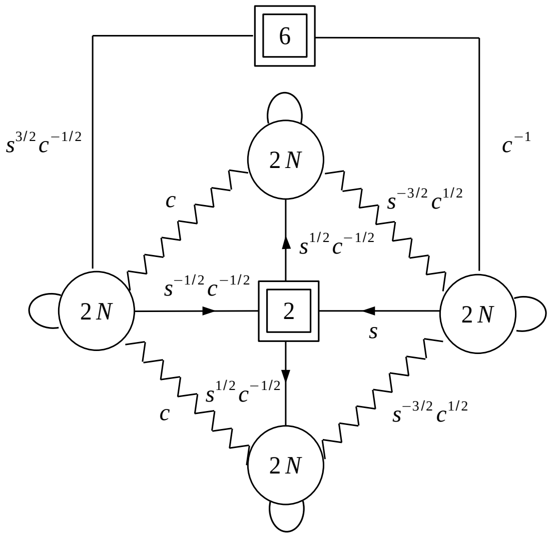

Figure 3.19: Gluing $2n$ $(1,1,1,1,1,1,0,0)$ tubes we obtain a torus with n units of flux preserving $SU(2)_t \times E_6 \times SU(2) \times U(1)$. Here we are depicting the case $n = 1$.

where we redefined the fugacities for the abelian symmetries with respect to the ones used in Figure 3.18 according to

$$c = m^{\frac{3}{2}} w^{-1}, \qquad s = m^{-\frac{1}{2}} w^{-1}, \tag{3.120}$$

to isolate $U(1)_w$ which is the one enhancing to $SU(2)$. In (3.119) we indicate by $(\cdot,\cdot)$ the characters of $SU(2) \times E_6$, for example $(\mathbf{2},\mathbf{1}) = w^2 + w^{-2}$. Then the $SU(2)_v \times SU(6)_u$ fugacities re-organize in the index in terms of characters of E_6 according to the branching rules

$$(\mathbf{1},\overline{\mathbf{27}}) = (\mathbf{2},\mathbf{6})_{SU(2)_v \times SU(6)_u} \oplus (\mathbf{1},\overline{\mathbf{15}})_{SU(2)_v \times SU(6)_u}. \tag{3.121}$$

We can also use the index result to compare with the $6d$ prediction of the spectrum. Since we are considering torus compactifications with flux breaking $E_8 \to U(1)_v \times SU(2) \times E_6$ we expect that the contribution to the index corresponding to the $6d$ conserved currents and energy momentum tensor will appear in the pq term of the index in the representations involved in the branching rule

$$\mathbf{248} \to (\mathbf{1},\mathbf{1})^0 \oplus (\mathbf{3},\mathbf{1})^0 \oplus (\mathbf{1},\mathbf{78})^0 \oplus (\mathbf{1},\mathbf{27})^2 \oplus (\mathbf{1},\overline{\mathbf{27}})^{-2} \oplus (\mathbf{2},\mathbf{27})^{-1} \oplus (\mathbf{2},\overline{\mathbf{27}})^1 \oplus (\mathbf{2},\mathbf{1})^{\pm 3}, \tag{3.122}$$

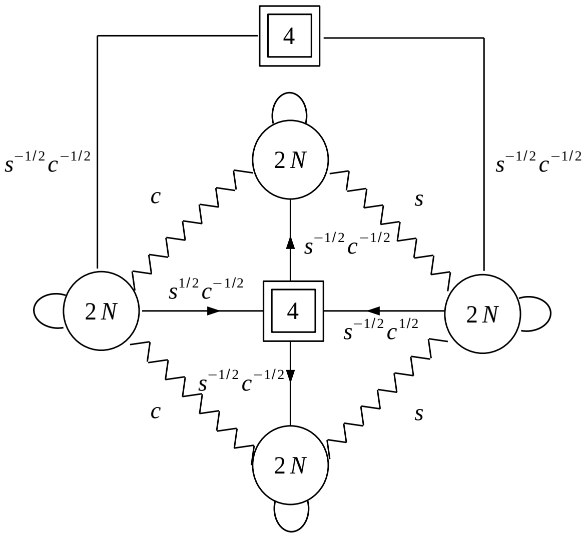

Figure 3.20: Gluing $2n$ $(1,1,1,1,0,0,0,0)$ tubes we obtain a torus with $z=n$ units of flux preserving $SU(2)_t\times SO(14)\times U(1)$. Here we are depicting the case $n=1$.

where the subscripts indicate the $U(1)_v$ charges. Indeed we see that in (3.119) at order pq we have operators in the $(\mathbf{1},\mathbf{27})^2$, $(\mathbf{1},\overline{\mathbf{27}})^{-2}$, $(\mathbf{2},\mathbf{27})^{-1}$, $(\mathbf{2},\overline{\mathbf{27}})^{1}$ and $(\mathbf{2},\mathbf{1})^3$. In particular, these appear with a coefficient determined by the value of the flux $z=n$ and their charge under $U(1)_v$. We again can think of

$$0=(\mathbf{1},\mathbf{1})^0+(\mathbf{3},\mathbf{1})^0+(\mathbf{1},\mathbf{78})^0+\mathbf{3}^0_{SU(2)_L}-\Big((\mathbf{1},\mathbf{1})^0+(\mathbf{3},\mathbf{1})^0+(\mathbf{1},\mathbf{78})^0+\mathbf{3}^0_{SU(2)_L}\Big) \tag{3.123}$$

as the cancellation of marginal operators and 4*d* conserved currents, which is compatible with the dimension of the conformal manifold predicted from 6*d*. We are again missing the $(\mathbf{2},\mathbf{1})^{-3}$ operator, and it will be interesting to understand the mechanism causing this.

Tori with $\mathcal{F}=(2n,2n,2n,2n,0,0,0,0)$

We can glue $2n$ tubes with $(1,1,1,1,0,0,0,0)$ fluxes given in Figure 3.13 with a trivial action of the $SO(16)$ Weyl group to construct tori with $z=n$ units of flux for the $U(1)$ whose commutant in E_8 is $SO(14)$ as shown in Figure 3.20.

Anomalies:

The conformal anomalies are given by:

$$a=\frac{1}{8}nN(3N+5)^{3/2}\,,\qquad c=\frac{1}{8}nN\sqrt{3N+5}(3N+7)\,. \tag{3.124}$$

This matches the six-dimensional prediction (3.3) for $SO(14)$ preserving n units of flux, that is with $\xi_G = 2$ and $z = n$.

Index:

Again we compute the index in the case of rank $N = 2$ for simplicity (for the case $N = 1$ see [116]) and generic flux. We use the six-dimensional R-charge $\hat{R}$ and rescale $t = \hat{t}^{\frac{1}{2}}$

$$\mathcal{I}^{(z=n)}_{N=2} = 1 + \cdots + pq(2z\mathbf{14}m^{-2} + z\mathbf{64}m^{-1} - z\mathbf{64}m) + \\ +pq(p+q)(2z\mathbf{14}m^{-2} + z\mathbf{64}m^{-1}) + (pq)^{\frac{3}{2}}(2z\mathbf{14}m^{-2} + z\mathbf{64}m^{-1})(\hat{t}^{\frac{1}{2}} + \hat{t}^{-\frac{1}{2}}) + \cdots . \tag{3.125}$$

where we redefined the fugacities for the abelian symmetries with respect to the ones used in Figure 3.13 according to

$$c = m\,w^{-1}, \qquad s = m\,w\,, \tag{3.126}$$

which is useful since the $U(1)_w$ symmetry is the one contributing to the enhancement to $SO(14)$ together with the two $SU(4)$ symmetries. Indeed, the corresponding fugacities re-organize in the index in terms of characters of $SO(14)$ according to the branching rules

$$\mathbf{14} = (\mathbf{1},\mathbf{1})^{\pm 2} \oplus (\mathbf{6},\mathbf{1})^0 \oplus (\mathbf{1},\mathbf{6})^0 \\ \mathbf{64} = (\mathbf{4},\overline{\mathbf{4}})^{\pm 1} \oplus (\overline{\mathbf{4}},\mathbf{4})^{\pm 1} . \tag{3.127}$$

We can also use the index result to compare with the $6d$ prediction of the spectrum. Since we are considering torus compactifications with flux breaking $E_8 \to U(1)_v \times SO(14)$ we expect that the contribution to the index of these operators will be in the representations appearing in

$$\mathbf{248} \to 1^0 \oplus \mathbf{91}^0 \oplus \mathbf{14}^{\pm 2} \oplus \mathbf{64}^{-} \oplus \overline{\mathbf{64}}^1 , \tag{3.128}$$

where the subscripts indicate the $U(1)_v$ charges. Indeed we see that in (3.126) at order pq we have operators in the $\mathbf{14}^2$, $\overline{\mathbf{64}}^1$ and $\mathbf{64}^{-1}$. We again can think of

$$0 = 1^0 + \mathbf{91}^0 + \mathbf{3}^0_{SU(2)_L} - \left(\mathbf{1}^0 + \mathbf{91}^0 + \mathbf{3}^0_{SU(2)_L}\right) \tag{3.129}$$

as the cancellation of marginal operators and conserved currents.

Tori with $\mathcal{F} = (2n, 2n, 0, 0, 0, 0, 0, 0)$ and the braid relation

Let us consider gluing two basic tubes with the element of $SO(16)$ Weyl symmetry group which flips six of its fugacities, that is

$$a = 1,2\ :\quad \eta_a = \xi_a\,, \qquad a = 3,\dots 8\ :\quad \eta_a = \frac{1}{\xi_a}\,. \tag{3.130}$$

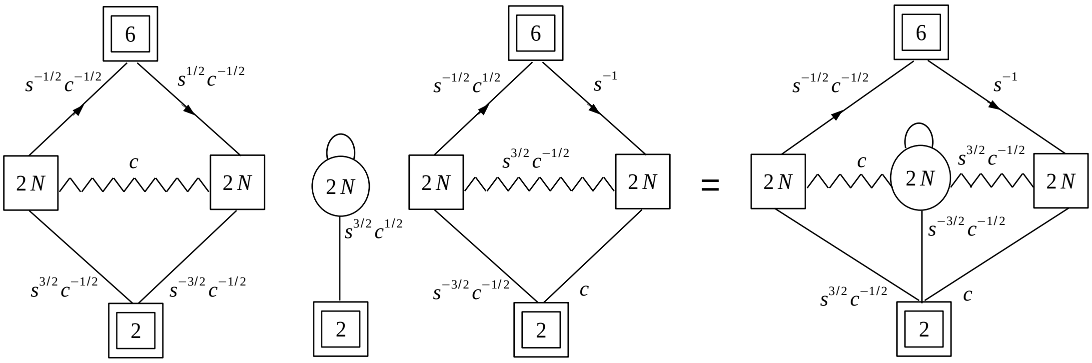

Figure 3.21: Gluing two basic tubes to form a tube with flux $(1,1,0,0,0,0,0,0)$.

We split the fugacities into $SU(2)_u \times SU(6)_v \times U(1)_s \times U(1)_c$ as,

$$a=1,2\ :\quad \eta_a = s^{-\frac{3}{2}}c^{-\frac{1}{2}}u_a\,,\qquad a=3\dots 8\ :\quad \eta_a = s^{\frac{3}{2}}c^{-\frac{1}{2}}v_a\,,\tag{3.131}$$

$$a=1,2\ :\quad \xi_a = s'^{-\frac{3}{2}}c'^{\frac{1}{2}}u'_a\,,\qquad a=3\dots 8\ :\quad \xi_a = s'^{\frac{3}{2}}c'^{-\frac{1}{2}}v'_a\,,\tag{3.132}$$

with $\prod_{a=1}^{2} u_a = \prod_{a=1}^{2} u'_a = \prod_{a=3}^{8} v_a = \prod_{a=3}^{8} v_a = 1$. Then the map between the fugacities η_a and ξ_a implies $c' = s^{\frac{3}{2}}c^{-\frac{1}{2}}$, $s' = s^{\frac{1}{2}}c^{\frac{1}{2}}$, $u_i = u'_i$ and $v_i = 1/v'_i$. Now six fugacities are flipped so the gluing will involve only two $USp(2N)$ fundamentals Φ_a, $a=1,2$, as shown in Figure 3.21.

Note that the flux is

$$\left(\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2}\right) + \left(\frac{1}{2},\frac{1}{2},-\frac{1}{2},-\frac{1}{2},-\frac{1}{2},-\frac{1}{2},-\frac{1}{2},-\frac{1}{2}\right) = (1,1,0,0,0,0,0,0)\,.\tag{3.133}$$

This is also an E_7 flux related to $(\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2})$ by the action of the Weyl symmetry. Thus in order for the picture to be consistent the torus theories obtained by gluing either type of tubes have to be equivalent. This is indeed the case due to the braid relation discussed in Section 3.3.5.

For example, as shown in Figure 3.22, if we glue two $(1,1,0,0,0,0,0,0)$ tubes and apply twice the braid relation we obtain the torus with $(1,1,1,1,1,1,1,1)$ flux, provided that we redefine the fugacities as

$$v_a = \tilde{v}_a s^{-1/4} c^{1/4}\,,\qquad u_a = \tilde{u}_a s^{3/4} c^{-3/4}\,,\tag{3.134}$$

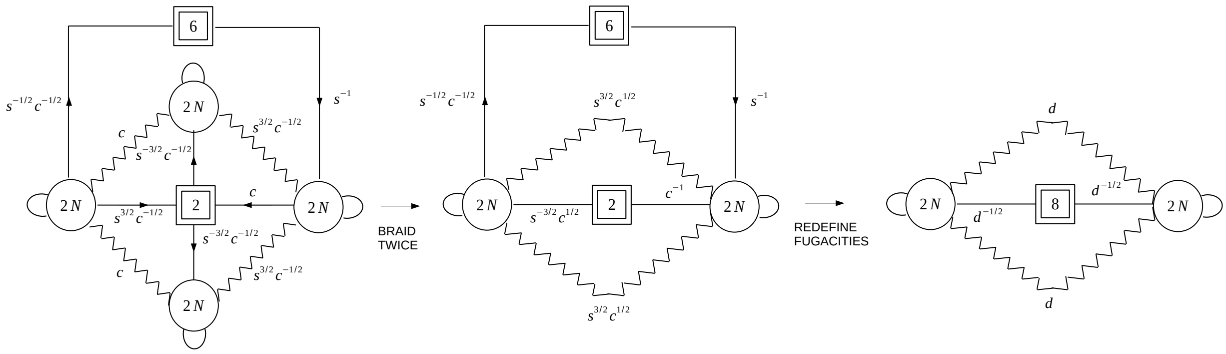

Figure 3.22: On the l.h.s. the torus with flux $(2,2,0,0,0,0,0,0)$. We apply twice the braid relation to obtain the quiver in the middle. Finally by redefining the fugacities we obtain the quiver on the r.h.s. corresponding to flux $(1,1,1,1,1,1,1,1)$.

which recombine into the $SU(8)_x$ fugacities $x_a = \tilde{u}_a$ for $a = 1,2$ and $x_a = \tilde{v}_a$ for $a = 3,\cdots 8$ satisfying $\prod_{a=1}^{8} x_a = 1$ and we define the remaining $U(1)$ fugacity $d = s^{3/2}c^{1/2}$.

Anomalies:

Gluing $2n$ tubes together we can easily compute the conformal anomalies and obtain that they give us

$$a = \frac{1}{8}\sqrt{\frac{1}{2}}nN(3N+5)^{3/2}\,, \qquad c = \frac{1}{8}\sqrt{\frac{1}{2}}nN\sqrt{3N+5}(3N+7)\,. \tag{3.135}$$

This matches the six-dimensional prediction (3.3) for E_7 preserving n units of flux, that is with $\xi_G = 1$ and $z = n$.

Index:

As we have seen the braid relation (3.66) guarantees that the index of this theory is the same as the one of the E_7 torus.

3.5 E-string compactifications on spheres

3.5.1 Derivation of the basic cap model

In this section we are going to construct the theory corresponding to the compactification of the rank N E-string theory on a one punctured sphere with some value of flux for the $E_8 \times SU(2)_L$ global symmetry, which we shall call the *cap model*. This will be, together with the basic tube theory that we studied in the previous section, one of the fundamental building blocks using which we will construct theories corresponding to compactifications on spheres with fluxes.

The key idea to derive the cap model is to start from the theory obtained by compactification on a tube and completely close one of the two punctures. The way the closure of the puncture is implemented in field theory is analogous to what is usually done in class $\mathcal{S}$ theories [10]. In that context, each puncture carries a flavor symmetry and thanks to $\mathcal{N}=2$ supersymmetry we have moment map operators that contain the conserved currents for such symmetries. The puncture can then be completely or partially closed by giving a nilpotent VEV to the moment map operator, which breaks the global symmetry of the puncture to some subgroup [153, 154]. In our case, since we only have $\mathcal{N}=1$ supersymmetry we don't have true moment map operators for the symmetry of the puncture, but we as we already discussed we still have some operators that transform non-trivially under this symmetry and to which we can give a VEV to break it. These operators can be identified with the $5d$ matter fields assigned Neumann boundary conditions at the puncture in the $5d$ effective description of the puncture, see *e.g.* [116]. We recall that the relevant $5d$ description of the rank N E-string is in terms of $USp(2N)$ $\mathcal{N}=1$ gauge theory with eight fundamental hypermultiplets as well as a hypermultiplet in traceless antisymmetric representation [132]. This matter content leads to a $USp(2N)$ global symmetry associated to the puncture in $4d$ and to an octet of $\mathcal{N}=1$ chiral operators in the fundamental representation of this symmetry as well as an $\mathcal{N}=1$ chiral operator in the traceless antisymmetric representation. With a little abuse of terminology, we will still refer to these operators as "moment maps".

Our starting point is thus the model obtained from the compactification of the rank N E-string theory on a tube with fluxes. We analyzed this at length in the previous section, where we saw that the simplest model is the one corresponding to flux $\mathcal{F}_{tube}=(0;\frac{1}{2};0,\cdots,0)$ in the overcomplete basis, where now we are also specifying in the first entry the flux for the Cartan of the $SU(2)_L$ part of the global symmetry, and that more general tubes associated to different choices of flux can be derived by various gluings of several copies of this basic one. Recall that the manifest non-anomalous global symmetry of the tube model is

$$USp(2N)_x\times USp(2N)_y\times U(1)_c\times U(1)_t\times U(1)_f\times SU(8)_u\,, \tag{3.136}$$

The two $USp(2N)$ symmetries are associated to the two punctures, while the rest, except for the $U(1)_f$ to be discussed momentarily, is the residual $6d$ symmetry that is preserved by the compactification. In particular $U(1)_t\subset SU(2)_L$, while $U(1)_c\times SU(8)_u\subset E_8$. In the next subsection we will see that in the construction of theories corresponding to $\mathbb{S}^2$ compactifications the $U(1)_f$ symmetry is to be identified, upon properly mixing it with other $U(1)$ symmetries, with the Cartan of the $SU(2)_{\text{ISO}}$ isometry of $\mathbb{S}^2$. In particular it is natural then to think about the $U(1)_f$ symmetry of the tube to be associated with its isometry, namely to be related to the KK symmetry of the compactification of the $6d$ SCFT on a circle[19]. We also recall here that the supersymmetric index of the tube theory is

$$\begin{aligned}\mathcal{I}_{tube}(\vec{x};\vec{y};c;t;f;\vec{u}) &= \prod_{i=1}^{N}\prod_{a=1}^{8}\Gamma_e\left((pq)^{\frac{1}{2}}c^{-\frac{1}{2}}f^{-\frac{1}{4}}u_a x_i^{\pm 1}\right)\Gamma_e\left((pq)^{\frac{1}{2}}c^{-\frac{1}{2}}f^{\frac{1}{4}}u_a^{-1}y_i^{\pm 1}\right)\times \\ &\times \mathcal{I}_{FE[USp(2N)]}(\vec{x};\vec{y};c;t)\,, \end{aligned} \quad (3.137)$$

The tube theory possesses two types of moment map operators that transform under the symmetries of the punctures:

- The chiral fields contained in the two octets M and M', which transform in the fundamental representation of $USp(2N)_x$ and $USp(2N)_y$ respectively;
- The operators $\mathsf{O_H}$ and C of $FE[USp(2N)]$, which transform in the traceless antisymmetric representation of $USp(2N)_x$ and $USp(2N)_y$ respectively.

We would like to give VEV or linear superpotential interaction to some of these operators so to completely break the symmetry of one of the two punctures, say $USp(2N)_x$. We will do so by first turning on linearly in the superpotential the traceless antisymmetric operator $\mathsf{O_H}$, which induces a VEV for the mesonic operator H. Such a VEV can at most break the $USp(2N)_x$ symmetry down to $SU(2)_v$. Hence, we will then need a VEV for one of the octet fields to further break $SU(2)_v$.

We first deform the tube model by turning on linearly in the superpotential the operator $\mathsf{O_H}$ as follows:

$$\delta\mathcal{W} = \mathsf{J}_N\mathsf{O_H}\,, \quad (3.138)$$

where

$$\mathsf{J}_N = \frac{i\sigma_2}{2}\otimes\left(J_N + J_N^T\right) \quad (3.139)$$

and J_N is the Jordan matrix of dimension N. Recall that the operator $\mathsf{O_H}$ of $FE[USp(2N)]$ is actually a matrix of singlets flipping the meson H. Hence, this deformation implies a VEV for H that can be understood by looking at the equations of motion of $\mathsf{O_H}$[20]

$$\langle\mathsf{H}\rangle = \mathsf{J}_N\,. \quad (3.140)$$

[19]Let us also mention again here that the $U(1)_f$ symmetry is broken by anomalies/superpotentials when one constructs theories corresponding to a torus. This breaking leaves behind a discrete group, whose order depends on the flux. It would be very interesting to understand precisely the relation between $U(1)_f$ and the KK symmetry and the nature of this discrete subgroup. We leave this for future investigations.

[20]A more general class of VEVs for the antisymmetric operators of $FE[USp(2N)]$ has been studied in [145]. We will study these other VEVs in the next chapter, while we will be only interested here in completely breaking the puncture symmetry, that is producing *no puncture*. We thus focus on the VEV preserving the minimal $SU(2)_v$ symmetry which we then completely break with an octet VEV.

We already mentioned this type of VEV in the $FE[USp(2N)]$ theory in Subsection 3.3.4. There we saw that the effect of this VEV can be more easily understood at the level of the supersymmetric index, where it imposes the following constraints on the fugacities:

$$x_i = t^{i-1}v, \quad i = 1, \cdots, N\,. \tag{3.141}$$

Using the identity (3.65), which was proven in Corollary 2.8 of [79], we find that the supersymmetric index of the tube theory after such a specification of the fugacities reduces to

$$\begin{aligned}
&\mathcal{I}_{tube}(v, t\,v, \cdots, t^{N-1}v; \vec{y}; c; t; f; \vec{u}) = \\
&\qquad = \prod_{i=1}^{N}\prod_{a=1}^{8}\Gamma_e\left((pq)^{\frac{1}{2}}c^{-\frac{1}{2}}u_a(t^{i-1}v)^{\pm1}\right)\Gamma_e\left((pq)^{\frac{1}{2}}c^{-\frac{1}{2}}u_a^{-1}y_i^{\pm1}\right) \\
&\qquad \times\prod_{j=2}^{N}\frac{1}{\Gamma_e\left(t^j\right)}\prod_{i=1}^{N}\frac{\Gamma_e\left(v\,c\,y_i^{\pm1}\right)\Gamma_e\left(v^{-1}c\,t^{1-N}y_i^{\pm1}\right)}{\Gamma_e\left(c^2t^{1-i}\right)}\,.
\end{aligned} \tag{3.142}$$

In order to make the residual $SU(2)_v$ symmetry manifest we have to perform the redefinition $v \to t^{\frac{1-N}{2}}v$

$$\begin{aligned}
&\mathcal{I}_{tube}(t^{\frac{1-N}{2}}v, t^{\frac{3-N}{2}}v, \cdots, t^{\frac{QN1}{2}}v; \vec{y}; c; t; f; \vec{u}) = \\
&\qquad = \prod_{i=1}^{N}\prod_{a=1}^{8}\Gamma_e\left((pq)^{\frac{1}{2}}c^{-\frac{1}{2}}t^{\frac{N-2i+1}{2}}v^{\pm1}f^{-\frac{1}{4}}u_a\right)\Gamma_e\left((pq)^{\frac{1}{2}}c^{-\frac{1}{2}}f^{\frac{1}{4}}u_a^{-1}y_i^{\pm1}\right)\times \\
&\qquad \times\prod_{j=2}^{N}\Gamma_e\left(pq\,t^{-j}\right)\prod_{i=1}^{N}\Gamma_e\left(pq\,c^{-2}t^{i-1}\right)\Gamma_e\left(c\,t^{\frac{1-N}{2}}v^{\pm1}y_i^{\pm1}\right)\,.
\end{aligned} \tag{3.143}$$

The next step consists of breaking also the $SU(2)_v$ symmetry by giving a VEV to one of the octet fields represented now in the index by $\prod_{i=1}^{N}\prod_{a=1}^{8}\Gamma_e\left((pq)^{\frac{1}{2}}c^{-\frac{1}{2}}t^{\frac{N-2i+1}{2}}v^{\pm1}f^{-\frac{1}{4}}u_a\right)$. It turns out that the correct choice to reproduce the anomalies of the cap model predicted from 6d is to give VEV to the field corresponding to $i = 1$ and any $a = 1, \cdots, 8$. For definiteness we shall choose $a = 8$. Then, the VEV implies the following constraint on the fugacities of the index:

$$v = (pq)^{\frac{1}{2}}c^{-\frac{1}{2}}t^{\frac{N-1}{2}}f^{-\frac{1}{4}}u_8\,. \tag{3.144}$$

The index (3.143) after giving such a VEV becomes[21]

$$\mathcal{I} \;=\; \prod_{j=2}^{N}\Gamma_e\left(t^{1-j}\right)\Gamma_e\left(pq\,t^{-j}\right)\prod_{i=1}^{N}\Gamma_e\left(pq\,c^{-2}t^{i-1}\right)\Gamma_e\left(pq\,c^{-1}t^{N-i}f^{-\frac{1}{2}}u_8^2\right)\times$$

[21]For future convenience we are not simplifying in these expressions the contributions of some of the massive fields.

$$\times \quad \Gamma_e\left((pq)^{\frac{1}{2}}c^{-\frac{1}{2}}f^{\frac{1}{4}}u_8^{-1}y_i^{\pm 1}\right)\Gamma_e\left((pq)^{\frac{1}{2}}c^{\frac{1}{2}}f^{-\frac{1}{4}}u_8y_i^{\pm 1}\right)\Gamma_e\left((pq)^{-\frac{1}{2}}c^{\frac{3}{2}}t^{1-N}f^{\frac{1}{4}}u_8^{-1}y_i^{\pm 1}\right)\times$$
$$\times \quad \prod_{a=1}^{7}\Gamma_e\left(pq\,c^{-1}t^{N-i}f^{-\frac{1}{2}}u_8u_a\right)\Gamma_e\left(t^{1-i}u_8^{-1}u_a\right)\Gamma_e\left((pq)^{\frac{1}{2}}c^{-\frac{1}{2}}f^{\frac{1}{4}}u_a^{-1}y_i^{\pm 1}\right)\,. \tag{3.145}$$

This is not the index of the cap model that we are looking for yet. Indeed, in general one may need to introduce additional gauge singlet chiral fields that flip some of the operators of the theory in order to get the correct model corresponding to the compactification of the $6d$ theory. The singlets that need to be added can be worked out by requiring that the anomalies of the resulting model match those predicted from $6d$ for a compactification on a one-punctured sphere with some value of the flux, whose expression we gave in Subsection 3.2.1. This was already noticed in Section 6 of [116] for the rank $N=1$ case, where the symmetry carried by each puncture is just $SU(2)$ so only the octet VEV was needed in order to close it. It turns out that in the higher rank case some of the operators that we need to flip are just straightforward generalizations of those worked out for rank 1, while the others only appear for $N>1$ [97]. The complete list of singlets that we have to add is the following (again encoded in their contributions to the index):

$$\begin{aligned}
&\prod_{i=1}^{N}\prod_{a=1}^{7}\Gamma_e\left(pq\,t^{i-1}u_8u_a^{-1}\right)\\
&\prod_{i=1}^{N}\Gamma_e\left(c\,t^{i-N}f^{\frac{1}{2}}u_8^{-2}\right)\\
&\prod_{j=2}^{N}\Gamma_e\left(pq\,t^{j-1}\right)\Gamma_e\left(t^{j}\right)\\
&\Gamma_e\left(t\right)^{N-1}\prod_{i<j}^{Q}\Gamma_e\left(t\,y_i^{\pm 1}y_j^{\pm 1}\right)\,.
\end{aligned} \tag{3.146}$$

In addition, we make a shift of fugacity $c\to cf^{\frac{1}{2}}$ for later convenience. The index of the resulting cap model is thus

$$\begin{aligned}
\mathcal{I}_{cap}(\vec{y};c;t;f;\vec{u};u_8) = &\underbrace{\Gamma_e\left(t\right)^{N-1}\prod_{i<j}^{N}\Gamma_e\left(t\,y_i^{\pm 1}y_j^{\pm 1}\right)}_{A}\prod_{i=1}^{N}\underbrace{\Gamma_e\left(pq\,c^{-2}t^{i-1}f^{-1}\right)}_{\beta_i}\times\\
&\times\underbrace{\Gamma_e\left((pq)^{-\frac{1}{2}}c^{\frac{3}{2}}t^{1-N}fu_8^{-1}y_i^{\pm 1}\right)}_{P}\underbrace{\Gamma_e\left((pq)^{\frac{1}{2}}c^{-\frac{1}{2}}u_8^{-1}y_i^{\pm 1}\right)}_{L_8}\underbrace{\Gamma_e\left((pq)^{\frac{1}{2}}c^{\frac{1}{2}}u_8y_i^{\pm 1}\right)}_{K}\times\\
&\times\prod_{a=1}^{7}\underbrace{\Gamma_e\left(pq\,c^{-1}t^{N-i}f^{-1}u_8u_a\right)}_{R^{(i)a}}\underbrace{\Gamma_e\left((pq)^{\frac{1}{2}}c^{-\frac{1}{2}}u_a^{-1}y_i^{\pm 1}\right)}_{L_a}\,.
\end{aligned} \tag{3.147}$$

We can check that the model we obtained after the VEVs and the addition of the singlets indeed corresponds to the compactification of the rank N E-string theory on a sphere with one puncture for some value of flux by computing its anomalies and comparing with those that can be predicted from $6d$, which we reviewed in Subsection 3.2.1 and derived in Appendix D. This anomaly matching is a necessary condition for the duality between the $4d$ model we propose and the $6d$ theory compactified on the $2d$ surface. In addition, we also provide further evidence in the next subsection that one can construct various $4d$ models corresponding to the E-string theory on a sphere with different fluxes by gluing the cap models we propose here. Those $4d$ models exhibit the expected (enhanced) symmetries and the spectrums of operators perfectly consistent with the $6d$ theory compactified on a sphere with a given flux, which is strong evidence of our conjecture that the proposed cap model indeed corresponds to the E-string theory compactified on a one-punctured sphere, or a cap.

Remembering the constraint $u_8 = \prod_{a=1}^{7} u_a^{-1}$, we first find the following linear anomalies for the $U(1)$'s in the Cartan of the original $6d$ $E_8 \times SU(2)_L$ symmetry:

$$\begin{aligned} &\mathrm{Tr}\, U(1)_R = 2N(N+1), \quad \mathrm{Tr}\, U(1)_t = (N-1)(4N+1), \\ &\mathrm{Tr}\, U(1)_c = -13N, \qquad \mathrm{Tr}\, U(1)_{u_a} = -6N \quad a = 1, \cdots, 7\,. \end{aligned} \tag{3.148}$$

For the cubic non-mixed anomalies we have

$$\begin{aligned} &\mathrm{Tr}\, U(1)_R^3 = -2N(N+1)^2, \quad \mathrm{Tr}\, U(1)_t^3 = (N-1)(2N^2+1), \\ &\mathrm{Tr}\, U(1)_c^3 = -10N, \qquad \mathrm{Tr}\, U(1)_{u_a}^3 = -6N\,. \end{aligned} \tag{3.149}$$

Next, we list all the cubic mixed anomalies for the abelian symmetries

$$\begin{aligned} &\mathrm{Tr}\, U(1)_R^2 U(1)_t = -\frac{2}{3}N(N^2-1), \ \mathrm{Tr}\, U(1)_R^2 U(1)_c = \frac{3}{2}N(N+1), \\ &\mathrm{Tr}\, U(1)_R^2 U(1)_{u_a} = N(N+1), \ \mathrm{Tr}\, U(1)_t^2 U(1)_R = \frac{2}{3}N(N^2-1), \\ &\mathrm{Tr}\, U(1)_t^2 U(1)_c = -\frac{3}{2}N(N-1), \ \mathrm{Tr}\, U(1)_t^2 U(1)_{u_a} = -N(N-1), \\ &\mathrm{Tr}\, U(1)_c^2 U(1)_R = N(N+1), \ \mathrm{Tr}\, U(1)_c^2 U(1)_t = N(N-1), \\ &\mathrm{Tr}\, U(1)_c^2 U(1)_{u_a} = -2N, \ \mathrm{Tr}\, U(1)_{u_a}^2 U(1)_R = N(N+1), \\ &\mathrm{Tr}\, U(1)_{u_a}^2 U(1)_t = N(N-1), \ \mathrm{Tr}\, U(1)_{u_a}^2 U(1)_c = -4N, \\ &\mathrm{Tr}\, U(1)_{u_a} U(1)_{u_b}^2 = -3N, \quad \mathrm{Tr}\, U(1)_R U(1)_{u_a} U(1)_{u_b} = \frac{1}{2}N(N+1), \\ &\mathrm{Tr}\, U(1)_t U(1)_{u_a} U(1)_{u_b} = \frac{1}{2}N(N-1), \quad \mathrm{Tr}\, U(1)_c U(1)_{u_a} U(1)_{u_b} = -2N \qquad a \neq b \\ &\mathrm{Tr}\, U(1)_{u_a} U(1)_{u_b} U(1)_{u_d} = -2N, \qquad a \neq b \neq d \neq a\,. \end{aligned} \tag{3.150}$$

Finally, we have the anomalies between these $U(1)$ symmetries and the $USp(2N)$ symmetry of the puncture

$$\begin{aligned}&\operatorname{Tr} U(1)_R USp(2N)^2 = -\frac{N+1}{2}, \quad \operatorname{Tr} U(1)_t USp(2N)^2 = -\frac{N-1}{2},\\ &\operatorname{Tr} U(1)_c USp(2N)^2 = -1, \quad \operatorname{Tr} U(1)_{u_a} USp(2N)^2 = 0\,.\end{aligned} \tag{3.151}$$

All of these anomalies perfectly match those that one can compute from $6d$ using equations (3.4)-(3.6)-(3.7)-(3.8) for a sphere with one puncture and flux $\mathcal{F}_{cap} = \left(-\frac{1}{2}; \frac{3}{4}; \frac{1}{8}, \cdots, \frac{1}{8}, -\frac{7}{8}\right)$. Comparing with the flux of the original tube model $\mathcal{F}_{tube} = (0; \frac{1}{2}; 0, \cdots, 0)$, we notice that the effect of the VEVs has been to shift the entries of the flux vector. In particular, $n_c \to n_c + \frac{1}{4}$, $n_8 \to n_8 - \frac{7}{8}$ and $n_a \to n_a + \frac{1}{8}$ for $a = 1, \cdots, 7$. This is the same shift for the E_8 fluxes found in [116] for the rank $N = 1$ case, so we interpret it as the effect of the octet VEV. Moreover, we also have that the flux for $SU(2)_L$ has been shifted by $n_t \to n_t - \frac{1}{2}$, which we instead interpret as the effect of the antisymmetric VEV.

In order to interpret the model we obtained, it is useful to redefine the fugacities in the index (3.147) in such a way that they conform to the new manifest global symmetry, which is

$$USp(2N) \times SU(7)_u \times U(1)_x \times U(1)_c \times U(1)_t \times U(1)_f\,. \tag{3.152}$$

This is achieved by the shifts

$$\begin{aligned} u_{a=1,\dots,7} &\longrightarrow x^{-1} u_a\,,\\ u_8 &\longrightarrow x^7 \end{aligned} \tag{3.153}$$

with the new u_a on the right hand side satisfying $\prod_{a=1}^7 u_a = 1$. The index thus reads

$$\begin{aligned}&\mathcal{I}_{cap}(\vec{y}; c; t; f; x^{-1}u_1, \cdots, x^{-1}u_7; x) =\\ &\times \underbrace{\Gamma_e(t)^{N-1} \prod_{i<j}^N \Gamma_e\left(t\, y_i^{\pm 1} y_j^{\pm 1}\right)}_{A} \prod_{i=1}^N \underbrace{\Gamma_e\left(pq\, c^{-2} t^{i-1} f^{-1}\right)}_{\beta_i} \underbrace{\Gamma_e\left((pq)^{-\frac{1}{2}} c^{\frac{3}{2}} t^{1-N} f x^{-7} y_i^{\pm 1}\right)}_{P} \times\\ &\times \underbrace{\Gamma_e\left((pq)^{\frac{1}{2}} c^{-\frac{1}{2}} x^{-7} y_i^{\pm 1}\right)}_{L_8} \underbrace{\Gamma_e\left((pq)^{\frac{1}{2}} c^{\frac{1}{2}} x^{7} y_i^{\pm 1}\right)}_{K} \times\\ &\times \prod_{a=1}^7 \underbrace{\Gamma_e\left(pq\, c^{-1} t^{N-i} f^{-1} x^{-6} u_a\right)}_{R^{(n)a}} \underbrace{\Gamma_e\left((pq)^{\frac{1}{2}} c^{-\frac{1}{2}} x\, u_a^{-1} y_i^{\pm 1}\right)}_{L_a}\,.\end{aligned} \tag{3.154}$$

Note that the cap model is a simple WZ model. Of course this flows to a collection of free fields in the IR. However, the various superpotentials of the model are constraining the global symmetry of the theory. When one glues the cap model to other theories by gauging the $USp(2N)$ symmetry these superpotentials become important. The massless fields of the WZ model can be schematically represented with the following quiver diagram:

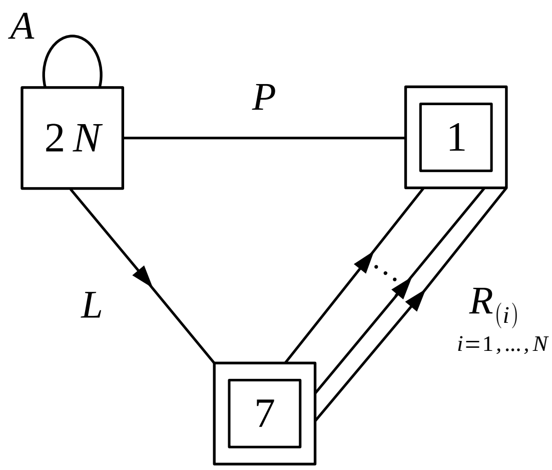

Figure 3.23: The cap theory. This is just a WZ model with the superpotentials detailed in (3.155).

Here as before the simple square box represents the $USp(2N)$ flavor symmetry of the remaining puncture, while double square boxes represent unitary flavor symmetries out of which we have to mod out an overall $U(1)$. In particular the square box with the 1 inside represents the $U(1)_x$ flavor symmetry associated to the fugacity x in the index, while the square box with the 7 inside represents the $SU(7)_u$ flavor symmetry associated to the fugacities u_a in the index. On top of the fields represented in the quiver, we also have the singlets under the non-abelian symmetries β_i and the two massive fields L_8 and K, which we prefer to keep and not integrate out in order to write the superpotential in a more pleasant way:

$$\mathcal{W}_{cap} = \sum_{i=1}^{N}\sum_{a=1}^{7} \text{Tr}_y \left(R^{(i)a} A^{N-i} P L_a\right) + \sum_{i=1}^{N} \text{Tr}_y \left(\beta_i A^{N-i} P K\right) + K L_8 \,. \tag{3.155}$$

Integrating L_8 and K out we obtain the field content of the previous quiver and the interaction superpotential between the remaining massless fields. We keep here the massive fields as using those it will be more convenient to define the procedure of gluing the cap to other models.

3.5.2 Sphere compactifications

In the previous subsection we have obtained the model corresponding to the one-punctured sphere and here we will use it to construct several examples of theories which we will associate to compactifications on $\mathbb{S}^2$. In particular we will see how various dualities and emergence of symmetry phenomena naturally arise in this construction. Moreover we will directly identify the $U(1)_f$ symmetry with the Cartan generator of the $SU(2)_{\rm ISO}$ geometric symmetry of $\mathbb{S}^2$[22].

$SU(8)\times U(1)^2$ and $SU(8)\times SU(2)\times U(1)$ sphere models

Our first example is the $4d$ model corresponding to the compactification of the rank N E-string theory on a sphere with flux in the $SU(8)\times U(1)_c$ overcomplete basis

$$\mathcal{F}=\left(-1;\frac{3}{2};\frac{1}{4},\frac{1}{4},\frac{1}{4},\frac{1}{4},\frac{1}{4},\frac{1}{4},\frac{1}{4},-\frac{7}{4}\right). \tag{3.156}$$

We choose to switch to the complete basis since the gluing rules we consider will be written in a simpler way than in the overcomplete basis. Using the transformation rule between the two bases, which we recall are

$$n_a^{SO(16)}=n^{U(1)}+2n_a^{SU(8)}\,,\qquad\qquad a=1,\dots,8\,, \tag{3.157}$$

the flux (3.156) can be written in the $SO(16)$ basis as follows:

$$\mathcal{F}=(-1;2,2,2,2,2,2,2,-2)\,. \tag{3.158}$$

One can easily check that this flux preserves $SU(8)\times U(1)_b\times U(1)_t\subset E_8\times SU(2)_L$.

A sphere with such a flux can be constructed from two basic caps, whose flux we recall is given by

$$\mathcal{F}^{(1)}=\mathcal{F}^{(2)}=\left(-\frac{1}{2};\frac{3}{4};\frac{1}{8},\frac{1}{8},\frac{1}{8},\frac{1}{8},\frac{1}{8},\frac{1}{8},\frac{1}{8},-\frac{7}{8}\right) \tag{3.159}$$

in the overcomplete basis or

$$\mathcal{F}^{(1)}=\mathcal{F}^{(2)}=\left(-\frac{1}{2};1,1,1,1,1,1,1,-1\right) \tag{3.160}$$

in the complete basis. We remind the reader that the geometric action of gluing has the following field theoretic interpretation. We take two theories with punctures with symmetry $USp(2N)$ and associated "moment map" operators M_a and M'_a respectively. Then for each component of the moment map we have the choice whether to Φ-glue or S-glue. The former choice amounts to adding a chiral field Φ_a in the fundamental representation of $USp(2N)$

[22]For simplicity we will only consider gluing pair of cap theories together in various ways without introducing tube theories. Technically this makes it simpler to perform explicitly various index checks. As the tube theory has one of the $USp(2N)$ symmetries emergent in the IR, adding it would entail gauging of IR emergent symmetries: this is not an issue conceptually, but as we refrain from doing so the dualities we will arrive at are standard dualities between completely Lagrangian theories.

and turning on the superpotential

$$\Delta W_a = \Phi_a \cdot (M_a - M'_a) \,. \tag{3.161}$$

The latter choice amounts only to turning on a superpotential without adding any additional field

$$\Delta W_a = M_a \cdot M'_a \,. \tag{3.162}$$

Similar operations can be performed for the traceless antisymmetric operators. The fundamental moment maps are charged under the Cartan of $SO(16)$ while the traceless antisymmetric moment map is charged under the Cartan of $SU(2)_L$. To obtain the fluxes of the glued theory we join the fluxes of the two theories depending on the gluing. If a moment map component is S-glued we subtract the fluxes of the symmetry it is charged under while we add them for Φ-gluing

$$n'_a = \begin{cases} n_a^{(1)} + n_a^{(2)} \,, & a \in \Phi \,, \\ n_a^{(1)} - n_a^{(2)} \,, & a \in S \,, \end{cases} \tag{3.163}$$

where Φ and S denotes the Φ-gluing and the S-gluing respectively. The $SO(16)$ basis is thus more convenient to discuss the gluings. Hence, the two basic caps glued with the Φ-gluing for all the moment map components simply leads to a sphere with the flux

$$\mathcal{F} = \mathcal{F}^{(1)} + \mathcal{F}^{(2)} = (-1; 2, 2, 2, 2, 2, 2, 2, -2) \,, \tag{3.164}$$

which is exactly (3.158).

The resulting 4d model is thus expected to preserve $SU(8)_v \times U(1)_b \times U(1)_t$, which stems from $E_8 \times SU(2)_t$ in 6d. Moreover, on top of $SU(8)_v \times U(1)_b \times U(1)_t$, this theory also exhibits the $SU(2)_f$ symmetry which we will argue comes from the isometry of the two-sphere on which we compactified the 6d theory. Therefore, the total global symmetry of the model is given by

$$SU(8)_v \times SU(2)_f \times U(1)_b \times U(1)_t \,, \tag{3.165}$$

which we will confirm using the supersymmetric index.

To construct the model corresponding to the flux (3.158), we first recall that the basic cap is given by the WZ model with the superpotential

$$\mathcal{W}_{cap} = \sum_{i=1}^{N} \sum_{a=1}^{7} \text{Tr}_z \left(R^{(i)a} A^{N-i} P L_a \right) + \sum_{i=1}^{N} \text{Tr}_z \left(\beta_i A^{N-i} P K \right) + K L_8 \,, \tag{3.166}$$

which preserves

$$USp(2N) \times SU(7)_u \times U(1)_x \times U(1)_c \times U(1)_t \times U(1)_f \,. \tag{3.167}$$

The index of the basic cap is given by

$$\begin{aligned}
&\mathcal{I}_{cap}(\vec{y};c;t;f;u_1,\dots,u_7;x) = \\
&= \underbrace{\Gamma_e\left(t\right)^{N-1}\prod_{i<j}^{N}\Gamma_e\left(ty_i^{\pm1}y_j^{\pm1}\right)}_{A}\prod_{i=1}^{N}\underbrace{\Gamma_e\left(pqt^{i-1}c^{-2}f^{-1}\right)}_{\beta_i}\underbrace{\prod_{i=1}^{N}\Gamma_e\left((pq)^{-\frac{1}{2}}t^{1-N}c^{\frac{3}{2}}fx^{-7}y_i^{\pm1}\right)}_{P}\times \\
&\times\prod_{a=1}^{7}\underbrace{\prod_{i=1}^{N}\Gamma_e\left((pq)^{\frac{1}{2}}c^{-\frac{1}{2}}xu_a^{-1}y_i^{\pm1}\right)}_{L_a}\prod_{i=1}^{N}\prod_{a=1}^{7}\underbrace{\Gamma_e\left(pqt^{i-1}c^{-1}f^{-1}x^6u_a\right)}_{R^{(i)\,a}}\times \\
&\times\underbrace{\prod_{i=1}^{N}\Gamma_e\left((pq)^{\frac{1}{2}}c^{-\frac{1}{2}}x^{-7}y_i^{\pm1}\right)}_{L_8}\underbrace{\prod_{i=1}^{Q}\Gamma_e\left((pq)^{\frac{1}{2}}c^{\frac{1}{2}}x^{7}y_i^{\pm1}\right)}_{K}\,,
\end{aligned} \tag{3.168}$$

where recall that we have redefined

$$\begin{aligned}
u_{a=1,\dots,7} &\longrightarrow x^{-1}u_a\,, \\
u_8 &\longrightarrow x^{7}\,,
\end{aligned} \tag{3.169}$$

with new u_a on the right hand side satisfying $\prod_{a=1}^{7}u_a = 1$.

We then glue two caps using the Φ-gluing, namely we introduce $\hat{A}$, $\Phi^{1,\dots,8}$ and the superpotential

$$\mathcal{W}_{glue} = \mathrm{Tr}_z\left[\hat{A}\cdot(A-\tilde{A})\right] + \sum_{b=1}^{8}\Phi^b\left(L_b-\tilde{L}_b\right) \tag{3.170}$$

and gauge the puncture symmetry $USp(2N)$. The entire superpotential of the glued theory is thus given by

$$\begin{aligned}
\mathcal{W} &= \mathcal{W}_{cap} + \tilde{\mathcal{W}}_{cap} + \mathcal{W}_{glue} = \\
&= \sum_{i=1}^{N}\sum_{a=1}^{7}\mathrm{Tr}_z\left(R^{(i)a}A^{N-i}PL_a\right) + \sum_{i=1}^{N}\mathrm{Tr}_z\left(\beta_iA^{N-i}PK\right) + KL_8 + \\
&\quad + \sum_{i=1}^{N}\sum_{a=1}^{7}\mathrm{Tr}_z\left(\tilde{R}^{(i)a}\tilde{A}^{N-i}\tilde{P}\tilde{L}_a\right) + \sum_{i=1}^{N}\mathrm{Tr}_z\left(\tilde{\beta}_i\tilde{A}^{N-i}\tilde{P}\tilde{K}\right) + \tilde{K}\tilde{L}_8 + \\
&\quad + \mathrm{Tr}_z\left[\hat{A}\cdot(A-\tilde{A})\right] + \sum_{b=1}^{8}\Phi^b\left(L_b-\tilde{L}_b\right),
\end{aligned} \tag{3.171}$$

	$USp(2N)$	$SU(8)_v$	$SU(2)_f$	$U(1)_b$	$U(1)_t$	$U(1)_R$
$S^{(i)}$	$\mathbf{1}$	$\mathbf{8}$	$\mathbf{2}$	-3	$i-1$	2
Q	$\mathbf{2N}$	$\overline{\mathbf{8}}$	$\mathbf{1}$	-1	0	1
P	$\mathbf{2N}$	$\mathbf{1}$	$\mathbf{2}$	4	$1-N$	-1
A	$\mathbf{N(2N-1)-1}$	$\mathbf{1}$	$\mathbf{1}$	0	1	0

Table 3.3: The matter content of the $SU(8)\times SU(2)\times U(1)^2$ model and the corresponding transformation rules under the global symmetry.

where Tr_z is the trace over the gauged puncture symmetry $USp(2N)$. Note that the superpotential contains some massive fields. Once we integrate them out, the superpotential becomes

$$\begin{aligned}\mathcal{W} = &\sum_{n=1}^{Q}\sum_{b=1}^{7}\text{Tr}_z\left(R^{(i)b}A^{N-i}PL_b\right)+\sum_{i=1}^{N}\sum_{b=1}^{7}\text{Tr}_z\left(\tilde{R}^{(i)b}A^{N-i}\tilde{P}L_b\right)+\\ &+\frac{1}{3}\sum_{i=1}^{N}\text{Tr}_z\left(\beta_i A^{N-i}P(K-\tilde{K}-\Phi)\right)+\frac{1}{3}\sum_{i=1}^{N}\text{Tr}_z\left(\tilde{\beta}_i A^{N-i}\tilde{P}(\tilde{K}-K+\Phi)\right). \qquad (3.172)\end{aligned}$$

One can check that the superpotential (3.172) actually preserves

$$SU(8)_v\times SU(2)_f\times U(1)_b\times U(1)_t\,, \qquad (3.173)$$

which is consistent with the 6*d* prediction. The symmetry (3.173) can be made manifest by rewriting the superpotential as follows:

$$\mathcal{W}=\sum_{i=1}^{N}\sum_{b=1}^{8}\sum_{\alpha=\pm}\text{Tr}_z\left(S^{(i)b}{}_{\alpha}A^{N-i}P^{\alpha}Q_b\right)\,, \qquad (3.174)$$

where we have defined

$$S^{(i)b}{}_{+}=\begin{cases}R^{(i)b}\,, & b=1,\dots,7\,,\\ \beta_i\,, & b=8\,,\end{cases}$$

$$\begin{aligned}S^{(i)b}{}_{-}&=\begin{cases}\tilde{R}^{(i)b}\,, & b=1,\dots,7\,,\\ -\tilde{\beta}_n\,, & b=8\,,\end{cases}\\ Q_b&=\begin{cases}L_b\,, & b=1,\dots,7\,,\\ \frac{K-\tilde{K}-\Phi}{3}\,, & b=8\,,\end{cases}\\ P^{+}&=P\,,\\ P^{-}&=\tilde{P}\,. \end{aligned} \qquad (3.175)$$

The resulting model is the $USp(2N)$ gauge theory with one traceless antisymmetric A, 10 fundamentals $(Q_b; P^{\pm})$ and $16N$ gauge singlets $S^{(i)}{}^{b}_{\pm}$. The charges of each chiral multiplet are presented in Table 3.3. As we mentioned, $SU(2)_f$ doesn't come from the symmetry of $6d$ E-string but originates from the isometry of the compactifying two-sphere, which we will show using anomalies shortly.

Now let us evaluate the superconformal index. The index of this model is given by

$$\begin{aligned}\mathcal{I}_{(-1;2^7,-2)} = &\oint \mathrm{d}z_N\, \Gamma_e\left(pqt^{-1}\right)^{N-1} \prod_{i<j}^{N} \Gamma_e\left(pqt^{-1} z_i^{\pm 1} z_j^{\pm 1}\right) \prod_{i=1}^{N}\prod_{a=1}^{8} \Gamma_e\left((pq)^{\frac{1}{2}} c^{\frac{1}{2}} u_a z_i^{\pm 1}\right) \times \\ &\times \mathcal{I}_{cap}(\vec{z}; c; t; f; u_1, \ldots, u_7; x) \times \mathcal{I}_{cap}(\vec{z}; c; t; f^{-1}; u_1, \ldots, u_7; x) = \\ = &\prod_{i=1}^{N} \Gamma_e\left(pqt^{i-1} c^{-2} f^{\pm 1}\right) \prod_{i=1}^{N}\prod_{a=1}^{7} \Gamma_e\left(pqt^{i-1} c^{-1} f^{\pm 1} x^6 u_a\right) \times \\ &\times \oint \mathrm{d}z_N\, \Gamma_e(t)^{N-1} \prod_{i<j}^{N} \Gamma_e\left(t z_i^{\pm 1} z_j^{\pm 1}\right) \prod_{i=1}^{N} \Gamma_e\left((pq)^{-\frac{1}{2}} t^{1-N} c^{\frac{3}{2}} f^{\pm 1} x^{-7} z_i^{\pm 1}\right) \times \\ &\times \prod_{a=1}^{7}\prod_{i=1}^{Q} \Gamma_e\left((pq)^{\frac{1}{2}} c^{-\frac{1}{2}} x u_a^{-1} z_i^{\pm 1}\right) \prod_{i=1}^{N} \Gamma_e\left((pq)^{\frac{1}{2}} c^{\frac{1}{2}} x^7 z_i^{\pm 1}\right),\end{aligned} \tag{3.176}$$

where all the massive contributions are canceled out. As we have just observed, while the basic cap preserves $SU(7)_u \times U(1)_x \times U(1)_c$, the sphere model preserves not only $SU(7)_u \times U(1)_x \times U(1)_c$ but also $SU(8)_v \times U(1)_b$. Therefore, in terms of the $SU(8)_v \times U(1)_b$ fugacities which are defined by

$$\begin{aligned} v_{a=1,\ldots,7} &= c^{\frac{1}{8}} x^{\frac{3}{4}} u_a\,, \\ v_8 &= c^{-\frac{7}{8}} x^{-\frac{21}{4}}\,, \\ b &= c^{\frac{3}{8}} x^{-\frac{7}{4}}\,, \end{aligned} \tag{3.177}$$

the index is written as

$$\begin{aligned}\mathcal{I}_{(-1;2^7,-2)} = &\prod_{i=1}^{N}\prod_{a=1}^{8} \underbrace{\Gamma_e\left(pqt^{i-1} b^{-3} f^{\pm 1} v_a\right)}_{S^{(i)}{}^{a}_{\pm}} \oint \mathrm{d}z_N\, \underbrace{\Gamma_e(t)^{N-1} \prod_{i<j}^{N} \Gamma_e\left(t z_i^{\pm 1} z_j^{\pm 1}\right)}_{A} \times \\ &\times \underbrace{\prod_{i=1}^{N} \Gamma_e\left((pq)^{-\frac{1}{2}} t^{1-N} b^4 f^{\pm 1} z_i^{\pm 1}\right)}_{P^{\pm}} \prod_{a=1}^{8} \underbrace{\prod_{i=1}^{N} \Gamma_e\left((pq)^{\frac{1}{2}} b^{-1} v_a^{-1} z_i^{\pm 1}\right)}_{Q_a},\end{aligned} \tag{3.178}$$

which preserves the symmetry

$$SU(8)_v \times SU(2)_f \times U(1)_b \times U(1)_t\,. \tag{3.179}$$

As we already anticipated, we claim that the $SU(2)_f$ symmetry descends from the isometry of the two-sphere on which we compactified the $6d$ theory, which manifests itself in $4d$ as a flavor symmetry. To support this claim, we can compute the anomalies for this symmetry in our four-dimensional model and check that they match those of the $\mathbb{S}^2$ isometry which can be computed from the 8-form anomaly polynomial of the original $6d$ theory. The anomalies for the flavor symmetries in arbitrary even dimensions that come from the isometries of the compactification manifold can be computed following the strategy of [130]. In Appendix D we apply it to the compactification of the rank N E-string theory on a two-sphere, so to find the mixed anomalies between its $SU(2)_{\text{ISO}}$ isometry and the $U(1)$ symmetries in the Cartan of the $E_8 \times SU(2)_L$ global symmetry of the $6d$ theory. In particular, in equation (D.20) we give the anomalies in the basis for the $U(1)$ symmetries that correspond to the Cartan of the subgroup $U(1)_c \times SU(8)_u \subset E_8$. Hence, in order to compare (D.20) with the anomalies of the $4d$ model we are considering in this section we first need to translate back the fugacities appearing in the index (3.178) in terms of the original $U(1)_c \times SU(8)_u$ fugacities using (3.169)-(3.177). By doing so, we find the following anomalies for $SU(2)_f$:

$$\begin{aligned}
&\text{Tr}\left(SU(2)_f^2 U(1)_R\right) = N(N+1), \quad &&\text{Tr}\left(SU(2)_f^2 U(1)_t\right) = N(N-1), \\
&\text{Tr}\left(SU(2)_f^2 U(1)_c\right) = -3N, &&\text{Tr}\left(SU(2)_f^2 U(1)_{u_a}\right) = -2N\,.
\end{aligned} \tag{3.180}$$

These perfectly match the anomalies (D.20) that we can compute from $6d$ for the value of the flux (3.164). In addition, one can see that the mixing of the original $U(1)_f$ from (3.137) with $U(1)_c$ that we did above (3.147) and that gives the correct anomalies matches the mixing prediction given in (D.21).

While the model we have considered is obtained from the Φ-gluing for the octet moment maps as well as the antisymmetric moment maps, one can also consider the S-gluing for the antisymmetric moment maps[23]. The corresponding flux is given by

$$\begin{aligned}
\mathcal{F} &= (-1/2-(-1/2); 1+1, 1+1, 1+1, 1+1, 1+1, 1+1, 1+1, -1+(-1)) = \\
&= (0; 2, 2, 2, 2, 2, 2, 2, -2)\,,
\end{aligned} \tag{3.181}$$

which now has the vanishing $U(1)_t$ flux. As a result, this model is supposed to inherit the full $SU(2)_L$ symmetry of E-string rather than just its $U(1)_t$ subgroup. In addition, this model also has a geometric $SU(2)_f$ symmetry coming from the isometry of the two-sphere. However, those $SU(2)_L$ and $SU(2)_f$ symmetries are not manifest; only $U(1)_t \times U(1)_f \subset SU(2)_L \times SU(2)_f$ is visible in the Lagrangian description.

[23] The S-gluing for the octet moment maps will be considered in the subsequent examples.

Since we now take the S-gluing for the antisymmetric moment maps, we introduce the superpotential[24]

$$\mathcal{W}_{glue} = \text{Tr}_z\left(A\cdot\tilde{A}\right) + \sum_{b=1}^{8}\Phi^b\left(L_b - \tilde{L}_b\right) \tag{3.182}$$

without the additional chiral superfield $\hat{A}$. The total superpotential is then

$$\mathcal{W} = \sum_{i=1}^{N}\sum_{b=1}^{8}\text{Tr}_z\left(S^{(i)b}A^{N-i}PQ_b\right) + \sum_{i=1}^{N}\sum_{b=1}^{8}\text{Tr}_z\left(\tilde{S}^{(i)b}\tilde{A}^{N-i}\tilde{P}Q_b\right) + \text{Tr}_z\left(A\cdot\tilde{A}\right)\,, \tag{3.183}$$

where we have defined as before

$$\begin{aligned} S^{(i)b} &= \begin{cases} R^{(i)b}\,, & b=1,\dots,7\,,\\ \beta_i\,, & b=8\,,\end{cases}\\ \tilde{S}^{(i)b} &= \begin{cases} \tilde{R}^{(i)b}\,, & b=1,\dots,7\,,\\ -\tilde{\beta}_i\,, & b=8\,,\end{cases}\\ Q_b &= \begin{cases} L_b\,, & b=1,\dots,7\,,\\ \frac{K-\tilde{K}-\Phi}{3}\,, & b=8\,.\end{cases}\end{aligned} \tag{3.184}$$

Note that we don't integrate out A and $\tilde{A}$ for simplicity of the superpotential. Unlike (3.174), $(S^{(i)b},\tilde{S}^{(i)b})$ and $(P,\tilde{P})$ do not form doublets of any $SU(2)$ because $A\neq\tilde{A}$. Thus, the manifest symmetry is only

$$SU(8)_v\times U(1)_f\times U(1)_b\times U(1)_t\,. \tag{3.185}$$

Nevertheless, using the supersymmetric index, we find that $U(1)_f\times U(1)_t$ is actually enhanced to $SU(2)_f\times SU(2)_L$ at low energies. The full global symmetry is thus

$$SU(8)_v\times SU(2)_f\times U(1)_b\times SU(2)_L\,. \tag{3.186}$$

The transformation rules of each chiral multiplet under the manifest global symmetry are presented in Table 3.4.

[24]Notice that since the antisymmetric chirals are massive in this case, the theory is just $USp(2N)$ with some fundamental chirals, so the rank N can't be too large. Specifically, since we have 10 fundamental chirals, the theory is free for $N=3$, as it can be seen from the fact that it's Intriligator–Pouliot dual [81] is just a WZ model, and it is SUSY breaking for $N>3$ (notice that in this case the dual would have negative rank). In the following we will study in more details the cases $N=1,2$ by computing their supersymmetric indices.

	$USp(2N)$	$SU(8)_v$	$U(1)_f$	$U(1)_b$	$U(1)_t$	$U(1)_R$
$S^{(i)}$	$\mathbf{1}$	$\mathbf{8}$	$2-i$	-3	$i-1$	2
$\tilde{S}^{(i)}$	$\mathbf{1}$	$\mathbf{8}$	$i-2$	-3	$1-i$	$2i$
Q	$\mathbf{2N}$	$\overline{\mathbf{8}}$	0	-1	0	1
P	$\mathbf{2N}$	$\mathbf{1}$	$Q-2$	4	$1-Q$	-1
$\tilde{P}$	$\mathbf{2N}$	$\mathbf{1}$	$2-Q$	4	$Q-1$	$1-2Q$
A	$\mathbf{N(2N-1)-1}$	$\mathbf{1}$	-1	0	1	0
$\tilde{A}$	$\mathbf{N(2N-1)-1}$	$\mathbf{1}$	1	0	-1	2

Table 3.4: The matter content of the $SU(8)\times SU(2)^2\times U(1)$ model and the corresponding transformation rules under the global symmetry.

Let us evaluate the index of the model corresponding to the flux (3.181), which is given by the following matrix integral:

$$\begin{aligned}\mathcal{I}_{(0;2^7,-2)} = &\oint \mathrm{d}z_N \prod_{i=1}^{N}\prod_{a=1}^{8}\Gamma_e\left((pq)^{\frac{1}{2}}c^{\frac{1}{2}}u_a z_i^{\pm1}\right)\times\\ &\times\mathcal{I}_{cap}(\vec{z};c;tf;f;u_1,\ldots,u_7;x)\times\mathcal{I}_{cap}(\vec{z};c;pqt^{-1}f^{-1};f^{-1};u_1,\ldots,u_7;x)\,,\end{aligned} \tag{3.187}$$

where we have made a shift of fugacity $t\to tf$. If we use the $SU(8)_v\times U(1)_b\supset SU(7)_u\times U(1)_x\times U(1)_c$ fugacities defined in (3.177), the index is written as

$$\begin{aligned}\mathcal{I}_{(0;2^7,-2)} = &\prod_{i=1}^{N}\prod_{a=1}^{8}\underbrace{\Gamma_e\left(pqt^{i-1}b^{-3}f^{2-i}v_a\right)}_{S^{(i)a}}\prod_{i=1}^{N}\prod_{a=1}^{8}\underbrace{\Gamma_e\left(pq\left(pqt^{-1}\right)^{i-1}b^{-3}f^{i-2}v_a\right)}_{\tilde{S}^{(i)a}}\times\\ &\times\oint\mathrm{d}z_N\,\underbrace{\Gamma_e\left(tf^{-1}\right)^{N-1}\prod_{i<j}^{N}\Gamma_e\left(tf^{-1}z_i^{\pm1}z_j^{\pm1}\right)}_{A}\times\underbrace{\Gamma_e\left(pqt^{-1}f\right)^{N-1}\prod_{i<j}^{N}\Gamma_e\left(pqt^{-1}fz_i^{\pm1}z_j^{\pm1}\right)}_{\tilde{A}}\times\\ &\times\prod_{i=1}^{N}\underbrace{\Gamma_e\left((pq)^{-\frac{1}{2}}t^{1-N}b^4f^{N-2}z_i^{\pm1}\right)}_{P}\prod_{i=1}^{N}\underbrace{\Gamma_e\left((pq)^{-\frac{1}{2}}\left(pqt^{-1}\right)^{1-N}b^4f^{2-N}z_i^{\pm1}\right)}_{\tilde{P}}\times\\ &\times\prod_{i=1}^{N}\prod_{a=1}^{8}\underbrace{\Gamma_e\left((pq)^{\frac{1}{2}}b^{-1}v_a^{-1}z_n^{\pm1}\right)}_{Q_a}\,.\end{aligned} \tag{3.188}$$

The corresponding theory is the $USp(2N)$ gauge theory with 10 fundamental chirals $(Q_a;P,\tilde{P})$, two massive traceless antisymmetric chirals $(A,\tilde{A})$, $16N$ gauge singlets $(S^{(i)a};\tilde{S}^{(i)a})$ and the superpotential (3.183).

