STRONG UNIQUE CONTINUATION AND LOCAL ASYMPTOTICS AT THE BOUNDARY FOR FRACTIONAL ELLIPTIC EQUATIONS

ALESSANDRA DE LUCA, VERONICA FELLI AND STEFANO VITA

Abstract. We study local asymptotics of solutions to fractional elliptic equations at boundary points, under some outer homogeneous Dirichlet boundary condition. Our analysis is based on a blow-up procedure which involves some Almgren type monotonicity formulæ and provides a classification of all possible homogeneity degrees of limiting entire profiles. As a consequence, we establish a strong unique continuation principle from boundary points.

1. Introduction and main results

Let $N \geq 2$ and $s \in (0,1)$. Dealing with nontrivial solutions to the following fractional equation

$$(-\Delta)^s u = hu \qquad \text{in } \Omega \tag{1.1}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded domain, we are interested in a strong unique continuation property and local asymptotics of solutions at boundary points where the domain is locally $C^{1,1}$ and some outer homogeneous Dirichlet boundary condition is prescribed.

A family of solutions to some elliptic equation is said to satisfy the *strong unique continuation property* if no element of the family has a zero of infinite order, except for the null function.

Asymptotic expansions of solutions to fractional elliptic equations at interior points of the domain were derived in [13], even in the presence of singular homogeneous potentials, by combining Almgren type monotonicity formulas with blow-up arguments; as a relevant byproduct of such sharp asymptotic analysis, in [13] unique continuation principles were established. The difficulty of defining a suitable Almgren's type frequency function in a non-local setting was overcome in [13] by considering the Caffarelli-Silvestre extension [6], which provides an equivalent formulation of the fractional equation as a local problem in one dimension more. For local problems such as second order elliptic equations, the classical approach developed by Garofalo and Lin [23] allows deriving unique continuation directly from doubling conditions obtained as a consequence of the boundedness of an Almgren type frequency function. In the fractional case instead, the monotonicity formula and the doubling type conditions obtained in [13] imply unique continuation properties only for the extended local problem and not for the fractional one; then in [13] the further step of classification of blow-up limits is performed in order to derive first asymptotic estimates, and then unique continuation principles, in the spirit of [19, 20], see also [14].

Since [13], the literature devoted to unique continuation for fractional problems has flourished producing many important results in several directions; we mention, among others, [31] for unique continuation in presence of rough potentials by Carleman estimates, [41] for fractional operators with variable coefficients, and [17, 18, 22, 32, 33, 34, 40] for higher order fractional problems.

The aim of the present paper is to extend the results of [13] to boundary points of the domain, i.e. to establish sharp asymptotics and unique continuation from boundary points for fractional equations of type (1.1). Possible loss of regularity and unavoidable interference with the geometry of the domain make the derivation of monotonicity formulas around boundary points, and consequently the proof of unique continuation, more difficult and, at the same time, produce new interesting phenomena: in particular, in [16] it was shown that, under homogeneous Dirichlet boundary conditions, the possible vanishing rates of solutions at conical boundary points depend of the opening of the vertex. Unique continuation from the boundary for elliptic equations was also investigated in [2, 3, 26, 39] under homogeneous Dirichlet conditions and in [10, 38] under Neumann type conditions. Furthermore, we refer to [15] for unique continuation from Dirichlet-Neumann junctions for planar mixed boundary value problems and to [9] for unique continuation from the edge of a crack.

Date: January 13, 2022.

2020 Mathematics Subject Classification: 31B25, 35R11, 35C20.

Keywords: Fractional elliptic equations; unique continuation; monotonicity formula; boundary behavior of solutions.

The related problem of regularity up to the boundary for solutions to fractional elliptic problems was studied in [28, 30]. We also mention the paper [4], where quantitative upper and lower estimates at the boundary were discussed for nonnegative solutions to semilinear nonlocal elliptic equations, giving us a motivation to search for sharp asymptotics at boundary points; see also [21] for boundary asymptotics of s-harmonic functions with applications to the thin one-phase problem.

In order to give a suitable weak formulation of (1.1), we introduce the functional space $\mathcal{D}^{s,2}(\mathbb{R}^N)$, defined as the completion of $C_c^\infty(\mathbb{R}^N)$ with respect to the scalar product

$$(u,v)_{\mathcal{D}^{s,2}(\mathbb{R}^N)} := \int_{\mathbb{R}^N} |\xi|^{2s}\, \widehat{u}(\xi)\overline{\widehat{v}(\xi)}\, d\xi \tag{1.2}$$

and the associated norm $\|u\|_{\mathcal{D}^{s,2}(\mathbb{R}^N)} = \big((u,u)_{\mathcal{D}^{s,2}(\mathbb{R}^N)}\big)^{1/2}$, where $\hat{u}$ denotes the unitary Fourier transform of u in $\mathbb{R}^N$, i.e.

$$\widehat{u}(\xi) = \mathcal{F}u(\xi) := \frac{1}{(2\pi)^{N/2}} \int_{\mathbb{R}^N} e^{-ix\cdot\xi} u(x)\, dx.$$

The fractional Laplacian $(-\Delta)^s$ can be defined as the Riesz isomorphism of $\mathcal{D}^{s,2}(\mathbb{R}^N)$ with respect to the scalar product (1.2), i.e.

$${}_{(\mathcal{D}^{s,2}(\mathbb{R}^N))^*}\langle(-\Delta)^s u, v\rangle_{\mathcal{D}^{s,2}(\mathbb{R}^N)} = (u,v)_{\mathcal{D}^{s,2}(\mathbb{R}^N)}$$

for all $u,v \in \mathcal{D}^{s,2}(\mathbb{R}^N)$. Then we can define a weak solution to (1.1) as a function $u \in \mathcal{D}^{s,2}(\mathbb{R}^N)$ satisfying

$$(u,\varphi)_{\mathcal{D}^{s,2}(\mathbb{R}^N)} = \int_\Omega h(x)u(x)\varphi(x)\, dx, \text{ for all } \varphi \in C_c^\infty(\Omega). \tag{1.3}$$

As far as the potential term is concerned, we assume that

$$\text{there exists } p > \frac{N}{2s} \quad \text{such that} \quad h \in W^{1,p}(\Omega). \tag{1.4}$$

We observe that the right hand side of (1.3) is well defined in view of assumption (1.4), Hölder's inequality, and the following well-known Sobolev-type inequality

$$S_{N,s}\|u\|^2_{L^{2^*(s)}(\mathbb{R}^N)} \le \|u\|^2_{\mathcal{D}^{s,2}(\mathbb{R}^N)}, \tag{1.5}$$

where $S_{N,s}$ is a positive constant depending only on N and s and

$$2^*(s) = \frac{2N}{N-2s}, \tag{1.6}$$

see [8].

Let $u \in \mathcal{D}^{s,2}(\mathbb{R}^N)$ be a solution to (1.3). Let us assume that there exists a boundary point $x_0 \in \partial\Omega$ such that the boundary $\partial\Omega$ is of class $C^{1,1}$ in a neighbourhood of x_0, i.e. there exist $R>0$ and $g \in C^{1,1}(\mathbb{R}^{N-1})$ such that, choosing a proper coordinate system $(x',x_N) \in \mathbb{R}^{N-1}\times\mathbb{R}$,

$$\begin{aligned} &B'_R(x_0)\cap\Omega = \{(x',x_N) \in B'_R(x_0) : x_N < g(x')\} \quad \text{and} \\ &B'_R(x_0)\cap\partial\Omega = \{(x',x_N) \in B'_R(x_0) : x_N = g(x')\}, \end{aligned} \tag{1.7}$$

where $B'_R(x_0) = \{x \in \mathbb{R}^N : |x-x_0| < R\}$ is the ball in $\mathbb{R}^N$ centered at x_0 with radius R. We prescribe for the solution u a local outer homogeneous Dirichlet boundary condition, i.e

$$u = 0 \quad \text{a.e. in } \Omega^c \cap B'_R(x_0). \tag{1.8}$$

By the extension technique introduced in [6], by adding an additional space variable $t \in [0,+\infty)$, we can reformulate the nonlocal problem (1.1) as a local degenerate or singular problem on the half space

$$\mathbb{R}^{N+1}_+ = \mathbb{R}^N \times (0,+\infty).$$

We denote the total variable $z = (x,t) \in \mathbb{R}^N \times (0,+\infty)$, with $x = (x',x_N) = (x_1,...,x_{N-1},x_N)$, and define $\mathcal{D}^{1,2}(\mathbb{R}^{N+1}_+, t^{1-2s}\, dz)$ as the completion of $C_c^\infty(\overline{\mathbb{R}^{N+1}_+})$ with respect to the norm

$$\|U\|_{\mathcal{D}^{1,2}(\mathbb{R}^{N+1}_+, t^{1-2s}\, dz)} = \sqrt{\int_{\mathbb{R}^{N+1}_+} t^{1-2s}|\nabla U(x,t)|^2\, dx\, dt}.$$

