VANISHING VISCOSITY AND BACKWARD EULER APPROXIMATIONS FOR CONSERVATION LAWS WITH DISCONTINUOUS FLUX

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Abstract. Solutions to a class of one dimensional conservation laws with discontinuous flux are constructed relying on the Crandall-Liggett theory of nonlinear contractive semigroups [14, 21], with a vanishing viscosity approach. The solutions to the corresponding viscous conservation laws are studied using the Backward Euler approximations. We prove their convergence to a *unique* vanishing viscosity solution to the Cauchy problem for the non viscous equations as the viscous parameter tends to zero. This approach allows to avoid the technicalities in existing literature such as traces, Riemann problems, interfaces conditions, compensated compactness and entropy inequalities. Consequently we establish our result under very mild assumptions on the flux, with only a requirement on the smoothness with respect to the unknown variable and a condition that allows the application of the maximum principle.

Key words. Scalar Conservation Laws, discontinuous flux, vanishing viscosity, nonlinear semigroups, backward Euler approximation.

AMS subject classifications. 35L65, 35R05

1. Introduction .

We consider the Cauchy problem for the scalar conservation law

$$u_t + f(x,u)_x = 0, \tag{1.1}$$

with initial data

$$u(0,x) = \bar{u}(x). \tag{1.2}$$

In the simpler case where $f = f(u)$ is independent of x, solutions have been constructed by a variety of techniques [22, 23, 41]. In particular, in [20] it was proved that the abstract theory of nonlinear contractive semigroups developed by Crandall and Liggett [21] can indeed be applied to scalar conservation laws, and yields the same solutions obtained by Kruzhkov [41] as vanishing viscosity limits. While these approaches are effective even for multi–dimensional scalar conservation laws, their exploitation is harder when the flux depends explicitly on the time and space variables (t,x) in a discontinuous way. Aim of the present paper is to develop a semigroup approach for the one–dimensional case in the more general context where the flux function $f = f(x,u)$ is allowed to depend on x in a discontinuous way, by extending the classical results of [20, 21].

We consider the following hypotheses on the flux f:

f0) i) $x \mapsto f(x,\omega)$ is in $\mathbf{L}^\infty(\mathbb{R},\mathbb{R})$ for any $\omega \in \mathbb{R}$; $\omega \mapsto f(x,\omega)$ is smooth for any $x \in \mathbb{R}$;

ii) there exists a constant $L \geq 0$ such that, for any fixed $x \in \mathbb{R}$:

$$|f(x,\omega_1) - f(x,\omega_2)| \leq L\,|\omega_1 - \omega_2|, \qquad \text{for any } \omega_1,\omega_2 \in \mathbb{R};$$

iii) there exists a constant $L_1 \geq 0$ such that,

$$\int_{\mathbb{R}} |f(x,0)|\; dx \leq L_1.$$

f1) The flux f satisfies **f0)**, and has the following form

$$f(x,\omega) = \begin{cases} f_l(\omega) & \text{if } x \leq 0, \\ f_r(\omega) & \text{if } x > 0, \end{cases}$$

where f_l and f_r are smooth functions satisfying

$$f_l(0) = f_r(0) = 0, \qquad f_l(1) = f_r(1).$$

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Scalar conservation laws with discontinuous flux arise in many applications where the conservation laws describe physical models in rough media. Examples include but are not limited to traffic flow with rough road condition and various polymer flooding models in two phase flow in porous media. Beginning with the work by Isaacson & Temple [31, 32, 33, 52] and by Risebro and collaborators [26, 27, 39], scalar conservation laws with discontinuous coefficients have become the topic of a vast literature [1, 7, 8, 10, 19, 24, 25, 26, 34, 40, 43, 48].

The existence of solutions for (1.1) can be established through a compactness argument on a family of approximate solutions. These approximations can be constructed by mollification of the flux [10, 39, 45], by wave front tracking [25, 26, 27, 28, 40], by Godunov's method [2, 4, 32, 38, 42], and by several other numerical schemes [16, 37, 48, 53, 54]. We would also like to mention the recent related results on existence of solutions for Cauchy problems for models of polymer flooding [50] and slow erosion in granular flow [49].

In a general setting, the solutions to the conservation law (1.1) can be obtained as limits of two combined approximations:

$$u_t + f^{\delta_n}(x,u)_x = \varepsilon_n u_{xx}. \tag{1.3}$$

Here $\varepsilon_n u_{xx}$ is a viscosity term, and $f^{\delta_n}(x,u)$ is a mollified flux which is smooth in x. As $n \to \infty$, one takes the double limits $\varepsilon_n \to 0$ and $\delta_n \to 0$ (where $f^{\delta_n} \to f^0 = f$). It is important to observe that in general these two limits do not commute. Indeed, one can let $\varepsilon_n, \delta_n \to 0$ keeping the ratio $\kappa = \delta_n/\varepsilon_n$ constant. A detailed study of viscous traveling waves in [29, 51] reveals that, for the same initial data, infinitely many limit solutions of (1.3) can exist, depending on the ratio κ. The uniqueness of the double-limit solution is proved in [51] only under some additional monotonicity conditions on the flux function and on the mollification f^{δ_n}.

In this paper we set $\delta_n \equiv 0$, and consider the viscous approximation to (1.1):

$$u_t + f(x,u)_x = \varepsilon u_{xx}, \tag{1.4}$$

for small $\varepsilon > 0$. A Backward Euler scheme is adopted to generate approximate solutions to the viscous equation (1.4). Using the results in [14, 21] and relying on a detailed study of the Backward Euler approximations, we establish existence and uniqueness of the vanishing viscosity limit, as $\varepsilon \to 0$.

We remark that the backward Euler approximation was recently implemented in [13], to construct a semigroup of solutions to a conservation law with nonlocal flux, modeling slow erosion phenomena in granular flow.

In the literature, uniqueness of solutions to (1.1) is usually obtained through specific entropy conditions, possibly supplemented with interface conditions at the point where the flux is discontinuous, satisfied by the limit of the approximate solutions, see [3, 8, 16, 17, 25, 37, 38, 39, 41]. It must be noted that, when the flux is discontinuous in the variable x, different entropy conditions or interface conditions may lead to different solutions to (1.1). This is also indicated by the non-uniqueness of the double limits for (1.3), studied in [51]. A systematic study of the various entropy conditions that can be imposed on the solutions to (1.1), leading to different semigroups of solutions, can be found in [8].

In addition to the vanishing viscosity approach, an additional approach to obtain uniqueness is available in the literature, utilizing the so called *adapted entropies*. The basic concept was first introduced in [9], and then further extended and applied in [11, 15, 18, 46]. Under further restrictions on the flux function, the adapted entropy inequality can be applied to multi-dimensional problems [15, 30]. However, with the exception of some very particular fluxes, the solutions selected by the adapted entropies in [9] are NOT the vanishing viscosity solutions obtained by letting $\varepsilon \to 0$ in (1.4). Some preliminary analysis shows that the adapted entropy concept corresponds to taking $\varepsilon_n \to 0$ first, then taking $\delta_n \to 0$ in (1.3). A detailed discussion can be found in Section 6, where counter examples and analysis for selected examples are provided, and more observations are made.

The novelty of our approach lies mainly on the techniques applied to the problem, i.e. the application of the Brezis & Pazy convergence result [14] to obtain the existence and uniqueness of vanishing viscosity solutions to conservation laws with discontinuous fluxes. Compactness and entropy arguments are only used to study solutions to the resolvent equations, constructing the approximate and the limit semigroups (see Section 4). This involves solutions to ordinary differential equations, depending only on the variable x. With this approach and using the result in [14], we prove directly the strong convergence of the semigroups generated by (1.4) to a unique semigroup generated by (1.1), without the need of additional entropy conditions. In this way, we obtain the uniqueness and strong convergence results without any additional hypothesis on the flux. We list a few comparisons with some existing literature.

- We do not require the nondegeneracy condition, which is usually required for compensated compactness arguments [35, 37].
- We do not have requirement on the shape of the graph of the flux, which is usually needed by the arguments based on BV bounds [3, 16, 25, 31, 38]. In particular, we do not exclude the presence of an infinite number of maxima/minima or flux crossings, which was required by [38].
- We study the convergence, as $\varepsilon \to 0$, of solutions to (1.4) directly, without mollifying the flux as it is done, for instance, in [18, 35, 36]. Therefore we avoid the problem of choosing the relative ratio of convergence between the mollification parameter and the viscosity.

In this paper we establish the existence and uniqueness of solution for the conservation law where the flux is discontinuous at one location. Such a result can serve as a building block for equations where the discontinuities in the flux function form a more complex pattern. Indeed, the result in this paper is utilized as the starting point for the recent paper [12], where the existence and uniqueness of the vanishing viscosity limit is extended to one dimensional scalar conservation laws with regulated flux. To be precise, in [12] we prove the existence and uniqueness of the limit as $\varepsilon \to 0$ of the solution u^ε to

$$\begin{cases} u_t + f(t,x,u)_x \;=\; \varepsilon u_{xx}\,, \\ u(0,x) \;=\; u_0(x). \end{cases}$$

Here the mapping $(t,x) \mapsto f$ is a regulated function in two dimensions (see Definition 1.1 in [12]), which can be highly discontinuous in the (t,x)–plane. Specially, this result can be applied directly to the existence and uniqueness of solutions to the triangular system

$$\begin{cases} u_t + f(v,u)_x \;=\; 0\,, \\ v_t + g(v)_x \;=\; 0, \end{cases} \qquad \begin{cases} u(0,x) = u_0(x), \\ v(0,x) = v_0(x), \end{cases}$$

as the vanishing viscosity solution of

$$\begin{cases} u_t + f(v,u)_x \;=\; \varepsilon u_{xx}\,, \\ v_t + g(v)_x \;=\; 0, \end{cases} \qquad \begin{cases} u(0,x) = u_0(x), \\ v(0,x) = v_0(x), \end{cases}$$

under mild assumptions on the flux g and the initial data $v_0(x)$. We refer to [12] for details.

The remainder of the paper is organized as follows. In Section 2 we review classical results on non linear semigroups that are used in the other sections. In Section 3 we study the *resolvent equation*

$$u + \lambda\,[f(x,u)_x - \varepsilon u_{xx}] = w$$

for the viscous problem and prove that, under the assumption **f0)**, it has a unique solution $u = J_\lambda^\varepsilon w$. Furthermore, according to [21], the operator J_λ^ε generates a non linear semigroup S_t^ε of weak solutions for the viscous equation (1.4). In Section 4, under the hypothesis **f1)**, we show that $J_\lambda^\varepsilon w$, as $\varepsilon \to 0$, converges to a unique limit $J_\lambda w$ which solves

$$u + \lambda f(x,u)_x = w.$$

In Section 5 we apply the results in [21] to show that J_λ generates a non linear semigroup S_t whose trajectories are solutions to (1.1). Then [14] is applied to show that S_t^ε converges to S_t uniformly for t in compact sets. See the diagram in Figure 1. In Section 6 we discuss in some detail the adapted entropies introduced in [9], to illustrate their difference from vanishing viscosity solutions. Finally, several examples and counterexamples related to the generation of non linear semigroups are presented in Section 7, together with some final remarks.

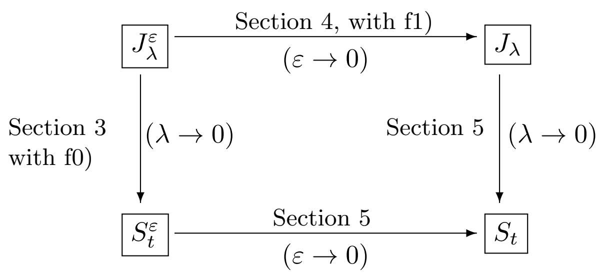

FIG. 1. *A diagram for the structure of the paper.*

2. Review on contractive semigroups generated by backward Euler operator. We first give a brief review on the main results in [14, 21], which are important to the analysis in this paper. Let X be a Banach space with norm $\|\cdot\|$, and let A be a possibly nonlinear, multivalued map that we view as a subset of $X \times X$. The set Au, the domain of A and its range are defined as

$$\begin{aligned} Au = &\ \{v \in X : (u,v) \in A\}, \\ \mathcal{D}(A) = &\ \{u \in X : \ \ Au \neq \emptyset\ \}, \\ \mathcal{R}(A) = &\ \bigcup_{u \in \mathcal{D}(A)} Au. \end{aligned} \tag{2.1}$$

We say that the operator A is **accretive** if

$$v_1 \in Au_1, \quad v_2 \in Au_2, \quad \lambda > 0 \quad \Longrightarrow \quad \big\|(u_1 + \lambda v_1) - (u_2 + \lambda v_2)\big\| \ \geq \ \|u_1 - u_2\|\,. \tag{2.2}$$

Consider the abstract Cauchy problem

$$\frac{d}{dt}u + Au \ \ni \ 0\,, \qquad\qquad u(0) = \bar{u}\,. \tag{2.3}$$

We define its **Backward Euler operator** J_λ by setting

$$(w,u) \in J_\lambda \quad \text{if and only if} \quad u \in \mathcal{D}(A) \ \text{ and there exists } v \in Au \text{ such that } u + \lambda v = w. \tag{2.4}$$

If A is accretive, because of (2.2), J_λ is a single valued map. Fix a time step $\lambda > 0$, we consider the approximation

$$u(t+\lambda) \ = \ J_\lambda u(t)\,. \tag{2.5}$$

Approximate solutions to (2.3) can be constructed by time iterations with the Backward Euler operator. For time interval $[0,\tau]$ and n time steps, one computes

$$u(\tau) \ \approx \ (J_\lambda)^n \bar{u}\,, \qquad \lambda = \tau/n.$$

The Backward Euler operator J_λ has the following properties.

LEMMA 2.1. *[21, Lemma 1.2] Let A be an accretive operator on the Banach space X, and assume that there exists $\lambda_0 > 0$ such that*

$$\mathcal{D}(J_\lambda) = \mathcal{R}(I + \lambda A) \ \supseteq \ \overline{\mathcal{D}(A)} \qquad\qquad \forall \lambda \in\,]0, \lambda_0]\,. \tag{2.6}$$

Then, for $\lambda, \mu \in\,]0, \lambda_0]$ the following holds.