As we mentioned before, we will see using the supersymmetric index that $U(1)_f$ and $U(1)_t$ are enhanced in the IR to $SU(2)_f$ and $SU(2)_L$ respectively. Moreover, as in the previous example, we claim that $SU(2)_f$ descends from the isometry of the two-sphere on which we performed the compactification. This can be checked by computing the anomalies for $U(1)_f$ in $4d$ and comparing with those predicted from $6d$ (D.20). From our $4d$ Lagrangian description we find

$$\begin{aligned} &\mathrm{Tr}\left(U(1)_f^2 U(1)_R\right) = -\frac{4}{3}N(N+1)(N-4), \qquad \mathrm{Tr}\left(U(1)_f^2 U(1)_t\right) = 0, \\ &\mathrm{Tr}\left(U(1)_f^2 U(1)_c\right) = 3N(N-5), \qquad \mathrm{Tr}\left(U(1)_f^2 U(1)_{u_a}\right) = 2N(N-5)\,, \end{aligned} \tag{3.189}$$

with all the other anomalies non-quadratic in $U(1)_f$ being zero, in agreement with the claim that this symmetry is enhanced to $SU(2)_f$ in the IR. Notice that the mixed anomaly with $U(1)_t$ is also zero, again in agreement with it being enhanced to $SU(2)_L$. Remembering that the anomalies for an $SU(2)$ symmetry are related to those for its $U(1)$ Cartan by

$$\mathrm{Tr}\left(U(1)^2 U(1)_i\right) = 4\,\mathrm{Tr}\left(SU(2)^2 U(1)_i\right) \tag{3.190}$$

since 4 is the embedding index of $U(1)$ inside $SU(2)$, we can perfectly match the anomalies (3.189) computed in $4d$ with those predicted from $6d$ (D.20) for the value of the flux (3.181). In addition, we find that the mixing of the original $U(1)_f$ from (3.137) with the other $U(1)$'s giving the correct anomalies matches the mixing prediction in (D.21).

We will now compute the index for small values of N to see explicitly the various enhancements of symmetry.

Rank 1

Now we compute the indices (3.176) and (3.187) for $N=1$. Note that there is no distinction between (3.176) and (3.187) for $N=1$ because the traceless antisymmetric representation of $USp(2N)$ is trivial in this case. Also $U(1)_t$ decouples for $N=1$ because no fields are charged under it. The remaining abelian symmetry is $U(1)_b$, whose mixing coefficient with the R-symmetry can be determined by the a-maximization [144]. In general, once the mixing coefficients R_a are determined for a set of abelian symmetries $\prod_a U(1)_a$, the R-charge we use for the expansion of the index is given by

$$R = R_0 + \sum_a R_a Q_a\,, \tag{3.191}$$

where Q_a is the $U(1)_a$ charge and R_0 is the trial R-charge we have used to define the index formula. For example, the index (3.176) is defined with the trial R-charge given in Table 3.3. Given the $U(1)_a$ fugacity t_a, the mixing of R-symmetry with $U(1)_a$ is realized by a shift of the fugacity $t_a \to t_a(pq)^{R_a/2}$.

In the current example, the mixing coefficient of $U(1)_b$ is given by

$$R_b \approx 0.4269\,. \tag{3.192}$$

To avoid clutter due to the irrational value of the mixing coefficient, we will use the following approximate rational value:

$$R_b = \frac{3}{7} \tag{3.193}$$

to evaluate the index, which shouldn't affect the contribution of the conserved current we are interested in because it is in the adjoint representation of the symmetry group and hence independent of $U(1)$ mixing coefficients. In addition, the exact value of the R-charge can be easily implemented if necessary by shifting $U(1)$ fugacities as explained above. With this choice of the mixing coefficient, the index for $N = 1$ is given by

$$\begin{aligned}
&\mathcal{I}^{N=1}_{(-1;2^7,-2)} = \\
&= 1 + b^{-3}\mathbf{2}_{SU(2)}\mathbf{8}_{SU(8)}(pq)^{\frac{5}{14}} + b^{-2}\overline{\mathbf{28}}_{SU(8)}(pq)^{\frac{4}{7}} + \\
&+ \left(b^8 + b^{-6}\left(\mathbf{3}_{SU(2)}\mathbf{36}_{SU(8)} + \mathbf{28}_{SU(8)}\right)\right)(pq)^{\frac{5}{7}} + b^{-3}\mathbf{2}_{SU(2)}\mathbf{8}_{SU(8)}(pq)^{\frac{5}{14}}(p+q) + \\
&+ b^{-5}\mathbf{2}_{SU(2)}\overline{\mathbf{216}}_{SU(8)}(pq)^{\frac{13}{14}} + \left(-\mathbf{63}_{SU(8)} - \mathbf{3}_{SU(2)} - 1\right)pq + \cdots\,.
\end{aligned} \tag{3.194}$$

We highlighted the negative contributions at order pq, which correspond to the conserved current multiplet [141]. We find that they are in the adjoint representation of

$$SU(8)_v \times SU(2)_f \times U(1)_b\,. \tag{3.195}$$

We can also check the presence of the operators expected from $6d$ according to the general formula (3.14). For the case at hand, we need to consider the branching rule for the adjoint representation of E_8 with respect to its $SU(8) \times U(1)$ subgroup:

$$\mathbf{248} \to \mathbf{1}^0 \oplus \mathbf{63}^0 \oplus \overline{\mathbf{56}}^1 \oplus \mathbf{56}^{-1} \oplus \mathbf{28}^2 \oplus \overline{\mathbf{28}}^{-2} \oplus \overline{\mathbf{8}}^3 \oplus \mathbf{8}^{-3}\,. \tag{3.196}$$

The $6d$ R-symmetry assigns R-charge 1 to the octet fields Q_a and so it is related to the R-symmetry we used for computing the index (3.194) by the shift $b \to b(pq)^{-\frac{3}{14}}$. Moreover, the $U(1)$ inside the $6d$ global symmetry for which we turned on a unit of flux $F = 1$[25] is related to the $U(1)_b$ symmetry of our $4d$ model. With this dictionary, we can immediately identify all the states appearing in (3.196) that contribute to the index (3.194) up to the

[25]We choose to work in a normalization for this $U(1)$ such that the minimal flux is 1. Our model is the one with the minimal value of flux that preserves $SU(8) \times U(1)$, since it was constructed by gluing two caps together without inserting any tube in the middle, which would have increased the flux.

order we evaluated it

$$\begin{aligned}
\mathbf{8}^{-3} \quad &\rightarrow \quad 2b^{-3}\chi_{\mathbf{8}}^{SU(8)}(pq)^{\frac{5}{14}} \\
\overline{\mathbf{28}}^{-2} \quad &\rightarrow \quad b^{-2}\chi_{\mathbf{28}}^{SU(8)}(pq)^{\frac{4}{7}} \\
\mathbf{1}^0 \oplus \mathbf{63}^0 \quad &\rightarrow \quad -(\chi_{\mathbf{63}}^{SU(8)}+1)pq\,.
\end{aligned} \tag{3.197}$$

Notice that the state $\mathbf{56}^{-1}$ doesn't contribute with any operator since in this case $-1-qF=0$. Moreover, the state $\mathbf{8}^{-3}$ contributes with two operators, which are distinguished in our index (3.232) by the quantum number for the geometric $SU(2)_f$ symmetry.

Rank 2

Next let us compute the indices for $N=2$. Now the abelian symmetry is $U(1)_b \times U(1)_t$. We use the mixing coefficients

$$R_t \approx 0.2892 \approx \frac{2}{7}\,, \qquad R_b \approx 0.4707 \approx \frac{7}{15}\,, \tag{3.198}$$

which give rise to the following expansion of the index:

$$\begin{aligned}
\mathcal{I}^{N=2}_{(-1;2^7,-2)} = {} & 1 + t^2(pq)^{\frac{2}{7}} + b^{-3}\mathbf{2}_{SU(2)}\mathbf{8}_{SU(8)}(pq)^{\frac{3}{10}} + tb^{-3}\mathbf{2}_{SU(2)}\mathbf{8}_{SU(8)}(pq)^{\frac{31}{70}} + \\
& + b^{-2}\overline{\mathbf{28}}_{SU(8)}(pq)^{\frac{8}{15}} + t^4(pq)^{\frac{4}{7}} + t^{-2}b^8(pq)^{\frac{61}{105}} + t^2b^{-3}\mathbf{2}_{SU(2)}\mathbf{8}_{SU(8)}(pq)^{\frac{41}{70}} + \\
& + b^{-6}\left(\mathbf{28}_{SU(8)} + \mathbf{3}_{SU(2)}\mathbf{36}_{SU(8)}\right)(pq)^{\frac{3}{5}} + tb^{-2}\overline{\mathbf{28}}_{SU(8)}(pq)^{\frac{71}{105}} + t^{-1}b^8(pq)^{\frac{76}{105}} + \\
& + t^3b^{-3}\mathbf{2}_{SU(2)}\mathbf{8}_{SU(8)}(pq)^{\frac{51}{70}} + tb^{-6}\left(1+\mathbf{3}_{SU(2)}\right)\left(\mathbf{28}_{SU(8)} + \mathbf{36}_{SU(8)}\right)(pq)^{\frac{26}{35}} + \\
& + t^2(pq)^{\frac{2}{7}}(p+q) + \cdots + \left(-\mathbf{63}_{SU(8)} - \mathbf{3}_{SU(2)} - 2\right)pq + \cdots\,.
\end{aligned} \tag{3.199}$$

We find that the contribution of the conserved current, which is highlighted in blue, is in the adjoint representation of

$$SU(8)_v \times SU(2)_f \times U(1)_t \times U(1)_b\,. \tag{3.200}$$

This is the same as the manifest symmetry of the Lagrangian description we found. Again, on top of checking that the global symmetry of the model is the one preserved by the flux, we can also check the presence of the operators coming from the $6d$ conserved currents. In this case we also have operators charged under $U(1)_t$ coming from the conserved current for $SU(2)_L$, according to the branching rule

$$\mathbf{3}_{SU(2)_L} \rightarrow \mathbf{1}^0 \oplus \mathbf{1}^{\pm 2}\,. \tag{3.201}$$

The $6d$ R-symmetry assigns R-charge 1 to the fundamentals Q_a and also to the antisymmetric field A. Hence, it is related to the R-symmetry we used for computing the index (3.199) by the shifts $b \to b(pq)^{-\frac{7}{30}}$ and $t \to t(pq)^{\frac{5}{14}}$. Moreover, the $U(1) \subset SU(2)_L$ for which we turned on flux $F_t = -1$ is related to the $U(1)_t$ symmetry of our $4d$ model, while the $U(1) \subset E_8$ for which we turned flux $F = 1$ is related to $U(1)_b$. With this dictionary, we can immediately identify all the states appearing in (3.196) and (3.201) that contribute to the index (3.199) up to the order we evaluated it:

$$\begin{aligned}
&\mathbf{8}^{(-3,0)} \quad \to \quad 2b^{-3}\chi_{\mathbf{8}}^{SU(8)}(pq)^{\frac{3}{10}} \\
&\overline{\mathbf{28}}^{(-2,0)} \quad \to \quad b^{-2}\overline{\mathbf{28}}_{SU(8)}(pq)^{\frac{8}{15}} \\
&\mathbf{1}^{(0,2)} \quad \to \quad t^2(pq)^{\frac{2}{7}} \\
&2\times\mathbf{1}^{(0,0)}\oplus\mathbf{63}^{(0,0)} \quad \to \quad -(\mathbf{63}_{SU(8)}+2)pq\,,
\end{aligned} \tag{3.202}$$

where at the exponent of the states on the left hand side we reported, in order, the charges under $U(1)_b$ and $U(1)_t$. Notice again that the state $\mathbf{8}^{(-3,0)}$ contributes with two operators, which transform as a doublet under the geometric $SU(2)_f$ symmetry.

In fact, one can see that there are operators violating the unitarity bound $R = \frac{2}{3}$, which corresponds to $(pq)^{\frac{1}{3}}$. For instance, the first nontrivial term in (3.199), which corresponds to $\mathrm{Tr}A^2$, is below this bound. Repeating the a-maximization after flipping a unitarity violating operator whenever it appears, we have found that the decoupled operators are $\mathrm{Tr}A^2$ and $S^{(1)}$, which correspond to the first two nontrivial terms of the index (3.199) respectively. We have also found that the remaining interacting sector still exhibits the same symmetry as (3.200).

On the other hand, the index (3.187) with the S-gluing is evaluated with different mixing coefficients

$$R_t = 1\,, \qquad R_b \approx 0.6321\,, \tag{3.203}$$

where the latter is approximated by

$$R_b = \frac{7}{13}\,. \tag{3.204}$$

The resulting index is

$$\begin{aligned}
\mathcal{I}^{N=2}_{(0;2^7,-2)} = 1 &+ b^8(pq)^{\frac{2}{13}} + b^{-3}\mathbf{2}_{SU(2)}\mathbf{8}_{SU(8)}(pq)^{\frac{5}{26}} + b^{16}(pq)^{\frac{4}{13}} + b^5\mathbf{2}_{SU(2)}\mathbf{8}_{SU(8)}(pq)^{\frac{9}{26}} + \\
&+ b^{-6}\left(\mathbf{28}_{SU(8)} + \mathbf{3}_{SU(2)}\mathbf{36}_{SU(8)}\right)(pq)^{\frac{5}{13}} + \left(b^{24} + b^{-2}\overline{\mathbf{28}}_{SU(8)}\right)(pq)^{\frac{6}{13}} + \\
&+ b^{13}\mathbf{2}_{SU(2)}\mathbf{8}_{SU(8)}(pq)^{\frac{1}{2}} + b^2\left(\mathbf{28}_{SU(8)} + \mathbf{3}_{SU(2)}\mathbf{36}_{SU(8)}\right)(pq)^{\frac{7}{13}} +
\end{aligned}$$

$$
\begin{aligned}
&+ b^{-9}\left(\mathbf{2}_{SU(2)}\mathbf{168}_{SU(8)} + \mathbf{4}_{SU(2)}\mathbf{120}_{SU(8)}\right)(pq)^{\frac{15}{26}} + \left(b^{32} + b^{6}\overline{\mathbf{28}}_{SU(8)}\right)(pq)^{\frac{8}{13}} + \\
&+ b^{8}(pq)^{\frac{2}{13}}(p+q) + \left(b^{21}\mathbf{2}_{SU(2)}\mathbf{8}_{SU(8)} + b^{-5}\mathbf{2}_{SU(2)}\left(\overline{\mathbf{8}}_{SU(8)} + \overline{\mathbf{216}}_{SU(8)}\right)\right)(pq)^{\frac{17}{26}} + \\
&+ \cdots + 2b^{13}\mathbf{2}_{SU(2)}\mathbf{8}_{SU(8)}(pq)^{\frac{1}{2}}(p+q) + \left(-\mathbf{63}_{SU(8)} - \mathbf{3}_{SU(2)} - \mathbf{3}_{SU(2)_t} - 1 + \right. \\
&\left. + b^{26}\left(\mathbf{28}_{SU(8)} + \mathbf{3}_{SU(2)}\mathbf{36}_{SU(8)}\right) + \mathbf{720}_{SU(8)} + \mathbf{3}_{SU(2)}\mathbf{945}_{SU(8)}\right) pq + \dots\,, \qquad (3.205)
\end{aligned}
$$

where one can see that the manifest symmetry $SU(8)_v \times U(1)_f \times U(1)_t \times U(1)_b$ is indeed enhanced to

$$
SU(8)_v \times SU(2)_f \times SU(2)_t \times U(1)_b\,. \qquad (3.206)
$$

Again, the operators appearing in the index confirm our expectation from $6d$. With respect to the case of Φ-gluing, the flux preserves the $SU(2)_L$ $6d$ symmetry this time, which is related to $SU(2)_t$ of our $4d$ model. Moreover, in this case the $6d$ R-symmetry is related to the one we used for computing the index (3.205) by the shift $b \to b(pq)^{-\frac{7}{26}}$. With this dictionary, we can immediately identify all the states that are expected from the $6d$ conserved currents according to the branching rule (3.196) in the index (3.205) up to the order we evaluated it:

$$
\begin{aligned}
&(\mathbf{8},\mathbf{1})^{-3} &\to\quad & 2b^{-3}\mathbf{8}_{SU(8)}(pq)^{\frac{5}{26}} \\
&(\overline{\mathbf{28}},\mathbf{1})^{-2} &\to\quad & b^{-2}\overline{\mathbf{28}}_{SU(8)}(pq)^{\frac{6}{13}} \\
&(\mathbf{1},\mathbf{1})^{0} \oplus (\mathbf{63},\mathbf{1})^{0} \oplus (\mathbf{1},\mathbf{3})^{0} &\to\quad & -\left(\mathbf{63}_{SU(8)} + \mathbf{3}_{SU(2)_t} + 1\right) pq\,, \qquad (3.207)
\end{aligned}
$$

where in the states on the left hand side we reported, in order, the representations under $SU(8)$ and $SU(2)_L$. Notice again that the state $(\mathbf{8},\mathbf{1})^{-3}$ contributes with two operators, which transform as a doublet under the geometric $SU(2)_f$ symmetry.

We have some operators violating the unitarity bound $R \geq \frac{2}{3}$:

$$
\mathrm{Tr}P\tilde{P}\,, \qquad (S^{(1)}, \tilde{S}^{(1)})\,, \qquad (3.208)
$$

which correspond to the first two nontrivial terms of the index (3.205). Once those are flipped, the interacting sector turns out to be independent of $SU(2)_f$ and only exhibits

$$
SU(8) \times SU(2)_t \times U(1)_b\,. \qquad (3.209)
$$

Note that we still have the non-trivial enhancement from $U(1)_t$ to $SU(2)_t$. On the other hand, the geometric $SU(2)_f$ symmetry is realized in the decoupled sector because $(S^{(1)}, \tilde{S}^{(1)})$ form a doublet of $SU(2)_f$.

$E_6 \times SU(2) \times U(1)^2$ sphere model

The next example is the model corresponding to the compactification of the E-string theory on a sphere with flux

$$\mathcal{F} = (-1; 0, 0, 2, 2, 2, 2, 2, -2) , \tag{3.210}$$

which preserves $E_6 \times SU(2)_v \times U(1)_b \times U(1)_t \subset E_8 \times SU(2)_L$. Such flux can be achieved by gluing two basic caps, taking the S-gluing for the first two octet moment maps[26] and the Φ-gluing for the other octet moment maps and the antisymmetric moment maps[27]. As this model also turns out to have the geometric $SU(2)_f$ symmetry, the total global symmetry of the theory is given by

$$E_6 \times SU(2)_v \times SU(2)_f \times U(1)_b \times U(1)_t \,. \tag{3.211}$$

First recall that the basic cap is given by the WZ model with the superpotential

$$\mathcal{W}_{cap} = \sum_{i=1}^{N}\sum_{a=1}^{7} \text{Tr}_z \left(R^{(i)a} A^{N-i} P L_a\right) + \sum_{i=1}^{N} \text{Tr}_z \left(\beta_i A^{N-i} P K\right) + K L_8 \,, \tag{3.212}$$

which preserves

$$USp(2N) \times SU(7)_u \times U(1)_x \times U(1)_c \times U(1)_t \times U(1)_f \,. \tag{3.213}$$

As we mentioned, we take the S-gluing for $L_{1,2}$ and the Φ-gluing for $L_{3,\ldots,8}$ and A. Namely, we introduce $\hat{A}$, $\Phi^{3,\ldots,8}$ and the superpotential

$$\mathcal{W}_{glue} = \text{Tr}_z \left[\hat{A} \cdot (A - \tilde{A})\right] + \sum_{a=1}^{2} L_a \tilde{L}^a + \sum_{b=3}^{8} \Phi^b \left(L_b - \tilde{L}_b\right) \tag{3.214}$$

[26] In this and the next example, we choose an even number of octet moment maps to be S-glued. This is because if an odd number of octet moment maps are S-glued, the resulting theory has the $USp(2N)$ gauge group with an odd number of fundamental chirals, which lead to an inconsistency of the theory due to the Witten anomaly [152]. In general, one has to take the gluing rule in such a way that the resulting theory is a consistent anomaly-free theory.

[27] One may wonder about the possibility for $N > 1$ of performing an S-gluing for the antisymmetric moment maps, which would give flux $n_t = 0$ that preserves the $SU(2)_L$ symmetry. It turns out that the resulting model doesn't flow to an SCFT. In order to avoid this problem one should consider a sphere compactification with higher values of the fluxes n_c, n_a, which can be achieved by connecting the two caps with an even number of tubes in the middle. The resulting model is too complicated to analyze with the supersymmetric index, so we will neglect this possibility in what follows.

and gauge the puncture symmetry $USp(2N)$. The entire superpotential of the glued theory is given by

$$\begin{aligned}\mathcal{W} = \mathcal{W}_{cap} + \tilde{\mathcal{W}}_{cap} + \mathcal{W}_{glue} = \\ = \sum_{i=1}^{N}\sum_{a=1}^{7} \text{Tr}_z \left(R^{(i)a} A^{N-i} P L_a\right) + \sum_{i=1}^{N} \text{Tr}_z \left(\beta_i A^{N-i} P K\right) + K L_8 + \\ + \sum_{i=1}^{N}\sum_{a=1}^{2} \text{Tr}_z \left(\tilde{R}^{(i)}{}_a \tilde{A}^{N-i} \tilde{P} \tilde{L}^a\right) + \sum_{i=1}^{N}\sum_{a=3}^{7} \text{Tr}_z \left(\tilde{R}^{(i)a} \tilde{A}^{N-i} \tilde{P} \tilde{L}_a\right) + \sum_{i=1}^{N} \text{Tr}_z \left(\tilde{\beta}_i \tilde{A}^{N-i} \tilde{P} \tilde{K}\right) + \\ + \tilde{K}\tilde{L}_8 + \text{Tr}_z \left[\hat{A} \cdot (A - \tilde{A})\right] + \sum_{a=1}^{2} L_a \tilde{L}^a + \sum_{b=3}^{8} \Phi^b \left(L_b - \tilde{L}_b\right) ,\end{aligned} \tag{3.215}$$

where Tr_z is the trace over the gauged puncture symmetry $USp(2N)$. After integrating out the massive fields, we obtain the $USp(2N)$ gauge theory with one traceless antisymmetric A, 8 fundamentals $(Q_b; P^{\pm})$, $16N$ gauge singlets $(R^{(i)a}, \tilde{R}^{(i)}{}_a; S^{(i)b}{}_{\pm})$ and the superpotential

$$\mathcal{W} = \sum_{i=1}^{N}\sum_{b=3}^{8}\sum_{\alpha=\pm} \text{Tr}_z \left(S^{(i)b}{}_{\alpha} A^{N-i} P^{\alpha} Q_b\right) - \sum_{i,j=1}^{N}\sum_{a=1}^{2} \text{Tr}_z \left(A^{2N-i-j} P^+ P^- R^{(i)a} \tilde{R}^{(j)}{}_a\right) , \tag{3.216}$$

where we have defined

$$\begin{aligned} S^{(i)b}{}_{+} &= \begin{cases} R^{(i)b} , & b = 3, \ldots, 7 , \\ \beta_i , & b = 8 , \end{cases} \\ S^{(nib}{}_{-} &= \begin{cases} \tilde{R}^{(i)b} , & b = 3, \ldots, 7 , \\ -\tilde{\beta}_i , & b = 8 , \end{cases} \\ Q_b &= \begin{cases} L_b , & b = 3, \ldots, 7 , \\ \frac{K - \tilde{K} - \Phi}{3} , & b = 8 , \end{cases} \\ P^+ &= P , \\ P^- &= \tilde{P} . \end{aligned} \tag{3.217}$$

The superpotential (3.216) preserves

$$SU(2)_v \times SU(6)_w \times SU(2)_d \times SU(2)_f \times U(1)_b \times U(1)_t . \tag{3.218}$$

The transformation rules of each chiral multiplet under the manifest global symmetry are presented in Table 3.5. In particular, using the supersymmetric index, we will show that $SU(6)_w \times SU(2)_d$ is enhanced to E_6. Thus, the enhanced global symmetry of the theory is

	$USp(2N)$	$SU(2)_v$	$SU(6)_w$	$SU(2)_d$	$SU(2)_f$	$U(1)_b$	$U(1)_t$	$U(1)_R$
$S^{(i)}$	**1**	**1**	**6**	**2**	**1**	-2	$i-1$	2
$(R^{(i)}, \tilde{R}^{(i)})$	**1**	**2**	**1**	**1**	**2**	-3	$i-1$	2
Q	**2N**	**1**	$\overline{\mathbf{6}}$	**1**	**1**	-1	0	1
P	**2N**	**1**	**1**	**2**	**1**	3	$1-N$	-1
A	$\mathbf{N(2N-1)-1}$	**1**	**1**	**1**	**1**	0	1	0

Table 3.5: The matter content of the $E_6 \times SU(2)^2 \times U(1)^2$ model and the corresponding transformation rules under the global symmetry.

given by

$$E_6 \times SU(2)_v \times SU(2)_f \times U(1)_b \times U(1)_t \,. \tag{3.219}$$

To evaluate the index, we first note that the gluing we take breaks $SU(7)_u \times U(1)_x$ of the basic cap into $SU(2)_v \times SU(5)_u \times U(1)_y \times U(1)_x$. Thus, we need to redefine the fugacities as follows:

$$\begin{aligned} u_1 &\longrightarrow x^{-1}y^{-5}v\,, \\ u_2 &\longrightarrow x^{-1}y^{-5}v^{-1}\,, \\ u_{b=3,\ldots,7} &\longrightarrow x^{-1}y^2u_b\,, \\ u_8 &\longrightarrow x^7\,, \end{aligned} \tag{3.220}$$

where u_b on the right hand side satisfies $\prod_{b=3}^7 u_b = 1$. The octet fugacities of the caps are then given by

$$\eta_a = \left(c^{\frac{1}{2}}x^{-1}y^{-5}v, c^{\frac{1}{2}}x^{-1}y^{-5}v^{-1}; c^{\frac{1}{2}}x^{-1}y^2u_{a-2}; c^{\frac{1}{2}}x^7\right) \tag{3.221}$$

for the left cap and

$$\xi_a = \left(c'^{\frac{1}{2}}x'^{-1}y'^{-5}v', c'^{\frac{1}{2}}x'^{-1}y'^{-5}v'^{-1}; c'^{\frac{1}{2}}x'^{-1}y'^2u'_{a-2}; c'^{\frac{1}{2}}x'^7\right) \tag{3.222}$$

for the right cap. Since we take the S-gluing for the first two components and the Φ-gluing for the others, we impose the conditions $\eta_a = 1/\xi_a$ for $a = 1$, 2 and $\eta_a = \xi_a$ otherwise, which are solved by

$$\begin{aligned} c' &= c^{\frac{1}{2}}xy^5\,, \\ x' &= c^{\frac{1}{28}}x^{\frac{13}{14}}y^{-\frac{5}{14}}\,, \\ y' &= c^{\frac{1}{7}}x^{-\frac{2}{7}}y^{-\frac{3}{7}}\,, \end{aligned}$$

$$
\begin{aligned}
v' &= v^{-1}\,, \\
u'_b &= u_b\,.
\end{aligned}
\tag{3.223}
$$

With this identification, we obtain the index of the theory on a sphere with the flux (3.210), which is given by

$$
\begin{aligned}
&\mathcal{I}_{(-1;0^2,2^5,-2)} = \\
&= \oint \mathrm{d}z_N\, \Gamma_e\left(pqt^{-1}\right)^{N-1} \prod_{i<j}^{N} \Gamma_e\left(pqt^{-1} z_i^{\pm 1} z_j^{\pm 1}\right) \prod_{i=1}^{N}\prod_{a=3}^{8} \Gamma_e\left((pq)^{\frac{1}{2}} c^{\frac{1}{2}} u_a z_i^{\pm 1}\right) \times \\
&\quad \times \mathcal{I}_{cap}(\vec{z}; c; t; f; u_1, \ldots, u_7; x) \times \\
&\quad \times \left(\mathcal{I}_{cap}(\vec{z}; c; t; f^{-1}; u_1, \ldots, u_7; x)\Big|_{c \to c^{\frac{1}{2}} x y^5,\, x \to c^{\frac{1}{28}} x^{\frac{13}{14}} y^{-\frac{5}{14}},\, y \to c^{\frac{1}{7}} x^{-\frac{2}{7}} y^{-\frac{3}{7}},\, v \to v^{-1}} \right)\,,
\end{aligned}
\tag{3.224}
$$

where (3.220) is understood for u_a. Furthermore, introducing the following redefinition of the fugacities:

$$
\begin{aligned}
w_{a=1,\ldots,5} &= c^{\frac{1}{6}} x y^{\frac{1}{3}} u_{a+2}\,, \\
w_6 &= c^{-\frac{5}{6}} x^{-5} y^{-\frac{5}{3}}\,, \\
b &= c^{\frac{1}{3}} x^{-2} y^{\frac{5}{3}}\,, \\
d &= c^{-\frac{1}{2}} x y^5 f^{-1}\,,
\end{aligned}
\tag{3.225}
$$

one can see that the index has the manifest $SU(2)_v \times SU(6)_w \times SU(2)_d \times SU(2)_f \times U(1)_b \times U(1)_t$ symmetry as follows:

$$
\begin{aligned}
&\mathcal{I}_{(-1;0^2,2^5,-2)} = \\
&= \prod_{i=1}^{N} \underbrace{\Gamma_e\left(pqt^{i-1} b^{-3} f^{\pm 1} v^{\pm 1}\right)}_{(R^{(i)\pm},\pm \tilde{R}^{(i)}{}_{\mp})} \prod_{i=1}^{N}\prod_{a=1}^{6} \underbrace{\Gamma_e\left(pqt^{i-1} b^{-2} d^{\pm 1} w_a\right)}_{S_{\pm}^{(i)a}} \times \\
&\times \oint \mathrm{d}z_N\, \underbrace{\Gamma_e(t)^{N-1} \prod_{i<j} \Gamma_e\left(t z_i^{\pm 1} z_j^{\pm 1}\right)}_{A} \prod_{i=1}^{N}\prod_{a=1}^{6} \underbrace{\Gamma_e\left((pq)^{\frac{1}{2}} b^{-1} w_a^{-1} z_i^{\pm 1}\right)}_{Q_a} \prod_{i=1}^{N} \underbrace{\Gamma_e\left((pq)^{-\frac{1}{2}} t^{1-N} b^3 d^{\pm 1} z_i^{\pm 1}\right)}_{P^{\pm}}.
\end{aligned}
\tag{3.226}
$$

We comment again that the $SU(2)_f$ symmetry descends from the isometry of the $\mathbb{S}^2$ on which we performed the compactification. The anomalies for this symmetry computed using our $4d$ Lagrangian description are

$$
\mathrm{Tr}\left(SU(2)_f^2 U(1)_R\right) = \frac{N(N+1)}{2}, \quad \mathrm{Tr}\left(SU(2)_f^2 U(1)_t\right) = \frac{N(N-1)}{2},
$$

$$
\begin{aligned}
&\mathrm{Tr}\left(SU(2)_f^2 U(1)_c\right) = -N, \quad \mathrm{Tr}\left(SU(2)_f^2 U(1)_{u_a}\right) = -\frac{N}{2} \quad a = 1, 2, \\
&\mathrm{Tr}\left(SU(2)_f^2 U(1)_{u_a}\right) = -N \quad a = 3, \cdots, 7\,,
\end{aligned} \tag{3.227}
$$

where we went back from the $U(1)$'s used to write the index (3.226) to those parametrizing the Cartan of $U(1)_c \times SU(8)_u \subset E_8$ using (3.220)-(3.225). These anomalies perfectly match those that we can compute from $6d$ (D.20) for the value of the flux (3.210). In this case as well, the mixing of the original $U(1)_f$ from (3.137) with the other $U(1)$'s that gives the correct anomalies matches the mixing prediction in (D.21).

While the manifest UV symmetry visible in the integral expression is

$$
SU(2)_v \times SU(6)_w \times SU(2)_d \times SU(2)_f \times U(1)_b \times U(1)_t\,, \tag{3.228}
$$

the expanded index will show that the global symmetry is enhanced to

$$
E_6 \times SU(2)_v \times SU(2)_f \times U(1)_b \times U(1)_t\,. \tag{3.229}
$$

Notice also that the $SU(2)_v$ and the $SU(2)_f$ symmetries only act on the singlets $R^{(i)\pm}$ and $\tilde{R}^{(i)}{}_{\pm}$. Removing these fields we get a model with $E_6 \times U(1)_b \times U(1)_t$ symmetry. This turns out to be dual under a generalization of the Intriligator–Pouliot duality [81] with antisymmetric matter to the model which was observed to have E_6 enhancement in [155] for $N = 1$ and in [156] for arbitrary N up to some extra flips of operators neutral under $SU(2)_v \times SU(2)_f$. Indeed, applying the a-maximization, we will see that the flipped operators will decouple in the IR as their R-charges fall below the unitarity bound. Then the interacting sector corresponds exactly to the E_6 models in [155] and [156].

Let us again analyze the index of low values of N in more details.

Rank 1

We first compute the index (3.226) for $N = 1$ with the $U(1)_b$ mixing coefficient

$$
R_b \approx 0.5117\,, \tag{3.230}
$$

which is approximated by the following rational value:

$$
R_b = \frac{1}{2}\,. \tag{3.231}
$$

The resulting index is

$$
\begin{aligned}
&\mathcal{I}^{N=1}_{(-1;0^2,2^5,-2)} = \\
&= 1 + b^{-3}\mathbf{2}_{SU(2)_v}\mathbf{2}_{SU(2)_f}(pq)^{\frac{1}{4}} + \left(b^6 + b^{-6}\left(1 + \mathbf{3}_{SU(2)_v}\mathbf{3}_{SU(2)_f}\right) + b^{-2}\overline{\mathbf{27}}_{E_6}\right)(pq)^{\frac{1}{2}} +
\end{aligned}
$$

$$
\begin{aligned}
&+b^{-3}\mathbf{2}_{SU(2)_v}\mathbf{2}_{SU(2)_f}(pq)^{\frac{1}{4}}(p+q)+\\
&+\left(b^{-9}\left(\mathbf{2}_{SU(2)_v}\mathbf{2}_{SU(2)_f}+\mathbf{4}_{SU(2)_v}\mathbf{4}_{SU(2)_f}\right)+b^{-5}\mathbf{2}_{SU(2)_v}\mathbf{2}_{SU(2)_f}\overline{\mathbf{27}}_{E_6}\right)(pq)^{\frac{3}{4}}+\\
&+\left(b^{6}+b^{-6}\left(1+\mathbf{2}_{SU(3)_v}\right)\left(1+\mathbf{3}_{SU(2)_f}\right)+b^{-2}\overline{\mathbf{27}}_{E_6}\right)(pq)^{\frac{1}{2}}(p+q)+\\
&+\left(-\mathbf{78}_{E_6}-\mathbf{3}_{SU(2)_v}-\mathbf{3}_{SU(2)_f}-1+b^{12}+b^{-12}\left(1+\mathbf{3}_{SU(2)_v}\mathbf{3}_{SU(2)_f}+\mathbf{5}_{SU(2)_v}\mathbf{5}_{SU(2)_f}\right)\right.\\
&\left.+b^{-8}\left(1+\mathbf{3}_{SU(2)_v}\mathbf{3}_{SU(2)_f}\right)\overline{\mathbf{27}}_{E_6}+b^{-4}\mathbf{351'}_{E_6}\right)pq+\cdots\,, \qquad (3.232)
\end{aligned}
$$

where the E_6 characters are written in terms of the fugacities w_a and d. The negative terms at order pq highlighted in blue represent the conserved currents for the global symmetry of the theory. We can see that the IR global symmetry is indeed

$$E_6\times SU(2)_v\times SU(2)_f\times U(1)_b\,. \qquad (3.233)$$

Also in this case we can check the presence of gauge invariant operators that descend from the 6d conserved currents. For rank 1 we focus on the branching rule of the adjoint representation of E_8 under the $E_6\times SU(2)\times U(1)$ subgroup:

$$\mathbf{248}\to(\mathbf{1},\mathbf{1})^0\oplus(\mathbf{1},\mathbf{3})^0\oplus(\mathbf{78},\mathbf{1})^0\oplus(\overline{\mathbf{27}},\mathbf{2})^1\oplus(\mathbf{27},\mathbf{2})^{-1}\oplus(\mathbf{27},\mathbf{1})^2\oplus(\overline{\mathbf{27}},\mathbf{1})^{-2}\oplus(\mathbf{1},\mathbf{2})^{\pm2}\,. \qquad (3.234)$$

The 6d R-symmetry is related to the one we used to compute the index (3.232) by the shift $b\to b(pq)^{\frac{1}{4}}$. Moreover, the $U(1)$ inside E_8 for which we turned on a unit of flux is related to $U(1)_b$. With this dictionary, we can identify the states appearing in (3.234) in the index (3.232) up to the order we evaluated it:

$$
\begin{aligned}
(\mathbf{1},\mathbf{2})^{-3} &\quad\to\quad 2b^{-3}\mathbf{2}_{SU(2)_v}(pq)^{\frac{1}{4}}\\
(\overline{\mathbf{27}},\mathbf{1})^{-2} &\quad\to\quad b^{-2}\overline{\mathbf{27}}_{E_6}(pq)^{\frac{1}{2}}\\
(\mathbf{1},\mathbf{1})^0\oplus(\mathbf{1},\mathbf{3})^0\oplus(\mathbf{78},\mathbf{1})^0 &\quad\to\quad -\left(\mathbf{78}_{E_6}+\mathbf{3}_{SU(2)_v}+1\right)pq\,. \qquad (3.235)
\end{aligned}
$$

Notice again that, similarly to the example of the previous subsection, the state $(\mathbf{1},\mathbf{2})^{-3}$ contributes with two operators, which transform as a doublet under the geometric $SU(2)_f$.

The a-maximization tells us that the following operators decouple in the IR:

$$(R^{(1)\pm},\tilde{R}^{(1)\pm})\,, \qquad (3.236)$$

which are in the $(\mathbf{2},\mathbf{2})$ of $SU(2)_v\times SU(2)_f$. In fact, as we mentioned, the singlets $(R^{(1)\pm},\tilde{R}^{(1)\pm})$ are the only chiral multiplets charged under $SU(2)_v\times SU(2)_f$. Thus, the remaining interacting sector is independent of $SU(2)_v\times SU(2)_f$. Indeed, the interacting sector is the Intriligator–Pouliot dual theory [81] of the E_6 model in [155], which was observed to have the enhanced

IR symmetry

$$E_6 \times U(1)_b \,. \tag{3.237}$$

Rank 2

For rank $N = 2$ we compute the index (3.226) with the following mixing coefficients:

$$R_t \approx 0.2221 \approx \frac{1}{4}, \qquad R_b \approx 5539 \approx \frac{4}{7} \,. \tag{3.238}$$

The expansion of the index up to order pq is much more complicated than for the rank 1 case. For simplicity we only show the first few orders and the order pq containing the conserved currents, but we checked that also the other terms organize into characters of the expected global symmetry

$$\begin{aligned}
&\mathcal{I}^{N=2}_{(-1;0^2,2^5,-2)} = \\
&= 1 + b^{-3}\mathbf{2}_{SU(2)_v}\mathbf{2}_{SU(2)_f}(pq)^{\frac{1}{7}} + t^2(pq)^{\frac{1}{4}} + b^{-3}t\mathbf{2}_{SU(2)_v}\mathbf{2}_{SU(2)_f}(pq)^{\frac{15}{56}} + \\
&+ b^{-6}\left(1 + \mathbf{3}_{SU(2)_v}\mathbf{3}_{SU(2)_f}\right)(pq)^{\frac{2}{7}} + b^{-3}t^2\mathbf{2}_{SU(2)_v}\mathbf{2}_{SU(2)_f}(pq)^{\frac{11}{28}} + \\
&+ b^{-6}t\left(1 + \mathbf{3}_{SU(2)_v}\right)\left(1 + \mathbf{3}_{SU(2)_f}\right)(pq)^{\frac{23}{56}} + \\
&+ \left(b^{-9}\left(\mathbf{2}_{SU(2)_v}\mathbf{2}_{SU(2)_f} + \mathbf{4}_{SU(2)_v}\mathbf{4}_{SU(2)_f}\right) + b^{-2}\overline{\mathbf{27}}_{E_6}\right)(pq)^{\frac{3}{7}} + b^6t^{-2}(pq)^{\frac{13}{28}} + t^4(pq)^{\frac{1}{2}} + \\
&+ \cdots + t^4(pq)^{\frac{1}{2}}(p+q) + \left(-\mathbf{78}_{E_6} - \mathbf{3}_{SU(2)_v} - \mathbf{3}_{SU(2)_f} - 2 + 1 + t^8 + \mathbf{3}_{SU(2)_v}\mathbf{3}_{SU(2)_f} + \right. \\
&+ b^{-21}\left(\mathbf{2}_{SU(2)_v}\mathbf{2}_{SU(2)_f} + \mathbf{4}_{SU(2)_v}\mathbf{4}_{SU(2)_f} + \mathbf{6}_{SU(2)_v}\mathbf{6}_{SU(2)_f} + \mathbf{8}_{SU(2)_v}\mathbf{8}_{SU(2)_f}\right) + \\
&\left. + b^{-14}\left(1 + \mathbf{3}_{SU(2)_v}\mathbf{3}_{SU(2)_f} + \mathbf{5}_{SU(2)_v}\mathbf{5}_{SU(2)_f}\right)\overline{\mathbf{27}}_{E_6} + b^{-7}\mathbf{2}_{SU(2)_v}\mathbf{2}_{SU(2)_f}\left(\mathbf{351'}_{E_6} + \mathbf{27}_{E_6}\right)\right)pq + \\
&+ \cdots \,.
\end{aligned} \tag{3.239}$$

Again we can see characters for the E_6 symmetry that is enhanced from the manifest $SU(6)_w \times SU(2)_d$. Moreover, from the negative terms at order pq highlighted in blue we can see the conserved currents for

$$E_6 \times SU(2)_v \times SU(2)_f \times U(1)_b \times U(1)_t \tag{3.240}$$

as expected. Note that there should be at least two $U(1)$ currents because the theory already exhibits $U(1)_b$ and $U(1)_t$. Also in this case we can check the presence of gauge invariant operators that descend from the 6d conserved currents. On top of those that we found for rank 1 coming from the E_8 conserved currents, we now expect also operators coming from the $SU(2)_L$ conserved currents, according to the branching rule of its adjoint representation (3.201). This time the 6d R-symmetry is related to the one we used to compute the index (3.239) by the shifts $b \to b(pq)^{-\frac{2}{7}}$ and $t \to t(pq)^{\frac{3}{8}}$. Moreover, the $U(1)$ inside $SU(2)_L$ for which we turned on a flux -1 coincides with the $U(1)_t$ of our $4d$ model. With this dictionary, we can identify the states appearing in (3.234) and (3.201) in the index (3.239) up to the order we evaluated it:

$$\begin{aligned}
&(\mathbf{1},\mathbf{2})^{(-3,0)} \quad \to \quad 2b^{-3}\mathbf{2}_{SU(2)_v}(pq)^{\frac{1}{7}}\\
&(\overline{\mathbf{27}},\mathbf{1})^{-2} \quad \to \quad b^{-2}\overline{\mathbf{27}}_{E_6}(pq)^{\frac{3}{7}}\\
&(\mathbf{1},\mathbf{1})^{(0,2)} \quad \to \quad t^2(pq)^{\frac{1}{4}}\\
&2\times(\mathbf{1},\mathbf{1})^{(0,0)}\oplus(\mathbf{1},\mathbf{3})^{(0,0)}\oplus(\mathbf{78},\mathbf{1})^{(0,0)} \quad \to \quad -\Big(\mathbf{78}_{E_6}+\mathbf{3}_{SU(2)_v}+2\Big)\,pq\,,
\end{aligned} \tag{3.241}$$

where at the exponent of the states on the left hand side we reported, in order, the charges under $U(1)_b$ and $U(1)_t$. Notice again that the state $(\mathbf{1},\mathbf{2})^{(-3,1)}$ contributes with two operators, which transform as a doublet under the geometric $SU(2)_f$.

We have found that the decoupled operators for $N=2$ are given by

$$(R^{(1)\pm},\tilde{R}^{(1)\pm})\,,\qquad \mathrm{Tr}A^2\,,\qquad (R^{(2)\pm},\tilde{R}^{(2)\pm})\,, \tag{3.242}$$

which correspond to the first three non-trivial terms of the index (3.239) respectively. Note that $(R^{(i)\pm},\tilde{R}^{(i)\pm})$ for $i=1,\,2$ are in the $(\mathbf{2},\mathbf{2})$ of $SU(2)_v\times SU(2)_f$. The remaining interacting sector is dual to the E_6 model considered in [156], which was shown to have the enhanced IR symmetry

$$E_6\times U(1)_b\times U(1)_t\,. \tag{3.243}$$

$SO(14)\times U(1)^2$ sphere model I

We next consider the model corresponding to the flux

$$\mathcal{F}=(-1;0,0,0,0,2,2,2,-2)\ , \tag{3.244}$$

which preserves $SO(14)\times U(1)_b\times U(1)_t\subset E_8\times SU(2)_L$. This model can be obtained by gluing two basic caps with the S-gluing for the first four octet moment maps and the Φ-gluing for the other octet moment maps and the antisymmetric moment maps. From the flux (3.244) we expect that the theory has the global symmetry

$$SO(14)\times U(1)_b\times U(1)_t\,. \tag{3.245}$$

Again we start from the basic cap, the WZ model with the superpotential

$$\mathcal{W}_{cap} = \sum_{i=1}^{N}\sum_{a=1}^{7} \mathrm{Tr}_z \left(R^{(i)a} A^{N-i} P L_a\right) + \sum_{i=1}^{N} \mathrm{Tr}_z \left(\beta_i A^{N-i} P K\right) + K L_8\,, \tag{3.246}$$

which preserves

$$USp(2N) \times SU(7)_u \times U(1)_x \times U(1)_c \times U(1)_t \times U(1)_f\,. \tag{3.247}$$

Since we want to take the S-gluing for $L_{1,\dots,4}$ and the Φ-gluing for $L_{5,\dots,8}$ and A, we introduce $\hat{A}$, $\Phi^{5,\dots,8}$ and the superpotential

$$\mathcal{W}_{glue} = \mathrm{Tr}_z \left[\hat{A} \cdot (A - \tilde{A})\right] + \sum_{a=1}^{4} L_a \tilde{L}^a + \sum_{b=5}^{8} \Phi^b \left(L_b - \tilde{L}_b\right) \tag{3.248}$$

and gauge the puncture symmetry $USp(2N)$. The total superpotential of the glued theory is given by

$$\begin{aligned}
\mathcal{W} = {} & \mathcal{W}_{cap} + \tilde{\mathcal{W}}_{cap} + \mathcal{W}_{glue} = \\
= {} & \sum_{i=1}^{N}\sum_{a=1}^{7} \mathrm{Tr}_z \left(R^{(i)a} A^{N-i} P L_a\right) + \sum_{i=1}^{N} \mathrm{Tr}_z \left(\beta_i A^{N-i} P K\right) + K L_8 + \\
& + \sum_{i=1}^{N}\sum_{a=1}^{4} \mathrm{Tr}_z \left(\tilde{R}^{(i)}{}_a \tilde{A}^{N-i} \tilde{P} \tilde{L}^a\right) + \sum_{i=1}^{N}\sum_{a=5}^{7} \mathrm{Tr}_z \left(\tilde{R}^{(i)a} \tilde{A}^{N-i} \tilde{P} \tilde{L}_a\right) + \sum_{i=1}^{N} \mathrm{Tr}_z \left(\tilde{\beta}_i \tilde{A}^{N-i} \tilde{P} \tilde{K}\right) + \\
& + \tilde{K}\tilde{L}_8 + \mathrm{Tr}_z \left[\hat{A} \cdot (A - \tilde{A})\right] + \sum_{a=1}^{4} L_a \tilde{L}^a + \sum_{b=5}^{8} \Phi^b \left(L_b - \tilde{L}_b\right)\,,
\end{aligned} \tag{3.249}$$

which becomes

$$\mathcal{W} = \sum_{i=1}^{N}\sum_{b=5}^{8}\sum_{\alpha=\pm} \mathrm{Tr}_z \left(S^{(i)b}{}_\alpha A^{N-i} P^\alpha Q_b\right) - \sum_{i,j=1}^{N}\sum_{a=1}^{4} \mathrm{Tr}_z \left(A^{2N-i-j} P^+ P^- R^{(i)a} \tilde{R}^{(j)}{}_a\right) \tag{3.250}$$

once we integrate out the massive fields. We have defined

$$S^{(i)b}{}_+ = \begin{cases} R^{(i)b}\,, & b = 5,\dots,7\,, \\ \beta_i\,, & b = 8\,, \end{cases}$$

$$S^{(i)b}{}_- = \begin{cases} \tilde{R}^{(i)b}\,, & b = 5,\dots,7\,, \\ -\tilde{\beta}_i\,, & b = 8\,, \end{cases}$$

$$Q_b = \begin{cases} L_b\,, & b = 5,\dots,7\,, \\ \frac{K-\tilde{K}-\Phi}{3}\,, & b = 8\,, \end{cases}$$

	$USp(2N)$	$SU(4)_v$	$SU(4)_w$	$U(1)_d$	$U(1)_f$	$U(1)_b$	$U(1)_t$	$U(1)_R$
$S^{(i)}_{\pm}$	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{4}$	± 2	± 1	-1	$i-1$	2
$R^{(i)}$	$\mathbf{1}$	$\mathbf{4}$	$\mathbf{1}$	1	0	-2	$i-1$	2
$\tilde{R}^{(i)}$	$\mathbf{1}$	$\overline{\mathbf{4}}$	$\mathbf{1}$	-1	0	-2	$i-1$	2
Q	$\mathbf{2N}$	$\mathbf{1}$	$\overline{\mathbf{4}}$	0	0	-1	0	1
$P^{\pm}$	$\mathbf{2N}$	$\mathbf{1}$	$\mathbf{1}$	∓ 2	∓ 1	2	$1-Q$	-1
A	$\mathbf{N(2N-1)-1}$	$\mathbf{1}$	$\mathbf{1}$	0	0	0	1	0

Table 3.6: The matter content of the $SO(14)\times U(1)^2$ model I and the corresponding transformation rules under the manifest global symmetry.

$$P^+ = P\,,$$
$$P^- = \tilde{P}\,. \tag{3.251}$$

Note that the superpotential (3.250) preserves[28]

$$SU(4)_v \times SU(4)_w \times U(1)_d \times U(1)_f \times U(1)_b \times U(1)_t \tag{3.253}$$

and the various chiral multiplets transform under it as in Table 3.6.

Interestingly, although the $S^{(i)}$ and P are charged under $U(1)_f$, we will see that it is not a faithful symmetry because no gauge invariant operator is charged under this $U(1)_f$. In addition, we will also see that $SU(4)_v \times SU(4)_w \times U(1)_d$ is enhanced to $SO(14)$. Thus, the global symmetry of the theory is given by

$$SO(14)\times U(1)_b \times U(1)_t\,. \tag{3.254}$$

We should comment that, as we will see later on, this theory flows to a WZ model according to the duality of [86] that we already presented in the Outlook 2.5 for the second chapter. The symmetry (3.254) is the one preserved by the superpotential of the dual WZ model:

$$\mathcal{W} = \sum_{k+l+m+n=2N+1}\sum_{a=1}^{14} A^{(k)}P^{(l)}Q_a^{(m)}Q_a^{(n)}\,, \tag{3.255}$$

[28]While we focus on the global symmetry (3.253) for simplicity, the full symmetry preserved by the superpotential (3.250) is

$$SO(8)_s \times SU(4)_w \times SU(2)_g \times U(1)_b \times U(1)_t \tag{3.252}$$

where $SO(8)_s \times SU(2)_g$ is enhanced from $SU(4)_v \times U(1)_d \times U(1)_f$. The fugacity map between the two groups is given by

$$s_a = dv_a\,,$$
$$g = d^2 f\,.$$

where the dual fields can be identified with the operators of the current model as follows:

$$\begin{aligned} A^{(k)} &\quad\longleftrightarrow\quad \mathrm{Tr}A^k\,, \\ P^{(l)} &\quad\longleftrightarrow\quad \mathrm{Tr}\left[A^{l-1}P^+P^-\right]\,, \\ Q^{(m)} &\quad\longleftrightarrow\quad R^{(m)},\, \tilde{R}^{(m)},\, \mathrm{Tr}\left[A^{m-1}Q_{[a}Q_{b]}\right] \end{aligned} \tag{3.256}$$

Since the theory is a WZ model, the superpotential (3.255) is irrelevant and all the fields in (3.256) become free in the IR. Accordingly, neglecting the superpotential the global symmetry becomes much larger. We will nevertheless keep the superpotential (3.255) because we want to focus on the smaller symmetry (3.254) so to show explicitly that our model conforms to the expectations from $6d$. The true larger symmetry is not in contrast with the $6d$ prediction and it is accidental from this point of view.

Before evaluating the index, we should note that the gluing breaks $SU(7)_u \times U(1)_x$ of the basic cap into $SU(4)_v \times SU(3)_u \times U(1)_y \times U(1)_x$. Thus, we need to redefine the fugacities as follows:

$$\begin{aligned} u_{a=1,\ldots,4} &\quad\longrightarrow\quad x^{-1}y^{-3}v_a\,, \\ u_{b=5,\ldots,7} &\quad\longrightarrow\quad x^{-1}y^{4}u_b\,,\quad, \\ u_8 &\quad\longrightarrow\quad x^7 \end{aligned}$$

where v_a and u_b on the right hand side satisfy $\prod_{a=1}^4 v_a = 1$ and $\prod_{b=5}^7 u_b = 1$ respectively. The octet fugacities of the caps are then given by

$$\eta_a = \left(c^{\frac{1}{2}}x^{-1}y^{-3}v_a; c^{\frac{1}{2}}x^{-1}y^4u_{a-4}; c^{\frac{1}{2}}x^7\right) \tag{3.257}$$

for the left cap and

$$\xi_a = \left(c'^{\frac{1}{2}}x'^{-1}y'^{-3}v_a; c'^{\frac{1}{2}}x'^{-1}y'^4u'_{a-4}; c'^{\frac{1}{2}}x'^7\right) \tag{3.258}$$

for the right cap. Since we take the S-gluing for the first four and the Φ-gluing for the others, we impose the conditions $\eta_a = 1/\xi_a$ for $a = 1, \ldots, 4$ and $\eta_a = \xi_a$ otherwise, which are solved by

$$\begin{aligned} c' &= x^2y^6\,, \\ x' &= c^{\frac{1}{14}}x^{\frac{6}{7}}y^{-\frac{3}{7}}\,, \\ y' &= c^{\frac{1}{7}}x^{-\frac{2}{7}}y^{\frac{1}{7}}\,, \\ v'_a &= v_a^{-1}\,, \\ u'_b &= u_b\,. \end{aligned} \tag{3.259}$$

With this identification, we obtain the index of the theory with the flux (3.244), which is given by

$$\begin{aligned}&\mathcal{I}_{(-1;0^4,2^3,-2)} = \\ &= \oint \mathrm{d}z_N\, \Gamma_e\left(pqt^{-1}\right)^{N-1} \prod_{i<j}^N \Gamma_e\left(pqt^{-1}z_i^{\pm1}z_j^{\pm1}\right) \prod_{i=1}^N \prod_{a=5}^8 \Gamma_e\left((pq)^{\frac{1}{2}}c^{\frac{1}{2}}u_a z_i^{\pm1}\right) \times \\ &\quad\times \mathcal{I}_{cap}(\vec{z};c;t;f;u_1,\dots,u_7;x)\times \\ &\quad\times \left(\mathcal{I}_{cap}(\vec{z};c;t;f^{-1};u_1,\dots,u_7;x)\Big|_{c\to x^2y^6,x\to c^{\frac{1}{14}}x^{\frac{6}{7}}y^{-\frac{3}{7}},y\to c^{-\frac{1}{7}}x^{\frac{2}{7}}y^{-\frac{1}{7}},v_a\to v_a^{-1}}\right)\,,\end{aligned} \tag{3.260}$$

where (3.257) is understood for u_a. Furthermore, if we introduce the following redefinition of the fugacities:

$$\begin{aligned} w_{a=1,2,3} &= c^{\frac{1}{4}}x^{\frac{3}{2}}yu_a\,, \\ w_4 &= c^{-\frac{3}{4}}x^{-\frac{9}{2}}y^{-3}\,, \\ b &= c^{\frac{1}{4}}x^{-\frac{5}{2}}y^3\,, \\ d &= c^{-\frac{1}{2}}xy^3f^{-1}\,, \end{aligned} \tag{3.261}$$

the index is written with the manifest $SU(4)_v \times SU(4)_w \times U(1)_d \times U(1)_f \times U(1)_b \times U(1)_t$ symmetry as follows:

$$\begin{aligned}\mathcal{I}_{(-1;0^4,2^3,-2)} &= \prod_{i=1}^N \underbrace{\prod_{a=1}^4 \Gamma_e\left(pqt^{i-1}b^{-2}(dv_a)^{\pm1}\right)}_{R^{(i)},\,\tilde{R}^{(i)}} \prod_{i=1}^N \prod_{a=1}^4 \underbrace{\Gamma_e\left(pqt^{i-1}b^{-1}(d^2f)^{\pm1}w_a\right)}_{S_\pm^{(i)\,a}} \times \\ &\times \oint \mathrm{d}z_N\, \underbrace{\Gamma_e\left(t\right)^{N-1} \prod_{i<j}^N \Gamma_e\left(tz_i^{\pm1}z_j^{\pm1}\right)}_{A} \prod_{i=1}^N \prod_{a=1}^4 \underbrace{\Gamma_e\left((pq)^{\frac{1}{2}}b^{-1}w_a^{-1}z_i^{\pm1}\right)}_{Q_a} \times \\ &\times \prod_{i=1}^N \underbrace{\Gamma_e\left((pq)^{-\frac{1}{2}}t^{1-N}b^2(d^2f)^{\pm1}z_i^{\pm1}\right)}_{P^{\mp}}\,.\end{aligned} \tag{3.262}$$

Although the integrand depends on f, by explicitly expanding the index, we are able to see that $U(1)_f$ is not a faithful symmetry because the index is independent of f. As we will check, the faithful symmetry of the theory is thus

$$SO(14) \times U(1)_b \times U(1)_t\,. \tag{3.263}$$

The fact that the $U(1)_f$ symmetry is not faithful manifests itself also in the vanishing of all the anomalies involving it. Moreover, we again want to interpret $U(1)_f$ as the symmetry descending from the isometry of the two-sphere. The two statements are indeed compatible, since we find that all of the anomalies for this symmetry computed from 6d (D.20) accidentally vanish for the value of the flux (3.244). As before, the mixing of the original $U(1)_f$ from (3.137) with the other $U(1)$'s that gives the correct anomalies matches the predicted mixing given in (D.21).

We now discuss the index for the low values of N in some detail.

Rank 1

To see the enhanced global symmetry, let us expand the the index (3.262) for some low values of the rank N. Firstly, for $N = 1$, we take the mixing coefficient of $U(1)_b$ with the R-symmetry as

$$R_b = \frac{2}{3}\,, \tag{3.264}$$

which is the value determined by the a-maximization. Note that this is the exact value rather than the approximate one and that this already suggests that the theory is dual to a IR free WZ model. With this mixing coefficient, the expansion of the index is given by

$$\begin{aligned}\mathcal{I}^{N=1}_{(-1;0^4,2^3,-2)} = 1 &+ \left(b^4 + b^{-2}\mathbf{14}_{SO(14)}\right)(pq)^{\frac{1}{3}} + \left(b^8 + b^{-4}\mathbf{104}_{SO(14)}\right)(pq)^{\frac{2}{3}} + \\ &+ \left(b^4 + b^{-2}\mathbf{14}_{SO(14)}\right)(pq)^{\frac{1}{3}}(p+q) + \\ &+ \left(-\mathbf{91}_{SO(14)} - 1 + b^{12} + b^{-6}\mathbf{546}_{SO(14)}\right)pq + \cdots\,,\end{aligned} \tag{3.265}$$

where $\mathbf{m}_{SO(14)}$ is the $\mathbf{m}$-dimensional character of $SO(14)$ written in terms of v_a, w_a and d. For example, the $\mathbf{14}$ of $SO(14)$ is decomposed into

$$\mathbf{14} \quad \longrightarrow \quad (\mathbf{4},\mathbf{1})^1 \oplus (\overline{\mathbf{4}},\mathbf{1})^{-1} \oplus (\mathbf{1},\mathbf{6})^0 \tag{3.266}$$

under $SU(4)_v \times SU(4)_w \times U(1)_d$ and the character is written accordingly. Furthermore, we see that the contribution of the current multiplet, which is highlighted in blue, is in the adjoint representation of

$$SO(14) \times U(1)_b \tag{3.267}$$

as expected.

Also in this case we can check the presence of gauge invariant operators that descend from the 6d conserved currents. For rank 1 we focus on the branching rule of the adjoint representation of E_8 under the $SO(14) \times U(1)$ subgroup

$$\mathbf{248} \rightarrow \mathbf{1}^0 \oplus \mathbf{91}^0 \oplus \mathbf{64}^1 \oplus \overline{\mathbf{64}}^{-1} \oplus \mathbf{14}^{\pm 2}\,. \tag{3.268}$$

The 6d R-symmetry is related to the one we used to compute the index (3.265) by the shift $b \to b(pq)^{-\frac{1}{3}}$. Moreover, the $U(1)$ inside E_8 for which we turned on a unit of flux is related to $U(1)_b$. With this dictionary, we can identify the states appearing in (3.268) in the index (3.265) up to the order we evaluated it

$$\begin{aligned} \mathbf{14}^{-2} \quad &\to \quad b^{-2}\mathbf{14}_{SO(14)}(pq)^{\frac{1}{3}} \\ \mathbf{1}^0 \oplus \mathbf{91}^0 \quad &\to \quad -\left(\mathbf{91}_{SO(14)}+1\right)pq\,. \end{aligned} \tag{3.269}$$

We notice that in this case there is no state contributing with more than one operator and which can form representations of $U(1)_f$. This is compatible with our finding that $U(1)_f$ is not a symmetry of the IR theory.

Note that in this model the chiral operators given by the first non-trivial term of the index (3.265) hit the unitarity bound. In fact these operators become free fields in the IR and are the full content of the theory in the IR, as we remarked below eq. (3.256). This model flows indeed to a WZ model with the superpotential

$$\mathcal{W}^{N=1} = \sum_{a=1}^{14} P^{(1)} Q_a^{(1)} Q_a^{(1)}\,, \tag{3.270}$$

which preserves the symmetry (3.267).

Rank 2

Next we compute the index (3.262) for $N=2$, where we take the mixing coefficients as

$$R_t = \frac{1}{7}\,, \qquad R_b = \frac{7}{10}\,, \tag{3.271}$$

which approximate the irrational values determined by the a-maximization

$$R_t \approx 0.1442\,, \qquad R_b \approx 0.7045\,. \tag{3.272}$$

With those mixing coefficients, the index is given by

$$\begin{aligned} \mathcal{I}^{N=2}_{(-1;0^4,2^3,-2)} = {} & 1 + t^2(pq)^{\frac{1}{7}} + b^4t^{-2}(pq)^{\frac{9}{35}} + t^4(pq)^{\frac{2}{7}} + b^{-2}\mathbf{14}_{SO(14)}(pq)^{\frac{3}{10}} + b^4t^{-1}(pq)^{\frac{23}{70}} + \\ & + b^{-2}t\mathbf{14}_{SO(14)}(pq)^{\frac{13}{35}} + b^4(pq)^{\frac{2}{5}} + t^6(pq)^{\frac{3}{7}} + b^{-2}t^2\mathbf{14}_{SO(14)}(pq)^{\frac{31}{70}} + b^4t(pq)^{\frac{33}{70}} + \\ & + \left(b^8t^{-4} + b^{-2}t^3\mathbf{14}_{SO(14)}\right)(pq)^{\frac{18}{35}} + b^4t^2(pq)^{\frac{19}{35}} + b^2t^{-2}\mathbf{14}_{SO(14)}(pq)^{\frac{39}{70}} + \\ & + t^8(pq)^{\frac{4}{7}} + \left(b^8t^{-3} + b^{-2}t^4\mathbf{14}_{SO(14)}\right)(pq)^{\frac{41}{70}} + b^{-4}(1+\mathbf{104}_{SO(14)})(pq)^{\frac{3}{5}} + \\ & + b^4t^3(pq)^{\frac{43}{70}} + b^2t^{-1}\mathbf{14}_{SO(14)}(pq)^{\frac{22}{35}} + t^2(pq)^{\frac{1}{7}}(p+q) + \\ & + \cdots + \left(-\mathbf{91}_{SO(14)} - 2 + t^{14} + \mathbf{104}_{SO(14)}\right)pq + \cdots\,, \end{aligned} \tag{3.273}$$

where we see the contribution of the conserved current for

$$SO(14) \times U(1)_b \times U(1)_t \,. \tag{3.274}$$

Also in this case we can check the presence of gauge invariant operators that descend from the 6*d* conserved currents. On top of those that we found for rank 1 coming from the E_8 conserved current, we now expect also operators coming from the $SU(2)_L$ conserved current, according to the branching rule of its adjoint representation (3.201). This time the 6*d* R-symmetry is related to the one we used to compute the index (3.273) by the shifts $b \to b(pq)^{-\frac{7}{20}}$ and $t \to t(pq)^{\frac{3}{7}}$. Moreover, the $U(1)$ inside $SU(2)_L$ for which we turned on a flux -1 coincides with the $U(1)_t$ symmetry of our 4*d* model. With this dictionary, we can identify the states appearing in (3.268) and (3.201) in the index (3.273) up to the order we evaluated it

$$\begin{array}{lcl} \mathbf{14}^{(-2,0)} & \to & b^{-2}\mathbf{14}_{SO(14)}(pq)^{\frac{3}{10}} \\ \mathbf{1}^{(0,2)} & \to & t^2(pq)^{\frac{1}{7}} \\ 2\times\mathbf{1}^{(0,0)}\oplus\mathbf{91}^{(0,0)} & \to & -\left(\mathbf{91}_{SO(14)}+2\right)pq \,, \end{array} \tag{3.275}$$

where at the exponent of the states on the left hand side we reported, in order, the charges under $U(1)_b$ and $U(1)_t$. Again there is no state contributing with more than one operator and which can form representations of $U(1)_f$. This is compatible with our finding that $U(1)_f$ is not a symmetry of the IR theory.

Also for $N = 2$ the theory flows to a free theory. One can see that the first non-trivial term of the index (3.273), which corresponds to $\text{Tr}A^2$, is below the unitarity bound. Once we flip this operator, the resulting theory only includes chiral operators hitting the unitarity bound $R = \frac{2}{3}$. Therefore, this models flows to a free theory in the IR where all the chiral operators become free fields. The symmetry (3.274) captured by the index should be understood, like the in $N = 1$ case, as the symmetry preserved during the flow to the free model.

$SO(14) \times U(1)^2$ sphere model II and Csaki–Skiba–Schmaltz duality

Lastly, we analyze an example corresponding to the flux

$$\mathcal{F} = (-1; 1, 1, 1, 1, 1, 1, 3, -1) \,, \tag{3.276}$$

which also preserves $SO(14) \times U(1)_b \times U(1)_t \subset E_8 \times SU(2)_L$ as in the previous case. Most importantly, the flux (3.276) is equivalent to (3.244) up to Weyl reflections of E_8. Therefore, from the 6*d* perspective they stem from the same 6*d* theory compactified on a sphere with the same flux, which thus leads to a duality between the resulting 4*d* theories. Indeed, we will see that the theory with the flux (3.276) is the WZ model dual to the $USp(2N)$ theory in the previous example and that the duality is nothing but the confining duality for $USp(2N)$ with one antisymmetric and six fundamental chirals of [86] that we mentioned in the Outlook 2.5 for the second chapter.