It is well-known that there exists a continuous trace map $\mathrm{Tr}:\mathcal{D}^{1,2}(\mathbb{R}^{N+1}_+,t^{1-2s}\,dz)\to\mathcal{D}^{s,2}(\mathbb{R}^N)$ which is onto, see [5]. By [6], for every $u\in\mathcal{D}^{s,2}(\mathbb{R}^N)$, the minimization problem

$$\min\left\{\|W\|^2_{\mathcal{D}^{1,2}(\mathbb{R}^{N+1}_+,t^{1-2s}\,dz)}:\,W\in\mathcal{D}^{1,2}(\mathbb{R}^{N+1}_+,t^{1-2s}\,dz),\ \mathrm{Tr}\,W=u\right\}$$

admits a unique minimizer $U=\mathcal{H}(u)\in\mathcal{D}^{1,2}(\mathbb{R}^{N+1}_+,t^{1-2s}\,dz)$, which can be obtained by convoluting u with the Poisson kernel of the half-space $\mathbb{R}^{N+1}_+$ and weakly solves

$$\begin{cases}-\mathrm{div}(t^{1-2s}\nabla U)=0 & \text{in }\mathbb{R}^{N+1}_+,\\ -\lim_{t\to0^+}t^{1-2s}\partial_tU=\kappa_s(-\Delta)^su & \text{in }\mathbb{R}^N\times\{0\},\end{cases}$$

where

$$\kappa_s=\frac{\Gamma(1-s)}{2^{2s-1}\Gamma(s)}>0,$$

i.e.

$$\int_{\mathbb{R}^{N+1}_+}t^{1-2s}\nabla\mathcal{H}(u)(x,t)\cdot\nabla W(x,t)\,dx\,dt=\kappa_s(u,\mathrm{Tr}\,W)_{\mathcal{D}^{s,2}(\mathbb{R}^N)}\quad\text{for all }W\in\mathcal{D}^{1,2}(\mathbb{R}^{N+1}_+,t^{1-2s}\,dz).$$

As a consequence, $u\in\mathcal{D}^{s,2}(\mathbb{R}^N)$ is a solution (1.3) if and only if its extension $U=\mathcal{H}(u)$ weakly solves

$$\begin{cases}-\mathrm{div}\left(t^{1-2s}\nabla U\right)=0 & \text{in }\mathbb{R}^{N+1}_+,\\ \mathrm{Tr}U=u & \text{in }\mathbb{R}^N\times\{0\},\\ -\lim_{t\to0^+}t^{1-2s}\partial_tU=\kappa_shu & \text{in }\Omega\times\{0\},\end{cases}\tag{1.9}$$

in a weak sense, i.e

$$\int_{\mathbb{R}^{N+1}_+}t^{1-2s}\nabla U(x,t)\cdot\nabla\phi(x,t)\,dx\,dt=\kappa_s\int_\Omega hu\,\mathrm{Tr}\,\phi\,dx\tag{1.10}$$

for every $\phi\in C^\infty_c(\overline{\mathbb{R}^{N+1}_+})$ with $\mathrm{Tr}\,\phi\in C^\infty_c(\Omega)$.

The asymptotic behaviour at $x_0\in\partial\Omega$ of solutions to (1.9), and consequently to (1.1), will turn out to be related to eigenvalues and eigenfunctions of the following weighted spherical eigenvalue problem with mixed Dirichlet-Neumann boundary conditions

$$\begin{cases}-\mathrm{div}_{\mathbb{S}^N}\left(\theta_{N+1}^{1-2s}\nabla_{\mathbb{S}^N}\psi\right)=\theta_{N+1}^{1-2s}\mu\psi & \text{in }\mathbb{S}^N_+,\\ \psi=0 & \text{on }\mathbb{S}^{N-1}\cap\{\theta_N\ge0\},\\ \lim_{\theta_{N+1}\to0^+}\theta_{N+1}^{1-2s}\nabla_{\mathbb{S}^N}\psi\cdot\nu=0 & \text{on }\mathbb{S}^{N-1}\cap\{\theta_N<0\},\end{cases}\tag{1.11}$$

on the half-sphere

$$\mathbb{S}^N_+=\{(\theta_1,\dots,\theta_N,\theta_{N+1})\in\mathbb{S}^N:\theta_{N+1}>0\},$$

where $\nu=(0,0,\dots,0,-1)$ and $\partial\mathbb{S}^N_+=\mathbb{S}^{N-1}\times\{0\}$ is identified with $\mathbb{S}^{N-1}$. In order to write the variational formulation of (1.11), we define $H^1(\mathbb{S}^N_+,\theta_{N+1}^{1-2s}dS)$ as the completion of $C^\infty(\overline{\mathbb{S}^N_+})$ with respect to the norm

$$\|\psi\|_{H^1(\mathbb{S}^N_+,\theta_{N+1}^{1-2s}dS)}=\left(\int_{\mathbb{S}^N_+}\theta_{N+1}^{1-2s}\left(|\nabla_{\mathbb{S}^N}\psi(\theta)|^2+\psi^2(\theta)\right)dS\right)^{1/2},$$

where dS denotes the volume element on N-dimensional spheres. Let $\mathcal{H}_0$ be the closure of $C^\infty_c(\overline{\mathbb{S}^N_+}\setminus S_1^+)$ in $H^1(\mathbb{S}^N_+,\theta_{N+1}^{1-2s}dS)$, where $S_1^+=\{(\theta',\theta_N,0)\in\mathbb{S}^{N-1}:\theta_N\ge0\}$. We say that $\mu\in\mathbb{R}$ is an eigenvalue of (1.11) if there exists $\psi\in\mathcal{H}_0\setminus\{0\}$ such that

$$\int_{\mathbb{S}^N_+}\theta_{N+1}^{1-2s}\nabla_{\mathbb{S}^N}\psi\cdot\nabla_{\mathbb{S}^N}\phi\,dS=\mu\int_{\mathbb{S}^N_+}\theta_{N+1}^{1-2s}\psi\phi\,dS\quad\text{for any }\phi\in\mathcal{H}_0.\tag{1.12}$$

By classical spectral theory, problem (1.11) admits a diverging sequence of real eigenvalues with finite multiplicity $\{\mu_k\}_{k\ge0}$. In Appendix B we obtain the following explicit formula for such eigenvalues

$$\mu_k=(k+s)(k+N-s),\quad k\in\mathbb{N}.\tag{1.13}$$

For all $k\in\mathbb{N}$, let $M_k\in\mathbb{N}\setminus\{0\}$ be the multiplicity of the eigenvalue μ_k and $\{Y_{k,m}\}_{m=1,2,\dots,M_k}$ be a $L^2(\mathbb{S}^N_+,\theta_{N+1}^{1-2s}dS)$-orthonormal basis of the eigenspace of problem (1.11) associated to μ_k. In particular,

$$\{Y_{k,m}:k\in\mathbb{N},\ m=1,...,M_k\}\tag{1.14}$$

is an orthonormal basis of $L^2(\mathbb{S}^N_+, \theta_{N+1}^{1-2s} dS)$.

Remark 1.1. It is worth highlighting the fact that eigenfunctions of problem (1.11) cannot vanish identically on $\mathbb{S}^{N-1} \cap \{\theta_N < 0\}$, i.e. on the boundary portion where a Neumann homogeneous condition is imposed. Indeed, if an eigenfunction ψ associated to the eigenvalue $\mu_k = (k+s)(k+N-s)$ vanishes on $\mathbb{S}^{N-1} \cap \{\theta_N < 0\}$, then the function $\Psi(\rho\theta) = \rho^{k+s}\psi(\theta)$ would be a weak solution to the equation $\operatorname{div}(t^{1-2s}\nabla\Psi) = 0$ in $\mathbb{R}^{N-1} \times (-\infty, 0) \times (0, +\infty)$ satisfying both Dirichlet and weighted Neumann homogeneous boundary conditions on $\mathbb{R}^{N-1} \times (-\infty, 0) \times \{0\}$; then its trivial extension to $\mathbb{R}^{N-1} \times (-\infty, 0) \times \mathbb{R}$ would violate the unique continuation principle for elliptic equations with Muckenhoupt weights proved in [39] (see also [23], [35, Corollary 3.3], and [31, Proposition 2.2]).

Our first result is a sharp description of the asymptotic behaviour of solutions to (1.1) at a boundary point.

Theorem 1.2. *Let Ω be a bounded domain in $\mathbb{R}^N$ such that there exist $g \in C^{1,1}(\mathbb{R}^{N-1})$, $x_0 \in \partial\Omega$ and $R > 0$ satisfying (1.7). Let h satisfy (1.4) and $u \in \mathcal{D}^{s,2}(\mathbb{R}^N)$, $u \not\equiv 0$, be a weak solution to (1.1) in the sense of (1.3), satisfying (1.8). Then there exists $k_0 \in \mathbb{N}$ and an eigenfunction Y of problem (1.11) associated to the eigenvalue $\mu_{k_0} = (k_0+s)(k_0+N-s)$ such that*

$$\frac{u(x_0+\lambda x)}{\lambda^{k_0+s}} \to |x|^{k_0+s} Y\left(\tfrac{x}{|x|}, 0\right) \quad \text{in } H^s(B_1') \text{ as } \lambda \to 0^+,$$

where $H^s(B_1')$ is the usual fractional Sobolev space on the N-dimensional unit ball $B_1' = B_1'(0)$.

Theorem 1.2 will be proved as a consequence of the following description of the asymptotic behaviour of nontrivial solutions to (1.9) near $x_0 \in \partial\Omega$.