(i) *The operator* J_λ *is single-valued. Indeed, for* $u_1, u_2 \in \mathcal{D}(J_\lambda)$

$$\|J_\lambda u_1 - J_\lambda u_2\| \;\leq\; \|u_1 - u_2\| \,. \tag{2.7}$$

(ii) *For* $u \in \mathcal{D}(A)$ *one has*

$$\frac{1}{\lambda}\,\|J_\lambda u - u\| \;\leq\; \|Au\| \;\doteq\; \inf\{\|v\|\,,\quad v \in Au\}. \tag{2.8}$$

(iii) *If* n *is a positive integer and* $u \in \mathcal{D}(J_\lambda)$*, then*

$$\|(J_\lambda)^n u - u\| \;\leq\; n\,\|J_\lambda u - u\| \,. \tag{2.9}$$

(iv) *For any* $u \in \mathcal{D}(J_\lambda)$*, the "resolvent formula" holds:*

$$\tilde{u} \;\doteq\; \frac{\mu}{\lambda}u + \frac{\lambda-\mu}{\lambda}J_\lambda u \in \mathcal{D}(J_\mu) \qquad \text{and} \qquad J_\lambda u = J_\mu\left(\tilde{u}\right). \tag{2.10}$$

Here and in the following I denotes the identity operator. The Backward Euler approximation converges to a limit as $n \to \infty$. The limit solution generates a semigroup of contractions, as shown in this elegant result by Crandall & Liggett [21].

Theorem 2.2. *[21, Theorem I] Let* A *be an accretive operator on the Banach space* X*, and* J_λ *the corresponding Backward-Euler operator. Assume that there exists* $\lambda_0 > 0$ *such that*

$$\mathcal{D}\left(J_\lambda\right) = \mathcal{R}(I + \lambda A) \;\supseteq\; \overline{\mathcal{D}(A)} \qquad\qquad \forall \lambda \in]0, \lambda_0]\,. \tag{2.11}$$

Then the following holds.

(I) *For every initial datum* $u \in \overline{\mathcal{D}(A)}$ *and every* $t \geq 0$ *the limit*

$$S_t u \;\doteq\; \lim_{n\to\infty}\left(J_{t/n}\right)^n u \tag{2.12}$$

is well defined.

(II) *The family of operators* $\{S_t\,;\ t \geq 0\}$ *defined at (2.12) is a continuous semigroup of contractions on the set* $\overline{\mathcal{D}(A)}$*. Namely*

(i) *For all* $t, s \geq 0$ *and* $u \in \overline{\mathcal{D}(A)}$ *one has* $S_0 u = u$, $S_t S_s u = S_{t+s}u$.

(ii) *For every* $u \in \overline{\mathcal{D}(A)}$ *the map* $t \mapsto S_t u$ *is continuous.*

(iii) *For every* $u_1, u_2 \in \overline{\mathcal{D}(A)}$ *and every* $t \geq 0$ *one has* $\|S_t u_1 - S_t u_2\| \leq \|u_1 - u_2\|$.

Consider a family of accretive operators A^σ, and the corresponding semigroups S^σ. As shown by Brezis and Pazy [14], the limit $\lim_{\sigma\to 0} S_t^\sigma x$ exists if one has the convergence of the corresponding Backward Euler operators J_λ^σ.

Theorem 2.3. *[14, Theorem 3.1] Let* A, A^σ, $\sigma \in]0, \sigma_0]$ *be accretive operators such that*

$$\mathcal{D}\left(J_\lambda\right) = \mathcal{R}(I + \lambda A) \supseteq \overline{\mathcal{D}(A)}\,, \quad \mathcal{D}\left(J_\lambda^\sigma\right) = \mathcal{R}(I + \lambda A^\sigma) \supseteq \overline{\mathcal{D}(A^\sigma)}\,, \quad \forall \lambda \in]0, \lambda_0],\ \forall \sigma \in]0, \sigma_0].$$

Let S, S^σ *be the corresponding semigroups (Theorem 2.2), and call*

$$D \;\doteq\; \bigcap_{\sigma > 0} \overline{\mathcal{D}(A^\sigma)} \cap \mathcal{D}(A).$$

If the corresponding Backward Euler operators satisfy

$$\lim_{\sigma\to 0} J_\lambda^\sigma u \;=\; J_\lambda u \qquad \forall u \in \overline{D}\,,\ \forall \lambda \in]0, \lambda_0], \tag{2.13}$$

then

$$\lim_{\sigma\to 0} S_t^\sigma u \;=\; S_t u \qquad\qquad \forall u \in \overline{D},\ \forall t \geq 0, \tag{2.14}$$

and the limit is uniform for t *in bounded intervals.*

3. The resolvent equation for the viscous problem. In this section we assume hypothesis **f0)** for the flux f, and study the resolvent equation for the viscous conservation law (1.4). We establish suitable properties so that the classical results stated in Section 2 can be applied. To this end, we consider

$$u_t + [f(x,u) - \varepsilon u_x]_x \ni 0$$

and define the non linear map $A^\varepsilon \subset \mathbf{L^1}(\mathbb{R},\mathbb{R}) \times \mathbf{L^1}(\mathbb{R},\mathbb{R})$ as

$$(u,v) \in A^\varepsilon \quad \text{if and only if} \quad u,\ v \in \mathbf{L^1}(\mathbb{R},\mathbb{R}) \text{ and } [f(x,u) - \varepsilon u_x]_x = v. \tag{3.1}$$

The domain of the map is

$$\mathcal{D}(A^\varepsilon) = \left\{u \in \mathbf{L^1}(\mathbb{R},\mathbb{R}) : [f(x,u) - \varepsilon u_x]_x \in \mathbf{L^1}(\mathbb{R},\mathbb{R})\right\}. \tag{3.2}$$

We also use the notation $A^\varepsilon u$ to denote v in (3.1), since A^ε is a single valued operator.

Recall that I is the identity, λ is any positive real number and $\mathcal{R}(B)$ denotes the range of a map B. We consider the resolvent equation

$$u + \lambda A^\varepsilon u = w \quad \text{i.e.} \quad u + \lambda [f(x,u) - \varepsilon u_x]_x = w, \tag{3.3}$$

where w is any given function in $\mathbf{L^1}(\mathbb{R},\mathbb{R})$.

We begin with the definition of weak solution to (3.3).

Definition 3.1. *A function $u \in \mathbf{L^1_{loc}}(\mathbb{R},\mathbb{R})$ is a* weak solution *to (3.3), with data $w \in \mathbf{L^1}(\mathbb{R},\mathbb{R})$, if for any test function $\phi \in \mathbf{C}^2_c(\mathbb{R},\mathbb{R})$ it holds*

$$\int_{\mathbb{R}} \left[\frac{u(x) - w(x)}{\lambda}\phi(x) - f(x,u)\phi'(x) - \varepsilon u(x)\phi''(x)\right] dx = 0.$$

We now introduce the definitions of the upper and lower solutions to (3.3).

Definition 3.2. *Let $\Omega \subset \mathbb{R}$ be open and w be any function in $\mathbf{L^1_{loc}}(\Omega,\mathbb{R})$. The function $u \in \mathbf{L^1_{loc}}(\Omega,\mathbb{R})$ is a **lower solution** to (3.3) in Ω if*

$$u + \lambda [f(x,u) - \varepsilon u_x]_x \le w$$

*holds in Ω in the sense of distributions. On the other hand, the function $u \in \mathbf{L^1_{loc}}(\Omega,\mathbb{R})$ is an **upper solution** to (3.3) in Ω if*

$$u + \lambda [f(x,u) - \varepsilon u_x]_x \ge w$$

holds instead.

Lower and upper solutions to (3.3) satisfy the following maximum principle.

Theorem 3.3. *Let $u_1,\ u_2 \in \mathbf{L^1_{loc}}(\Omega,\mathbb{R})$ be respectively a lower and an upper solution to (3.3) in the open set Ω with right hand sides respectively equal to $w_1,\ w_2 \in \mathbf{L^1_{loc}}(\Omega,\mathbb{R})$:*

$$\begin{cases} u_1 + \lambda [f(x,u_1) - \varepsilon u_{1,x}]_x \le w_1, \\ u_2 + \lambda [f(x,u_2) - \varepsilon u_{2,x}]_x \ge w_2. \end{cases} \tag{3.4}$$

Let $a, b \in \mathbb{R} \cup \{\pm\infty\}$ be such that $]a,b[\subset \Omega$ and

$$\liminf_{x \to a^+} [u_1(x) - u_2(x)] \le 0, \quad \liminf_{x \to b^-} [u_1(x) - u_2(x)] \le 0, \tag{3.5}$$

then the inequality

$$\int_a^b [u_1(x) - u_2(x)]^+ \, dx \le \int_a^b [w_1(x) - w_2(x)]^+ \, dx \tag{3.6}$$

holds, where $[\cdot]^+$ denotes the positive part of a real number: $[t]^+ = \max\{0,t\}$. In particular, if $w_1 \le w_2$ holds in $]a,b[$, then $u_1 \le u_2$ holds in the same interval.

Proof. Define the function $v = u_1 - u_2$. Subtracting the inequalities in (3.4) we have

$$v + \lambda \left[f\left(x, u_1\right) - f\left(x, u_2\right) - \varepsilon v_x\right]_x \leq w_1 - w_2, \tag{3.7}$$

in the space of distributions. Therefore, the distribution

$$w_1 - w_2 - v - \lambda \left[f\left(x, u_1\right) - f\left(x, u_2\right) - \varepsilon v_x\right]_x$$

is non negative and consequently a positive Radon measure on Ω (see [47, Theorem 2.14]). Since $w_1 - w_2 - v$ is a locally integrable function, $\left[f\left(x, u_1\right) - f\left(x, u_2\right) - \varepsilon v_x\right]_x$ is a signed Radon measure. Therefore

$$f\left(x, u_1\right) - f\left(x, u_2\right) - \varepsilon v_x \in \mathbf{BV_{loc}}\left(\Omega, \mathbb{R}\right) \subset \mathbf{L^1_{loc}}\left(\Omega, \mathbb{R}\right).$$

Hypothesis **f0)** implies that $f(x, u_1), f(x, u_2) \in \mathbf{L^1_{loc}}\left(\Omega, \mathbb{R}\right)$, leading to $v_x \in \mathbf{L^1_{loc}}\left(\Omega, \mathbb{R}\right)$. Thus, v is locally absolutely continuous in Ω.

If $v \leq 0$ holds in $]a, b[$, there is nothing to prove. Otherwise, let $]\alpha, \beta[$ be any connected component of the open set

$$V = \left\{x \in \left]a, b\right[: \ v(x) > 0\right\}.$$

Note that we do not exclude the possibilities $\alpha = -\infty$ and $\beta = +\infty$. Now, hypothesis (3.5) and the continuity of v in Ω imply

$$\liminf_{x \to \alpha^+} v(x) = 0, \quad \liminf_{x \to \beta^-} v(x) = 0, \quad v(x) > 0 \text{ for all } x \in \left]\alpha, \beta\right[. \tag{3.8}$$

Let $\mathcal{L}$ be the intersection of the Lebesgue points in Ω of the functions $f\left(x, u_1\right)$, $f\left(x, u_2\right)$ and v_x. Fix $\eta,\ \xi \in \mathcal{L}$ such that $\alpha < \eta < \xi < \beta$, we evaluate the measures in (3.7) over the set $]\eta, \xi[$ and obtain

$$\begin{aligned} &\int_\eta^\xi v(x)\ dx + \lambda \left[f\left(\xi, u_1(\xi)\right) - f\left(\xi, u_2(\xi)\right)\right] \\ &\qquad - \lambda \left[f\left(\eta, u_1(\eta)\right) - f\left(\eta, u_2(\eta)\right)\right] - \lambda\varepsilon \left[v_x\left(\xi\right) - v_x\left(\eta\right)\right] \\ &\qquad \leq \int_\eta^\xi \left[w_1(x) - w_2(x)\right]\ dx \leq \int_\eta^\xi \left[w_1(x) - w_2(x)\right]^+\ dx. \end{aligned} \tag{3.9}$$

Using **f0)** and recalling that $v > 0$ in $]\alpha, \beta[$, the above inequality becomes

$$\int_\eta^\xi v(x)\ dx \leq \lambda \left[\varepsilon v_x\left(\xi\right) + Lv\left(\xi\right)\right] + \lambda \left[-\varepsilon v_x\left(\eta\right) + Lv\left(\eta\right)\right] + \int_\eta^\xi \left[w_1(x) - w_2(x)\right]^+\ dx. \tag{3.10}$$

Now we claim that

$$\liminf_{\substack{\eta \to \alpha^+ \\ \eta \in \mathcal{L}}} \left[-\varepsilon v_x\left(\eta\right) + Lv\left(\eta\right)\right] \leq 0, \qquad \liminf_{\substack{\xi \to \beta^- \\ \xi \in \mathcal{L}}} \left[\varepsilon v_x\left(\xi\right) + Lv\left(\xi\right)\right] \leq 0. \tag{3.11}$$

Indeed, by contradiction, suppose that the second inequality in (3.11) were not true (the other case being similar). Then there should exist $\gamma > 0$ and $\xi_o < \beta$ such that $\varepsilon v_x\left(\xi\right) + Lv\left(\xi\right) \geq \gamma$ for all $\xi \in (\xi_o, \beta) \cap \mathcal{L}$. Solving this differential inequality with $v\left(\xi_o\right) > 0$ as initial data we have

$$v\left(\xi\right) \geq \left[v\left(\xi_o\right) - \frac{\gamma}{L}\right] e^{-\frac{L}{\varepsilon}\left(\xi - \xi_o\right)} + \frac{\gamma}{L}.$$

If in this last inequality we take the lower limit as $\xi \to \beta$ we have (including the case $\beta = +\infty$):

$$\liminf_{\xi \to \beta^-} v\left(\xi\right) \geq v\left(\xi_o\right) e^{-\frac{L}{\varepsilon}\left(\beta - \xi_o\right)} + \frac{\gamma}{L}\left(1 - e^{-\frac{L}{\varepsilon}\left(\beta - \xi_o\right)}\right) > 0$$

which contradicts (3.8).

Now we take the lower limits in (3.10) as $\eta \to \alpha^+$ and $\xi \to \beta^-$. Using (3.11) we obtain

$$\int_\alpha^\beta v(x)\, dx \leq \int_\alpha^\beta [w_1(x) - w_2(x)]^+ \, dx.$$

Finally, writing V as the union of its connected components $V = \bigcup_{i=1}^N (\alpha_i, \beta_i)$ (with $N = +\infty$ if there are countable many connected components), we compute

$$\begin{aligned}\int_a^b [u_1(x) - u_2(x)]^+ \, dx &= \int_a^b [v(x)]^+ \, dx = \int_V v(x)\, dx \\ &= \sum_{i=1}^N \int_{\alpha_i}^{\beta_i} v(x)\, dx \leq \sum_{i=1}^N \int_{\alpha_i}^{\beta_i} [w_1(x) - w_2(x)]^+ \, dx \leq \int_a^b [w_1(x) - w_2(x)]^+ \, dx,\end{aligned} \tag{3.12}$$

proving the theorem. ∎

If the functions u and w in (3.3) are integrable over $\mathbb{R}$, then the operator A^ε defined in (3.1) is accretive. Indeed we have the following result.