This model can be simply obtained from the cap theory by closing the remaining puncture. First recall that the index of the cap is given by

$$\begin{aligned}&\mathcal{I}_{cap}(\vec{y};c;t;f;u_1,\dots,u_7;x)=\\&=\Gamma_e\left(t\right)^{N-1}\prod_{i<j}^{N}\Gamma_e\left(ty_i^{\pm1}y_j^{\pm1}\right)\prod_{i=1}^{N}\Gamma_e\left(pqt^{i-1}c^{-2}f^{-1}\right)\prod_{i=1}^{N}\Gamma_e\left((pq)^{-\frac{1}{2}}t^{1-N}c^{\frac{3}{2}}fu_8^{-1}y_i^{\pm1}\right)\times\\&\times\prod_{i=1}^{N}\prod_{a=1}^{7}\Gamma_e\left(pqt^{i-1}c^{-1}f^{-1}u_8u_a\right)\prod_{i=1}^{N}\prod_{a=1}^{7}\Gamma_e\left((pq)^{\frac{1}{2}}c^{-\frac{1}{2}}u_a^{-1}y_i^{\pm1}\right)\,.\end{aligned}\tag{3.277}$$

One can repeat the procedure in Subsection 3.5.1 now specializing $y_i=wt^{-\frac{N-2i+1}{2}}$ and $w=(pq)^{\frac{1}{2}}c^{-\frac{1}{2}}t^{\frac{N-1}{2}}u_7^{-1}$ with extra singlets provided to ensure the anomaly matching. The resulting sphere model has the index

$$\begin{aligned}\mathcal{I}_{(-1;1^6,3,-1)}=&\prod_{j=2}^{N}\Gamma_e\left(t^j\right)\prod_{i=1}^{N}\Gamma_e\left(pqt^{i-1}c^{-2}f^{-1}\right)\Gamma_e\left(pqt^{i-1}y^{-2}\right)\Gamma_e\left((pq)^{-1}t^{-2N+1+i}c^2fy^2\right)\times\\&\times\prod_{i=1}^{N}\prod_{a=1}^{6}\Gamma_e\left(pqt^{i-1}c^{-1}f^{-1}y^{-1}xu_a\right)\prod_{i=1}^{N}\prod_{a=1}^{6}\Gamma_e\left(pqt^{i-1}c^{-1}y^{-1}x^{-1}u_a^{-1}\right)\,,\end{aligned}\tag{3.278}$$

where we have made the following redefinition of the fugacities:

$$\begin{aligned}u_{a=1,\dots,6}&\longrightarrow x^{-\frac{1}{2}}u_a\,,\\u_7&\longrightarrow x^{\frac{3}{2}}y\,,\\u_8&\longrightarrow x^{\frac{3}{2}}y^{-1}\,,\end{aligned}\tag{3.279}$$

with new u_a on the right hand side satisfying $\prod_{a=1}^6 u_a=1$. This choice of the fugacities makes only $SU(6)_u\times U(1)_x\times U(1)_y$ manifest. However, as we mentioned, the full symmetry of the theory is supposed to be $SO(14)\times U(1)_b\times U(1)_t$. Indeed, one can introduce the $SO(14)\times U(1)_b$ fugacities defined by

$$\begin{aligned}v_{a=1,\dots,6}&=xu_af^{-\frac{1}{2}}\,,\\v_7&=cy^{-1}f^{\frac{1}{2}}\,,\\b&=c^{\frac{1}{2}}y^{\frac{1}{2}}f^{\frac{1}{4}}\end{aligned}\tag{3.280}$$

so that the index is written in the $SO(14)\times U(1)_b\times U(1)_t$ symmetric way. The mixing of $U(1)_f$ with the other $U(1)$'s reflected in (3.280) exactly matches the predicted mixing given in (D.21). After mixing the dependence of f completely disappears, which is consistent our

	$SO(14)$	$U(1)_b$	$U(1)_t$	$U(1)_R$
$Q^{(n)}$	$\mathbf{14}$	-2	$n-1$	2
$P^{(n)}$	$\mathbf{1}$	4	$-2Q+1+n$	-2
$A^{(n)}$	$\mathbf{1}$	0	n	0

Table 3.7: The matter content of the $SO(14)\times U(1)^2$ model II and the corresponding transformation rules under the global symmetry.

claim that this is not a faithful symmetry of the model and with the fact that we expect this $U(1)_f$ to descend from the isometry of the two-sphere, whose anomalies computed from 6d (D.20) vanish for the value of the flux (3.276). With those fugacities in (3.280), the index is now written as

$$\mathcal{I}_{(-1;1^6,3,-1)}=\prod_{j=2}^{N}\underbrace{\Gamma_e\left(t^j\right)}_{A^{(j)}}\prod_{i=1}^{N}\underbrace{\Gamma_e\left((pq)^{-1}t^{-2N+1+i}b^4\right)}_{P^{(i)}}\prod_{i=1}^{N}\underbrace{\prod_{a=1}^{7}\Gamma_e\left(pqt^{i-1}b^{-2}v_a^{\pm1}\right)}_{Q^{(i)}}\,. \tag{3.281}$$

From the index, one can read that the corresponding 4d theory is a WZ model with the superpotential

$$\mathcal{W}=\sum_{k+l+m+n=2N+1}\sum_{a=1}^{14}A^{(k)}P^{(l)}Q_a^{(m)}Q_a^{(n)}\,, \tag{3.282}$$

with $k=2,\dots,N$ and $l,m,n=1,\dots,N$. The superpotential (3.282) preserves

$$SO(14)_v\times U(1)_b\times U(1)_t \tag{3.283}$$

as expected from the 6d perspective. The transformation rules of each chiral multiplet under this global symmetry are presented in Table 3.7. Note as this is a WZ model it flows to a collection of free chiral fields with the above symmetry being a subgroup of the symmetry of the free theory in the IR.

One can also explicitly expand the index (3.281). With the R-symmetry mixing coefficient of $U(1)_b$

$$R_b=\frac{2}{3}\,, \tag{3.284}$$

the index (3.281) for $N=1$ reads

$$\begin{aligned}\mathcal{I}^{N=1}_{(-1;1^6,3,-1)}=&\,1+\left(b^4+b^{-2}\mathbf{14}_{SO(14)}\right)(pq)^{\frac{1}{3}}+\left(b^8+b^{-4}\mathbf{104}_{SO(14)}\right)(pq)^{\frac{2}{3}}+\\&+\left(b^4+b^{-2}\mathbf{14}_{SO(14)}\right)(pq)^{\frac{1}{3}}(p+q)+\\&+\left(-\mathbf{91}_{SO(14)}-1+b^{12}+b^{-6}\mathbf{546}_{SO(14)}\right)pq+\dots\,,\end{aligned} \tag{3.285}$$

which is exactly the same as the index (3.265) of the $SO(14) \times U(1)_b$ model I. Indeed, the two models for general N have been described as dual theories [86], whose index agreement was also shown in [157]. We presented the statement of this duality in the Outlook 2.5 for the second chapter, where we already mentioned that we would have given a novel way to understand this duality from the 6d perspective. We have shown that both theories are obtained by compactifying the same E-string theory on a sphere with the same flux because the flux (3.276) is equivalent to (3.244) up to Weyl reflections of the E_8 symmetry of the 6d theory. This leads to a duality between the resulting 4d theories, both of which should exhibit at least the global symmetry $SO(14) \times U(1)_b \times U(1)_t$.

Also in this case, as for the dual model corresponding to the flux (3.244), all of the anomalies for the $U(1)_f$ symmetry trivially vanish because all the chiral multiplets are neutral under $U(1)_f$. This is again compatible with the interpretation of this $U(1)_f$ as the symmetry descending from the isometry of the two-sphere, since we find that all of the anomalies for this symmetry computed from 6d (D.20) accidentally vanish also for the value of the flux (3.276).

3.6 Outlook

In this chapter we have seen a powerful approach to find new symmetry enhancements and dualities for 4d $\mathcal{N} = 1$ theories, which is based on the compactification of 6d $\mathcal{N} = (1, 0)$ SCFTs on Riemann surfaces with fluxes. In particular, we focused on the compactifications of the rank-N E-string theory on surfaces that can be built from tubes and caps, that is tori and spheres, with various values of flux. With this strategy we managed to find several four-dimensional models enjoying global symmetry enhancements, in some cases even to exceptional groups, and dualities, such as the braid relation that generalizes the Seiberg duality and the confining duality for $USp(2N)$ with one antisymmetric and six fundamental chirals. An important role in the construction of the E-string models was played by the $E[USp(2N)]$ theory. We have seen that this theory is a 4d avatar of the $M[SU(N)]$ and the $T[SU(N)]$ theories and we discussed several properties that it enjoys at low energies, most importantly its symmetry enhancement and its duality web.

There are several directions that one may follow starting from the topic of this chapter. One of these is to investigate some aspects of the E-string compactifications that are still not fully understood. For example, we have discussed the basic tube model from which we constructed more tubes, caps, tori and spheres, but in order to construct the most general Riemann surface we need another ingredient: the trinion theory associated to the compactification on a sphere with three punctures. Two trinions for two different choices of flux have been found for the rank $N = 1$ E-string theory in [124, 125], but the higher rank trinion is still unknown.

Another open problem regarding E-string compactifications is related to punctures. Here we only discussed maximal punctures, that is punctures that carry a $USp(2N)$ global symmetry, but one may also consider other types of punctures. From the field theory point of view we expect that to partially close the punctures, that is to break the symmetry that they carry to $USp(2N) \to \prod_k USp(2n_k)$, we should give a VEV to the antisymmetric operators of $E[USp(2N)]$. Such VEVs have been studied in detail [145] and we will encounter them in the next chapter. What is missing is instead the $6d$ prediction for such models. Specifically, it is not clear yet what should be the contribution to the anomalies of such non-maximal punctures.

Another possible interesting line of research is to try to apply the same strategy but to the compactification of $5d$ $\mathcal{N}=1$ SCFTs on Riemann surfaces with fluxes so to engineer $3d$ $\mathcal{N}=2$ theories. This possibility has been recently investigated in [83], where the compactifications of the $5d$ rank-1 E_{N_f+1} Seiberg SCFTs [131] on tubes and tori have been studied. We said that the realm of $6d$ $\mathcal{N}=(1,0)$ SCFTs is wide, but the one of $5d$ $\mathcal{N}=1$ SCFTs is even larger. Indeed, if we compactify a $6d$ $\mathcal{N}=(1,0)$ SCFT on a circle we obtain a $5d$ $\mathcal{N}=1$ SCFT and from this we can flow via mass deformations to other $5d$ theories that are not the dimensional reduction of any $6d$ theory. Understanding how these mass deformations are mapped in the $3d$ $\mathcal{N}=2$ theories is a first non-trivial task. Moreover, in three dimensions there are new ingredients that are not present in four dimensions, namely Chern–Simons interactions for the gauge fields and monopole operators that may be turned on in the superpotential. This makes the analysis of the compactifications from $5d$ to $3d$ more complicated, but also more interesting. The outcome consists also in this different setup of a series of $3d$ $\mathcal{N}=2$ theories enjoying non-trivial symmetry enhancements and dualities. The latter can in some cases be explained in terms of known fundamental dualities, such as the Aharony duality.

One may also wonder if one can further relate the findings of this chapter with those of the previous one. Indeed, we found a $6d$ origin for some $4d$ results that can be considered as an uplift of part of those that we encountered in $3d$ in Chapter 2. Specifically, we saw that the $E[USp(2N)]$ theory appearing in the basic E-string tube is the $4d$ uplift of the $M[SU(N)]$ theory. Moreover, we gave a $6d$ origin in terms of sphere compactifications with fluxes related by an element of the E_8 Weyl group of the confining duality for the $USp(2N)$ gauge theory with one antisymmetric and six fundamental chirals, which can be reduced to the $3d$ confining duality for the $U(N)$ gauge theory with one adjoint and one fundamental chiral that we studied in detail in the previous chapter. It is then natural to ask whether the other dualities we discussed in Chapter 2 admit a $4d$ uplift and if this can be equivalently derived from the compactification of some $6d$ SCFT. The $4d$ version of the recombination and of the rank stabilization duality will be discussed in [92], but it is not clear yet if these can be understood from a six-dimensional point of view.

Finally, there are various interesting questions concerning the $E[USp(2N)]$ theory. The first one is if it can be understood as a domain wall interpolating between $5d$ $\mathcal{N}=1$ theories.

It is known that the $T[SU(N)]$ theory is the S-duality domain wall for the $4d$ $\mathcal{N}=4$ $SU(N)$ SYM theory [12] and, since $E[USp(2N)]$ is a $4d$ ancestor of $T[SU(N)]$, it is natural to expect that also $E[USp(2N)]$ can be understood as a domain wall. In fact, this interpretation will be strengthened even more in [158], where it will be shown that $E[USp(2N)]$ enjoys many properties that are characteristic of the S element of $SL(2,\mathbb{Z})$. We can already give one example of this: the braid duality we saw in Subsection 3.3.5 admits a $3d$ limit to the duality relating the gluing of two $T[SU(N)]$ tails with a CS level -1 to a single $T[SU(N)]$ with some background CS for its non-abelian global symmetries, which is known to correspond to the relation $(ST)^3=-1$ of the $SL(2,\mathbb{Z})$ generators where S is identified with $T[SU(N)]$ and T with the introduction of a CS level [12, 159]. Moreover, the E-string perspective suggests us that the basic tube model, that we recall is composed of the $E[USp(2N)]$ plus two octets of chirals, is a domain wall interpolating between two copies of the $5d$ $USp(2N)$ gauge theory with one antisymmetric and eight fundamental hypermultiplets. This is actually how the tube model was conjectured for the rank $N=1$ case in [116].

Finally, we would like to remark that the mirror-like duality of $E[USp(2N)]$ is very peculiar. Indeed, it represents the first example of a $4d$ duality that reduces in the $3d$ limit to an instance of mirror symmetry, specifically the mirror self-duality of $T[SU(N)]$, while no other $3d$ mirror duality had a known $4d$ origin before. The topic of the next chapter will be based on this observation and we will find an infinite class of four-dimensional theories that enjoy similar mirror-like dualities.

Chapter 4

Mirror dualities in $4d$

In this chapter we will introduce a class of $4d$ $\mathcal{N}=1$ quiver gauge theories that we call $E_\rho^\sigma[USp(2N)]$ theories and which are related in pairs by a novel type of IR duality. In order to derive these, we will go back to a bottom-up approach, which is thus more similar in spirit to the one of Chapter 2. Indeed, these new four-dimensional dualities are an uplift of the three-dimensional mirror symmetry [13]. The content of this paper is mostly taken from [145].

4.1 The general idea

The starting point of this chapter is the observation that the self-duality of the $E[USp(2N)]$ theory that we defined "mirror-like" in Section 3.3 represents a genuinely new type of IR duality in four dimensions under several aspects. The name "mirror-like" is due to the fact that it shares many features with the $3d$ self-duality of $T[SU(N)]$ under mirror symmetry. We also saw that one can explicitly derive the $3d$ duality from the $4d$ one by considering a flow across dimensions and suitable mass deformations. The peculiarity of this duality is that it exchanges the two $USp(2N)$ symmetries of $E[USp(2N)]$, in the same way as mirror symmetry exchanges the two $SU(N)$ symmetries of $T[SU(N)]$. Both in the $4d$ and in the $3d$ case, the duality is intertwined with a symmetry enhancement that the theories enjoy: for $E[USp(2N)]$ one $USp(2N)$ symmetry is enhanced from the $SU(2)$ symmetries of the saw, while for $T[SU(N)]$ the $SU(N)$ symmetry of the Coulomb branch is enhanced from the $U(1)$ topological symmetries. For this reason, it is useful to think of the saw of $E[USp(2N)]$ as implementing a symmetry which is the analogue of the topological symmetry in $3d$.

Another crucial observation related to this is that the mirror self-duality of $E[USp(2N)]$ is the first case of a $4d$ IR duality that can be reduced to an instance of mirror symmetry in $3d$[1]. As we mentioned in several occasions, it is possible to reduce a $4d$ duality to a $3d$ duality and this strategy has been intensively used both to connect known $3d$ dualities to

[1] A derivation of a $3d$ mirror duality from $6d$ has been discussed in [159].

$4d$ dualities and to discover new ones [5, 6, 160–163, 41, 164, 37, 165, 166]. Nevertheless, this has been possible only for the so called *Seiberg-like* dualities in three-dimensions, whose structure is identical to the four-dimensional Seiberg duality and its variants. In particular, these dualities don't exchange Higgs and Coulomb branch, which is instead the distinctive feature of mirror symmetry. In contrast, no $4d$ ancestor of a mirror duality was known before the $E[USp(2N)]$ theory was introduced. This naturally leads us to ask the following question:

> *Is it possible to uplift* all *the mirror dualities, or at least those between Lagrangian theories, to four dimensions?*

In this chapter we will try to partially answer this questions by finding the $4d$ version of a particular class of mirror dualities in $3d$. Specifically, we will consider the set of $3d$ $\mathcal{N}=4$ linear quiver gauge theories with unitary gauge nodes schematically represented in Figure 4.1. As we did in the Introduction when we reviewed the $T[SU(N)]$ theory, we will use an $\mathcal{N}=2$ notation to describe these theories, in which a $\mathcal{N}=4$ vector multiplet is described by a $\mathcal{N}=2$ vector multiplet and an adjoint chiral represented with an arc in the quiver, while a $\mathcal{N}=4$ hypermultiplet is decomposed into a pair of chirals in complex conjugate representation represented by double lines in the quiver. This choice is due to the fact that, as we already saw for $E[USp(2N)]$, the $4d$ uplift of these theories will only have $\mathcal{N}=1$ supersymmetry. Recall indeed that $\mathcal{N}=4$ in $3d$ is recovered only after the real mass deformation that relates $M[SU(N)]$ and $T[SU(N)]$.

This class of theories is usually denoted by $T^\sigma_\rho[SU(N)]$ [12] since all the data of the theory can be encoded in two partitions of N that we denote by σ and ρ and which we write as

$$\rho=\left[N^{l_N},\dots,1^{l_1}\right]\,,\qquad \sigma=\left[N^{k_N},\dots,1^{k_1}\right]\,,\tag{4.1}$$

where some of the l_n, k_m integers can be zero and must satisfy the conditions

$$\begin{gathered}\sum_{n=1}^N n\times l_n=\sum_{m=1}^N m\times k_m=N\,,\\ L=l_1+\cdots+l_N\,,\qquad K=k_1+\cdots+k_N\,.\end{gathered}\tag{4.2}$$

Since the theory has $\mathcal{N}=4$ supersymmetry, all we need to specify the theory are the ranks for the gauge nodes N_i and for the flavor nodes M_i, which can be extracted from the partitions according to

$$\begin{aligned}&M_{L-i}=k_i\,,\\ &N_{L-i}=\sum_{j=i+1}^L\rho_j-\sum_{j=i+1}^N(j-i)k_j\,.\end{aligned}\tag{4.3}$$

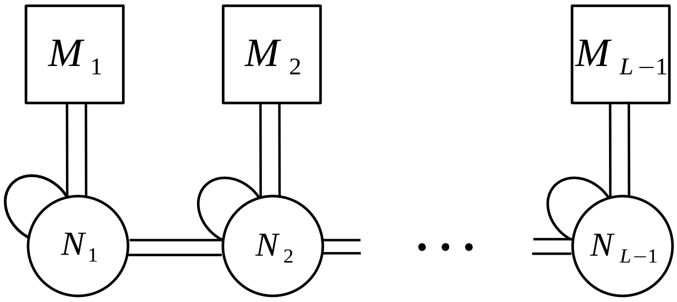

Figure 4.1: The quiver description of the $T_\rho^\sigma[SU(N)]$ theory. The ranks of the gauge and flavor nodes are determined in terms of the partitions σ and ρ as in eq. (4.3). As usual our notation is that for the $3d$ quivers all the nodes correspond to unitary groups, in contrast to the $4d$ quivers which have symplectic groups.

The $T[SU(N)]$ theory corresponds to the particular case in which both partitions are trivial $\sigma = \rho = [1^N]$.

The $T_\rho^\sigma[SU(N)]$ theories are related in pairs by mirror symmetry which exchanges the two partitions, namely $T_\rho^\sigma[SU(N)]$ is mirror dual to $T_\sigma^\rho[SU(N)]$. This comes about with a necessary enhancement of the topological symmetry that acts on the CB, similarly to what happens for $T[SU(N)]$. Indeed, the full global symmetry of $T_\rho^\sigma[SU(N)]$ is

$$S\left(\prod_{i=1}^{N} U(k_i)\right) \times S\left(\prod_{i=1}^{N} U(l_i)\right) , \tag{4.4}$$

where the integers l_n, k_m are the same appearing in the partitions (4.1). The factor $S(\prod_{i=1}^N U(k_i))$ corresponds to the flavor symmetry acting on the HB and it is manifest in the Lagrangian description. The factor $S(\prod_{i=1}^N U(l_i))$ is instead enhanced from the topological symmetries. This pattern of symmetry enhancement is compatible with the action of mirror symmetry which exchanges HB and CB as well as the two partitions.

It is also important to mention that the $T_\rho^\sigma[SU(N)]$ theory can be realized on a brane set-up [14, 12] with N D3-branes suspended between K D5-branes and L NS5-branes, where K and L are the lengths of the partitions σ and ρ respectively. The integers σ_i in $\sigma = [\sigma_1, \cdots, \sigma_K]$ are the net number of D3-branes ending on the D5-branes going from the interior to the exterior of the configuration, while the integers ρ_i in $\rho = [\rho_1, \cdots, \rho_L]$ are the net number of D3-branes ending on the NS5-branes again going from the interior to the exterior.

Our main goal will be to find a $4d$ analogue of the $T_\rho^\sigma[SU(N)]$ theories and their associated mirror dualities. For this purpose, it will turn out to be useful the fact that the $T_\rho^\sigma[SU(N)]$ theory can be reached from the $T[SU(N)]$ theory by giving nilpotent VEVs to the Higgs and the Coulomb branch moment maps $\mathcal{H}$ and $\mathcal{C}$ labelled by σ and ρ respectively. These VEVs initiate sequential Higgs mechanisms which are quite intricate to follow[2]. Indeed one typically relies on the brane realization of the theory. Here we propose an alternative procedure to systematically derive $T_\rho^\sigma[SU(N)]$ theories from $T[SU(N)]$ which is based on field theory methods only. This is going to be crucial for us, since a brane set-up for $E[USp(2N)]$ is not known at the moment, so we will need to apply the same field theory procedure in $4d$ to construct a new family of $4d$ theories, which we name $E_\rho^\sigma[USp(2N)]$ theories, enjoying mirror-like dualities.

Our approach relies on the duality web of $T[SU(N)]$ that we reviewed in the Introduction, see Figure 1.2. In order to study the nilpotent VEV of $T[SU(N)]$, we first notice that it can be implemented by adding singlets flipping some components of its moment maps and by turning them on linearly in the superpotential. The F-term equations of the singlets then fix the VEV of these components of the moment maps to a non-vanishing value. Remember also that $FFT[SU(N)]$ is defined as $T[SU(N)]$ with the addition of two matrices of singlets $\mathcal{O}_\mathcal{H}$ and $\mathcal{O}_\mathcal{C}$ in the adjoint representations of the two $SU(N)$ global symmetries that flip the moment map operators. Hence, the IR theory obtained turning on a VEV in $T[SU(N)]$ is equivalently reached by deforming $FFT[SU(N)]$ by a linear superpotential in some of the components of $\mathcal{O}_\mathcal{H}$ and $\mathcal{O}_\mathcal{C}$ and by removing those that become free after the deformation. That is, we claim that by deforming $FFT[SU(N)]$ by

$$\delta\mathcal{W}_{FF} = \text{Tr}_X\left[\left(\mathcal{J}_\sigma + \mathcal{S}_\sigma\right)\mathcal{O}_\mathcal{H}\right] + \text{Tr}_Y\left[\left(\mathcal{J}_\rho + \mathcal{T}_\rho\right)\mathcal{O}_\mathcal{C}\right]\,, \tag{4.5}$$

where $\mathcal{J}_\sigma$ and $\mathcal{J}_\rho$ are block diagonal Jordan matrices encoding the VEV, while $\mathcal{S}_\sigma$ and $\mathcal{T}_\rho$ are matrices of gauge singlets (both of these will be described in more details later), we flow to $T_\rho^\sigma[SU(N)]$ as shown in the bottom left corner of Figure 4.2.

Instead of following the effects of this deformation, which is in general quite difficult especially in case of VEVs for the monopole operators, we can use the flip-flip duality, under which we recall that the moment maps $\mathcal{H}$ and $\mathcal{C}$ are mapped to the singlets $\mathcal{O}_\mathcal{H}$ and $\mathcal{O}_\mathcal{C}$ as we saw in (1.16). In this way, the deformation we are considering is turned into a deformation of $T[SU(N)]$ linear in the entries of the moment maps, that is, in this frame rather than turning on VEVs, we turn on mass and monopole deformations

$$\delta\mathcal{W} = \text{Tr}_X\left[\left(\mathcal{J}_\sigma + \mathcal{S}_\sigma\right)\mathcal{H}\right] + \text{Tr}_Y\left[\left(\mathcal{J}_\rho + \mathcal{T}_\rho\right)\mathcal{C}\right]\,. \tag{4.6}$$

This deformation triggers a flow to theory $\mathcal{T}$, in the upper left corner of Figure 4.2, which is flip-flip dual to $T_\rho^\sigma[SU(N)]$. We will show that moving along the vertical edge of the web from $\mathcal{T}$ to $T_\rho^\sigma[SU(N)]$ by means of the flip-flip duality is equivalent to iteratively applying a combination of the Aharony and the one-monopole duality that we saw in Subsection 2.3.3.

[2] In [167] the VEVs were implemented at the level of the Hilbert series by means of a residue procedure.

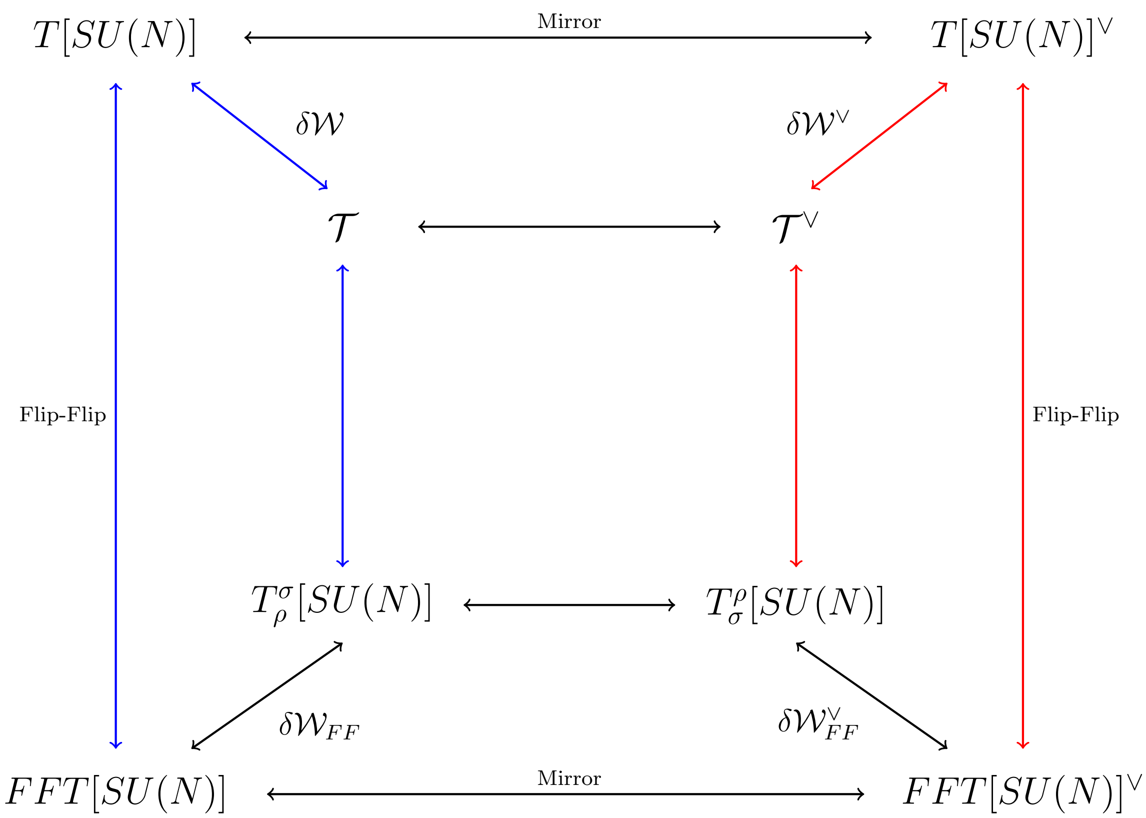

Figure 4.2: Deformed duality web for $T[SU(N)]$. In blue we highlighted the path that we will use to reach the $T_\rho^\sigma[SU(N)]$ theory, while in red we have the one for its mirror dual $T_\sigma^\rho[SU(N)]$.

This procedure is similar to the derivation of the flip-flip duality of the original $T[SU(N)]$ theory by iterating the Aharony duality which we already mentioned. Flowing from $T[SU(N)]$ to $\mathcal{T}$ and then moving to the dual $T_\rho^\sigma[SU(N)]$ allows us to bypass the study of the sequential Higgs mechanism initiated by the VEVs, which, in the case of monopole VEV, is particularly complicated. This path to reach the $T_\rho^\sigma[SU(N)]$ theory is highlighted in blue in Figure 4.2.

We can then apply the same procedure to the mirror dual frame. The $T_\sigma^\rho[SU(N)]$ theory can be obtained by deforming $FFT[SU(N)]^\vee$ by a linear superpotential

$$\delta\mathcal{W}_{FF}^\vee = \mathrm{Tr}_Y\left[\left(\mathcal{J}_\rho + \mathcal{T}_\rho\right)\mathcal{O}_\mathcal{H}^\vee\right] + \mathrm{Tr}_X\left[\left(\mathcal{J}_\sigma + \mathcal{S}_\sigma\right)\mathcal{O}_\mathcal{C}^\vee\right] , \tag{4.7}$$

as shown in the bottom right corner of Figure 4.2, which corresponds, in the flip-flip dual frame, to a deformation of $T[SU(N)]^\vee$ by

$$\delta\mathcal{W}^\vee = \mathrm{Tr}_Y\left[\left(\mathcal{J}_\rho + \mathcal{T}_\rho\right)\mathcal{H}^\vee\right] + \mathrm{Tr}_X\left[\left(\mathcal{J}_\sigma + \mathcal{S}_\sigma\right)\mathcal{C}^\vee\right] . \tag{4.8}$$

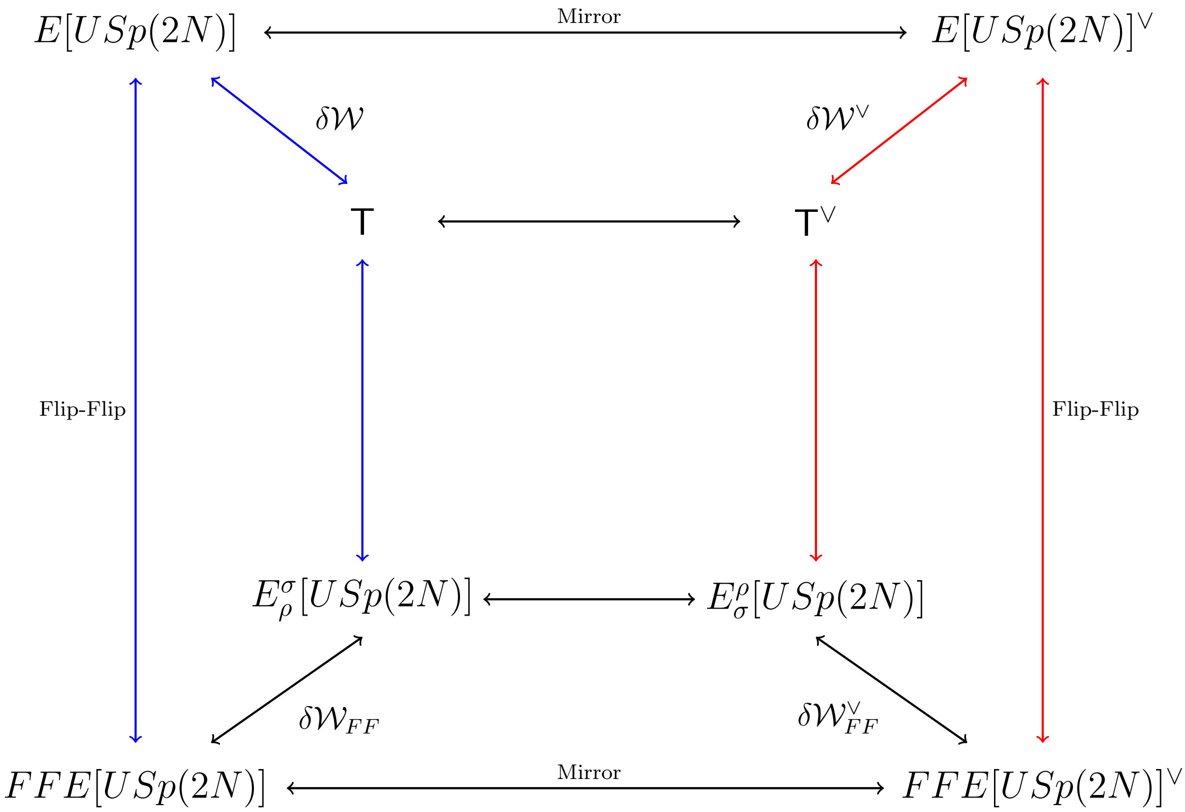

Figure 4.3: Deformed duality web for $E[USp(2N)]$. In blue we highlighted the path that we will use to reach the $E_\rho^\sigma[USp(2N)]$ theory, while in red we have the one for its mirror dual $E_\sigma^\rho[USp(2N)]$.

This deformation triggers a flow to the theory $\mathcal{T}^\vee$, in the upper right corner of Figure 4.2, which is flip-flip dual to $T_\sigma^\rho[SU(N)]$. Again, we will go from $\mathcal{T}^\vee$ to $T_\sigma^\rho[SU(N)]$ by iteratively applying the Aharony duality and the one-monopole duality. This path to reach the $T_\sigma^\rho[SU(N)]$ theory is highlighted in red in Figure 4.2.

In analogy with the $3d$ case, we would like now to consider deformations of the $E[USp(2N)]$ theory triggered by VEVs of the operators H and C, since these descend to the moment maps $\mathcal{H}$ and $\mathcal{C}$ of $T[SU(N)]$. This will lead us to a new class of theories that we call $E_\rho^\sigma[USp(2N)]$ and that are related in pairs by mirror-like dualities. Studying the Higgsing initiated by such VEVs is however quite tricky and in the $4d$ case we don't have a brane realization for $E[USp(2N)]$. Hence, the field theory procedure we just described in $3d$ will be very useful for this purpose. This can indeed be exported to $4d$ since we recall that also $E[USp(2N)]$ enjoys a duality web which is very similar to the one of $T[SU(N)]$ and which we summarized in Figure 3.7. The VEVs we are interested in, when mapped in the various duality frames, lead to the deformed duality web represented in Figure 4.3, which we will use to obtain the $E_\rho^\sigma[USp(2N)]$ theories and their mirror dualities.

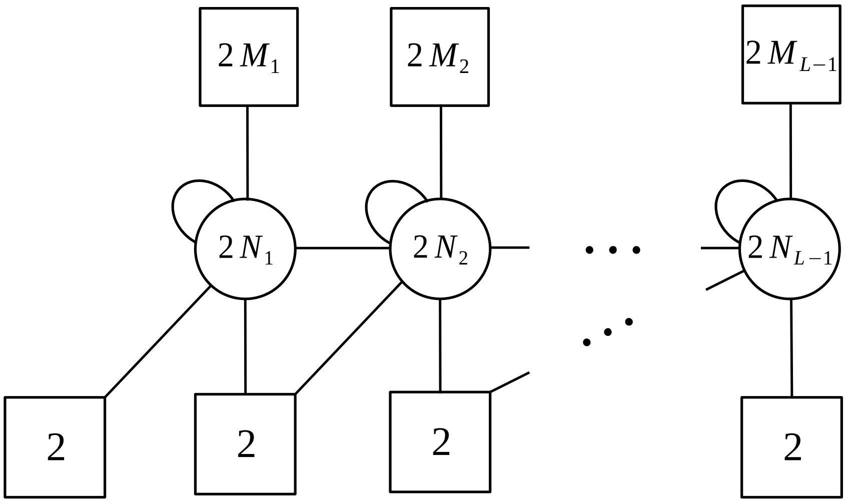

Figure 4.4: The quiver description of the $E_\rho^\sigma[USp(2N)]$ theory. The drawing is schematic, as we are note specifying the singlets and the superpotential of the theory. The ranks of the gauge and flavor nodes are determined in terms of the partitions σ and ρ as in eq. (4.3). As usual our notation is that for the $3d$ quivers all the nodes correspond to unitary groups, in contrast to the $4d$ quivers which have symplectic groups.

More precisely, we name $E_\rho^\sigma[USp(2N)]$ the theories obtained turning on VEVs for C and H labelled by partitions ρ and σ of N. They are the quiver theories with $USp(2n)$ gauge and flavor nodes depicted in Figure 4.4, where the ranks N_i and M_i are related to the data of the partitions σ and ρ as in (4.3). There are also additional singlet fields and various superpotential terms that should be included in the definition of the theory, on which we will give more details later. Because of the VEV, the two $USp(2N)$ global symmetries of $E[USp(2N)]$ are broken to subgroups, according to the particular partitions chosen. Moreover, as a consequence of the duality web we have that $E_\rho^\sigma[USp(2N)]$ is dual to $E_\sigma^\rho[USp(2N)]$. This duality is a $4d$ version of the mirror duality between $T_\rho^\sigma[SU(N)]$ and $T_\sigma^\rho[SU(N)]$. It implies that the $SU(2)$ symmetries of the saw of $E_\rho^\sigma[USp(2N)]$ can be collected into groups that are enhanced at low energies to $\prod_{i=1}^N USp(2l_i)$, so the total full IR global symmetry is

$$\prod_{i=1}^N USp(2k_i) \times \prod_{i=1}^N USp(2l_i) \times U(1)^2\,, \tag{4.9}$$

where the first non-abelian factor is the manifest flavor symmetry, while the abelian symmetries are the same $U(1)_t$ and $U(1)_c$ that we had in $E[USp(2N)]$. Once again, we stress the similarity between this enhancement and the enhancement of the topological symmetry in $3d$. In fact, the $SU(2)$ symmetries of the saw in $4d$ and the topological symmetries in $3d$ are related to each other by the flow from $E^{\rho}_{\sigma}[USp(2N)]$ to $T^{\sigma}_{\rho}[SU(N)]$, which works exactly in the same way as the flow from $E[USp(2N)]$ to $T[SU(N)]$ that we saw in Subsections 2.4.1 and 3.3.3.

We will start by giving more details on how to recover the $T^{\sigma}_{\rho}[SU(N)]$ theories and their mirror dualities by exploiting the duality web of $T[SU(N)]$. After having discussed our strategy, we will explicitly work out some concrete examples, one corresponding to the partitions $\sigma = [1^N]$ and $\rho = [N-1, 1]$ and the other to the partitions $\sigma = [1^4]$ and $\rho = [2, 1^2]$. Once again we will perform all the necessary manipulations at the level of the $\mathbb{S}^3_b$ partition function. The case $\sigma = [1^N]$ and $\rho = [N-1, 1]$, in particular, leads to the so-called *abelian mirror symmetry*, which relates SQED with N flavors to a linear quiver with $N-1$ $U(1)$ gauge nodes and one extra flavor at each of the two ends. It is known that this duality can be derived by iterating the fundamental case of $N = 1$, which says that $U(1)$ with one flavor is dual to a free hyper, by means of a *piecewise procedure* that was first discussed in [168].

We will then move to the four-dimensional set-up and apply the same strategy to arrive at the $E^{\sigma}_{\rho}[USp(2N)]$ theories and their mirror dualities. Again we will discuss the general strategy and then some explicit case. Specifically, we will consider the cases $\sigma = [1^N]$ and $\rho = [N]$, $\sigma = [1^N]$ and $\rho = [N-1, 1]$, $\sigma = [1^4]$ and $\rho = [2^2]$, $\sigma = [1^4]$ and $\rho = [2, 1^2]$ and finally $\sigma = \rho = [2^3, 1]$. In $4d$ we will perform all the necessary manipulations at the level of the $\mathbb{S}^3 \times \mathbb{S}^1$ partition function. In particular, the case $\sigma = [1^N]$ and $\rho = [N-1, 1]$ will provide a $4d$ uplift of the abelian mirror symmetry that we just mentioned. Interestingly, we are able to derive this duality by iterating piecewise the one for $N = 1$ also in this $4d$ set-up, similarly to what happens in $3d$. This derivation is given in Appendix C.3.

4.2 $3d$ mirror symmetry and $T^{\sigma}_{\rho}[SU(N)]$ theories using the web

4.2.1 The strategy

$T^{\sigma}_{\rho}[SU(N)]$ can be obtained as a deformation of $T[SU(N)]$ corresponding to giving nilpotent VEVs labelled by partitions σ and ρ of N to the moment maps:

$$\langle \mathcal{H} \rangle = \mathcal{J}_{\sigma}\,, \qquad \langle \mathcal{C} \rangle = \mathcal{J}_{\rho}\,, \tag{4.10}$$

where $\mathcal{J}_\rho$ and $\mathcal{J}_\sigma$ are $N \times N$ block diagonal matrices with each block being a Jordan matrix that can be uniquely determined after specifying the partitions σ and ρ

$$\mathcal{J}_\rho = \bigoplus_{i=1}^{L} \mathbb{J}_{\rho_i} = \left(\begin{array}{c|c|c|c} \mathbb{J}_{\rho_1} & 0_{\rho_1\times\rho_2} & \cdots & 0_{\rho_1\times\rho_L} \\ \hline 0_{\rho_2\times\rho_1} & \mathbb{J}_{\rho_2} & \cdots & 0_{\rho_2\times\rho_L} \\ \hline & & \ddots & \\ \hline 0_{\rho_L\times\rho_1} & 0_{\rho_L\times\rho_2} & \cdots & \mathbb{J}_{\rho_L} \end{array}\right), \quad \mathbb{J}_{\rho_i} = \underbrace{\begin{pmatrix} 0 & 1 & \dots & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & & & \vdots \\ 0 & 0 & \dots & 0 & 0 \end{pmatrix}}_{\rho_i} \tag{4.11}$$

These VEVs trigger a sequential Higgsing. The Higgsing procedure is in general very difficult to study, in particular when the VEV is for the monopole operators contained in $\mathcal{C}$.

As we explained in the previous section, we will follow an alternative procedure based on the duality web of $T[SU(N)]$ of Figure 1.2. First of all we observe that the VEV can be implemented by adding two sets of N^2-1 flipping fields $\mathcal{O}_\mathcal{H}$ and $\mathcal{O}_\mathcal{C}$ that couple to the meson and monopole matrices, which is the same as considering $FF[TSU(N)]$, and turning on linearly in the superpotential some of their entries, depending on the partitions σ and ρ. Some of the components of $\mathcal{O}_\mathcal{H}$ and $\mathcal{O}_\mathcal{C}$ remain massless and correspond to a decoupled free sector of the low energy theory. Hence, we remove them by adding some additional singlets $\mathcal{S}_\sigma$ and $\mathcal{T}_\rho$ that flip them [153, 154, 49]. In order to do so, $\mathcal{S}_\sigma$ and $\mathcal{T}_\rho$ have to be $N \times N$ traceless matrices whose transposes commute with the Jordan matrices $\mathcal{J}_\sigma$ and $\mathcal{J}_\rho$ respectively.

For a generic nilpotent VEV, the deformation taking $FF[TSU(N)]$ to $T_\rho^\sigma[SU(N)]$ is

$$\delta\mathcal{W}_{FF} = Tr_X\left[(\mathcal{J}_\sigma + \mathcal{S}_\sigma)\,\mathcal{O}_\mathcal{H}\right] + \mathrm{Tr}_Y\left[(\mathcal{J}_\rho + \mathcal{T}_\rho)\,\mathcal{O}_\mathcal{C}\right] . \tag{4.12}$$

Using the operator map (1.16) we can then translate the deformation of $FF[TSU(N)]$ into a deformation of $T[SU(N)]$ which is linear in some of the components of $\mathcal{H}$ and $\mathcal{C}$

$$\delta\mathcal{W} = \mathrm{Tr}_X\left[(\mathcal{J}_\sigma + \mathcal{S}_\sigma)\,\mathcal{H}\right] + \mathrm{Tr}_Y\left[(\mathcal{J}_\rho + \mathcal{T}_\rho)\,\mathcal{C}\right] . \tag{4.13}$$

This is a mass and linear monopole deformation of $T[SU(N)]$ that leads to an IR theory that we denoted with $\mathcal{T}$ in Figure 4.2. This deformation is easier to study than the VEV of $T[SU(N)]$, but the price we have to pay is that we end up not directly with $T_\rho^\sigma[SU(N)]$ but its flip-flip dual $\mathcal{T}$.

We propose that to implement the flip-flip duality moving from $\mathcal{T}$ to $T_\rho^\sigma[SU(N)]$ we can generalize the strategy to move from $T[SU(N)]$ to $FFT[SU(N)]$, where one has to apply iteratively the Aharony duality[3]. Here since some of the nodes will have a linear monopole

[3] We actually saw this derivation for the flip-flip duality of $E[USp(2N)]$ by iteration of the Intriligator–Pouliot duality in Subsection 3.3.2 and in Appendix C.1, but the derivation in 3d works exactly in the same way. See [145, 82] for more details.

superpotential we will use a combination of the Aharony duality and the one-monopole duality, depending on whether a monopole is turned on in the superpotential at the node we are considering.

For simplicity we will now restrict to the case where one of the two partitions is trivial. We first consider the case where $\sigma = [1^N]$, which corresponds to turning on a nilpotent VEV labelled by a partition ρ for the CB moment map $\mathcal{C}$ leading to $T_\rho[SU(N)]$. In the flip-flip dual frame, this deformation corresponds to the following deformation of $T[SU(N)]$:

$$\delta\mathcal{W} = \mathrm{Tr}_X\left[\left(\mathcal{J}_{[1^N]} + \mathcal{S}_{[1^N]}\right)\mathcal{H}\right] + \mathrm{Tr}_Y\left[\left(J_\rho + \mathcal{T}_\rho\right)\mathcal{C}\right] . \tag{4.14}$$

Here $\mathcal{J}_{[1^N]}$ is the null matrix, while $\mathcal{S}_{[1^N]}$ and $\mathcal{T}_\rho$ are matrices of gauge singlets whose transposes commute with $\mathcal{J}_{[1^N]}$ and $\mathcal{J}_\rho$ respectively, so in particular $\mathcal{S}_{[1^N]}$ is an arbitrary $N \times N$ traceless matrix which is completely flipping the HB moment map $\mathcal{H}$.

This deformation leads to theory $\mathcal{T}$ whose global symmetry will be the product of $SU(N)_X$ and of the subgroup of $SU(N)_Y$ preserved by the VEV, which can be at most broken to $S(U(1)^L)$ when all the entries ρ_i of the partition are different. Instead, when some of the entries coincide the corresponding $U(1)$ factors combine and are enhanced in the infrared[4]. More precisely, for a generic partition of the form $\rho = [N^{l_N}, \cdots, 1^{l_1}]$ the IR CB global symmetry will be broken to[5]

$$SU(N)_Y \to S\left(\prod_{i=1}^{N} U(l_i)\right) , \tag{4.15}$$

which is precisely the CB symmetry of $T_\rho[SU(N)]$. Correspondingly at the level of partition functions we will introduce the following fugacities:

$$Y_i, \quad \text{with} \quad i = 1, \cdots, N \quad \to \quad Y^{(1)}_{i_1}, Y^{(2)}_{i_2}, \cdots \quad \text{with} \quad i_s = 1, \cdots, l_s \tag{4.16}$$

and similarly, when also σ is non-trivial, we introduce

$$X_j, \quad \text{with} \quad j = 1, \cdots, N \quad \to \quad X^{(1)}_{j_1}, X^{(2)}_{j_2}, \cdots \quad \text{with} \quad j_r = 1, \cdots, k_r . \tag{4.17}$$

We can then reach $T_\rho[SU(N)]$ implementing the flip-flip duality by applying sequentially the Aharony and the one-monopole duality. Below we illustrate this procedure in the case of a next-to-maximal VEV corresponding to the partition $\rho = [N-1, 1]$ and for the partition $\rho = [2, 1^2]$.

[4]This happens because some nodes are *balanced*, that is the number of flavors attached to it is twice the rank of the group.

[5]Notice that when we write the partition as $\rho = [N^{l_N}, \cdots, 1^{l_1}]$, some of the l_i will in general be zero. The corresponding factor in the CB global symmetry is just an empty group.

On the mirror dual side, we will have a nilpotent VEV labelled by a partition ρ for the HB moment map $\mathcal{H}^\vee$ leading to $T^\rho[SU(N)]$. In the flip-flip dual frame this VEV corresponds to the following deformation of $T[SU(N)]^\vee$:

$$\delta\mathcal{W}^\vee = \mathrm{Tr}_Y\left[\left(\mathcal{J}_\rho + \mathcal{T}_\rho\right)\mathcal{H}^\vee\right] + \mathrm{Tr}_X\left[\left(\mathcal{J}_{[1^N]} + \mathcal{S}_{[1^N]}\right)\mathcal{C}^\vee\right] . \tag{4.18}$$

Since this is a purely massive deformation we can find a Lagrangian description for the theory $\mathcal{T}^\vee$ which we flow to by integrating out the massive fields. Specifically, $\mathcal{T}^\vee$ is the same quiver as $T[SU(N)]^\vee$ but with less flavors attached to the last $U(N-1)$ node. The number of the remaining massless flavors coincides with the length L of the partition ρ and each of them interacts with a different power of the adjoint chiral $\Phi^{(N-1)}$ of the last gauge node. Because of this superpotential coupling the HB $SU(N)_Y$ global symmetry of $T[SU(N)]^\vee$ will be generically broken down to $S(U(1)^L)$, but if some of the ρ_i are equal we can form blocks of chirals transforming under a larger symmetry group since they interact with the same power of $\Phi^{(N-1)}$. Hence, for a partition of the form $\rho = [N^{l_N}, \cdots, 1^{l_1}]$ the resulting interaction is

$$\begin{aligned}\mathrm{Tr}_{N-1}\left[\Phi^{(N-1)}\left(\mathrm{Tr}_Y\,\tilde{q}^{(N-1,N)}q^{(N-1,N)}\right)\right] \quad \to \quad & \sum_{i=1}^{L}\mathrm{Tr}_{N-1}\left[\tilde{q}_i\left(\Phi^{(N-1)}\right)^{\rho_i}q_i\right] \\ &= \sum_{m=1}^{N}\mathrm{Tr}_{N-1}\left[\left(\Phi^{(N-1)}\right)^m \mathrm{Tr}_{Y^{(m)}}\left(\tilde{q}_m q_m\right)\right] ,\end{aligned} \tag{4.19}$$

where we renamed as $q_m, \tilde{q}_m$ the massless chirals at the $U(N-1)$ gauge node in the fundamental and anti-fundamental representation of each $U(l_m)$ factor, with $m = 1, \cdots, N$. In particular, for the values of m for which $l_m = 0$ we don't have any chiral field. We also introduced the notation $\mathrm{Tr}_{Y^{(i)}}$ for the trace over the i-th factor in this global symmetry group.

The full superpotential will be

$$\begin{aligned}\mathcal{W}_{\mathcal{T}^\vee} \;=\;& \mathcal{W}_{T[SU(N-1)]} - \mathrm{Tr}_{N-1}\left(\Phi^{(N-1)}\,\mathrm{Tr}_{N-2}\,\tilde{q}^{(N-2,N-1)}q^{(N-2,N-1)}\right) \\ &+ \sum_{m=1}^{N}\mathrm{Tr}_{N-1}\left[\left(\Phi^{(N-1)}\right)^m \mathrm{Tr}_{Y^{(m)}}\left(\tilde{q}_m q_m\right)\right] + \mathrm{Tr}_Y\left(\mathcal{T}_\rho\,\mathcal{H}^\vee\right)\big|_{eom} + \mathrm{Tr}_X\left(\mathcal{S}_{[1^N]}\,\mathcal{C}^\vee\right)\end{aligned} \tag{4.20}$$

and the flavor symmetry will be

$$S\left(\prod_{i=1}^{N}U(l_i)\right) , \tag{4.21}$$

The subscript *eom* refers to the fact that after imposing the F-terms equations only some of the components of $\mathcal{H}^\vee$ will survive.

From $\mathcal{T}^\vee$ we can reach $T^\rho[SU(N)]$ by implementing the flip-flip duality, which in this case is equivalent to applying the Aharony duality only since we have no monopole superpotential. Below we illustrate this procedure for the partitions $\rho = [N-1,1]$ and $\rho = [2,1^2]$.

4.2.2 Some examples

Example I: $\rho = [N-1,1]$ **and** $\sigma = [1^N]$

Flow to $T_{[N-1,1]}[SU(N)]$

We define theory $\mathcal{T}$ as the theory obtained from $T[SU(N)]$ via the deformation

$$\delta\mathcal{W} = \mathrm{Tr}_X\left[\left(\mathcal{J}_{[1^N]} + \mathcal{S}_{[1^N]}\right)\mathcal{H}\right] + \mathrm{Tr}_Y\left[\left(\mathcal{J}_{[N-1,1]} + \mathcal{T}_{[N-1,1]}\right)\mathcal{C}\right] . \tag{4.22}$$

The matrix $J_{[1^N]}$ is simply the null matrix and, consequently, $\mathcal{S}_{[1^N]}$ is a generic $N \times N$ traceless matrix. Instead by requiring that the transpose of $\mathcal{T}_{[N-1,1]}$ commutes with $J_{[N-1,1]}$ we find its non-vanishing entries

$$\mathcal{J}_{[N-1,1]} + \mathcal{T}_{[N-1,1]} = \left(\begin{array}{cccc|c} \mathcal{T}_1 & 1 & \cdots & 0 & 0 \\ \mathcal{T}_2 & \mathcal{T}_1 & 1 & & \vdots \\ \vdots & \ddots & \ddots & 1 & 0 \\ \mathcal{T}_{N-1} & & \mathcal{T}_2 & \mathcal{T}_1 & \mathcal{T}_+ \\ \hline \mathcal{T}_- & 0 & \cdots & 0 & -(N-1)\mathcal{T}_1 \end{array}\right) . \tag{4.23}$$

More explicitly, the superpotential deformation is

$$\delta\mathcal{W} = \mathrm{Tr}_X\left(\mathcal{S}_{[1^N]}\mathcal{H}\right) + \mathrm{Tr}_Y\left(\mathcal{T}_{[N-1,1]}\mathcal{C}\right) + \mathfrak{M}^{(1,0,\cdots,0)} + \mathfrak{M}^{(0,1,0\cdots,0)} + \cdots + \mathfrak{M}^{(0,\cdots,1,0)} . \tag{4.24}$$

The linear monopole deformation at the first $N-2$ nodes breaks the topological and the axial symmetries to a combination, implying the constraint on the fugacities

$$Y_i - Y_{i-1} = 2m_A \quad \text{for} \quad i = 2,\cdots N-1 , \tag{4.25}$$

which can be solved by

$$Y_i = Y_1 + 2(i-1)m_A, \qquad i = 1,\cdots N-1 . \tag{4.26}$$

From this we can easily determine the charges of the singlets $\mathcal{T}_i$ and $\mathcal{T}^\pm$. Before imposing the constraint on the fugacities the charges of the entry (i,j) of the moment map matrix $\mathcal{C}$ under the Cartan $\prod_{i=1}^{N-1} U(1)_{Y_i} \subset SU(N)_Y$ and under $U(1)_{m_A}$ can be read off from the coefficients of Y_i and m_A in the combination

$$Y_j - Y_i - 2m_A\,. \tag{4.27}$$

Imposing the constraint (4.26) on this combination we can extract the charges under the residual symmetry $SU(N)_X \times U(1)_Y \times U(1)_{m_A}$, where $U(1)_Y$ is a combination of $U(1)_{Y_1}$, $U(1)_{Y_N}$ and $U(1)_{m_A}$

	$U(1)_Y$	$SU(N)_X$	$U(1)_{m_A}$	$U(1)_R$
$\mathcal{T}_i$	0	$\mathbf{1}$	$2i$	$2ri$
$\mathcal{T}_-$	-1	$\mathbf{1}$	N	Nr
$\mathcal{T}_+$	1	$\mathbf{1}$	N	Nr
$\mathcal{S}_{[1^N]}$	0	$\mathbf{N^2-1}$	-2	$2-2r$

From theory $\mathcal{T}$ we want to move along the vertical edge of the web and reach $T_{[N-1,1]}[SU(N)]$. This is achieved by applying iteratively either the Aharony or the one-monopole duality, depending on whether the node we are considering has a linear monopole superpotential or not. In this case, we apply $N-2$ times the one-monopole duality starting from the first node until we reach the $U(N-2)$ node. Since this duality is always applied to a $U(n)$ gauge node with $n+1$ flavors, which corresponds to the case dual to a WZ model, its effect is to sequentially confine the nodes of the quiver. This phenomenon is known as sequential confinement [49, 169, 20].

In particular the effect of the linear monopole deformation in (4.24), but without the first two terms involving the singlets $\mathcal{T}_i$, $\mathcal{T}_\pm$ and $\mathcal{S}_{[1^N]}$, was analyzed in great detail in [20]. There it was shown that after confining the first $N-2$ nodes one reaches a $U(N-1)$ theory with N flavors and superpotential

$$\mathcal{W} = -\sum_{k=1}^{N-1} \frac{(-1)^k}{k} \gamma_k \operatorname{Tr}[\mathcal{Q}^k]\,, \tag{4.28}$$

where the singlets γ_k flip the traces of powers of the meson matrix $\mathcal{Q}$ and have R-charge $R[\gamma_k] = 2(1-kr)$. The chiral ring of this theory in addition to the γ_k contains the fundamental $U(N-1)$ monopoles with $R[\mathfrak{M}^\pm] = 2 - Nr$ and the traceless meson matrix $\mathcal{Q} - \frac{\operatorname{Tr}\mathcal{Q}}{N}$ of R-charge $2r$.

To complete our flip-flip prescription we need to apply the Aharony duality to the remaining $U(N-1)$ node. We arrive at a $U(1)$ theory with N flavors and three sets of singlets: $\sigma^\pm$ with R-charge $2-Nr$ flipping the fundamental $U(1)$ monopoles, F_{ij} with R-charge $2r$ flipping the meson matrix (with trace) and singlets θ_k with $k = 1, \cdots N-1$, with R-charge $2-2rk$ flipping the traces of powers of the matrix F_{ij}.

When we consider the full deformation in (4.24), including singlets $\mathcal{T}_i, \mathcal{T}^\pm$ and $\mathcal{S}_{[1^N]}$, the singlets $\sigma^\pm$, θ_k and the traceless part of F_{ij} becomes massive. The trace part of F_{ij}, which we call $\Phi = \text{Tr}(F)$, instead reconstructs the $\mathcal{N}=4$ superpotential

$$\mathcal{W}_{T_{[N-1,1]}[SU(N)]} = \Phi \sum_{i=1}^{N} \tilde{P}^i P_i \,, \tag{4.29}$$

so we arrive at theory $T_{[N-1,1]}[SU(N)]$ which is $\mathcal{N}=4$ SQED with N flavors.

Flow to $T^{[N-1,1]}[SU(N)]$

Theory $\mathcal{T}^\vee$, the mirror dual of $\mathcal{T}$, is obtained by the following deformation of $T[SU(N)]^\vee$:

$$\delta\mathcal{W}^\vee = \text{Tr}_Y\left[\left(\mathcal{J}_{[N-1,1]} + \mathcal{T}_{[N-1,1]}\right)\mathcal{H}^\vee\right] + \text{Tr}_X\left[\left(\mathcal{J}_{[1^N]} + \mathcal{S}_{[1^N]}\right)\mathcal{C}^\vee\right] . \tag{4.30}$$

We can integrate out the massive fields to get a quiver theory with increasing ranks of the gauge groups as in $T[SU(N)]$, but with only two flavors at the end of the tail which interact differently with the adjoint chiral of the $U(N-1)$ gauge node, plus some residual flipping fields originally coming from $\mathcal{S}_{[1^N]}$ and $\mathcal{T}_{[N-1,1]}$

$$\begin{aligned}
\mathcal{W}_{\mathcal{T}^\vee} &= \mathcal{W}_{T[SU(N-1)]^\vee} - \text{Tr}_{N-1}\left(\Phi^{(N-1)}\,\text{Tr}_{N-2}\,\tilde{q}^{(N-2,N-1)}q^{(N-2,N-1)}\right) \\
&+ \text{Tr}_{N-1}\left[\tilde{q}_1\Phi^{(N-1)}q_1 + \tilde{q}_2\left(\Phi^{(N-1)}\right)^{N-1} q_2 + \mathcal{T}_-\,\tilde{q}_1 q_2 + \mathcal{T}_+\tilde{q}_2 q_1 + \right. \\
&+ \left. \sum_{i=1}^{N-1} \mathcal{T}_i\,\tilde{q}_2\left(\Phi^{(N-1)}\right)^{i-1} q_2\right] + \text{Tr}_X\left[\mathcal{S}_{[1^N]}\,\mathcal{C}_{\mathcal{T}^\vee}\right] ,
\end{aligned} \tag{4.31}$$

where $\mathcal{C}_{\mathcal{T}^\vee}$ is the CB moment map of theory $\mathcal{T}^\vee$, which is constructed as in $T[SU(N)]$.

To reach $T^{[N-1,1]}[SU(N)]$ we now have to implement the flip-flip duality which amounts to applying the Aharony duality sequentially. This derivation is carried out explicitly at the level of the sphere partition function in the $N=3$ case in Appendix B.3, while here we only discuss its main steps which are sketched in Figure 4.5:

- At the first iteration we start from the $U(1)$ gauge node and proceed applying the Aharony duality along the tail. Since the first $N-2$ nodes are $U(n)$ nodes with $2n$ flavors, the gauge group doesn't change when we apply the duality and because of the charge assignments no new links between the nodes are created. The last $U(N-1)$ node however sees N flavors, so when we apply Aharony duality it becomes a $U(1)$ gauge node. A new link is created connecting one of the two flavor nodes (the blue one in the picture) to the second last gauge node.

- At the second iteration we start again from the leftmost $U(1)$ gauge node and go along the whole tail, but this time we stop at the second last node. Because of the result of the previous iteration, this is now a $U(N-2)$ gauge node with $N-1$ flavors, so when we apply Aharony duality it becomes a $U(1)$ node. Now the blue flavor node gets

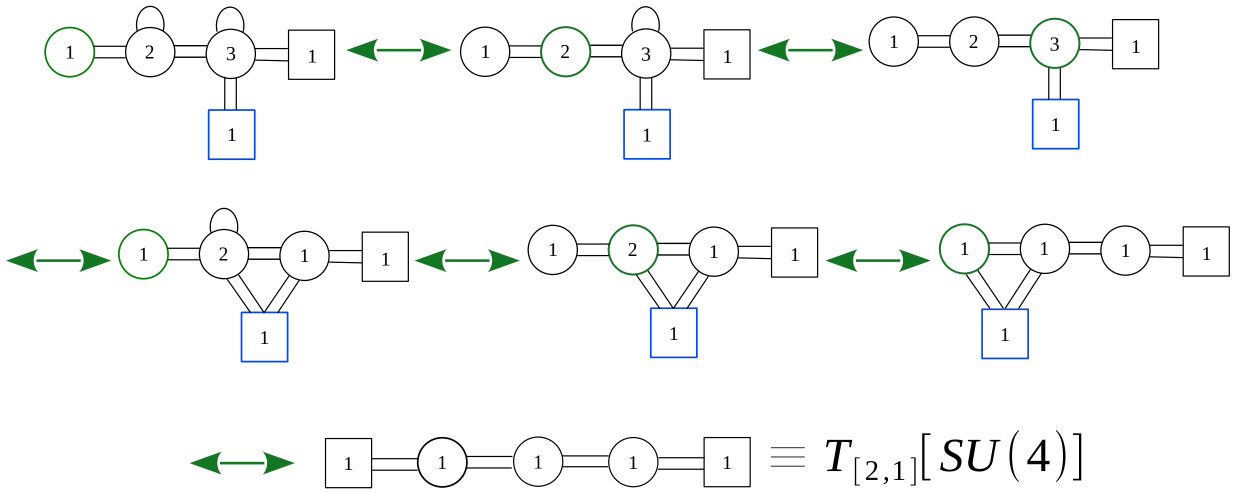

Figure 4.5: Quiver representation of the iterative application of the Aharony duality in the case $N = 4$. We highlighted in green the gauge node to which we apply the duality at each step. We only sketch the main steps and neglect gauge singlets; taking into account the $\mathcal{S}_{[1^N]}$ and $\mathcal{T}_{[N-1,1]}$ singlets from the beginning, all the remaining ones are only those corresponding to adjoint chirals for $U(1)$ gauge nodes.

attached to the $U(N-2)$ gauge node, while the link with the rightmost $U(1)$ gauge node is removed.

- We iterate this procedure $N-1$ times, meaning that we apply the Aharony duality $N(N-1)/2$ times and we arrive to the abelian $U(1)^{N-1}$ linear quiver with exactly $\mathcal{N} = 4$ superpotential.

- There are no extra singlets, since they became massive because of $\mathcal{S}_{[1^N]}$ and $\mathcal{T}_{[N-1,1]}$.

The final results is a linear quiver with $N-1$ $U(1)$ gauge nodes, connected by bifundamental flavors $p^{(i-1,i)}$, $\tilde{p}^{(i-1,i)}$. The first and last nodes are also connected to fundamental flavors $p^{(0,1)}$, $\tilde{p}^{(0,1)}$ and $p^{(N-1,N)}$, $\tilde{p}^{(N-1,N)}$. The superpotential consists of the standard $\mathcal{N} = 4$ interaction with the adjoint chiral fields

$$\mathcal{W}_{T^{[N-1,1]}[SU(N)]} = \sum_{i=1}^{N-1} \Phi^{(i)} \left(\tilde{p}^{(i,i+1)}p^{(i,i+1)} + \tilde{p}^{(i-1,i)}p^{(i-1,i)}\right) . \qquad (4.32)$$

This theory is indeed dual to the $\mathcal{N} = 4$ SQED with N flavors according to abelian mirror symmetry and it corresponds to $T^{[N-1,1]}[SU(N)]$.

Example II: $\rho = [2, 1^2]$ **and** $\sigma = [1^4]$

Flow to $T_{[2,1^2]}[SU(4)]$

We start analyzing the VEV for the CB moment map as a monopole deformation in the flip-flip dual theory plus flipping fields

$$\delta\mathcal{W} = \mathrm{Tr}_X \left[\left(\mathcal{J}_{[1^4]} + \mathcal{S}_{[1^4]} \right) \mathcal{H} \right] + \mathrm{Tr}_Y \left[\left(\mathcal{J}_{[2,1^2]} + \mathcal{T}_{[2,1^2]} \right) \mathcal{C} \right] . \tag{4.33}$$

In this case the Jordan matrix encoding the nilpotent deformation is

$$\mathcal{J}_{[2,1^2]} = \left(\begin{array}{cc|c|c} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 \end{array} \right) \tag{4.34}$$

and consequently the matrix of singlets that we need to add is

$$\mathcal{T}_{[2,1^2]} = \left(\begin{array}{cc|c|c} \alpha_2 & 0 & 0 & 0 \\ \alpha_1 & \alpha_2 & \tilde{\gamma}_1 & \tilde{\gamma}_2 \\ \hline \gamma_1 & 0 & \beta_{33} & \beta_{34} \\ \hline \gamma_2 & 0 & \beta_{43} & -2\alpha_2 - \beta_{33} \end{array} \right) , \tag{4.35}$$

while $\mathcal{S}_{[1^4]}$ is an arbitrary 4×4 traceless matrix. Hence, the deformation $\delta\mathcal{W}$ corresponds to turning on linearly the positive fundamental monopole of the first $U(1)$ gauge node of $T[SU(4)]$

$$\mathcal{W}_{\mathcal{T}} = \mathcal{W}_{T[SU(4)]} + \mathfrak{M}^{(1,0,0)} + \mathrm{Tr}_X \left(\mathcal{S}_{[1^4]}\, \mathcal{H} \right) + \mathrm{Tr}_Y \left(\mathcal{T}_{[2,1^2]}\, \mathcal{C} \right) . \tag{4.36}$$

This monopole deformation breaks the $SU(4)_Y$ global symmetry down to $U(1)_{Y^{(1)}} \times SU(2)_{Y^{(2)}}$.