Theorem 1.3. *Let Ω be a bounded domain in $\mathbb{R}^N$ such that there exist $g \in C^{1,1}(\mathbb{R}^{N-1})$, $x_0 \in \partial\Omega$ and $R > 0$ satisfying (1.7). Let h satisfy (1.4) and $U \in \mathcal{D}^{1,2}(\mathbb{R}^{N+1}_+, t^{1-2s}\, dz)$ be a weak solution to (1.9) in the sense of (1.10), with $U \not\equiv 0$ and $\operatorname{Tr} U = u$ satisfying (1.8). Then there exists $k_0 \in \mathbb{N}$ and an eigenfunction Y of problem (1.11) associated to the eigenvalue $\mu_{k_0} = (k_0+s)(k_0+N-s)$ such that, letting $z_0 = (x_0, 0)$,*

$$\frac{U(z_0+\lambda z)}{\lambda^{k_0+s}} \to |z|^{k_0+s} Y(z/|z|) \quad \text{in } H^1(B_1^+, t^{1-2s} dz) \text{ as } \lambda \to 0^+,$$

where $H^1(B_1^+, t^{1-2s} dz)$ is the weighted Sobolev space on the half ball B_1^+ defined in Section 2.

Actually the proof of Theorem 1.3 contains a more precise characterization of the angular limit profile Y, as a linear combination of the orthonormalized eigenfunctions $\{Y_{k_0,m}\}_{m=1,2,\dots,M_{k_0}}$ of (1.11) associated to the eigenvalue μ_{k_0} with coefficients explicitly given by formula (4.62).

The salient consequence of the precise asymptotic expansions described above is the following *strong unique continuation principle* for problems (1.1) and (1.9), whose proof follows straightforwardly from Theorems 1.2 and 1.3, taking into account Remark 1.1.

Corollary 1.4.

(i) *Under the same assumptions as in Theorems 1.2, let $u \in \mathcal{D}^{s,2}(\mathbb{R}^N)$ be a weak solution to (1.1) (in the sense of (1.3)) satisfying (1.8) and such that $u(x) = O(|x-x_0|^k)$ as $x \to x_0$, for any $k \in \mathbb{N}$. Then $u \equiv 0$ in $\mathbb{R}^N$.*
(ii) *Under the same assumptions as in Theorems 1.3, let $U \in \mathcal{D}^{1,2}(\mathbb{R}^{N+1}_+, t^{1-2s}\, dz)$ be a weak solution to (1.9) (in the sense of (1.10)) with $\operatorname{Tr} U = u$ satisfying (1.8) and such that $U(z) = O(|z-z_0|^k)$ as $z \to z_0$, for any $k \in \mathbb{N}$. Then $U \equiv 0$ in $\mathbb{R}^{N+1}_+$.*

The proof of Theorem 1.3 makes use of the procedure developed in [13, 19, 20] and consisting in a fine blow-up analysis for scaled solutions based on sharp energy estimates, obtained as a consequence of the existence of the limit of an Almgren type frequency function. Such method is applied to an equivalent auxiliary problem obtained by straightening the boundary of the domain Ω through a diffeomorphic deformation, which is inspired by [2] and built specifically to ensure that the extended equation is conserved by reflection through a straightened vertical boundary, see Section 2.1.

Significant additional difficulties arise with respect to the non-fractional or interior setting. Since the optimal regularity of solutions to (1.1) is s-Hölder continuity, see [30], solutions to (1.9) could have singular gradient at $\partial\Omega$, which represents for problem (1.9) the interface between mixed Dirichlet and Neumann boundary conditions. In fact, problems with mixed boundary conditions raise delicate regularity issues, which turn out to be more difficult in dimension $N \geq 2$ due to the positive dimension of the junction set and some role played by the geometry of the domain.

In particular, we remark that the regularity results known for non-fractional problems (for which solutions are smooth up to the boundary) or for interior points (see e.g. [25]) are not available here, even excluding a neighborhood of the boundary point if $N \geq 2$. Therefore, for problems that, like (1.9), are characterized by mixed boundary conditions in dimension $N \geq 2$, the development of a monotonicity argument around points located at Dirichlet-Neumann junctions presents substantial new difficulties with respect to both the case treated in [13] of boundary points around which a Neumann condition is given and the case treated in [15] of mixed conditions in dimension $N = 1$.

In the present paper, the difficulties related to lack of regularity at Dirichlet-Neumann junctions are overcome by a double approximation procedure: by approximating the potential h with potentials vanishing near the boundary and the Dirichlet N-dimensional region with smooth $(N+1)$-sets with straight vertical boundary, we will be able to construct a sequence of approximating solutions (see Section 2.3) which enjoy enough regularity to derive Pohozaev type identities, needed to obtain Almgren type monotonicity formulæ and consequently to perform blow-up analysis. We mention that a similar approximation procedure was developed in [9] for a class of elliptic equations in a domain with a crack.

The paper is structured as follows. In Section 2, after introducing a suitable functional setting for the study of the extended problem (1.9), we present an equivalent auxiliary problem obtained by straightening the boundary; then, after providing some Hardy-Sobolev and coercivity type inequalities, in Subsection 2.3 we perform the approximation procedure which allows us to establish a Pohozaev type inequality. Section 3 is devoted to the proof of an Almgren type monotonicity formula, which is the key tool for the blow-up analysis carried out in Section 4. Finally, in Appendix A we present some boundary regularity results for singular/degenerate equations in cylinders, while in Appendix B we prove (1.13), through a classification of possible homogeneity degrees of homogeneous solutions to (B.1).

2. Functional setting and construction of the approximating domains

Let us call the total variable $z = (x,t) \in \mathbb{R}^N \times \mathbb{R}$, with $x = (x', x_N) = (x_1, ..., x_{N-1}, x_N)$. We set the following notations for all $r > 0$:

$$B_r = \{z \in \mathbb{R}^{N+1} : |z| < r\}, \quad B_r^+ = B_r \cap \mathbb{R}^{N+1}_+,$$
$$B_r' = \{(x,t) \in B_r : t = 0\}, \quad \partial^+ B_r^+ = \partial B_r \cap \mathbb{R}^{N+1}_+.$$

In the sequel B_r' will be sometimes identified with the ball in $\mathbb{R}^N$ centered at 0 with radius r. The weighted Sobolev space $H^1(B_r^+, t^{1-2s}\,dz)$ in the extension context is defined as the completion of $C^\infty(\overline{B_r^+})$ with respect to the norm

$$\|U\|_{H^1(B_r^+, t^{1-2s}\,dz)} = \sqrt{\int_{B_r^+} t^{1-2s}\left(|U|^2 + |\nabla U|^2\right) dz}.$$

It is well known, see e.g. [25, Proposition 2.1], that there exists a well-defined continuous trace map $\mathrm{Tr} : H^1(B_r^+, t^{1-2s}\,dz) \to L^{2^*(s)}(B_r')$; in particular there exists a positive constant $C_{N,s}$ depending only on N and s such that, for all $r > 0$ and $U \in H^1(B_r^+, t^{1-2s}\,dz)$,

$$\|\mathrm{Tr}(U)\|^2_{L^{2^*(s)}(B_r')} \leq C_{N,s} \int_{B_r^+} t^{1-2s}\left(r^{-2}|U(z)|^2 + |\nabla U(z)|^2\right) dz. \tag{2.1}$$

We are interested in boundary qualitative properties of solutions to (1.1) close to a fixed point $x_0 \in \partial\Omega$. Without loss of generality, up to translation and rotation, we can assume $x_0 = 0$ and consider the extension $U = \mathcal{H}(u)$ which, under assumptions (1.7) and (1.8), solves

$$\begin{cases} -\mathrm{div}\left(t^{1-2s}\nabla U\right) = 0 & \text{in } B_R^+, \\ -\lim_{t\to 0^+} t^{1-2s}\partial_t U = \kappa_s h u & \text{in } \Gamma^-_{g,R} := \{(x', x_N, 0) \in B_R' : x_N < g(x')\}, \\ U = 0 & \text{in } \Gamma^+_{g,R} := \{(x', x_N, 0) \in B_R' : x_N \geq g(x')\}, \end{cases} \tag{2.2}$$

for some $R > 0$, where g is the function which locally parametrizes the boundary $\partial\Omega$ around $x_0 = 0$ according to (1.7), with

$$g \in C^{1,1}(\mathbb{R}^{N-1}), \quad g(0) = 0 \quad \text{and} \quad \nabla g(0) = 0. \tag{2.3}$$

The suitable weighted Sobolev space for energy solutions to (2.2) is $H^1_{\Gamma^+_{g,R}}(B_R^+, t^{1-2s}\,dz)$, defined as the closure of $C_c^\infty(\overline{B_R^+} \setminus \Gamma^+_{g,R})$ in $H^1(B_R^+, t^{1-2s}\,dz)$. By energy solution to (2.2) we mean a function $U \in H^1_{\Gamma^+_{g,R}}(B_R^+, t^{1-2s}\,dz)$ such that

$$\int_{B_R^+} t^{1-2s}\nabla U(x,t)\cdot\nabla\phi(x,t)\,dz - \kappa_s\int_{\Gamma^-_{g,R}} h\,\mathrm{Tr}\,U\,\mathrm{Tr}\,\phi\,dx = 0 \quad \text{for all } \phi \in C_c^\infty(B_R^+ \cup \Gamma^-_{g,R}).$$

2.1. **A diffeomorphism to straighten the boundary.** We follow the construction in [2]. Let us consider the following set of variables $(y,t) \in \mathbb{R}^N \times [0,+\infty)$, with $y = (y', y_N) = (y_1, ..., y_{N-1}, y_N)$. Let $\rho \in C_c^\infty(\mathbb{R}^{N-1})$ be such that $\rho \geq 0$, $\mathrm{supp}(\rho) \subset B_1'$ and $\int_{\mathbb{R}^{N-1}} \rho(y')\,dy' = 1$. For every $\delta > 0$ we define

$$\rho_\delta(y') = \delta^{-N+1}\rho\left(\frac{y'}{\delta}\right).$$

Let us define also, for every $j = 1, ..., N-1$,

$$G_j(y', y_N) = \begin{cases} \left(\rho_{y_N} * \partial_{y_j} g\right)(y') & \text{if } y' \in \mathbb{R}^{N-1},\ y_N > 0, \\ \partial_{y_j} g(y') & \text{if } y' \in \mathbb{R}^{N-1},\ y_N = 0, \end{cases}$$

where $*$ denotes the convolution product.