Corollary 3.4. *Let* $u_1,\ u_2,\ w_1,\ w_2 \in \mathbf{L^1}(\mathbb{R}, \mathbb{R})$ *satisfy*

$$\begin{cases} u_1 + \lambda \left[f(x, u_1) - \varepsilon u_{1,x}\right]_x = w_1, \\ u_2 + \lambda \left[f(x, u_2) - \varepsilon u_{2,x}\right]_x = w_2, \end{cases} \tag{3.13}$$

in the sense of distribution, then the following inequalities hold:

$$\int_{\mathbb{R}} [u_1(x) - u_2(x)]^+ \, dx \leq \int_{\mathbb{R}} [w_1(x) - w_2(x)]^+ \, dx, \tag{3.14}$$

$$\int_{\mathbb{R}} |u_1(x) - u_2(x)| \, dx \leq \int_{\mathbb{R}} |w_1(x) - w_2(x)| \, dx. \tag{3.15}$$

Proof. According to Definition 3.2, u_1 and u_2 are both lower and upper solutions to (3.3) with right hand side respectively w_1 and w_2. Since they are integrable, we have

$$\liminf_{x \to \pm\infty} [u_1(x) - u_2(x)] \leq 0.$$

Theorem 3.3 with $a = -\infty$ and $b = +\infty$ can be applied to get (3.14). Changing the role of u_1 and u_2 allows us to obtain (3.15). ∎

In the following theorem we establish some properties of the Backward Euler operator $J_\lambda^\varepsilon = (I + \lambda A^\varepsilon)^{-1}$ defined in (2.4) and show that $\mathcal{R}(I + \lambda A^\varepsilon) = \mathbf{L^1}(\mathbb{R}, \mathbb{R})$.

Theorem 3.5. *Suppose the flux $f(x, \omega)$ satisfies hypothesis **f0)**. Then for any $\lambda,\ \varepsilon > 0$, $w \in \mathbf{L^1}(\mathbb{R}, \mathbb{R})$, there exists a unique weak solution $u = J_\lambda^\varepsilon w \in \mathbf{L^1}(\mathbb{R}, \mathbb{R})$ to equation (3.3). The maps $A^\varepsilon : \mathcal{D}(A^\varepsilon) \to \mathbf{L^1}(\mathbb{R}, \mathbb{R})$ and $J_\lambda^\varepsilon : \mathbf{L^1}(\mathbb{R}, \mathbb{R}) \to \mathbf{L^1}(\mathbb{R}, \mathbb{R})$ satisfy*

- *(i)* $J_\lambda^\varepsilon w_1 \leq J_\lambda^\varepsilon w_2$ *whenever* $w_1 \leq w_2$, *(monotonicity);*
- *(ii)* $\int_{\mathbb{R}} J_\lambda^\varepsilon w \, dx = \int_{\mathbb{R}} w \, dx$ *for any* $w \in \mathbf{L^1}(\mathbb{R}, \mathbb{R})$, *(conservation);*
- *(iii)* $\|J_\lambda^\varepsilon w_1 - J_\lambda^\varepsilon w_2\|_{\mathbf{L^1}(\mathbb{R},\mathbb{R})} \leq \|w_1 - w_2\|_{\mathbf{L^1}(\mathbb{R},\mathbb{R})}$, *(contraction property);*
- *(iv)* $\overline{\mathcal{D}(A^\varepsilon)} = \mathbf{L^1}(\mathbb{R}, \mathbb{R})$ *(density of the domain).*

Proof. The uniqueness, the monotonicity (i) and the contraction property (iii) are direct consequences of Corollary 3.4. We now show the existence. For $\lambda > 0$, $x \in \mathbb{R}$, we consider the traditional convolution kernel

$$H(\lambda, x) = \frac{1}{2\sqrt{\lambda}} e^{-\frac{|x|}{\sqrt{\lambda}}}. \tag{3.16}$$

It has the following properties:

$$
\begin{cases}
H_x(\lambda,x) = -\frac{1}{\sqrt{\lambda}}\,\mathrm{sign}(x)\,H(\lambda,x)\,,\\
H_{xx}(\lambda,\cdot) = \frac{1}{\lambda}\left(H(\lambda,\cdot)-\delta_0\right),\\
\lim_{\lambda\to0^+} H(\lambda,\cdot)=\delta_0, \quad \text{where } \delta_0 \text{ is the unit mass at } x=0,\\
\left\|H(\lambda,\cdot)\right\|_{\mathbf{L^1}(\mathbb{R},\mathbb{R})}=1, \qquad \left\|H_x(\lambda,\cdot)\right\|_{\mathbf{L^1}(\mathbb{R},\mathbb{R})}=\frac{1}{\sqrt{\lambda}}.
\end{cases}
\tag{3.17}
$$

Fix $\varepsilon,\lambda>0$ and for any $w\in\mathbf{L^1}(\mathbb{R},\mathbb{R})$ we define the Lipschitz continuous map $\Lambda_\lambda^w : \mathbf{L^1}(\mathbb{R},\mathbb{R})\to\mathbf{L^1}(\mathbb{R},\mathbb{R})$ as

$$
\left[\Lambda_\lambda^w(u)\right](x)=\int_{\mathbb{R}} H(\lambda\varepsilon,x-y)\,w(y)\,dy-\int_{\mathbb{R}}\lambda H_x(\lambda\varepsilon,x-y)\,f(y,u(y))\,dy. \tag{3.18}
$$

By properties (3.17), it follows that $u\in\mathbf{L^1}(\mathbb{R},\mathbb{R})$ is a weak solution to (3.3) if and only if $u=\Lambda_\lambda^w(u)$. Moreover, one has

$$
\left\|\Lambda_\lambda^w(u_1)-\Lambda_\lambda^w(u_2)\right\|_{\mathbf{L^1}(\mathbb{R},\mathbb{R})}\le L\sqrt{\frac{\lambda}{\varepsilon}}\left\|u_1-u_2\right\|_{\mathbf{L^1}(\mathbb{R},\mathbb{R})}, \qquad \text{for any } u_1,u_2\in\mathbf{L^1}(\mathbb{R},\mathbb{R}).
$$

Set $\lambda_o=\frac{\varepsilon}{2L^2}>0$, so that $L\sqrt{\frac{\lambda_o}{\varepsilon}}<1$. Then, for any $\lambda\in]0,\lambda_o]$, Λ_λ^w is a strict contraction in $\mathbf{L^1}(\mathbb{R},\mathbb{R})$. As a consequence, it has a unique fixed point $u=\Lambda_\lambda^w u$, which we denote by $J_\lambda^\varepsilon w$. We conclude that $\mathcal{R}(I+\lambda A^\varepsilon)=\mathbf{L^1}(\mathbb{R},\mathbb{R})$ holds for any $\lambda\in]0,\lambda_o]$, and the domain of A^ε is not empty.

Using the contraction property of $J_{\lambda_o}^\varepsilon$ and a classical argument that we repeat here for completeness (see [44, Lemma 2.13]), we prove that $\mathcal{R}(I+\lambda A^\varepsilon)=\mathbf{L^1}(\mathbb{R},\mathbb{R})$ for any $\lambda>0$. Indeed, fix $\lambda>\lambda_o$, $w\in\mathbf{L^1}(\mathbb{R},\mathbb{R})$, we need to show that there is a function $u\in\mathcal{D}(A^\varepsilon)$ that satisfies

$$
(I+\lambda A^\varepsilon)\,u=w.
$$

Multiplying this equality by $\frac{\lambda_o}{\lambda}$, algebraic manipulations give

$$
(I+\lambda_o A^\varepsilon)\,u=\left(1-\frac{\lambda_o}{\lambda}\right)u+\frac{\lambda_o}{\lambda}w.
$$

By the surjectivity of $(I+\lambda_o A^\varepsilon)$, the above equation is equivalent to the fixed point equation $u=T_w u$, where the map $T_w:\mathbf{L^1}(\mathbb{R},\mathbb{R})\to\mathbf{L^1}(\mathbb{R},\mathbb{R})$ is defined by

$$
T_w u=J_{\lambda_o}^\varepsilon\left(\left(1-\frac{\lambda_o}{\lambda}\right)u+\frac{\lambda_o}{\lambda}w\right).
$$

Since $J_{\lambda_o}^\varepsilon$ is a contraction, we compute

$$
\left\|T_w u_1-T_w u_2\right\|_{\mathbf{L^1}(\mathbb{R},\mathbb{R})}\le\left(1-\frac{\lambda_o}{\lambda}\right)\left\|u_1-u_2\right\|_{\mathbf{L^1}(\mathbb{R},\mathbb{R})}.
$$

One concludes that T_w is a strict contraction in $\mathbf{L^1}(\mathbb{R},\mathbb{R})$, and hence it has a unique fixed point $u=T_w u$.

To prove (ii), it is enough to integrate over $\mathbb{R}$ the identity $u=\Lambda_\lambda^w u$ and apply Fubini's theorem.

It remains to prove (iv), the density of the domain of A^ε. Fix $w\in\mathbf{L^1}(\mathbb{R},\mathbb{R})$ and observe that $u_\lambda=J_\lambda^\varepsilon w\in\mathcal{D}(A^\varepsilon)$ for any $\lambda>0$. Hence it suffices to show that $u_\lambda\to w$ as $\lambda\to0$. Fix a point $\bar u\in\mathcal{D}(A^\varepsilon)$. The contraction property of J_λ^ε and (2.8) imply

$$
\begin{aligned}
\left\|u_\lambda\right\|_{\mathbf{L^1}(\mathbb{R},\mathbb{R})}&\le\left\|J_\lambda^\varepsilon w-J_\lambda^\varepsilon\bar u\right\|_{\mathbf{L^1}(\mathbb{R},\mathbb{R})}+\left\|J_\lambda^\varepsilon\bar u-\bar u\right\|_{\mathbf{L^1}(\mathbb{R},\mathbb{R})}+\left\|\bar u\right\|_{\mathbf{L^1}(\mathbb{R},\mathbb{R})}\\
&\le\left\|w-\bar u\right\|_{\mathbf{L^1}(\mathbb{R},\mathbb{R})}+\lambda\left\|A^\varepsilon\bar u\right\|_{\mathbf{L^1}(\mathbb{R},\mathbb{R})}+\left\|\bar u\right\|_{\mathbf{L^1}(\mathbb{R},\mathbb{R})}\\
&\le C,
\end{aligned}
\tag{3.19}
$$

where C is a constant independent of $\lambda \in]0,1]$. Using **f0)**, and the fact that u_λ is the unique fixed point of Λ_λ^w, we compute

$$
\begin{aligned}
\|u_\lambda - w\|_{\mathbf{L^1}(\mathbb{R},\mathbb{R})} &= \|\Lambda_\lambda^w(u_\lambda) - w\|_{\mathbf{L^1}(\mathbb{R},\mathbb{R})} \\
&\le \int_{\mathbb{R}^2} H(\lambda\varepsilon, x-y)\,|w(x)-w(y)|\;dxdy + \int_{\mathbb{R}^2} \lambda\,|H_x(\lambda\varepsilon, x-y)|\,|f(y,u_\lambda(y))|\;dxdy \\
&\le \int_{\mathbb{R}} H(\lambda\varepsilon,\xi)\left[\int_{\mathbb{R}} |w(x)-w(x-\xi)|\;dx\right]d\xi + \int_{\mathbb{R}} \frac{\lambda}{\sqrt{\lambda\varepsilon}}\left[|f(y,0)| + L\,|u_\lambda(y)|\right]\;dy \qquad (3.20)\\
&\le \int_{\mathbb{R}} H(\lambda\varepsilon,\xi)\left[\int_{\mathbb{R}} |w(x)-w(x-\xi)|\;dx\right]d\xi + \sqrt{\frac{\lambda}{\varepsilon}}\left[L_1 + L\cdot C\right] \\
&\xrightarrow{\lambda\to 0} 0,
\end{aligned}
$$

completing the proof. ∎

We are now ready to apply Theorem 2.2.

THEOREM 3.6. *If the flux f satisfies the hypothesis **f0)**, then the operator A^ε defined in (3.1) generates (in the sense of Theorem 2.2) a non linear continuous semigroup $S_t^\varepsilon : \mathbf{L^1}(\mathbb{R},\mathbb{R}) \to \mathbf{L^1}(\mathbb{R},\mathbb{R})$ of contractions. For any $\bar u \in \mathbf{L^1}(\mathbb{R},\mathbb{R})$, the trajectory of the semigroup $u(t,x) = (S_t^\varepsilon \bar u)(x)$ belongs to $\mathbf{C^0}\left([0,+\infty), \mathbf{L^1}(\mathbb{R},\mathbb{R})\right)$ and is a weak solutions to the parabolic equation (1.4).*

Proof. Theorem 3.5 guarantees that A^ε satisfies the hypotheses of Theorem 2.2. Therefore it generates a continuous semigroup $S_t^\varepsilon : \overline{\mathcal{D}(A^\varepsilon)} \to \overline{\mathcal{D}(A^\varepsilon)}$ of contractions with $\overline{\mathcal{D}(A^\varepsilon)} = \mathbf{L^1}(\mathbb{R},\mathbb{R})$. For any $\bar u \in \mathbf{L^1}(\mathbb{R},\mathbb{R})$, the trajectory of the semigroup $u(t,x) = (S_t^\varepsilon \bar u)(x)$ belongs to $\mathbf{C^0}\left([0,+\infty), \mathbf{L^1}(\mathbb{R},\mathbb{R})\right)$. The trajectory u can be obtained (see [44, Theorem 4.2]) as the limit in $\mathbf{L^1}$ of approximations

$$u(t) = \lim_{\lambda\to 0} u_\lambda(t), \qquad u_\lambda(t) = (J_\lambda^\varepsilon)^{\left[\frac{t}{\lambda}\right]}\bar u.$$

By the definition of the resolvent J_λ^ε, the approximations $u_\lambda(t)$ solve

$$\frac{u_\lambda(t,x) - u_\lambda(t-\lambda,x)}{\lambda} + \left[f(x,u_\lambda(t,x)) - \varepsilon u_{\lambda,x}(t,x)\right]_x = 0, \quad t \ge \lambda.$$

We multiply this equation by a test function with compact support in $]0,+\infty[\times\mathbb{R}$, and perform integrations by parts. Taking the limit $\lambda \to 0$, one shows that u is a weak solution to the parabolic problem (1.4). ∎

4. The vanishing viscosity limit for the Backward Euler operator. In this section we study the vanishing viscosity limit $\varepsilon \to 0$ in (1.4), where we assume the hypotheses **f1)** on the flux f. Under **f1)**, the region $[0,1]$ is invariant for (3.3). We introduce the domain

$$D \;\doteq\; \left\{w \in \mathbf{L^1}(\mathbb{R},\mathbb{R}) :\; 0 \le w \le 1\right\}. \tag{4.1}$$

If the source term w in (3.3) is in D, then $\underline{u}(x) = 0$ and $\overline{u}(x) = 1$ are respectively a lower and an upper solution to (3.3). An application of Theorem 3.3 shows that $J_\lambda^\varepsilon w \in D$.