In terms of the real masses Y_i, the superpotential term we added implies the constraint

$$Y_2 = Y_1 + 2m_A \,. \tag{4.37}$$

Moreover, it will be useful to also redefine the Y_1 real mass by

$$Y_1 \to Y_1 - m_A \,. \tag{4.38}$$

The residual symmetry is then parametrized by

$$\begin{aligned} Y^{(1)} &= Y_1 \\ Y^{(2)}_1 &= Y_3 + Y_1 \\ Y^{(2)}_2 &= Y_4 + Y_1 \,, \end{aligned} \tag{4.39}$$

The charges and representations of the chiral fields of the theory are the same as those of $T[SU(4)]$ since the deformation only affected the monopole operators. The gauge singlets in $\mathcal{T}_{[2,1^2]}$ transform under the global symmetries as follows:[6]

	$SU(4)_X$	$U(1)_{Y_1}$	$SU(2)_{Y_3,Y_4}$	$U(1)_{m_A}$	$U(1)_{R_0}$
α_1	$\mathbf{1}$	0	$\mathbf{1}$	4	0
α_2	$\mathbf{1}$	0	$\mathbf{1}$	2	0
β	$\mathbf{1}$	0	$\mathbf{3}$	2	0
$\gamma,\,\tilde\gamma$	$\mathbf{1}$	±1	$\mathbf{2}$	3	0
$\mathcal{S}_{[1^4]}$	$\mathbf{15}$	0	$\mathbf{1}$	-2	2

where $U(1)_{Y_1}$ and $SU(2)_{Y_3,Y_4}$ denote the symmetries after imposing the superpotential constraint (4.37)–(4.38), but before the redefinition (4.39). This will be performed at the very end of the derivation of the flip-flip dual of theory $\mathcal{T}$, coinciding with $T_{[2,1^2]}[SU(4)]$.

We can study the deformation at the level of the $\mathbb{S}^3_b$ partition function of the theory $\mathcal{T}$, which can be obtained imposing (4.37) and (4.38) on $\mathcal{Z}_{T[SU(4)]}$ (recall that the partition function of $T[SU(N)]$ was defined in (1.10))

$$\begin{aligned} \mathcal{Z}_{\mathcal{T}} = \mathcal{B} \int \mathrm{d}\vec{z}^{(3)}_3\, \mathrm{e}^{2\pi i (Y_3-Y_4)\sum_{k=1}^3 z^{(3)}_k} \prod_{k,l=1}^{3} s_b\left(-i\frac{Q}{2} + (z^{(3)}_k - z^{(3)}_l) + 2m_A\right) \\ \times \prod_{k=1}^{3}\prod_{i=1}^{4} s_b\left(i\frac{Q}{2} \pm (z^{(3)}_k - X_i) - m_A\right) \int \mathrm{d}\vec{z}^{(2)}_2\, \mathrm{e}^{2\pi i (Y_1-Y_3+m_A)\sum_{a=1}^2 z^{(2)}_a} \\ \times \prod_{a,b=1}^{2} s_b\left(-i\frac{Q}{2} + (z^{(2)}_a - z^{(2)}_b) + 2m_A\right) \prod_{a=1}^{2}\prod_{k=1}^{3} s_b\left(i\frac{Q}{2} \pm (z^{(2)}_a - z^{(3)}_k) - m_A\right) \\ \times \int \mathrm{d}z^{(1)}_1\, \mathrm{e}^{-4\pi i m_A z^{(1)}} s_b\left(-i\frac{Q}{2} + 2m_A\right) \prod_{a=1}^{2} s_b\left(i\frac{Q}{2} \pm (z^{(1)} - z^{(2)}_a) - m_A\right) \,, \end{aligned} \tag{4.40}$$

[6]With β we collectively denote the singlets β_{33}, β_{34}, β_{43} that form a triplet of the $SU(2)$. Similarly γ, $\tilde\gamma$ are made of the singlets γ_1, γ_2, $\tilde\gamma_1$, $\tilde\gamma_2$ and transform as two doublets under the $SU(2)$.

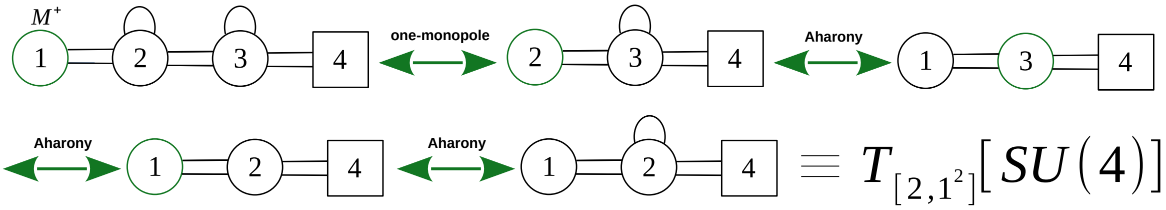

Figure 4.6: Quiver representation of the sequential application of the Aharony and the one-monopole duality that leads to $T_{[2,1^2]}[SU(4)]$ starting from its flip-flip dual $\mathcal{T}$.

where $\mathcal{B}$ is the contribution of the singlets

$$\begin{aligned}\mathcal{B} &= \prod_{i,j=1}^{4} s_b\left(-i\frac{Q}{2}+(X_i-X_j)+2m_A\right)s_b\left(i\frac{Q}{2}-2m_A\right)s_b\left(i\frac{Q}{2}-4m_A\right)\\ &\times\prod_{\alpha,\beta=3}^{4} s_b\left(i\frac{Q}{2}+(Y_\alpha-Y_\beta)-2m_A\right)\prod_{\alpha=3}^{4} s_b\left(i\frac{Q}{2}\pm(Y_1-Y_\alpha)-3m_A\right)\,. \end{aligned} \tag{4.41}$$

As mentioned in our previous general discussion, from $\mathcal{T}$ we can reach the flip-flip dual theory $T_{[2,1^2]}[SU(4)]$ by sequentially applying the Aharony and the one-monopole duality. We show this explicitly for this particular case at the level of the sphere partition function in Appendix B.4, while here we only outline the main steps of the derivation sketched in Figure 4.6.

We begin by applying the one-monopole duality to the $U(1)$ gauge node in (4.40). This node confines yielding a quiver theory with no monopoles turned on

$$\begin{aligned}\mathcal{Z}_{\mathcal{T}} &= \mathcal{B}\,s_b\left(-i\frac{Q}{2}+2m_A\right)s_b\left(-i\frac{Q}{2}+4m_A\right)\int \mathrm{d}\vec{z}_3^{(3)}\,\mathrm{e}^{2\pi i(Y_3-Y_4)\sum_{k=1}^{3}z_k^{(3)}}\\ &\times\prod_{k,l=1}^{3} s_b\left(-i\frac{Q}{2}+(z_k^{(3)}-z_l^{(3)})+2m_A\right)\prod_{k=1}^{3}\prod_{i=1}^{4} s_b\left(i\frac{Q}{2}\pm(z_k^{(3)}-X_i)-m_A\right)\\ &\times\int \mathrm{d}\vec{z}_2^{(2)}\,\mathrm{e}^{2\pi i(Y_1-Y_3)\sum_{a=1}^{2}z_a^{(2)}}\prod_{a=1}^{2}\prod_{k=1}^{3} s_b\left(i\frac{Q}{2}\pm(z_a^{(2)}-z_k^{(3)})-m_A\right)\,. \end{aligned} \tag{4.42}$$

From this frame we proceed by iteratively applying the Aharony duality until we reach the flip-flip dual frame[7]

$$\begin{aligned}\mathcal{Z}_{\mathcal{T}} = &\int \mathrm{d}\vec{z}_2^{(3)}\, \mathrm{e}^{2\pi i(2Y^{(1)}-Y_1^{(2)})\sum_{k=1}^2 z_i^{(3)}} \prod_{k,l=1}^{2} s_b\left(i\frac{Q}{2}+(z_k^{(3)}-z_l^{(3)})-m_A\right)\\ &\times \prod_{k=1}^{2}\prod_{i=1}^{4} s_b\left(\pm(z_k^{(3)}+X_i)+m_A\right)\int \mathrm{d}z_1^{(2)}\, \mathrm{e}^{2\pi i(Y_1^{(2)}-Y_2^{(2)})z^{(2)}} s_b\left(i\frac{Q}{2}-2m_A\right)\\ &\times \prod_{k=1}^{2} s_b\left(\pm(z^{(2)}-z_k^{(3)})+m_A\right) = \mathcal{Z}_{T_{[2,1^2]}[SU(4)]}(\vec{X};\vec{Y}^{(2)},Y^{(1)};m_A)\,. \end{aligned} \tag{4.43}$$

In this last expression we also introduced the proper $U(1)_{Y^{(1)}}\times SU(2)_{Y^{(2)}}$ fugacities defined in (4.39). This is precisely the partition function of $T_{[2,1^2]}[SU(4)]$.

Flow to $T^{[2,1^2]}[SU(4)]$

We now move to analyzing the deformation in the mirror dual theory. This corresponds to a VEV for the HB moment map which we can study as a mass deformation of $T[SU(4)]^\vee$ plus flipping fields

$$\delta\mathcal{W}^\vee = \mathrm{Tr}_Y\left[\left(\mathcal{J}_{[2,1^2]}+\mathcal{T}_{[2,1^2]}\right)\mathcal{H}^\vee\right]+\mathrm{Tr}_X\left[\left(\mathcal{J}_{[1^4]}+\mathcal{S}_{[1^4]}\right)\mathcal{C}^\vee\right]\,, \tag{4.44}$$

where $\mathcal{T}_{[2,1^2]}$ is the matrix (4.35). The mass deformation breaks the $SU(4)_Y$ global symmetry associated to the HB of $T[SU(4)]^\vee$ down to $U(1)_{Y^{(1)}}\times SU(2)_{Y^{(2)}}$. We parametrize these symmetries with the fugacities $Y^{(1)}$, $Y_\alpha^{(2)}$ defined as in (4.37)–(4.38)–(4.39). After integrating out the massive fields, we end up with a quiver similar to $T[SU(4)]^\vee$, but with only three flavors at the end of the tail coupling to different powers of the adjoint chiral field of the last node and extra flipping fields

$$\begin{aligned}\mathcal{W}_{\mathcal{T}^\vee} \;=\;& \mathcal{W}_{T[SU(3)]}-\mathrm{Tr}_3\left(\Phi^{(3)}\,\mathrm{Tr}_2\,\tilde{q}^{(2,3)}q^{(2,3)}\right)+\mathrm{Tr}_3\left(\Phi^{(3)}\tilde{q}_1q_1\right)+\\ +\;& \mathrm{Tr}_3\left[\left(\Phi^{(3)}\right)^2\mathrm{Tr}_{Y^{(2)}}\left(\tilde{q}_2q_2\right)\right]+\mathrm{Tr}_Y\left(\mathcal{T}_{[2,1^2]}\mathcal{H}^\vee\right)\Big|_{eom}+\mathrm{Tr}_X\left(\mathcal{S}_{[1^4]}\mathcal{C}^\vee\right)\,. \end{aligned} \tag{4.45}$$

[7] Note that as a consequence of the sequential application of the Aharony and the one-monopole duality, the fugacities for the topological symmetries are permuted and appear in the opposite order compared to the definition of the original $T[SU(4)]$ partition function. For this reason, we call the index (4.43) as $\mathcal{Z}_{T_{[2,1^2]}[SU(4)]}(\vec{X};\vec{Y}^{(2)},Y^{(1)};m_A)$ instead of $\mathcal{Z}_{T_{[2,1^2]}[SU(4)]}(\vec{X};Y^{(1)},\vec{Y}^{(2)};m_A)$. Indeed we can't use the $SU(4)_Y$ Weyl symmetry to reorder the two sets of fugacities $Y^{(1)}$ and $Y^{(2)}$ since this is not a symmetry of $T_{[2,1^2]}[SU(4)]$.

where $\mathrm{Tr}_{Y^{(2)}}$ is the trace with respect to the $SU(2)_{Y^{(2)}}$ symmetry which is manifest in this frame of the web and

$$\begin{aligned}\mathrm{Tr}_Y\left(\mathcal{T}_{[2,1^2]}\mathcal{H}^\vee\right)\Big|_{eom} &= \alpha_1\,\mathrm{Tr}_3\left(\tilde{q}_1 q_1\right)+\alpha_2\,\mathrm{Tr}_3\,\mathrm{Tr}_{Y^{(2)}}\left(\tilde{q}_2 q_2\right)\\ &+ \mathrm{Tr}_{Y^{(2)}}\left(\beta\,\mathcal{H}^{(2)}\right)+\mathrm{Tr}_{Y^{(2)}}\left[\gamma\,\mathrm{Tr}_3\left(\tilde{q}_2 q_1\right)\right]+\mathrm{Tr}_{Y^{(2)}}\left[\tilde{\gamma}\,\mathrm{Tr}_3\left(\tilde{q}_1 q_2\right)\right]\,,\end{aligned} \tag{4.46}$$

where we defined the $SU(2)_{Y^{(2)}}$ moment map

$$\mathcal{H}^{(2)}=\mathrm{Tr}_3\left(\tilde{q}_2 q_2\right)-\frac{1}{2}\,\mathrm{Tr}_{Y^{(2)}}\,\mathrm{Tr}_3\left(\tilde{q}_2 q_2\right)\,. \tag{4.47}$$

The three-sphere partition function of this theory can be obtained from the one of $T[SU(4)]^\vee$ imposing the constraint on the fugacities (4.37) and (4.38), simplifying the contribution of the massive fields thanks to the relation $s_b\left(x\right)s_b\left(-x\right)=1$ and adding the contribution of the singlets $\mathcal{T}_{[2,1^2]}$ and $\mathcal{S}_{[1^N]}$

$$\begin{aligned}\mathcal{Z}_{\mathcal{T}^\vee}=\mathcal{B}\int \mathrm{d}\vec{z}_3^{(3)}\,\mathrm{e}^{2\pi i(X_3-X_4)\sum_{k=1}^3 z_k^{(3)}}\prod_{k,l=1}^{3}s_b\left(i\frac{Q}{2}+(z_k^{(3)}-z_l^{(3)})-2m_A\right)\\ \times\prod_{k=1}^{3}s_b\left(\pm(z_k^{(3)}-Y_1)+2m_A\right)\prod_{\alpha=3}^{4}s_b\left(\pm(z_k^{(3)}-Y_\alpha)+m_A\right)\int \mathrm{d}\vec{z}_2^{(2)}\,\mathrm{e}^{2\pi i(X_2-X_3)\sum_{a=1}^2 z_a^{(2)}}\\ \times\prod_{a,b=1}^{2}s_b\left(i\frac{Q}{2}+(z_a^{(2)}-z_b^{(2)})-2m_A\right)\prod_{a=1}^{2}\prod_{k=1}^{3}s_b\left(\pm(z_a^{(2)}-z_k^{(3)})+m_A\right)\\ \times\int \mathrm{d}z_1^{(1)}\,\mathrm{e}^{2\pi i(X_1-X_2)z^{(1)}}s_b\left(i\frac{Q}{2}-2m_A\right)\prod_{a=1}^{2}s_b\left(\pm(z^{(1)}-z_a^{(2)})+m_A\right)\,.\end{aligned} \tag{4.48}$$

where $\mathcal{B}$ is the contribution of the singlets defined in (4.41).

Again we want to find the flip-flip dual frame of this theory since we know that it will coincide with $T^{[2,1^2]}[SU(4)]$ and we claim that it can be obtained by sequentially applying the Aharony duality only, as in this case there is no monopole superpotential. This derivation is carried out explicitly for this particular case at the level of the sphere partition function in Appendix B.4, while here we just report the final result, where we introduced the new

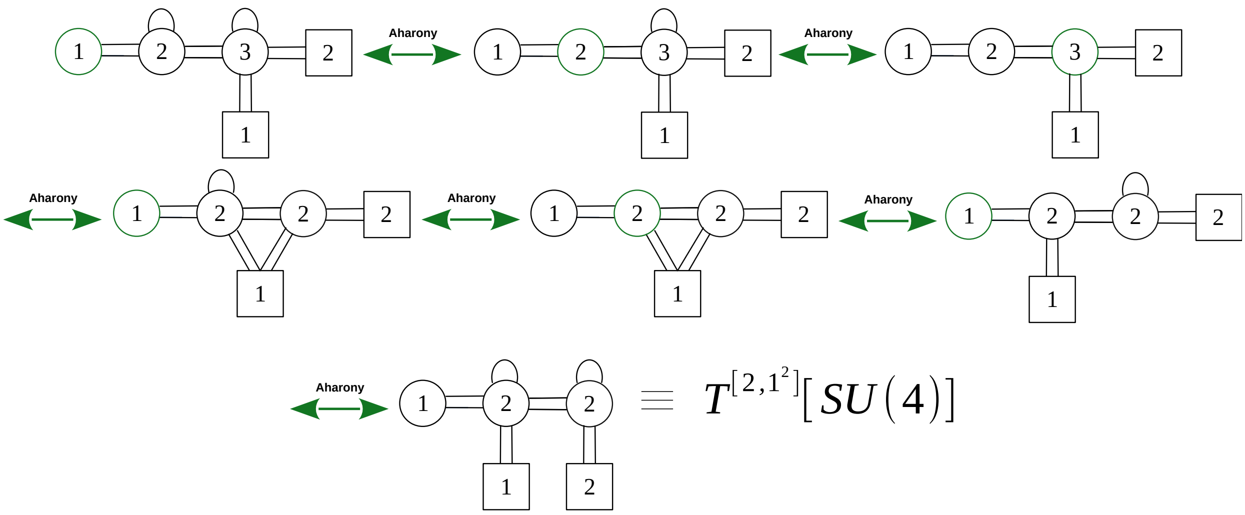

Figure 4.7: Quiver representation of the sequential application of the Aharony duality that leads to $T^{[2,1^2]}[SU(4)]$ starting from its flip-flip dual $\mathcal{T}^\vee$.

fugacities (4.39)[8]

$$\begin{aligned}
\mathcal{Z}_{\mathcal{T}^\vee} \;=\;& e^{4\pi i(X_1+X_2)Y^{(1)}}\int d\vec{z}^{(3)}_2\, e^{2\pi i(X_1-X_2)\sum_{k=1}^2 z_k^{(3)}}\prod_{k,l=1}^{2} s_b\left(-i\frac{Q}{2}+(z_k^{(3)}-z_l^{(3)})+2m_A\right)\\
&\times\prod_{k=1}^2\prod_{\alpha=1}^2 s_b\left(i\frac{Q}{2}\pm(z_k^{(3)}+Y_\alpha^{(2)})-m_A\right)\int d\vec{z}^{(2)}_2\, e^{2\pi i(X_2-X_3)\sum_{a=1}^2 z_a^{(2)}}\\
&\times\prod_{a,b=1}^{2} s_b\left(-i\frac{Q}{2}+(z_a^{(2)}-z_b^{(2)})+2m_A\right)\prod_{a=1}^{2} s_b\left(i\frac{Q}{2}\pm(z_a^{(2)}+Y^{(1)})-m_A\right)\\
&\times\prod_{k=1}^{2} s_b\left(i\frac{Q}{2}\pm(z_a^{(2)}-z_k^{(3)})-m_A\right)s_b\left(-i\frac{Q}{2}+2m_A\right)\int dz_1^{(1)}\, e^{2\pi i(X_3-X_4)z^{(1)}}\\
&\times\prod_{a=1}^{2} s_b\left(i\frac{Q}{2}\pm(z^{(1)}-z_a^{(2)})-m_A\right)=\mathcal{Z}_{T^{[2,1^2]}[SU(4)]}(Y^{(1)},\vec{Y}^{(2)};\vec{X};i\frac{Q}{2}-m_A)\,. \qquad (4.49)
\end{aligned}$$

This is precisely the partition function of $T^{[2,1^2]}[SU(4)]$, which is the quiver theory depicted at the end of Figure 4.7 where all the fields interact with the $\mathcal{N}=4$ superpotential. The presence of the contact terms in the prefactor is essential in order for the partition function of $T_{[2,1^2]}[SU(4)]$ in (4.43) to match with the one of $T^{[2,1^2]}[SU(4)]$ in (4.49). Indeed, from the equality of the partition functions (1.14) of $T[SU(4)]$ and $T[SU(4)]^\vee$ and the results of the manipulations we just explained it follows the equality of the partition functions

[8]Again, the labelling of the topological parameters X_i is in the opposite order compared to the original $T[SU(4)]^\vee$ partition function. This time, however, the permutations of X_i belong to the Weyl symmetry of the $SU(4)_X$ global symmetry. Thus, the partition function is invariant under such permutations, so we just call it $\mathcal{Z}_{T^{[2,1^2]}[SU(4)]}(Y^{(1)},\vec{Y}^{(2)};\vec{X};i\frac{Q}{2}-m_A)$ without specifying a particular order of X_i.

associated to the mirror symmetry relating $T_{[2,1^2]}[SU(4)]$ and $T^{[2,1^2]}[SU(4)]$

$$\mathcal{Z}_{T_{[2,1^2]}[SU(4)]}(\vec{X};\vec{Y}^{(2)},Y^{(1)};m_A)=\mathcal{Z}_{T^{[2,1^2]}[SU(4)]}(Y^{(1)},\vec{Y}^{(2)};\vec{X};i\frac{Q}{2}-m_A)\,, \tag{4.50}$$

where the parameter m_A is mapped to $i\frac{Q}{2}-m_A$ across the duality, as required by mirror symmetry (1.12).

4.3 4d mirror dualities and $E_\rho^\sigma[USp(2N)]$ theories using the web

4.3.1 The strategy

Now we would like to apply the same strategy and exploit the duality web of $E[USp(2N)]$ to find a class of 4d $\mathcal{N}=1$ theories which we call $E_\rho^\sigma[USp(2N)]$ enjoying mirror-like dualities. Before starting, we have to make a comment on conventions. For this chapter, we will define the $E[USp(2N)]$ theory including the β_i singlets only for $i=1,\cdots,N-1$, that is compared to the definition we gave in Section 3.3 we remove the singlet β_N that flips the meson constructed with the last diagonal of the saw. This is just for simplicity, since it allows us to fit $E[USp(2N)]$ inside the class of the $E_\rho^\sigma[USp(2N)]$ theories as the case $\sigma=\rho=[1^N]$ according to the definition that we will give for them. Notice that the singlet β_N is trivially mapped to itself across all the dualities of the $E[USp(2N)]$ duality web (see eqs. (3.26)-(3.33)-(3.36)), so removing it doesn't affect in any way our analysis.

The idea is to turn on VEVs labelled by partitions $\rho=[\rho_1,\dots,\rho_N]=\left[N^{l_N},\dots,1^{l_1}\right]$ and $\sigma=[\sigma_1,\dots,\sigma_N]=\left[N^{k_N},\dots,1^{k_1}\right]$ for the operators C and H. Remember that the operators H and C reduce in the 3d limit followed by suitable real mass deformations to the 3d moment maps $\mathcal{H}$ and $\mathcal{C}$. It is then easy to guess which 4d deformations of $E[USp(2N)]$ reduce in the 3d limit to the nilpotent deformations depending on the partitions ρ and σ of $SU(N)$ we turned on for $T[SU(N)]$. These are the deformations we are looking for and they correspond to the following VEVs:

$$\langle\mathsf{H}\rangle=\mathsf{J}_\sigma\,,\qquad\langle\mathsf{C}\rangle=\mathsf{J}_\rho\,, \tag{4.51}$$

where J_σ and J_ρ are the antisymmetric matrices

$$\mathsf{J}_\rho=\frac{1}{2}\left(J_\rho-J_\rho^T\right)\,, \tag{4.52}$$

where

$$J_\rho=i\sigma_2\otimes\left(\mathbb{J}_{\rho_1}\oplus\cdots\oplus\mathbb{J}_{\rho_L}\right) \tag{4.53}$$

and $\mathbb{J}_{\rho_i}$ are the Jordan matrices we defined in (4.11)[9]. We call $E_\rho^\sigma[USp(2N)]$ the theories we reach at the end of the flow triggered by such VEVs, after suitably removing some extra massless fields, as we will discuss.

Again we can think that the VEVs for H and C are implemented by F-term equations when we turn on linear deformations in $\mathsf{O_H}$ and in $\mathsf{O_C}$ in the flip-flip frame. We can then use the same strategy described in the $3d$ case, but this time using the $4d$ duality web of Figure 3.7 and map these deformations across flip-flip duality, so that they become mass deformations of $E[USp(2N)]$. Finally we move back to the flip-flip dual frame, using sequentially the Intriligator–Pouliot duality to reach $E_\rho^\sigma[USp(2N)]$.

More precisely we consider the following deformation of $E[USp(2N)]$:

$$\delta\mathcal{W} = \mathrm{Tr}_x\left[(\mathsf{J}_\sigma + \mathsf{S}_\sigma)\cdot\mathsf{H}\right] + \mathrm{Tr}_y\left[(\mathsf{J}_\rho + \mathsf{T}_\rho)\cdot\mathsf{C}\right] + \sum_{\{(i,j)\neq(1,1)|1\leq i\leq\sigma_j,\,1\leq j\leq\rho_i\}} \mathsf{O}_B^{ij}\, B_{j,N-i+1}\,. \tag{4.54}$$

We have introduced extra gauge singlet chiral multiplets flipping some operators of the original $E[USp(2N)]$ theory that would represent a massless free sector of the theory after the deformation. Note that the role of S_σ and T_ρ is the same as that of $\mathcal{S}_\sigma$ and $\mathcal{T}_\rho$ in $3d$, which flip part of the antisymmetric mesonic operators remaining massless in the presence of the mass terms, but in $4d$ they are determined requiring that they are traceless antisymmetric matrices commuting with the matrices J_σ and J_ρ respectively. In addition, there are other gauge singlet fields O_B^{ij} which flip the operators B_{ij} we defined in (3.26)[10].

The superpotential (4.54) triggers a flow to a new theory T. Due to this superpotential term, the $USp(2N)_x$ global symmetry of the original $E[USp(2N)]$ theory is now broken to

$$USp(2N)_x \quad\longrightarrow\quad \prod_{m=1}^{N} USp(2k_m)_{x^{(m)}}\,. \tag{4.55}$$

Likewise, the $USp(2N)_y$ global symmetry is also broken to

$$USp(2N)_y \quad\longrightarrow\quad \prod_{n=1}^{N} USp(2l_n)_{y^{(n)}}\,. \tag{4.56}$$

[9]Notice that the VEVs we are considering are not labelled by partitions of $USp(2N)$, but by partitions of the $SU(N)$ part of $U(1)\times SU(N)\subset USp(2N)$. This choice is due to the fact that we want to mimic the deformations we perform in $3d$ and find models that reduce to $T_\rho^\sigma[SU(N)]$.

[10]These extra O_B^{ij} singlets were absent in the $3d$ case. Indeed, they are charged under $U(1)_c$, which means that they are massive and integrated out in the limit leading to $T[SU(N)]$.

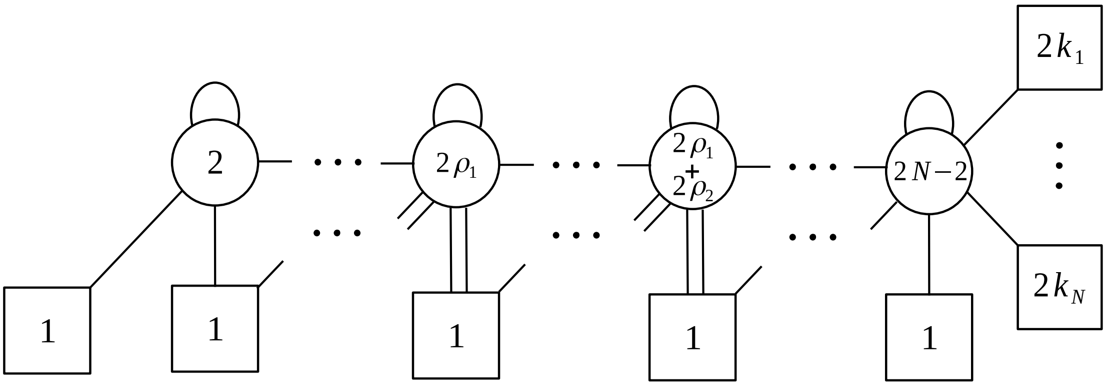

Figure 4.8: The quiver diagram representation of the deformed theory T. We have double lines in the saw only for the gauge nodes at positions ρ_1, $\rho_1+\rho_2$, $\dots$, $\sum_{i=1}^{N-1}\rho_i$. The mirror-like dual theory, which is denoted by $\mathsf{T}^\vee$, has the same diagram with ρ and σ exchanged.

This IR symmetry will become manifest in the mirror dual Lagrangian. Correspondingly at the level of supersymmetric indices we will introduce the following fugacities

$$\begin{aligned} &x_i, \quad \text{with} \quad i=1,\cdots,N \quad \rightarrow \quad x^{(1)}_{i_1}, x^{(2)}_{i_2},\cdots \quad \text{with} \quad i_m=1,\cdots,k_m \\ &y_i, \quad \text{with} \quad i=1,\cdots,N \quad \rightarrow \quad y^{(1)}_{i_1}, y^{(2)}_{i_2},\cdots \quad \text{with} \quad i_n=1,\cdots,l_n\,. \end{aligned} \tag{4.57}$$

We denote by $\text{Tr}_{x^{(m)}}$ and $\text{Tr}_{y^{(n)}}$ respectively the traces over $USp(2k_m)_{x^{(m)}}$ and $USp(2l_n)_{y^{(n)}}$ indices.

Moreover, the mass terms in (4.54) make some of the chiral multiplets of $E[USp(2N)]$ massive and being integrated out. First, let us look at the chirals in the saw. Due to the mass terms, only the followings among the original set of $D^{(i)}$ and $V^{(i)}$ remain massless:

$$\begin{aligned} &D^{(i)}_1\,, \qquad i=\rho_1,\,\rho_1+\rho_2,\,\dots,\,N\,, \\ &D^{(i)}_2\,, \qquad i=1,\,\dots,\,N\,, \\ &V^{(i)}_1\,, \qquad i=1,\,\dots,\,N\,, \\ &V^{(i)}_2\,, \qquad i=\rho_1,\,\rho_1+\rho_2,\,\dots,\,\sum_{n=1}^{L-1}\rho_n\,. \end{aligned} \tag{4.58}$$

Second, in $E[USp(2N)]$ there are $2N$ fundamental chirals $Q^{(N-1,N)}$ attached to the last gauge node. Again due to the mass terms in (4.54), only $2K$ of them remain massless. We rename as $Q_m, \tilde{Q}_m$ the massless chirals at the $USp(2(N-1))$ gauge node in the fundamental representation of each $USp(2k_m)$ factor, with $m=1,\cdots,N$. In particular, for the values of m for which $k_m=0$ we don't have any chiral field. Their interaction with the antisymmetric

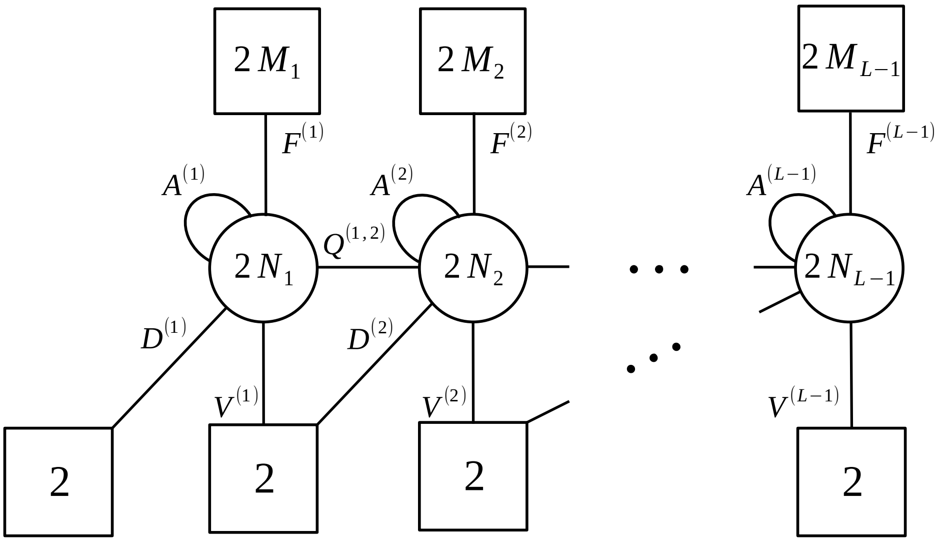

Figure 4.9: The $E^{\sigma}_{\rho}[USp(2N)]$ quiver diagram. To avoid cluttering the drawing the gauge singlet γ_{nj} and $\pi^{(i,j)}$ are not shown in this diagram.

$A^{(N-1)}$ is

$$\mathrm{Tr}_{N-1}\left[A^{(N-1)}\mathsf{H}\right] \quad \longrightarrow \quad \sum_{m=1}^{N}\mathrm{Tr}_{N-1}\left[\left(A^{(N-1)}\right)^{m}\mathrm{Tr}_{x^{(m)}}\,Q_m Q_m\right]\,. \tag{4.59}$$

The quiver diagram of T is drawn in Figure 4.8.

At this point we can go from T to $E^{\sigma}_{\rho}[USp(2N)]$ by iteratively applying Intriligator–Pouliot dualities to move to the flip-flip dual frame. The quiver diagram representation of the $E^{\sigma}_{\rho}[USp(2N)]$ theory is shown in Figure 4.9. There are also two sets of gauge singlets: the chirals γ_{nj} which are also singlets under the non-abelian global symmetries and the chirals $\pi^{(i,j)}$ that transform non-trivially under the non-abelian symmetries. To avoid cluttering Figure 4.9 we did not draw the gauge singlets (but we will do so in the examples we will present). The flavor nodes in the top line and the gauge nodes in the middle line are $USp(2n)$ groups with ranks determined by the partitions ρ and σ as for $T^{\sigma}_{\rho}[SU(N)]$

$$\begin{aligned} M_{L-i} &= k_i\,, \\ N_{L-i} &= \sum_{j=i+1}^{L}\rho_j - \sum_{j=i+1}^{N}(j-i)k_j\,. \end{aligned} \tag{4.60}$$

The $E_\rho^\sigma[USp(2N)]$ superpotential is given by

$$\begin{aligned}\mathcal{W}_{E_\rho^\sigma[USp(2N)]} = &\sum_{n=1}^{L-1} \mathrm{Tr}_n \left[A^{(n)} \left(\mathrm{Tr}_{n+1}\, \mathbb{Q}^{(n,n+1)} - \mathrm{Tr}_{n-1}\, \mathbb{Q}^{(n-1,n)} + \mathrm{Tr}_{x^{(n)}}\, F^{(n)} F^{(n)}\right)\right] + \\ &+ \sum_{n=1}^{L-2} \mathrm{Tr}_n\, \mathrm{Tr}_{n+1} \left[V_{[1}^{(n)} Q^{(n,n+1)} D_{2]}^{(n+1)}\right] + \\ &+ \sum_{n=1}^{L-1} \sum_{j=1}^{N_n - N_{n-1}} \gamma_{nj}\, \mathrm{Tr}_n \left[\left(A^{(n)}\right)^{j-1} D_{[1}^{(n)} D_{2]}^{(n)}\right] + \\ &+ \sum_{i=1}^{L-1} \sum_{j=i+1}^{L} \left(\prod_{k=1}^{j-1} \mathrm{Tr}_k\right) \mathrm{Tr}_{x^{(i)}} \left[F^{(i)} \left(\prod_{l=i}^{j-2} Q^{(l,l+1)}\right) V_{[1}^{(j-1)} \pi_{2]}^{(i,j)}\right],\end{aligned} \tag{4.61}$$

where we defined $N_0 = 0$. We also recall that the Tr_n traces are taken over the n-th gauge node. Notice the interaction terms for the gauge singlets. In particular, the singlets γ_{nj} couple to the n-th diagonal meson dressed by the $(j-1)$-th power of the antisymmetric chiral $A^{(n)}$, with $j = 1, \cdots, N_n - N_{n-1}$. This means that the maximum power of the dressing is given by how much the rank of the n-th gauge group jumps when compared to the $(n-1)$-th one. Moreover, we have singlets $\pi^{(i,j)}$ connecting the i-th $USp(2M_i)$ flavor node to all the j-th $SU(2)$ nodes of the saw sitting to its right, that is $j = i+1, \cdots, L$. The $\pi^{(i,j)}$ singlets play a key role in the enhancement of the non-abelian global symmetry since they enter the superpotential by flipping gauge invariant operators which do not respect the enhanced symmetry.

The IR non-anomalous global symmetry of $E_\rho^\sigma[USp(2N)]$ is

$$\prod_{m=1}^{N} USp(2k_m)_{x^{(m)}} \times \prod_{n=1}^{N} USp(2l_n)_{y^{(n)}} \times U(1)_c \times U(1)_t\,. \tag{4.62}$$

Indeed, one can verify that the constraints coming from the superpotential (4.61) and from the requirement that the NSVZ beta-functions vanish at each gauge node fix all the R-charges of the chiral fields up to two parameters, which correspond to the mixing coefficients $\mathfrak{c}$ and $\mathfrak{t}$ with $U(1)_c$ and $U(1)_t$. For what concerns the non-abelian part, the global symmetry $USp(2N)_x \times USp(2N)_y$ of the original $E[USp(2N)]$ theory is broken to

$$USp(2N)_x \times USp(2N)_y \quad \longrightarrow \quad \prod_{m=1}^{N} USp(2k_m)_{x^{(m)}} \times \prod_{n=1}^{N} USp(2l_n)_{y^{(n)}}\,, \tag{4.63}$$

where, like the original $E[USp(2N)]$ theory, only $USp(2)^{l_n} \subset USp(2l_n)_{y^{(n)}}$ is manifest in the quiver gauge theory description.

Let's now consider the mirror dual frame where, because of the operator map (3.33), the deformation superpotential (4.54) becomes

$$\delta\mathcal{W}^\vee = \text{Tr}_x\left[(J_\sigma + \mathsf{S}_\sigma)\cdot\mathsf{C}^\vee\right] + \text{Tr}_y\left[(J_\rho + \mathsf{T}_\rho)\cdot\mathsf{H}^\vee\right] + \sum_{\{(i,j)\neq(1,1)|1\leq i\leq\sigma_j,\,1\leq j\leq\rho_i\}} \mathsf{O}_B^{ij}\,B^\vee_{i,N-j+1}\,. \tag{4.64}$$

This deformation triggers a flow from $E[USp(2N)]^\vee$ to $\mathsf{T}^\vee$ which contains gauge singlets S_σ, T_ρ and O_B, which are mapped to the same gauge singlets in T.

Next we take the flip-flip duality on $\mathsf{T}^\vee$. This leads to the mirror dual of $E_\rho^\sigma[USp(2N)]$, denoted by $E_\sigma^\rho[USp(2N)]$. Indeed, $E_\rho^\sigma[USp(2N)]$ and $E_\sigma^\rho[USp(2N)]$ have the same global symmetry as well as the same operator spectrum. In the following we will illustrate this construction in various examples.

4.3.2 Some examples

Example I: $\rho = [N]$ **and** $\sigma = [1^N]$

Flow to $E_{[N]}[USp(2N)]$

In this case, the superpotential deformation triggering the flow to theory T is given by

$$\delta\mathcal{W} = \text{Tr}_x\left[\mathsf{S}_{[1^N]}\cdot\mathsf{H}\right] + \text{Tr}_y\left[\mathsf{T}_{[N]}\cdot\mathsf{C}\right] + \sum_{i=1}^{N-1}\text{Tr}_i\left[D_1^{(i)}V_2^{(i)}\right] + \sum_{j=2}^{N}\mathsf{O}_B^{1j}\,\beta_{N-j+1}\,, \tag{4.65}$$

where $\mathsf{S}_{[1^N]}$ is an arbitrary $2N\times 2N$ antisymmetric matrix and $\mathsf{T}_{[N]}$ is determined requiring that it is traceless antisymmetric and that it commutes with $\mathsf{J}_{[N]}$

$$\mathsf{T}_{[N]} = \begin{pmatrix} 0 & -\mathsf{T}^{(2)T} & \cdots & -\mathsf{T}^{(N)T} \\ \mathsf{T}^{(2)} & 0 & \cdots & -\mathsf{T}^{(N-1)T} \\ \vdots & \ddots & \ddots & \vdots \\ \mathsf{T}^{(N)} & \cdots & \mathsf{T}^{(2)} & 0 \end{pmatrix}, \tag{4.66}$$

where each $\mathsf{T}^{(i)}$ is a 2×2 matrix with a single non-zero element

$$\mathsf{T}^{(i)} = \begin{pmatrix} 0 & 0 \\ \mathsf{t}^{(i)} & 0 \end{pmatrix}. \tag{4.67}$$

Note that the flavor indices 1, 2 of $D_1^{(i)}$ and $V_2^{(i)}$ do not belong to the same $SU(2)$; $D_1^{(i)}$ is charged under the i-th $SU(2)$ in the saw while $V_2^{(i)}$ is charged under the $(i+1)$-th $SU(2)$.

It turns out that this deformation breaks the $USp(2N)_y$ symmetry of the original $E[USp(2N)]$ to $SU(2)_y$. The deformation also makes $D_1^{(i)}$ and $V_2^{(i)}$ massive for $i = 1,\ldots,N-1$.

We obtain theory T by integrating out those massive fields. In theory T each gauge node except the last one now has only two fundamental chirals while the last gauge node has $2N+2$ fundamental chirals in addition to the bifundamental and antisymmetric chirals which remain the same.

Now to reach $E_{[N]}[USp(2N)]$ we need to implement the flip-flip duality by sequentially applying the Intriligator–Pouliot duality on each gauge node starting from the left. The first gauge node is $USp(2)$ with a total of 6 fundamental chirals, the antisymmetric is a gauge singlet so we can apply directly the Intriligator–Pouliot duality. As the $USp(2)$ theory with 6 chirals is dual to a WZ model with 15 chirals, the leftmost gauge node is confined once the duality is applied. Some of the 15 chirals make massive the traceless part of the antisymmetric chiral of the next $USp(4)$ gauge node, while the others partially cancel with the singlets $\mathsf{S}_{[1^N]}$, $\mathsf{T}_{[N]}$ and O_B^{1j}. Now the $USp(4)$ node has 8 chirals and is also confined when we apply the Intriligator–Pouliot duality. Proceeding to the right we see that the entire chain of gauge nodes is sequentially confined leaving a set of chirals at the end. So the $E_{[N]}[USp(2N)]$ theory will be a Wess–Zumino model.

This procedure of applying sequential Intriligator–Pouliot dualities can be realized at the level of the index. The mass deformation $\sum_{i=1}^{N-1}\mathrm{Tr}_i\left[D_1^{(i)}V_2^{(i)}\right]$ in (4.65) imposes the constraints on the fugacities of the saw

$$\frac{y_{i+1}}{y_i}=t, \qquad i=1,\cdots,N-1\,, \tag{4.68}$$

which can be solved with

$$y_i=t^{i-1}a, \qquad i=1,\cdots,N\,. \tag{4.69}$$

For our purpose, it is convenient to use $y=at^{\frac{N-1}{2}}=y_it^{\frac{N-2i+1}{2}}$, which makes the unbroken $SU(2)_y\subset USp(2N)_y$ manifest. The extra chirals we introduce give rise to the following index contributions:

$$\begin{aligned}
\mathsf{S}_{[1^N]} &\quad\longrightarrow\quad \Gamma_e\left(pqt^{-1}\right)^{N-1}\prod_{i<j}^{N}\Gamma_e\left(pqt^{-1}x_i^{\pm1}x_j^{\pm1}\right)\,,\\
\mathsf{T}_{[N]} &\quad\longrightarrow\quad \prod_{j=2}^{N}\Gamma_e\left(t^j\right)\,, \qquad \mathsf{O}_B^{1j}\quad\longrightarrow\quad\Gamma_e\left(t^{1-j}c^2\right)\,.
\end{aligned} \tag{4.70}$$

Hence, the complete index of theory T is given by[11]

$$\mathcal{I}_{\mathsf{T}}(\vec{x};y;c;t)=\Gamma_e\left(pqt^{-1}\right)^{N-1}\prod_{i<j}^{N}\Gamma_e\left(pqt^{-1}x_i^{\pm1}x_j^{\pm1}\right)\prod_{j=2}^{N}\Gamma_e\left(t^j\right)\Gamma_e\left(t^{1-j}c^2\right)\times$$

[11]We recall that now the index of $E[USp(2N)]$ is defined without the contribution of the singlet β_N, which is represented by $\Gamma_e\left(pqc^{-2}\right)$.

$$\times\, \mathcal{I}_{E[USp(2N)]}\left(\vec{x}; t^{\frac{1-N}{2}}y, t^{\frac{3-N}{2}}y, \cdots, t^{\frac{N-1}{2}}y; c; t\right). \tag{4.71}$$

The sequential confinement of the tail then translates in the identity

$$\begin{aligned}&\mathcal{I}_{E[USp(2N)]}\left(\vec{x}; t^{N-1}u, t^{N-2}u, \cdots, u; c; t\right) = \\ &= \Gamma_e\left(c^2\right)\Gamma_e\left(t\right)^N \prod_{i<j}^N \Gamma_e\left(t\, x_i^{\pm1}x_j^{\pm1}\right)\prod_{i=1}^N \frac{\Gamma_e\left(u\, c\, x_i^{\pm1}\right)\Gamma_e\left(\frac{c}{u\, t^{N-1}}x_i^{\pm1}\right)}{\Gamma_e\left(t^{1-i}c^2\right)\Gamma_e\left(t^i\right)}.\end{aligned} \tag{4.72}$$

This is another form of the equality (3.65) that we already encountered and which we recall was proven by Rains in Corollary 2.8 of [79]. Putting this back into $\mathcal{I}_{\mathsf{T}}$ with $u = t^{\frac{1-N}{2}}y$[12], we obtain the identity

$$\mathcal{I}_{\mathsf{T}}(\vec{x}; y; c; t) = \prod_{i=1}^N \Gamma_e\left(y^{\pm1}t^{-\frac{N-1}{2}}cx_i^{\pm1}\right) = \mathcal{I}_{E_{[N]}[USp(2N)]}(\vec{x}; y; c; t)\,. \tag{4.73}$$

As expected, $E_{[N]}[USp(2N)]$ is a WZ model with $2N$ chirals, which are a bifundamental of $USp(2N)_x \times SU(2)_y$. One can see that the new fugacity y makes the $SU(2)_y$ symmetry manifest.

Flow to $E^{[N]}[USp(2N)]$

Now let us examine this confinement on the mirror side. The superpotential deformation triggering the flow to theory $\mathsf{T}^\vee$ is given by

$$\begin{aligned}\delta\mathcal{W}^\vee =\ & \mathrm{Tr}_x\left[\mathsf{S}_{[1^N]}\cdot\mathsf{C}^\vee\right] + \mathrm{Tr}_y\left[\mathsf{T}_{[N]}\cdot\mathsf{H}^\vee\right] + \sum_{n=1}^{N-1}\mathrm{Tr}_{N-1}\left[q_{2n-1}^{(N-1,N)}q_{2n+2}^{(N-1,N)}\right] + \\ &+ \sum_{j=2}^N \mathsf{O}_B^{1j}\,\mathrm{Tr}_{N-1}\left[\left(A^{(N-1)}\right)^{j-2}v_{[1}^{(N-1)}v_{2]}^{(N-1)}\right],\end{aligned} \tag{4.74}$$

which makes $q^{(N-1,N)}$ massive except $q_2^{(N-1,N)}$ and $q_{2N-1}^{(N-1,N)}$. Integrating out the massive $q^{(N-1,N)}$, we reach theory $\mathsf{T}^\vee$, which is mirror-like dual to theory T.

$\mathsf{T}^\vee$ differs from $E[USp(2N)]$ only by the fact that there are just two chirals attached to the last gauge node. Now to reach $E^{[N]}[USp(2N)]$ we can implement the flip-flip duality by sequentially applying the Intriligator–Pouliot duality on each gauge node starting from the leftmost $USp(2)$ node and proceeding along the tail. Since the first $N-2$ nodes are $USp(2n)$ with $4n+4$ chirals, their rank does not change when we apply the Intriligator–Pouliot duality. However, when we act one the last gauge node, which is $USp(2(N-1))$ with $2n+2$ chirals, it confines. At the second iteration we start again from the leftmost $USp(2)$ node but when we

[12] Notice that to apply (4.72) we need to use the $USp(2N)_y$ Weyl symmetry of $E[USp(2N)]$ to reorder the fugacities.

reach the $USp(2(N-2))$ node it confines. In this way the quiver is confined from the right until we are left with the same gauge singlets as in (4.73), that is we reach the $E^{[N]}[USp(2N)]$ WZ model. This is the 4d version of a similar confinement of $T[SU(N)]$ when a monopole superpotential is turned on at each gauge node which was studied in [169].

Example II: $\rho = [N-1,1]$ **and** $\sigma = [1^N]$

Flow to $E_{[N-1,1]}[USp(2N)]$

The deformation leading to theory T is given in this case by

$$\delta\mathcal{W} = \mathrm{Tr}_x\left[\mathsf{S}_{[1^N]}\cdot\mathsf{H}\right] + \mathrm{Tr}_y\left[\mathsf{T}_{[N-1,1]}\cdot\mathsf{C}\right] + \sum_{i=1}^{N-2}\mathrm{Tr}_i\left[D_1^{(i)}V_2^{(i)}\right] + \sum_{j=2}^{N-1}\mathsf{O}_B^{1j}\,\beta_{N-j+1}\,, \qquad (4.75)$$

where $\mathsf{S}_{[1^N]}$ is again an arbitrary $2N\times 2N$ skew-symmetric matrix and $\mathsf{T}_{[N-1,1]}$ is given by

$$\mathsf{T}_{[N-1,1]} = \begin{pmatrix} \mathsf{T}_{11}^{(1)} & -\mathsf{T}_{11}^{(2)T} & \cdots & -\mathsf{T}_{11}^{(N-1)T} & -\mathsf{T}_{N1}^{(1)\,T} \\ \mathsf{T}_{11}^{(2)} & \mathsf{T}_{11}^{(1)} & \cdots & -\mathsf{T}_{11}^{(N-2)T} & \vdots \\ \vdots & \ddots & \ddots & \vdots & 0 \\ \mathsf{T}_{11}^{(N-1)} & \cdots & \mathsf{T}_{11}^{(2)} & \mathsf{T}_{11}^{(1)} & \mathsf{T}_{1N}^{(1)} \\ \mathsf{T}_{N1}^{(1)} & 0 & \cdots & -\mathsf{T}_{1N}^{(1)\,T} & -(N-1)\mathsf{T}_{11}^{(1)} \end{pmatrix}, \qquad (4.76)$$

where each $T_{ij}^{(n)}$ is a 2×2 matrix of the form:

$$\mathsf{T}_{11}^{(1)} = \begin{pmatrix} 0 & -\mathsf{t}_{11}^{(1)} \\ \mathsf{t}_{11}^{(1)} & 0 \end{pmatrix},$$

$$\mathsf{T}_{ij}^{(n)} = \begin{cases} \begin{pmatrix} 0 & 0 \\ \mathsf{t}_{ii}^{(n)} & 0 \end{pmatrix}, & i=j,\ n\neq 1\,, \\ \begin{pmatrix} \mathsf{r}_{ij}^{(n)} & 0 \\ \mathsf{s}_{ij}^{(n)} & 0 \end{pmatrix}, & i>j,\ n\neq 1\,, \\ \begin{pmatrix} 0 & 0 \\ \mathsf{u}_{ij}^{(n)} & \mathsf{w}_{ij}^{(n)} \end{pmatrix}, & i<j,\ n\neq 1\,. \end{cases} \qquad (4.77)$$

One can write down the supersymmetric index of the theory T by constraining the fugacities of the index of $E[USp(2N)]$. The deformation (4.75) demands the following conditions on the $USp(2N)_y$ fugacities:

$$\frac{y_{i+1}}{y_i} = t\,, \qquad i=1,\ldots,N-2\,, \qquad (4.78)$$

which are satisfied by

$$y_i = t^{i-1}a\,, \qquad i = 1, \ldots, N-1\,. \tag{4.79}$$

For later convenience, we introduce the new fugacities

$$\begin{aligned} y_i &= t^{i-\frac{N}{2}}y^{(1)}\,, \qquad i = 1, \ldots, N-1\,, \\ y_N &= y^{(2)}\,, \end{aligned} \tag{4.80}$$

which will make the unbroken $USp(2)_{y^{(1)}} \times USp(2)_{y^{(2)}} \subset USp(2N)_y$ manifest in the index. The extra chiral singlets we introduce then give rise to the following index contributions:

$$\begin{aligned} \mathsf{S}_{[1^N]} \quad &\longrightarrow \quad \Gamma_e\left(pqt^{-1}\right)^{N-1}\prod_{i<j}^{N}\Gamma_e\left(pqt^{-1}x_i^{\pm1}x_j^{\pm1}\right)\,, \\ \mathsf{T}_{[N-1,1]} \quad &\longrightarrow \quad \Gamma_e\left(t^{\frac{N}{2}}y^{(1)\pm1}y^{(2)\pm1}\right)\prod_{i=1}^{N-1}\Gamma_e\left(t^i\right)\,, \\ \mathsf{O}_B^{1j} \quad &\longrightarrow \quad \Gamma_e\left(t^{1-j}c^2\right)\,. \end{aligned} \tag{4.81}$$

Substituting them into the recursive definition of the index of the $E[USp(2N)]$ theory, we obtain the index of theory T as follows:

$$\begin{aligned} &\mathcal{I}_{\mathsf{T}}\left(\vec{x}; y^{(1)}, y^{(2)}; c; t\right) = \\ &= \Gamma_e\left(pqt^{-1}\right)^{N-1}\prod_{i<j}^{N}\Gamma_e\left(pqt^{-1}x_i^{\pm1}x_j^{\pm1}\right)\Gamma_e\left(t^{\frac{N}{2}}y^{(1)\pm1}y^{(2)\pm1}\right)\prod_{i=1}^{N-1}\Gamma_e\left(t^i\right) \\ &\quad\times\prod_{j=2}^{N-1}\Gamma_e\left(t^{1-j}c^2\right)\mathcal{I}_{E[USp(2N)]}\left(\vec{x}; t^{-\frac{N}{2}+1}y^{(1)}, t^{-\frac{N}{2}+2}y^{(1)}, \cdots, t^{\frac{N}{2}-1}y^{(1)}, y^{(2)}; c; t\right) = \\ &= \Gamma_e\left(pqt^{-1}\right)^{N-1}\prod_{i<j}^{N}\Gamma_e\left(pqt^{-1}x_i^{\pm1}x_j^{\pm1}\right)\Gamma_e\left(t^{\frac{N}{2}}y^{(1)\pm1}y^{(2)\pm1}\right)\prod_{i=1}^{N-1}\Gamma_e\left(t^i\right)\times \\ &\quad\times\prod_{j=2}^{N-1}\Gamma_e\left(t^{1-j}c^2\right)\frac{\prod_{i=1}^{N}\Gamma_e\left(c\,y^{(2)\pm1}x_i^{\pm1}\right)}{\Gamma_e\left(t^{-1}c^2\right)}\oint \mathrm{d}\vec{z}^{(N-1)}_{N-1}\,\Gamma_e\left(pqt^{-1}\right)^{N-1}\times \\ &\quad\times\prod_{a<b}^{N-1}\Gamma_e\left(pqt^{-1}z_a^{(N-1)\pm1}z_b^{(N-1)\pm1}\right)\prod_{a=1}^{N-1}\frac{\prod_{i=1}^{N}\Gamma_e\left(t^{1/2}z_a^{(N-1)\pm1}x_i^{\pm1}\right)}{\Gamma_e\left(t^{1/2}c\,y^{(2)\pm1}z_a^{(N-1)\pm1}\right)}\times \\ &\quad\times\mathcal{I}_{E[USp(2(N-1))]}\left(z_1^{(N-1)}, \cdots, z_{N-1}^{(N-1)}; t^{-\frac{N}{2}+1}y^{(1)}, t^{-\frac{N}{2}+2}y^{(1)}, \cdots, t^{\frac{N}{2}-1}y^{(1)}; t^{-1/2}c; t\right)\,. \end{aligned} \tag{4.82}$$

At this stage, one can see that there is an $SU(2)$ symmetry for $y^{(2)}$ while it is not clear whether or not we have an enhanced $SU(2)$ symmetry for $y^{(1)}$.

To reach $E_{[N-1,1]}[USp(2N)]$ we need to implement the flip-flip duality by applying iteratively the IP duality. We can recycle some of the previous calculations noting that the last factor of the integrand is the index of $E[USp(2N-2)]$ with the specialization of parameters leading to the evaluation formula (4.72) as we discussed in the previous example. Taking this into account, we obtain

$$\begin{aligned}\mathcal{I}_{\mathsf{T}} = &\Gamma_e\left(pqt^{-1}\right)^{N-1}\prod_{i<j}^{N}\Gamma_e\left(pqt^{-1}x_i^{\pm1}x_j^{\pm1}\right)\Gamma_e\left(t^{\frac{N}{2}}y^{(1)\pm1}y^{(2)\pm1}\right)\times\\ &\times\frac{\prod_{i=1}^N\Gamma_e\left(c\,y^{(2)\pm1}x_i^{\pm1}\right)}{\Gamma_e\left(t^{-N+1}c^2\right)}\oint \mathrm{d}\vec{z}_{N-1}^{(N-1)}\prod_{a=1}^{N-1}\Gamma_e\left(t^{-\frac{N-1}{2}}cy^{(1)\pm1}z_a^{(N-1)\pm1}\right)\\ &\times\Gamma_e\left(pqt^{-1/2}c^{-1}y^{(2)\pm1}z_a^{(N-1)\pm1}\right)\prod_{i=1}^{N}\Gamma_e\left(t^{1/2}z_a^{(N-1)\pm1}x_i^{\pm1}\right)\,,\end{aligned} \tag{4.83}$$

where the $SU(2)$ symmetry for $y^{(1)}$ is now manifest.

This is the index of a $USp(2(N-1))$ theory with $2N+4$ favors and various flipping fields. To complete the derivation of the flip-flip duality we need to apply the Intriligator–Pouliot duality one more time and we obtain

$$\begin{aligned}\mathcal{I}_{\mathsf{T}} = &\frac{\prod_{i=1}^N\Gamma_e\left(t^{-\frac{N}{2}+1}cy^{(1)\pm1}x_i^{\pm1}\right)}{\Gamma_e\left(p^{-1}q^{-1}tc^2\right)}\oint \mathrm{d}\vec{z}_1^{(1)}\,\Gamma_e\left(t\right)\Gamma_e\left(p^{1/2}q^{1/2}t^{\frac{N-1}{2}}c^{-1}y^{(1)\pm1}z^{(1)\pm1}\right)\times\\ &\times\Gamma_e\left(p^{-1/2}q^{-1/2}t^{1/2}cy^{(2)\pm1}z^{(1)\pm1}\right)\prod_{i=1}^{N}\Gamma_e\left(p^{1/2}q^{1/2}t^{-1/2}x_i^{\pm1}z^{(1)\pm1}\right)=\\ &=\mathcal{I}_{E_{[N-1,1]}[USp(2N)]}\left(\vec{x};y^{(2)},y^{(1)};c;t\right)\,.\end{aligned} \tag{4.84}$$

The $E_{[N-1,1]}[USp(2N)]$ theory is a $USp(2)$ theory with $2N+4$ fundamental chirals and some additional singlets, which are shown in Figure 4.10.[13] From the index (4.84), one can read off the charges of each chiral multiplet and the available superpotential. For example, one can see that there is a singlet γ_{11}, whose index contribution is $\Gamma_e\left(p^{-1}q^{-1}tc^2\right)^{-1}$, flipping the diagonal meson $\mathrm{Tr}_1 D^{(1)}_{[1}D^{(1)}_{2]}$ where $D^{(1)}$ contributes to the index by $\Gamma_e\left(p^{-1/2}q^{-1/2}t^{1/2}cy_2^{\pm1}z^{(1)\pm1}\right)$. The total superpotential of $E_{[N-1,1]}[USp(2N)]$ is given by

$$\mathcal{W}_{E_{[N-1,1]}[USp(2N)]} = \mathrm{Tr}_1\mathrm{Tr}_x\left[A^{(1)}F^{(1)}F^{(1)}\right]+\mathrm{Tr}_1\mathrm{Tr}_x\left[F^{(1)}V^{(1)}_{[1}\pi^{(1,2)}_{2]}\right]+\gamma_{11}\mathrm{Tr}_2\left[D^{(1)}_{[1}D^{(1)}_{2]}\right]\,. \tag{4.85}$$

[13]Note that, as a consequence of the sequential application of the Intriligator–Pouliot duality, the fugacities are permuted and the two nodes in the saw are labeled by $y^{(2)}$ and $y^{(1)}$ respectively, from the left, which is the opposite labelling compared to the definition of the original $E[USp(2N)]$ index. For this reason, we call the index (4.84) as $\mathcal{I}_{E_{[N-1,1]}[USp(2N)]}(\vec{x};y^{(2)},y^{(1)};t,c)$ instead of $\mathcal{I}_{E_{[N-1,1]}[USp(2N)]}(\vec{x};y^{(1)},y^{(2)};t,c)$. Indeed we can't use the $USp(2N)$ Weyl symmetry to reorder the two set of fugacities $y^{(1)}$ and $y^{(2)}$. This is similar to what we saw in the $3d$ case.

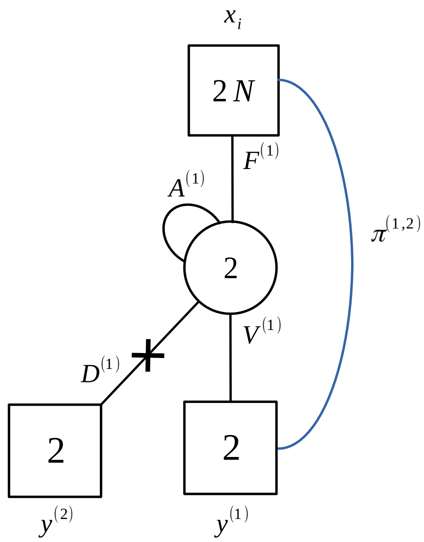

Figure 4.10: The quiver diagram representation of $E_{[N-1,1]}[USp(2N)]$. We also explicitly draw all the singlet fields. The crosses as usual denote singlets flipping the corresponding mesons.

We can work out some interesting gauge invariant operators

$$\begin{aligned}
\Pi^{(1)} &= \pi^{(1,2)}\,, \\
\Pi^{(2)} &= \mathrm{Tr}_1\left[D^{(1)}F^{(1)}\right]\,, \\
\mathsf{C}^{(1)} &= \mathrm{Tr}_1 A^{(1)}\,, \\
\mathsf{C}^{(1,2)} &= \mathrm{Tr}_1\left[D^{(1)}V^{(1)}\right]\,, \\
\mathsf{H} &= \mathrm{Tr}_1\left[F^{(1)}F^{(1)}\right]\,.
\end{aligned} \tag{4.86}$$

Recall that the global symmetry of $E_{[N-1,1]}[USp(2N)]$ includes $USp(2N)_x \times USp(2)_{y^{(1)}} \times USp(2)_{y^{(2)}}$ rather than $USp(2N)_x \times USp(4)_y$ unless $N = 2$. Indeed, we find that the would-be antisymmetric operator of $USp(4)_y$ is decomposed into one singlet operator and one bifundamental operator between $USp(2)_{y^{(1)}} \times USp(2)_{y^{(2)}}$, which are denoted by $\mathsf{C}^{(1)}$ and $\mathsf{C}^{(1,2)}$ respectively. Also each $\Pi^{(i)}$ is a bifundamental operator between $USp(2N)_x \times USp(2)_{y^{(i)}}$. As expected, $\mathsf{C}^{(1)}$ and $\mathsf{C}^{(1,2)}$ have different $U(1)$ global charges unless $N = 2$, and so do $\Pi^{(1)}$ and $\Pi^{(2)}$. Thus, only $USp(2)_{y^{(1)}} \times USp(2)_{y^{(2)}} \subset USp(4)_y$ is preserved. On the other hand, H is an antisymmetric operator respecting the entire $USp(2N)_x$ symmetry.

This is the $4d$ version of the result we found in the $3d$ Example I for the same choice of partitions. There is a crucial difference between $3d$ and $4d$, though. Indeed, while in $3d$ because of the fact that the gauge coupling classically has positive mass dimension all gauge theories are expected to flow to an interacting SCFT in the IR if the number of flavors is sufficiently large, in $4d$ this depends on the sign of the β-function since the gauge coupling is classically marginal. In particular, in $4d$ the one-loop β-function is negative and the theory is asymptotically free only if the number of flavors is not too large, which is the opposite of what happens in $3d$. For the present case, the $3d$ theory is interacting if $N > 1$ and it is dual to a free hyper if $N = 1$. Instead in $4d$, notice that $E_{[N-1,1]}[USp(2N)]$ is asymptotically free only when $N < 4$. Among these three cases, $N = 1$ is the confining case while $N = 2$ is the self-dual case of Intriligator–Pouliot duality and $N = 3$ will give us a genuinely new duality between interacting SCFTs. Let us comment that, even though for some range of the parameters we will find dualities between IR free theories, these are still of interest. Indeed, the associated integral identities for the supersymmetric indices that we will find are still non-trivial. Moreover, upon compactification to $3d$ they give rise to known mirror dualities between interacting theories, so they constitute $4d$ ancestors of them.

In the rest of this section, we will mostly focus on the $N = 3$ case although the mathematical identities of the supersymmetric indices hold beyond $N = 3$.

Flow to $E^{[N-1,1]}[USp(2N)]$

Now let us consider the mass deformation in the mirror dual frame. In this dual theory, the superpotential deformation (4.75) is mapped to

$$\delta\mathcal{W}^\vee = \text{Tr}_x\left[\mathsf{S}_{[1^N]}\cdot\mathsf{C}^\vee\right] + \text{Tr}_y\left[\mathsf{T}_{[N-1,1]}\cdot\mathsf{H}^\vee\right] + \sum_{n=1}^{N-2}\text{Tr}_{N-1}q_{2n-1}^{(N-1,N)}q_{2n+2}^{(N-1,N)} + \\ + \sum_{j=2}^{N-1}\mathsf{O}_B^{1j}\,\text{Tr}_{N-1}\left[\left(A^{(N-1)}\right)^{j-2}v_{[1}^{(N-1)}v_{2]}^{(N-1)}\right]\,, \tag{4.87}$$

which makes $q_n^{(N-1,N)}$ massive except $n = 2,\, 2N-3,\, 2N-1,\, 2N$. The extra singlets we introduce are denoted by the same letters as in the original side.