It is easy to verify that, for all $j = 1, \dots, N-1$, $G_j \in C^\infty(\mathbb{R}^N_+)$, G_j is Lipschitz continous in $\overline{\mathbb{R}^N_+}$, and $\frac{\partial G_j}{\partial y_i} \in L^\infty(\mathbb{R}^N_+)$ for every $i \in \{1, \dots, N\}$. Moreover, for all $j = 1, \dots, N-1$ and $i = 1, \dots, N$,

$$y_N \frac{\partial G_j}{\partial y_i} \quad \text{is Lipschitz continuous in } \overline{\mathbb{R}^N_+}.$$

As a consequence, we have that, letting

$$\widetilde{G}_j : \mathbb{R}^N \to \mathbb{R}, \quad \widetilde{G}_j(y', y_N) := G_j(y', |y_N|)$$

and

$$\psi_j : \mathbb{R}^N \to \mathbb{R}, \quad \psi_j(y', y_N) = y_j - y_N \widetilde{G}_j(y', y_N),$$

$\widetilde{G}_j$ is Lipschitz continuous in $\mathbb{R}^N$ and $\psi_j \in C^{1,1}(\mathbb{R}^N)$ (i.e. ψ_j is continuously differentiable with Lipschitz gradient) for all $j = 1, \dots, N-1$. Let

$$\widetilde{G}(y', y_N) = (\widetilde{G}_1(y', y_N), \widetilde{G}_2(y', y_N), \dots, \widetilde{G}_{N-1}(y', y_N))$$

and denote as $J_{\widetilde{G}}(y', y_N)$ the Jacobian matrix of $\widetilde{G}$ at (y', y_N). Then $J_{\widetilde{G}} \in L^\infty(\mathbb{R}^N, \mathbb{R}^{N(N-1)})$ and

$$|\widetilde{G}(y', y_N) - \nabla g(y')| \leq C\,|y_N| \quad \text{for all } (y', y_N) \in \mathbb{R}^N, \tag{2.4}$$

for some constant $C > 0$ independent of (y', y_N).

Let us consider the local diffeomorphism $F\colon \mathbb{R}^{N+1} \to \mathbb{R}^{N+1}$ defined as

$$F(y', y_N, t) = (\psi_1(y', y_N), ..., \psi_{N-1}(y', y_N), y_N + g(y'), t). \tag{2.5}$$

We observe that F is of class $C^{1,1}$ and $F(y', 0, 0) = (y', g(y'), 0)$, namely F^{-1} is straightening the boundary of the set $\{(x', x_N, 0) : x_N < g(x')\}$.

Direct computations and (2.4) yield that

$$
\begin{aligned}
J_F(y',y_N,t) &= J(y',y_N) \\
&= \begin{pmatrix}
1-y_N\frac{\partial\widetilde G_1}{\partial y_1} & -y_N\frac{\partial\widetilde G_1}{\partial y_2} & \cdots & -y_N\frac{\partial\widetilde G_1}{\partial y_{N-1}} & -\widetilde G_1-y_N\frac{\partial\widetilde G_1}{\partial y_N} & 0\\
-y_N\frac{\partial\widetilde G_2}{\partial y_1} & 1-y_N\frac{\partial\widetilde G_2}{\partial y_2} & \cdots & -y_N\frac{\partial\widetilde G_2}{\partial y_{N-1}} & -\widetilde G_2-y_N\frac{\partial\widetilde G_2}{\partial y_N} & 0\\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots\\
-y_N\frac{\partial\widetilde G_{N-1}}{\partial y_1} & -y_N\frac{\partial\widetilde G_{N-1}}{\partial y_2} & \cdots & 1-y_N\frac{\partial\widetilde G_{N-1}}{\partial y_{N-1}} & -\widetilde G_{N-1}-y_N\frac{\partial\widetilde G_{N-1}}{\partial y_N} & 0\\
\frac{\partial g}{\partial y_1}(y') & \frac{\partial g}{\partial y_2}(y') & \cdots & \frac{\partial g}{\partial y_{N-1}}(y') & 1 & 0\\
0 & 0 & \cdots & 0 & 0 & 1
\end{pmatrix}\\
&= \left(\begin{array}{c|c|c}
\mathrm{Id}_{N-1}-y_NJ_{\widetilde G} & -\nabla g(y')+O(y_N) & \mathbf 0\\ \hline
(\nabla g(y'))^T & 1 & 0\\ \hline
\mathbf 0^T & 0 & 1
\end{array}\right),
\end{aligned}
\tag{2.6}
$$

where $\nabla g(y')$ is meant as a column vector in $\mathbb R^{N-1}$, $\mathbf 0$ is the null column vector in $\mathbb R^{N-1}$ and $(\nabla g(y'))^T,\mathbf 0^T$ are their transpose; from now on, the notation $O(y_N)$ will be used to denote blocks of matrices with all entries being $O(y_N)$ as $y_N\to0$ uniformly with respect to y' and t.

From (2.3) and the fact that $g\in C^{1,1}(\mathbb R^{N-1})$ it follows that $\nabla g(y')=O(|y'|)$ as $|y'|\to0$, then

$$\det J(y',y_N)=1+|\nabla g(y')|^2+O(y_N)=1+O(|y'|^2)+O(y_N) \tag{2.7}$$

as $y_N\to0$ and $|y'|\to0$.

In particular we have that $\det J_F(0)=1\neq0$; therefore, by the Inverse Function Theorem, F is invertible in a neighbourhood of the origin, i.e. there exists $R_1>0$ such that

$$\alpha(y',y_N):=\det J(y',y_N)>0 \quad \text{in } B'_{R_1} \tag{2.8}$$

and F is a diffeomorphism of class $C^{1,1}$ from B_{R_1} to $\mathcal U=F(B_{R_1})$ for some $\mathcal U$ open neighbourhood of 0 such that $\mathcal U\subset B_R$. Furthermore

$$F^{-1}(\mathcal U\cap\Gamma^-_{g,R})=\Gamma^-_{R_1}\quad\text{and}\quad F^{-1}(\mathcal U\cap\Gamma^+_{g,R})=\Gamma^+_{R_1},$$

where, for all $r>0$, we denote

$$\Gamma^-_r:=\{(y',y_N,0)\in B'_r: y_N<0\},\quad \Gamma^+_r:=\{(y',y_N,0)\in B'_r: y_N\ge0\}.$$

Since

$$F^{-1}\in C^{1,1}(\mathcal U,B_{R_1}),\quad F\in C^{1,1}(B_{R_1},\mathcal U),\quad F(0)=F^{-1}(0)=0,\quad J_F(0)=J_{F^{-1}}(0)=\mathrm{Id}_{N+1},$$

we have that

$$J_{F^{-1}}(x)=\mathrm{Id}_{N+1}+O(|x|)\quad\text{and}\quad F^{-1}(x)=x+O(|x|^2)\quad\text{as } |x|\to0, \tag{2.9}$$

$$J_F(y)=\mathrm{Id}_{N+1}+O(|y|)\quad\text{and}\quad F(y)=y+O(|y|^2)\quad\text{as } |y|\to0. \tag{2.10}$$

If U is a solution to (2.2), then $W=U\circ F$ is solution to

$$\begin{cases}-\mathrm{div}\left(t^{1-2s}A\nabla W\right)=0 & \text{in } B^+_{R_1},\\ \lim_{t\to0^+}\left(t^{1-2s}A\nabla W\cdot\nu\right)=\kappa_s\tilde h\ \mathrm{Tr}\,W & \text{in } \Gamma^-_{R_1},\\ W=0 & \text{in } \Gamma^+_{R_1},\end{cases} \tag{2.11}$$

where $\nu=(0,0,\dots,0,-1)$ is the vertical downward unit vector, A is the $(N+1)\times(N+1)$ variable coefficient matrix (not depending on t) given by

$$A(y)=(J(y))^{-1}((J(y))^{-1})^T|\det J(y)|, \tag{2.12}$$

and

$$\tilde h(y)=\alpha(y)h(F(y,0)),\quad y\in\Gamma^-_{R_1}.$$

Equation (2.11) is meant in a weak sense, i.e. W belongs to $H^1_{\Gamma^+_{R_1}}(B^+_{R_1}, t^{1-2s}\,dz)$ (defined as the closure of $C^\infty_c(\overline{B^+_{R_1}} \setminus \Gamma^+_{R_1})$ in $H^1(B^+_{R_1}, t^{1-2s}\,dz)$) and satisfies

$$\int_{B^+_{R_1}} t^{1-2s} A(y)\nabla W(y,t)\cdot\nabla\phi(y,t)\,dz - \kappa_s \int_{\Gamma^-_{R_1}} \tilde h \operatorname{Tr} W \operatorname{Tr}\phi\,dy = 0 \tag{2.13}$$

for all $\phi \in C^\infty_c(B^+_{R_1} \cup \Gamma^-_{R_1})$.