Hypothesis **f1)** implies additional regularity on the solutions to (3.3).

LEMMA 4.1. *Suppose $f(x,\omega)$ satisfies **f1)**. If $w \in \mathbf{L^1_{loc}}(\mathbb{R},\mathbb{R})$, then a function $u \in \mathbf{L^1_{loc}}(\mathbb{R},\mathbb{R})$ is a weak solution to (3.3) if and only if the following three conditions are satisfied:*
(i) $u \in \mathbf{W^{2,1}_{loc}}(\mathbb{R}\setminus\{0\},\mathbb{R}) \cap \mathbf{W^{1,1}_{loc}}(\mathbb{R},\mathbb{R})$;
(ii) in $\mathbb{R}\setminus\{0\}$, u is a weak (Sobolev) solution to (3.3);
(iii) the two limits $\lim_{x\to 0^\pm} u_x(x) = u_x(0\pm)$ exist and they satisfy

$$f_r(u(0)) - f_l(u(0)) = \varepsilon\left(u_x(0+) - u_x(0-)\right). \tag{4.2}$$

Moreover, we have

$$f(x,u) - \varepsilon u_x \in \mathbf{W^{1,1}_{loc}}(\mathbb{R},\mathbb{R}). \tag{4.3}$$

Proof. Suppose first that $u \in \mathbf{L^1_{loc}}(\mathbb{R},\mathbb{R})$ is a weak solution to (3.3) in $\mathbb{R}$. Then it satisfies

$$\lambda \left[f(x,u) - \varepsilon u_x\right]_x = w - u \in \mathbf{L^1_{loc}}(\mathbb{R},\mathbb{R}) \ \Rightarrow f(x,u) - \varepsilon u_x \in \mathbf{W^{1,1}_{loc}}(\mathbb{R},\mathbb{R}) \tag{4.4}$$

proving (4.3). It further shows that $f(x,u) - \varepsilon u_x$ is continuous in $\mathbb{R}$. Since $f(x,u) \in \mathbf{L^1_{loc}}(\mathbb{R},\mathbb{R})$, (4.4) also implies $u_x \in \mathbf{L^1_{loc}}(\mathbb{R},\mathbb{R})$ and consequently $u \in \mathbf{W^{1,1}_{loc}}(\mathbb{R},\mathbb{R})$. Therefore both u and $f(x,u) - \varepsilon u_x$ are continuous in $\mathbb{R}$, and we have

$$\lim_{x\to 0-} \left[f(x,u) - \varepsilon u_x\right] = f_l\left(u(0)\right) - \varepsilon u_x(0-) = \lim_{x\to 0+} \left[f(x,u) - \varepsilon u_x\right] = f_r\left(u(0)\right) - \varepsilon u_x(0+),$$

concluding (iii).

Consider now the domain $]-\infty, 0[$, where $f(x,u) = f_l(u) \in \mathbf{W^{1,1}_{loc}}(]-\infty,0[\,,\mathbb{R})$. Then (4.4) imply $u_x \in \mathbf{W^{1,1}_{loc}}(]-\infty,0[\,,\mathbb{R})$ and hence $u \in \mathbf{W^{2,1}_{loc}}(]-\infty,0[\,,\mathbb{R})$. The same argument holds in the domain $]0,+\infty[$, proving (i) and (ii).

Suppose now that $u \in \mathbf{L^1_{loc}}(\mathbb{R},\mathbb{R})$ satisfies (i), (ii), (iii). In $\mathbb{R}\setminus\{0\}$, equation (3.3) is equivalent to $\left[f(x,u) - \varepsilon u_x\right]_x = w - u$ and (iii) implies that $f(x,u) - \varepsilon u_x$ is continuous at $x = 0$. Therefore u is a weak solution to (3.3) on $\mathbb{R}$. ∎

Corollary 4.2. *For $k \in \mathbb{N}$, if $w \in \mathbf{C^k}(\mathbb{R},\mathbb{R})$, f_l, $f_r \in \mathbf{C^{k+1}}(\mathbb{R},\mathbb{R})$, and $u \in \mathbf{L^1_{loc}}(\mathbb{R},\mathbb{R})$ is a weak solution to (3.3), then $u \in \mathbf{C^{k+2}}(]-\infty,0[\,\cup\,]0,+\infty[\,,\mathbb{R})$.*

Proof. In $]-\infty,0[$, we have that $u_{xx} = \frac{1}{\varepsilon}\left(u + f_l'\left(u\right)u_x - w\right)$ holds. This relation, starting with the initial regularity given by Lemma 4.1 item (i), by induction proves the result. The same holds in $]0,+\infty[$. ∎

The next Lemma shows that the total variation of $J^\varepsilon_\lambda w$ is uniformly bounded with respect to the parameter ε.

Lemma 4.3. *Under the hypothesis **f1)**, the map J^ε_λ defined in Theorem 3.5 satisfies*

$$\operatorname{Tot.Var.}\{J^\varepsilon_\lambda w\} \le 2 + \operatorname{Tot.Var.}\{w\}, \qquad \textit{for all } w \in D. \tag{4.5}$$

Proof. Consider first $w \in \mathbf{C^\infty_c}(\mathbb{R},\mathbb{R})$ and define $u = J^\varepsilon_\lambda w$. Lemma 4.1 and Corollary 4.2 imply that u is smooth in $\mathbb{R}\setminus\{0\}$ and continuous in $\mathbb{R}$. We claim the following:

- If $\bar{x} \neq 0$ is a point of local maximum for u, then $u(\bar{x}) \le w(\bar{x})$.
- If $\hat{x} \neq 0$ is a point of local minimum for u, then $u(\hat{x}) \ge w(\hat{x})$.

Indeed, consider a local max $\bar{x} > 0$ (the case $\bar{x} < 0$ being completely similar). We have

$$u_x(\bar{x}) = 0, \quad u_{xx}(\bar{x}) \le 0,$$

so

$$w(\bar{x}) - u(\bar{x}) = \lambda\left[(f_r)'(u(\bar{x}))u_x(\bar{x}) - \varepsilon u_{xx}(\bar{x})\right] \ge 0.$$

Fix $\gamma < \operatorname{Tot.Var.}\{u\}$ and points $x_0 < x_1 < \ldots < x_{J-1} < 0 < x_J < \ldots < x_N$ such that

$$\gamma < \sum_{i=1}^{J-1}\left|u\left(x_i\right) - u\left(x_{i-1}\right)\right| + \left|u(0) - u\left(x_{J-1}\right)\right| + \left|u(0) - u\left(x_J\right)\right| + \sum_{i=J+1}^{N}\left|u\left(x_i\right) - u\left(x_{i-1}\right)\right|.$$

It is not restrictive to assume that $w\left(x_0\right) = w\left(x_N\right) = 0$, and that the points x_i, for $1 \le i \le J-1$, are alternatively points of local maximum and minimum for u beginning with a maximum at x_1 while, for $J \le i \le N-1$, they are alternatively point of local maximum and minimum beginning with a maximum at x_{N-1}. Therefore we have $\left|u\left(x_i\right) - u\left(x_{i-1}\right)\right| \le \left|w\left(x_i\right) - w\left(x_{i-1}\right)\right|$ for $0 \le i \le N$, with $i \neq J$ which implies

$$\gamma < \sum_{i=1}^{J-1}\left|w\left(x_i\right) - w\left(x_{i-1}\right)\right| + 1 + 1 + \sum_{i=J+1}^{N}\left|w\left(x_i\right) - w\left(x_{i-1}\right)\right| \le \operatorname{Tot.Var.}\{w\} + 2.$$

This proves the assertion because of the arbitrariness of $\gamma < \operatorname{Tot.Var.}\{u\}$.

Finally, given any $w \in D$ there exists a sequence $w_\nu \in \mathbf{C_c^\infty}(\mathbb{R},\mathbb{R}) \cap D$ converging to w in $\mathbf{L^1}(\mathbb{R},\mathbb{R})$ such that $\mathrm{Tot.Var.}\{w_\nu\} \leq \mathrm{Tot.Var.}\{w\}$. The continuity of J_λ^ε and the lower semicontinuity of the total variation imply

$$\begin{aligned}\mathrm{Tot.Var.}\{J_\lambda^\varepsilon w\} &\leq \liminf_{\nu\to+\infty} \mathrm{Tot.Var.}\{J_\lambda^\varepsilon w_\nu\} \leq 2 + \liminf_{\nu\to+\infty} \mathrm{Tot.Var.}\{w_\nu\} \\ &\leq 2 + \mathrm{Tot.Var.}\{w\}.\end{aligned} \tag{4.6}$$

□

The previous Lemma yields the compactness of the family $\{J_\lambda^\varepsilon w\}_{\varepsilon>0}$ whenever w has bounded total variation. The limit is unique due to the following characterization.

THEOREM 4.4. *Given $w \in D \cap \mathbf{BV}(\mathbb{R},\mathbb{R})$, from any sequence $\varepsilon_\nu \to 0$, we can extract a subsequence ε_{ν_j} such that $J_\lambda^{\varepsilon_{\nu_j}} w$ converges pointwise to a function $u \in \mathbf{BV}(\mathbb{R},\mathbb{R})$ which satisfies*

$$u + \lambda f(x,u)_x = w. \tag{4.7}$$

Furthermore the following entropy inequality holds in the space of distributions

$$\lambda\delta_0 \int_0^{u(0)} \eta''(\omega)\left[f_r(\omega) - f_l(\omega)\right]\, d\omega + \lambda q(x,u)_x + \eta'(u)\left[u - w\right] \leq 0, \tag{4.8}$$

where δ_0 is the unit mass at the origin, η is any smooth convex function, q is defined by

$$q(x,\omega) = \int_0^\omega \eta'(\bar\omega)\, f_{\bar\omega}(x,\bar\omega)\, d\bar\omega.$$

Moreover, if u is discontinuous at x_o with $u^\pm = u(x_o\pm)$, we must have

$$f\left(x_o-, u^-\right) = f\left(x_o+, u^+\right) \doteq \bar f, \tag{4.9}$$

and the following entropy conditions:

1. *if $u^- < u^+$ and $x_o \neq 0$ then*
$$f(x_o,k) \geq \bar f, \text{ for all } k \in \left[u^-, u^+\right];$$
2. *if $u^- > u^+$ and $x_o \neq 0$ then*
$$f(x_o,k) \leq \bar f, \text{ for all } k \in \left[u^+, u^-\right];$$
3. *if $u^- < u^+$ and $x_o = 0$ then there exists $u^* \in [u^-, u^+]$ such that*
$$\begin{cases} f_l(k) \geq \bar f, \text{ for all } k \in [u^-, u^*], \\ f_r(k) \geq \bar f, \text{ for all } k \in [u^*, u^+]; \end{cases}$$
4. *if $u^- > u^+$ and $x_o = 0$ then there exists $u^* \in [u^+, u^-]$ such that*
$$\begin{cases} f_r(k) \leq \bar f, \text{ for all } k \in [u^+, u^*], \\ f_l(k) \leq \bar f, \text{ for all } k \in [u^*, u^-]. \end{cases}$$

Proof. The proof takes a few steps.

Step 1. Define $u^\varepsilon = J_\lambda^\varepsilon w$. By Lemma 4.3, Tot. Var.$\{u^\varepsilon\}$ is bounded uniformly in ε. Therefore there exists a subsequence $u^{\varepsilon_{\nu_j}}$ which converges *pointwise* to a function $u \in \mathbf{BV}(\mathbb{R},\mathbb{R})$ with $0 \leq u \leq 1$. To simplify the notation we denote $u^\varepsilon = u^{\varepsilon_{\nu_j}}$. By definition of $J_\lambda^\varepsilon w$, u^ε is a weak solution to (3.3). Passing to the limit as $\varepsilon \to 0$ in (3.3) we immediately obtain (4.7).

Step 2. By Lemma 4.1, given any smooth convex function $\eta(\xi)$, the composition $\eta(u^\varepsilon)$ is in $\mathbf{W^{1,1}_{loc}}(\mathbb{R},\mathbb{R})$ with $\eta(u^\varepsilon)_x = \eta'(u^\varepsilon)\, u^\varepsilon_x$. Multiplying (3.3) by the continuous function $\eta'(u^\varepsilon)$ we obtain

$$\eta'(u^\varepsilon)\,\lambda f(x,u^\varepsilon)_x - \lambda\varepsilon\eta'(u^\varepsilon)\, u^\varepsilon_{xx} + \eta'(u^\varepsilon)\left[u^\varepsilon - w\right] = 0. \tag{4.10}$$

By Lemma 4.1, $u_x^\varepsilon \in \mathbf{W^{1,1}_{loc}}(\mathbb{R}\setminus\{0\},\mathbb{R})$ with a possible discontinuity at $x=0$, therefore $u_x^\varepsilon \in \mathbf{BV_{loc}}(\mathbb{R},\mathbb{R})$. Since $\eta'(u^\varepsilon)$ is locally Lipschitz we obtain by Leibniz rule ([5, Proposition 3.2])

$$[\eta(u^\varepsilon)_x]_x = [\eta'(u^\varepsilon)u_x^\varepsilon]_x = \eta''(u^\varepsilon)(u_x^\varepsilon)^2 + \eta'(u^\varepsilon)u_{xx}^\varepsilon.$$

Using this equality, (4.10) becomes

$$\eta'(u^\varepsilon)\lambda f(x,u^\varepsilon)_x - \lambda\varepsilon\eta(u^\varepsilon)_{xx} + \eta'(u^\varepsilon)[u^\varepsilon - w] = -\lambda\varepsilon\eta''(u^\varepsilon)(u_x^\varepsilon)^2 \le 0.$$

In the space of distribution, $\lambda\varepsilon\eta(u^\varepsilon)_{xx} \to 0$ and $\eta'(u^\varepsilon)[u^\varepsilon - w] \to \eta'(u)[u-w]$ as $\varepsilon \to 0$. It remains to show the weak convergence of the measure $\eta'(u^\varepsilon)f(x,u^\varepsilon)_x$.