The supersymmetric index of the theory $\mathsf{T}^\vee$ can be obtained from that of $E[USp(2N)]^\vee$ taking into account the extra singlet contributions (4.81) and by imposing the fugacity conditions (4.78)-(4.79)

$$\mathcal{I}_{\mathsf{T}^\vee}\left(\vec{x};y^{(1)},y^{(2)};c;t\right) = \\ = \Gamma_e\left(pqt^{-1}\right)^{N-1}\prod_{i<j}^{N}\Gamma_e\left(pqt^{-1}x_i^{\pm1}x_j^{\pm1}\right)\Gamma_e\left(t^{\frac{N}{2}}y^{(1)\pm1}y^{(2)\pm1}\right)\prod_{i=1}^{N-1}\Gamma_e\left(t^i\right)\prod_{j=2}^{N-1}\Gamma_e\left(t^{1-j}c^2\right)\times \\ \times\mathcal{I}_{E[USp(2N)]^\vee}\left(\vec{x};t^{-\frac{N}{2}+1}y^{(1)},t^{-\frac{N}{2}+2}y^{(1)},\cdots,t^{\frac{N}{2}-1}y^{(1)},y^{(2)};c;t\right) = \\ = \Gamma_e\left(pqt^{-1}\right)^{N-1}\prod_{i<j}^{N}\Gamma_e\left(pqt^{-1}x_i^{\pm1}x_j^{\pm1}\right)\Gamma_e\left(t^{\frac{N}{2}}y^{(1)\pm1}y^{(2)\pm1}\right)\prod_{i=1}^{N-1}\Gamma_e\left(t^i\right)\prod_{j=2}^{N-1}\Gamma_e\left(t^{1-j}c^2\right)\times$$

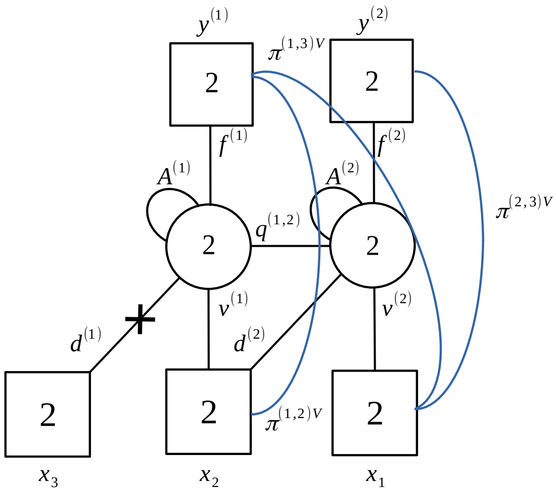

Figure 4.11: The quiver diagram representation of $E^{[2,1]}[USp(6)]$. The fugacity corresponding to each flavor node is also indicated.

$$\times\,\Gamma_e\left(p^2q^2t^{-1}c^{-2}\right)\Gamma_e\left(c\,x_N^{\pm1}y^{(2)\pm1}\right)\prod_{i=1}^{N-1}\Gamma_e\left(c\,x_N^{\pm1}\left(t^{i-\frac{N}{2}}y^{(1)}\right)^{\pm1}\right)\times$$
$$\times\oint \mathrm{d}\vec{z}^{(N-1)}_{N-1}\,\Gamma_e\left(t\right)^{N-1}\prod_{a<b}^{N-1}\Gamma_e\left(tz_a^{(N-1)\pm1}z_b^{(N-1)\pm1}\right)\times$$
$$\times\frac{\prod_{a=1}^{N-1}\Gamma_e\left(p^{1/2}q^{1/2}t^{-1/2}z_a^{(N-1)\pm1}y^{(2)\pm1}\right)\Gamma_e\left(p^{1/2}q^{1/2}t^{-\frac{N-1}{2}}z_a^{(N-1)\pm1}y^{(1)\pm1}\right)}{\prod_{a=1}^{N-1}\Gamma_e\left(p^{1/2}q^{1/2}t^{-1/2}c\,x_N^{\pm1}z_a^{(N-1)\pm1}\right)}\times$$
$$\times\,\mathcal{I}_{E[USp(2(N-1))]}\left(z_1^{(N-1)},\cdots,z_{N-1}^{(N-1)};x_1,\cdots,x_{N-1};p^{-1/2}q^{-1/2}t^{1/2}c;pq/t\right)\,. \qquad (4.88)$$

We see that the theory $\mathsf{T}^\vee$ is basically the same quiver theory as $E[USp(2N)]^\vee$ but there are only 4 fundamental chirals attached to the $(N-1)$-th gauge node on top of those of the saw. Two of these 4 chirals couple to $A^{(N-1)}$, while the other two couple to $\left(A^{(N-1)}\right)^{N-1}$.

Now we need to implement the flip-flip duality as a chain of sequential Intriligator–Pouliot dualities. In Appendix C.2 we do this at the level of the supersymmetric index for the $N=3$ case obtaining[14]:

$$\mathcal{I}_{\mathsf{T}^\vee}=\Gamma_e(t^{-1/2}cx_1^{\pm1}y^{(1)\pm1})\Gamma_e(t^{-1/2}cx_2^{\pm1}y^{(1)\pm1})\Gamma_e(cx_1^{\pm1}y^{(2)\pm1})\Gamma_e(pqt^2c^{-2})\times$$
$$\times\oint \mathrm{d}\vec{z}_1^{(1)}\,\mathrm{d}\vec{z}_1^{(2)}\,\Gamma_e\left(pqt^{-1}\right)^2\Gamma_e(t^{1/2}z^{(1)\pm1}y^{(1)\pm1})\times$$

[14] Again, the labelling of the saw by the x_i fugacities is in the opposite order compared to the original $E[USp(2N)]^\vee$ index. This time, however, the permutations of x_i belong to the Weyl group of the $USp(6)_x$ global symmetry. Thus, the index is invariant under such permutations, so we just call the index $\mathcal{I}_{E^{[2,1]}[USp(6)]}(y^{(1)},y^{(2)};\vec{x};c;pq/t)$ without specifying a particular order of x_i. Notice that this is the same as what happened in $3d$.

$$\begin{aligned}
&\times \Gamma_e(pqc^{-1}x_2^{\pm1}z^{(1)\pm1})\Gamma_e(t^{-1}cx_3^{\pm1}z^{(1)\pm1})\Gamma_e(t^{1/2}z^{(2)\pm1}y^{(2)\pm1})\times \\
&\times \Gamma_e(pqt^{-1/2}c^{-1}x_1^{\pm1}z^{(2)\pm1})\Gamma_e(t^{-1/2}cx_2^{\pm1}z^{(2)\pm1})\Gamma_e(t^{1/2}z^{(1)\pm1}z^{(2)\pm1}) = \\
&= \mathcal{I}_{E^{[2,1]}[USp(6)]}\left(y^{(1)}, y^{(2)}; \vec{x}; c; pq/t\right) . \qquad (4.89)
\end{aligned}$$

One can read off the matter content of $E^{[2,1]}[USp(6)]$ from the index (4.89), which is shown in Figure 4.11. In particular, there is a single flipping field $\gamma_{11}^{\vee}$, denoted by a cross in Figure 4.11, which flips the diagonal meson $\text{Tr}_1 d_{[1}^{(1)} d_{2]}^{(1)}$. The total superpotential is given by

$$\begin{aligned}
&\mathcal{W}_{E^{[2,1]}[USp(6)]} = \\
&= \text{Tr}_1\left[A^{(1)}\left(\text{Tr}_2\, q^{(1,2)}q^{(1,2)} + \text{Tr}_{y_1} f^{(1)}f^{(1)}\right)\right] + \text{Tr}_2\left[A^{(2)}\left(\text{Tr}_1\, q^{(1,2)}q^{(1,2)} + \text{Tr}_{y_2} f^{(2)}f^{(2)}\right)\right] + \\
&\quad + \text{Tr}_1\text{Tr}_2\left[v_{[1}^{(1)}q^{(1,2)}d_{2]}^{(2)}\right] + \text{Tr}_1\text{Tr}_{y_2}\left[f^{(1)}v_{[1}^{(1)}\pi_{2]}^{(1,2)\vee}\right] + \text{Tr}_1\text{Tr}_2\text{Tr}_{y_2}\left[f^{(1)}q^{(1,2)}v_{[1}^{(2)}\pi_{2]}^{(1,3)\vee}\right] + \\
&\quad + \text{Tr}_2\text{Tr}_{y_2}\left[f^{(2)}v_{[1}^{(2)}\pi_{2]}^{(2,3)\vee}\right] + \gamma_{11}^{\vee}\,\text{Tr}_1\left[d_{[1}^{(1)}d_{2]}^{(1)}\right] . \qquad (4.90)
\end{aligned}$$

Some examples of gauge invariant operators are as follows:

$$\begin{aligned}
\Pi^{(1)\vee} &= \left(\pi^{(1,3)\vee}, \pi^{(1,2)\vee}, \text{Tr}_1\left[d^{(1)}f^{(1)}\right]\right) , \\
\Pi^{(2)\vee} &= \left(\pi^{(2,3)\vee}, \text{Tr}_2\left[d^{(2)}f^{(2)}\right], \text{Tr}_1\text{Tr}_2\left[d^{(1)}q^{(1,2)}f^{(2)}\right]\right) , \\
\mathsf{H}^{(1)\vee} &= \text{Tr}_1\left[f^{(1)}f^{(1)}\right] = \mathsf{H}^{(2)\vee} = \text{Tr}_2\left[f^{(2)}f^{(2)}\right] , \\
\mathsf{H}^{(1,2)\vee} &= \text{Tr}_1\text{Tr}_2\left[f^{(1)}q^{(1,2)}f^{(2)}\right] , \qquad (4.91) \\
\mathsf{C}^{\vee} &= \begin{pmatrix} i\sigma_2\text{Tr}_1 A^{(1)} & \text{Tr}_1 d^{(1)}v^{(1)} & \text{Tr}_1\text{Tr}_2 d^{(1)}q^{(1,2)}v^{(2)} \\ -\text{Tr}_1 d^{(1)}v^{(1)} & i\sigma_2\text{Tr}_2 A^{(2)} & \text{Tr}_2 d^{(2)}v^{(2)} \\ -\text{Tr}_1\text{Tr}_2 d^{(1)}q^{(1,2)}v^{(2)} & -\text{Tr}_2 d^{(2)}v^{(2)} & -i\sigma_2\text{Tr}_1 A^{(1)} - i\sigma_2\text{Tr}_2 A^{(2)} \end{pmatrix} .
\end{aligned}$$

Each $\Pi^{(i)\vee}$ is a bifundamental operator between $USp(6)_x \times USp(2)_{y^{(i)}}$. Note that the superpotential (4.90) is crucial to realize the non-abelian part of the global symmetry, $USp(6)_x \times USp(2)_{y^{(1)}} \times USp(2)_{y^{(2)}}$, because other bifundamental operators $\text{Tr}_1 f^{(1)}v^{(1)}$, $\text{Tr}_1 f^{(1)}q^{(1,2)}v^{(2)}$ and $\text{Tr}_2 f^{(2)}v^{(2)}$, which do not respect this symmetry, are flipped by $\pi^{(1,2)\vee}$, $\pi^{(1,3)\vee}$ and $\pi^{(2,3)\vee}$ respectively and thus are trivial in the chiral ring. Each $\mathsf{H}^{(i)\vee}$ is an $USp(2)_{y^{(i)}}$ antisymmetric, i.e. a singlet operator. Note that $\mathsf{H}^{(1)\vee}$ and $\mathsf{H}^{(2)\vee}$ are identified due to the superpotential, which implies that

$$\text{Tr}_1\left[f_{[1}^{(1)}f_{2]}^{(1)}\right] \sim \left[\text{Tr}_1\text{Tr}_2 q^{(1,2)}q^{(1,2)}\right] \sim \text{Tr}_2\left[f_{[1}^{(2)}f_{2]}^{(2)}\right] . \qquad (4.92)$$

Moreover, $\mathsf{H}^{(1,2)\vee}$ is a bifundamental operator between $USp(2)_{y^{(1)}} \times USp(2)_{y^{(2)}}$. Lastly $\mathsf{C}^{\vee}$ is an $USp(6)_x$ antisymmetric operator.

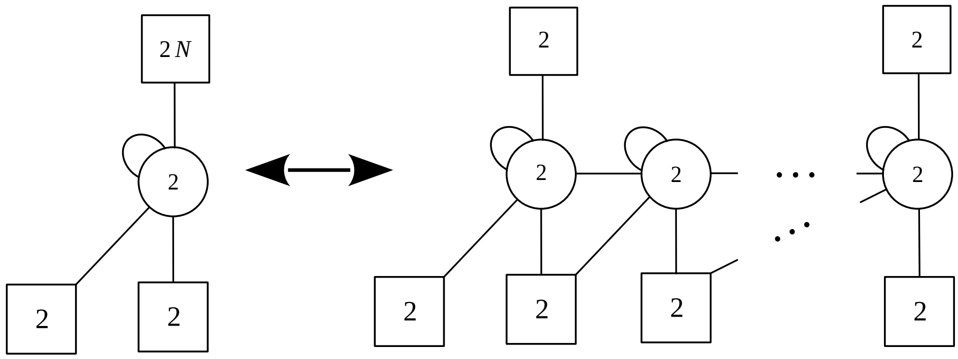

Figure 4.12: Duality between $E_{[N-1,1]}[USp(2N)]$ and $E^{[N-1,1]}[USp(2N)]$. Here we are not drawing the gauge singlets. The number of $SU(2)$ gauge nodes in the theory on the right is $N-1$.

We also find the map of the operators between $E_{[2,1]}[USp(2N)]$ and $E^{[2,1]}[USp(2N)]$

$$\begin{aligned} \Pi^{(1)} &\longleftrightarrow \Pi^{(1)\vee}\,, \\ \Pi^{(2)} &\longleftrightarrow \Pi^{(2)\vee}\,, \\ \mathsf{C}^{(1)} &\longleftrightarrow \mathsf{H}^{(1)\vee} = \mathsf{H}^{(2)\vee}\,, \\ \mathsf{C}^{(1,2)} &\longleftrightarrow \mathsf{H}^{(1,2)\vee}\,, \\ \mathsf{H} &\longleftrightarrow \mathsf{C}^{\vee}\,. \end{aligned} \tag{4.93}$$

This is compatible with $E_{[2,1]}[USp(2N)]$ and $E^{[2,1]}[USp(2N)]$ having the same low-lying operator spectrum, which respects the same global symmetry.

Although here we only considered the $N=3$ case, we checked that the supersymmetric index identity holds for higher N as well. The mirror duality between $E^{[N-1,1]}[USp(2N)]$ and $E_{[N-1,1]}[USp(2N)]$ for arbitrary N is represented in Figure 4.12 in a simplified version where we omit gauge singlets. This is the $4d$ analogue of the $3d$ abelian mirror duality[15]. As shown in [168], the abelian $3d$ mirror symmetry for SQED with N flavors can be derived from the basic duality between SQED with one flavor and the XYZ model with a piecewise procedure (see also [170] for an implementation of this procedure at the level of the three-sphere partition function and [171] for the $\mathcal{N}=2$ case). Interestingly, we can do the same in $4d$ and derive the duality 4.12 with a similar piecewise procedure, where the role of the basic duality is now played by the Intriligator–Pouliot duality in the confining case of $USp(2)$ with 6 chirals dual to a WZ model of 15 chiral fields. We show this at the level of the index in the $N=3$ case in Appendix C.3.

[15]See [64] for the $2d$ $\mathcal{N}=(0,2)$ reduction of this $4d$ $\mathcal{N}=1$ duality and for an analogue of the piecewise derivation in that context.

Example III: $\rho = [2^2]$ **and** $\sigma = [1^4]$

Flow to $E_{[2^2]}[USp(8)]$

Starting from $E[USp(8)]$ we introduce the superpotential (4.54) with $\rho = [2^2]$ and $\sigma = [1^4]$, which includes the mass terms

$$\delta\mathcal{W} = \dots + \mathrm{Tr}_1 D^{(1)}_{[1} V^{(1)}_{2]} + \mathrm{Tr}_3 D^{(3)}_{[1} V^{(3)}_{2]} + \dots \,, \tag{4.94}$$

which lead to the following constraints on fugacities:

$$y_1 = t^{-\frac{1}{2}} y^{(1)}_1 \,, \qquad y_2 = t^{\frac{1}{2}} y^{(1)}_1 \,, \qquad y_3 = t^{-\frac{1}{2}} y^{(1)}_2 \,, \qquad y_4 = t^{\frac{1}{2}} y^{(1)}_2 \,. \tag{4.95}$$

For simplicity, we will omit the superscript (1) of the new variables $y^{(1)}_i$, which should not be confused with the original variables y_i. We also introduce a set of extra flipping fields, which contribute to the index as follows:

$$\begin{aligned}
\mathsf{S}_{[1^4]} \quad &\longrightarrow \quad \Gamma_e\left(pqt^{-1}\right)^3 \prod_{i<j}^{4} \Gamma_e\left(pqt^{-1} x_i^{\pm 1} x_j^{\pm 1}\right) \,, \\
\mathsf{T}_{[2,2]} \quad &\longrightarrow \quad \Gamma_e\left(t\right) \Gamma_e\left(t^2\right)^2 \prod_{i=1}^{2} \Gamma_e\left(t^i y_1^{\pm 1} y_2^{\pm 1}\right) \,, \\
\mathsf{O}^{12}_B \quad &\longrightarrow \quad \Gamma_e\left(t^{-1} c^2\right) \,.
\end{aligned} \tag{4.96}$$

After integrating out the massive fields and applying sequentially the Intriligator–Pouliot duality we obtain that the index of the $E_{[2^2]}[USp(8)]$ theory is as follows:

$$\begin{aligned}
\mathcal{I}_{\mathsf{T}} &= \Gamma_e\left(pqt^{-1}\right)^2 \prod_{i<j}^{4} \Gamma_e\left(pqt^{-1} x_i^{\pm 1} x_j^{\pm 1}\right) \Gamma_e\left(t^2\right)^2 \prod_{i=1}^{2} \Gamma_e\left(t^i y_1^{\pm 1} y_2^{\pm 1}\right) \Gamma_e\left(t^{-1} c^2\right) \times \\
&\quad \times \mathcal{I}_{E[USp(8)]}(\vec{x}; \vec{y}; c; t)\Big|_{y_1 \to t^{-\frac{1}{2}} y_1,\, y_2 \to t^{\frac{1}{2}} y_1,\, y_3 \to t^{-\frac{1}{2}} y_2,\, y_4 \to t^{\frac{1}{2}} y_2} = \\
&= \Gamma_e\left(p^2 q^2 c^{-2}\right) \Gamma_e\left(p^2 q^2 t^{-1} c^{-2}\right) \prod_{i=1}^{4} \Gamma_e\left(t^{-1/2} c y_1^{\pm 1} x_i^{\pm 1}\right) \times \\
&\quad \times \oint \mathrm{d}\vec{z}^{(1)}_2 \, \Gamma_e\left(t\right)^2 \prod_{a<b}^{2} \Gamma_e\left(t z_a^{(1)\pm 1} z_b^{(1)\pm 1}\right) \prod_{a=1}^{2} \Gamma_e\left(p^{-1/2} q^{-1/2} c y_2^{\pm 1} z_a^{(1)\pm 1}\right) \times \\
&\quad \times \prod_{a=1}^{2} \prod_{i=1}^{4} \Gamma_e\left(p^{1/2} q^{1/2} t^{-1/2} z_a^{(1)\pm 1} x_i^{\pm 1}\right) \Gamma_e\left(p^{1/2} q^{1/2} t c^{-1} y_1^{\pm 1} z_a^{(1)\pm 1}\right) = \\
&= \mathcal{I}_{E_{[2^2]}[USp(8)]}(\vec{x}; \vec{y}; c; t) \,.
\end{aligned} \tag{4.97}$$

From the supersymmetric index (4.97) one can read off the matter content of the $E_{[2^2]}[USp(8)]$ theory, which we represent using the quiver diagram of Figure 4.13. Furthermore, we can

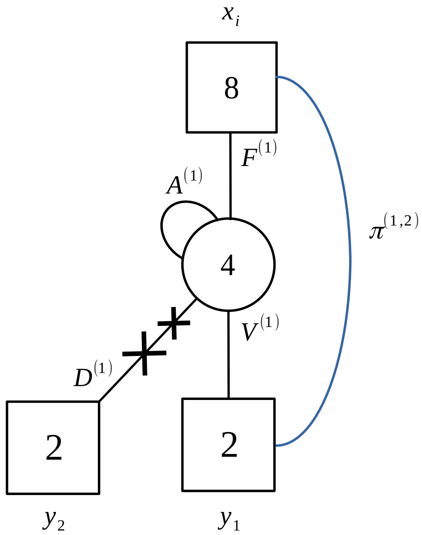

Figure 4.13: The quiver diagram representation of $E_{[2^2]}[USp(8)]$. Two crosses with different sizes on top of the diagonal line denote the singlets γ_{11} and γ_{12}, which flip $\mathrm{Tr}_1\left[D^{(1)}_{[1}D^{(1)}_{2]}\right]$ and $\mathrm{Tr}_1\left[A^{(1)}D^{(1)}_{[1}D^{(1)}_{2]}\right]$ respectively.

also read off the total superpotential of $E_{[2^2]}[USp(8)]$, which is given by

$$\begin{aligned}\mathcal{W}_{E_{[2^2]}[USp(8)]} =& \mathrm{Tr}_1\mathrm{Tr}_x\left[A^{(1)}F^{(1)}F^{(1)}\right] + \mathrm{Tr}_1\mathrm{Tr}_x\left[F^{(1)}v^{(1)}_{[1}\pi^{(1,2)}_{2]}\right] + \\ &+ \gamma_{11}\,\mathrm{Tr}_1 D^{(1)}_{[1}D^{(1)}_{2]} + \gamma_{12}\,\mathrm{Tr}_1\left[A^{(1)}D^{(1)}_{[1}D^{(1)}_{2]}\right] .\end{aligned} \tag{4.98}$$

One can see that the superpotential involves a set of gauge singlet operators, which contribute to the resulting index (4.97) by

$$\begin{aligned}&\pi^{(1,2)} \longrightarrow \prod_{i=1}^{4}\Gamma_e\left(t^{-1/2}cy_1^{\pm1}x_i^{\pm1}\right) , \\ &\gamma_{11} \longrightarrow \Gamma_e\left(p^2q^2c^{-2}\right) , \qquad \gamma_{12} \longrightarrow \Gamma_e\left(p^2q^2t^{-1}c^{-2}\right) .\end{aligned} \tag{4.99}$$

The non-abelian global symmetry of $E_{[2^2]}[USp(8)]$ is $USp(8)_x\times USp(4)_y$. A few examples of gauge invariant operators respecting this symmetry are as follows:

$$\begin{aligned}&\Pi = \left(\pi^{(1,2)}, \mathrm{Tr}_1\left[D^{(1)}F^{(1)}\right]\right) , \qquad \mathsf{H} = \mathrm{Tr}_1\left[F^{(1)}F^{(1)}\right] , \\ &\mathsf{C} = \begin{pmatrix} i\sigma_2\mathrm{Tr}_1 A^{(1)} & \mathrm{Tr}_1\left[D^{(1)}V^{(1)}\right] \\ -\mathrm{Tr}_1\left[D^{(1)}V^{(1)}\right] & -i\sigma_2\mathrm{Tr}_1 A^{(1)}\end{pmatrix} ,\end{aligned} \tag{4.100}$$

where Π is a bifundamental between $USp(8)_x\times USp(4)_y$, while H and C are antisymmetrics of $USp(8)_x$ and $USp(4)_y$ respectively.

Flow to $E^{[2^2]}[USp(8)]$

Let's now look at the mirror side. The deformation (4.94) is mapped to a deformation of $E[USp(8)]^\vee$ which includes the mass terms

$$\delta\mathcal{W} = \cdots + q_1^{(3,4)} q_4^{(3,4)} + q_5^{(3,4)} q_8^{(3,4)} + \ldots\,, \tag{4.101}$$

implying the constraints on fugacities

$$y_1 = t^{-1/2} y_1^{(1)}\,, \qquad y_2 = t^{1/2} y_1^{(1)}\,, \qquad y_3 = t^{-1/2} y_2^{(1)}\,, \qquad y_4 = t^{1/2} y_2^{(1)}\,. \tag{4.102}$$

As before we will omit the superscript (1) of $y_i^{(1)}$. Taking into account the contributions of the extra flipping fields (4.96) and applying sequentially the Intriligator–Pouliot duality we obtain the supersymmetric index of $E^{[2^2]}[USp(8)]$:

$$\begin{aligned}
&\mathcal{I}_{E^{[2^2]}[USp(8)]}(\vec{y};\vec{x};c;pq/t) = \\
&= \prod_{i=1}^{2} \Gamma_e\left(t^{-1/2} c x_1^{\pm1} y_i^{\pm1}\right) \Gamma_e\left(t^{-1/2} c x_2^{\pm1} y_i^{\pm1}\right) \Gamma_e\left(pqt^3 c^{-2}\right) \Gamma_e\left(pqt^2 c^{-2}\right) \times \\
&\quad \times \oint \mathrm{d}\vec{z}_1^{(1)}\, \mathrm{d}\vec{z}_2^{(2)}\, \mathrm{d}\vec{z}_1^{(3)}\, \Gamma_e\left(pqt^{-1}\right)^4 \prod_{a<b}^{2} \Gamma_e\left(pqt^{-1} z_a^{(2)\pm1} z_b^{(2)\pm1}\right) \times \\
&\quad \times \Gamma_e\left(t^{-3/2} c x_4^{\pm1} z^{(1)\pm1}\right) \prod_{a=1}^{2} \Gamma_e\left(t^{-1} c x_3^{\pm1} z_a^{(2)\pm1}\right) \Gamma_e\left(t^{-1/2} c x_2^{\pm1} z^{(3)\pm1}\right) \times \\
&\quad \times \prod_{a=1}^{2} \Gamma_e\left(t^{1/2} z^{(1)\pm1} z_a^{(2)\pm1}\right) \Gamma_e\left(t^{1/2} z_a^{(2)\pm1} z^{(3)\pm1}\right) \prod_{i=1}^{2} \Gamma_e\left(t^{1/2} z_a^{(2)\pm1} y_i^{\pm1}\right) \times \\
&\quad \times \Gamma_e\left(pqt^{1/2} c^{-1} x_3^{\pm1} z^{(1)\pm1}\right) \prod_{a=1}^{2} \Gamma_e\left(pqc^{-1} x_2^{\pm1} z_a^{(2)\pm1}\right) \Gamma_e\left(pqt^{-1/2} c^{-1} x_1^{\pm1} z^{(3)\pm1}\right)\,.
\end{aligned} \tag{4.103}$$

Starting from the identity for the mirror-like duality of $E[USp(8)]$ we have derived a new identity for the duality between $E_{[2^2]}[USp(8)]$ and $E^{[2^2]}[USp(8)]$:

$$\mathcal{I}_{E_{[2^2]}[USp(8)]}(\vec{x};\vec{y};c;t) = \mathcal{I}_{E^{[2^2]}[USp(8)]}(\vec{y};\vec{x};c;pq/t)\,. \tag{4.104}$$

The quiver diagram of $E^{[2^2]}[USp(8)]$ can be read from (4.103) and is represented in Figure 4.14. The superpotential of $E^{[2^2]}[USp(8)]$ is given by

$$\begin{aligned}
&\mathcal{W}_{E^{[2^2]}[USp(8)]} = \\
&= \mathrm{Tr}_1 \mathrm{Tr}_2 \left[A^{(1)} q^{(1,2)} q^{(1,2)}\right] + \mathrm{Tr}_2 \left[A^{(2)} \left(\mathrm{Tr}_1\, q^{(1,2)} q^{(1,2)} + \mathrm{Tr}_y f^{(2)} f^{(2)} + \mathrm{Tr}_3 q^{(2,3)} q^{(2,3)}\right)\right] +
\end{aligned}$$

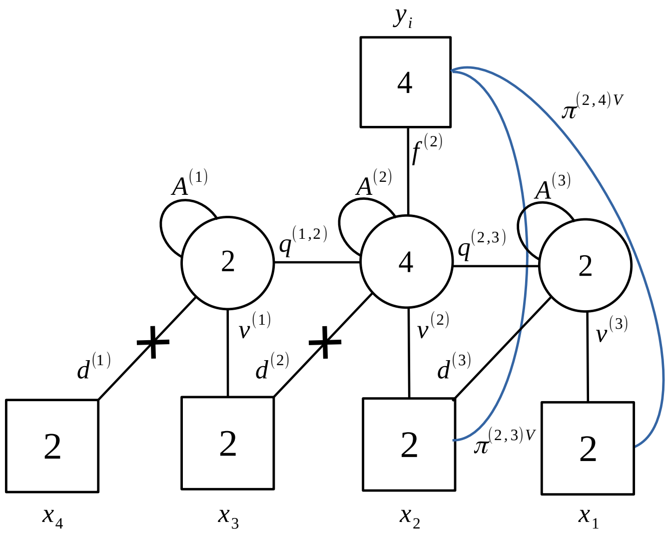

Figure 4.14: The quiver diagram representation of $E^{[2^2]}[USp(8)]$. Two flipping fields $\gamma_{11}^\vee$ and $\gamma_{11}^\vee$, denoted by crosses, flip $\mathrm{Tr}_1\left[d_{[1}^{(1)}d_{2]}^{(1)}\right]$ and $\mathrm{Tr}_2\left[d_{[1}^{(2)}d_{2]}^{(2)}\right]$ respectively.

$$
\begin{aligned}
&+\mathrm{Tr}_2\mathrm{Tr}_3\left[A^{(3)}q^{(2,3)}q^{(2,3)}\right]+\mathrm{Tr}_1\mathrm{Tr}_2\left[v_{[1}^{(1)}q^{(1,2)}d_{2]}^{(2)}\right]+\mathrm{Tr}_2\mathrm{Tr}_3\left[v_{[1}^{(2)}q^{(2,3)}d_{2]}^{(3)}\right]+\\
&+\mathrm{Tr}_2\mathrm{Tr}_y\left[f^{(2)}v_{[1}^{(2)}\pi_{2]}^{(2,3)\vee}\right]+\mathrm{Tr}_2\mathrm{Tr}_3\mathrm{Tr}_y\left[f^{(2)}q^{(2,3)}v_{[1}^{(3)}\pi_{2]}^{(2,4)\vee}\right]+\sum_{i=1}^{2}\gamma_{i1}^\vee\,\mathrm{Tr}_i\left[d_{[1}^{(i)}d_{2]}^{(i)}\right]\,,
\end{aligned}
\tag{4.105}
$$

which involves the gauge singlet operators whose index contributions are as follows:

$$
\begin{aligned}
\pi^{(2,3)\vee} &\longrightarrow \prod_{i=1}^{2}\Gamma_e\left(t^{-1/2}cx_2^{\pm1}y_i^{\pm1}\right)\,,\\
\pi^{(2,4)\vee} &\longrightarrow \prod_{i=1}^{2}\Gamma_e\left(t^{-1/2}cx_1^{\pm1}y_i^{\pm1}\right)\,,\\
\gamma_{11}^\vee &\longrightarrow \Gamma_e\left(pqt^3c^{-2}\right)\,,\qquad \gamma_{21}^\vee \longrightarrow \Gamma_e\left(pqt^2c^{-2}\right)\,.
\end{aligned}
\tag{4.106}
$$

One can also construct gauge invariant operators transforming non-trivially under the non-abelian global symmetry. For example,

$$
\Pi^\vee=\left(\pi^{(2,4)\vee},\pi^{(2,3)\vee},\mathrm{Tr}_2\left[d^{(2)}f^{(2)}\right],\mathrm{Tr}_1\mathrm{Tr}_2\left[d^{(1)}q^{(1,2)}f^{(2)}\right]\right)\,,\qquad \mathsf{H}^\vee=\mathrm{Tr}_2\left[f^{(2)}f^{(2)}\right]\,,
$$

$$
\mathsf{C}^\vee=\begin{pmatrix}
i\sigma_2\mathrm{Tr}_1A^{(1)} & \mathrm{Tr}_1d^{(1)}v^{(1)} & \mathrm{Tr}_1\mathrm{Tr}_2d^{(1)}q^{(1,2)}v^{(2)} & \mathrm{Tr}_1\mathrm{Tr}_2\mathrm{Tr}_3d^{(1)}q^{(1,2)}q^{(2,3)}v^{(3)}\\
-\mathrm{Tr}_1d^{(1)}v^{(1)} & i\sigma_2\mathrm{Tr}_2A^{(2)} & \mathrm{Tr}_2d^{(2)}v^{(2)} & \mathrm{Tr}_2\mathrm{Tr}_3d^{(2)}q^{(2,3)}v^{(3)}\\
-\mathrm{Tr}_1\mathrm{Tr}_2d^{(1)}q^{(1,2)}v^{(2)} & -\mathrm{Tr}_2d^{(2)}v^{(2)} & i\sigma_2\mathrm{Tr}_3A^{(3)} & \mathrm{Tr}_3d^{(3)}v^{(3)}\\
-\mathrm{Tr}_1\mathrm{Tr}_2\mathrm{Tr}_3d^{(1)}q^{(1,2)}q^{(2,3)}v^{(3)} & -\mathrm{Tr}_2\mathrm{Tr}_3d^{(2)}q^{(2,3)}v^{(3)} & -\mathrm{Tr}_3d^{(3)}v^{(3)} & -i\sigma_2\sum_{i=1}^3\mathrm{Tr}_iA^{(i)}
\end{pmatrix}\,,
\tag{4.107}
$$

which are mapped to operators of $E_{[2^2]}[USp(8)]$ as follows:

$$\begin{aligned} \Pi &\longleftrightarrow \Pi^\vee\,, \\ \mathsf{H} &\longleftrightarrow \mathsf{C}^\vee\,, \\ \mathsf{C} &\longleftrightarrow \mathsf{H}^\vee\,. \end{aligned} \tag{4.108}$$

Note that $\Pi^\vee$ is a bifundamental between $USp(8)_x \times USp(4)_y$, while $\mathsf{H}^\vee$ and $\mathsf{C}^\vee$ are antisymmetrics of $USp(4)_y$ and $USp(8)_x$ respectively.

Example IV: $\rho = [2,1^2]$ **and** $\sigma = [1^4]$

Flow to $E_{[2,1^2]}[USp(8)]$

We now consider a deformation of $E[USp(8)]$ corresponding to $\rho = [2,1^2]$ and $\sigma = [1^4]$, which includes a mass term

$$\delta\mathcal{W} = \cdots + \mathrm{Tr}_1 D^{(1)}_{[1} V^{(1)}_{2]} + \dots\,, \tag{4.109}$$

which relates y_1 and y_2 as follows:

$$y_1 = t^{-\frac{1}{2}} y^{(1)}\,, \qquad y_2 = t^{\frac{1}{2}} y^{(1)}\,. \tag{4.110}$$

For later convenience, we also rename y_3 and y_4 as

$$y_3 = y^{(2)}_1\,, \qquad y_4 = y^{(2)}_2\,. \tag{4.111}$$

The extra flipping fields we introduce in this case are

$$\begin{aligned} \mathsf{S}_{[1^4]} &\longrightarrow \Gamma_e\left(pqt^{-1}\right)^3 \prod_{n<m}^4 \Gamma_e\left(pqt^{-1} x_n^{\pm1} x_m^{\pm1}\right)\,, \\ \mathsf{T}_{[2,1^2]} &\longrightarrow \Gamma_e(t)^2\,\Gamma_e\left(t^2\right) \prod_{i=1}^2 \Gamma_e\left(t^{\frac{3}{2}} y^{(1)\pm1} y_i^{(2)\pm1}\right) \Gamma_e\left(t y_1^{(2)\pm1} y_2^{(2)\pm1}\right)\,, \\ \mathsf{O}_B^{12} &\longrightarrow \Gamma_e\left(t^{-1}c^2\right)\,. \end{aligned} \tag{4.112}$$

After applying sequentially the Intriligator–Pouliot duality, we obtain the supersymmetric index of $E_{[2,1^2]}[USp(8)]$:

$$\begin{aligned} &\mathcal{I}_{E_{[2,1^2]}[USp(8)]}\left(\vec{x};\vec{y}^{(2)},y^{(1)};c;t\right) = \\ &= \Gamma_e\left(p^3q^3t^{-2}c^{-2}\right)\Gamma_e\left(p^2q^2t^{-1}c^{-2}\right) \prod_{i=1}^4 \Gamma_e\left(t^{-1/2} c y^{(1)\pm1} x_i^{\pm1}\right) \times \end{aligned}$$

$$\begin{aligned}&\times \oint \mathrm{d}\vec{z}_1^{(1)}\,\mathrm{d}\vec{z}_2^{(2)}\,\Gamma_e\left(t\right)^3\prod_{a<b}^{2}\Gamma_e\left(tz_a^{(2)\pm1}z_b^{(2)\pm1}\right)\times\\
&\times\Gamma_e\left(p^{-1}q^{-1}tcz^{(1)\pm1}y_2^{(2)\pm1}\right)\prod_{a=1}^{2}\Gamma_e\left(p^{-1/2}q^{-1/2}t^{1/2}cy_1^{(2)\pm1}z_a^{(2)\pm1}\right)\times\\
&\times\prod_{a=1}^{2}\Gamma_e\left(p^{1/2}q^{1/2}t^{-1/2}z^{(1)\pm1}z_a^{(2)\pm1}\right)\prod_{a=1}^{2}\prod_{i=1}^{4}\Gamma_e\left(p^{1/2}q^{1/2}t^{-1/2}z_a^{(2)\pm1}x_i^{\pm1}\right)\times\\
&\times\Gamma_e\left(pqc^{-1}y_1^{(2)\pm1}z^{(1)\pm1}\right)\prod_{a=1}^{2}\Gamma_e\left(p^{1/2}q^{1/2}tc^{-1}y^{(1)\pm1}z_a^{(2)\pm1}\right)\,. \qquad (4.113)\end{aligned}$$

The quiver diagram of $E_{[2,1^2]}[USp(8)]$ is drawn in Figure 4.15, which can be worked out from the supersymmetric index (4.113). The total superpotential of $E_{[2,1^2]}[USp(8)]$ is given by

$$\begin{aligned}\mathcal{W}_{E_{[2,1^2]}[USp(8)]} =\ &\mathrm{Tr}_1\mathrm{Tr}_2\left[A^{(1)}Q^{(1,2)}Q^{(1,2)}\right]+\mathrm{Tr}_2\left[A^{(2)}\left(\mathrm{Tr}_1Q^{(1,2)}Q^{(1,2)}+\mathrm{Tr}_xF^{(2)}F^{(2)}\right)\right]+\\
&+\mathrm{Tr}_1\mathrm{Tr}_2\left[V_{[1}^{(1)}Q^{(1,2)}D_{2]}^{(2)}\right]+\mathrm{Tr}_2\mathrm{Tr}_x\left[F^{(2)}V_{[1}^{(2)}\pi_{2]}^{(2,3)}\right]+\sum_{i=1}^{2}\gamma_{i1}\,\mathrm{Tr}_iD_{[1}^{(i)}D_{2]}^{(i)}\,. \qquad (4.114)\end{aligned}$$

One can see that the superpotential involves a set of gauge singlet operators, which contribute to the index (4.97) by

$$\begin{aligned}&\pi^{(2,3)} \longrightarrow \prod_{i=1}^{4}\Gamma_e\left(t^{-1/2}cy_1^{\pm1}x_i^{\pm1}\right)\,,\\
&\gamma_{11} \longrightarrow \Gamma_e\left(p^3q^3t^{-2}c^{-2}\right)\,,\qquad \gamma_{21} \longrightarrow \Gamma_e\left(p^2q^2t^{-1}c^{-2}\right)\,.\end{aligned} \qquad (4.115)$$

The non-abelian global symmetry of $E_{[2,1^2]}[USp(8)]$ is $USp(8)_x\times USp(2)_{y^{(1)}}\times USp(4)_{y^{(2)}}$. Some interesting examples of gauge invariant operators, which respect this symmetry are

$$\begin{aligned}&\Pi^{(1)}=\pi^{(2,3)}\,,\qquad \Pi^{(2)}=\left(\mathrm{Tr}_2\left[D^{(2)}F^{(2)}\right],\mathrm{Tr}_1\mathrm{Tr}_2\left[D^{(1)}Q^{(1,2)}F^{(2)}\right]\right)\,,\\
&\mathsf{H}=\mathrm{Tr}_2\left[F^{(2)}F^{(2)}\right]\,,\qquad \mathsf{C}^{(1)}=A^{(2)}\,,\\
&\mathsf{C}^{(2)}=\begin{pmatrix} i\sigma_2\mathrm{Tr}_1A^{(1)} & \mathrm{Tr}_1D^{(1)}V^{(1)}\\ -\mathrm{Tr}_1D^{(1)}V^{(1)} & -i\sigma_2\mathrm{Tr}_1A^{(1)}\end{pmatrix}\,,\\
&\mathsf{C}^{(1,2)}=\mathrm{Tr}_1\mathrm{Tr}_2\left[D^{(1)}Q^{(1,2)}V^{(2)}\right]\,,\end{aligned} \qquad (4.116)$$

where $\Pi^{(i)}$ is a bifundamental between $USp(8)_x\times USp(2l_i)_{y^{(i)}}$ with $l_1=1$ and $l_2=2$, H and $\mathsf{C}^{(i)}$ are antisymmetrics of $USp(8)_x$ and $USp(2l_i)_{y^{(i)}}$ respectively, and lastly $C^{(1,2)}$ is a bifundamental between $USp(2)_{y^{(1)}}\times USp(4)_{y^{(2)}}$.

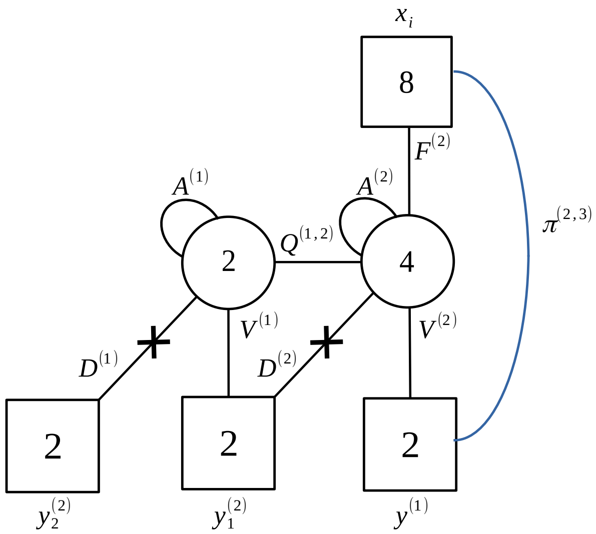

Figure 4.15: The quiver diagram representation of $E_{[2,1^2]}[USp(8)]$. Two flipping fields γ_{11} and γ_{21}, denoted by crosses, flip the operators $\mathrm{Tr}_1\left[D^{(1)}_{[1}D^{(1)}_{2]}\right]$ and $\mathrm{Tr}_2\left[D^{(2)}_{[1}D^{(2)}_{2]}\right]$ respectively.

Flow to $E^{[2,1^2]}[USp(8)]$

On the mirror side we have the deformation superpotential

$$\delta\mathcal{W} = \cdots + q_1^{(3,4)} q_4^{(3,4)} + \dots . \tag{4.117}$$

This imposes the following constraint on the fugacities appearing in the index of$E[USp(8)]^\vee$:

$$y_1 = t^{-\frac{1}{2}} y^{(1)} , \qquad y_2 = t^{\frac{1}{2}} y^{(1)} , \quad y_3 = y_1^{(2)} , \qquad y_4 = y_2^{(2)} . \tag{4.118}$$

We also introduce the extra flipping fields given in (4.112). After sequentially applying the Intriligator–Pouliot duality we obtain the index of the $E^{[2,1^2]}[USp(8)]$ theory

$$\mathcal{I}_{E^{[2,1^2]}[USp(8)]}\left(y^{(1)}, \vec{y}^{(2)}; \vec{x}; c; pqt^{-1}\right) =$$
$$= \Gamma_e\left(t^{-1/2} c x_1^{\pm1} y^{(1)\pm1}\right) \prod_{i=1}^{2} \Gamma_e\left(c x_1^{\pm1} y_i^{(2)\pm1}\right) \Gamma_e\left(t^{-1/2} c x_2^{\pm1} y^{(1)\pm1}\right) \Gamma_e\left(pqt^3 c^{-2}\right) \Gamma_e\left(pqt^2 c^{-2}\right) \times$$
$$\times \oint \mathrm{d}\vec{z}_1^{(1)}\, \mathrm{d}\vec{z}_2^{(2)}\, \mathrm{d}\vec{z}_2^{(3)}\, \Gamma_e\left(pqt^{-1}\right)^5 \prod_{a<b}^{2} \Gamma_e\left(pqt^{-1} z_a^{(2)\pm1} z_b^{(2)\pm1}\right) \prod_{\alpha<\beta}^{2} \Gamma_e\left(pqt^{-1} z_\alpha^{(3)\pm1} z_\beta^{(3)\pm1}\right) \times$$
$$\times \Gamma_e\left(t^{-3/2} c x_4^{\pm1} z^{(1)\pm1}\right) \prod_{a=1}^{2} \Gamma_e\left(t^{-1} c x_3^{\pm1} z_a^{(2)\pm1}\right) \prod_{\alpha=1}^{2} \Gamma_e\left(t^{-1/2} c x_2^{\pm1} z_\alpha^{(3)\pm1}\right) \times$$

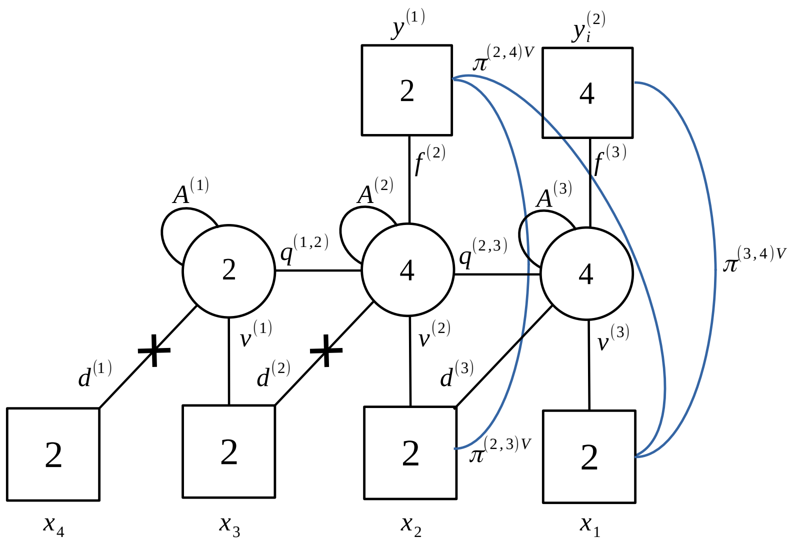

Figure 4.16: The quiver diagram representation of $E^{[2,1^2]}[USp(8)]$. Two flipping fields $\gamma_{11}^\vee$ and $\gamma_{21}^\vee$, denoted by crosses, flip the operators $\mathrm{Tr}_1\left[d^{(1)}_{[1}d^{(1)}_{2]}\right]$ and $\mathrm{Tr}_2\left[d^{(2)}_{[1}d^{(2)}_{2]}\right]$ respectively.

$$\times\prod_{a=1}^{2}\Gamma_e\left(t^{1/2}z^{(1)\pm1}z_a^{(2)\pm1}\right)\prod_{a=1}^{2}\prod_{\alpha=1}^{2}\Gamma_e\left(t^{1/2}z_a^{(2)\pm1}z_\alpha^{(3)\pm1}\right)\times$$
$$\times\prod_{a=1}^{2}\Gamma_e\left(t^{1/2}z_a^{(2)\pm1}y^{(1)\pm1}\right)\prod_{\alpha=1}^{2}\prod_{i=1}^{2}\Gamma_e\left(t^{1/2}z_\alpha^{(3)\pm1}y_i^{(2)\pm1}\right)\times$$
$$\times\Gamma_e\left(pqt^{1/2}c^{-1}x_3^{\pm1}z^{(1)\pm1}\right)\prod_{a=1}^{2}\Gamma_e\left(pqc^{-1}x_2^{\pm1}z_a^{(2)\pm1}\right)\prod_{\alpha=1}^{2}\Gamma_e\left(pqt^{-1/2}c^{-1}x_1^{\pm1}z_\alpha^{(3)\pm1}\right), \tag{4.119}$$

We then have shown the equality of indices

$$\mathcal{I}_{E_{[2,1^2]}[USp(8)]}(\vec{x};\vec{y}^{(2)},y^{(1)};c;t)=\mathcal{I}_{E^{[2,1^2]}[USp(8)]}(y^{(1)},\vec{y}^{(2)};\vec{x};c;pq/t)\,. \tag{4.120}$$

The quiver diagram read from the index (4.119) is shown in Figure 4.16. The superpotential of $E^{[2,1^2]}[USp(8)]$ is

$$\begin{aligned}&\mathcal{W}_{E^{[2,1^2]}[USp(8)]}=\\&\mathrm{Tr}_1\mathrm{Tr}_2\left[A^{(1)}q^{(1,2)}q^{(1,2)}\right]+\mathrm{Tr}_2\left[A^{(2)}\left(\mathrm{Tr}_1\,q^{(1,2)}q^{(1,2)}+\mathrm{Tr}_{y^{(1)}}f^{(2)}f^{(2)}+\mathrm{Tr}_3q^{(2,3)}q^{(2,3)}\right)\right]+\\&+\mathrm{Tr}_3\left[A^{(3)}\left(\mathrm{Tr}_2\,q^{(2,3)}q^{(2,3)}+\mathrm{Tr}_{y^{(2)}}f^{(3)}f^{(3)}\right)\right]+\mathrm{Tr}_1\mathrm{Tr}_2\left[v^{(1)}_{[1}q^{(1,2)}d^{(2)}_{2]}\right]+\mathrm{Tr}_2\mathrm{Tr}_3\left[v^{(2)}_{[1}q^{(2,3)}d^{(3)}_{2]}\right]+\\&+\mathrm{Tr}_2\mathrm{Tr}_{y^{(1)}}\left[f^{(2)}v^{(2)}_{[1}\pi^{(2,3)\vee}_{2]}\right]+\mathrm{Tr}_2\mathrm{Tr}_3\mathrm{Tr}_{y^{(1)}}\left[f^{(2)}q^{(2,3)}v^{(3)}_{[1}\pi^{(2,4)\vee}_{2]}\right]+\mathrm{Tr}_3\mathrm{Tr}_{y^{(2)}}\left[f^{(3)}v^{(3)}_{[1}\pi^{(3,4)\vee}_{2]}\right]\\&+\sum_{i=1}^{2}\gamma_{i1}^\vee\,\mathrm{Tr}_i\left[d^{(i)}_{[1}d^{(i)}_{2]}\right],\end{aligned} \tag{4.121}$$

which involves a set of gauge singlet operators, which contribute to the index (4.119) by

$$\begin{aligned}
\pi^{(2,3)\vee} &\longrightarrow \Gamma_e\left(t^{-1/2}cx_2^{\pm1}y^{(1)\pm1}\right)\,,\\
\pi^{(2,4)\vee} &\longrightarrow \Gamma_e\left(t^{-1/2}cx_1^{\pm1}y^{(1)\pm1}\right)\,,\\
\pi^{(3,4)\vee} &\longrightarrow \prod_{i=1}^{2}\Gamma_e\left(cx_1^{\pm1}y_i^{(2)\pm1}\right)\,,\\
\gamma_{11}^{\vee} &\longrightarrow \Gamma_e\left(pqt^3c^{-2}\right)\,,\\
\gamma_{21}^{\vee} &\longrightarrow \Gamma_e\left(pqt^2c^{-2}\right)\,.
\end{aligned} \tag{4.122}$$

We also exhibit some gauge invariant operators

$$\begin{aligned}
&\Pi^{(1)\vee} = \left(\pi^{(2,4)\vee},\pi^{(2,3)\vee},\mathrm{Tr}_2\left[d^{(2)}f^{(2)}\right],\mathrm{Tr}_1\mathrm{Tr}_2\left[d^{(1)}q^{(1,2)}f^{(2)}\right]\right)\,,\\
&\Pi^{(2)\vee} = \left(\pi^{(3,4)\vee},\mathrm{Tr}_3\left[d^{(3)}f^{(3)}\right],\mathrm{Tr}_2\mathrm{Tr}_3\left[d^{(2)}q^{(2,3)}f^{(3)}\right],\mathrm{Tr}_1\mathrm{Tr}_2\mathrm{Tr}_3\left[d^{(1)}q^{(1,2)}q^{(2,3)}f^{(3)}\right]\right)\,,\\
&\mathsf{H}^{(1)\vee} = \mathrm{Tr}_2\left[f^{(2)}f^{(2)}\right]\,,\\
&\mathsf{H}^{(2)\vee} = \mathrm{Tr}_3\left[f^{(3)}f^{(3)}\right]\,,\\
&\mathsf{H}^{(1,2)\vee} = \mathrm{Tr}_2\mathrm{Tr}_3\left[f^{(2)}q^{(2,3)}f^{(3)}\right]
\end{aligned} \tag{4.123}$$

and

$$\mathsf{C}^{\vee} = \begin{pmatrix}
i\sigma_2\mathrm{Tr}_1A^{(1)} & \mathrm{Tr}_1d^{(1)}v^{(1)} & \mathrm{Tr}_1\mathrm{Tr}_2d^{(1)}q^{(1,2)}v^{(2)} & \mathrm{Tr}_1\mathrm{Tr}_2\mathrm{Tr}_3d^{(1)}q^{(1,2)}q^{(2,3)}v^{(3)}\\
-\mathrm{Tr}_1d^{(1)}v^{(1)} & i\sigma_2\mathrm{Tr}_2A^{(2)} & \mathrm{Tr}_2d^{(2)}v^{(2)} & \mathrm{Tr}_2\mathrm{Tr}_3d^{(2)}q^{(2,3)}v^{(3)}\\
-\mathrm{Tr}_1\mathrm{Tr}_2d^{(1)}q^{(1,2)}v^{(2)} & -\mathrm{Tr}_2d^{(2)}v^{(2)} & i\sigma_2\mathrm{Tr}_3A^{(3)} & \mathrm{Tr}_3d^{(3)}v^{(3)}\\
-\mathrm{Tr}_1\mathrm{Tr}_2\mathrm{Tr}_3d^{(1)}q^{(1,2)}q^{(2,3)}v^{(3)} & -\mathrm{Tr}_2\mathrm{Tr}_3d^{(2)}q^{(2,3)}v^{(3)} & -\mathrm{Tr}_3d^{(3)}v^{(3)} & -i\sigma_2\sum_{i=1}^{3}\mathrm{Tr}_iA^{(i)}
\end{pmatrix}\,, \tag{4.124}$$

where $\Pi^{(i)}$ is a bifundamental between $USp(8)_x\times USp(2l_i)_{y^{(i)}}$ with $l_1=1$ and $l_2=2$, $\mathsf{C}^{\vee}$ and $\mathsf{H}^{(i)\vee}$ are antisymmetrics of $USp(8)_x$ and $USp(2l_i)_{y^{(i)}}$ respectively, and lastly $H^{(1,2)\vee}$ is a bifundamental between $USp(2)_{y^{(1)}}\times USp(4)_{y^{(2)}}$. Note that the non-abelian global symmetry of $E^{[2,1^2]}[USp(8)]$ is $USp(8)_x\times USp(2)_{y^{(1)}}\times USp(4)_{y^{(2)}}$. The operators of $E^{[2,1^2]}[USp(8)]$ are mapped to those of $E_{[2,1^2]}[USp(8)]$ as follows:

$$\begin{aligned}
\Pi^{(1)} &\longleftrightarrow \Pi^{(1)\vee}\,,\\
\Pi^{(2)} &\longleftrightarrow \Pi^{(2)\vee}\,,\\
\mathsf{H} &\longleftrightarrow \mathsf{C}^{\vee}\,,\\
\mathsf{C}^{(1)} &\longleftrightarrow \mathsf{H}^{(1)\vee}\,,\\
\mathsf{C}^{(2)} &\longleftrightarrow \mathsf{H}^{(2)\vee}\,,\\
\mathsf{C}^{(1,2)} &\longleftrightarrow \mathsf{H}^{(1,2)\vee}\,.
\end{aligned} \tag{4.125}$$

Example V: $\rho = \sigma = [2^3, 1]$

So far we focused on cases with only one non-trivial partition, however we checked that our construction consistently produces mirror pairs of theories also when both ρ and σ are non-trivial (we checked this for all partitions up to $N = 14$). Here we exhibit one particular example with $N = 7$ and $\rho = \sigma = [2^3, 1]$, which corresponds to a self-duality. This example exhibits diverse increments of the gauge rank along the tail, so one can see how such different rank increments affect the number of the flipping fields in the resulting $E_\rho^\sigma[SU(N)]$ theory.

We start with the $E[USp(14)]$ theory and introduce the deformation (4.54) for $\rho = \sigma = [2^3, 1]$. This deformation enforces the following specialization of fugacities, now both for $\vec{x}$ and for $\vec{y}$:

$$\begin{aligned}
&x_1 = t^{-\frac{1}{2}} x_1^{(1)}, \quad x_2 = t^{\frac{1}{2}} x_1^{(1)}, \quad x_3 = t^{-\frac{1}{2}} x_2^{(1)}, \quad x_4 = t^{\frac{1}{2}} x_2^{(1)}, \quad x_5 = t^{-\frac{1}{2}} x_3^{(1)}, \quad x_6 = t^{\frac{1}{2}} x_3^{(1)}, \\
&y_1 = t^{-\frac{1}{2}} y_1^{(1)}, \quad y_2 = t^{\frac{1}{2}} y_1^{(1)}, \quad y_3 = t^{-\frac{1}{2}} y_2^{(1)}, \quad y_4 = t^{\frac{1}{2}} y_2^{(1)}, \quad y_5 = t^{-\frac{1}{2}} y_3^{(1)}, \quad y_6 = t^{\frac{1}{2}} y_3^{(1)}.
\end{aligned} \tag{4.126}$$

We also rename x_7 and y_7 as follows:

$$x_7 = x_1^{(2)}, \qquad y_7 = y_1^{(2)}. \tag{4.127}$$

Then those new variables will be the fugacities for the enhanced non-abelian global symmetry in the IR, which is $USp(6)_{x^{(1)}} \times USp(2)_{x^{(2)}} \times USp(6)_{y^{(1)}} \times USp(2)_{y^{(2)}}$ for $\rho = \sigma = [2^3, 1]$.

In addition, we introduce the extra singlets, which contribute to the index as follows:

$$\begin{aligned}
\mathsf{S}_{[2^3,1]} \quad &\longrightarrow \quad \Gamma_e\left(p^2q^2t^{-2}\right)^3 \Gamma_e\left(pqt^{-1}\right)^2 \prod_{i<j}^{3} \Gamma_e\left(p^2q^2t^{-2} x_i^{(1)\pm1} x_j^{(1)\pm1}\right) \times \\
&\times \prod_{i<j}^{3} \Gamma_e\left(pqt^{-1} x_i^{(1)\pm1} x_j^{(1)\pm1}\right) \prod_{i=1}^{3} \Gamma_e\left(p^{3/2}q^{3/2}t^{-\frac{3}{2}} x_i^{(1)\pm1} x_1^{(2)\pm1}\right), \\
\mathsf{T}_{[2^3,1]} \quad &\longrightarrow \quad \Gamma_e\left(t^2\right)^3 \Gamma_e(t)^2 \prod_{i<j}^{3} \Gamma_e\left(t^2 y_i^{(1)\pm1} y_j^{(1)\pm1}\right) \times \\
&\times \prod_{i<j}^{3} \Gamma_e\left(t y_i^{(1)\pm1} y_j^{(1)\pm1}\right) \prod_{i=1}^{3} \Gamma_e\left(t^{3/2} y_i^{(1)\pm1} y_1^{(2)\pm1}\right), \\
\mathsf{O}_B^{12} \quad &\longrightarrow \quad \Gamma_e\left(t^{-1}c^2\right), \\
\mathsf{O}_B^{21} \quad &\longrightarrow \quad \Gamma_e\left(p^{-1}q^{-1}tc^2\right), \\
\mathsf{O}_B^{22} \quad &\longrightarrow \quad \Gamma_e\left(p^{-1}q^{-1}c^2\right).
\end{aligned} \tag{4.128}$$

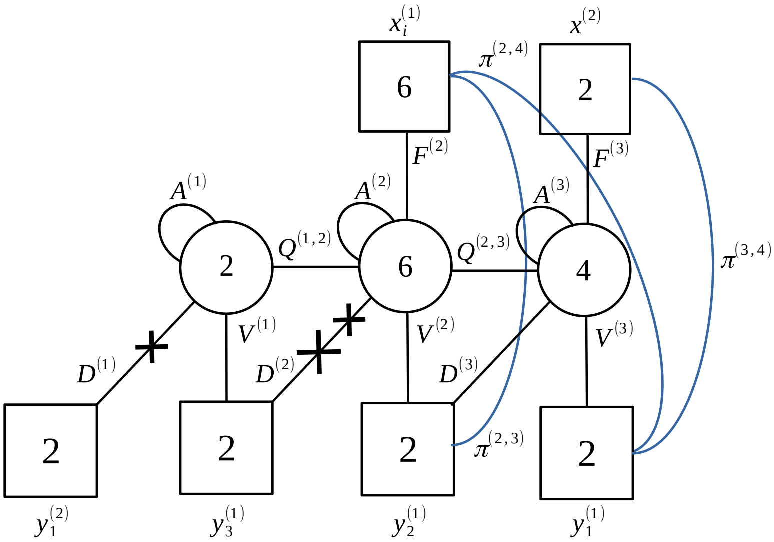

Figure 4.17: The quiver diagram representation of $E_{[2^3,1]}^{[2^3,1]}[USp(14)]$. Three flipping fields γ_{11}, γ_{21} and γ_{22}, denoted by crosses with two different sizes, flip the operators $\mathrm{Tr}_1 D_{[1}^{(1)} D_{2]}^{(1)}$, $\mathrm{Tr}_2 D_{[1}^{(2)} D_{2]}^{(2)}$ and $\mathrm{Tr}_2 A^{(2)} D_{[1}^{(2)} D_{2]}^{(2)}$ respectively.

Adding the singlets and applying sequentially the Intriligator–Pouliot duality we obtain the index of the $E_{[2^3,1]}^{[2^3,1]}[USp(14)]$ theory

$$
\begin{aligned}
&\mathcal{I}_{E_{[2^3,1]}[USp(14)]}\left(\vec{x}^{(1)}, x^{(2)}; y^{(2)}, \vec{y}^{(1)}; t; c\right) = \\
&= \Gamma_e\left(p^4q^4t^{-3}c^{-2}\right)\Gamma_e\left(p^3q^3t^{-2}c^{-2}\right)\Gamma_e\left(p^3q^3t^{-1}c^{-2}\right) \\
&\quad\times \prod_{i=1}^{3}\Gamma_e\left(p^{-1/2}q^{-1/2}cy_2^{(1)\pm1}x_i^{(1)\pm1}\right)\prod_{i=1}^{3}\Gamma_e\left(p^{-1/2}q^{-1/2}cy_1^{(1)\pm1}x_i^{(1)\pm1}\right)\Gamma_e\left(t^{-1/2}cy_1^{(1)\pm1}x_1^{(2)\pm1}\right)\times \\
&\quad\times \oint \mathrm{d}\vec{z}_1^{(1)}\,\mathrm{d}\vec{z}_3^{(2)}\,\mathrm{d}\vec{z}_2^{(3)}\,\Gamma_e(t)^6\prod_{a<b}^{3}\Gamma_e\left(tz_a^{(2)\pm1}z_b^{(2)\pm1}\right)\prod_{\alpha<\beta}^{2}\Gamma_e\left(tz_\alpha^{(3)\pm1}z_\beta^{(3)\pm1}\right)\times \\
&\quad\times \Gamma_e\left(p^{-3/2}q^{-3/2}t^{3/2}cz^{(1)\pm1}y_1^{(2)\pm1}\right)\prod_{a=1}^{3}\Gamma_e\left(p^{-1}q^{-1}t^{1/2}cy_3^{(1)\pm1}z_a^{(2)\pm1}\right)\prod_{\alpha=1}^{2}\Gamma_e\left(p^{-1/2}q^{-1/2}cy_2^{(1)\pm1}z_\alpha^{(3)\pm1}\right)\times \\
&\quad\times \prod_{a=1}^{3}\Gamma_e\left(p^{1/2}q^{1/2}t^{-1/2}z^{(1)\pm1}z_a^{(2)\pm1}\right)\prod_{a=1}^{3}\prod_{\alpha=1}^{2}\Gamma_e\left(p^{1/2}q^{1/2}t^{-1/2}z_a^{(2)\pm1}z_\alpha^{(3)\pm1}\right)\times \\
&\quad\times \Gamma_e\left(p^{3/2}q^{3/2}c^{-1}y_3^{(1)\pm1}z^{(1)\pm1}\right)\prod_{a=1}^{3}\Gamma_e\left(pqt^{1/2}c^{-1}y_2^{(1)\pm1}z_a^{(2)\pm1}\right)\prod_{\alpha=1}^{2}\Gamma_e\left(p^{1/2}q^{1/2}tc^{-1}y_1^{(1)\pm1}z_\alpha^{(3)\pm1}\right)\times \\
&\quad\times \prod_{a=1}^{3}\prod_{i=1}^{3}\Gamma_e\left(p^{1/2}q^{1/2}t^{-1/2}z_a^{(2)\pm1}x_i^{(1)\pm1}\right)\prod_{\alpha=1}^{2}\Gamma_e\left(p^{1/2}q^{1/2}t^{-1/2}z_\alpha^{(3)\pm1}x_1^{(2)\pm1}\right)\,,
\end{aligned}
\tag{4.129}
$$

from which one can read off the matter content and the superpotential. The matter content is conveniently represented using the quiver diagram, which is drawn in Figure 4.17. In particular we find the gauge singlets

$$
\begin{aligned}
\pi^{(2,3)} &\longrightarrow \prod_{i=1}^{3}\Gamma_e\left(p^{-1/2}q^{-1/2}cy_2^{(1)\pm 1}x_i^{(1)\pm 1}\right)\,,\\
\pi^{(2,4)} &\longrightarrow \prod_{i=1}^{3}\Gamma_e\left(p^{-1/2}q^{-1/2}cy_1^{(1)\pm 1}x_i^{(1)\pm 1}\right)\,,\\
\pi^{(3,4)} &\longrightarrow \Gamma_e\left(t^{-1/2}cy_1^{(1)\pm 1}x_1^{(2)\pm 1}\right)\,,\\
\gamma_{11} &\longrightarrow \Gamma_e\left(p^4q^4t^{-3}c^{-2}\right)\,,\\
\gamma_{21} &\longrightarrow \Gamma_e\left(p^3q^3t^{-2}c^{-2}\right)\,,\\
\gamma_{22} &\longrightarrow \Gamma_e\left(p^3q^3t^{-1}c^{-2}\right)\,,
\end{aligned}
\tag{4.130}
$$

and the superpotential

$$
\begin{aligned}
&\mathcal{W}_{E_{[2^3,1]}[USp(14)]} = \\
&\mathrm{Tr}_1\mathrm{Tr}_2\left[A^{(1)}Q^{(1,2)}Q^{(1,2)}\right] + \mathrm{Tr}_2\left[A^{(2)}\left(\mathrm{Tr}_1 Q^{(1,2)}Q^{(1,2)} + \mathrm{Tr}_{x^{(1)}}F^{(2)}F^{(2)} + \mathrm{Tr}_3 Q^{(2,3)}Q^{(2,3)}\right)\right] + \\
&+ \mathrm{Tr}_3\left[A^{(3)}\left(\mathrm{Tr}_2 Q^{(2,3)}Q^{(2,3)} + \mathrm{Tr}_{x^{(2)}}F^{(3)}F^{(3)}\right)\right] + \mathrm{Tr}_1\mathrm{Tr}_2\left[V_{[1}^{(1)}Q^{(1,2)}D_{2]}^{(2)}\right] + \\
&+ \mathrm{Tr}_2\mathrm{Tr}_3\left[V_{[1}^{(2)}Q^{(2,3)}D_{2]}^{(3)}\right] + \mathrm{Tr}_2\mathrm{Tr}_{x^{(1)}}\left[F^{(2)}V_{[1}^{(2)}\pi_{2]}^{(2,3)}\right] + \mathrm{Tr}_2\mathrm{Tr}_3\mathrm{Tr}_{x^{(1)}}\left[F^{(2)}Q^{(2,3)}V_{[1}^{(3)}\pi_{2]}^{(2,4)}\right] \\
&+ \mathrm{Tr}_3\mathrm{Tr}_{x^{(2)}}\left[F^{(3)}V_{[1}^{(3)}\pi_{2]}^{(3,4)}\right] + \sum_{i=1}^{2}\sum_{j=1}^{i}\gamma_{ij}\,\mathrm{Tr}_i\left[(A^{(i)})^{j-1}D_{[1}^{(i)}D_{2]}^{(i)}\right]\,.
\end{aligned}
\tag{4.131}
$$

where, as before, the subscripts 1, 2 denote the flavor indices for the corresponding $SU(2)$ in the saw. This superpotential is perfectly consistent with the general form of the $E_\rho^\sigma[USp(2N)]$ theory given by (4.61).