We observe that A is symmetric and, in view of (2.9)–(2.10), uniformly elliptic if R_1 is chosen sufficiently small; furthermore A has $C^{0,1}$ coefficients. We also remark that, under assumption (1.4),

$$\tilde h \in W^{1,p}(\Gamma^-_{R_1}). \tag{2.14}$$

From (2.6) it follows that

$$J^{-1} = \left(\begin{array}{c|c} M^{-1} & \mathbf{0} \\ \hline \mathbf{0}^T & 1 \end{array}\right),$$

where $\mathbf{0}$ is the null column vector in $\mathbb{R}^N$ and

$$M = M(y', y_N) = \left(\begin{array}{c|c} \mathrm{Id}_{N-1} - y_N J_{\tilde G} & -\nabla g(y') + O(y_N) \\ \hline (\nabla g(y'))^T & 1 \end{array}\right). \tag{2.15}$$

From (2.6) and (2.8) one can deduce that

$$\det M(y', y_N) = \alpha(y', y_N) > 0 \quad \text{in } B'_{R_1}. \tag{2.16}$$

Let us define

$$B(y', y_N) := \det M(y', y_N)(M(y', y_N))^{-1}.$$

By (2.15) and a direct calculation we have that

$$B = \left(\begin{array}{cccc|c} 1+\sum\limits_{j\neq 1}|\frac{\partial g}{\partial y_j}|^2 + O(y_N) & -\frac{\partial g}{\partial y_1}\frac{\partial g}{\partial y_2} + O(y_N) & \cdots & -\frac{\partial g}{\partial y_1}\frac{\partial g}{\partial y_{N-1}} + O(y_N) & \\ -\frac{\partial g}{\partial y_2}\frac{\partial g}{\partial y_1} + O(y_N) & 1+\sum\limits_{j\neq 2}|\frac{\partial g}{\partial y_j}|^2 + O(y_N) & \cdots & -\frac{\partial g}{\partial y_2}\frac{\partial g}{\partial y_{N-1}} + O(y_N) & \nabla g + O(y_N) \\ \vdots & \vdots & \ddots & \vdots & \\ -\frac{\partial g}{\partial y_{N-1}}\frac{\partial g}{\partial y_1} + O(y_N) & -\frac{\partial g}{\partial y_{N-1}}\frac{\partial g}{\partial y_2} + O(y_N) & \cdots & 1+\sum\limits_{j\neq N-1}|\frac{\partial g}{\partial y_j}|^2 + O(y_N) & \\ \hline \multicolumn{4}{c|}{-(\nabla g)^T + O(y_N)} & 1 + O(y_N) \end{array}\right). \tag{2.17}$$

Then $(J(y))^{-1}$ can be rewritten as follows

$$(J(y))^{-1} = \left(\begin{array}{c|c} \frac{1}{\alpha(y)}B(y) & \mathbf{0} \\ \hline \mathbf{0}^T & 1 \end{array}\right),$$

thus from (2.12) it turns out that

$$A(y) = \left(\begin{array}{c|c} D(y) & 0 \\ \hline 0 & \alpha(y) \end{array}\right), \tag{2.18}$$

where $D = \frac{1}{\alpha}BB^T$. From (2.17), (2.7), and (2.8) it follows that

$$D(y', y_N) = \left(\begin{array}{c|c} \mathrm{Id}_{N-1} + O(|y'|^2) + O(y_N) & O(y_N) \\ \hline O(y_N) & 1 + O(|y'|^2) + O(y_N) \end{array}\right), \tag{2.19}$$

where here $O(y_N)$, respectively $O(|y'|^2)$, denotes blocks of matrices with all entries being $O(y_N)$ as $y_N \to 0$, respectively $O(|y'|^2)$ as $|y'| \to 0$. In particular we have that

$$A(y) = \mathrm{Id}_{N+1} + O(|y|) \quad \text{as } |y| \to 0. \tag{2.20}$$

Let us define, for every $z = (y,t) \in B_{R_1}$,

$$\beta(z) = \frac{A(y)z}{\mu(z)} = (\beta'(z), \beta_{N+1}(z)) = \left(\frac{D(y)y}{\mu(z)}, \frac{\alpha(y)t}{\mu(z)}\right), \tag{2.21}$$

where

$$\mu(z) = \frac{A(y)z \cdot z}{|z|^2}, \quad z \neq 0. \tag{2.22}$$

We observe that, possibly choosing R_1 smaller, β is well-defined, since $\mu(z) > 0$ in B_{R_1}, and

$$\|A(y)\|_{\mathcal{L}(\mathbb{R}^{N+1},\mathbb{R}^{N+1})} \leq 2 \quad \text{for all } y \in B'_{R_1}. \tag{2.23}$$

Moreover, for every $\xi = (\xi_1, \dots, \xi_N, \xi_{N+1}) \in \mathbb{R}^{N+1}$ and $y \in B'_{R_1}$, we define $dA(y)\,\xi\,\xi \in \mathbb{R}^{N+1}$ as the vector in $\mathbb{R}^{N+1}$ with i-th component, for $i = 1, ..., N+1$, given by

$$(dA(y)\,\xi\,\xi)_i = \sum_{j,k=1}^{N+1} \partial_{z_i} a_{jk}(y)\,\xi_j \xi_k. \tag{2.24}$$

Lemma 2.1. *Let μ be as in* (2.22) *and A as in* (2.12)*. Then*

$$\mu(z) = 1 + O(|z|) \quad \text{as } |z| \to 0^+ \tag{2.25}$$

and

$$\nabla\mu(z) = O(1) \quad \text{as } |z| \to 0^+. \tag{2.26}$$

Proof. Estimate (2.25) follows directly from (2.22) and (2.20). In order to prove (2.26), we differentiate (2.22), obtaining that, for all $z = (y,t) \in B_{R_1}$,

$$\nabla\mu(z) = -2\frac{(A(y)z \cdot z)z}{|z|^4} + \frac{dA(y)zz}{|z|^2} + 2\frac{A(y)z}{|z|^2} = -2\frac{\mu(z)z}{|z|^2} + \frac{dA(y)zz}{|z|^2} + 2\frac{A(y)z}{|z|^2}.$$

Noting that $dA(y)zz = O(|z|^2)$ as $|z| \to 0$, since the matrix A has Lipschitz coefficients, and using (2.25) and (2.20), we deduce that

$$\nabla\mu(z) = -\frac{2z}{|z|^2}[1 + O(|z|)] + O(1) + \frac{2}{|z|^2}[z + O(|z|^2)] = O(1)$$

as $|z| \to 0^+$, thus proving (2.26). □

Lemma 2.2. *Let β be as in* (2.21) *and A as in* (2.12)*. Then we have that, as $|z| \to 0^+$,*

$$\beta(z) = z + O(|z|^2) = O(|z|), \tag{2.27}$$

$$J_\beta(z) = A(y) + O(|z|) = \mathrm{Id}_{N+1} + O(|z|), \tag{2.28}$$

$$\mathrm{div}\beta(z) = N + 1 + O(|z|). \tag{2.29}$$

Proof. The result follows by combining (2.25), (2.26) and (2.20). □

2.2. **Some inequalities.** We recall from [13, Lemma 2.4] the following Hardy type inequality with boundary terms, which will be used throughout the paper.

Lemma 2.3. *For all $r > 0$ and $w \in H^1(B_r^+, t^{1-2s}\,dz)$*

$$\left(\frac{N-2s}{2}\right)^2 \int_{B_r^+} t^{1-2s} \frac{w^2(z)}{|z|^2}\,dz \leq \int_{B_r^+} t^{1-2s} \left(\nabla w(z) \cdot \frac{z}{|z|}\right)^2 dz + \left(\frac{N-2s}{2r}\right) \int_{\partial^+ B_r^+} t^{1-2s} w^2 dS.$$

In order to prove a coercivity-type inequality, we provide the following Sobolev type inequality with boundary terms (see Lemma 2.6 in [13]).

Lemma 2.4. *There exists $\tilde{S}_{N,s} > 0$ such that, for all $r > 0$ and $V \in H^1(B_r^+, t^{1-2s}\,dz)$,*

$$\left(\int_{B'_r} |\operatorname{Tr} V|^{2^*(s)} dy\right)^{\frac{2}{2^*(s)}} \leq \tilde{S}_{N,s} \left[\frac{N-2s}{2r} \int_{\partial^+ B_r^+} t^{1-2s} V^2 dS + \int_{B_r^+} t^{1-2s} |\nabla V|^2 dz\right]. \tag{2.30}$$

Exploiting (2.30) we now prove the following inequality which will be useful in the sequel.

Lemma 2.5. *For every $\bar\alpha > 0$, there exists $r(\bar\alpha) \in (0, R_1)$ such that, for any $0 < r \le r(\bar\alpha)$, $\zeta \in L^p(B'_{R_1})$ such that $\|\zeta\|_{L^p(B'_{R_1})} \le \bar\alpha$ and $V \in H^1(B_r^+, t^{1-2s}\,dz)$,*

$$\int_{B_r^+} t^{1-2s} A\nabla V\cdot\nabla V\,dz - \kappa_s \int_{B'_r} \zeta|\operatorname{Tr} V|^2 dy + \frac{N-2s}{2r}\int_{\partial^+ B_r^+} t^{1-2s}\mu V^2\,dS \tag{2.31}$$
$$\ge \tilde C_{N,s}\left(\int_{B_r^+} t^{1-2s}|\nabla V|^2\,dz + \left(\int_{B'_r}|\operatorname{Tr} V|^{2^*(s)}dy\right)^{\frac{2}{2^*(s)}}\right),$$

for some positive constant $\tilde C_{N,s} > 0$ depending only on N and s.