We define the notations

$$q(x,\omega) = \begin{cases} q_l(\omega) & \text{for } x \le 0, \\ q_r(\omega) & \text{for } x > 0, \end{cases}$$

where

$$q_l(\omega) = \int_0^\omega \eta'(\bar\omega) f_l'(\bar\omega)\, d\bar\omega, \qquad q_r(\omega) = \int_0^\omega \eta'(\bar\omega) f_r'(\bar\omega)\, d\bar\omega.$$

Fix a test function φ. Observe that $\eta'(u^\varepsilon)f(x,u^\varepsilon)_x$ has a Dirac mass at the origin. We compute the duality product

$$\begin{aligned}
&\langle \eta'(u^\varepsilon)f(x,u^\varepsilon)_x, \varphi\rangle \\
&= \eta'(u^\varepsilon(0))[f_r(u^\varepsilon(0)) - f_l(u^\varepsilon(0))]\varphi(0) + \int_{-\infty}^0 q_l(u^\varepsilon)_x\ \varphi\, dx + \int_0^{+\infty} q_r(u^\varepsilon)_x\ \varphi\, dx \\
&= \eta'(u^\varepsilon(0))[f_r(u^\varepsilon(0)) - f_l(u^\varepsilon(0))]\varphi(0) + [q_l(u^\varepsilon(0)) - q_r(u^\varepsilon(0))]\varphi(0) \\
&\quad - \int_{-\infty}^0 q_l(u^\varepsilon)\varphi_x\, dx - \int_0^{+\infty} q_r(u^\varepsilon)\varphi_x\, dx \\
&= \Big\{\eta'(u^\varepsilon(0))[f_r(u^\varepsilon(0)) - f_l(u^\varepsilon(0))] + [q_l(u^\varepsilon(0)) - q_r(u^\varepsilon(0))]\Big\}\varphi(0) \\
&\quad - \int_{\mathbb{R}} q(x,u^\varepsilon)\varphi_x\, dx.
\end{aligned} \tag{4.11}$$

Using $f_l(0) = f_r(0) = 0$ and integration by parts, we obtain

$$\begin{aligned}
&\eta'(u^\varepsilon(0))[f_r(u^\varepsilon(0)) - f_l(u^\varepsilon(0))] + [q_l(u^\varepsilon(0)) - q_r(u^\varepsilon(0))] \\
&\qquad = \eta'(u^\varepsilon(0))[f_r(u^\varepsilon(0)) - f_l(u^\varepsilon(0))] + \int_0^{u^\varepsilon(0)} \eta'(\omega)[f_l'(\omega) - f_r'(\omega)]\, d\omega \\
&\qquad = \int_0^{u^\varepsilon(0)} \eta''(\omega)[f_r(\omega) - f_l(\omega)]\, d\omega.
\end{aligned} \tag{4.12}$$

This gives

$$\begin{aligned}
\langle \eta'(u^\varepsilon)f(x,u^\varepsilon)_x, \varphi\rangle &= \varphi(0)\int_0^{u^\varepsilon(0)} \eta''(\xi)[f_r(\xi) - f_l(\xi)]\, d\xi - \int_{\mathbb{R}} q(x,u^\varepsilon)\varphi_x\, dx \\
&= \langle \delta_0 \int_0^{u^\varepsilon(0)} \eta''(\xi)[f_r(\xi) - f_l(\xi)]\, d\xi + q(x,u^\varepsilon)_x, \varphi\rangle.
\end{aligned} \tag{4.13}$$

Finally, since u^ε converges pointwise to u and is uniformly bounded, we have that the convergence

$$\delta_0 \int_0^{u^\varepsilon(0)} \eta''(\omega)[f_r(\omega) - f_l(\omega)]\, d\omega + q(x,u^\varepsilon)_x \to \delta_0 \int_0^{u(0)} \eta''(\omega)[f_r(\omega) - f_l(\omega)]\, d\omega + q(x,u)_x$$

holds in the space of the distributions, completing the proof of (4.8).

Step 3. The entropy conditions follow from the entropy inequality (4.8). Indeed, assume that u has a jump at x_o with $u^{\pm} = u(x_o\pm)$. Suppose $u^- < u^+$ while the other case being completely similar. Since $\eta'(u)(u-w)$ is absolutely continuous with respect to the Lebesgue measure, computing the measure of (4.8) at the singleton $\{x_o\}$ we obtain

$$\delta_0(\{x_o\}) \int_0^{u(0)} \eta''(\omega)\left[f_r(\omega) - f_l(\omega)\right] \, d\omega + q\left(x_o+, u^+\right) - q\left(x_o-, u^-\right) \leq 0. \tag{4.14}$$

For $k \in [0,1]$ and $i \in \mathbb{N} \setminus \{0\}$, we consider the following family of smooth convex functions $\eta_{k,i}$ and the corresponding fluxes $q_{k,i}$:

$$\eta_{k,i}(\omega) = \sqrt{\frac{1}{i} + (\omega - k)^2}, \qquad q_{k,i}(x,\omega) = \int_0^{\omega} \eta'_{k,i}(\bar\omega)\, f_{\bar\omega}(x,\bar\omega) \, d\bar\omega.$$

We have that, as $i \to +\infty$:

$$\begin{aligned} \eta_{k,i}(\omega) &\to |\omega - k| && \text{uniformly,} \\ \eta'_{k,i}(\omega) &\to \operatorname{sign}(\omega - k) && \text{pointwise,} \\ \eta''_{k,i}(\omega) &\to 2\delta_k && \text{weakly}^* \text{ in the space of Radon measures,} \\ q_{k,i}(x,\omega) &\to \int_0^{\omega} \operatorname{sign}(\bar\omega - k)\, f_{\bar\omega}(x,\bar\omega)\, d\bar\omega && \text{pointwise.} \end{aligned} \tag{4.15}$$

Here δ_k is the unit mass centered at $\omega = k$.

We now substitute $\eta_{k,i}$ and $q_{k,i}$ in (4.14) and take the limit as $i \to +\infty$. We obtain, for any $k \notin \{0, u(0)\}$:

$$\begin{aligned} &2\delta_0(\{x_o\})\, \chi_{[0,u(0)]}(k)\left[f_r(k) - f_l(k)\right] \\ &\quad + \int_0^{u^+} \operatorname{sign}(\omega - k)\, f_\omega(x_o+, \omega)\, d\omega - \int_0^{u^-} \operatorname{sign}(\omega - k)\, f_\omega(x_o-, \omega)\, d\omega \leq 0. \end{aligned} \tag{4.16}$$

Since the second and the third terms in the left hand side of (4.16) are continuous with respect to k, it must hold for any $u(0), k \in [0,1]$.

Step 4. Suppose $x_o > 0$ (the case $x_o < 0$ is completely similar). Then (4.16) becomes (recall that we assume $u^- < u^+$)

$$\int_{u^-}^{u^+} \operatorname{sign}(\omega - k)\, f'_r(\omega)\, d\omega \leq 0, \qquad \text{for any } k \in [0,1]. \tag{4.17}$$

Evaluating (4.17) at $k = 0$ gives $f_r(u^+) - f_r(u^-) \leq 0$, while at $k = 1$ it gives $f_r(u^+) - f_r(u^-) \geq 0$, thus we conclude (4.9). Letting $k \in [u^-, u^+]$, (4.17) becomes $\bar f \leq f_r(k)$ which proves 1.

Finally we consider the case $x_o = 0$ where $\delta_0(x_o) = 1$. Then, (4.16) becomes

$$2\chi_{[0,u(0)]}(k)\left[f_r(k) - f_l(k)\right] + \int_0^{u^+} \operatorname{sign}(\omega - k)\, f'_r(\omega)\, d\omega - \int_0^{u^-} \operatorname{sign}(\omega - k)\, f'_l(\omega)\, d\omega \leq 0.$$

Setting $k = 0$ and $k = 1$ in the above inequality we obtain

$$f_l\left(u^-\right) = f_r\left(u^+\right) = \bar f,$$

proving (4.9). Then, with $k \in [u^-, u^+]$ we get

$$\bar f \leq -\chi_{[0,u(0)]}(k)\left[f_r(k) - f_l(k)\right] + f_r(k) \qquad \text{for any } k \in \left[u^-, u^+\right].$$

Letting

$$u^* = \begin{cases} u^- & \text{if } u(0) \leq u^-, \\ u(0) & \text{if } u^- \leq u(0) \leq u^+, \\ u^+ & \text{if } u(0) \geq u^+, \end{cases}$$

this proves 3. This proof for 2. and 4. is completely similar. □

We now establish the uniqueness of the vanishing viscosity limit for backward Euler operator J^ε_λ.

THEOREM 4.5. *For any $w \in D$, $J^\varepsilon_\lambda w$ converges in $\mathbf{L^1}(\mathbb{R},\mathbb{R})$ to a unique limit $J_\lambda w \in D$ as $\varepsilon \to 0$, the map $J_\lambda : D \to D$ being a contraction.*

Proof. Suppose first $w \in \mathbf{C^\infty_c}(\mathbb{R},\mathbb{R})$, where the support of w is contained in $[-M, M]$ for some $M > 0$. Define

$$\overline{u}(x) = e^{\gamma(x+M)}.$$

For $\gamma > 0$ sufficiently small independently of $\varepsilon \in (0, 1)$, we have, for all $x \in]-\infty, -M[$,

$$\overline{u} + \lambda \left[f(x, \overline{u}) - \varepsilon \overline{u}_x\right]_x = \overline{u}\left[1 + \lambda\gamma\left(f'_l(\overline{u}) - \varepsilon\gamma\right)\right] \geq 0 = w.$$

Therefore, for $\gamma > 0$ small, in $]-\infty, -M[$, $\overline{u}$ is an upper solution to (3.3) satisfying

$$\liminf_{x\to-\infty}\left[u(x) - \overline{u}(x)\right] = 0, \quad \liminf_{x\to-M^-}\left[u(x) - \overline{u}(x)\right] = u(-M) - 1 \leq 0,$$

where $u(x) = \left(J^\varepsilon_\lambda w\right)(x)$. Applying Theorem 3.3, we have that $0 \leq J^\varepsilon_\lambda w \leq \overline{u}$ in $]-\infty, -M[$. A similar argument holds in the interval $]M, +\infty[$.

Hence, for any $\varepsilon \in (0, 1)$ we have

$$0 \leq J^\varepsilon_\lambda w \leq \min\left\{1, e^{\gamma(x+M)}, e^{-\gamma(x-M)}\right\} \in \mathbf{L^1}(\mathbb{R},\mathbb{R}). \tag{4.18}$$

By Theorem 4.4, for any sequence $\varepsilon_\nu \to 0$ there exists a subsequence ε_{ν_j} such that $u_{\nu_j} = J^{\varepsilon_{\nu_j}}_\lambda w$ converges pointwise in $\mathbb{R}$ to a function u, and we have $0 \leq u \leq 1$. Using (4.18), the dominated convergence theorem implies that the pointwise limit u is in $\mathbf{L^1}(\mathbb{R},\mathbb{R})$ and that $J^{\varepsilon_{\nu_j}}_\lambda w$ converges to u in $\mathbf{L^1}(\mathbb{R},\mathbb{R})$. The limit u is a weak solution to (4.7) and must satisfy all the properties in Theorem 4.4.

We use contradiction to prove uniqueness of the limit. Assume that there are two limit functions u and v, which satisfy all the properties of Theorem 4.4. Since they have bounded total variation, we consider their left continuous representatives. This choice does not change at any point the left and right limits, therefore (4.9) and 1. 2. 3. 4. in Theorem 4.4 continue to hold. Suppose that there exists a point x_o such that $u(x_o) < v(x_o)$. Define (see Figure 2):

$$\begin{aligned} a &= \inf\left\{x \leq x_o : \ u(\xi) < v(\xi) \ \text{ for any } \xi \in]x, x_o]\right\}, \\ b &= \sup\left\{x \geq x_o : \ u(\xi) \leq v(\xi) \ \text{ for any } \xi \in [x_o, x]\right\}. \end{aligned} \tag{4.19}$$

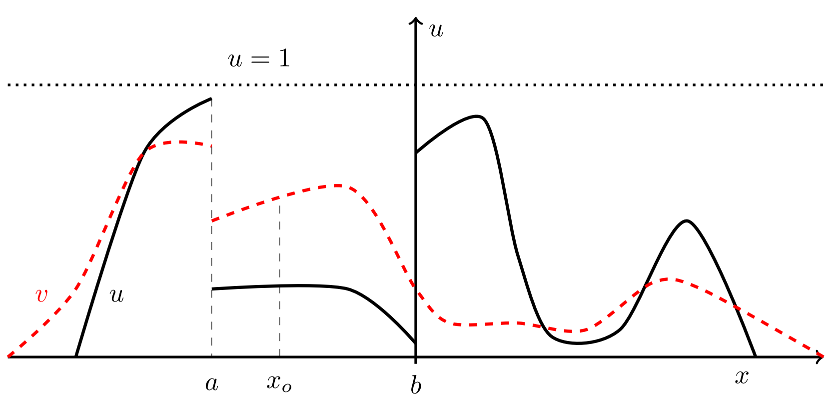

FIG. 2. *If there exists a point x_o such that $v(x_o) > u(x_o)$ a contradiction is reached.*

We have $a < x_o \leq b$, and

$$\begin{array}{ll} u(x) < v(x), \quad \text{for any } x \in]a, x_o]; & u(x) \leq v(x), \quad \text{for any } x \in]x_o, b], \\ v(a+) - u(a+) \geq 0, & v(b-) - u(b-) \geq 0, \\ v(a-) - u(a-) \leq 0, \ \text{for } a \neq -\infty; & v(b+) - u(b+) \leq 0, \ \text{for } b \neq +\infty. \end{array} \tag{4.20}$$

Observe that $f(x,u)$ and $f(x,v)$ are absolutely continuous thanks to (4.7). Hence integrating over the interval $]a, x_o[$ the identity

$$v - u = \lambda \left[f(x,u) - f(x,v) \right]_x ,$$

we get

$$\begin{aligned} 0 < \int_a^{x_o} [v-u] \ dx \le \int_a^b [v-u] \ dx = \lambda \int_a^b \left[f(x,u) - f(x,v) \right]_x \ dx \\ = \lambda \left[f(b-, u(b-)) - f(b-, v(b-)) \right] - \lambda \left[f(a+, u(a+)) - f(a+, v(a+)) \right]. \end{aligned} \tag{4.21}$$

We claim that the entropy conditions of Theorem 4.4 imply

$$f(b-, u(b-)) - f(b-, v(b-)) \le 0 \text{ and } f(a+, u(a+)) - f(a+, v(a+)) \ge 0. \tag{4.22}$$

The claim leads to the contradiction

$$0 < \int_a^{x_o} [v-u] \ dx \le 0.$$

Thus, $J_\lambda^\varepsilon w$ converges in $\mathbf{L^1}(\mathbb{R}, \mathbb{R})$ to a unique limit $J_\lambda w$ that satisfies all the properties of Theorem 4.4.