4.4 Outlook

In this chapter we introduced a class of $4d$ $\mathcal{N}=1$ theories that we denoted by $E_\rho^\sigma[USp(2N)]$. These admit a Lagrangian description in terms of linear quivers with symplectic gauge nodes and with matter chiral fields in the antisymmetric, bifundamental, fundamental and singlet representations, as schematically depicted in Figure 4.9. Moreover, we showed that these theories are related in pairs by a mirror duality which exchanges ρ and σ. The name "mirror" is due to the fact that it can be understood as a four-dimensional ancestor of mirror symmetry for the $3d$ $\mathcal{N}=4$ $T_\rho^\sigma[SU(N)]$ theories. These are quiver gauge theories with unitary gauge nodes and matter hypermultiplets in the bifundamental and fundamental representation, and the $E_\rho^\sigma[USp(2N)]$ reduce to them upon circle compactification and a suitable Coulomb branch VEV accompanied by real mass deformations. We derived the mirror dualities, both in $3d$ and in $4d$, by developing a purely field theoretic strategy based on the duality web of

$E[USp(2N)]$ and $T[SU(N)]$ and on the fact that certain VEVs of these theories make them flow to $E_\rho^\sigma[USp(2N)]$ and $T_\rho^\sigma[SU(N)]$, respectively.

There are several open questions related to the topic of this chapter. Most of them revolve around the idea of drawing more and more analogies between three and four dimensions. There are indeed many results that are known for the $T_\rho^\sigma[SU(N)]$ theories and the findings of this chapter naturally lead us to wonder whether similar results are true also for the $E_\rho^\sigma[USp(2N)]$ theories. For example, we mentioned that the $T_\rho^\sigma[SU(N)]$ theories admit an Hanany–Witten brane set-up, while we have no brane realization of the four-dimensional $E_\rho^\sigma[USp(2N)]$ theories. It would thus be very interesting to find one, since it could help us understand new things about these new $4d$ theories.

Another possible line of investigation regards the structure of the moduli space of the $E_\rho^\sigma[USp(2N)]$ theories. Indeed, the $T_\rho^\sigma[SU(N)]$ moduli space is known to have a neat description in terms of hyperKähler quotients [172]. It would be interesting to understand if also the moduli space of $E_\rho^\sigma[USp(2N)]$ possesses some interesting geometric structure. To this purpose, one possibility would be to investigate limits of the superconformal index of $E_\rho^\sigma[USp(2N)]$ that are analogues of the Higgs and Coulomb limits of the superconformal index of $T_\rho^\sigma[SU(N)]$ studied in [173]. In addition, the Coulomb limit of the superconformal index of $T_\rho^\sigma[SU(N)]$ takes the form of Hall–Littlewood polynomials [174], so a possible $4d$ version of this limit for the superconformal index of $E_\rho^\sigma[USp(2N)]$ may lead to an interesting generalization of these polynomials.

For the $T_\rho^\sigma[SU(N)]$ theories the holographic dual solutions are also known. These were constructed in [175] building on earlier works in [176, 177] and they were tested by matching the free energies in [178]. It would be interesting to understand whether also $E_\rho^\sigma[USp(2N)]$ admits a holographic dual and, in case of a positive answer, what it is. One can easily check that the conformal anomalies of $E[USp(2N)]$ are such that $a = c$ at large N, which suggests that a holographic dual for this theory does exist.

On top of trying to understand whether $E_\rho^\sigma[USp(2N)]$ possesses some of these properties, one may also wonder if a $4d$ uplift of other instances of $3d$ mirror symmetry exists, such as the mirror dualities for circular quivers or for linear quivers but with different gauge groups rather than unitary. One can for example consider the $T_\rho^\sigma[G]$ theories, which admit a quiver description containing alternating symplectic and orthogonal gauge nodes when G is one of the classical groups $SO(N)$ or $USp(2N)$. These are still related in pairs by the exchange of ρ and σ under mirror symmetry. There are two possibilities for the $4d$ uplifts of these theories and their mirror dualities. The first one is that in $4d$ we should consider a new class of quiver gauge theories. The second one is that their four-dimensional ancestors are still the $E_\rho^\sigma[USp(2N)]$ theories, but in order to get the $T_\rho^\sigma[G]$ theories with G different from $SU(N)$ we may need after the $3d$ reduction a different combination of Coulomb branch VEV and real mass deformation that leads to a different pattern of breaking of the symplectic gauge groups.

Finally, we are currently working on a new field theoretic derivation of mirror symmetry, both in $3d$ and in $4d$. More precisely, one can derive a large class of mirror dualities by locally dualizing the fields in the quiver using some properties of $E[USp(2N)]$ in $4d$ and $T[SU(N)]$ in $3d$, where the latter can be obtained as a limit of the former. Remember that the $3d$ mirror dualities can be understood from the brane set-ups of the theories as the action of the $SL(2,\mathbb{Z})$ symmetry of Type IIB on the various 5-branes contained in it. The local dualizations in the quiver mimic these transformations of the branes under $SL(2,\mathbb{Z})$. In this way, one can derive in field theory the $3d$ dualities for both linear and circular quivers with unitary nodes and possibly with Chern–Simons levels[16] that descend from $SL(2,\mathbb{Z})$ in Type IIB, as well as their $4d$ counterparts. The same strategy can also be used to derive in field theory the $SL(2,\mathbb{Z})$ dualities of the $3d$ S-fold SCFTs [179–187] and their $4d$ uplifts which haven't appeared in the literature yet. Let us mention that a similar idea was proposed first in $3d$ in [188] for quivers with gauge nodes of the same rank and in [189] for more general quivers. In those references, some fundamental identities for the matrix integral of the $\mathbb{S}^3$ partition function that involved the $T[SU(N)]$ theory were proven, which can be interpreted as the $SL(2,\mathbb{Z})$ transformations of the various 5-branes, and they were used to derive the integral identities for the dualities of the quiver gauge theories. In our approach we are able to give a field theory interpretation of these moves by understanding the identities associated to the 5-brane transformations as some basic dualities enjoyed by $E[USp(2N)]$, which plays the role of the S generator of $SL(2,\mathbb{Z})$ in the same way as $T[SU(N)]$ is usually interpreted as the S operator. By applying locally in the quiver these dualities in a similar way to how the transformations of the 5-branes are applied to the brane set-up of the $3d$ theories, one obtains the aforementioned algorithmic field theory derivation of mirror symmetry. We will present the properties of $E[USp(2N)]$ that can be understood as the $SL(2,\mathbb{Z})$ operations in [158]. Then, we will apply them to derive the $3d$ and $4d$ mirror dualities in [80]. One outcome of this analysis that is interesting from a conceptual point of view is that all the properties of $E[USp(2N)]$ that we will need to obtain the mirror dualities can be derived by just applying the Intriligator–Pouliot duality, with manipulations that are similar to those we saw extensively in this chapter, and those of $T[SU(N)]$ can be obtained as a limit as usual. This implies that mirror symmetry, both in $3d$ and $4d$, can be derived from the Intriligator–Pouliot duality alone, even if in an intricate way.

[16] Chern–Simons levels can be induced by introducing $(1,k)$ 5-branes in the brane set-up. These types of branes are produced when acting with the T^k element of $SL(2,\mathbb{Z})$ on an NS5-brane.

Appendix A

Supersymmetric partition functions conventions

A.1 Special functions

A.1.1 Multiple-sine functions

In order to introduce the multiple-sine function S_r (for more details on these and other special functions see [65]), we first need to define the multiple-gamma function Γ_r

$$\Gamma_r(z|\vec{\omega}) = \exp\left(\frac{\partial}{\partial s}\,\zeta_r(z,s|\vec{\omega})|_{s=0}\right)\,, \tag{A.1}$$

where ζ_r is the multiple-zeta function

$$\zeta_r(z,s|\vec{\omega}) = \sum_{n_1,\cdots,n_r=0}^{\infty} \frac{1}{(n_1\omega_1+\cdots n_r\omega_r+z)^s} \tag{A.2}$$

Then, the multiple-sine function is defined as

$$S_r(z|\vec{\omega}) = \Gamma_r(z|\vec{\omega})^{-1}\Gamma_r(|\vec{\omega}|-z|\vec{\omega})^{(-1)^r}\,, \tag{A.3}$$

where $|\vec{\omega}| = \omega_1+\cdots+\omega_r$.

The multiple-gamma function has poles at $z\in\mathbb{Z}_{\leq 0}$. This implies that the multiple-sine function has zeroes at these points. Depending on r being even or odd, the function S_r may have poles or additional zeroes at $z=|\vec{\omega}|-\mathbb{Z}_{\leq 0}$.

In Section 2.3.4 we used several useful properties of these special functions. One of them is the periodicity property

$$S_r(z+\omega_j|\vec{\omega}) = \frac{S_r(z|\vec{\omega})}{S_{r-1}(z|\vec{\omega}/\omega_j)}\,, \tag{A.4}$$

where $\vec{\omega}/\omega_j = (\omega_1,\cdots,\omega_{j+1},\omega_{j+1},\cdots,\omega_r)$. Another important property is the reflection property

$$S_r(z|\vec{\omega})S_r(|\vec{\omega}|-z|\vec{\omega})^{(-1)^r} = 1\,, \tag{A.5}$$

In particular, we needed this in the cases $r=2,3$, since the partition functions on $\mathbb{S}^3$ and $\mathbb{S}^5$ are written in terms of S_2 and S_3 functions respectively

$$S_3\left(z|\omega_1,\omega_2,\omega_3\right)=S_3\left(|\vec{\omega}|-z|\omega_1,\omega_2,\omega_3\right), \qquad S_2\left(z\right)=S_2\left(|\vec{\omega}|-z|\omega_1,\omega_2\right)^{-1}. \tag{A.6}$$

In Section 2.3.3, we actually wrote the partition function on the squashed three-sphere S_b^3 in terms of a related function

$$s_b\left(x\right)=S_2\left(\frac{Q}{2}-ix|b,b^{-1}\right) \tag{A.7}$$

For this special function, the reflection property (A.6) reads

$$s_b\left(x\right)s_b\left(-x\right)=1, \tag{A.8}$$

which encodes at the level of partition functions the fact that two chiral fields χ_1, χ_2 become massive and are integrated out anytime a superpotential term of the form $\mathcal{W}=\chi_1\chi_2$ is turned on.

The multiple-sine function S_r also possesses an interesting factorization property that the reader can find for generic r in [65]. For our purposes, we only needed it in the case $r=3$, where it reads

$$\begin{aligned} S_3\left(x|\omega_1,\omega_2,\omega_3\right) &= \mathrm{e}^{-i\frac{\pi}{3!}B_{33}(x)}\left(\mathrm{e}^{\frac{2\pi i}{e_3}x};q^{-1},t\right)_1\left(\mathrm{e}^{\frac{2\pi i}{e_3}x};q^{-1},t\right)_2\left(\mathrm{e}^{\frac{2\pi i}{e_3}x};q^{-1},t\right)_3 \\ &\equiv \mathrm{e}^{-i\frac{\pi}{3!}B_{33}(x)}\Big|\Big|\left(\mathrm{e}^{\frac{2\pi i}{e_3}x};q^{-1},t\right)\Big|\Big|_S^3, \end{aligned} \tag{A.9}$$

where

$$q=\mathrm{e}^{-2\pi i\frac{e_1}{e_3}}, \qquad t=\mathrm{e}^{2\pi i\frac{e_2}{e_3}} \tag{A.10}$$

and the parameters e_i are chosen differently in each of the three sectors according to 2.3. In the above expression, the double q-Pochhammer symbol is defined as

$$(x;q,t)=\prod_{m,n=0}^{\infty}\left(1-xq^mt^n\right). \tag{A.11}$$

This possesses the analytic continuation property

$$(Aq^mt^n;q,t)=\frac{1}{(Aq^{m-1}t^n;q^{-1},t)}. \tag{A.12}$$

A.1.2 Υ_β function

The contribution of a 5d $\mathcal{N}=1$ hypermultiplet to the partition function on $\mathbb{S}^4\times\mathbb{S}^1$ is written in terms of the Υ_β, which can be defined as (for more details we refer the reader to [78])

$$\Upsilon_\beta(x|\epsilon_1,\epsilon_2)=(1-\mathrm{e}^{\beta})^{-\frac{1}{\epsilon_1\epsilon_2}\left(x-\frac{\epsilon_1+\epsilon_2}{2}\right)^2}\prod_{n_1,n_2=0}^{\infty}\frac{(1-\mathrm{e}^{\beta(x+n_1\epsilon_1+n_2\epsilon_2)})(1-\mathrm{e}^{\beta(\epsilon_1+\epsilon_2-x+n_1\epsilon_1+n_2\epsilon_2)})}{(1-\mathrm{e}^{\beta\left(\frac{\epsilon_1+\epsilon_2}{2}+n_1\epsilon_1+n_2\epsilon_2\right)})^2}. \tag{A.13}$$

This is a q-deformed version of the function Υ in terms of which the three-point function of Liouville theory (2.16) is written and to which it reduces in the $\beta \to 0$ limit

$$\Upsilon_\beta(x|\epsilon_1, \epsilon_2) \underset{\beta \to 0}{\longrightarrow} \Upsilon(x|\epsilon_1, \epsilon_2)\,. \tag{A.14}$$

From the gauge theory point of view, this limit corresponds to the dimensional reduction from $\mathbb{S}^4 \times \mathbb{S}^1$ to $\mathbb{S}^4$.

The Υ_β function possesses some interesting periodicity and factorization property that allow us to analytically continue the partition function of the WZ model on $\mathbb{S}^2 \times \mathbb{S}^1$ to the partition function of T_2 on $\mathbb{S}^4 \times \mathbb{S}^1$ and to factorize it in two copies of $\mathcal{Z}_{top}$ (2.84). The periodicity property reads

$$\Upsilon_\beta(x + \epsilon_1|\epsilon_1, \epsilon_2) = \left(\frac{1 - \mathrm{e}^{\beta}}{1 - \mathrm{e}^{\beta\epsilon}}\right)^{1-\epsilon_2^{-1}x} \gamma_{\beta\epsilon_2}(x\epsilon_2^{-1})\Upsilon_\beta(x|\epsilon_1, \epsilon_2)\,, \tag{A.15}$$

where

$$\gamma_\beta(x) = (1 - \mathrm{e}^{\beta})^{1-2x} \frac{\left(\mathrm{e}^{1-\beta x}; \mathrm{e}^{\beta}\right)_\infty}{\left(\mathrm{e}^{\beta x}; \mathrm{e}^{\beta}\right)_\infty} \tag{A.16}$$

we recall being the contribution of a chiral field to the partition function of a theory on $\mathbb{S}^2 \times \mathbb{S}^1$. Instead, the factorization property is

$$\Upsilon_\beta(x|\epsilon_1, \epsilon_2) = (1 - \mathrm{e}^{\beta})^{-\frac{1}{\epsilon_1\epsilon_2}\left(x - \frac{\epsilon_1+\epsilon_2}{2}\right)^2} \left|\left|\frac{\left(\mathrm{e}^{-\beta x}; q, t\right)}{\left(\sqrt{\frac{t}{q}}; q, t\right)}\right|\right|_{id}^2\,, \tag{A.17}$$

where the id-norm is defined as

$$||(z; q, t)||_{id}^2 \equiv (z; q, t) \left(z^{-1}; q^{-1}, t^{-1}\right) \equiv (z; q, t) \left(\tilde{z}; \tilde{q}, \tilde{t}\right)\,. \tag{A.18}$$

and we defined the parameters

$$q = \mathrm{e}^{-\beta\epsilon_1}, \qquad t = \mathrm{e}^{\beta\epsilon_2}\,. \tag{A.19}$$

A.1.3 Elliptic gamma function

Another important special function which is involved in the integral representation of the $\mathbb{S}^3 \times \mathbb{S}^1$ partition function is the elliptic gamma function

$$\Gamma_e(z) \equiv \Gamma\left(z; p, q\right) \;= \textstyle\prod_{n,m=1}^{\infty} \frac{1-p^n q^m z^{-1}}{1-p^{n+1}q^{m+1}z}\,. \tag{A.20}$$

One useful property of this function is the following asymptotic behaviour:

$$\lim_{r\to 0} \Gamma_e\left(\mathrm{e}^{2\pi i r x}; \mathrm{e}^{-2\pi r b}, \mathrm{e}^{-2\pi r b^{-1}}\right) = \mathrm{e}^{-\frac{i\pi}{6r}\left(i\frac{Q}{2} - x\right)} s_b\left(i\frac{Q}{2} - x\right)\,. \tag{A.21}$$

In the main text we used this to study the limit from the $\mathbb{S}^3 \times \mathbb{S}^1$ to the $\mathbb{S}_b^3$ partition function, where the parameter r that is sent to zero is interpreted as the radius of the $\mathbb{S}^1$. Another very useful property is

$$\Gamma_e\left(x\right)\Gamma_e\left(pq\,x^{-1}\right)=1\,, \tag{A.22}$$

which encodes at the level of partition functions the fact that two chiral fields χ_1, χ_2 become massive and are integrated out anytime a superpotential term of the form $\mathcal{W}=\chi_1\chi_2$ is turned on.

A.2 $\mathbb{S}^2\times\mathbb{S}^1$ partition function

In this appendix, we briefly review some basic facts about the $\mathbb{S}^2\times\mathbb{S}^1$ partition function, which is also known as the $3d$ supersymmetric index. When computed with the superconformal R-symmetry it coincides with the superconformal index [28, 190, 191, 29, 30, 52]. The index is defined as a trace over states on $\mathbb{S}^2\times\mathbb{R}$. The standard definition is the following:

$$\mathcal{I}(x,\vec{\mu})\;=\;\mathrm{Tr}\left[(-1)^{2J_3}x^{\Delta+J_3}\prod_i\mu_i^{T_i}\right]\,, \tag{A.23}$$

where Δ is the energy in units of the $\mathbb{S}^2$ radius (for superconformal field theories, Δ is related to the conformal dimension), J_3 is the Cartan generator of the Lorentz $SO(3)$ isometry of $\mathbb{S}^2$ and T_i are charges under non-R global symmetries. The index only receives contributions from the states that satisfy

$$\Delta-R-J_3=0\,, \tag{A.24}$$

where R is the R-charge.

The $3d$ supersymmetric index also admits an integral representation that is obtained by considering it as the supersymmetric partition function on $\mathbb{S}^2\times\mathbb{S}^1$ and computing it with localization techniques

$$\mathcal{I}(x;\{\vec{\mu},\vec{n}\})\;=\;\sum_{\vec{m}}\frac{1}{|\mathcal{W}_{\vec{m}}|}\oint_{\mathbb{T}^{\mathrm{rk}G}}\prod_{i=1}^{\mathrm{rk}G}\frac{\mathrm{d}z_{\,a}}{2\pi i z_a}Z_{\mathrm{cl}}\,Z_{\mathrm{vec}}\,Z_{\mathrm{mat}}\,, \tag{A.25}$$

where we denoted by $\vec{z}$ the fugacities parametrizing the maximal torus of the gauge group and by $\vec{m}$ the corresponding GNO magnetic fluxes on $\mathbb{S}^2$. The integration contour is taken to be the unit circle $\mathbb{T}$ for each integration variable and the prefactor $|\mathcal{W}_{\vec{m}}|$ is the dimension of the Weyl group of the residual gauge symmetry in the monopole background labelled by the configuration of magnetic fluxes $\vec{m}$. We also use $\{\vec{\mu},\vec{n}\}$ to denote possible fugacities and fluxes for the background vector multiplets associated with global symmetries, respectively. The different contributions to the integrand of (A.25) are:

- the contribution from the classical action of CS and BF interactions

$$Z_{\mathrm{cl}}\;=\;\prod_{i=1}^{\mathrm{rk}G}\omega^{m_i}z_i^{k\,m_i+\mathfrak{n}}\,, \tag{A.26}$$

  where $\mathrm{rk}G$ is the rank of the gauge group G and we denoted with k the CS level and with ω and $\mathfrak{n}$ the fugacity and the background flux for the global symmetry;

- the contribution of the $\mathcal{N}=2$ vector multiplet

$$Z_{\text{vec}} = \prod_{\alpha\in\mathfrak{g}} x^{-\frac{|\alpha(\vec{m})|}{2}}\left(1-(-1)^{\alpha(\vec{m})}\vec{z}^{\alpha}x^{|\alpha(\vec{m})|}\right), \tag{A.27}$$

  where α are roots in the gauge algebra $\mathfrak{g}$ and we are using the short-hand notations

$$\vec{z}^{\alpha} = \prod_{i=1}^{\text{rk}G} z_i^{\alpha_i}, \quad \alpha(\vec{m}) = \sum_{i=1}^{\text{rk}G}\alpha_i m_i, \quad |\alpha(\vec{m})| = \sum_{i=1}^{\text{rk}G}\alpha_i m_i\,; \tag{A.28}$$

- the contribution of an $\mathcal{N}=2$ chiral field transforming in some representation $\mathcal{R}$ and $\mathcal{R}_F$ of the gauge and the flavour symmetry respectively and with R-charge r

$$Z_{mat} = \prod_{\rho\in\mathcal{R}}\prod_{\tilde{\rho}\in\mathcal{R}_F}\left(\vec{z}^{\rho}\,\vec{\mu}^{\tilde{\rho}}\,x^{r-1}\right)^{-\frac{|\rho(\vec{m})+\tilde{\rho}(\vec{n})|}{2}}\times$$
$$\times\frac{\left((-1)^{\rho(\vec{m})+\tilde{\rho}(\vec{n})}\,\vec{z}^{-\rho}\,\vec{\mu}^{-\tilde{\rho}}\,x^{2-r+|\rho(\vec{m})+\tilde{\rho}(\vec{n})|};x^2\right)_\infty}{\left((-1)^{\rho(\vec{m})+\tilde{\rho}(\vec{n})}\,\vec{z}^{\rho}\,\vec{\mu}^{\tilde{\rho}}\,x^{r+|\rho(\vec{m})+\tilde{\rho}(\vec{n})|};x^2\right)_\infty}, \tag{A.29}$$

  where ρ and $\tilde{\rho}$ are the weights of $\mathcal{R}$ and $\mathcal{R}_F$ respectively.

Even though we didn't use this in the text, it is useful to apply the index to $3d$ superconformal field theories. In which case, the index keeps track of the short multiplets of the theory, up to recombination. It proves useful to compute the index perturbatively by expanding the integrand in the fugacity x and taking the gauge projection $\oint\frac{d\vec{z}}{2\pi i\vec{z}}$ at each order. Turning off the background fluxes for the global symmetries, we obtain a result which is a power series in x

$$\mathcal{I}(x,\{\vec{\mu},\vec{n}=0\}) = \sum_{p=0}^{\infty}\chi_p(\vec{\mu})\,x^p \tag{A.30}$$

where $\chi_p(\vec{\mu})$ is the character of some representation of the global symmetry of the theory. As demonstrated in [192] (see also [193, 141]), one can study the contribution of superconformal multiplets to each order of x in the power series. Since the classification of the shortening conditions for $3d$ superconformal algebras is known [194, 195], it is possible to obtain useful information about the superconformal theory in question using the power series of the index, especially its lowest orders. The coefficients of x correspond to the $\mathcal{N}=2$ relevant operators, contributing with only a positive sign. The coefficient of x^2 receives a contribution from the $\mathcal{N}=2$ marginal operators, contributing with a positive sign, and the conserved currents, contributing with a negative sign. This is similar to what happens for the $4d$ index, which instead we intensively used to perform this type of analysis in the main text.

There exists also a slightly different definition of the index in the literature, see for example [29, 30]. As explained in [52], this differs from the one we just reviewed for the fact that in the trace definition (A.23) the factor $(-1)^{2J_3}$ is replaced by $(-1)^F$. This is the definition that we actually used in the main text, especially in Chapter 2 to make contact with the free field correlators. With this other convention, the various contributions to the integral representation of the index become:

- the contribution from the classical action of CS and BF interactions

$$Z_{\text{cl}} = \prod_{i=1}^{\text{rk}G}\omega^{m_i}(-z_i)^{k\,m_i}, \tag{A.31}$$

where we turned off the background magnetic flux for the topological symmetry;

- the contribution of the vector multiplet

$$Z_{\text{vec}} = \prod_{\alpha\in\mathfrak{g}} x^{-\frac{|\alpha(\vec{m})|}{2}}\left(1-\vec{z}^{\alpha}x^{|\alpha(\vec{m})|}\right),; \tag{A.32}$$

- the contribution of a chiral multiplet

$$Z_{mat} = \prod_{\rho\in\mathcal{R}}\prod_{\tilde{\rho}\in\mathcal{R}_F}\left((-\vec{z})^{\rho}\,\vec{\mu}^{\tilde{\rho}}\,x^{r-1}\right)^{-\frac{|\rho(\vec{m})+\tilde{\rho}(\vec{n})|}{2}}\frac{(\vec{z}^{-\rho}\,\vec{\mu}^{-\tilde{\rho}}\,x^{2-r+|\rho(\vec{m})+\tilde{\rho}(\vec{n})|};x^2)_\infty}{(\vec{z}^{\rho}\,\vec{\mu}^{\tilde{\rho}}\,x^{r+|\rho(\vec{m})+\tilde{\rho}(\vec{n})|};x^2)_\infty}\,. \tag{A.33}$$

This definition usually differs from the one we saw before by simply a redefinition of the fugacity ω for the topological symmetry $\omega \to (-1)^m\omega$ [52]. Indeed, when the index is computed as a power series in x, one gets different signs for some of the terms appearing in the expansion, specifically those carrying the fugacity ω, when using this different convention compared to the previous one. Hence, this alternative index is not suitable for the analysis of the various superconformal multiplets we mentioned before.

A.3 $\mathbb{S}^3_b$ partition function

The partition function of a $3d$ $\mathcal{N}=2$ theory on the three-sphere was first computed using localization techniques in [17]. The set-up considered was that of the theory on a round sphere, namely with trivial squashing parameter $b=1$, and with canonical assignment of R-charges to the chiral fields, namely R-charge $\frac{1}{2}$. This result was later generalized in [16] to the case of generic R-charges and in [18, 19] to the case of a squashed sphere $\mathbb{S}^3_b$, which can be parametrized as

$$b^2|z_1| + \frac{1}{b^2}|z_2| = 1, \qquad z_1, z_2 \in \mathbb{C}\,. \tag{A.34}$$

The result is a matrix integral with the following form:

$$Z(\vec{m},\eta,k) = \frac{1}{|\mathcal{W}|}\int_{-\infty}^{+\infty}\prod_{a=1}^{\text{rk}G}\mathrm{d}s_a\,Z_{cl}\,Z_{vec}\,Z_{chir}\,. \tag{A.35}$$

where $|\mathcal{W}|$ is the dimension of the Weyl group associated to the gauge group G. The different contributions to the integrand of (A.35) are:

- the contribution from the classical action of CS and BF interactions

$$Z_{cl} = \mathrm{e}^{2\pi i\eta\sum_{a=1}^{\text{rk}G}s_a}\mathrm{e}^{-i\pi k\sum_{a=1}^{\text{rk}G}s_a^2}\,, \tag{A.36}$$

where rk$G$ is the rank of the gauge group G and we denoted with k the CS level and with η the FI parameter;

- the contribution of the $\mathcal{N}=2$ vector multiplet

$$Z_{vec} = \frac{1}{\prod_{\alpha>0}s_b\left(i\frac{Q}{2}+\alpha(s)\right)}\,, \tag{A.37}$$

where α are the positive roots of the gauge algebra $\mathfrak{g}$ and we are using the short-hand notations

$$\alpha(s) = \prod_{a=1}^{\mathrm{rk}G} \alpha_a s_a \,; \tag{A.38}$$

- the contribution of an $\mathcal{N}=2$ chiral field transforming in some representation $\mathcal{R}$ and $\mathcal{R}_F$ of the gauge and the flavour symmetry respectively and with R-charge r

$$Z_{chir} = \prod_{\rho\in\mathcal{R}_G}\prod_{\tilde\rho\in\mathcal{R}_F} s_b\left(i\frac{Q}{2} - \rho(s) - \tilde\rho(m) - i\frac{Q}{2}r\right)\,, \tag{A.39}$$

where ρ and $\tilde\rho$ are the weights of $\mathcal{R}$ and $\mathcal{R}_F$ respectively.

As for the index, one can also extract very non-trivial information about the IR SCFT to which the theory flows from the $\mathbb{S}^3$ partition function, that is with no squashing $b=1$. Again, we didn't use this technology in the main text, but it is useful to keep in mind this possible application of the sphere partition function. For example, the $\mathbb{S}^3$ partition function is very useful for finding the superconformal R-symmetry of an IR SCFT with a weakly-coupled UV description, as well as the central charges associated with its global symmetry currents. These latter quantities appear in the correlation functions of such currents, and serve as nontrivial CFT data of the theory. We point out that when $b=1$ the double-sine function that appears in the $\mathbb{S}_b^3$ partition function reduces to the Jafferis function

$$s_b\left(i\frac{Q}{2}(1-r) - u\right) = \exp\left[l\left(1-r+iu\right)\right]\,, \tag{A.40}$$

which is defined as [16]

$$l\left(z\right) = -z\log\left(1-e^{2\pi i z}\right) + \frac{i}{2}\left[\pi z^2 + \frac{1}{\pi}\mathrm{Li}_2\left(e^{2\pi i z}\right)\right] - \frac{i\pi}{12}\,. \tag{A.41}$$

To be concrete, let us consider the flat space two-point function (at separated points) of a $U(1)$ global symmetry current J^μ. Conformal invariance then restricts it to take the following form:

$$\langle J^\mu\left(x\right) J^\nu\left(0\right)\rangle = \frac{C}{16\pi^2}\left(\delta^{\mu\nu}\partial^2 - \partial^\mu\partial^\nu\right)\frac{1}{x^2}\,. \tag{A.42}$$

The number C, which is positive in a unitary theory, is defined as the corresponding central charge.

To see how we can use the $\mathbb{S}^3$ partition function in order to find the IR R-symmetry and the central charges, let us consider the space of R-symmetries parametrized by the mixing coefficients with all abelian flavor symmetries $U(1)_I$. That is, we consider the trial R-symmetry

$$R\left(t\right) = R_0 + \sum_I t^I Q_I\,, \tag{A.43}$$

where R_0 is some reference R-symmetry and t^I and Q_I are the mixing coefficients and charges of $U(1)_I$, respectively. The $\mathbb{S}^3$ partition function, denoted by Z, is then a function of t when evaluated with respect to the R-symmetry $R(t)$. As shown in [16, 57], the value of t that minimizes $|Z(t)|$ is the one corresponding to the R-symmetry which appears in the $\mathcal{N}=2$ superconformal algebra. As a result, using this Z-minimization principle one is able to find the IR R-symmetry if the partition function $Z(t)$ is known.

This principle can also be formulated in terms of the real part of the free energy

$$\mathrm{Re}\, F(t) = -\mathrm{Re} \log Z\,. \tag{A.44}$$

In this case, the superconformal R-symmetry locally maximizes $\mathrm{Re}\, F(t)$ over the space of trial R-symmetries $R(t)$. Denoting the corresponding value of t by t_{SC}, we therefore have

$$\frac{\partial}{\partial t^I} \mathrm{Re}\, F \Big|_{t=t_{SC}} = 0\,. \tag{A.45}$$

The second derivative of $\mathrm{Re}\, F(t)$ at t_{SC} also turns out to have an interesting meaning, and in fact encodes the central charge C defined in (A.42). More explicitly, it is given by [57] (see also [196])

$$\left(\frac{\partial}{\partial t^I}\right)^2 \mathrm{Re}\, F \Big|_{t=t_{SC}} = -\frac{\pi^2}{2} C_I\,, \tag{A.46}$$

where C_I is the central charge of $U(1)_I$. We see that in addition to the superconformal R-symmetry, the $\mathbb{S}^3$ partition function can also be used to compute the global symmetry central charges of the IR SCFT.

A.4 $\mathbb{S}^3 \times \mathbb{S}^1$ partition function

In this appendix we briefly summarize the basic notion of the $\mathbb{S}^3 \times \mathbb{S}^1$ partition function of an $\mathcal{N} = 1$ theory, which is also known as the $4d$ supersymmetric index. This coincides with the superconformal index [139, 138, 140] when computed with the superconformal R-symmetry; see also [146] for a more comprehensive review. We follow closely the exposition of the latter reference.

The index of a $4d$ $\mathcal{N} = 1$ SCFT is a refined Witten index of the theory quantized on $\mathbb{S}^3 \times \mathbb{R}$,

$$\mathcal{I} = \mathrm{Tr}(-1)^F e^{-\beta\delta} e^{-\mu_i \mathcal{M}_i}, \qquad \delta = \frac{1}{2}\left\{\mathcal{Q}, \mathcal{Q}^\dagger\right\}\,, \tag{A.47}$$

where $\mathcal{Q}$ is one of the Poincaré supercharges, $\mathcal{Q}^\dagger = \mathcal{S}$ is the conjugate conformal supercharge, $\mathcal{M}_i$ are $\mathcal{Q}$-closed conserved charges and μ_i are their chemical potentials. All the states contributing to the index with non-vanishing weight have $\delta = 0$; this renders the index independent of β.

For $\mathcal{N} = 1$ SCFTs, the supercharges are

$$\left\{\mathcal{Q}_\alpha,\, \mathcal{S}^\alpha = \mathcal{Q}^{\dagger\alpha} \widetilde{\mathcal{Q}}_{\dot\alpha},\, \widetilde{\mathcal{S}}^{\dot\alpha} = \widetilde{\mathcal{Q}}^{\dagger\dot\alpha}\right\}\,, \tag{A.48}$$

where $\alpha = \pm$ and $\dot\alpha = \dot\pm$ are respectively the $SU(2)_1$ and $SU(2)_2$ indices of the isometry group $Spin(4) = SU(2)_1 \times SU(2)_2$ of $\mathbb{S}^3$. For definiteness, let us choose $\mathcal{Q} = \widetilde{\mathcal{Q}}_{\dot -}$. With this particular choice, it is common to define the index as

$$\mathcal{I}\,(p, q) = \mathrm{Tr}(-1)^F p^{j_1+j_2+\frac{1}{2}r} q^{j_2-j_1+\frac{1}{2}r}\,. \tag{A.49}$$

where p and q are fugacities associated with the supersymmetry preserving squashing of the $\mathbb{S}^3$ [140]. Indeed, even if the dimension of the bosonic part of the $4d$ $\mathcal{N} = 1$ superconformal algebra is four, the number of independent fugacities that we can turn on in the index is two because of the constraints $\delta = 0$ and $[\mathcal{M}_i, \mathcal{Q}] = 0$. A possible choice for the combinations of the bosonic generators that satisfy these requirements is $\pm j_1 + j_2 + \frac{r}{2}$, where j_1 and j_2 are the

Cartan generators of $SU(2)_1$ and $SU(2)_2$, and r is the generator of the $U(1)_r$ R-symmetry. Another parametrization of the fugacities which is common in the literature is

$$t = (pq)^{\frac{1}{2}} \ , \qquad y = \left(\frac{p}{q}\right)^{\frac{1}{2}} \ . \tag{A.50}$$

This is useful for computing the index perturbatively, namely as a power series in the fugacity t.

The index counts gauge invariant operators that can be constructed from modes of the fields. The latter are usually referred to as "letters" in the literature. The single-letter index for a vector multiplet and a chiral multiplet $\chi(\mathbf{R})$ transforming in the $\mathbf{R}$ representation of the gauge×flavour group is

$$\begin{aligned} i_V\left(p,q,U\right) &= \frac{2pq-p-q}{(1-p)(1-q)}\chi_{adj}\left(U\right), \\ i_{\chi(r)}\left(p,q,U,V\right) &= \frac{(pq)^{\frac{1}{2}r}\chi_{\mathbf{R}}\left(U,V\right)-(pq)^{\frac{2-r}{2}}\chi_{\bar{\mathbf{R}}}\left(U,V\right)}{(1-p)(1-q)}\,, \end{aligned} \tag{A.51}$$

where $\chi_{\mathbf{R}}\left(U,V\right)$ and $\chi_{\bar{\mathbf{R}}}\left(U,V\right)$ denote the characters of $\mathbf{R}$ and the conjugate representation $\bar{\mathbf{R}}$, with U and V gauge and flavour group matrices, respectively.

The index can then be obtained by symmetrizing of all of such letters and then projecting them to gauge singlets by integrating over the Haar measure of the gauge group. This takes the general form

$$\mathcal{I}\left(p,q,V\right) = \int [\mathrm{d}U] \prod_k \mathrm{PE}\left[i_k\left(p,q,U,V\right)\right] \ , \tag{A.52}$$

where k labels the different multiplets in the theory, and $\mathrm{PE}[i_k]$ is the plethystic exponential of the single-letter index of the k-th multiplet, responsible for generating the symmetrization of the letters. It is defined by

$$\mathrm{PE}\left[i_k\left(p,q,U,V\right)\right] = \exp\left[\sum_{n=1}^{\infty}\frac{1}{n}i_k\left(p^n,q^n,U^n,V^n\right)\right] \ . \tag{A.53}$$

For definiteness, let us discuss a specific example of the $SU(N_c)$ gauge group. The contribution of a chiral superfield in the fundamental representation $\mathbf{N_c}$ or anti-fundamental representation $\bar{\mathbf{N}}_\mathbf{c}$ of $SU(N_c)$ with R-charge r can be written in terms of elliptic gamma functions, as follows:

$$\mathrm{PE}\left[i_{\chi(\mathbf{N_c})}\left(p,q,U\right)\right] = \prod_{i=1}^{N_c}\Gamma_e\left((pq)^{\frac{r}{2}}z_i\right), \quad \mathrm{PE}\left[i_{\chi(\bar{\mathbf{N}}_\mathbf{c})}\left(t,y,U\right)\right] = \prod_{i=1}^{N_c}\Gamma_e\left((pq)^{\frac{r}{2}}z_i^{-1}\right) \ , \tag{A.54}$$

where $\{z_i\}$, with $i=1,...,N_c$ and $\prod_{i=1}^{N_c} z_i = 1$, are the fugacities parametrizing the Cartan subalgebra of $SU(N_c)$. We will also use the shorthand notation

$$\Gamma_e\left(uz^{\pm n}\right) = \Gamma_e\left(uz^{n}\right)\Gamma_e\left(uz^{-n}\right) \ . \tag{A.55}$$

On the other hand, the contribution of the vector multiplet in the adjoint representation of $SU(N_c)$, together with the $SU(N_c)$ Haar measure, is

$$\frac{\kappa^{N_c-1}}{N_c!}\oint_{\mathbb{T}^{N_c}}\prod_{i=1}^{N_c-1}\frac{\mathrm{d}z_i}{2\pi i z_i}\prod_{k\neq\ell}^{N_c}\frac{1}{\Gamma_e(z_k z_\ell^{-1})}\cdots\,, \tag{A.56}$$

where the dots denote that it will be used in addition to the full matter multiplets transforming in representations of the gauge group. The integration contour is taken over the maximal torus of the gauge group and κ is the index of $U(1)$ free vector multiplet defined as

$$\kappa=(p;p)_\infty(q;q)_\infty\,, \tag{A.57}$$

where we recall the definition of the Q-Pochhammer symbol $(a;b)=\prod_{n=0}^{\infty}(1-ab^n)$.

In case of a $USp(2N_c)$ gauge group, instead, the contribution of a chiral multiplet in the fundamental representation and with R-charge r is

$$\mathrm{PE}\left[i_{\chi(\mathbf{N_c})}(p,q,U)\right]\quad=\prod_{i=1}^{N_c}\Gamma_e\left((pq)^{\frac{r}{2}}z_i^{\pm1}\right)\,, \tag{A.58}$$

while the full contribution of the vector multiplet in the adjoint representation together with the matching Haar measure and the projection to gauge singlets can be written as

$$\frac{\kappa^{N_c}}{2^{N_c}N_c!}\oint_{\mathbb{T}^{N_c}}\prod_{i=1}^{N_c}\frac{dz_i}{2\pi i z_i}\prod_{k<\ell}^{N_c}\frac{1}{\Gamma_e(z_k^{\pm1}z_\ell^{\pm1})}\prod_{k=1}^{N_c}\frac{1}{\Gamma_e(z_k^{\pm2})}\cdots\,. \tag{A.59}$$

At the superconformal fixed point, we employ the superconformal symmetry to extract the information about the states. Although the index counts states up to cancellations due to recombinations of various short superconformal multiplets to long multiplets, it has been shown in [141] that at low orders of the expansion in t the index reliably contains information about certain important operators. In particular, at order $t^2=pq$, one obtains the difference between the marginal operators and the conserved currents. We extensively utilize the result of the computation at this order in the main text.

Appendix B

$\mathbb{S}^3_b$ partition function computations

B.1 Derivation of the equality (2.160) for the recombination duality for $N=3$

The identity for the partition functions of the recombination duality (2.160) can be proven by applying iteratively the one for the Aharony duality (2.69) following the procedure we schematically described in Subsubsection 2.4.2. Here we use that strategy explicitly at the level of the $\mathbb{S}^3_b$ partition function. For definiteness we focus on the case $N=3$, which is general enough to understand all the details of the derivation.

We start considering the partition function of the $G[U(3)]$ theory, where for simplicity we remove the contribution of the singlets β_i. Consequently according to the operator map (2.159), we expect that in the recombination dual frame the singlets $\beta_{L,a}$ are removed, while the singlets $\beta_{R,i}$ are restored. So our starting point is

$$\begin{aligned}
\mathcal{Z} \;=\;& \int \frac{\mathrm{d}u_1^{(3)}\,\mathrm{d}u_2^{(3)}\,\mathrm{d}u_3^{(3)}}{3!}\mathrm{e}^{2\pi i\zeta\sum_a u_a^{(3)}}\frac{\prod_{a=1}^{3} s_b\left(i\frac{Q}{2}\pm u_a^{(3)}-\mu\right)}{\prod_{a<b}^{3} s_b\left(i\frac{Q}{2}\pm(u_a^{(3)}-u_b^{(3)})\right)}\times\\
\times\;& s_b\left(i\frac{Q}{2}\pm(u^{(3)}-z_3)-\Delta\right)\int\frac{\mathrm{d}u_1^{(2)}\,\mathrm{d}u_2^{(2)}}{2}\frac{\prod_{i,j=1}^{2} s_b\left(i\frac{Q}{2}+(u_i^{(2)}-u_j^{(2)})-2m_A\right)}{s_b\left(i\frac{Q}{2}\pm(u_1^{(2)}-u_2^{(2)})\right)}\times\\
\times\;& \prod_{i=1}^{2} s_b\left(\pm(u_i^{(2)}-z_3)+\Delta-m_A\right)s_b\left(iQ\pm(u_i^{(2)}-z_2)-\Delta-m_A\right)\times\\
\times\;& \prod_{a=1}^{3} s_b\left(\pm(u_i^{(2)}-u_a^{(3)})+m_A\right)s_b\left(i\frac{Q}{2}-2m_A\right)\int \mathrm{d}u^{(1)}\, s_b\left(-i\frac{Q}{2}\pm(u^{(1)}-z_2)+\Delta\right)\times\\
\times\;& s_b\left(\frac{3}{2}iQ\pm(u^{(1)}-z_1)-\Delta-2m_A\right)\prod_{i=1}^{2} s_b\left(\pm(u^{(1)}-u_i^{(2)})+m_A\right)\,. \qquad \text{(B.1)}
\end{aligned}$$

We first want to apply Aharony duality to the $U(3)$ integral

$$I_3 = \int \frac{du_1^{(3)}\,du_2^{(3)}\,du_3^{(3)}}{3!} e^{2\pi i\zeta\sum_a u_a^{(3)}} \frac{\prod_{a=1}^3 s_b\left(i\frac{Q}{2} \pm u_a^{(3)} - \mu\right)}{\prod_{a<b}^3 s_b\left(i\frac{Q}{2} \pm (u_a^{(3)} - u_b^{(3)})\right)} \times$$
$$\times s_b\left(i\frac{Q}{2} \pm (u^{(3)} - z_3) - \Delta\right) \prod_{i=1}^2 s_b\left(\pm(u_i^{(2)} - u_a^{(3)}) + m_A\right) . \tag{B.2}$$

Using (2.69), we can rewrite it as a one-dimensional integral

$$\begin{aligned} I_3 &= e^{2\pi i\zeta(\sum_i u_i^{(2)}+z_3)} s_b\left(i\frac{Q}{2} \pm \zeta + \mu + \Delta - 2m_A\right) s_b\left(i\frac{Q}{2} - 2\mu\right) s_b\left(i\frac{Q}{2} - 2\Delta\right) \times \\ &\times s_b\left(i\frac{Q}{2} \pm z_3 - \mu - \Delta\right) \prod_{i=1}^2 s_b\left(\pm(u_i^{(2)} - z_3) - \Delta + m_A\right) s_b\left(\pm u_i^{(2)} - \mu + m_A\right) \times \\ &\times \prod_{i,j=1}^2 s_b\left(-i\frac{Q}{2} + (u_i^{(2)} - u_j^{(2)}) + 2m_A\right) \int dv^{(1)}\, e^{-2\pi i\zeta v^{(1)}} s_b\left(\pm v^{(1)} + \mu\right) \times \\ &\times s_b\left(\pm(v^{(1)} - z_3) + \Delta\right) \prod_{i=1}^2 s_b\left(i\frac{Q}{2} \pm (v^{(1)} - u_i^{(2)}) - m_A\right) . \end{aligned} \tag{B.3}$$

Notice the contact term between the topological fugacity ζ and the real masses $u_i^{(2)}$ for the $U(2)$ gauge symmetry. When we plug this back into the partition function (B.2), this has the effect of introducing an FI contribution in the $U(2)$ integral that was not present before because of the monopole superpotential term $\mathfrak{M}^{(0,\pm1,0)}$ that breaks the topological symmetry at this node. This means that applying Aharony duality we restored the topological symmetry of the $U(2)$ node and, since the corresponding monopole operators are charged under this symmetry, they can't be in the superpotential anymore. Moreover, the FI parameters of the $du^{(2)}$ and the $dv^{(1)}$ integral are opposite, which is compatible with a monopole superpotential term of the form $\mathfrak{M}^{(0,\pm1,\pm1)}$ that breaks the two topological symmetries of the corresponding gauge nodes to the anti-diagonal combination $U(1)_\zeta$. If we also use the property of the double-sine functions

$$s_b(x)\, s_b(-x) = 1\,, \tag{B.4}$$

which is the analogue from the point of view of partition functions of the fact that some fields have become massive and are integrated out, we see that plugging (B.2) into (B.1) many of the contributions cancel and we get

$$\begin{aligned} \mathcal{Z} &= e^{2\pi i\zeta z_3} s_b\left(i\frac{Q}{2} \pm z_3 - \mu - \Delta\right) s_b\left(i\frac{Q}{2} \pm \zeta + \mu + \Delta - 2m_A\right) s_b\left(i\frac{Q}{2} - 2\mu\right) \times \\ &\times s_b\left(i\frac{Q}{2} - 2\Delta\right) \int \frac{du_1^{(2)}\,du_2^{(2)}}{2} e^{2\pi i\zeta\sum_i u_i^{(2)}} \frac{\prod_{i=1}^2 s_b\left(\pm u_i^{(2)} - \mu + m_A\right)}{s_b\left(i\frac{Q}{2} \pm (u_1^{(2)} - u_2^{(2)})\right)} \times \\ &\times s_b\left(iQ \pm (u_i^{(2)} - z_2) - \Delta - m_A\right) s_b\left(i\frac{Q}{2} - 2m_A\right) \times \end{aligned}$$

$$\begin{aligned}
&\times \int \mathrm{d}u^{(1)}\, s_b\left(-i\frac{Q}{2} \pm (u^{(1)} - z_2) + \Delta\right) s_b\left(\frac{3}{2}iQ \pm (u^{(1)} - z_1) - \Delta - 2m_A\right) \times \\
&\times \prod_{i=1}^{2} s_b\left(\pm(u^{(1)} - u_i^{(2)}) + m_A\right) \int \mathrm{d}v^{(1)}\, \mathrm{e}^{-2\pi i\zeta v^{(1)}} \prod_{i=1}^{2} s_b\left(i\frac{Q}{2} \pm (v^{(1)} - u_i^{(2)})\right) \times \\
&\times s_b\left(\pm v^{(1)} + \mu\right) s_b\left(\pm(v^{(1)} - z_3) + \Delta\right) . \qquad \text{(B.5)}
\end{aligned}$$

If we reintroduce the contribution of the flipping fields β_i

$$\beta_i \;\to\; \prod_{i=1}^{3} s_b\left(-i\frac{Q}{2} + 2\Delta - 2(i-3)\left(i\frac{Q}{2} - m_A\right)\right) \qquad \text{(B.6)}$$

on both sides of the identity we found, we recover (2.160) in the case $N = 3$ and $k = 1$. Indeed, we can reconstruct the prefactor Λ_1^3 as well as the partition functions of the $G[U(2)]$ and $G[U(1)]$ glued together.

Since the contribution of the adjoint chiral canceled and since we have restored the FI contribution, we are allowed to apply (2.69) on the $U(2)$ integral

$$\begin{aligned}
I_2 \;=\; &\int \frac{\mathrm{d}u_1^{(2)}\,\mathrm{d}u_2^{(2)}}{2} \mathrm{e}^{2\pi i\zeta\sum_i u_i^{(2)}} \frac{\prod_{i=1}^{2} s_b\left(\pm u_i^{(2)} - \mu + m_A\right)}{s_b\left(i\frac{Q}{2} \pm (u_1^{(2)} - u_2^{(2)})\right)} \times \\
\times\;& s_b\left(iQ \pm (u_i^{(2)} - z_2) - \Delta - m_A\right) s_b\left(\pm(u^{(1)} - u_i^{(2)}) + m_A\right) s_b\left(i\frac{Q}{2} \pm (v^{(1)} - u_i^{(2)})\right) . \qquad \text{(B.7)}
\end{aligned}$$

Doing so, we don't replace it with a lower dimensional one as in the previous iteration, but with another two-dimensional integral. This is due to the fact that we reached the configuration with minimal rank and that N is odd in this case

$$\begin{aligned}
I_2 \;=\; &\mathrm{e}^{2\pi i\zeta(z_2+u^{(1)}+v^{(1)})} s_b\left(-i\frac{Q}{2} \pm \zeta + \mu + \Delta\right) s_b\left(-i\frac{Q}{2} - 2\mu + 2m_A\right) \times \\
\times\;& s_b\left(\frac{3}{2}iQ - 2\Delta - 2m_A\right) s_b\left(i\frac{Q}{2} \pm z_2 - \mu - \Delta\right) s_b\left(-i\frac{Q}{2} \pm u^{(1)} - \mu + 2m_A\right) \times \\
\times\;& s_b\left(i\frac{Q}{2} \pm (u^{(1)} - z_2) - \Delta\right) s_b\left(\pm v^{(1)} - \mu\right) s_b\left(iQ \pm (v^{(1)} - z_2) - \Delta - 2m_A\right) \times \\
\times\;& \int \frac{\mathrm{d}v_1^{(2)}\,\mathrm{d}v_2^{(2)}}{2} \mathrm{e}^{-2\pi i\zeta\sum_i v_i^{(2)}} \frac{\prod_{i=1}^{2} s_b\left(i\frac{Q}{2} \pm v^{(2)} + \mu - m_A\right) s_b\left(i\frac{Q}{2} \pm (v_i^{(2)} - u^{(1)}) - m_A\right)}{s_b\left(i\frac{Q}{2} \pm (v_1^{(2)} - v_2^{(2)})\right)} \times \\
\times\;& s_b\left(-i\frac{Q}{2} \pm (v_i^{(2)} - z_2) + \Delta + m_A\right) s_b\left(\pm(v_i^{(2)} - v^{(1)} + m_A)\right) . \qquad \text{(B.8)}
\end{aligned}$$

The contact term has the effect of removing the FI contribution from the $v^{(1)}$ integral and of producing one in the $u^{(1)}$ integral. This means that we broke the topological symmetry on the right $U(1)$ node and turned on a monopole superpotential for it, while we did the opposite on the left $U(1)$ node. Moreover, the FI parameters of the $U(2)$ node and of the left $U(1)$ node are opposite, meaning that a monopole superpotential of the form $\mathfrak{M}^{(\pm1,\pm1,0)}$ is turned on. Plugging (B.8) into (B.5) and simplifying the contributions of the massive fields, we get

$$\begin{aligned}
\mathcal{Z}_3 \;&=\; \Lambda_2^3(m_A,\Delta,\zeta,\mu)e^{2\pi i\zeta(z_2+z_3)}\prod_{n=2}^{3}s_b\left(i\frac{Q}{2}\pm z_n-\mu-\Delta\right)\times\\
&\times\;\int du^{(1)}\,e^{2\pi i\zeta u^{(1)}}s_b\left(-i\frac{Q}{2}\pm u^{(1)}-\mu+2m_A\right)s_b\left(\frac{3}{2}iQ\pm(u^{(1)}-z_1)-\Delta-2m_A\right)\times\\
&\times\;\int\frac{dv_1^{(2)}\,dv_2^{(2)}}{2}e^{-2\pi i\zeta\sum_i v_i^{(2)}}\frac{\prod_{i=1}^{2}s_b\left(i\frac{Q}{2}\pm v^{(2)}+\mu-m_A\right)}{s_b\left(i\frac{Q}{2}\pm(v_1^{(2)}-v_2^{(2)})\right)}\times\\
&\times\;s_b\left(i\frac{Q}{2}\pm(v_i^{(2)}-u^{(1)})-m_A\right)s_b\left(-i\frac{Q}{2}\pm(v_i^{(2)}-z_2)+\Delta+m_A\right)\times\\
&\times\;s_b\left(i\frac{Q}{2}-2m_A\right)\int dv^{(1)}\,s_b\left(iQ\pm(v^{(1)}-z_2)-\Delta-2m_A\right)\times\\
&\times\;s_b\left(\pm(v^{(1)}-z_3)+\Delta\right)\prod_{i=1}^{2}s_b\left(\pm(v_i^{(2)}-v^{(1)}+m_A)\right),
\end{aligned}\tag{B.9}$$

where

$$\begin{aligned}
\Lambda_2^3(m_A,\Delta,\zeta,\mu)\;&=\;s_b\left(i\frac{Q}{2}\pm\zeta+\mu+\Delta-2m_A\right)s_b\left(-i\frac{Q}{2}\pm\zeta+\mu+\Delta\right)\times\\
&\times\;s_b\left(i\frac{Q}{2}-2\mu\right)s_b\left(-i\frac{Q}{2}-2\mu+2m_A\right)\times\\
&\times\;s_b\left(i\frac{Q}{2}-2\Delta\right)s_b\left(\frac{3}{2}iQ-2\Delta-2m_A\right).
\end{aligned}\tag{B.10}$$

If we reintroduce the contribution of the flipping fields β_i on both sides, we recover (2.160) in the case $N=3$ and $k=2$.

Finally, we can apply (2.69) on the $u^{(1)}$ integral

$$\begin{aligned}
I_1\;&=\;\int du^{(1)}\,e^{2\pi i\zeta u^{(1)}}s_b\left(-i\frac{Q}{2}\pm u^{(1)}-\mu+2m_A\right)\times\\
&\times\;s_b\left(\frac{3}{2}iQ\pm(u^{(1)}-z_1)-\Delta-2m_A\right)\prod_{i=1}^{2}s_b\left(i\frac{Q}{2}\pm(v_i^{(2)}-u^{(1)})-m_A\right).
\end{aligned}\tag{B.11}$$

Since we passed the configuration of minimal rank, we get a three-dimensional integral

$$\begin{aligned}
I_1\;&=\;e^{2\pi i\zeta(z_3+\sum_i v_i^{(2)})}s_b\left(-\frac{3}{2}iQ\pm\zeta+\mu+\Delta+2m_A\right)s_b\left(-\frac{3}{2}iQ-2\mu+4m_A\right)\times\\
&\times\;s_b\left(\frac{5}{2}iQ-2\Delta-4m_A\right)s_b\left(i\frac{Q}{2}\pm z_1-\mu-\Delta\right)\prod_{i,j=1}^{2}s_b\left(i\frac{Q}{2}+(v_i^{(2)}-v_j^{(2)})-2m_A\right)\times\\
&\times\;\prod_{i=1}^{2}s_b\left(-i\frac{Q}{2}\pm v_i^{(2)}-\mu+m_A\right)s_b\left(\frac{3}{2}iQ\pm(v_i^{(2)}-z_1)-\Delta-3m_A\right)\times
\end{aligned}$$

$$\times \quad \int \frac{\mathrm{d}v_1^{(3)}\,\mathrm{d}v_2^{(3)}\,\mathrm{d}v_3^{(3)}}{3!} \mathrm{e}^{-2\pi i\zeta\sum_a v_a^{(3)}} \frac{\prod_{a=1}^3 s_b\left(iQ \pm v_a^{(3)} + \mu - 2m_A\right)}{\prod_{a<b}^3 s_b\left(i\frac{Q}{2} \pm (v_a^{(3)} - v_b^{(3)})\right)} \times$$
$$\times \quad \prod_{i=1}^{2} s_b\left(\pm(v_a^{(3)} - v_i^{(2)}) + m_A\right) s_b\left(-iQ \pm (v_a^{(3)} - z_1) + \Delta + 2m_A\right) . \qquad \text{(B.12)}$$

If we substitute this into (B.9), we finally arrive at

$$\begin{aligned}
\mathcal{Z}_3 \;=\; & \Lambda_3^3(m_A,\Delta,\zeta,\mu) \prod_{n=1}^{3} \mathrm{e}^{2\pi i\zeta z_n} s_b\left(i\frac{Q}{2} \pm z_n - \mu - \Delta\right) \times \\
\times\; & \int \frac{\mathrm{d}v_1^{(3)}\,\mathrm{d}v_2^{(3)}\,\mathrm{d}v_3^{(3)}}{3!} \mathrm{e}^{-2\pi i\zeta\sum_a v_a^{(3)}} \frac{\prod_{a=1}^3 s_b\left(iQ \pm v_a^{(3)} + \mu - 2m_A\right)}{\prod_{a<b}^3 s_b\left(i\frac{Q}{2} \pm (v_a^{(3)} - v_b^{(3)})\right)} \times \\
\times\; & s_b\left(-iQ \pm (v_a^{(3)} - z_1) + \Delta + 2m_A\right) \times \\
\times\; & \int \frac{\mathrm{d}v_1^{(2)}\,\mathrm{d}v_2^{(2)}}{2} \frac{\prod_{i,j=1}^2 s_b\left(i\frac{Q}{2} + (v_i^{(2)} - v_j^{(2)}) - 2m_A\right)}{s_b\left(i\frac{Q}{2} \pm (v_1^{(2)} - v_2^{(2)})\right)} \times \\
\times\; & \prod_{i=1}^{2} s_b\left(\frac{3}{2}iQ \pm (v_i^{(2)} - z_1) - \Delta - 3m_A\right) s_b\left(-i\frac{Q}{2} \pm (v_i^{(2)} - z_2) + \Delta + m_A\right) \times \\
\times\; & \prod_{a=1}^{3} s_b\left(\pm(v_a^{(3)} - v_i^{(2)}) + m_A\right) \times \\
\times\; & s_b\left(i\frac{Q}{2} - 2m_A\right) \int \mathrm{d}v^{(1)}\, s_b\left(iQ \pm (v^{(1)} - z_2) - \Delta - 2m_A\right) \times \\
\times\; & s_b\left(\pm(v^{(1)} - z_3) + \Delta\right) \prod_{i=1}^{2} s_b\left(\pm(v^{(1)} - v_i^{(2)}) + m_A\right) , \qquad \text{(B.13)}
\end{aligned}$$

where

$$\begin{aligned}
\Lambda_3^3(m_A,\Delta,\zeta,\mu) \;=\; & \prod_{n=1}^{3} s_b\left(\pm\zeta + \mu + \Delta - m_A + (3-2n)\left(i\frac{Q}{2} - m_A\right)\right) \times \\
\times\; & s_b\left(i\frac{Q}{2} - 2\mu - 2(n-1)\left(i\frac{Q}{2} - m_A\right)\right) \times \\
\times\; & s_b\left(i\frac{Q}{2} - 2\Delta + 2(n-1)\left(i\frac{Q}{2} - m_A\right)\right) . \qquad \text{(B.14)}
\end{aligned}$$

If we reintroduce the contribution of the flipping fields β_i on both sides, we recover (2.160) in the case $N=3$ and $k=3$.

B.2 Derivation of the equality (2.181) for the rank stabilization duality for $k=1,2$

In this section we prove analytically the equality of the partition functions (2.181) for the rank stabilization duality for low number of flavors, namely $k=1,2$. This can be done through iterative applications of some basic dualities, which are the one-monopole duality and the Aharony duality we presented in Subsection 2.3.3. The derivation highly relies on a *stabilization* property of the theory, which holds for $k < N$. We say that the theory is *stable* if, after applying to it some of the fundamental dualities, we recover the same theory but with the rank decreased by one unit and possibly some modification in the parameters of the theory, such as the number of gauge singlets. In Subsection 2.3.3, we showed that the $U(N)$ theory with one adjoint and one fundamental flavor, which corresponds to the case $k = 0$, is stable and this allowed us to reduce it to a WZ model. We will see that for a higher number of flavors Theory A is not itself stable, but with some initial manipulations we can find a dual frame which actually is. From this point, one can significantly simplify the integrals using the stabilization property and get the partition function of the claimed dual.

Before starting, we quote here the identity for the$\mathbb{S}^3_b$ partition functions associated to the two-monopole duality we saw in Subsection 2.3.2, since we are going to need it in a couple of occasions:

$$\begin{aligned}
\mathcal{Z}_{\mathcal{T}_1} &= \frac{1}{N_c!}\int\prod_{i=1}^{N_c}\mathrm{d}x_i\,\frac{\prod_{i=1}^{N_c}\prod_{a=1}^{N_f}s_b\left(i\frac{Q}{2}\pm(x_i+M_a)-\mu_a\right)}{\prod_{i<j}^{N_c}s_b\left(i\frac{Q}{2}\pm(x_i-x_j)\right)} = \\
&= \frac{1}{(N_f-N_c-2)!}\prod_{a,b=1}^{N_f}s_b\left(i\frac{Q}{2}-(\mu_a+\mu_b-M_a+M_b)\right)\times \\
&\times\int\prod_{i=1}^{N_f-N_c-2}\mathrm{d}x_i\,\frac{\prod_{i=1}^{N_f-N_c-2}\prod_{a=1}^{N_f}s_b\left(\pm(x_i-M_a)+\mu_a\right)}{\prod_{i<j}^{N_f-N_c-2}s_b\left(i\frac{Q}{2}\pm(x_i-x_j)\right)} = \mathcal{Z}_{\mathcal{T}_2}\,, \qquad \text{(B.15)}
\end{aligned}$$

where M_a, μ_a are real masses corresponding to the Cartan subalgebra of the diagonal and the anti-diagonal combinations of the two $SU(N_f)$ flavor symmetries. Hence, the vector masses sum to zero $\sum M_a = 0$, while the axial masses have to satisfy the constraint

$$2\sum_{a=1}^{N_f}\mu_a = iQ(N_f-N_c-1)\,. \qquad \text{(B.16)}$$

This balancing conditions is a consequence of the monopole superpotential, which is responsible for breaking the $U(1)$ axial symmetry and the topological symmetry.