Proof. Let us estimate from below each term in the left hand side of (2.31). To this aim, exploiting (2.20), we can choose $r_1 \in (0, R_1)$ such that, for all $0 < r \le r_1$ and $V \in H^1(B_r^+, t^{1-2s}\,dz)$,

$$\int_{B_r^+} t^{1-2s} A\nabla V\cdot\nabla V\;dz \ge \frac12\int_{B_r^+} t^{1-2s}|\nabla V|^2\;dz. \tag{2.32}$$

Furthermore, thanks to (2.25), we can assert that $\mu \ge 1/4$ in B_r if $0 < r \le r_2$, for some $r_2 \in (0, R_1)$. Hence, using (2.30), we deduce that, for all $0 < r \le r_2$ and $V \in H^1(B_r^+, t^{1-2s}\,dz)$,

$$\frac{N-2s}{2r}\int_{\partial^+ B_r^+} t^{1-2s}\mu V^2\,dS \ge \frac{1}{4\tilde S_{N,s}}\left(\int_{B'_r}|\operatorname{Tr} V|^{2^*(s)}dy\right)^{\frac{2}{2^*(s)}} - \frac14\int_{B_r^+} t^{1-2s}|\nabla V|^2\,dz. \tag{2.33}$$

By Hölder's inequality, we infer that, for all $r \in (0, R_1)$, $V \in H^1(B_r^+, t^{1-2s}\,dz)$, and $\zeta \in L^p(B'_{R_1})$ such that $\|\zeta\|_{L^p(B'_{R_1})} \le \bar\alpha$,

$$\int_{B'_r}\zeta\,|\operatorname{Tr} V|^2\,dy \le \tilde c_{N,s,p}\, r^{\bar\varepsilon}\|\zeta\,\|_{L^p(B'_{R_1})}\left(\int_{B'_r}|\operatorname{Tr} V|^{2^*(s)}dy\right)^{\frac{2}{2^*(s)}} \tag{2.34}$$
$$\le \tilde c_{N,s,p}\bar\alpha\, r^{\bar\varepsilon}\left(\int_{B'_r}|\operatorname{Tr} V|^{2^*(s)}dy\right)^{\frac{2}{2^*(s)}}$$

for some $\tilde c_{N,s,p} > 0$ (depending only on p, N, s), where

$$\bar\varepsilon = \frac{2sp-N}{p} > 0. \tag{2.35}$$

Selecting $r_3 = r_3(\bar\alpha) \in (0, R_1)$ such that

$$\kappa_s\tilde c_{N,s,p}\bar\alpha r^{\bar\varepsilon} \le \frac{1}{8\tilde S_{N,s}} \quad \text{for all } 0 < r \le r_3 \tag{2.36}$$

and combining (2.32), (2.33), and (2.34), we obtain that, for all $0 < r \le r(\bar\alpha) := \min\{r_1, r_2, r_3\}$ and $V \in H^1(B_r^+, t^{1-2s}\,dz)$,

$$\int_{B_r^+} t^{1-2s} A\nabla V\cdot\nabla V\,dz - \kappa_s \int_{B'_r} \zeta|\operatorname{Tr} V|^2 dy + \frac{N-2s}{2r}\int_{\partial^+ B_r^+} t^{1-2s}\mu V^2\,dS$$
$$\ge \frac14\int_{B_r^+} t^{1-2s}|\nabla V|^2 dz + \left(\frac{1}{4\tilde S_{N,s}} - \kappa_s\tilde c_{N,s,p}\bar\alpha\, r^{\bar\varepsilon}\right)\left(\int_{B'_r}|\operatorname{Tr} V|^{2^*(s)}dy\right)^{\frac{2}{2^*(s)}}$$
$$\ge \tilde C_{N,s}\left(\int_{B_r^+} t^{1-2s}|\nabla V|^2 dz + \left(\int_{B'_r}|\operatorname{Tr} V|^{2^*(s)}dy\right)^{\frac{2}{2^*(s)}}\right),$$

where $\tilde C_{N,s} := \min\left\{\frac14, \frac{1}{8\tilde S_{N,s}}\right\}$, thus proving (2.31). □

Remark 2.6. For $\bar\alpha > 0$, let $r(\bar\alpha)$ and $\tilde c_{N,s,p}$ be as in Lemma 2.5 and let $\zeta \in L^p(B'_{R_1})$ be such that $\|\zeta\|_{L^p(B'_{R_1})} \le \bar\alpha$. Then, for every $r \in (0, r(\bar\alpha)]$ and $V \in H^1(B_r^+, t^{1-2s}\,dz)$, we have that

$$\int_{B'_r}\zeta|\operatorname{Tr} V|^2 dy \le \tilde S_{N,s}\tilde c_{N,s,p} r^{\bar\varepsilon}\bar\alpha\frac{2(N-2s)}{r}\int_{\partial^+ B_r^+} t^{1-2s}\mu V^2 dS + \frac{1}{8\kappa_s}\int_{B_r^+} t^{1-2s}|\nabla V|^2\;dz. \tag{2.37}$$

Proof. Applying (2.34) and (2.30), we obtain (2.37), taking into account that, for all $0 < r \le r(\bar{\alpha})$, (2.36) holds and $\mu \ge 1/4$. □

2.3. **Construction of the approximating sequence and convergence.** The main difficulty in the proof of a Pohozaev type identity for problem (2.11), which is needed to differentiate the Almgren quotient, relies in a substancial lack of regularity at Dirichlet-Neumann junctions. Here we face this difficulty by a double approximation procedure, involving both the potential h and the N-dimensional region $\Gamma^+_{R_1}$ where the solution to (2.11) is forced to vanish.

Let $\eta \in C^\infty([0,+\infty))$ be such that

$$\eta \equiv 1 \text{ in } [0,1/2], \quad \eta \equiv 0 \text{ in } [1,+\infty), \quad 0 \le \eta \le 1 \quad \text{and} \quad \eta' \le 0. \tag{2.38}$$

Let

$$f : [0,+\infty) \to \mathbb{R}, \quad f(t) = \eta(t) + (1-\eta(t))t^{1/4}.$$

We observe that

$$f \in C^\infty([0,+\infty)), \quad f(t) = 1 \text{ for all } t \in [0,1/2], \quad \text{and} \quad f(t) - 4t\, f'(t) \ge 0 \text{ for all } t \ge 0. \tag{2.39}$$

Furthermore

$$f(t) \ge \frac{1}{2} \quad \text{and} \quad f(t) \ge t^{1/4} \quad \text{for all } t \ge 0. \tag{2.40}$$

For every $n \in \mathbb{N} \setminus \{0\}$, we consider the function

$$f_n(t) = \frac{f(nt)}{n^{1/8}}.$$

Then, (2.39) implies that

$$f_n \in C^\infty([0,+\infty)), \quad f_n(t) = n^{-1/8} \text{ for all } t \in [0,1/2n], \quad f_n(t) - 4t\, f_n'(t) \ge 0 \text{ for all } t \ge 0, \tag{2.41}$$

whereas (2.40) yields

$$f_n(t) \ge \frac{1}{2} n^{-1/8} \quad \text{and} \quad f_n(t) \ge n^{1/8} t^{1/4} \quad \text{for all } t \ge 0. \tag{2.42}$$

By (2.14) and density of smooth functions in Sobolev spaces, there exists a sequence of potential terms $h_n \in C^\infty\big(\overline{\Gamma^-_{R_1}}\big)$ such that

$$h_n \to \tilde{h} \quad \text{in } W^{1,p}(\Gamma^-_{R_1}). \tag{2.43}$$

Let

$$\bar{\alpha}_0 = \sup_n \|h_n\|_{L^p(\Gamma^-_{R_1})} \tag{2.44}$$

and set

$$R_0 = r(\bar{\alpha}_0) \tag{2.45}$$

according to the notation introduced in Lemma 2.5.

Remark 2.7. Because of the above choice of R_0, we have that (2.31) holds with any $\zeta \in L^p(B'_{R_1})$ such that $|\zeta| \le |h_n|$ a.e. (being h_n trivially extended in $B'_{R_1} \setminus \Gamma^-_{R_1}$), for any $n \in \mathbb{N} \setminus \{0\}$, $r \le R_0$, and for all $V \in H^1(B^+_r, t^{1-2s}\, dz)$.