Finally we take any function $w \in D$ and fix $\gamma > 0$. Take $w_\gamma \in \mathbf{C_c^\infty}(\mathbb{R}, \mathbb{R}) \cap D$ such that $\|w_\gamma - w\|_{\mathbf{L^1}(\mathbb{R},\mathbb{R})} < \gamma$. Then, using the contraction property of J_λ^ε, we have

$$\|J_\lambda^\varepsilon w - J_\lambda^\mu w\|_{\mathbf{L^1}(\mathbb{R},\mathbb{R})} \le 2\gamma + \|J_\lambda^\varepsilon w_\gamma - J_\lambda^\mu w_\gamma\|_{\mathbf{L^1}(\mathbb{R},\mathbb{R})} ,$$

so that

$$\limsup_{\varepsilon,\mu \to 0} \|J_\lambda^\varepsilon w - J_\lambda^\mu w\|_{\mathbf{L^1}(\mathbb{R},\mathbb{R})} \le 2\gamma.$$

This proves that $J_\lambda^\varepsilon w$ is a Cauchy sequence in the complete metric space D, hence it converges in D to a unique limit $J_\lambda w$. Consequently J_λ is also a contraction.

It remains to prove the claim (4.22). We prove only the first inequality, while the second being completely similar. If $b = +\infty$ then we have

$$f(b-, u(b-)) - f(b-, v(b-)) = f_r(0) - f_r(0) = 0.$$

If $b \in]0, +\infty[$ then

$$f(b-, u(b-)) - f(b-, v(b-)) = f_r(u(b-)) - f_r(v(b-)).$$

Denote $u^\pm = u(b\pm), v^\pm = v(b\pm)$, and suppose $u^- \le u^+$ (the other case being similar). By (4.9) we have

$$\bar{f}_u \doteq f_r(u^-) = f_r(u^+), \quad \bar{f}_v \doteq f_r(v^-) = f_r(v^+), \quad f_r(u(b-)) - f_r(v(b-)) = \bar{f}_u - \bar{f}_v.$$

If one of the two states v^- or v^+ belongs to the interval $[u^-, u^+]$, then 1. in Theorem 4.4 implies $\bar{f}_v \ge \bar{f}_u$. If none of them is on the interval $[u^-, u^+]$, by (4.20) we have $v^+ \le u^- \le u^+ \le v^-$. By point 2. in Theorem 4.4 applied to the function v we obtain again $\bar{f}_u \le \bar{f}_v$, proving the claim. The case $b < 0$ is completely similar.

Finally we consider the case $b = 0$. Again, suppose $u^- \le u^+$ and let u^* be the state in point 3. of Theorem 4.4. If either $v^- \in [u^-, u^*]$ or $v^+ \in [u^*, u^+]$ then 3. in Theorem 4.4 implies $\bar{f}_v \ge \bar{f}_u$. If neither $v^- \in [u^-, u^*]$ nor $v^+ \in [u^*, u^+]$, then by (4.20) we have $v^+ \le u^* \le v^-$. This relation, together with point 4. in Theorem 4.4 applied to the function v gives again $\bar{f}_u \le \bar{f}_v$ since there exists a point, namely $u^* \in [v^+, v^-]$, such that $\bar{f}_v \ge \min\{f_l(u^*), f_r(u^*)\} \ge \bar{f}_u$. This completes the proof for the claim (4.22). □

5. The vanishing viscosity limit for the generated semigroups. In this section we apply Theorems 2.2 and 2.3 to approximate the semigroup generated by

$$u_t + f(x,u)_x = 0 \tag{5.1}$$

with the semigroups generated by the parabolic evolution equations

$$u_t + f(x,u)_x \ = \ \varepsilon\, u_{xx}, \tag{5.2}$$

where the flux f satisfies **f1)**.

Theorem 4.5 implies

$$\lim_{\varepsilon\to 0} J_\lambda^\varepsilon w = J_\lambda w, \qquad \text{for any} \qquad w \in D,\ \lambda > 0, \tag{5.3}$$

J_λ being a family of contractions in D with D defined in (4.1). Therefore we can define the (possibly multivalued) map

$$A = \left\{ \left(J_\lambda w, \frac{1}{\lambda}\left(w - J_\lambda w\right)\right) :\ w \in D,\ \lambda > 0 \right\} \subset D \times \mathbf{L^1}(\mathbb{R},\mathbb{R}). \tag{5.4}$$

Remark 5.1. Recalling (2.1), the domain of A is given by:

$$\mathcal{D}(A) = \{u \in D : \ \text{ there exist } w \in D,\ \lambda > 0 \text{ such that } u = J_\lambda w\}. \tag{5.5}$$

Therefore, if $u \in \mathcal{D}(A)$, then, for some $w \in \mathcal{D}$, $\lambda > 0$: $u = J_\lambda w = \lim_{\varepsilon\to 0} u^\varepsilon$ with $u^\varepsilon = J_\lambda^\varepsilon w$. Since J_λ^ε is the resolvent of A^ε, the function u^ε solves $u^\varepsilon + \lambda\left[f(x,u^\varepsilon) - \varepsilon u_x^\varepsilon\right]_x = w$. Taking the weak limit of this equation as $\varepsilon \to 0$ we have that u is a weak solution to $u + \lambda f(x,u)_x = w$. This implies $w - J_\lambda w = w - u = \lambda f(x,u)_x$, therefore the operator A is single valued with

$$Au = f(x,u)_x \in \mathbf{L^1}(\mathbb{R},\mathbb{R}) \ \text{ for any } u \in \mathcal{D}(A). \tag{5.6}$$

The operator A is a candidate as a generator for the evolution equation (5.1).

Take now $u \in D \cap \mathbf{C_c^\infty}(\mathbb{R}\setminus\{0\},\mathbb{R})$, satisfying $u(x) < 1$ for any $x \in \mathbb{R}$. Then, for a suitably small $\lambda > 0$ and all $\varepsilon \in]0,1[$, we have

$$w^\varepsilon = u + \lambda\left[f(x,u) - \varepsilon u_x\right]_x \in \mathbf{C_c^\infty}(\mathbb{R}\setminus\{0\},\mathbb{R}) \cap D.$$

Moreover w^ε converges to $w = u + \lambda f(x,u)_x \in D$ in $\mathbf{L^1}(\mathbb{R},\mathbb{R})$ as $\varepsilon \to 0$. Since $u = J_\lambda^\varepsilon w^\varepsilon$ we compute

$$\|u - J_\lambda w\|_{\mathbf{L^1}(\mathbb{R},\mathbb{R})} = \lim_{\varepsilon\to 0} \|J_\lambda^\varepsilon w^\varepsilon - J_\lambda^\varepsilon w\|_{\mathbf{L^1}(\mathbb{R},\mathbb{R})} \le \lim_{\varepsilon\to 0}\|w^\varepsilon - w\|_{\mathbf{L^1}(\mathbb{R},\mathbb{R})} = 0.$$

This means that $u = J_\lambda w$ and hence $u \in \mathcal{D}(A)$. This implies $\overline{\mathcal{D}(A)} = D$ i.e. the domain of A is dense in D.

THEOREM 5.2. *The map A defined in* (5.4) *(or, alternatively in* (5.5), (5.6)*) generates a unique continuous semigroup of contractions $S_t : D \to D$ whose trajectories are weak solutions to* (5.1)*. Moreover, let $S_t^\varepsilon : D \to D$ be the semigroup generated by A^ε in* (3.1)*, then the following limit holds*

$$S_t \bar{u} \ = \ \lim_{\varepsilon\to 0} S_t^\varepsilon \bar{u}, \qquad \textit{for all } \bar{u} \in D \textit{ uniformly on bounded } t \textit{ intervals.} \tag{5.7}$$

Proof. Take any $w \in D$, $\lambda > 0$, by definition (5.5), $J_\lambda w \in \mathcal{D}(A)$, therefore, using (5.4) we compute

$$(I + \lambda A)\, J_\lambda w = J_\lambda w + \lambda \frac{1}{\lambda}\left(w - J_\lambda w\right) = w.$$

Therefore, for any $\lambda > 0$, we have $\overline{\mathcal{D}(A)} = D \subset \mathcal{R}(I + \lambda A)$. The previous equality also shows that the resolvent of A is J_λ which is a contraction on D. By the Crandall & Liggett generation theorem, Theorem 2.2, the map A generate a semigroup of contractions S_t defined on D.

We are now in a position to apply the result by Brezis and Pazy. Since $\lim_{\varepsilon\to 0} J_\lambda^\varepsilon w = J_\lambda w$ for any $w \in D = \overline{\mathcal{D}(A)}$ and $\lambda > 0$ with $\mathcal{D}(A) \subset \overline{\mathcal{D}(A^\varepsilon)} = \mathbf{L^1}(\mathbb{R},\mathbb{R})$, Theorem 2.3 implies (5.7) for the corresponding semigroups. Finally, passing to the limit in the weak formulation for (5.2), the trajectories of S_t are weak solutions to (5.1). ∎

6. Counter examples on adapted entropies and their applications. We first observe that the entropy solutions selected by the adapted entropies approach [9, 11, 15, 18, 46] are not, in general, the vanishing viscosity limits. We consider the example given in [9, Section 5], the paper where the concept was originally introduced. One considers a conservation law

$$u_t + f(x,u)_x = 0, \qquad \text{where} \quad f(x,u) = \begin{cases} f_l(u) = \dfrac{1}{2}(u-1)^2, & x \le 0, \\ f_r(u) = \dfrac{1}{2}u^2, & x > 0. \end{cases} \tag{6.1}$$

See Figure 3 for an illustration of the graphs for the flux functions f_l and f_r.

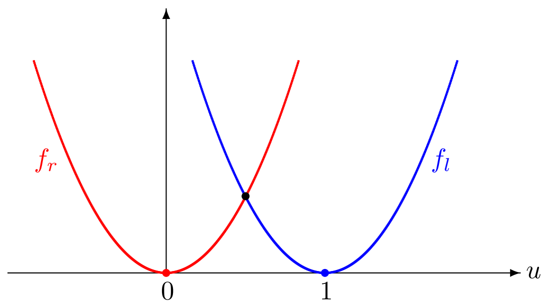

FIG. 3. *Graphs for the flux functions f_l and f_r.*

Below we give several examples, on various cases and aspects of this problem.

Example 6.1. In this example we show that the solution that satisfies the adapted entropy is different from the one obtained by vanishing viscosity. Consider the Riemann problem

$$u_t + f(x,u)_x = 0, \qquad u(0,x) = \begin{cases} u_l = \frac{1}{2}, & x \le 0, \\ u_r = \frac{1}{2}, & x > 0, \end{cases} \tag{6.2}$$

and the corresponding viscous equation

$$u_t + f(x,u)_x = \varepsilon u_{xx}, \qquad u(0,x) = \frac{1}{2}. \tag{6.3}$$

We observe that the graphs of f_l and f_r intersect at $u = \frac{1}{2}$ where $f_l(\frac{1}{2}) = f_r(\frac{1}{2})$. Therefore the constant function $u^\varepsilon(t,x) = \frac{1}{2}$ is the solution of the Cauchy problem for the viscous equation (6.3) for any $\varepsilon > 0$. Hence as $\varepsilon \to 0+$, the solution u^ε converges strongly to the constant function $u(t,x) = \frac{1}{2}$, which is the vanishing viscosity solution for the non-viscous equation with the same initial data (6.2).

However, the solution selected by the adapted entropy with initial condition (6.2) is different. From formula [9, (5.7)], we see that the adapted entropy solution consists of three parts:

- a rarefaction wave with negative characteristic speed, solving the Riemann problem

$$u_t + f_l(u)_x = 0, \qquad u(0,x) = \begin{cases} \frac{1}{2}, & x < 0, \\ 1, & x > 0, \end{cases}$$

- a discontinuity at $x = 0$, with the traces $u(t, 0-) = 1$ and $u(t, 0+) = 0$, and
- a rarefaction wave with positive characteristic speed, solving the Riemann problem

$$u_t + f_r(u)_x = 0, \qquad u(0,x) = \begin{cases} 0, & x < 0, \\ \frac{1}{2}, & x > 0. \end{cases}$$

Example 6.2. We now show that, a discontinuity satisfying the adapted entropy condition can not be obtained as the vanishing viscosity limit of a viscous traveling wave. Formula [9, (5.7)] further implies that, for the Cauchy problem with the initial condition

$$u(0,x) = \begin{cases} 1, & x < 0, \\ 0, & x > 0, \end{cases} \tag{6.4}$$

the adapted entropy solution is stationary in time, i.e.

$$u(t,x) = u(0,x) = \begin{cases} 1, & x < 0, \\ 0, & x > 0, \end{cases}, \qquad \forall t \geq 0. \tag{6.5}$$

We claim that the solution (6.5) can not be obtained by vanishing viscosity of a viscous traveling wave for the viscous equation

$$u_t + f(x,u)_x = \varepsilon u_{xx}. \tag{6.6}$$

Indeed, fix $\varepsilon > 0$, and let U be a monotone stationary viscous profile for (6.6), satisfying the asymptotic limits

$$\lim_{x\to-\infty} U(x) = 1, \qquad \lim_{x\to+\infty} U(x) = 0. \tag{6.7}$$

Then, U satisfies the ODE

$$\varepsilon U' = f(x,U). \tag{6.8}$$

With the asymptotic conditions (6.7) we seek monotonically decreasing solutions, i.e. $U'(x) \leq 0$ for all $x \in \mathbb{R}$. However from Figure 3 it is clear that, for every U between 0 and 1, we have both $f_l(U) > 0$ and $f_r(U) > 0$, therefore $f(x,U) > 0$ and thus $U'(x) > 0$ for all $x \in \mathbb{R}$ and $U \in (0,1)$, a contradiction. We conclude that no stationary, monotonically decreasing viscous wave profiles can exist with the asymptotic conditions (6.7), proving the claim.

Remark 6.3. We remark that a vanishing viscosity Riemann solver was constructed in [29] and a rigorous proof was given. Following the algorithm in [29], the unique vanishing viscosity solution for the Riemann problem in Example 6.2 consists of (i) a shock from $u = 1$ to $u = \frac{1}{2}$ with negative wave speed for $x < 0$, (ii) $u(t, 0-) = u(t, 0+) = \frac{1}{2}$ at $x = 0$, and (iii) a shock from $u = \frac{1}{2}$ to $u = 0$ with positive wave speed for $x > 0$.