Two flavors

We start considering the partition function of the $k = 1$ case without the contribution of the b-fields, which we will add at the end for simplicity

$$\begin{aligned}
\mathcal{Z}^1_N(z,\tau,\zeta,\mu) &\equiv \frac{1}{N!}\int\prod_{\alpha=1}^{N}\mathrm{d}u_\alpha\,\mathrm{e}^{2\pi i\zeta\sum_\alpha u_\alpha}\frac{\prod_{\alpha,\beta=1}^{N}s_b\left(i\frac{Q}{2}+(u_\alpha-u_\beta)-2\tau\right)}{\prod_{\alpha<\beta}^{N}s_b\left(i\frac{Q}{2}\pm(u_\alpha-u_\beta)\right)}\times \\
&\times\prod_{\alpha=1}^{N}s_b\left(i\frac{Q}{2}\pm u_\alpha-\mu\right)s_b\left(\pm(u_\alpha-z)+\tau\right)\,. \qquad \text{(B.17)}
\end{aligned}$$

We use the same deconfinement technique we used in Subsection 2.3.3 for the case $k = 0$, that is we start by replacing the contribution of the adjoint chiral with an auxiliary $U(N-1)$

integral using the one-monopole duality (2.67)

$$\mathcal{Z}_N^1(z,\tau,\zeta,\mu) = s_b\left(i\frac{Q}{2} - 2N\tau\right)\frac{1}{(N-1)!}\int\prod_{\alpha'=1}^{N-1}\mathrm{d}w_{\alpha'}\,\frac{\mathrm{e}^{-2\pi i N\tau\sum_{\alpha'}w_{\alpha'}}}{\prod_{\alpha'<\beta'}^{N-1}s_b\left(i\frac{Q}{2}\pm(w_{\alpha'}-w_{\beta'})\right)}\times$$
$$\times\frac{1}{N!}\int\prod_{\alpha=1}^{N}\mathrm{d}u_\alpha\,\mathrm{e}^{2\pi i(\zeta-(N-1)\tau)\sum_\alpha u_\alpha}\frac{\prod_{\alpha=1}^{N}s_b\left(i\frac{Q}{2}\pm u_\alpha-\mu\right)s_b\left(\pm(u_\alpha-z)+\tau\right)}{\prod_{\alpha<\beta}^{N}s_b\left(i\frac{Q}{2}\pm(u_\alpha-u_\beta)\right)}\times$$
$$\times\prod_{\alpha=1}^{N}\prod_{\alpha'=1}^{N-1}s_b\left(i\frac{Q}{2}\pm(u_\alpha+w'_\alpha)-\tau\right). \tag{B.18}$$

This corresponds to the partition function of an auxiliary $U(N-1)\times U(N)$ quiver gauge theory with a single fundamental monopole turned on at the $U(N)$ node. Then, we apply the Aharony duality on the original integral. In contrast to the $k=0$ case, because of the extra flavor, the identity (2.69) is not an evaluation formula, but it actually yields a $U(1)$ integral

$$\mathcal{Z}_N^1(z,\tau,\zeta,\mu) = \mathrm{e}^{2\pi i(\zeta-(N-1)\tau)z}s_b\left(\pm z-\mu+\tau\right)s_b\left(i\frac{Q}{2}-2N\tau\right)\times$$
$$\times s_b\left(i\frac{Q}{2}-2\mu\right)s_b\left(-\zeta+\mu+(2N-3)\tau\right)s_b\left(\zeta+\mu-\tau\right)\times$$
$$\times s_b\left(-i\frac{Q}{2}+2\tau\right)\int\mathrm{d}u\,\mathrm{e}^{2\pi i(\zeta-(N-1)\tau)u}s_b\left(\pm u+\mu\right)s_b\left(i\frac{Q}{2}\pm(u+z)-\tau\right)\times$$
$$\times\frac{1}{(N-1)!}\int\prod_{\alpha=1}^{N-1}\mathrm{d}w_\alpha\,\mathrm{e}^{-2\pi i(\zeta+\tau)\sum_\alpha w_\alpha}\frac{\prod_{\alpha,\beta=1}^{N-1}s_b\left(i\frac{Q}{2}+(w_\alpha-w_\beta)-2\tau\right)}{\prod_{\alpha<\beta}^{N-1}s_b\left(i\frac{Q}{2}\pm(w_\alpha-w_\beta)\right)}\times$$
$$\times\prod_{\alpha=1}^{N-1}s_b\left(i\frac{Q}{2}\pm w_\alpha-\mu-\tau\right)s_b\left(\pm(w_\alpha-u)+\tau\right). \tag{B.19}$$

Notice that the contact terms predicted by Aharony duality had the effect of restoring the topological symmetry at the $U(N-1)$ node and thus of removing the monopole superpotential (see [24] for a more exhaustive discussion of this phenomenon).

From (B.19) we can also see that the original integral was not in a stabilized form since its structure has changed after the application of these two fundamental dualities. Nevertheless, after performing the change of variables $w_i \leftrightarrow -w_i$, we see that in (B.19) the last integral has the form of the original integral, but with shifted parameters, so we can still write an iterative relation:

$$\mathcal{Z}_N^1(z,\tau,\zeta,\mu) = \mathrm{e}^{2\pi i(\zeta-(N-1)\tau)z}s_b\left(\pm z-\mu+\tau\right)s_b\left(i\frac{Q}{2}-2N\tau\right)\times$$
$$\times s_b\left(i\frac{Q}{2}-2\mu\right)s_b\left(-\zeta+\mu+(2N-3)\tau\right)s_b\left(\zeta+\mu-\tau\right)s_b\left(-i\frac{Q}{2}+2\tau\right)\times$$
$$\times\int\mathrm{d}u\,\mathrm{e}^{2\pi i(\zeta-(N-1)\tau)u}s_b\left(\pm u+\mu\right)s_b\left(i\frac{Q}{2}\pm(u+z)-\tau\right)\mathcal{Z}_{N-1}^1(u,\tau,\zeta+\tau,\mu+\tau). \tag{B.20}$$

With this identity, we can show that the integral that is stabilized is actually (B.19). Indeed, if we repeat the two previous steps, that is we iterate (B.20), we produce a second $U(1)$

integral

$$\begin{aligned}\mathcal{Z}_N^1(z,\tau,\zeta,\mu) &= \mathrm{e}^{2\pi i(\zeta-(N-1)\tau)z} s_b\left(\pm z-\mu+\tau\right)\prod_{j=1}^{2} s_b\left(i\frac{Q}{2}-2(N-j+1)\tau\right)\times\\ &\times s_b\left(i\frac{Q}{2}-2\mu-2(j-1)\tau\right) s_b\left(-\zeta+\mu+(2N-2j-1)\tau\right) s_b\left(\zeta+\mu+(2j-3)\tau\right)\times\\ &\times s_b\left(-i\frac{Q}{2}+2\tau\right)\int \mathrm{d}w\, \mathrm{e}^{2\pi i(\zeta-(N-3)\tau)w} s_b\left(\pm w+\mu+\tau\right)\mathcal{Z}_{N-2}^1(w,\tau,\zeta+2\tau,\mu+2\tau)\times\\ &\times s_b\left(-i\frac{Q}{2}+2\tau\right)\int \mathrm{d}u\, \mathrm{e}^{-4\pi i\tau u} s_b\left(i\frac{Q}{2}\pm(u-w)-\tau\right) s_b\left(i\frac{Q}{2}\pm(u+z)-\tau\right),\end{aligned} \tag{B.21}$$

but the u-integral can now be evaluated applying the one-monopole duality (2.67) in the confining case:

$$\begin{aligned}\mathcal{Z}_N^1(z,\tau,\zeta,\mu) &= \mathrm{e}^{2\pi i(\zeta-(N-2)\tau)z} s_b\left(\pm z-\mu+\tau\right)\prod_{j=1}^{2} s_b\left(i\frac{Q}{2}-2(N-j+1)\tau\right)\times\\ &\times s_b\left(i\frac{Q}{2}-2\mu-2(j-1)\tau\right) s_b\left(-\zeta+\mu+(2N-2j-1)\tau\right) s_b\left(\zeta+\mu+(2j-3)\tau\right)\times\\ &\times s_b\left(-i\frac{Q}{2}+4\tau\right)\int \mathrm{d}w\, \mathrm{e}^{2\pi i(\zeta-(N-2)\tau)w} s_b\left(\pm w+\mu+\tau\right) s_b\left(i\frac{Q}{2}\pm(w+z)-2\tau\right)\times\\ &\times\mathcal{Z}_{N-2}^1(w,\tau,\zeta+2\tau,\mu+2\tau)\,.\end{aligned} \tag{B.22}$$

Hence, we recover precisely the same structure of (B.19), but with a lower rank, some extra gauge singlets and a shift of the parameters. In particular, the shift of the FI parameter indicates that the oppositely charged fundamental monopoles have different topological charge and that charge conjugation is broken in this frame. This explicitly shows that (B.19) was indeed stable under the sequential application of one-monopole and Aharony dualities.

We can use this stabilization property to significantly simplify the integral. If we iterate (B.20) and (2.67) n times, we get indeed

$$\begin{aligned}\mathcal{Z}_N^1(z,\tau,\zeta,\mu) &= \mathrm{e}^{2\pi i(\zeta-(N-n)\tau)z} s_b\left(\pm z-\mu+\tau\right)\prod_{j=1}^{n} s_b\left(i\frac{Q}{2}-2(N-j+1)\tau\right)\times\\ &\times s_b\left(-\zeta+\mu+(2N-2j-1)\tau\right) s_b\left(\zeta+\mu+(2j-3)\tau\right) s_b\left(i\frac{Q}{2}-2\mu-2(j-1)\tau\right)\times\\ &\times s_b\left(-i\frac{Q}{2}+2n\tau\right)\int \mathrm{d}u\, \mathrm{e}^{2\pi i(\zeta-(N-n)\tau)u} s_b\left(\pm u+\mu+(n-1)\tau\right) s_b\left(i\frac{Q}{2}\pm(u+z)-n\tau\right)\times\\ &\times\mathcal{Z}_{N-n}^1(u,\tau,\zeta+n\tau,\mu+n\tau)\,.\end{aligned} \tag{B.23}$$

In particular, if we set $n = N$ in the above expression, the original gauge node is completely confined

$$\begin{aligned}
&\mathcal{Z}^1_N(z,\tau,\zeta,\mu) = \mathrm{e}^{2\pi i\zeta z} s_b\left(\pm z-\mu+\tau\right)\prod_{j=1}^{N} s_b\left(i\frac{Q}{2}-2j\tau\right)\times\\
&\times s_b\left(-\zeta+\mu+(2N-2j-1)\tau\right)s_b\left(\zeta+\mu+(2j-3)\tau\right)s_b\left(i\frac{Q}{2}-2\mu-2(j-1)\tau\right)\times\\
&\times s_b\left(-i\frac{Q}{2}+2N\tau\right)\int \mathrm{d}u\, \mathrm{e}^{2\pi i\zeta u} s_b\left(\pm u+\mu+(N-1)\tau\right)s_b\left(i\frac{Q}{2}\pm(u+z)-N\tau\right).
\end{aligned} \tag{B.24}$$

Notice that the FI parameter of the remaining $U(1)$ node is no longer shifted. This means that the oppositely charged monopole operators have the same quantum numbers under all the global symmetries and that charge conjugation, which was broken in all the previous auxiliary dual frames, has been restored.

The partition function that we obtained is that of $G[U(1)]$ with some extra gauge singlets. In order to write the result in the desired form, we apply the Aharony duality to the $U(1)$ integral. This gives back another $U(1)$ integral, but with different parameters and some of the extra gauge singlets flipped away. Essentially, what we are doing is applying the recombination duality we discussed in Subsection 2.4.2 in the particular case $N = 1$ and $k = 1$. This operation might seem trivial in this case, but it occurs also for the cases of higher number of flavors k as we will see later for $k = 2$. If we also add back the contribution of the $N-1$ b-fields, the final result coincides with (2.181) for $k = 1$

$$\begin{aligned}
\mathcal{Z}_{\mathcal{T}_A} &= \prod_{j=1}^{N-1} s_b\left(-i\frac{Q}{2}+2j\tau\right)\mathcal{Z}^1_N(z,\tau,\zeta,\mu) =\\
&= s_b\left(i\frac{Q}{2}-2N\tau\right)\prod_{j=1}^{N-1} s_b\left(i\frac{Q}{2}-2\mu-2(j-1)\tau\right)\times\\
&\times\ s_b\left(-\zeta+\mu+(2N-2j-1)\tau\right)\prod_{j=2}^{N} s_b\left(\zeta+\mu+(2j-3)\tau\right)\times\\
&\times\ \int \mathrm{d}u\, \mathrm{e}^{2\pi i\zeta u} s_b\left(i\frac{Q}{2}\pm u-\mu-(N-1)\tau\right)s_b\left(\pm(u-z)+N\tau\right) =\\
&= \prod_{j=1}^{N-1} s_b\left(i\frac{Q}{2}-2\mu-2(j-1)\tau\right)\times\\
&\times\ s_b\left(-\zeta+\mu+(2N-2j-1)\tau\right)\prod_{j=2}^{N} s_b\left(\zeta+\mu+(2j-3)\tau\right)\times\\
&\times\ \mathcal{Z}_{G[U(1)]}(z;\zeta;\mu+(N-1)\tau;i\frac{Q}{2}-N\tau;i\frac{Q}{2}-\tau) = \mathcal{Z}_{\mathcal{T}_B}\,.
\end{aligned} \tag{B.25}$$

A useful integral identity

In order to write the matrix integral for the $k = 2$ case in a stable form, we will make use of the following integral identity[1]:

$$\begin{aligned}
Z &= \int \frac{\mathrm{d}u_1\,\mathrm{d}u_2}{2} \mathrm{e}^{-4\pi i\tau(u_1+u_2)} \frac{\prod_{\alpha,\beta=1}^2 s_b\left(-i\frac{Q}{2} + (u_\alpha - u_\beta) + 2\tau\right)}{s_b\left(i\frac{Q}{2} \pm (u_1 - u_2)\right)} \times \\
&\quad \times \prod_{\alpha,\beta=1}^{2} s_b\left(i\frac{Q}{2} \pm (u_\alpha - w_\beta) - \tau\right) s_b\left(i\frac{Q}{2} \pm (u_\alpha - z_\beta) - \tau\right) = \\
&= \mathrm{e}^{-2\pi i\tau(z_1+z_2+w_1+w_2)} s_b\left(-i\frac{Q}{2} + 4\tau\right) s_b\left(-\frac{3}{2}iQ + 6\tau\right) s_b\left(i\frac{Q}{2} - 2\tau\right) \times \\
&\quad \times \prod_{\alpha,\beta=1}^{2} s_b\left(i\frac{Q}{2} \pm (w_\alpha - w_\beta) - 2\tau\right) \prod_{\alpha=1}^{2} s_b\left(i\frac{Q}{2} \pm (w_\alpha - z_1) - 2\tau\right) \times \\
&\quad \times \int \mathrm{d}u\, s_b\left(\pm(u + z_1) + \tau\right) s_b\left(iQ \pm (u + z_2) - 3\tau\right) \prod_{\alpha=1}^{2} s_b\left(\pm(u + w_\alpha) + \tau\right) .
\end{aligned} \tag{B.26}$$

This identity can be proven as follows. We first apply the identity (B.15) of the $\mathbb{S}_b^3$ partition functions for the two-monopole duality we saw in Subsection 2.3.2 in the particular confining case $N_c = 1$ and $N_f = 3$. This identity can also be written in the form of a star-triangle relation also known as *ultimate penthagon identity* [171, 197]

$$\begin{aligned}
&\int \mathrm{d}s\, D_{p_1}(s - z_1) D_{p_2}(s - z_2) D_{p_3}(s - z_3) = \\
&\qquad = \prod_{i=1}^{3} s_b(p_i - p'_i)\, D_{p'_3}(z_1 - z_2) D_{p'_2}(z_1 - z_3) D_{p'_1}(z_2 - z_3) \,,
\end{aligned} \tag{B.27}$$

where we defined

$$D_\alpha(x) = s_b\left(i\frac{Q}{2} + \alpha + x\right) s_b\left(i\frac{Q}{2} + \alpha - x\right) \tag{B.28}$$

and the parameters on the two sides of the identity are related by

$$p'_i = -i\frac{Q}{2} - p_i \,. \tag{B.29}$$

Moreover, the identity (B.27) holds provided that the following condition should be satisfied[2]

$$\sum_i p_i = -iQ/2 \quad \Leftrightarrow \quad \sum_i p'_i = -iQ \,. \tag{B.30}$$

[1]It would be interesting to interpret this identity as well as similar ones, whose 2d versions appear in the CFT literature, as dualities for theories with monopole superpotential and both adjoint and fundamental matter. We leave this for future investigations.

[2]This is the analogue of the balancing condition (2.56) due to the two-monopole superpotential, but at the level of the $\mathbb{S}_b^3$ partition function.

The idea is to use it to rewrite the following combination of double-sine functions

$$\mathcal{B} = s_b\left(-i\frac{Q}{2} \pm (u_1 - u_2) + 2\tau\right) s_b\left(i\frac{Q}{2} \pm (u_1 - z_2) - \tau\right) s_b\left(i\frac{Q}{2} \pm (u_2 - z_2) - \tau\right) . \tag{B.31}$$

One can indeed verify that the constraint (B.30) is satisfied for this choice. In this way, the contribution of the adjoint chiral Φ disappears, but at the price of introducing an additional $U(1)$ integral

$$\begin{aligned} Z \;&=\; s_b\left(-\frac{3}{2}iQ + 4\tau\right) \int \mathrm{d}s\, s_b\left(iQ \pm (s - z_2) - 2\tau\right) \int \frac{\mathrm{d}u_1\,\mathrm{d}u_2}{2} \mathrm{e}^{-4\pi i\tau(u_1+u_2)} \times \\ &\times \; \frac{\prod_{\alpha=1}^{2} s_b\left(i\frac{Q}{2} \pm (u_\alpha - z_1) - \tau\right) s_b\left(\pm(u_\alpha - s) + \tau\right) \prod_{\beta=1}^{2} s_b\left(i\frac{Q}{2} \pm (u_\alpha - w_\beta) - \tau\right)}{s_b\left(i\frac{Q}{2} \pm (u_1 - u_2)\right)} . \end{aligned} \tag{B.32}$$

Now we can replace the original integral with a lower dimensional one applying the one-monopole duality (2.67). This gives

$$\begin{aligned} Z \;&=\; \mathrm{e}^{-2\pi i\tau(z_1+w_1+w_2)} s_b\left(-\frac{3}{2}iQ + 4\tau\right) s_b\left(-i\frac{Q}{2} + 4\tau\right) \times \\ &\times \prod_{\alpha,\beta=1}^{2} s_b\left(i\frac{Q}{2} + (w_\alpha - w_\beta) - 2\tau\right) \prod_{\alpha=1}^{2} s_b\left(i\frac{Q}{2} \pm (w_\alpha - z_1) - 2\tau\right) \times \\ &\times \int \mathrm{d}u\, \mathrm{e}^{i\pi(iQ-4\tau)u} s_b\left(\pm(u + z_1) + \tau\right) \prod_{\alpha=1}^{2} s_b\left(\pm(u + w_\alpha) + \tau\right) \times \\ &\times \int \mathrm{d}s\, \mathrm{e}^{i\pi(iQ-6\tau)s} s_b\left(iQ \pm (s - z_2) - 2\tau\right) s_b\left(i\frac{Q}{2} \pm (u + s) - \tau\right) . \end{aligned} \tag{B.33}$$

Finally, we can use again the one-monopole duality (2.67) to get rid of the auxiliary ds integral since in this case it becomes an evaluation formula and obtain the desired result.

Three flavors

Again, we start considering the partition function of Theory A in the $k = 2$ case without the contribution of the b-fields

$$\begin{aligned} \mathcal{Z}_N^2(z_a, \tau, \zeta, \mu) \;&\equiv\; \frac{1}{N!} \int \prod_{\alpha=1}^{N} \mathrm{d}u_\alpha\, \mathrm{e}^{2\pi i\zeta \sum_\alpha u_\alpha} \frac{\prod_{\alpha,\beta=1}^{N} s_b\left(i\frac{Q}{2} + (u_\alpha - u_\beta) - 2\tau\right)}{\prod_{\alpha<\beta}^{N} s_b\left(i\frac{Q}{2} \pm (u_\alpha - u_\beta)\right)} \times \\ &\times \prod_{\alpha=1}^{N} s_b\left(i\frac{Q}{2} \pm u_\alpha - \mu\right) \prod_{a=1}^{2} s_b\left(\pm(u_\alpha - z_a) + \tau\right) . \end{aligned} \tag{B.34}$$

The first manipulations are still the same, that is we use the one-monopole duality (2.67) to deconfine the adjoint chiral and replace its contribution with an auxiliary $U(N-1)$ integral

$$\begin{aligned}\mathcal{Z}_N^2(z_a,\tau,\zeta,\mu) \;=\; & s_b\left(i\frac{Q}{2}-2N\tau\right)\frac{1}{(N-1)!}\int\prod_{\alpha'=1}^{N-1}\mathrm{d}w_{\alpha'}\,\frac{\mathrm{e}^{-2\pi iN\tau\sum_{\alpha'}w_{\alpha'}}}{\prod_{\alpha'<\beta'}^{N-1}s_b\left(i\frac{Q}{2}\pm(w_{\alpha'}-w_{\beta'})\right)}\times\\ &\times\frac{1}{N!}\int\prod_{\alpha=1}^{N}\mathrm{d}u_\alpha\,\mathrm{e}^{2\pi i(\zeta-(N-1)\tau)\sum_\alpha u_\alpha}\frac{\prod_{\alpha=1}^N s_b\left(i\frac{Q}{2}\pm u_\alpha-\mu\right)}{\prod_{\alpha<\beta}^N s_b\left(i\frac{Q}{2}\pm(u_\alpha-u_\beta)\right)}\times\\ &\times\prod_{a=1}^{2}s_b\left(\pm(u_\alpha-z_a)+\tau\right)\prod_{\alpha'=1}^{N-1}s_b\left(i\frac{Q}{2}\pm(u_\alpha+w_{\alpha'})-\tau\right)\end{aligned}\tag{B.35}$$

Then, we reduce again the rank of the original integral using Aharony duality (2.69)

$$\begin{aligned}&\mathcal{Z}_N^2(z_a,\tau,\zeta,\mu)=\mathrm{e}^{2\pi i(\zeta-(N-1)\tau)\sum_a z_a}\prod_{a,b=1}^{2}s_b\left(-i\frac{Q}{2}+(z_a-z_b)+2\tau\right)\times\\ &\times\prod_{a=1}^{2}s_b\left(\pm z_a-\mu+\tau\right)s_b\left(i\frac{Q}{2}-2N\tau\right)s_b\left(i\frac{Q}{2}-2\mu\right)\times\\ &\times s_b\left(-\zeta+\mu+2(N-2)\tau\right)s_b\left(\zeta+\mu-2\tau\right)\int\frac{\mathrm{d}u_1\,\mathrm{d}u_2}{2}\mathrm{e}^{2\pi i(\zeta-(N-1)\tau)\sum_a u_a}\times\\ &\times\frac{\prod_{a=1}^2 s_b\left(\pm u_a+\mu\right)\prod_{b=1}^2 s_b\left(i\frac{Q}{2}\pm(u_a+z_b)-\tau\right)}{s_b\left(i\frac{Q}{2}\pm(u_1-u_2)\right)}\times\\ &\times\frac{1}{(N-1)!}\int\prod_{\alpha=1}^{N-1}\mathrm{d}w_\alpha\,\mathrm{e}^{-2\pi i(\zeta+\tau)\sum_\alpha w_\alpha}\frac{\prod_{\alpha,\beta=1}^{N-1}s_b\left(i\frac{Q}{2}\pm(w_\alpha-w_\beta)-2\tau\right)}{\prod_{\alpha<\beta}^{N-1}s_b\left(i\frac{Q}{2}\pm(w_\alpha-w_\beta)\right)}\times\\ &\times\prod_{\alpha=1}^{N-1}s_b\left(i\frac{Q}{2}\pm w_\alpha-\mu-\tau\right)\prod_{a=1}^{2}s_b\left(\pm(w_\alpha-u_a)+\tau\right)\,.\end{aligned}\tag{B.36}$$

In the case $k=1$ that we considered in the previous section, it was at this point that we reached the stable form of the integral. This is not true anymore and we actually need some extra work to get the stable integral. Indeed, we can still recognize in the last integral of (B.36) the same original structure and this allows us to write the iterative relation

$$\begin{aligned}&\mathcal{Z}_N^2(z_a,\tau,\zeta,\mu)=\mathrm{e}^{2\pi i(\zeta-(N-1)\tau)\sum_a z_a}\prod_{a,b=1}^{2}s_b\left(-i\frac{Q}{2}+(z_a-z_b)+2\tau\right)\times\\ &\times\prod_{a=1}^{2}s_b\left(\pm z_a-\mu+\tau\right)s_b\left(i\frac{Q}{2}-2N\tau\right)s_b\left(i\frac{Q}{2}-2\mu\right)\times\\ &\times s_b\left(-\zeta+\mu+2(N-2)\tau\right)s_b\left(\zeta+\mu-2\tau\right)\int\frac{\mathrm{d}u_1\,\mathrm{d}u_2}{2}\mathrm{e}^{2\pi i(\zeta-(N-1)\tau)\sum_a u_a}\times\\ &\times\frac{\prod_{a=1}^2 s_b\left(\pm u_a+\mu\right)\prod_{b=1}^2 s_b\left(i\frac{Q}{2}\pm(u_a+z_b)-\tau\right)}{s_b\left(i\frac{Q}{2}\pm(u_1-u_2)\right)}\mathcal{Z}_{N-1}^2(u_a,\tau,\zeta+\tau,\mu+\tau)\,,\end{aligned}\tag{B.37}$$

but if we iterate this identity once we get

$$\begin{aligned}\mathcal{Z}_N^2(z_a,\tau,\zeta,\mu) &= \mathrm{e}^{2\pi i(\zeta-(N-1)\tau)\sum_a z_a}\prod_{a,b=1}^2 s_b\left(-i\frac{Q}{2}+(z_a-z_b)+2\tau\right)\times\\ &\times\prod_{a=1}^2 s_b\left(\pm z_a-\mu+\tau\right)\prod_{j=1}^2 s_b\left(i\frac{Q}{2}-2(N-j+1)\tau\right)\times\\ &\times s_b\left(i\frac{Q}{2}-2\mu-2(j-1)\tau\right)s_b\left(-\zeta+\mu+2(N-j-1)\tau\right)s_b\left(\zeta+\mu+2(j-2)\tau\right)\times\\ &\times\int\frac{\mathrm{d}w_1\,\mathrm{d}w_2}{2}\mathrm{e}^{2\pi i(\zeta-(N-3)\tau)\sum_a w_a}\frac{\prod_{a=1}^2 s_b\left(\pm w_a+\mu+\tau\right)}{s_b\left(i\frac{Q}{2}\pm(w_1-w_2)\right)}\mathcal{Z}_{N-2}^2(w_a,\tau,\zeta+2\tau,\mu+2\mu)\times\\ &\times\int\frac{\mathrm{d}u_1\,\mathrm{d}u_2}{2}\mathrm{e}^{-4\pi i\tau\sum_a u_a}\frac{\prod_{a,b=1}^2 s_b\left(-i\frac{Q}{2}\pm(u_a-u_b)+2\tau\right)}{s_b\left(i\frac{Q}{2}\pm(u_1-u_2)\right)}\times\\ &\times\prod_{a,b=1}^2 s_b\left(i\frac{Q}{2}\pm(u_a-w_b)-\tau\right)s_b\left(i\frac{Q}{2}\pm(u_a+z_b)-\tau\right)\,,\end{aligned}\tag{B.38}$$

and now there is no evaluation formula for any of the two $U(2)$ integrals which allows us to get back to an integral of the form of (B.37). This shows that the integral is not stable yet. Instead, we can at this point apply the basic identity we proved in the previous subsection (B.26)

$$\begin{aligned}\mathcal{Z}_N^2(z_a,\tau,\zeta,\mu) &= \mathrm{e}^{2\pi i(\zeta-(N-2)\tau)\sum_a z_a}\prod_{a,b=1}^2 s_b\left(-i\frac{Q}{2}+(z_a-z_b)+2\tau\right)\times\\ &\times\prod_{a=1}^2 s_b\left(\pm z_a-\mu+\tau\right)\prod_{j=1}^2 s_b\left(i\frac{Q}{2}-2(N-j+1)\tau\right)\times\\ &\times s_b\left(i\frac{Q}{2}-2\mu-2(j-1)\tau\right)s_b\left(-\zeta+\mu+2(N-j-1)\tau\right)s_b\left(\zeta+\mu+2(j-2)\tau\right)\times\\ &\times s_b\left(-i\frac{Q}{2}+4\tau\right)s_b\left(-\frac{3}{2}iQ+6\tau\right)s_b\left(i\frac{Q}{2}-2\tau\right)\times\\ &\times\int\frac{\mathrm{d}w_1\,\mathrm{d}w_2}{2}\mathrm{e}^{2\pi i(\zeta-(N-2)\tau)\sum_a w_a}\frac{\prod_{a,b=1}^2 s_b\left(i\frac{Q}{2}+(w_a-w_b)-2\tau\right)}{s_b\left(i\frac{Q}{2}\pm(w_1-w_2)\right)}\times\\ &\times\prod_{a=1}^2 s_b\left(\pm w_a+\mu+\tau\right)s_b\left(i\frac{Q}{2}\pm(w_a-z_1)-2\tau\right)\mathcal{Z}_{N-2}^2(w_a,\tau,\zeta+2\tau,\mu+2\tau)\times\\ &\times\int\mathrm{d}u\,s_b\left(\pm(u+z_1)+\tau\right)s_b\left(iQ\pm(u+z_2)-3\tau\right)\prod_{a=1}^2 s_b\left(\pm(u+w_a)+\tau\right)\,.\end{aligned}\tag{B.39}$$

This is the integral that is actually stable. To see this, we apply again (B.37)

$$\mathcal{Z}_N^2(z_a,\tau,\zeta,\mu) = \mathrm{e}^{2\pi i(\zeta-(N-2)\tau)\sum_a z_a} \prod_{a,b=1}^{2} s_b\left(-i\frac{Q}{2}+(z_a-z_b)+2\tau\right)\prod_{a=1}^{2} s_b\left(\pm z_a-\mu+\tau\right)\times$$
$$\times\prod_{j=1}^{3} s_b\left(i\frac{Q}{2}-2(N-j+1)\tau\right) s_b\left(i\frac{Q}{2}-2\mu-2(j-1)\tau\right) s_b\left(-\zeta+\mu+2(N-j-1)\tau\right)\times$$
$$\times s_b\left(\zeta+\mu+2(j-2)\tau\right) s_b\left(-i\frac{Q}{2}+4\tau\right) s_b\left(-\frac{3}{2}iQ+6\tau\right) s_b\left(i\frac{Q}{2}-2\tau\right)\times$$
$$\times\int\frac{\mathrm{d}u_1\,\mathrm{d}u_2}{2}\mathrm{e}^{2\pi i(\zeta-(N-5)\tau)\sum_a u_a}\frac{\prod_{a=1}^{2} s_b\left(\pm u_a+\mu+2\tau\right)}{s_b\left(i\frac{Q}{2}\pm(u_1-u_2)\right)}\mathcal{Z}_{N-3}^2(u_a,\tau,\zeta+3\tau,\mu+3\tau)\times$$
$$\times\int\mathrm{d}u\, s_b\left(\pm(u-z_1)+\tau\right) s_b\left(iQ\pm(u-z_2)-3\tau\right)\times$$
$$\times\int\frac{\mathrm{d}w_1\,\mathrm{d}w_2}{2}\mathrm{e}^{-6\pi i\tau\sum_a w_a}\frac{\prod_{a=1}^{2} s_b\left(i\frac{Q}{2}\pm(w_a+z_1)-2\tau\right) s_b\left(\pm(w_a+u)+\tau\right)}{s_b\left(i\frac{Q}{2}\pm(w_1-w_2)\right)}\times$$
$$\times\prod_{b=1}^{2} s_b\left(i\frac{Q}{2}\pm(w_a-u_b)-\tau\right)\,. \tag{B.40}$$

Then, we use the one-monopole duality (2.67) to replace the last integral with a $U(1)$ one

$$\mathcal{Z}_N^2(z_a,\tau,\zeta,\mu) = \mathrm{e}^{2\pi i(\zeta-(N-3)\tau)z_1}\mathrm{e}^{2\pi i(\zeta-(N-2)\tau)z_2} \prod_{a,b=1}^{2} s_b\left(-i\frac{Q}{2}+(z_a-z_b)+2\tau\right)\times$$
$$\times\prod_{a=1}^{2} s_b\left(\pm z_a-\mu+\tau\right)\prod_{j=1}^{3} s_b\left(i\frac{Q}{2}-2(N-j+1)\tau\right)\times$$
$$\times s_b\left(i\frac{Q}{2}-2\mu-2(j-1)\tau\right) s_b\left(-\zeta+\mu+2(N-j-1)\tau\right) s_b\left(\zeta+\mu+2(j-2)\tau\right)\times$$
$$\times s_b\left(-i\frac{Q}{2}+6\tau\right) s_b\left(-\frac{3}{2}iQ+6\tau\right) s_b\left(i\frac{Q}{2}-2\tau\right)\times$$
$$\times\int\frac{\mathrm{d}u_1\,\mathrm{d}u_2}{2}\mathrm{e}^{2\pi i(\zeta-(N-3)\tau)\sum_a u_a}\frac{\prod_{a,b=1}^{2} s_b\left(i\frac{Q}{2}+(u_a-u_b)-2\tau\right)}{s_b\left(i\frac{Q}{2}\pm(u_1-u_2)\right)}\times$$
$$\times\prod_{a=1}^{2} s_b\left(\pm u_a+\mu+2\tau\right) s_b\left(i\frac{Q}{2}\pm(u_a+z_1)-3\tau\right)\mathcal{Z}_{N-3}^2(u_a,\tau,\zeta+3\tau,\mu+3\tau)\times$$
$$\times\int\mathrm{d}w\,\mathrm{e}^{i\pi(iQ-6\tau)w} s_b\left(\pm(w-z_1)+2\tau\right)\prod_{a=1}^{2} s_b\left(\pm(w+u_a)+\tau\right)\times$$
$$\times\int\mathrm{d}u\,\mathrm{e}^{i\pi(iQ-8\tau)u} s_b\left(iQ\pm(u+z_2)-3\tau\right) s_b\left(i\frac{Q}{2}\pm(u+w)-\tau\right) \tag{B.41}$$

and finally we can evaluate the last $U(1)$ integral using again the one-monopole duality (2.67)

$$\begin{aligned}
\mathcal{Z}_N^2(z_a,\tau,\zeta,\mu) &= \mathrm{e}^{2\pi i(\zeta-(N-3)\tau)\sum_a z_a} \prod_{a,b=1}^{2} s_b\left(-i\frac{Q}{2}+(z_a-z_b)+2\tau\right) \prod_{a=1}^{2} s_b\left(\pm z_a-\mu+\tau\right)\times \\
&\times \prod_{j=1}^{3} s_b\left(i\frac{Q}{2}-2(N-j+1)\tau\right) s_b\left(i\frac{Q}{2}-2\mu-2(j-1)\tau\right) s_b\left(-\zeta+\mu+2(N-j-1)\tau\right)\times \\
&\times s_b\left(\zeta+\mu+2(j-2)\tau\right) s_b\left(-i\frac{Q}{2}+6\tau\right) s_b\left(-\frac{3}{2}iQ+8\tau\right) s_b\left(i\frac{Q}{2}-2\tau\right)\times \\
&\times \int \frac{du_1\,du_2}{\mathrm{e}}{}^{2\pi i(\zeta-(N-3)\tau)\sum_a u_a} \frac{\prod_{a,b=1}^{2} s_b\left(i\frac{Q}{2}+(u_a-u_b)-2\tau\right)}{s_b\left(i\frac{Q}{2}\pm(u_1-u_2)\right)}\times \\
&\times \prod_{a=1}^{2} s_b\left(\pm u_a+\mu+2\tau\right) s_b\left(i\frac{Q}{2}\pm(u_a+z_1)-3\tau\right) \mathcal{Z}_{N-3}^2(u_a,\tau,\zeta+3\tau,\mu+3\tau)\times \\
&\qquad \times \int dw\, s_b\left(\pm(w-z_1)+2\tau\right) s_b\left(iQ\pm(w-z_2)-4\tau\right) \prod_{a=1}^{2} s_b\left(\pm(w+u_a)+\tau\right)\,.
\end{aligned} \tag{B.42}$$

The result has exactly the same structure of (B.39), which means that the integral is now stable. Hence, we can iterate the last three steps n times to get

$$\begin{aligned}
\mathcal{Z}_N^2(z_a,\tau,\zeta,\mu) &= \mathrm{e}^{2\pi i(\zeta-(N-n)\tau)\sum_a z_a} \prod_{a,b=1}^{2} s_b\left(-i\frac{Q}{2}+(z_a-z_b)+2\tau\right)\times \\
&\times \prod_{a=1}^{2} s_b\left(\pm z_a-\mu+\tau\right) \prod_{j=1}^{n} s_b\left(i\frac{Q}{2}-2(N-j+1)\tau\right)\times \\
&\times s_b\left(i\frac{Q}{2}-2\mu-2(j-1)\tau\right) s_b\left(-\zeta+\mu+2(N-j-1)\tau\right) s_b\left(\zeta+\mu+2(j-2)\tau\right)\times \\
&\times s_b\left(-i\frac{Q}{2}+2n\tau\right) s_b\left(-\frac{3}{2}iQ+2(n+1)\tau\right) s_b\left(i\frac{Q}{2}-2\tau\right)\times \\
&\times \int \frac{du_1\,du_2}{\mathrm{e}}{}^{2\pi i(\zeta-(N-n)\tau)\sum_a u_a} \frac{\prod_{a,b=1}^{2} s_b\left(i\frac{Q}{2}+(u_a-u_b)-2\tau\right)}{s_b\left(i\frac{Q}{2}\pm(u_1-u_2)\right)}\times \\
&\times \prod_{a=1}^{2} s_b\left(\pm u_a+\mu+(n-1)\tau\right) s_b\left(i\frac{Q}{2}\pm(u_a+z_1)-n\tau\right) \mathcal{Z}_{N-n}^2(u_a,\tau,\zeta+n\tau,\mu+n\tau)\times \\
&\times \int dw\, s_b\left(\pm(w-z_1)+(n-1)\tau\right) s_b\left(iQ\pm(w-z_2)-(n+1)\tau\right) \prod_{a=1}^{2} s_b\left(\pm(w+u_a)+\tau\right)\,.
\end{aligned} \tag{B.43}$$

As in the previous cases, we can use the stabilization property of the integral to significantly simplify the result. Indeed, if we set $n = N$, the original $U(N)$ gauge node is completely confined

$$
\begin{aligned}
&\mathcal{Z}_N^2(z_a,\tau,\zeta,\mu) = \mathrm{e}^{2\pi i\zeta\sum_a z_a}\prod_{a,b=1}^{2} s_b\left(-i\frac{Q}{2}+(z_a-z_b)+2\tau\right)\prod_{a=1}^{2} s_b\left(\pm z_a-\mu+\tau\right)\times\\
&\times\prod_{j=1}^{N} s_b\left(i\frac{Q}{2}-2j\tau\right)s_b\left(i\frac{Q}{2}-2\mu-2(j-1)\tau\right)s_b\left(-\zeta+\mu+2(N-j-1)\tau\right)\times\\
&\times s_b\left(\zeta+\mu+2(j-2)\tau\right)s_b\left(-i\frac{Q}{2}+2N\tau\right)s_b\left(-\frac{3}{2}iQ+2(N+1)\tau\right)s_b\left(i\frac{Q}{2}-2\tau\right)\times\\
&\times\int\frac{\mathrm{d}u_1\,\mathrm{d}u_2}{\mathrm{e}}^{2\pi i\zeta\sum_a u_a}\frac{\prod_{a,b=1}^{2} s_b\left(i\frac{Q}{2}+(u_a-u_b)-2\tau\right)}{s_b\left(i\frac{Q}{2}\pm(u_1-u_2)\right)}\prod_{a=1}^{2} s_b\left(\pm u_a+\mu+(N-1)\tau\right)\times\\
&\times s_b\left(i\frac{Q}{2}\pm(u_a+z_1)-N\tau\right)\int\mathrm{d}w\, s_b\left(\pm(w-z_1)+(N-1)\tau\right)\times\\
&\times s_b\left(iQ\pm(w-z_2)-(N+1)\tau\right)\prod_{a=1}^{2} s_b\left(\pm(w+u_a)+\tau\right)\,.
\end{aligned}
\tag{B.44}
$$

This integral is not the partition function of $G[U(2)]$ yet because of the contribution of the adjoint chiral corresponding to the $U(2)$ node. This problem can be solved by simply applying the two-monopole duality (B.15) to the $U(1)$ integral

$$
\begin{aligned}
&\mathcal{Z}_N^2(z_a,\tau,\zeta,\mu) = \mathrm{e}^{2\pi i\zeta\sum_a z_a}\prod_{a=1}^{2} s_b\left(\pm z_a-\mu+\tau\right)\prod_{j=1}^{N} s_b\left(i\frac{Q}{2}-2(N-j+1)\tau\right)\times\\
&\times s_b\left(i\frac{Q}{2}-2\mu-2(j-1)\tau\right)s_b\left(-\zeta+\mu+2(N-j-1)\tau\right)s_b\left(\zeta+\mu+2(j-2)\tau\right)\times\\
&\times s_b\left(-i\frac{Q}{2}+2N\tau\right)s_b\left(-i\frac{Q}{2}+2(N-1)\tau\right)s_b\left(i\frac{Q}{2}-2\tau\right)\times\\
&\times\int\frac{\mathrm{d}u_1\,\mathrm{d}u_2}{2}\frac{\mathrm{e}^{2\pi i\zeta\sum_a u_a}}{s_b\left(i\frac{Q}{2}\pm(u_1-u_2)\right)}\prod_{a=1}^{2} s_b\left(\pm u_a+\mu+(N-1)\tau\right)\times\\
&\times s_b\left(i\frac{Q}{2}\pm(u_a+z_2)-N\tau\right)\int\mathrm{d}u\, s_b\left(i\frac{Q}{2}\pm(u+z_1)-(N-1)\tau\right)\times\\
&\times s_b\left(-i\frac{Q}{2}\pm(u+z_2)+(N+1)\tau\right)\prod_{a=1}^{2} s_b\left(i\frac{Q}{2}\pm(w+u_a)-\tau\right)\,.
\end{aligned}
\tag{B.45}
$$

Now we can apply the recombination duality (2.160) in the case $N=k=2$ to flip away some of the gauge singlets and obtain the desired form of the $G[U(2)]$. If we also restore the contribution of the $N-2$ b-fields, we get indeed

$$\begin{aligned}
\mathcal{Z}_{\mathcal{T}_A} &= \prod_{j=1}^{N-2} s_b\left(-i\frac{Q}{2}+2j\tau\right)\mathcal{Z}_N^2(z_a,\tau,\zeta,\mu) = \\
&= \prod_{a=1}^{2} s_b\left(i\frac{Q}{2}-2(N+a-2)\tau\right)\prod_{j=1}^{N-2} s_b\left(i\frac{Q}{2}-2\mu-2(j-1)\tau\right)\times \\
&\times\ s_b\left(-\zeta+\mu+2(N-j-1)\tau\right)\prod_{j=3}^{N} s_b\left(\zeta+\mu+2(2j-2)\tau\right)\times \\
&\times \int\frac{\mathrm{d}u_1\,\mathrm{d}u_2}{2}\mathrm{e}^{2\pi i\zeta\sum_a u_a}\frac{\prod_{a=1}^2 s_b\left(i\frac{Q}{2}\pm u_a-\mu-(N-2)\tau\right)}{s_b\left(i\frac{Q}{2}\pm(u_1-u_2)\right)}\times \\
&\times\ s_b\left(\pm(u_a-z_1)+(N-1)\tau\right)\int\mathrm{d}u\, s_b\left(\pm(u-z_1)-(N-2)\tau\right)\times \\
&\times\ s_b\left(\pm(u-z_2)+N\tau\right)\prod_{a=1}^{2} s_b\left(i\frac{Q}{2}\pm(u-u_a)-\tau\right) = \\
&= \prod_{j=1}^{N-2} s_b\left(i\frac{Q}{2}-2\mu-2(j-1)\tau\right)s_b\left(-\zeta+\mu+2(N-j-1)\tau\right)\times \\
&\times \prod_{j=3}^{N} s_b\left(\zeta+\mu+2(2j-2)\tau\right)\mathcal{Z}_{G[U(2)]}(\vec{z};\zeta;\mu+(N-2)\tau;i\frac{Q}{2}-(N-1)\tau;i\frac{Q}{2}-\tau) = \mathcal{Z}_{\mathcal{T}_B}\,,
\end{aligned} \tag{B.46}$$

which precisely corresponds to (2.181) in the case $k=2$.

B.3 Derivation of the flip-flip duality $\mathcal{T}^\vee \leftrightarrow T^{[N-1,1]}[SU(N)]$ for $N=3$

The starting point of the computation is the partition function of $T[SU(3)]$, to which we have to impose the constraint on the real masses (4.26) due to the superpotential deformation (4.22)

$$Y_2 = Y_1 + 2m_A\,. \tag{B.47}$$

We know that the effect of the massive deformation (4.22) is of making some of the flavors at the end of the tail of $T[SU(3)]^\vee$ massive. This is realized at the level of the partition function using the identity $s_b(x)\,s_b(-x)=1$. Denoting with $z_a^{(2)}$ the integration variables of the $U(2)$ gauge node, we have (recall the partition function of $T[SU(N)]$ (1.10))

$$\begin{aligned}
\prod_{i=1}^{3} s_b\left(\pm(z_a^{(2)}-Y_i)+m_A\right) &= s_b\left(z_a^{(2)}-Y_1+m_A\right)s_b\left(-z_a^{(2)}+Y_1+3m_A\right)s_b\left(\pm(z_a^{(2)}-Y_3)+m_A\right) \\
&\to s_b\left(\pm(z_a^{(2)}-Y_1)+2m_A\right)s_b\left(\pm(z_a^{(2)}-Y_3)+m_A\right)\,,
\end{aligned} \tag{B.48}$$

where at the last step we redefined

$$Y_1 \to Y_1 - m_A\,. \tag{B.49}$$

Hence, the partition function of the theory $\mathcal{T}^\vee$ is

$$\mathcal{Z}_{\mathcal{T}^\vee} = \mathcal{B}\int \mathrm{d}\vec{z}_2^{(2)}\, \mathrm{e}^{2\pi i(X_2-X_3)\sum_{a=1}^2 z_a^{(2)}} \prod_{a,b=1}^{2} s_b\left(i\frac{Q}{2}+(z_a^{(2)}-z_b^{(2)})-2m_A\right)\prod_{a=1}^{2} s_b\left(\pm(z_a^{(2)}-Y_1)+2m_A\right)$$
$$\times\, s_b\left(\pm(z_a^{(2)}-Y_3)+m_A\right)\int \mathrm{d}z_1^{(1)}\,\mathrm{e}^{2\pi i(X_1-X_2)z^{(1)}} s_b\left(i\frac{Q}{2}-2m_A\right)\prod_{a=1}^{2} s_b\left(\pm(z^{(1)}-z_a^{(2)})+m_A\right)\,, \tag{B.50}$$

where $\mathcal{B}$ denotes the contribution of the flipping fields $\mathcal{S}_{[1^3]}$ and $\mathcal{T}_i, \mathcal{T}, \tilde{\mathcal{T}}$ contained in $\mathcal{T}_{[2,1]}$

$$\begin{aligned}\mathcal{B} \;&=\; s_b\left(i\frac{Q}{2}-2m_A\right)^2 s_b\left(i\frac{Q}{2}-4m_A\right)s_b\left(i\frac{Q}{2}\pm(Y_1-Y_3)-3m_A\right)\\ &\times\;\prod_{i,j=1}^{3} s_b\left(-i\frac{Q}{2}+(X_i-X_j)+2m_A\right)\,.\end{aligned}\tag{B.51}$$

Since the adjoint chiral field at the $U(1)$ node is just a singlet, we can apply the Aharony duality at this node. Using (2.69) we find

$$\begin{aligned}\mathcal{Z}_{\mathcal{T}^\vee} \;&=\; \mathcal{B}s_b\left(i\frac{Q}{2}-2m_A\right)s_b\left(i\frac{Q}{2}\pm(X_1-X_2)-2m_A\right)\int \mathrm{d}\vec{z}_2^{(2)}\,\mathrm{e}^{2\pi i(X_1-X_3)\sum_{a=1}^2 z_a^{(2)}}\\ &\times\;\prod_{a=1}^{2} s_b\left(\pm(z_a^{(2)}-Y_1)+2m_A\right)s_b\left(\pm(z_a^{(2)}-Y_3)+m_A\right)\int \mathrm{d}z_1^{(1)}\,\mathrm{e}^{2\pi i(X_1-X_2)z^{(1)}}\\ &\times\;\prod_{a=1}^{2} s_b\left(i\frac{Q}{2}\pm(z^{(1)}+z_a^{(2)})-m_A\right)\,.\end{aligned}\tag{B.52}$$

This had the effect of removing the adjoint chiral of the adjacent $U(2)$ gauge node, so now we can apply the Aharony duality to it. In this case the rank of the group gets lowered by one unit

$$\begin{aligned}\mathcal{Z}_{\mathcal{T}^\vee} \;&=\; \mathcal{B}\,\mathrm{e}^{2\pi i(X_1-X_3)(Y_1+Y_3)}s_b\left(i\frac{Q}{2}-2m_A\right)s_b\left(-i\frac{Q}{2}+4m_A\right)s_b\left(i\frac{Q}{2}\pm(X_1-X_2)-2m_A\right)\\ &\times\; s_b\left(i\frac{Q}{2}\pm(X_1-X_3)-2m_A\right)s_b\left(-i\frac{Q}{2}\pm(Y_1-Y_3)+3m_A\right)\int \mathrm{d}z_1^{(2)}\,\mathrm{e}^{2\pi i(X_1-X_3)z^{(2)}}\\ &\times\; s_b\left(i\frac{Q}{2}\pm(z^{(2)}+Y_1)-2m_A\right)s_b\left(i\frac{Q}{2}\pm(z^{(2)}+Y_3)-m_A\right)\int \mathrm{d}z_1^{(1)}\,\mathrm{e}^{2\pi i(X_3-X_2)z^{(1)}}\\ &\times\; s_b\left(\pm(z^{(1)}+Y_1)+m_A\right)s_b\left(\pm(z^{(1)}-z^{(2)})+m_A\right)\,.\end{aligned}\tag{B.53}$$

The last step of the computation consists of applying the Aharony duality on the first $U(1)$ node once again. The various flipping fields produced in the derivation perfectly cancel with those contained in the prefactor $\mathcal{B}$ and we get

$$\begin{aligned}\mathcal{Z}_{\mathcal{T}^\vee} &= e^{2\pi i(X_1+X_2-2X_3)Y_1}e^{2\pi i(X_1-X_3)Y_3}\int dz_1^{(2)}\, e^{2\pi i(X_2-X_1)z^{(2)}}s_b\left(-i\frac{Q}{2}+2m_A\right)\\ &\times s_b\left(i\frac{Q}{2}\pm(z^{(2)}-Y_3)-m_A\right)\int dz_1^{(1)}\, e^{2\pi i(X_3-X_2)z^{(1)}}s_b\left(-i\frac{Q}{2}+2m_A\right)\\ &\times s_b\left(i\frac{Q}{2}\pm(z^{(1)}-Y_1)-m_A\right)s_b\left(i\frac{Q}{2}\pm(z^{(1)}-z_i^{(2)})-m_A\right).\end{aligned} \tag{B.54}$$

At this point we recall that Y_1 and Y_3 are not independent variables because of the original tracelessness condition $\sum_{i=1}^3 Y_i = 0$, which after the constraint (4.26) and the shift (4.38) becomes

$$2Y_1 + Y_3 = 0\,. \tag{B.55}$$

We parametrize the residual $U(1)_{Y^{(1)}}$ symmetry with

$$Y^{(1)} = Y_1 - Y_3 \tag{B.56}$$

and we also perform the change of variables $z^{(i)} \to z^{(i)} + Y^{(1)}/3$, so that

$$\begin{aligned}\mathcal{Z}_{\mathcal{T}^\vee} &= e^{2\pi i(-2X_1+X_2+X_3)\frac{Y^{(1)}}{3}}\int dz_1^{(2)}\, e^{2\pi i(X_2-X_1)z^{(2)}}s_b\left(-i\frac{Q}{2}+2m_A\right)\\ &\times s_b\left(i\frac{Q}{2}\pm\left(z^{(2)}+Y^{(1)}\right)-m_A\right)\int dz_1^{(1)}\, e^{2\pi i(X_3-X_2)z^{(1)}}s_b\left(-i\frac{Q}{2}+2m_A\right)\\ &\times s_b\left(i\frac{Q}{2}\pm z^{(1)}-m_A\right)s_b\left(i\frac{Q}{2}\pm(z^{(1)}-z_i^{(2)})-m_A\right) = \mathcal{Z}_{T^{[2,1]}[SU(3)]}\,.\end{aligned} \tag{B.57}$$

This coincides with the partition function of $T^{[2,1]}[SU(3)]$ which, from the deformation of the duality web of $T[SU(N)]$, we expect to be flip-flip dual to theory $\mathcal{T}$. The real masses X_n correspond to the $SU(3)_X$ global symmetry of $T^{[2,1]}[SU(3)]$ that enhances from the $U(1)^2$ topological symmetry that is manifest in the UV. Instead, the flavor symmetry of $T^{[2,1]}[SU(3)]$ is $U(1)_{Y^{(1)}}$. Hence, we showed that flip-flip duality is equivalent to sequentially applying the Aharony duality.

B.4 Derivation of the partition functions of $T_{[2,1^2]}[SU(4)]$ and its mirror dual

Flow to $T_{[2,1^2]}[SU(N)]$

As discussed in Subsection 4.2.1, the VEV for the CB moment map of $T[SU(4)]$ can be studied as a linear superpotential in $FFT[SU(4)]$ or, using flip-flip duality, as a monopole deformation of $T[SU(4)]$ with the addition of extra singlet fields flipping the components of the HB and CB moment maps that remain free after the VEV. Hence, in our computation we start from the partition function (1.10) of $T[SU(4)]$, impose the constraint on the fugacities

$$Y_2 = Y_1 + 2m_A \tag{B.58}$$

due to the monopole deformation (4.33), as well as the redefinition

$$Y_1 \to Y_1 - m_A \tag{B.59}$$

and add the contribution of the flipping fields

$$\begin{aligned}\mathcal{Z}_{\mathcal{T}} = \mathcal{B} \int \mathrm{d}\vec{z}_3^{(3)}\, \mathrm{e}^{2\pi i (Y_3 - Y_4)\sum_{k=1}^3 z_k^{(3)}} \prod_{k,l=1}^{3} s_b\left(-i\frac{Q}{2} + (z_k^{(3)} - z_l^{(3)}) + 2m_A\right) \\ \times \prod_{k=1}^{3}\prod_{i=1}^{4} s_b\left(i\frac{Q}{2} \pm (z_k^{(3)} - X_i) - m_A\right) \int \mathrm{d}\vec{z}_2^{(2)}\, \mathrm{e}^{2\pi i (Y_1 - Y_3 + m_A)\sum_{a=1}^2 z_a^{(2)}} \\ \times \prod_{a,b=1}^{2} s_b\left(-i\frac{Q}{2} + (z_a^{(2)} - z_b^{(2)}) + 2m_A\right) \prod_{a=1}^{2}\prod_{k=1}^{3} s_b\left(i\frac{Q}{2} \pm (z_a^{(2)} - z_k^{(3)}) - m_A\right) \\ \times \int \mathrm{d}z_1^{(1)}\, \mathrm{e}^{-4\pi i m_A z^{(1)}} s_b\left(-i\frac{Q}{2} + 2m_A\right) \prod_{a=1}^{2} s_b\left(i\frac{Q}{2} \pm (z^{(1)} - z_a^{(2)}) - m_A\right)\,, \end{aligned} \tag{B.60}$$

where $\mathcal{B}$ is the contribution of the singlets

$$\begin{aligned}\mathcal{B} \;=\; & \prod_{i,j=1}^{4} s_b\left(-i\frac{Q}{2} + (X_i - X_j) + 2m_A\right) s_b\left(i\frac{Q}{2} - 2m_A\right) s_b\left(i\frac{Q}{2} - 4m_A\right) \\ \times & \prod_{\alpha,\beta=3}^{4} s_b\left(i\frac{Q}{2} + (Y_\alpha - Y_\beta) - 2m_A\right) \prod_{\alpha=3}^{4} s_b\left(i\frac{Q}{2} \pm (Y_1 - Y_\alpha) - 3m_A\right)\,. \end{aligned} \tag{B.61}$$

As we explained in the main text, we first apply the integral identity for the one-monopole duality (2.67) to the $U(1)$ gauge node where the monopole superpotential is turned on. In this way, this node confines and we get the partition function of a dual frame of theory $\mathcal{T}$ where we have no monopole superpotential

$$\begin{aligned}\mathcal{Z}_{\mathcal{T}} \;=\; & \mathcal{B}\, s_b\left(-i\frac{Q}{2} + 2m_A\right) s_b\left(-i\frac{Q}{2} + 4m_A\right) \int \mathrm{d}\vec{z}_3^{(3)}\, \mathrm{e}^{2\pi i (Y_3 - Y_4)\sum_{k=1}^3 z_k^{(3)}} \\ \times & \prod_{k,l=1}^{3} s_b\left(-i\frac{Q}{2} + (z_k^{(3)} - z_l^{(3)}) + 2m_A\right) \prod_{k=1}^{3}\prod_{i=1}^{4} s_b\left(i\frac{Q}{2} \pm (z_k^{(3)} - X_i) - m_A\right) \\ \times & \int \mathrm{d}\vec{z}_2^{(2)}\, \mathrm{e}^{2\pi i (Y_1 - Y_3)\sum_{a=1}^2 z_a^{(2)}} \prod_{a=1}^{2}\prod_{k=1}^{3} s_b\left(i\frac{Q}{2} \pm (z_a^{(2)} - z_k^{(3)}) - m_A\right)\,. \end{aligned} \tag{B.62}$$

In order to find the flip-flip dual of $\mathcal{T}$, we now have to sequentially apply the integral identity for the Aharony duality (2.69). First we apply the duality to the $U(2)$ gauge node, whose rank decreases by one since we confined the previous node

$$\begin{aligned}\mathcal{Z}_{\mathcal{T}} = \mathcal{B}\, s_b\left(-i\frac{Q}{2} + 2m_A\right) s_b\left(-i\frac{Q}{2} + 4m_A\right) s_b\left(-i\frac{Q}{2} \pm (Y_1 - Y_3) + 3m_A\right) \\ \times \int \mathrm{d}\vec{z}_3^{(3)}\, \mathrm{e}^{2\pi i (Y_1 - Y_4)\sum_{k=1}^3 z_k^{(3)}} \prod_{k=1}^{3}\prod_{i=1}^{4} s_b\left(i\frac{Q}{2} \pm (z_k^{(3)} - X_i) - m_A\right) \\ \times \int \mathrm{d}z_1^{(2)}\, \mathrm{e}^{2\pi i (Y_1 - Y_3) z^{(2)}} \prod_{k=1}^{3} s_b\left(\pm(z^{(2)} + z_k^{(3)}) + m_A\right)\,. \end{aligned} \tag{B.63}$$

Now we can apply the Aharony duality on the $U(3)$ gauge node since its adjoint chiral became massive and was integrated out. The rank of the node decreases to two and we get

$$\begin{aligned}\mathcal{Z}_{\mathcal{T}} = \mathcal{B}\, \mathrm{e}^{2\pi i (Y_1-Y_4)\sum_{i=1} X_i} s_b\left(-i\frac{Q}{2}+2m_A\right)^2 s_b\left(-i\frac{Q}{2}+4m_A\right) s_b\left(-i\frac{Q}{2}\pm(Y_1-Y_3)+3m_A\right)\\ \times s_b\left(-i\frac{Q}{2}\pm(Y_1-Y_4)+3m_A\right)\prod_{i,j=1}^{4} s_b\left(i\frac{Q}{2}+(X_i-X_j)-2m_A\right)\int \mathrm{d}\vec{z}_2^{(3)}\, \mathrm{e}^{2\pi i (Y_1-Y_4)\sum_{k=1}^2 z_k^{(3)}}\\ \times \prod_{k=1}^{2}\prod_{i=1}^{4} s_b\left(\pm(z_k^{(3)}+X_i)+m_A\right)\int \mathrm{d}z_1^{(2)}\, \mathrm{e}^{2\pi i (Y_4-Y_3) z^{(2)}} \prod_{k=1}^{2} s_b\left(i\frac{Q}{2}\pm(z^{(2)}-z_k^{(3)})-m_A\right).\end{aligned} \tag{B.64}$$

Finally, we apply Aharony duality to the $U(1)$ gauge node. Simplifying the contributions of the singlets we produced in the derivation of the flip-flip dual with those contained in the prefactor $\mathcal{B}$ and performing the change of variable $z^{(2)} \to -z^{(2)}$ we get

$$\begin{aligned}\mathcal{Z}_{\mathcal{T}} = \mathrm{e}^{2\pi i (Y_1-Y_4)\sum_{i=1}^4 X_i}\int \mathrm{d}\vec{z}_2^{(3)}\, \mathrm{e}^{2\pi i (Y_1-Y_3)\sum_{k=1}^2 z_k^{(3)}} \prod_{k,l=1}^{2} s_b\left(i\frac{Q}{2}+(z_k^{(3)}-z_l^{(3)})-m_A\right)\\ \times \prod_{k=1}^{2}\prod_{i=1}^{4} s_b\left(\pm(z_k^{(3)}+X_i)+m_A\right)\int \mathrm{d}z_1^{(2)}\, \mathrm{e}^{2\pi i (Y_3-Y_4) z^{(2)}} s_b\left(i\frac{Q}{2}-2m_A\right)\prod_{k=1}^{2} s_b\left(\pm(z^{(2)}-z_k^{(3)})+m_A\right).\end{aligned} \tag{B.65}$$

Notice that the contact term is actually trivial, since the X_i parameters still parametrize the Cartan of the $SU(4)_X$ HB global symmetry. Moreover, we should recall that the original Y_i real masses were parametrizing the $SU(4)_Y$ CB global symmetry of $T[SU(4)]$, meaning that $\sum_{i=1}^4 Y_i = 0$. After imposing the condition $Y_2 = Y_1 + 2m_A$ and redefining $Y_1 \to Y_1 - m_A$, this translates into a condition for the real masses Y_1, Y_α of the remaining $U(1)\times SU(2)$ CB global symmetry

$$2Y_1 + \sum_{\alpha=3}^{4} Y_\alpha = 0\,. \tag{B.66}$$

This means that the proper $U(1)_{Y^{(1)}} \times SU(2)_{Y^{(2)}}$ fugacities are

$$\begin{aligned}Y^{(1)} &= Y_1\\ Y_1^{(2)} &= Y_3 + Y_1\\ Y_2^{(2)} &= Y_4 + Y_1\,,\end{aligned} \tag{B.67}$$

so that $\sum_{\alpha=1}^2 Y_\alpha^{(2)} = 0$. After this shift, we get

$$\begin{aligned}\mathcal{Z}_{\mathcal{T}} = \int \mathrm{d}\vec{z}_2^{(3)}\, \mathrm{e}^{2\pi i (2Y^{(1)}-Y_1^{(2)})\sum_{k=1}^2 z_k^{(3)}} \prod_{k,l=1}^{2} s_b\left(i\frac{Q}{2}+(z_k^{(3)}-z_l^{(3)})-m_A\right)\\ \times \prod_{k=1}^{2}\prod_{i=1}^{4} s_b\left(\pm(z_k^{(3)}+X_i)+m_A\right)\int \mathrm{d}z_1^{(2)}\, \mathrm{e}^{2\pi i (Y_1^{(2)}-Y_2^{(2)}) z^{(2)}} s_b\left(i\frac{Q}{2}-2m_A\right)\\ \times \prod_{k=1}^{2} s_b\left(\pm(z^{(2)}-z_k^{(3)})+m_A\right) = \mathcal{Z}_{T_{[2,1^2]}[SU(4)]}(\vec{X};\vec{Y}^{(2)},Y^{(1)};m_A)\,,\end{aligned} \tag{B.68}$$

where the contact term disappeared because of the tracelessness condition $\sum_{i=1}^4 X_i = 0$. This is precisely the partition function of $T_{[2,1^2]}[SU(4)]$, whose global symmetry is indeed $SU(4)_X \times U(1)_{Y^{(1)}} \times SU(2)_{Y^{(2)}}$ with the CB factor $U(1)_{Y^{(1)}} \times SU(2)_{Y^{(2)}}$ being enhanced at low energies.