Let us define, for all $n \in \mathbb{N} \setminus \{0\}$,

$$\gamma_n = \{(y', y_N, t) \in \overline{B^+_{R_0}} : \; y_N = f_n(t)\}, \tag{2.46}$$

with R_0 as in (2.45). If $(y', y_N, t) \in \gamma_n$, then from (2.42) it follows that

$$n^{1/8} t^{1/4} \le f_n(t) = y_N \le R_0,$$

thus proving that

$$t \le \frac{R_0^4}{\sqrt{n}} \quad \text{if} \quad (y', y_N, t) \in \gamma_n. \tag{2.47}$$

The approximating domains are defined as

$$\mathcal{U}_n = \{(y', y_N, t) \in B^+_{R_0} : y_N < f_n(t)\} \tag{2.48}$$

with topological boundary

$$\partial \mathcal{U}_n = \sigma_n \cup \gamma_n \cup \tau_n,$$

where γ_n has been defined in (2.46) and

$$\sigma_n = \left\{ (y', y_N) \in B'_{R_0} : y_N < \frac{1}{n^{1/8}} \right\}, \quad \tau_n = \{(y', y_N, t) \in \partial B_{R_0} : t \geq 0,\ y_N < f_n(t)\},$$

see Figure 1.

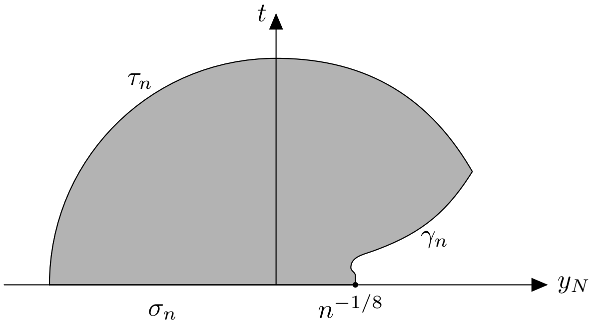

FIGURE 1. Vertical section of the approximating domain $\mathcal{U}_n$.

The functions f_n above have been constructed with the aim of making $\mathcal{U}_n$ satisfy the following geometric property, which will be needed in the proof of a monotonicity formula.

Lemma 2.8. *There exists $\bar{n} \in \mathbb{N} \setminus \{0\}$ such that, for all $n \geq \bar{n}$ and $z = (y, t) \in \gamma_n \cap B^+_{R_0}$,*

$$A(y)z \cdot \nu \geq 0 \quad \text{on } \gamma_n, \tag{2.49}$$

where γ_n has been defined in (2.46) *and $\nu = \nu(z)$ is the exterior unit normal vector at $z \in \partial \mathcal{U}_n$.*

Proof. For all $z = (y, t) \in \gamma_n \cap B^+_{R_0}$ we have that $\nu = \nu(z) = \frac{\mathfrak{n}}{|\mathfrak{n}|}$ where $\mathfrak{n} = (\mathbf{0}, 1, -f'_n(t))$. Hence, from (2.18) and (2.19) it follows that

$$A(y)(y, t) \cdot \mathfrak{n} = (D(y)y, \alpha(y)t) \cdot ((\mathbf{0}, 1), -f'_n(t)) = y_N(1 + O(|y'|) + O(y_N)) - \alpha(y)tf'_n(t).$$

Therefore, possibly choosing R_1 (and consequently R_0) smaller from the beginning and recalling (2.7)–(2.8), we obtain that

$$A(y)(y, t) \cdot \mathfrak{n} \geq \begin{cases} \frac{y_N}{2} - 2tf'_n(t) = \frac{1}{2}(f_n(t) - 4tf'_n(t)) & \text{if } f'_n(t) \geq 0 \\ \frac{y_N}{2} & \text{if } f'_n(t) \leq 0 \end{cases}$$

thus concluding that $A(y)(y, t) \cdot \mathfrak{n} \geq 0$ in view of (2.41). □

Now we construct a sequence U_n of solutions to some suitable approximating problems in the domains $\mathcal{U}_n$ defined in (2.48), which converges strongly to W in the weighted Sobolev space $H^1(B^+_{R_0}, t^{1-2s}\,dz)$. The functions U_n will be sufficiently regular to satisfy a Rellich-Nečas identity and make it integrable on $\mathcal{U}_n$, thus allowing us to obtain a Pohozaev type identity for U_n with some remainder terms produced by the transition of the boundary conditions, whose sign can anyway be understood thanks to the geometric property (2.49); therefore, passing to the limit in the Pohozaev identity satisfied by U_n, we end up with inequality (2.64) for W, which will be used to estimate from below the derivative of the Almgren frequency function (3.9) and then to prove that such frequency has a finite limit at 0 (Proposition 3.6).

Let $W \in H^1(B^+_{R_1}, t^{1-2s}\,dz)$ be a non-trivial energy solution to (2.11), in the sense clarified in (2.13). By density, there exists a sequence of functions $G_n \in C^\infty_c(\overline{B^+_{R_1}} \setminus \Gamma^+_{R_1})$ such that $G_n \to W$ strongly in $H^1(B^+_{R_1}, t^{1-2s}\,dz)$. Thanks to (2.47), without loss of generality we can assume that $G_n = 0$ on γ_n.

We construct a sequence of cut-off functions in the following way: letting $\eta \in C^\infty([0, +\infty))$ be as in (2.38), we define

$$\eta_n : \mathbb{R}^N \to \mathbb{R}, \quad \eta_n(y', y_N) = \begin{cases} 1 - \eta\left(-\frac{ny_N}{2}\right) & \text{if } y_N \leq 0, \\ 0 & \text{if } y_N > 0. \end{cases} \tag{2.50}$$

For any fixed $n \in \mathbb{N}$, we consider the following boundary value problem

$$\begin{cases} -\mathrm{div}\left(t^{1-2s} A \nabla U_n\right) = 0 & \text{in } \mathcal{U}_n, \\ \lim_{t\to 0^+}\left(t^{1-2s} A\nabla U_n \cdot \nu\right) = \kappa_s \eta_n h_n \operatorname{Tr} U_n & \text{in } \sigma_n, \\ U_n = G_n & \text{in } \tau_n \cup \gamma_n, \end{cases} \tag{2.51}$$

in a weak sense, i.e.

$$\begin{cases} U_n - G_n \in \mathcal{H}_n, \\ \displaystyle\int_{\mathcal{U}_n} t^{1-2s} A\nabla U_n \cdot \nabla \Phi \, dz - \kappa_s \int_{\sigma_n} \eta_n h_n \operatorname{Tr} U_n \, \operatorname{Tr}\Phi \, dy = 0 \quad \text{for all } \Phi \in \mathcal{H}_n, \end{cases} \tag{2.52}$$

where $\mathcal{H}_n$ is defined as the closure of $C_c^\infty(\mathcal{U}_n \cup \sigma_n)$ in $H^1(\mathcal{U}_n, t^{1-2s}\, dz)$. Existence of solutions to (2.52) and their convergence to W are established in the following proposition.

Proposition 2.9. *For any fixed $n \in \mathbb{N}$, there exists a unique solution U_n to* (2.52). *Moreover $U_n \to W$ strongly in $H^1(B^+_{R_0}, t^{1-2s}dz)$ (where U_n is extended trivially to zero in $B^+_{R_0} \setminus \mathcal{U}_n$).*

Proof. U_n solves (2.52) if and only if $V_n = U_n - G_n$ satisfies

$$V_n \in \mathcal{H}_n \quad \text{and} \quad b_n(V_n, \Phi) = \langle F_n, \Phi\rangle \quad \text{for all } \Phi \in \mathcal{H}_n, \tag{2.53}$$

where

$$b_n : \mathcal{H}_n \times \mathcal{H}_n \to \mathbb{R}, \quad b_n(V, \Phi) = \int_{\mathcal{U}_n} t^{1-2s} A\nabla V \cdot \nabla \Phi \, dz - \kappa_s \int_{\sigma_n} \eta_n h_n \operatorname{Tr} V \operatorname{Tr}\Phi \, dy \tag{2.54}$$

and

$$F_n : \mathcal{H}_n \to \mathbb{R}, \quad \langle F_n, \Phi\rangle = -\int_{\mathcal{U}_n} t^{1-2s} A\nabla G_n \cdot \nabla \Phi \, dz + \kappa_s \int_{\sigma_n} \eta_n h_n \operatorname{Tr} G_n \operatorname{Tr}\Phi \, dy. \tag{2.55}$$

From Hölder's inequality, (2.30), and the boundedness of $\{h_n\}$ and $\{G_n\}$ respectively in $L^p(\Gamma^-_{R_1})$ and in $H^1(B^+_{R_1}, t^{1-2s}\, dz)$, it follows that

$$|\langle F_n, \Phi\rangle| \le c\, \|\Phi\|_{\mathcal{H}_n} \quad \text{for all } \Phi \in \mathcal{H}_n \tag{2.56}$$

for some constant $c > 0$ which does not depend on n. In particular $F_n \in \mathcal{H}_n^\star$, being $\mathcal{H}_n^\star$ the dual space of $\mathcal{H}_n$, and $\|F_n\|_{\mathcal{H}_n^\star} \le c$ uniformly in n.