Example 6.4. It would be of interest to analyze rigorously the admissible solutions selected by the adapted entropies as limit solutions of some regularization (for example vanishing viscosity), a problem which is still open. Preliminarily, by comparing the above examples to Example 4.4 in [51], it appears that the adapted entropy condition selects the solution with $\kappa = \infty$, i.e., with $\varepsilon_n \equiv 0$ in (1.3). We now provide a simple proof for this claim in the setting of this example.

The flux in (6.1) can also be rewritten as

$$f(x,u) = \frac{(u - 1 + H(x))^2}{2},$$

where H is the Heaviside step function. Consider a smooth and monotone mollification H^δ such that

$$H^\delta(x) = \begin{cases} 0 & \text{for } x \leq -\delta, \\ 1 & \text{for } x \geq \delta, \end{cases} \qquad (H^\delta)'(x) \geq 0, \text{ for } |x| \leq \delta,$$

and

$$\lim_{\delta\to 0} H^\delta(x) = H(x) \qquad \text{pointwise } \forall x \in \mathbb{R} \setminus \{0\}.$$

We denote the mollified flux as

$$f^\delta(x,u) \doteq \frac{(u-1+H^\delta(x))^2}{2}.$$

Fix an x, we have

$$(f^\delta)_u(x,u) = u - 1 + H^\delta(x).$$

Therefore the minimum of the mapping $u \mapsto f^\delta$ is at

$$u_m(x) = 1 - H^\delta(x), \qquad \text{where} \quad f^\delta(x, u_m(x)) = 0 \quad \forall x \in \mathbb{R}.$$

One can readily verify that the smooth function u^δ defined as

$$u^\delta(t,x) = \begin{cases} 1, & x \le -\delta, \\ u_m(x) = 1 - H^\delta(x), & |x| \le \delta, \\ 0, & x \ge \delta, \end{cases}$$

is a stationary solution of the Cauchy problem for the conservation law

$$u_t + f^\delta(x,u)_x = 0$$

with initial condition $u^\delta(0,\cdot)$. Taking the limit $\delta \to 0$, we see that $u^\delta(t,\cdot)$ converges to $u(t,\cdot)$ in (6.5), which is the solution selected by the adapted entropies. This proves our claim.

The analysis in Example 6.4 applies only to this specific example, with the specific choice of mollification. A rigorous analysis for the general cases is beyond the scope of this paper, and could be the topic of a separated future work.

Remark 6.5. We remark that the adapted entropies require strong restrictions on the fluxes, even in one space dimension. In [9] (H3′), the flux can have at most one single minimum (or maximum) *at the same level* for any x. This restricts a direct application to models of traffic flow with rough road conditions and road junctions (see [25]), where a typical flux function is $f(x,u) = V(x)u(1-u)$ with V discontinuous. In contrast, our hypothesis allows the presence in the flux both at $x > 0$ and $x < 0$ of any number of maxima/minima at any number of different levels.

The adapted entropy concept is utilized in [46] to establish uniqueness of solutions for scalar conservation laws with discontinuous flux. The procedure introduced in [46, (1.5)] allows the study of rather general right and left flux functions, in the adapted entropies framework. Unfortunately, in the case where the right and left fluxes have extrema at different levels, one obtains non-physical solutions in applications. Below we give a concrete example.

Example 6.6. Consider the Riemann problem for traffic flow

$$\begin{cases} u_t + f\left(x,u\right)_x = 0, \\ u\left(0,x\right) = \frac{1}{2}, \end{cases}$$

for the flux

$$f\left(x,\omega\right) = \begin{cases} f_l\left(\omega\right) = 8\omega\left(1-\omega\right) & \text{if } x \le 0 \text{ and } \omega \in [0,1]\,, \\ f_r\left(\omega\right) = 4\omega\left(1-\omega\right) & \text{if } x > 0 \text{ and } \omega \in [0,1]\,. \end{cases}$$

Depending on the choice of g and β satisfying $f(x,\omega) = g\left(\beta\left(x,\omega\right)\right)$ as in [46] one can get only two types of solutions. The first one is a connection in the sense of [3, 17] with $A = 1$ and $B = 0$ as it has already been observed in [46, after (4.3)]. The other type of solutions take values outside the interval $[0,1]$ which is "nonphysical" since the conserved variable is a density function. Other types of solutions studied in the literature, such as the ones obtained with connections using different A and B values, or the one obtained by vanishing viscosity, do not satisfy the adapted entropy condition.

We finally remark that the solutions considered in [46] are not the vanishing viscosity solutions in general case, from the discussions in Examples 6.1-6.4.

7. Examples and concluding remarks.

Example 7.1. We first give several examples of the backward Euler operators for the non viscous conservation law

$$u_t + f(u)_x = 0.$$

If f_u has a fixed sign, say $f_u > 0$, then for any $\lambda > 0$ the backward Euler operator J_λ generates a continuous function u, even for discontinuous function of w. In this simpler case, the entropy condition is automatically satisfied, and the operator generates a Lipschitz semigroup of entropy weak solution for the conservation law [20]. However, when f_u changes sign, the backward Euler solution might not be unique, and entropy conditions (such as in Theorem 4.4) are required to single out the admissible solution.

To fix the idea, we consider the traffic flow model with $f(u) = u(1-u)$. Given w, the solution $u = J_\lambda w$ satisfies the ODE

$$u'(x) = \frac{w-u}{\lambda f'(u)} = \frac{w-u}{2\lambda(0.5-u)}. \tag{7.1}$$

If $w(x)$ is piecewise constant, the solution for the above ODE can be constructed explicitly on each interval where $w(x)$ is constant. One can then piece them together to form a solution on the whole real line.

We observe that u' blows up at $u = 0.5$, unless $w = 0.5$ also. When $w = 0.5$, we have $u' = 1/(2\lambda)$ if $u \neq 0.5$. Since u' can be anything at $u = 0.5$, we also have $u \equiv 0.5$ as a solution.

We consider 3 typical cases, where we use $\lambda = 0.5$.

Case 1. If we use the initial condition

$$w(x) = \begin{cases} 0.5 & (|x| > 1), \\ 1 & (-1 < x < 1), \end{cases}$$

the solution of the conservation law consists of a shock at $x = -1$ and a rarefaction at $x = 1$. Furthermore, we have $f'(u) \leq 0$ in the solution. Consequently, the backward Euler solution $u(x)$ contains no jump, see Figure 4 for a qualitative illustration. However, we observe vertical tangent at $x = 1$ where $u = 0.5$ and $f'(u) = 0$.

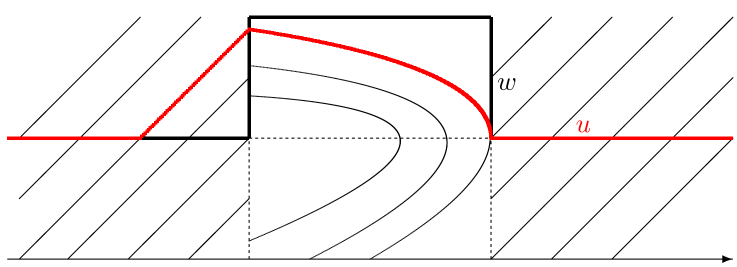

FIG. 4. *Case 1. Plots of $w(x)$ (thick black lines), possible solutions of (7.1) (thin black curves), and $u(x)$ (thick red curves) for the case with $f'(u) \leq 0$. Here $u(x)$ contains no jumps.*

Case 2. If we use the initial condition

$$w(x) = \begin{cases} 0.5 & (x < -1), \\ 1 & (-1 < x < 0), \\ 0.25 & (0 < x < 0.5), \\ 0.5 & (x > 0.5), \end{cases}$$

the solution for the conservation law consists of a transonic rarefaction initiated at $x = 0$. The backward Euler solution $u(x)$ contains no jumps, see Figure 5, although the gradient is infinite at $x = 0$ where $u = 0.5$.

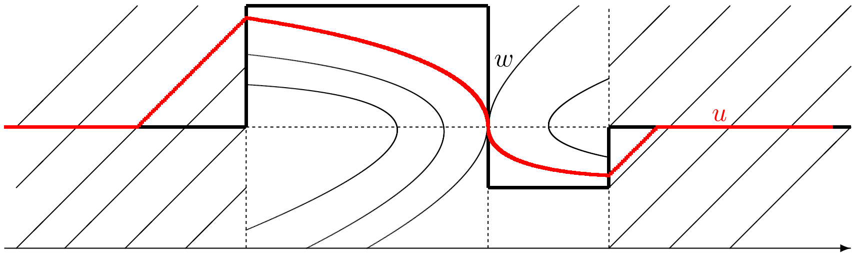

FIG. 5. *Case 2. Plots of $w(x)$ (thick black lines), possible solutions of (7.1) (thin black curves), and $u(x)$ (thick red curves) for the case with a transonic rarefaction at $x = 0$. The solution $u(x)$ contains no jumps.*

Case 3. If we use the initial data

$$w(x) = \begin{cases} 0.5 & (x < -1), \\ 0.25 & (-1 < x < 0), \\ 1 & (0 < x < 1), \\ 0.5 & (x > 1), \end{cases}$$

the solution of the conservation laws contains a transonic shock initiated at $x = 0$. The backward Euler solution $u(x)$ contains a jump, see Figure 6. Note that there are many places to insert the jump, if no entropy conditions are required. The unique location of the jump in $u(x)$ is determined by the entropy conditions in Theorem 4.4 point 1.

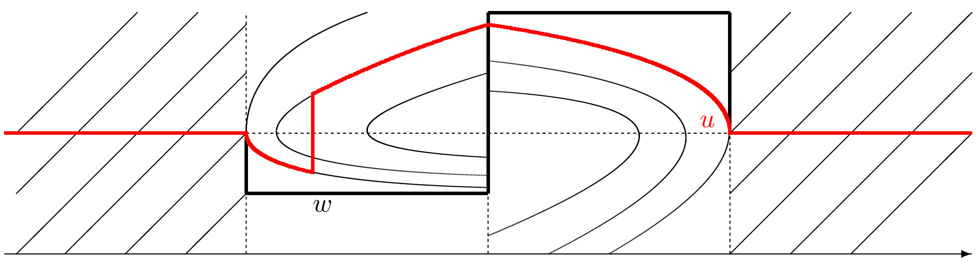

FIG. 6. *Case 3. Plots of $w(x)$ (thick black lines), possible solutions of (7.1) (thin black curves), and $u(x)$ (thick red curves) for the case with a transonic shock initiated at $x = 0$. The solution $u(x)$ has a jump. The location of the discontinuity is uniquely determined by the entropy conditions in Theorem 4.4.*

Example 7.2. We now give an example of the backward Euler operator for the non viscous conservation law with discontinuous flux. We consider

$$u_t + f(x, u)_x = 0, \qquad f(x, u) = \begin{cases} f_l(u) = u(1 - u) & (x < 0), \\ f_r(u) = 2u(1 - u) & (x > 0). \end{cases}$$

We use the following initial data:

$$w(x) = \begin{cases} 0.5 & (x < -1), \\ 0.4 & (-1 < x < 0), \\ 0.7 & (0 < x < 1), \\ 0.5 & (x > 1). \end{cases}$$

The solution $u = J_\lambda w$ satisfies the ODE

$$u'(x) = \frac{w - u}{\lambda f_u(x, u)}. \tag{7.2}$$

In the solution of the conservation law, we have rarefaction waves at $x = \pm 1$. The Riemann problem at $x = 0$ is solved with a stationary jump and a shock with positive speed. The backward Euler solutions without entropy conditions, are not unique. Applying the entropy conditions of Theorem 4.4, the solution $u(x)$ contains two discontinuities, as illustrated in Figure 7. The discontinuity at $x = 0$ satisfies the condition in point 4 of Theorem 4.4, while the location of the transonic shock satisfies point 1 of Theorem 4.4.

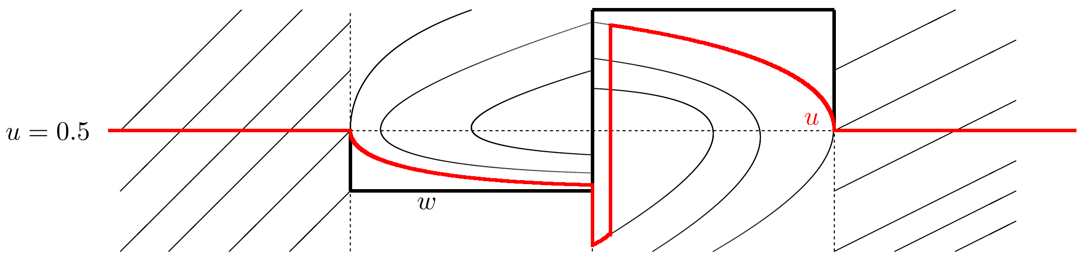

FIG. 7. *Plots of $w(x)$ (thick black lines), possible solutions of the ODE (7.2) (thin black curves), and $u(x)$ (thick red curves) for Example 7.2. The solution $u(x)$ contains two jumps, one at $x = 0$, and the other one represents the transonic shock. The location of the discontinuity is uniquely determined by the entropy conditions in Theorem 4.4.*

We now study the same phenomenon from the point of view of the non linear generator of the semigroup. At first sight, the evolution equation (5.1) should correspond to the operator B defined by

$$(u, v) \in B \quad \text{if and only if} \quad u,\ v \in \mathbf{L^1}(\mathbb{R}, \mathbb{R}) \text{ and } v = f(x, u)_x, \tag{7.3}$$

as in the definition of the operators A^ε. Unfortunately, as we have seen from the point of view of the backward Euler operator, the domain of B is "too big" and it is not an accretive operator, therefore the Crandall & Liggett generation theorem does not apply. We see this in a spatial homogeneous case.

Example 7.3. Consider (5.1) with $f_l(u) = f_r(u) = f(u) = u(1-u)$, i.e. a classical example for scalar conservation laws:

$$u_t + [u(1-u)]_x = 0. \tag{7.4}$$

Let $\phi(x) \in [0, 1]$ be a Lipschitz continuous function such that $\phi(x) = 0$ for $x \le -2$ and $\phi(x) = 1$ for $x \ge -1$ and define the following family of functions parametrized by $\gamma \in [0, 1]$ (See Figure 8):

$$u_\gamma(x) = \begin{cases} \phi(x) & \text{if } x \le -1, \\ 1 & \text{if } -1 < x \le -\gamma, \\ \frac{1}{2}\left(1 - \frac{x}{\gamma}\right) & \text{if } x \in \left]-\gamma, \gamma\right[, \\ 0 & \text{if } x \ge \gamma. \end{cases}$$

Since u_γ for $\gamma > 0$ is Lipschitz continuous, it belongs to the domain of B. When $\gamma = 0$, u_0 is discontinuous at $x = 0$, but $x \mapsto f(u_0)$ is smooth, therefore u_0 also belongs to the domain of B. But we will show that u_0 *does not* belong to the domain of A as defined in (5.5).