Flow to $T^{[2,1^2]}[SU(4)]$

As discussed in Subsection 4.2.1, on the mirror dual side we should consider the VEV for the HB moment map of $T[SU(4)]^\vee$, which can be studied as a linear superpotential in $FFT[SU(4)]^\vee$ or, using flip-flip duality, as a mass deformation of $T[SU(4)]^\vee$ with the addition of extra singlet fields flipping the components of the HB and CB moment maps that remain free after the vev. Hence, in our computation we start from the partition function of $T[SU(4)]^\vee$ and impose the constraint on the fugacities $Y_2 = Y_1 + 2m_A$ due to the mass deformation. Using the relation $s_b(x)\, s_b(-x) = 1$, we have that the contribution of some of the chiral fields attached to the last $U(3)$ gauge node cancel each other, meaning that they have become massive fields. Denoting with $z_k^{(3)}$ the integration variables of the $U(3)$ gauge node, we have

$$\begin{aligned}\prod_{i=1}^4 s_b\left(\pm(z_k^{(3)} - Y_i) + m_A\right) &= \prod_{\alpha=3}^4 s_b\left(\pm(z_k^{(3)} - Y_\alpha) + m_A\right) s_b\left(z_k^{(3)} - Y_1 + m_A\right) \\ &\times\ s_b\left(-z_k^{(3)} + Y_1 + 3m_A\right) \\ &\to \prod_{\alpha=3}^4 s_b\left(\pm(z_k^{(3)} - Y_\alpha) + m_A\right) s_b\left(\pm(z_k^{(3)} - Y_1) + 2m_A\right)\,,\end{aligned} \tag{B.69}$$

where at the last step we redefined

$$Y_1 \to Y_1 - m_A\,. \tag{B.70}$$

Thus, the starting point of our computation is the partition function of the theory $\mathcal{T}^\vee$

$$\begin{aligned}\mathcal{Z}_{\mathcal{T}^\vee} = \mathcal{B} \int \mathrm{d}\vec{z}_3^{(3)}\, \mathrm{e}^{2\pi i(X_3 - X_4)\sum_{k=1}^3 z_k^{(3)}} \prod_{k,l=1}^3 s_b\left(i\frac{Q}{2} + (z_k^{(3)} - z_l^{(3)}) - 2m_A\right) \\ \times \prod_{k=1}^3 s_b\left(\pm(z_k^{(3)} - Y_1) + 2m_A\right) \prod_{\alpha=3}^4 s_b\left(\pm(z_k^{(3)} - Y_\alpha) + m_A\right) \int \mathrm{d}\vec{z}_2^{(2)}\, \mathrm{e}^{2\pi i(X_2 - X_3)\sum_{a=1}^2 z_a^{(2)}} \\ \times \prod_{a,b=1}^2 s_b\left(i\frac{Q}{2} + (z_a^{(2)} - z_b^{(2)}) - 2m_A\right) \prod_{a=1}^2 \prod_{k=1}^3 s_b\left(\pm(z_a^{(2)} - z_k^{(3)}) + m_A\right) \int \mathrm{d}z_1^{(1)}\, \mathrm{e}^{2\pi i(X_1 - X_2)z^{(1)}} \\ \times s_b\left(i\frac{Q}{2} - 2m_A\right) \prod_{a=1}^2 s_b\left(\pm(z^{(1)} - z_a^{(2)}) + m_A\right)\,.\end{aligned} \tag{B.71}$$

Again we claim that in order to reach the flip-flip dual frame which corresponds to $T^{[2,1^2]}[SU(4)]$, we can iteratively apply the integral identity for the Aharony duality (2.69). We start from the $U(1)$ gauge node since its adjoint chiral field is just a singlet. This node has two flavors attached to it, so it remains a $U(1)$ node and we get

$$\begin{aligned}\mathcal{Z}_{\mathcal{T}^\vee} = &\mathcal{B}\, s_b\left(i\frac{Q}{2}-2m_A\right) s_b\left(i\frac{Q}{2}\pm(X_1-X_2)-2m_A\right)\int \mathrm{d}\vec{z}_3^{(3)}\, \mathrm{e}^{2\pi i(X_3-X_4)\sum_{k=1}^3 z_k^{(3)}}\\ &\times\prod_{k,l=1}^{3} s_b\left(i\frac{Q}{2}+(z_k^{(3)}-z_l^{(3)})-2m_A\right)\prod_{k=1}^{3} s_b\left(\pm(z_k^{(3)}-Y_1)+2m_A\right)\prod_{\alpha=3}^{4} s_b\left(\pm(z_k^{(3)}-Y_\alpha)+m_A\right)\\ &\times\int \mathrm{d}\vec{z}_2^{(2)}\, \mathrm{e}^{2\pi i(X_1-X_3)\sum_{a=1}^2 z_a^{(2)}}\prod_{a=1}^{2}\prod_{k=1}^{3} s_b\left(\pm(z_a^{(2)}-z_k^{(3)})+m_A\right)\int \mathrm{d}z_1^{(1)}\, \mathrm{e}^{2\pi i(X_1-X_2)z^{(1)}}\\ &\times\prod_{a=1}^{2} s_b\left(i\frac{Q}{2}\pm(z^{(1)}+z_a^{(2)})-m_A\right).\end{aligned} \tag{B.72}$$

Notice that this application of the Aharony duality had the effect of removing the adjoint chiral field for the next $U(2)$ gauge node, which allows us to apply the duality again on this second node. This is a $U(2)$ gauge node with four flavors, so its rank doesn't change

$$\begin{aligned}\mathcal{Z}_{\mathcal{T}^\vee} = &\mathcal{B}\, s_b\left(i\frac{Q}{2}-2m_A\right)^2 s_b\left(i\frac{Q}{2}\pm(X_1-X_2)-2m_A\right) s_b\left(i\frac{Q}{2}\pm(X_1-X_3)-2m_A\right)\\ &\times\int \mathrm{d}\vec{z}_3^{(3)}\, \mathrm{e}^{2\pi i(X_1-X_4)\sum_{k=1}^3 z_k^{(3)}}\prod_{k=1}^{3} s_b\left(\pm(z_k^{(3)}-Y_1)+2m_A\right)\prod_{\alpha=3}^{4} s_b\left(\pm(z_k^{(3)}-Y_\alpha)+m_A\right)\\ &\times\int \mathrm{d}\vec{z}_2^{(2)}\, \mathrm{e}^{2\pi i(X_1-X_3)\sum_{a=1}^2 z_a^{(2)}}\prod_{a=1}^{2}\prod_{k=1}^{3} s_b\left(i\frac{Q}{2}\pm(z_a^{(3)}+z_k^{(3)})-m_A\right)\int \mathrm{d}z_1^{(1)}\, \mathrm{e}^{2\pi i(X_3-X_2)z^{(1)}}\\ &\times\prod_{a=1}^{2} s_b\left(\pm(z^{(1)}-z_a^{(2)})+m_A\right).\end{aligned} \tag{B.73}$$

Again, since we removed the adjoint chiral field from the $U(3)$ node we can apply the Aharony duality to it. In this case the rank of the group decreases, since some of the flavors that used to be attached to it became massive, so this node is not balanced anymore. Hence, we get

$$\begin{aligned}\mathcal{Z}_{\mathcal{T}^\vee} = &\mathcal{B}\, \mathrm{e}^{2\pi i(X_1-X_4)(Y_1+\sum_{\alpha=1}^2 Y_\alpha)} s_b\left(i\frac{Q}{2}-2m_A\right)^2 s_b\left(-i\frac{Q}{2}+4m_A\right) s_b\left(i\frac{Q}{2}\pm(X_1-X_2)-2m_A\right)\\ &\times s_b\left(i\frac{Q}{2}\pm(X_1-X_3)-2m_A\right) s_b\left(i\frac{Q}{2}\pm(X_1-X_4)-2m_A\right)\prod_{\alpha,\beta=3}^{4} s_b\left(-i\frac{Q}{2}+(Y_\alpha-Y_\beta)+2m_A\right)\\ &\times\prod_{\alpha=3}^{4} s_b\left(-i\frac{Q}{2}\pm(Y_1-Y_\alpha)+3m_A\right)\int \mathrm{d}\vec{z}_2^{(3)}\, \mathrm{e}^{2\pi i(X_1-X_4)\sum_{k=1}^2 z_k^{(3)}}\prod_{k=1}^{2} s_b\left(i\frac{Q}{2}\pm(z_k^{(3)}+Y_1)-2m_A\right)\\ &\times\prod_{\alpha=3}^{4} s_b\left(i\frac{Q}{2}\pm(z_k^{(3)}+Y_\alpha)-m_A\right)\int \mathrm{d}\vec{z}_2^{(2)}\, \mathrm{e}^{2\pi i(X_4-X_3)\sum_{a=1}^2 z_a^{(2)}}\prod_{a,b=1}^{2} s_b\left(i\frac{Q}{2}+(z_a^{(2)}-z_b^{(2)})-2m_A\right)\\ &\times\prod_{a=1}^{2}\prod_{k=1}^{2} s_b\left(\pm(z_a^{(2)}-z_k^{(3)})+m_A\right)\int \mathrm{d}z_1^{(1)}\, \mathrm{e}^{2\pi i(X_3-X_2)z^{(1)}}\prod_{a=1}^{2} s_b\left(\pm(z^{(1)}-z_a^{(2)})+m_A\right).\end{aligned} \tag{B.74}$$

This concludes the first iteration of the sequential application of the Aharony duality along the whole tail. In the second iteration, we again sequentially apply the duality starting from the left $U(1)$ gauge node, but stopping at the second last node in order to restore the adjoint chiral at the $U(2)$ gauge node labelled by $\vec{z}^{(3)}$. From the first application of the Aharony duality we get

$$\begin{aligned}\mathcal{Z}_{\mathcal{T}^\vee} &= \mathcal{B}\,\mathrm{e}^{2\pi i(X_1-X_4)(Y_1+\sum_{\alpha=1}^{2}Y_\alpha)} s_b\left(i\frac{Q}{2}-2m_A\right)^2 s_b\left(-i\frac{Q}{2}+4m_A\right) s_b\left(i\frac{Q}{2}\pm(X_1-X_2)-2m_A\right)\\
&\times s_b\left(i\frac{Q}{2}\pm(X_1-X_3)-2m_A\right) s_b\left(i\frac{Q}{2}\pm(X_1-X_4)-2m_A\right) s_b\left(i\frac{Q}{2}\pm(X_2-X_3)-2m_A\right)\\
&\times \prod_{\alpha,\beta=3}^{4} s_b\left(-i\frac{Q}{2}+(Y_\alpha-Y_\beta)+2m_A\right)\prod_{\alpha=3}^{4} s_b\left(-i\frac{Q}{2}\pm(Y_1-Y_\alpha)+3m_A\right)\\
&\times \int \mathrm{d}\vec{z}_2^{(3)}\,\mathrm{e}^{2\pi i(X_1-X_4)\sum_{k=1}^{2}z_k^{(3)}}\prod_{k=1}^{2} s_b\left(i\frac{Q}{2}\pm(z_k^{(3)}+Y_1)-2m_A\right)\prod_{\alpha=3}^{4} s_b\left(i\frac{Q}{2}\pm(z_k^{(3)}+Y_\alpha)-m_A\right)\\
&\times \int \mathrm{d}\vec{z}_2^{(2)}\,\mathrm{e}^{2\pi i(X_4-X_2)\sum_{a=1}^{2}z_a^{(2)}}\prod_{a=1}^{2} s_b\left(\pm(z_a^{(2)}+Y_1)+m_A\right)\prod_{k=1}^{2} s_b\left(\pm(z_a^{(2)}-z_k^{(3)})+m_A\right)\\
&\times \int \mathrm{d}z_1^{(1)}\,\mathrm{e}^{2\pi i(X_3-X_2)z^{(1)}}\prod_{a=1}^{2} s_b\left(i\frac{Q}{2}\pm(z^{(1)}+z_a^{(2)})-m_A\right)\,.\end{aligned} \tag{B.75}$$

Now we apply the Aharony duality to the $U(2)$ gauge node labelled by $\vec{z}^{(2)}$

$$\begin{aligned}\mathcal{Z}_{\mathcal{T}^\vee} &= \mathcal{B}\,\mathrm{e}^{2\pi i(X_1+X_2-2X_4)Y_1}\mathrm{e}^{2\pi i(X_1-X_4)\sum_{\alpha=1}^{2}Y_\alpha} s_b\left(i\frac{Q}{2}-2m_A\right)^2 s_b\left(-i\frac{Q}{2}+4m_A\right)\\
&\times s_b\left(i\frac{Q}{2}\pm(X_1-X_2)-2m_A\right) s_b\left(i\frac{Q}{2}\pm(X_1-X_3)-2m_A\right) s_b\left(i\frac{Q}{2}\pm(X_1-X_4)-2m_A\right)\\
&\times s_b\left(i\frac{Q}{2}\pm(X_2-X_3)-2m_A\right) s_b\left(i\frac{Q}{2}\pm(X_2-X_4)-2m_A\right)\prod_{\alpha,\beta=3}^{4} s_b\left(-i\frac{Q}{2}+(Y_\alpha-Y_\beta)+2m_A\right)\\
&\times \prod_{\alpha=3}^{4} s_b\left(-i\frac{Q}{2}\pm(Y_1-Y_\alpha)+3m_A\right)\int \mathrm{d}\vec{z}_2^{(3)}\,\mathrm{e}^{2\pi i(X_1-X_2)\sum_{k=1}^{2}z_k^{(3)}}\prod_{k,l=1}^{2} s_b\left(-i\frac{Q}{2}+(z_k^{(3)}-z_l^{(3)})+2m_A\right)\\
&\times \prod_{k=1}^{2}\prod_{\alpha=3}^{4} s_b\left(i\frac{Q}{2}\pm(z_k^{(3)}+Y_\alpha)-m_A\right)\int \mathrm{d}\vec{z}_2^{(2)}\,\mathrm{e}^{2\pi i(X_4-X_2)\sum_{a=1}^{2}z_a^{(2)}}\prod_{a=1}^{2} s_b\left(i\frac{Q}{2}\pm(z_a^{(2)}-Y_1)-m_A\right)\\
&\times \prod_{k=1}^{2} s_b\left(i\frac{Q}{2}\pm(z_a^{(2)}+z_k^{(3)})-m_A\right)\int \mathrm{d}z_1^{(1)}\,\mathrm{e}^{2\pi i(X_3-X_4)z^{(1)}}\prod_{a=1}^{2} s_b\left(\pm(z^{(1)}-z_a^{(2)})+m_A\right)\,.\end{aligned} \tag{B.76}$$

This concludes also the second iteration. The last iteration only consists of applying the Aharony duality on the original $U(1)$ node, so to restore the adjoint chiral also at the $U(2)$ node labelled by $\vec{z}^{(2)}$. Simplifying the contributions of the many singlets we produced by the sequential application of the Aharony duality with those contained in the prefactor $\mathcal{B}$ and performing the change of variables $z_a^{(2)} \to -z_a^{(2)}$ we get

$$\begin{aligned}\mathcal{Z}_{\mathcal{T}^\vee} &= \mathrm{e}^{2\pi i(X_1+X_2-2X_4)Y_1}\mathrm{e}^{2\pi i(X_1-X_4)\sum_{\alpha=3}^{4}Y_\alpha}\int \mathrm{d}\vec{z}_2^{(3)}\,\mathrm{e}^{2\pi i(X_1-X_2)\sum_{k=1}^{2}z_k^{(3)}}\\
&\times \prod_{k,l=1}^{2} s_b\left(-i\frac{Q}{2}+(z_k^{(3)}-z_l^{(3)})+2m_A\right)\prod_{k=1}^{2}\prod_{\alpha=3}^{4} s_b\left(i\frac{Q}{2}\pm(z_k^{(3)}+Y_\alpha)-m_A\right)\end{aligned}$$

$$
\begin{aligned}
&\times \int \mathrm{d}\vec{z}_2^{(2)}\, \mathrm{e}^{2\pi i (X_2-X_3)\sum_{a=1}^2 z_a^{(2)}} \prod_{a,b=1}^{2} s_b\left(-i\frac{Q}{2}+(z_a^{(2)}-z_b^{(2)})+2m_A\right)\\
&\times \prod_{a=1}^{2} s_b\left(i\frac{Q}{2}\pm(z_a^{(2)}+Y_1)-m_A\right)\prod_{k=1}^{2} s_b\left(i\frac{Q}{2}\pm(z_a^{(2)}-z_k^{(3)})-m_A\right)\\
&\times\ s_b\left(-i\frac{Q}{2}+2m_A\right)\int \mathrm{d}z_1^{(2)}\, \mathrm{e}^{2\pi i(X_3-X_4)z^{(1)}}\prod_{a=1}^{2} s_b\left(i\frac{Q}{2}\pm(z^{(1)}-z_a^{(2)})-m_A\right).
\end{aligned}
\tag{B.77}
$$

At this point we implement the redefinition of the fugacities (B.67) and we also perform the change of variables $z^{(i)} \to z^{(i)} + Y^{(1)}$. By taking into account the tracelessness conditions $\sum_{i=1}^4 X_i = \sum_{\alpha=1}^2 Y_\alpha^{(2)} = 0$, we get

$$
\begin{aligned}
\mathcal{Z}_{\mathcal{T}^\vee} \ =\ & \mathrm{e}^{4\pi i (X_1+X_2)Y^{(1)}}\int \mathrm{d}\vec{z}_2^{(3)}\, \mathrm{e}^{2\pi i (X_1-X_2)\sum_{k=1}^2 z_k^{(3)}}\prod_{k,l=1}^{2} s_b\left(-i\frac{Q}{2}+(z_k^{(3)}-z_l^{(3)})+2m_A\right)\\
&\times \prod_{k=1}^{2}\prod_{\alpha=1}^{2} s_b\left(i\frac{Q}{2}\pm(z_k^{(3)}+Y_\alpha^{(2)})-m_A\right)\int \mathrm{d}\vec{z}_2^{(2)}\, \mathrm{e}^{2\pi i(X_2-X_3)\sum_{a=1}^2 z_a^{(2)}}\\
&\times \prod_{a,b=1}^{2} s_b\left(-i\frac{Q}{2}+(z_a^{(2)}-z_b^{(2)})+2m_A\right)\prod_{a=1}^{2} s_b\left(i\frac{Q}{2}\pm(z_a^{(2)}+Y^{(1)})-m_A\right)\\
&\times \prod_{k=1}^{2} s_b\left(i\frac{Q}{2}\pm(z_a^{(2)}-z_k^{(3)})-m_A\right) s_b\left(-i\frac{Q}{2}+2m_A\right)\int \mathrm{d}z_1^{(1)}\, \mathrm{e}^{2\pi i(X_3-X_4)z^{(1)}}\\
&\times \prod_{a=1}^{2} s_b\left(i\frac{Q}{2}\pm(z^{(1)}-z_a^{(2)})-m_A\right) = \mathcal{Z}_{T^{[2,1^2]}[SU(4)]}(Y^{(1)},\vec{Y}^{(2)};\vec{X};i\frac{Q}{2}-m_A)\,.
\end{aligned}
\tag{B.78}
$$

This is precisely the partition function of $T^{[2,1^2]}[SU(4)]$, whose global symmetry is indeed $U(1)_{Y^{(1)}} \times SU(2)_{Y^{(2)}} \times SU(4)_X$ with the CB factor $SU(4)_X$ being enhanced at low energies.

Combining the results (B.68) and (B.78) with the integral identity for the mirror self-duality of $T[SU(4)]$ (1.14) we get that the partition function of $T_{[2,1^2]}[SU(4)]$ coincides with that of $T^{[2,1^2]}[SU(4)]$ provided that $m_A \leftrightarrow i\frac{Q}{2} - m_A$ as expected from mirror symmetry (1.12)

$$
\mathcal{Z}_{T_{[2,1^2]}[SU(4)]}(\vec{X};\vec{Y}^{(2)},Y^{(1)};m_A) = \mathcal{Z}_{T^{[2,1^2]}[SU(4)]}(Y^{(1)},\vec{Y}^{(2)};\vec{X};i\frac{Q}{2}-m_A)\,.
\tag{B.79}
$$

Appendix C

$\mathbb{S}^3 \times \mathbb{S}^1$ partition function computations

The computations we perform in $4d$ at the level of the $\mathbb{S}^3 \times \mathbb{S}^1$ partition function make intensive use of the integral identity associated with the Intriligator–Pouliot duality [81]. This identity can be written as follows:

$$\oint \mathrm{d}\vec{w}_{N_c} \prod_{i=1}^{N_c} \prod_{a=1}^{2N_f} \Gamma_e\left(x_a w_i^{\pm 1}\right) = \prod_{a<b}^{2N_f} \Gamma_e\left(x_a x_b\right) \oint \mathrm{d}\vec{w}_{N_f-N_c-2} \prod_{i=1}^{N_f-N_c-2} \prod_{a=1}^{2N_f} \Gamma_e\left((pq)^{1/2} x_a^{-1} w_i^{\pm 1}\right)\,, \tag{C.1}$$

which holds provided that

$$\prod_{a=1}^{2N_f} x_a = (pq)^{N_f-N_c-1} \tag{C.2}$$

and where we recall that the integration measure is defined as

$$\mathrm{d}\vec{w}_N = \frac{\left[(p;p)(q;q)\right]^N}{2^N N!} \prod_{i=1}^{N} \frac{\mathrm{d}w_i}{2\pi i\, w_i} \frac{1}{\prod_{n=1}^{N} \Gamma_e\left(w_n^{\pm 2}\right) \prod_{n<m}^{N} \Gamma_e\left(w_n^{\pm 1} w_m^{\pm 1}\right)}\,. \tag{C.3}$$

This identity was proven in Theorem 3.1 of [90]. Notice that for $N_c = N$ and $N_f = N+2$ the dual theory is a WZ model of $(N+2)(2N+3)$ chiral fields and the identity (C.1) reduces to

$$\oint \mathrm{d}\vec{w}_N \prod_{i=1}^{N} \prod_{a=1}^{2N+4} \Gamma_e\left(x_a w_i^{\pm 1}\right) = \prod_{a<b}^{2N+4} \Gamma_e\left(x_a x_b\right)\,, \tag{C.4}$$

with the condition

$$\prod_{a=1}^{2N+4} x_a = pq\,, \tag{C.5}$$

which was first conjectured in [198]. The two conditions (C.2) and (C.5) are a consequence of of the cancellations of the NSVZ β-function, or equivalently of the mixed $U(1)_R$-gauge anomaly, when the theory is a gauge theory and of the superpotential when it is a WZ.

C.1 Derivation of the flip-flip duality of $E[USp(2N)]$

In Subsection 3.3.2 we briefly described how to derive the flip-flip duality of $E[USp(2N)]$ by using the Intriligator–Pouliot duality only. In this appendix, we use the supersymmetric index to show how to obtain $FFE[USp(2N)]$, the flip-flip dual of $E[USp(2N)]$, by sequential Intriligator–Pouliot dualities. As an explicit example, we take $N = 3$, which requires the Intriligator–Pouliot duality three times in total to obtain the flip-flip dual.

The superconformal index of $E[USp(6)]$ is given by

$$\begin{aligned}
&\mathcal{I}_{E[USp(6)]}(\vec{x};\vec{y};t,c) = \\
&= \frac{\prod_{n=1}^{3}\Gamma_e\left(c\,y_3^{\pm1}x_n^{\pm1}\right)}{\Gamma_e\left(t^{-2}c^2\right)\Gamma_e\left(t^{-1}c^2\right)\Gamma_e\left(c^2\right)}\oint \mathrm{d}\vec{w}_1^{(1)}\,\mathrm{d}\vec{w}_2^{(2)}\,\Gamma_e\left(pqt^{-1}\right)^3\prod_{i<j}^{2}\Gamma_e\left(pqt^{-1}w_i^{(2)\pm1}w_j^{(2)\pm1}\right)\times \\
&\quad\times\frac{\prod_{j=1}^{2}\Gamma_e\left(t^{1/2}w^{(1)\pm1}w_j^{(2)\pm1}\right)}{\Gamma_e\left(cy_2^{\pm1}w^{(1)\pm1}\right)}\frac{\prod_{i=1}^{2}\prod_{n=1}^{3}\Gamma_e\left(t^{1/2}w_i^{(2)\pm1}x_n^{\pm1}\right)}{\prod_{i=1}^{2}\Gamma_e\left(t^{1/2}cy_3^{\pm1}w_i^{(2)\pm1}\right)}\times \\
&\quad\times\Gamma_e\left(t^{-1}cy_1^{\pm1}w^{(1)\pm1}\right)\prod_{i=1}^{2}\Gamma_e\left(t^{-1/2}cy_2^{\pm1}w_i^{(2)\pm1}\right)\,.
\end{aligned}\tag{C.6}$$

As a first step, we apply the Intriligator–Pouliot duality on the leftmost node, which corresponds to the following identity:

$$\begin{aligned}
&\oint \mathrm{d}\vec{w}_1^{(1)}\,\Gamma_e(t^{-1}cy_1^{\pm1}w^{(1)\pm1})\Gamma_e(pqc^{-1}y_2^{\pm1}w^{(1)\pm1})\prod_{j=1}^{2}\Gamma_e(t^{1/2}w^{(1)\pm1}w_j^{(2)\pm1}) = \\
&= \Gamma_e(t^{-2}c^2)\Gamma_e(pqt^{-1}y_1^{\pm1}y_2^{\pm1})\prod_{j=1}^{2}\Gamma_e(t^{-1/2}cy_1^{\pm1}w_j^{(2)\pm1})\times \\
&\quad\times\Gamma_e(p^2q^2c^{-2})\prod_{j=1}^{2}\Gamma_e(pqt^{1/2}c^{-1}y_2^{\pm1}w_j^{(2)\pm1})\Gamma_e(t)^2\prod_{i<j}^{2}\Gamma_e(tw_i^{(2)\pm1}w_j^{(2)\pm1})\oint \mathrm{d}\vec{w}_1^{(1)}\times \\
&\quad\times\Gamma_e(p^{1/2}q^{1/2}tc^{-1}y_1^{\pm1}w^{(1)\pm1})\Gamma_e(p^{-1/2}q^{-1/2}cy_2^{\pm1}w^{(1)\pm1})\prod_{j=1}^{2}\Gamma_e(p^{1/2}q^{1/2}t^{-1/2}w^{(1)\pm1}w_j^{(2)\pm1})\,.
\end{aligned}\tag{C.7}$$

Next we apply the Intriligator–Pouliot duality on the middle gauge node. We thus collect the $z^{(2)}$-dependent factors and apply the following identity:

$$\begin{aligned}
&\oint \mathrm{d}\vec{w}_2^{(2)}\prod_{i=1}^{2}\prod_{n=1}^{3}\Gamma_e(t^{1/2}w_i^{(2)\pm1}x_n^{\pm1}) = \\
&\times\prod_{j=1}^{2}\Gamma_e(pqt^{-1/2}c^{-1}y_3^{\pm1}w_j^{(2)\pm1})\prod_{j=1}^{2}\Gamma_e(t^{-1/2}cy_1^{\pm1}w_j^{(2)\pm1})\prod_{j=1}^{2}\Gamma_e(p^{1/2}q^{1/2}t^{-1/2}w^{(1)\pm1}w_j^{(2)\pm1}) = \\
&= \Gamma_e\left(t\right)^2\prod_{m<n}^{3}\Gamma_e\left(tx_m^{\pm1}x_n^{\pm1}\right)\prod_{n=1}^{3}\Gamma_e(pqc^{-1}x_n^{\pm1}y_3^{\pm1})\prod_{n=1}^{3}\Gamma_e(cx_n^{\pm1}y_1^{\pm1})\times \\
&\quad\times\Gamma_e(p^2q^2t^{-1}c^{-2})\Gamma_e(pqt^{-1}y_3^{\pm1}y_1^{\pm1})\Gamma_e(p^{3/2}q^{3/2}t^{-1}c^{-1}y_3^{\pm1}w^{(1)\pm1})\times
\end{aligned}$$

$$\times\,\Gamma_e(t^{-1}c^2)\Gamma_e(p^{1/2}q^{1/2}t^{-1}cy_1^{\pm1}w^{(1)\pm1})\oint \mathrm{d}\vec{w}_2'^{(2)}\prod_{i=1}^{2}\prod_{n=1}^{3}\Gamma_e(p^{1/2}q^{1/2}t^{-1/2}w_i'^{(2)\pm1}x_n^{\pm1})\times$$
$$\times\prod_{i=1}^{2}\Gamma_e(p^{-1/2}q^{-1/2}t^{1/2}cy_3^{\pm1}w_i'^{(2)\pm1})\prod_{i=1}^{2}\Gamma_e(p^{1/2}q^{1/2}t^{1/2}c^{-1}y_1^{\pm1}w_i'^{(2)\pm1})\prod_{i=1}^{2}\Gamma_e(t^{1/2}w^{(1)\pm1}w_i'^{(2)\pm1})\,. \tag{C.8}$$

Lastly, we collect the $z^{(1)}$-dependent factors resulting from the previous two applications of the Intriligator–Pouliot duality, which become

$$\oint \mathrm{d}\vec{w}_1^{(1)}\,\Gamma_e(p^{-1/2}q^{-1/2}cy_2^{\pm1}w^{(1)\pm1})\Gamma_e(p^{3/2}q^{3/2}t^{-1}c^{-1}y_3^{\pm1}w^{(1)\pm1})\prod_{i=1}^{2}\Gamma_e(t^{1/2}w^{(1)\pm1}w_i'^{(2)\pm1})=$$
$$=\Gamma_e(p^{-1}q^{-1}c^2)\Gamma_e(pqt^{-1}y_2^{\pm1}y_3^{\pm1})\prod_{i=1}^{2}\Gamma_e(p^{-1/2}q^{-1/2}t^{1/2}cy_2^{\pm1}w_i'^{(2)\pm1})\Gamma_e(p^3q^3t^{-2}c^{-2})\times$$
$$\times\prod_{i=1}^{2}\Gamma_e(p^{3/2}q^{3/2}t^{-1/2}c^{-1}y_3^{\pm1}w_i'^{(2)\pm1})\Gamma_e(t)^2\prod_{i<j}^{2}\Gamma_e(tw_i'^{(2)\pm1}w_j'^{(2)\pm1})\oint \mathrm{d}\vec{w}_1'^{(1)}\times$$
$$\times\,\Gamma_e(pqc^{-1}y_2^{\pm1}w'^{(1)\pm1})\Gamma_e(p^{-1}q^{-1}tcy_3^{\pm1}w'^{(1)\pm1})\prod_{i=1}^{2}\Gamma_e(p^{1/2}q^{1/2}t^{-1/2}w'^{(1)\pm1}w_i'^{(2)\pm1})\,. \tag{C.9}$$

Combining all the remaining factors, we obtain the following expression for the entire supersymmetric index:

$$\mathcal{I}_{E[USp(6)]}(\vec{x};\vec{y};t,c)=$$
$$=\prod_{m<n}^{3}\Gamma_e\left(tx_m^{\pm}x_n^{\pm}\right)\prod_{m<n}^{3}\Gamma_e\left(pqt^{-1}y_m^{\pm}y_n^{\pm}\right)\times$$
$$\times\frac{\prod_{n=1}^{3}\Gamma_e(cy_1^{\pm1}x_n^{\pm1})}{\Gamma_e(p^{-2}q^{-2}t^2c^2)\Gamma_e(p^{-1}q^{-1}tc^2)\Gamma_e\left(c^2\right)}\oint \mathrm{d}\vec{w}_1'^{(1)}\,\mathrm{d}\vec{w}_2'^{(2)}\,\Gamma_e\left(t\right)^3\prod_{i<j}^{2}\Gamma_e\left(tw_i'^{(2)\pm1}w_j'^{(2)\pm1}\right)\times$$
$$\times\frac{\prod_{i=1}^{2}\Gamma_e(p^{1/2}q^{1/2}t^{-1/2}w'^{(1)\pm1}w_i'^{(2)\pm1})}{\Gamma_e(c^2y_2^{\pm1}w'^{(1)\pm1})}\frac{\prod_{i=1}^{2}\prod_{n=1}^{3}\Gamma_e(p^{1/2}q^{1/2}t^{-1/2}w_i'^{(2)\pm1}x_n^{\pm1})}{\prod_{i=1}^{2}\Gamma_e(p^{1/2}q^{1/2}t^{-1/2}cy_1^{\pm1}w_i'^{(2)\pm1})}\times$$
$$\times\,\Gamma_e(p^{-1}q^{-1}tcy_3^{\pm1}w'^{(1)\pm1})\prod_{i=1}^{2}\Gamma_e(p^{-1/2}q^{-1/2}t^{1/2}cy_2^{\pm1}w_i'^{(2)\pm1})=$$
$$=\mathcal{I}_{E[USp(6)]}(\vec{x};\vec{y};pq/t,c)\,. \tag{C.10}$$

This proves the index equality of the flip-flip duality of $E[USp(6)]$ and the sequential applications of the Intriligator–Pouliot duality. Note that while the variables y_n appear in the opposite way compared to the original definition, the index is invariant under such a shuffling of variables because it is a Weyl symmetry of the $USp(6)_y$ global symmetry.

C.2 Derivation of the index (4.89) of $E^{[2,1]}[USp(2N)]$

In this appendix we show how to obtain $E^{[2,1]}[USp(6)]$ from its flip-flip dual $\mathsf{T}^\vee$ by sequential applications of the Intriligator–Pouliot duality. We start with the index of the theory $\mathsf{T}^\vee$, which is given by (4.88). For $N = 3$, it is written as follows:

$$\begin{aligned}
&\mathcal{I}_{\mathsf{T}^\vee}\left(\vec{x}; y^{(1)}, y^{(2)}; c; t\right) = \Gamma_e\left(pqt^{-1}\right)^2 \prod_{i<j}^{3} \Gamma_e\left(pqt^{-1}x_i^{\pm1}x_j^{\pm1}\right)\Gamma_e\left(t^{\frac{3}{2}}y^{(1)\pm1}y^{(2)\pm1}\right)\prod_{i=1}^{2}\Gamma_e\left(t^i\right)\Gamma_e\left(t^{-1}c^2\right)\times \\
&\times \frac{\Gamma_e\left(c\,x_3^{\pm1}y^{(2)\pm1}\right)\prod_{i=1}^{2}\Gamma_e\left(c\,x_3^{\pm1}\left(t^{i-\frac{3}{2}}y^{(1)}\right)^{\pm1}\right)}{\Gamma_e\left(p^{-2}q^{-2}t^2c^2\right)\Gamma_e\left(p^{-1}q^{-1}tc^2\right)} \oint \mathrm{d}\vec{z}_1^{(1)}\,\mathrm{d}\vec{z}_2^{(2)}\,\Gamma_e\left(t\right)^3 \prod_{a<b}^{2}\Gamma_e(tz_a^{(2)\pm1}z_b^{(2)\pm1})\times \\
&\times \frac{\prod_{a=1}^{2}\Gamma_e\left(p^{1/2}q^{1/2}t^{-1/2}z^{(1)\pm1}z_a^{(2)\pm1}\right)\Gamma_e\left(p^{1/2}q^{1/2}t^{-1}z_a^{(2)\pm1}y^{(1)\pm1}\right)\Gamma_e\left(p^{1/2}q^{1/2}t^{-1/2}z_a^{(2)\pm1}y^{(1)\pm1}\right)}{\Gamma_e\left(cx_2^{\pm1}z^{(1)\pm1}\right)\prod_{a=1}^{2}\Gamma_e\left(p^{1/2}q^{1/2}t^{-1/2}c\,x_3^{\pm1}z_a^{(2)\pm1}\right)}\times \\
&\times \Gamma_e\left(p^{-1}q^{-1}tcx_1^{\pm1}z^{(1)\pm1}\right)\prod_{a=1}^{2}\Gamma_e\left(p^{-1/2}q^{-1/2}t^{1/2}cx_2^{\pm1}z_a^{(2)\pm1}\right)\,.
\end{aligned} \tag{C.11}$$

We first apply the Intriligator–Pouliot duality on the leftmost node, which corresponds to the following identity:

$$\begin{aligned}
&\oint \mathrm{d}\vec{z}_1^{(1)}\,\Gamma_e(p^{-1}q^{-1}tcx_1^{\pm1}z^{(1)\pm1})\Gamma_e(pqc^{-1}x_2^{\pm1}z^{(1)\pm1})\prod_{a=1}^{2}\Gamma_e(p^{1/2}q^{1/2}t^{-1/2}z^{(1)\pm1}z_a^{(2)\pm1}) = \\
&= \Gamma_e(p^{-2}q^{-2}t^2c^2)\Gamma_e(tx_1^{\pm1}x_2^{\pm1})\prod_{a=1}^{2}\Gamma_e(p^{-1/2}q^{-1/2}t^{1/2}cx_1^{\pm1}z_a^{(2)\pm1})\times \\
&\quad\times \Gamma_e(p^2q^2c^{-2})\prod_{a=1}^{2}\Gamma_e(p^{3/2}q^{3/2}t^{-1/2}c^{-1}x_2^{\pm1}z_a^{(2)\pm1})\Gamma_e(pqt^{-1})^2\prod_{a<b}^{2}\Gamma_e(pqt^{-1}z_a^{(2)\pm1}z_b^{(2)\pm1})\times \\
&\quad\times \oint \mathrm{d}\vec{z}_1^{(1)}\,\Gamma_e(p^{3/2}q^{3/2}t^{-1}c^{-1}x_1^{\pm1}z^{(1)\pm1})\Gamma_e(p^{-1/2}q^{-1/2}cx_2^{\pm1}z^{(1)\pm1})\prod_{a=1}^{2}\Gamma_e(t^{1/2}z^{(1)\pm1}z_a^{(2)\pm1})\,.
\end{aligned} \tag{C.12}$$

Next, we collect the $z^{(2)}$ dependent factors and apply the Intriligator–Pouliot duality again:

$$\begin{aligned}
&\oint \mathrm{d}\vec{z}_2^{(2)}\prod_{a=1}^{2}\Gamma_e(p^{1/2}q^{1/2}t^{-1}z_a^{(2)\pm1}y^{(1)\pm1})\Gamma_e(p^{1/2}q^{1/2}t^{-1/2}z_a^{(2)\pm1}y^{(2)\pm1})\times \\
&\times\prod_{a=1}^{2}\Gamma_e(p^{1/2}q^{1/2}t^{1/2}c^{-1}x_3^{\pm1}z_a^{(2)\pm1})\Gamma_e(p^{-1/2}q^{-1/2}t^{1/2}cx_1^{\pm1}z_a^{(2)\pm1})\Gamma_e(t^{1/2}z^{(1)\pm1}z_a^{(2)\pm1}) = \\
&= \Gamma_e(pqt^{-2})\Gamma_e(pqt^{-3/2}y^{(1)\pm1}y^{(2)\pm1})\Gamma_e(pqt^{-1/2}c^{-1}y^{(1)\pm1}x_3^{\pm1})\Gamma_e(t^{-1/2}cy^{(1)\pm1}x_1^{\pm1})\times \\
&\quad\times \Gamma_e(p^{1/2}q^{1/2}t^{-1/2}y^{(1)\pm1}z^{(1)\pm1})\Gamma_e(pqc^{-1}y^{(2)\pm1}x_3^{\pm1})\Gamma_e(cy^{(2)\pm1}x_1^{\pm1})\times \\
&\quad\times \Gamma_e(pqtc^{-2})\Gamma_e(tx_3^{\pm1}x_1^{\pm1})\Gamma_e(p^{1/2}q^{1/2}tc^{-1}x_3^{\pm1}z^{(1)\pm1})\times \\
&\quad\times \Gamma_e(p^{-1}q^{-1}tc^2)\Gamma_e(p^{-1/2}q^{-1/2}tcx_1^{\pm1}z^{(1)\pm1})\oint \mathrm{d}\vec{z}_1^{\prime(2)}\,\Gamma_e(tz^{\prime(2)\pm1}y^{(1)\pm1})\Gamma_e(t^{1/2}z^{\prime(2)\pm1}y^{(2)\pm1})\times \\
&\quad\times \Gamma_e(t^{-1/2}cx_3^{\pm1}z^{\prime(2)\pm1})\Gamma_e(pqt^{-1/2}c^{-1}x_1^{\pm1}z^{\prime(2)\pm1})\Gamma_e(p^{1/2}q^{1/2}t^{-1/2}z^{(1)\pm1}z^{\prime(2)\pm1})\,.
\end{aligned} \tag{C.13}$$

Lastly, we collect the $z^{(1)}$ dependent factors, which become

$$\begin{aligned}
&\oint \mathrm{d}\vec{z}_1^{(1)}\, \Gamma_e(p^{-1/2}q^{-1/2}cx_2^{\pm1}z^{(1)\pm1})\Gamma_e(p^{1/2}q^{1/2}tc^{-1}x_3^{\pm1}z^{(1)\pm1})\times\\
&\times \Gamma_e(p^{1/2}q^{1/2}t^{-1/2}z^{(1)\pm1}y^{(1)\pm1})\Gamma_e(p^{1/2}q^{1/2}t^{-1/2}z^{(1)\pm1}z'^{(2)\pm1}) =\\
&= \Gamma_e(p^{-1}q^{-1}c^2)\Gamma_e(tx_2^{\pm1}x_3^{\pm1})\Gamma_e(t^{-1/2}cx_2^{\pm1}y^{(1)\pm1})\Gamma_e(t^{-1/2}cx_2^{\pm1}z'^{(2)\pm1})\Gamma_e(pqt^2c^{-2})\times\\
&\quad\times \Gamma_e(pqt^{1/2}c^{-1}x_3^{\pm1}y^{(1)\pm1})\Gamma_e(pqt^{1/2}c^{-1}x_3^{\pm1}z'^{(2)\pm1})\Gamma_e(pqt^{-1})^2\Gamma_e(pqt^{-1}y^{(1)\pm1}z'^{(2)\pm1})\times\\
&\quad\times \oint \mathrm{d}\vec{z}_1'^{(1)}\, \Gamma_e(pqc^{-1}x_2^{\pm1}z'^{(1)\pm1})\Gamma_e(t^{-1}cx_3^{\pm1}z'^{(1)\pm1})\Gamma_e(t^{1/2}z'^{(1)\pm1}y^{(1)\pm1})\Gamma_e(t^{1/2}z'^{(1)\pm1}z'^{(2)\pm1})\,.
\end{aligned} \tag{C.14}$$

Combining all the remaining factors, we obtain the following expression for the entire supersymmetric index:

$$\begin{aligned}
&\mathcal{I}_{\mathsf{T}^\vee}\left(\vec{x};y^{(1)},y^{(2)};c;t\right) = \Gamma_e(t^{-1/2}cx_1^{\pm1}y^{(1)\pm1})\Gamma_e(t^{-1/2}cx_2^{\pm1}y^{(1)\pm1})\Gamma_e(cx_1^{\pm1}y^{(2)\pm1})\Gamma_e(pqt^2c^{-2})\times\\
&\times \oint \mathrm{d}\vec{z}_1'^{(1)}\,\mathrm{d}\vec{z}_1'^{(2)}\,\Gamma_e\left(pqt^{-1}\right)^2\Gamma_e(t^{1/2}z'^{(1)\pm1}y^{(1)\pm1})\times\\
&\times \Gamma_e(pqc^{-1}x_2^{\pm1}z'^{(1)\pm1})\Gamma_e(t^{-1}cx_3^{\pm1}z'^{(1)\pm1})\Gamma_e(t^{1/2}z'^{(2)\pm1}y^{(2)\pm1})\\
&\times \Gamma_e(pqt^{-1/2}c^{-1}x_1^{\pm1}z'^{(2)\pm1})\Gamma_e(t^{-1/2}cx_2^{\pm1}z'^{(2)\pm1})\Gamma_e(t^{1/2}z'^{(1)\pm1}z'^{(2)\pm1}) =\\
&= \mathcal{I}_{E[USp(6)]^{[2,1]}}\left(\vec{x};y^{(1)},y^{(2)};c;pq/t\right)\,,
\end{aligned} \tag{C.15}$$

which completes the derivation.

C.3 Alternative piecewise derivation of $E_{[N-1,1]}[USp(2N)] \leftrightarrow E^{[N-1,1]}[USp(2N)]$

In section 4.3.2, we derived the mirror-like duality between $E_{[N-1,1]}[USp(2N)]$ and $E^{[N-1,1]}[USp(2N)]$ using the $E[USp(2N)]$ duality web. In this appendix we provide an alternative derivation of this duality.

First we note that the $3d$ counterpart of this duality is the abelian mirror symmetry which maps the $3d$ SQED with N flavors to an abelian quiver of $N-1$ gauge nodes with one flavor attached to each end of the quiver. This abelian mirror can be obtained by sequential applications of the Aharony duality between the SQED with one flavor and the XYZ WZ model [168]. Accordingly one can expect that the $4d$ mirror-like duality between $E_{[N-1,1]}[USp(2N)]$ and $E^{[N-1,1]}[USp(2N)]$ is also obtained by sequential applications of the Intriligator–Pouliot duality in the confining case, which indeed turns out to be true. For example, this procedure for $N=3$ is shown in Figure C.1. In this appendix, we also exhibit the derivation of the duality in terms of the $4d$ superconformal index.

Let us start with $E^{[2,1]}[USp(6)]$, whose supersymmetric index is given by

$$\begin{aligned}
&\mathcal{I}_{E^{[2,1]}[USp(6)]}\left(y^{(1)},y^{(2)};\vec{x};c;pq/t\right) =\\
&= \Gamma_e(t^{-1/2}cx_1^{\pm1}y^{(1)\pm1})\Gamma_e(t^{-1/2}cx_2^{\pm1}y^{(1)\pm1})\Gamma_e(cx_1^{\pm1}y^{(2)\pm1})\Gamma_e(pqt^2c^{-2})\times\\
&\quad\times \oint \mathrm{d}\vec{z}_1^{(1)}\,\mathrm{d}\vec{z}_1^{(2)}\,\Gamma_e\left(pqt^{-1}\right)^2\Gamma_e(t^{1/2}z^{(1)\pm1}y^{(1)\pm1})\times
\end{aligned}$$

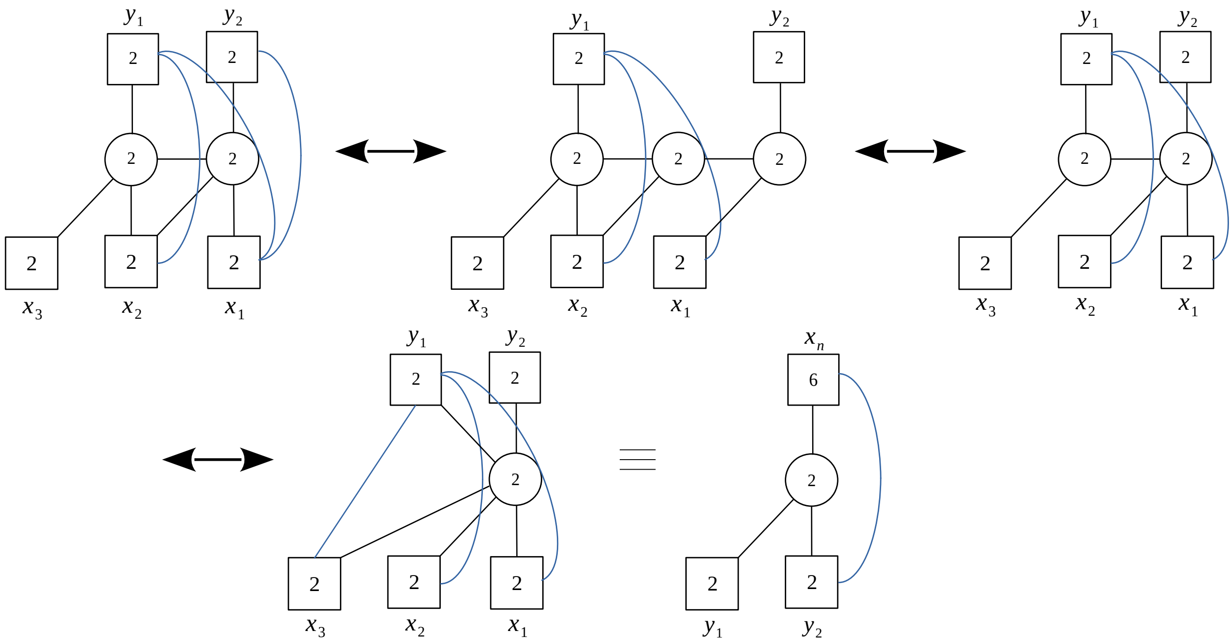

Figure C.1: A direct derivation of the $4d$ mirror-like duality between $E_{[N-1,1]}[USp(2N)]$ and $E^{[N-1,1]}[USp(2N)]$ using the Intriligator–Pouliot duality.

$$\begin{aligned}&\times \Gamma_e(pqc^{-1}x_2^{\pm 1}z^{(1)\pm 1})\Gamma_e(t^{-1}cx_3^{\pm 1}z^{(1)\pm 1})\Gamma_e(t^{1/2}z^{(2)\pm 1}y^{(2)\pm 1})\times \\ &\times \Gamma_e(pqt^{-1/2}c^{-1}x_1^{\pm 1}z^{(2)\pm 1})\Gamma_e(t^{-1/2}cx_2^{\pm 1}z^{(2)\pm 1})\Gamma_e(t^{1/2}z^{(1)\pm 1}z^{(2)\pm 1})\,. \end{aligned} \tag{C.16}$$

We can apply the Intriligator–Pouliot duality relating a WZ model with 15 chirals to the $USp(2)$ theory with six chirals to trade some of the chirals in (C.16) for a new $USp(2)$ gauge node:

$$\begin{aligned}&\Gamma_e(cx_1^{\pm 1}y^{(2)\pm 1})\Gamma_e(pqt^{-1/2}c^{-1}x_1^{\pm 1}z^{(2)\pm 1})\Gamma_e(t^{1/2}z^{(2)\pm 1}y^{(2)\pm 1}) = \\ &= \Gamma_e(p^2q^2t^{-1}c^{-2})\Gamma_e(t)\Gamma_e(c^2)\oint \mathrm{d}\vec{z}_1'^{(1)}\,\Gamma_e(p^{-1/2}q^{-1/2}t^{1/2}cz'^{(1)\pm 1}y^{(2)\pm 1})\times \\ &\quad \times \Gamma_e(p^{1/2}q^{1/2}t^{-1/2}z'^{(1)\pm 1}x_1^{\pm 1})\Gamma_e(p^{1/2}q^{1/2}c^{-1}z'^{(1)\pm 1}z^{(2)\pm 1})\,, \end{aligned} \tag{C.17}$$

in this way we obtain the second quiver in Figure C.1.

We then observe that collecting the factors depending on $z^{(2)}$, we can apply the Intriligator–Pouliot duality to confine the second node in the second quiver in Figure C.1

$$\begin{aligned}&\oint \mathrm{d}\vec{z}_1^{(2)}\,\Gamma_e(t^{1/2}z^{(1)\pm 1}z^{(2)\pm 1})\Gamma_e(t^{-1/2}cx_2^{\pm 1}z^{(2)\pm 1})\Gamma_e(p^{1/2}q^{1/2}c^{-1}z'^{(1)\pm 1}z^{(2)\pm 1}) = \\ &= \Gamma_e(cz^{(1)\pm 1}x_2^{\pm 1})\Gamma_e(p^{1/2}q^{1/2}t^{-1/2}x_2^{\pm 1}z'^{(1)\pm 1})\Gamma_e(p^{1/2}q^{1/2}t^{1/2}c^{-1}z^{(1)\pm 1}z'^{(1)\pm 1})\times \\ &\times \Gamma_e(t)\Gamma_e(t^{-1}c^2)\Gamma_e(pqc^{-2})\,, \end{aligned} \tag{C.18}$$

we then arrive at the third quiver in Figure C.1.

After this, we collect the factors depending on $z^{(1)}$ and apply again the Intriligator–Pouliot duality to confine this node

$$\begin{aligned}&\oint \mathrm{d}\vec{z}_1^{(1)}\,\Gamma_e(t^{1/2}z^{(1)\pm1}y^{(1)\pm1})\Gamma_e(t^{-1}cz^{(1)\pm1}x_3^{\pm1})\Gamma_e(p^{1/2}q^{1/2}t^{1/2}c^{-1}z^{(1)\pm1}z'^{(1)\pm1}) = \\ &= \Gamma_e(t^{-1/2}cx_3^{\pm1}y^{(1)\pm1})\Gamma_e(p^{1/2}q^{1/2}t^{-1/2}x_3^{\pm1}z'^{(1)\pm1})\Gamma_e(p^{1/2}q^{1/2}tc^{-1}y^{(1)\pm1}z'^{(1)\pm1})\times \\ &\quad\times \Gamma_e(t)\Gamma_e(t^{-2}c^2)\Gamma_e(pqtc^{-2})\,. \end{aligned} \tag{C.19}$$

Collecting the remaining factors, we obtain the partition function of the last quiver in Figure C.1

$$\begin{aligned}&\mathcal{I}_{E^{[2,1]}[USp(6)]}\left(y^{(1)},y^{(2)};\vec{x};c;pq/t\right) = \\ &= \frac{\prod_{i=1}^3\Gamma_e\left(t^{-1/2}cy^{(1)\pm1}x_i^{\pm1}\right)}{\Gamma_e\left(p^{-1}q^{-1}tc^2\right)}\oint \mathrm{d}\vec{z}_1'^{(1)}\,\Gamma_e\left(t\right)\Gamma_e\left(p^{1/2}q^{1/2}tc^{-1}y^{(1)\pm1}z'^{(1)\pm1}\right)\times \\ &\quad\times\Gamma_e\left(p^{-1/2}q^{-1/2}t^{1/2}cy^{(2)\pm1}z'^{(1)\pm1}\right)\prod_{i=1}^3\Gamma_e\left(p^{1/2}q^{1/2}t^{-1/2}x_i^{\pm1}z'^{(1)\pm1}\right) = \\ &= \mathcal{I}_{E_{[2,1]}[USp(6)]}\left(\vec{x};y^{(2)},y^{(1)};c;t\right)\,, \end{aligned} \tag{C.20}$$

which coincides with the supersymmetric index of $E_{[2,1]}[USp(6)]$ given in (4.84). Applying this procedure for generic N we can prove the identity between the indices of $\mathcal{I}_{E_{[N-1,1]}[USp(2N)]}$ and $\mathcal{I}_{E^{[N-1,1]}[USp(2N)]}$.

Appendix D

Anomalies from the $6d$ E-string theory

In this appendix we derive the predicted anomalies from $6d$ for the $4d$ models obtained compactifying the E-string theory on a generic Riemann surface of genus g with s punctures and a general flux for the $6d$ global symmetry. We also compute the anomalies for the flavor symmetry that descend from the $SU(2)_{\text{ISO}}$ isometry in the case in which the Riemann surface is $\mathbb{S}^2$.

We start by writing the $6d$ anomaly polynomial eight-form for the rank N E-string theory as given in [116]

$$\begin{aligned} I_8^{E-string} \quad = \quad & \frac{N\left(4N^2+6N+3\right)}{24}C_2^2\left(R\right)_{\mathbf{2}}+\frac{\left(N-1\right)\left(4N^2-2N+1\right)}{24}C_2^2\left(L\right)_{\mathbf{2}} \\ & -\frac{N\left(N^2-1\right)}{3}C_2\left(R\right)_{\mathbf{2}}C_2\left(L\right)_{\mathbf{2}}+\frac{\left(N-1\right)\left(6N+1\right)}{48}C_2\left(L\right)_{\mathbf{2}}p_1\left(T\right) \\ & -\frac{N\left(6N+5\right)}{48}C_2\left(R\right)_{\mathbf{2}}p_1\left(T\right)+\frac{N\left(N-1\right)}{120}C_2\left(L\right)_{\mathbf{2}}C_2\left(E_8\right)_{\mathbf{248}} \\ & -\frac{N\left(N+1\right)}{120}C_2\left(R\right)_{\mathbf{2}}C_2\left(E_8\right)_{\mathbf{248}}+\frac{N}{240}p_1\left(T\right)C_2\left(E_8\right)_{\mathbf{248}} \\ & +\frac{N}{7200}C_2\left(E_8\right)^2_{\mathbf{248}}+\left(30N-1\right)\frac{7p_1\left(T\right)-4p_2\left(T\right)}{5760}\,, \end{aligned} \tag{D.1}$$

where $C_n(G)_{\mathbf{R}}$ is the n-th Chern class of the group G in the representation $\mathbf{R}$. In particular, $C_2(R)_{\mathbf{2}}$ and $C_2(L)_{\mathbf{2}}$ stand for the second Chern class of $SU(2)_R$ and $SU(2)_L$, respectively, in the doublet representation. Moreover, $p_1(T)$ and $p_2(T)$ denote the first and second Pontryagin classes of the tangent bundle T.

We wish to write the anomalies under the decomposition

$$E_8 \to E_7 \times U(1)_c \to SU(8)_u \times U(1)_c \to \prod_{a=1}^{8} U(1)_{u_a} \times U(1)_c\,, \tag{D.2}$$

with the constraint that $U(1)_{u_8} = -\sum_{a=1}^{7} U(1)_{u_a}$. Thus, we first want to decompose the E_8 Chern classes to the Chern classes of E_7 and $U(1)_c$. Using the decomposition

$$\mathbf{248} \to (\mathbf{133}) \oplus (\mathbf{56})\left(c+c^{-1}\right) \oplus \left(c^2+1+c^{-2}\right)\,, \tag{D.3}$$

We can substitute the following Chern classes:

$$C_2\left(E_8\right)_{\mathbf{248}} = -60\, C_1^2\left(c\right) + C_2\left(E_7\right)_{\mathbf{133}} + 2\, C_2\left(E_7\right)_{\mathbf{56}} \,. \tag{D.4}$$

Next, we further break $E_7 \to SU(8)_u$ using the branching rules

$$\begin{aligned} \mathbf{56} &\to (\mathbf{28}) \oplus (\overline{\mathbf{28}}) \,, \\ \mathbf{133} &\to (\mathbf{63}) \oplus (\mathbf{70}) \,. \end{aligned} \tag{D.5}$$

Translating to the Chern classes substitutions

$$\begin{aligned} C_2\left(E_7\right)_{\mathbf{56}} &= C_2\left(SU\left(8\right)\right)_{\mathbf{28}} + C_2\left(SU\left(8\right)\right)_{\overline{\mathbf{28}}} = 12\, C_2\left(SU\left(8\right)\right)_{\mathbf{8}} \,, \\ C_2\left(E_7\right)_{\mathbf{133}} &= C_2\left(SU\left(8\right)\right)_{\mathbf{63}} + C_2\left(SU\left(8\right)\right)_{\mathbf{70}} = 36\, C_2\left(SU\left(8\right)\right)_{\mathbf{8}} \,, \end{aligned} \tag{D.6}$$

where in the second equalities we used the fact that $C_2\left(G\right)_{\mathbf{R_1}}/T_G\left(\mathbf{R_1}\right) = C_2\left(G\right)_{\mathbf{R_2}}/T_G\left(\mathbf{R_2}\right)$, with $T_G\left(\mathbf{R}\right)$ standing for the Dynkin index of the representation $\mathbf{R}$ of the group G (a.k.a. quadratic Casimir).

The final step in the decomposition is the one taking $SU(8)_u \to U(1)^7$ with the well known branching rule

$$\mathbf{8} \to \sum_{i=1}^{8} u_a \,, \tag{D.7}$$

with the constraint $u_8 = \prod_{i=1}^{7} u_i^{-1}$. In terms of Chern classes it translates into

$$C_2\left(SU\left(8\right)\right)_{\mathbf{8}} = -\frac{1}{2}\sum_{a=1}^{8} C_1^2\left(U(1)_{u_a}\right) \,. \tag{D.8}$$

Gathering all the decompositions together, we find that we simply need to substitute

$$C_2\left(E_8\right)_{\mathbf{248}} = -60\, C_1^2\left(c\right) - 30\sum_{a=1}^{8} C_1^2\left(U(1)_{u_a}\right) \tag{D.9}$$

in the above anomaly polynomial.

In the main text we also consider models with flux in the Cartan of $SU(2)_L$ global symmetry. In this case we should break $SU(2)_L \to U(1)_t$ by setting the Chern classes as follows:

$$C_2\left(L\right)_{\mathbf{2}} = -C_1^2\left(t\right) \,. \tag{D.10}$$

After the above substitutions, we obtain the 6*d* 8-form anomaly polynomial written in terms of Chern classes for the $U(1)$ symmetries in the Cartan of the 6*d* global symmetry, including the R-symmetry. The next step consists of compactifying the theory on a Riemann surface C_g of genus g. At the level of the Pontryagin classes this means

$$p_1(T_{6d}) = p_1(T_{4d}) + e^2, \qquad p_2(T_{6d}) = p_2(T_{4d}) + p_1(T_{4d})e^2 \,, \tag{D.11}$$

where e is the Euler class of the Riemann surface. We also want to turn on fluxes n_t for $U(1)_t$, n_c for $U(1)_c$ and n_a for $U(1)_{u_a}$ through the Riemann surface, meaning

$$\begin{aligned} C_1^{6d}(t) &= C_1^{4d}(t) - n_t \frac{e}{2(1-g)} \\ C_1^{6d}(c) &= C_1^{4d}(c) - n_c \frac{e}{2(1-g)} \\ C_1^{6d}(u_a) &= C_1^{4d}(u_a) - n_a \frac{e}{2(1-g)}\,, \end{aligned} \tag{D.12}$$

where the Chern classes for the $SU(8)$ Cartan satisfy $C_1^{4d}(u_8) = -\sum_{a=1}^{7} C_1^{4d}(u_a)$. For the R-symmetry we turn on a very specific flux which is the one needed to perform the topological twist that preserves half of the supercharges in the compactification. This amounts to

$$C_2^{6d}(R)_{\mathbf{2}} = -C_1^{6d}(R)^2, \qquad C_1^{6d}(R) = C_1^{4d}(R) - \frac{e}{2}\,. \tag{D.13}$$

When we compactify our theory on a generic Riemann surface, only the terms linear in e out of the full 8-form anomaly polynomial contribute and their contribution can be computed using the Gauss–Bonnet theorem

$$\int_{C_g} e = 2(1-g)\,. \tag{D.14}$$

In this way we get the 6-form anomaly polynomial I_6 of the $4d$ theory, out of which we can read all the anomalies for the $4d$ global symmetries that descend from $6d$

$$\begin{aligned} &\mathrm{Tr}\, U(1)_i = -24 I_6\big|_{C_1^{4d}(U(1)_i) p_1(T_{4d})}, \\ &\mathrm{Tr}\, U(1)_i U(1)_j U(1)_k = d_{ijk} I_6\big|_{C_1^{4d}(U(1)_i) C_1^{4d}(U(1)_j) C_1^{4d}(U(1)_k)}\,, \end{aligned} \tag{D.15}$$

where $d_{ijk} = m!$ with m the number of equal indices between i, j and k. Eventually we find the following $4d$ anomalies:

$$\begin{aligned} &\mathrm{Tr}\left(U(1)_R^3\right) = \left(g + \frac{s}{2} - 1\right) N \left(4N^2 + 6N + 3\right), \\ &\mathrm{Tr}\left(U(1)_R\right) = -\left(g + \frac{s}{2} - 1\right) N \left(6N + 5\right), \\ &\mathrm{Tr}\left(U(1)_c^3\right) = -12 N n_c, \quad \mathrm{Tr}\left(U(1)_c\right) = -12 N n_c\,, \\ &\mathrm{Tr}\left(U(1)_{u_a}^3\right) = -6N\left(n_a - n_8\right), \quad \mathrm{Tr}\left(U(1)_{u_a}\right) = -6N\left(n_a - n_8\right), \\ &\mathrm{Tr}\left(U(1)_R U(1)_c^2\right) = -2\left(g + \frac{s}{2} - 1\right) N\left(N+1\right), \quad \mathrm{Tr}\left(U(1)_R^2 U(1)_c\right) = 2N\left(N+1\right) n_c\,, \\ &\mathrm{Tr}\left(U(1)_R U(1)_{u_a}^2\right) = -2\left(g + \frac{s}{2} - 1\right) N\left(N+1\right), \\ &\mathrm{Tr}\left(U(1)_R^2 U(1)_{u_a}\right) = N\left(N+1\right)\left(n_a - n_8\right), \\ &\mathrm{Tr}\left(U(1)_R U(1)_{u_a} U(1)_{u_b}\right) = -\left(g + \frac{s}{2} - 1\right) N\left(N+1\right), \\ &\mathrm{Tr}\left(U(1)_c U(1)_{u_a}^2\right) = -4N n_c, \quad \mathrm{Tr}\left(U(1)_c^2 U(1)_{u_a}\right) = -2N\left(n_a - n_8\right), \end{aligned}$$

$$\begin{aligned}
&\mathrm{Tr}\left(U(1)_{u_a}U(1)_{u_b}^2\right) = -2N\left(n_a+n_b-2n_8\right), \quad \mathrm{Tr}\left(U(1)_cU(1)_{u_a}U(1)_{u_b}\right) = -2Nn_c\,,\\
&\mathrm{Tr}\left(U(1)_{u_a}U(1)_{u_b}U(1)_{u_d}\right) = -N\left(n_a+n_b+n_d-3n_8\right),\\
&\mathrm{Tr}\left(U(1)_t^3\right) = -\left(N-1\right)\left(4N^2-2N+1\right)n_t, \quad \mathrm{Tr}\left(U(1)_t\right) = -\left(N-1\right)\left(6N+1\right)n_t\,,\\
&\mathrm{Tr}\left(U(1)_RU(1)_t^2\right) = -\frac{4}{3}\left(g+\frac{s}{2}-1\right)N\left(N^2-1\right),\\
&\mathrm{Tr}\left(U(1)_R^2U(1)_t\right) = \frac{4}{3}N\left(N^2-1\right)n_t\,,\\
&\mathrm{Tr}\left(U(1)_tU(1)_{c/u_a}^2\right) = -2N\left(N-1\right)n_t, \quad \mathrm{Tr}\left(U(1)_t^2U(1)_c\right) = -2N\left(N-1\right)n_c\,,\\
&\mathrm{Tr}\left(U(1)_t^2U(1)_{u_a}\right) = -N\left(N-1\right)\left(n_a-n_8\right),\\
&\mathrm{Tr}\left(U(1)_tU(1)_{u_a}U(1)_{u_b}\right) = -N\left(N-1\right)n_t\,,
\end{aligned} \tag{D.16}$$

with the constraint $n_8 = -\sum_{a=1}^{7} n_a$ and where we also shifted $g \to g+\frac{s}{2}$ to include the contribution of possible s punctures that the Riemann surface may possess. The rest of the anomalies that don't appear vanish. Moreover, the 6d R-symmetry used to computed the anomalies and the 4d R-symmetry used in the main text are related by $R_{6d} = R_{4d} + q_t$, where q_t denotes the charge under $U(1)_t$.

When the Riemann surface possesses some isometry, this manifests itself as a flavor symmetry from the point of view of the 4d theory. We can then compute the anomalies for such a symmetry starting from the 8-form anomaly polynomial of the 6d theory [199] and for this we will follow the discussion of [130] (See [200–202] for earlier application in physics.). In the main text we are also interested in the case in which the surface is a two-sphere, whose isometry group is $SO(3)_{\mathrm{ISO}} \cong SU(2)_{\mathrm{ISO}}$. The anomalies for this symmetry can be computed following the same procedure that led us to (3.4) by just taking into account that, from the 8-form anomaly polynomial, we now receive contributions also from terms which are cubic in the Euler class e. This follows from the fact that now e fibers the surface in a non-trivial way over the 4d space. Their contribution can be computed using the Bott–Cattaneo formula [129], which in general states that

$$\int_{\mathbb{S}^2} e^{2s+1} = 2p_1(SO(3)_{\mathrm{ISO}})^s, \qquad \int_{\mathbb{S}^2} e^{2s} = 0\,, \tag{D.17}$$

where $p_1(SO(3)_{\mathrm{ISO}})$ is the first Pontryagin class of the real vector bundle for the $SO(3)_{\mathrm{ISO}}$ isometry of $\mathbb{S}^2$. In particular, for $s=0$ we recover the usual Gauss–Bonnet theorem (D.14) for the two-sphere, while for $s=1$ we get the formula that we need for evaluating the integral of e^3

$$\int_{\mathbb{S}^2} e^3 = -2C_2(SU(2)_{\mathrm{ISO}})_{\mathbf{3}}\,, \tag{D.18}$$

where we used that $p_1(SO(3)_{\mathrm{ISO}}) = -C_2(SU(2)_{\mathrm{ISO}})_{\mathbf{3}}$ when we think of $SO(3)_{\mathrm{ISO}} \cong SU(2)_{\mathrm{ISO}}$. In this way we get additional terms in the 6-form anomaly polynomial of the 4d theory that are proportional to $C_2(SU(2)_{\mathrm{ISO}})_{\mathbf{3}}$, from which we can read off the anomalies for the 4d flavor symmetry $SU(2)_{\mathrm{ISO}}$. Using the fact that

$$\mathrm{Tr}\left(G^2\,U(1)\right) = -T_G(\mathbf{R})I_6\big|_{C_2(G)_{\mathbf{R}}C_1(U(1))}\,, \tag{D.19}$$

and that the Dynkin index of the adjoint representation of $SU(2)_{\text{ISO}}$ is 2, we find

$$\begin{aligned}
\text{Tr}\left(SU(2)^2_{\text{ISO}}U(1)_R\right) &= \frac{N(N+1)}{12}\left(-8+6n_c^2+4(N-1)n_t^2-4N+3\sum_{a=1}^{8}n_a^2\right)\\
\text{Tr}\left(SU(2)^2_{\text{ISO}}U(1)_t\right) &= -\frac{N-1}{12}n_t\left(-1+2N(3n_c^2-5)+(4N^2-2N+1)n_t^2+\right.\\
&\qquad\left.+N(-4N+3\sum_{a=1}^{8}n_a^2)\right)\\
\text{Tr}\left(SU(2)^2_{\text{ISO}}U(1)_c\right) &= -\frac{N}{2}n_c\left(-3+2n_c^2+(N-1)n_t^2-N+\sum_{a=1}^{8}n_a^2\right)\\
\text{Tr}\left(SU(2)^2_{\text{ISO}}U(1)_{u_a}\right) &= -\frac{N}{4}(n_a-n_8)\left(-3+2n_c^2+(N-1)n_t^2-N+\sum_{a=1}^{8}n_a^2\right)\,.
\end{aligned} \tag{D.20}$$

Note that the anomalies of $SU(2)_{\text{ISO}}$ depend on fluxes qualitatively in a different way than the ones for other symmetries. Namely these are non-linear in the fluxes. In particular this implies that the correct relation of the $U(1)_f$ symmetry to the Cartan of $SU(2)_{\text{ISO}}$ depends on the fluxes.

We would like to give a prediction for the mixing of the $U(1)_f$ symmetry appearing in (3.137) with fugacity f and other $U(1)$ symmetries. Such mixing is required when generating a flux sphere in order to get an enhancement of symmetry to the isometry symmetry of a sphere given by $SU(2)_{\text{ISO}}$. Finding the mixing can be done in an algorithmic fashion by comparing the anomalies involving a generally mixed $U(1)_f$ with the ones predicted from $6d$ given in (D.20). Building a sphere involves generally two caps and a series of tubes in glued in between them. We will fix the initial isometry $U(1)_f$ symmetry on one of the caps to be the one naturally derived from the tube in (3.137) and set the rest of the $U(1)_f$ symmetries of the tubes and the other cap to mix with the other $U(1)$'s such that all the mixed gauge anomalies vanish. In this construction the mixing is given by

$$q_f^{mixed} = q_f + (n_t+1)q_t + (n_c-1)q_c + \sum_{i=1}^{7}\left(n_{a_i}-\frac{1}{4}\right)q_{a_i}, \tag{D.21}$$

where q_m denotes the charge of a chiral multiplet under $U(1)_m$, and n_m denote the flux of the sphere under $U(1)_m$. One can see that all the expected examples in Subsection 3.5 obey this mixing prescription.

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