For $\gamma \in [0, 1]$ we have (see Figure 9)

$$(Bu_\gamma)(x) = \begin{cases} f(\phi(x))_x & \text{if } x \le -1, \\ 0 & \text{if } -1 < x \le -\gamma, \\ -\dfrac{x}{2\gamma^2} & \text{if } -\gamma < x < \gamma, \\ 0 & \text{if } x \ge \gamma, \end{cases} \qquad (Bu_0)(x) = \begin{cases} f(\phi(x))_x & \text{if } x \le -1, \\ 0 & \text{if } -1 < x. \end{cases}$$

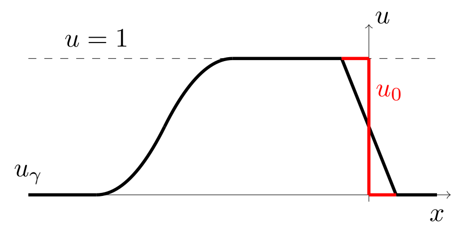

FIG. 8. *Graph of function $u_\gamma(x)$.*

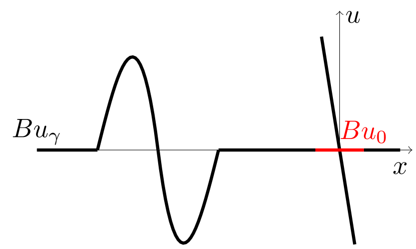

FIG. 9. *Graph of function $(Bu_\gamma)(x)$.*

Then for $\gamma > 0$, $\lambda \in]0,1]$, we have

$$\begin{aligned} \|u_\gamma - u_0\|_{\mathbf{L^1}(\mathbb{R},\mathbb{R})} &= 2\int_0^\gamma \frac{1}{2}\left(1-\frac{x}{\gamma}\right)\,dx = \frac{\gamma}{2}, \\ \|u_\gamma + \lambda B u_\gamma - (u_0 + \lambda B u_0)\|_{\mathbf{L^1}(\mathbb{R},\mathbb{R})} &= 2\int_0^\gamma \left|\frac{1}{2}\left(1-\frac{x}{\gamma}\right) - \lambda\frac{x}{2\gamma^2}\right|\,dx, \\ &= \int_0^\gamma \left|1 - \frac{x}{\gamma}\left(1+\frac{\lambda}{\gamma}\right)\right|\,dx = \frac{\gamma}{2}\cdot\frac{1+\lambda^2/\gamma^2}{1+\lambda/\gamma}. \end{aligned} \tag{7.5}$$

so that

$$\|u_\gamma + \lambda B u_\gamma - (u_0 + \lambda B u_0)\|_{\mathbf{L^1}(\mathbb{R},\mathbb{R})} = \frac{1+\lambda^2/\gamma^2}{1+\lambda/\gamma}\|u_\gamma - u_0\|_{\mathbf{L^1}(\mathbb{R},\mathbb{R})}.$$

Therefore choosing $\lambda \in \left]0,\frac{1}{2}\right]$ and $\gamma = 2\lambda$ we have

$$\|u_{2\lambda} + \lambda B u_{2\lambda} - (u_0 + \lambda B u_0)\|_{\mathbf{L^1}(\mathbb{R},\mathbb{R})} = \frac{5}{6}\|u_\gamma - u_0\|_{\mathbf{L^1}(\mathbb{R},\mathbb{R})}.$$

This shows that B is not accretive. Furthermore, it does not satisfies the broader condition [14, (1.1)]. Observe that an argument similar to the one in Remark 5.1 shows that all Lipschitz continuous functions in D are contained in $\mathcal{D}(A)$, therefore $u_\gamma \in \mathcal{D}(A)$ for any $\gamma \in]0,1]$, hence since A is accretive, u_0 *cannot* belong to the domain of A. On the other hand some computations show that

$$\|u_\gamma + \lambda B u_\gamma - (u_{\bar\gamma} + \lambda B u_{\bar\gamma})\|_{\mathbf{L^1}(\mathbb{R},\mathbb{R})} \geq \|u_\gamma - u_{\bar\gamma}\|_{\mathbf{L^1}(\mathbb{R},\mathbb{R})} \quad \text{for any } \gamma, \bar\gamma \in]0,1],$$

which is compatible with A being accretive.

Remark 7.4. It is well known that the solution $u(t,x)$ to the Cauchy problem for the evolution equation (7.4) develops discontinuities in finite time, even with smooth integrable initial data. If a discontinuity travels with a speed different from zero, then $[u(1-u)]_x = -u_t$ must contains a Dirac mass, hence the solution at time t is not contained in the domain $\mathcal{D}(A)$, see (5.6), of the generator of the evolution semigroup, but only in its closure $\overline{\mathcal{D}(A)} = D$. Therefore, this represents a very natural example of a non–linear semigroup for which the domain of its generator is not invariant.

We note that, in order to apply the generation theorem, the domain of B must be "reduced", and different "reductions" may lead to different generated semigroups. The reduction given by (5.5) leads to the semigroup of viscous approximations in Theorem 5.2. This reduction can also lead to Kružkov entropy inequalities, see [20] for the multidimensional case with smooth fluxes, or [6] for (5.1). Kružkov entropy inequalities can also be used to define different reductions which gives correspondingly different semigroups in [6], referred to as "germs".

What happens if the dependence of the flux f on the spatial variable x is more irregular? In [12], using Theorem 5.2 as a building block, existence and uniqueness of the vanishing viscosity limit for fluxes $f(t, x, \omega)$ with general **BV** regularity with respect to the variables (t, x) is obtained. The result in [12] is based on comparison estimates for solutions to the corresponding Hamilton–Jacobi equations.

The **BV** regularity on the flux is an essential assumption, as shown in the following counter example. Suppose that the map $x \mapsto f(x, \omega)$ is $\mathbf{L}^\infty$ but with unbounded variation. In this case the domain of the operator $Au = f(x, u)_x$ may not be dense in $\mathbf{L^1}$. For every $\varepsilon > 0$, the viscous approximations

$$u_t^\varepsilon + f(x, u^\varepsilon)_x \;=\; \varepsilon\, u_{xx}^\varepsilon\,, \qquad u^\varepsilon(0, x) = \bar{u}(x), \tag{7.6}$$

are still well defined for any initial data $\bar{u} \in \mathbf{L^1}(\mathbb{R}, \mathbb{R})$, according to Theorem 3.6. However, they may not converge to a (weakly) continuous function of time $t \mapsto u(t)$. In the next example, we show that one could have

$$\lim_{t \to 0+} \left(\lim_{\varepsilon \to 0} u^\varepsilon(t, \cdot) \right) \;\neq\; \bar{u}\,. \tag{7.7}$$

Example 7.5. Let $\mathbb{Q} = \{q_i\}_{i=1}^{+\infty}$ be an enumeration of the rational numbers, $\kappa > 0$, and V be the open set defined by

$$V = \bigcup_{i=1}^{+\infty} \left] q_i - \kappa 2^{-(i+1)}, q_i + \kappa 2^{-(i+1)} \right[,$$

so that $\operatorname{meas}(V) \leq \kappa$. Define the closed set $K = \mathbb{R} \setminus V$ and the function $\alpha = \chi_K$ i.e. the characteristic function of the set K. Observe that K is totally disconnected, that any rational number $q \in \mathbb{Q}$ has a neighborhood in which α is identically zero and that α has unbounded total variation on any interval with length greater than κ.

THEOREM 7.6. *Any weak solution $u \in \mathbf{L}^\infty([0, T] \times \mathbb{R}, \mathbb{R})$ to the conservation law*

$$u_t + f(x, u)_x = 0, \qquad f(x, u) \;=\; \alpha(x)\, u(1 - u) \tag{7.8}$$

such that the map $t \mapsto u(t, \cdot)$ is continuous from $[0, T]$ into $\mathbf{L}^\infty(\mathbb{R}, \mathbb{R})$ endowed with the weak topology is constant in time, $u(t, x) = \bar{u}(x)$, and must satisfy $\bar{u}(x) \in \{0, 1\}$ almost everywhere in K.*

Proof. For $\varepsilon > 0$, define the convolution kernels λ_ε as

$$\lambda_\varepsilon(x) = \frac{1}{\varepsilon} \lambda\left(\frac{x}{\varepsilon}\right), \quad \lambda \in \mathbf{C}^\infty(\mathbb{R}, [0, 1])\,, \quad \int_{\mathbb{R}} \lambda(x)\; dx = 1, \quad \lambda(x) = 0\; \forall x \notin [-1, 1]\,. \tag{7.9}$$

We let

$$a_\varepsilon(x) = \int_{-\infty}^{x} \lambda_\varepsilon(\xi)\; d\xi.$$

Fix two rational numbers $r < q$, a time $\tau > 0$, small positive $\varepsilon, \gamma > 0$ and evaluate (7.8) using the test function

$$\varphi(t, x) = \left(a_\varepsilon(x - r) - a_\varepsilon(x - q)\right)\left(a_\gamma(t - \gamma) - a_\gamma(t - \tau - \gamma)\right),$$

we get

(7.10)

$$\int_0^{\tau+2\gamma}\int_{r-\varepsilon}^{q+\varepsilon} u(t,x)\left(a_\varepsilon(x-r)-a_\varepsilon(x-q)\right)\left(\lambda_\gamma(t-\gamma)-\lambda_\gamma(t-\tau-\gamma)\right)\,dtdx$$
$$+\int_0^{\tau+2\gamma}\int_{r-\varepsilon}^{q+\varepsilon} f\left(x,u(t,x)\right)\left(\lambda_\varepsilon(x-r)-\lambda_\varepsilon(x-q)\right)\left(a_\gamma(t-\gamma)-a_\gamma(t-\tau-\gamma)\right)\,dtdx=0.$$

If ε is sufficiently small the supports of $\lambda_\varepsilon(x-r)$ and of $\lambda_\varepsilon(x-q)$ are contained in V where $f(x,u(t,x))$ vanishes. Therefore, the previous equality becomes

$$\int_0^{\tau+2\gamma}\int_{r-\varepsilon}^{q+\varepsilon} u(t,x)\left(a_\varepsilon(x-r)-a_\varepsilon(x-q)\right)\left(\lambda_\gamma(t-\gamma)-\lambda_\gamma(t-\tau-\gamma)\right)\,dtdx=0.$$

Letting ε tend to zero we obtain

$$\int_0^{2\gamma}\lambda_\gamma(t-\gamma)\int_r^q u(t,x)\,dxdt=\int_\tau^{\tau+2\gamma}\lambda_\gamma(t-\tau-\gamma)\int_r^q u(t,x)\,dxdt.$$

The weak* continuity assumption implies that the map $t\mapsto\int_r^q u(t,x)\,dx$ is continuous, therefore we can take the limit as $\gamma\to 0$ and get

$$\int_r^q u(\tau,x)\,dx=\int_r^q u(0,x)\,dx,\qquad\text{for any } r<q,\ \text{ with } r,q\in\mathbb{Q}\text{ and }\tau\in[0,T].$$

This implies that $u(t,x)=u(0,x)\doteq\bar{u}(x)$ must be constant in time as a function from $[0,T]$ into $\mathbf{L}^\infty(\mathbb{R},\mathbb{R})$. From (7.8), we have

$$\left[\alpha(x)\bar{u}(x)\left(1-\bar{u}(x)\right)\right]_x=0,\quad\Longrightarrow\quad\alpha(x)\bar{u}(x)\left(1-\bar{u}(x)\right)=C$$

for some constant $C\in\mathbb{R}$. But α vanishes on the set V, therefore $C=0$. Finally, since $\alpha(x)=1$ for any $x\in K$, then $\bar{u}(x)\in\{0,1\}$ a.e. $x\in K$. □

Remark 7.7. As a consequence of this theorem, only initial data $\bar{u}$ that satisfy $\bar{u}(x)\in\{0,1\}$ almost everywhere on K have a solution to the Cauchy problem

$$\begin{cases}u_t+f(x,u)_x=0\\ u(0,x)=\bar{u}(x)\end{cases}$$

which depends continuously on time. Hence, if $b-a>\kappa$, and $\bar{u}=\frac{1}{2}\chi_{[a,b]}$, the previous Cauchy problem cannot have a weak solution $u\in\mathbf{L}^\infty([0,T]\times\mathbb{R},\mathbb{R})$ such that the map $t\mapsto u(t,\cdot)$ is continuous from $[0,T]$ into $\mathbf{L}^\infty(\mathbb{R},\mathbb{R})$ endowed with the weak* topology.

We remark that the initial condition $\bar{u}=\frac{1}{2}\chi_{[a,b]}$ does not lie in the closure of the domain of the operator $Au\doteq f(x,u)_x$. This can be checked by showing that, if $\|u-\bar{u}\|_{\mathbf{L^1}(\mathbb{R},\mathbb{R})}<\rho$ with $0<\rho<\frac{b-a-\kappa}{4}$, then the function $x\mapsto f(x,u(x))$ has unbounded variation. Indeed, if $\|u-\bar{u}\|_{\mathbf{L^1}(\mathbb{R},\mathbb{R})}<\rho$, then, setting $B=\left\{x\in[a,b]:\frac{1}{4}\le u(x)\le\frac{3}{4}\right\}$

$$\operatorname{meas}(B)=b-a-\operatorname{meas}\left(\left\{x\in[a,b]:\left|u(x)-\frac{1}{2}\right|>\frac{1}{4}\right\}\right)\ge b-a-4\|u-\bar{u}\|_{\mathbf{L^1}_{\mathbf{L^1}(\mathbb{R},\mathbb{R})}}>\kappa.$$

So that $\operatorname{meas}(K\cap B)>0$. We now have

$$f(x,u(x))\begin{cases}\ge\frac{3}{16}&\text{if } x\in B\cap K\\ =0&\text{if } x\in V.\end{cases}$$

Since for any two points $x_1,x_2\in B\cap K$ we can find an interval contained in V between them, the total variation of $f(x,u)$ is infinite and $u\notin\mathcal{D}(A)$. It is thus clear that the classical theory of contractive semigroups [21] cannot be applied here.

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