The idea is to apply the Lax-Milgram Theorem. In order to do this, we remark that, using the Hardy inequality in Lemma 2.3, after extending functions V_n trivially to zero in $B^+_{R_0} \setminus \mathcal{U}_n$, the weighted L^2-norm of the gradient

$$\left(\int_{\mathcal{U}_n} t^{1-2s} |\nabla V_n|^2 dz\right)^{1/2}$$

turns out to be an equivalent norm in the space $\mathcal{H}_n$ that will be denoted as $\|\cdot\|_{\mathcal{H}_n}$. It follows that b_n is coercive: indeed, for every $V \in \mathcal{H}_n$, we have that

$$\begin{aligned} b_n(V,V) &= \int_{\mathcal{U}_n} t^{1-2s} A\nabla V \cdot \nabla V \, dz - \kappa_s \int_{\sigma_n} \eta_n h_n |\operatorname{Tr} V|^2 dy \\ &= \int_{B^+_{R_0}} t^{1-2s} A\nabla V \cdot \nabla V \, dz - \kappa_s \int_{B'_{R_0}} \eta_n h_n |\operatorname{Tr} V|^2 dy \\ &\ge \tilde{C}_{N,s} \int_{B^+_{R_0}} t^{1-2s} |\nabla V|^2 dz = \tilde{C}_{N,s} \int_{\mathcal{U}_n} t^{1-2s} |\nabla V|^2 dz = \tilde{C}_{N,s} \|V\|^2_{\mathcal{H}_n}, \end{aligned} \tag{2.57}$$

as a consequence of Lemma 2.5, with $\zeta = \eta_n h_n$, see Remark 2.7. Furthermore, from (2.23) and (2.37) it follows that

$$|b_n(V,W)| \le \left(2 + \frac{1}{8}\right) \|V\|_{\mathcal{H}_n} \|W\|_{\mathcal{H}_n} \le 3 \|V\|_{\mathcal{H}_n} \|W\|_{\mathcal{H}_n} \tag{2.58}$$

for all $V, W \in \mathcal{H}_n$. In particular b_n is continuous.

Hence, from (2.57), (2.58), and the Lax-Milgram Theorem we can conclude that there exists a unique $V_n \in \mathcal{H}_n$ solving (2.53), which implies also the existence and uniqueness of a solution U_n to (2.52). Moreover, combining (2.57) and (2.56) we also obtain that, extending V_n trivially to zero in $B^+_{R_0} \setminus \mathcal{U}_n$,

$$\|V_n\|_{H^1(B^+_{R_0},t^{1-2s}\,dz)} \leq \frac{c}{\tilde{C}_{N,s}} \quad \text{for all } n.$$

From this, it follows that there exist $V \in H^1(B^+_{R_0}, t^{1-2s}\,dz)$ and a subsequence $\{V_{n_k}\}$ of $\{V_n\}$ such that

$$V_{n_k} \rightharpoonup V \quad \text{weakly in } H^1(B^+_{R_0}, t^{1-2s}\,dz). \tag{2.59}$$

From the fact that $V_n \in \mathcal{H}_n$, it follows easily that V has null trace on $\partial^+ B^+_{R_0}$ and on $\Gamma^+_{R_0}$. Hence it can be taken as a test function in (2.13) yielding

$$\int_{B^+_{R_0}} t^{1-2s} A\nabla W \cdot \nabla V\,dz - \kappa_s \int_{\Gamma^-_{R_0}} \tilde{h}\,\mathrm{Tr}\,W\,\mathrm{Tr}\,V\,dy = 0. \tag{2.60}$$

Since $G_n \to W$ strongly in $H^1(B^+_{R_1}, t^{1-2s}\,dz)$, from (2.1) we deduce that $\mathrm{Tr}\,G_n \to \mathrm{Tr}\,W$ in $L^{2^*(s)}(B'_{R_1})$. (2.1) and (2.59) imply that $\mathrm{Tr}\,V_{n_k} \rightharpoonup \mathrm{Tr}\,V$ weakly in $L^{2^*(s)}(B'_{R_1})$. Furthermore $\eta_n h_n \to \tilde{h}$ in $L^{\frac{N}{2s}}(\Gamma^-_{R_1})$. Hence (2.60) yields

$$\begin{aligned}
0 &= \int_{B^+_{R_0}} t^{1-2s} A\nabla W \cdot \nabla V\,dz - \kappa_s \int_{\Gamma^-_{R_0}} \tilde{h}\,\mathrm{Tr}\,W\,\mathrm{Tr}\,V\,dy \\
&= \lim_{k\to\infty} \int_{B^+_{R_0}} t^{1-2s} A\nabla G_{n_k} \cdot \nabla V_{n_k}\,dz - \kappa_s \int_{\Gamma^-_{R_0}} \eta_{n_k} h_{n_k}\,\mathrm{Tr}\,G_{n_k}\,\mathrm{Tr}\,V_{n_k}\,dy \\
&= -\lim_{k\to\infty} \langle F_{n_k}, V_{n_k}\rangle = -\lim_{k\to\infty} b_{n_k}(V_{n_k}, V_{n_k})
\end{aligned}$$

thus concluding that $\|V_{n_k}\|_{H^1(B^+_{R_0},t^{1-2s}\,dz)} \to 0$ as $k \to +\infty$ in view of (2.57). Hence $V_{n_k} \to 0$ strongly in $H^1(B^+_{R_0}, t^{1-2s}\,dz)$ and $U_{n_k} = V_{n_k} + G_{n_k} \to W$ as $k \to +\infty$ strongly in $H^1(B^+_{R_0}, t^{1-2s}\,dz)$. By Urysohn's subsequence principle, we finally conclude that $U_n \to W$ in $H^1(B^+_{R_0}, t^{1-2s}\,dz)$ as $n \to +\infty$. □

2.4. **Pohozaev-type inequalities.** The aim of this section is to prove a Pohozaev-type inequality for the energy solution $W \in H^1(B^+_{R_1}, t^{1-2s}\,dz)$ to (2.11); in this situation we have to settle for an inequality instead of a classical Pohozev-type identity because of the mixed boundary conditions, which produce some extra singular terms with a recognizable sign when integrating the Rellich-Nečas identity.

The idea is to obtain the inequality as limit of ones for the approximating sequence U_n. For every $r \in (0, R_0)$, $n \in \mathbb{N}$ such that $n > r^{-8}$, and $\delta \in \left(0, \frac{1}{4n}\right)$, let us consider the following domain

$$O_{r,n,\delta} = \mathcal{U}_n \cap \{(y,t) \in B_r : t > \delta\}.$$

We note that, if $\delta \in \left(0, \frac{1}{4n}\right)$, then $f_n(t) = n^{-1/8}$ for $0 \leq t \leq 2\delta$, see (2.41). We can describe its topological boundary as $\partial O_{r,n,\delta} = \sigma_{r,n,\delta} \cup \gamma_{r,n,\delta} \cup \tau_{r,n,\delta}$, with

$$\sigma_{r,n,\delta} = \left\{(y', y_N, t) \in B_r : y_N < \frac{1}{n^{1/8}},\ t = \delta\right\}, \tag{2.61}$$

$$\gamma_{r,n,\delta} = \left\{(y', y_N, t) \in \overline{B^+_r} : y_N = f_n(t),\ t \geq \delta\right\}, \tag{2.62}$$

$$\tau_{r,n,\delta} = \left\{(y', y_N, t) \in \partial^+ B^+_r : y_N < f_n(t),\ t \geq \delta\right\}, \tag{2.63}$$

see Figure 2.
We also define

$$S^-_r = \{(y', y_N, t) \in \partial B_r : t = 0 \text{ and } y_N < 0\}.$$

Having the matrix A Lipschitz coefficients and being the equation satisfied in a smooth domain containing $O_{r,n,\delta}$, by classical elliptic regularity theory (see e.g. [24, Theorem 2.2.2.3]) we have that $U_n \in H^2(O_{r,n,\delta})$.

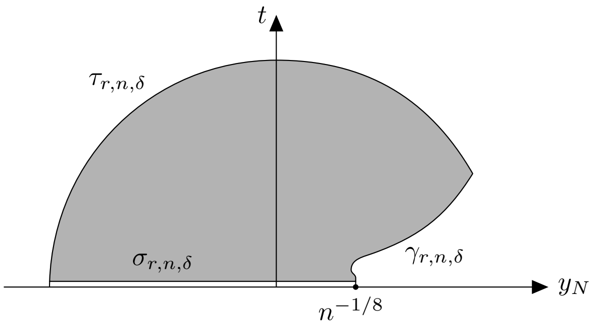

FIGURE 2. Vertical section of $O_{r,n,\delta}$.

Proposition 2.10 (Pohozaev-type inequality)**.** *Let $W \in H^1(B^+_{R_1}, t^{1-2s}\, dz)$ weakly solve (2.11). Then, for almost every $r \in (0, R_0)$,*

$$\begin{aligned}(2.64)\qquad &\frac{r}{2}\int_{\partial^+ B_r^+} t^{1-2s} A\nabla W\cdot\nabla W\, dS - r\int_{\partial^+ B_r^+} t^{1-2s}\frac{|A\nabla W\cdot\nu|^2}{\mu}\, dS\\ &+\frac{\kappa_s}{2}\int_{\Gamma_r^-}\Big(\nabla\tilde h\cdot\beta' + \tilde h\,\mathrm{div}\beta'\Big)|\mathrm{Tr}W|^2 dy - \frac{\kappa_s r}{2}\int_{S_r^-}\tilde h|\mathrm{Tr}W|^2 dS'\\ &\geq \frac12\int_{B_r^+} t^{1-2s}A\nabla W\cdot\nabla W\ \mathrm{div}\beta\, dz - \int_{B_r^+} t^{1-2s}J_\beta(A\nabla W)\cdot\nabla W\, dz\\ &+\frac12\int_{B_r^+} t^{1-2s}(dA\nabla W\nabla W)\cdot\beta\, dz + \frac{1-2s}{2}\int_{B_r^+} t^{1-2s}\frac{\alpha}{\mu}A\nabla W\cdot\nabla W\, dz\end{aligned}$$

and

$$(2.65)\qquad \int_{B_r^+} t^{1-2s}A\nabla W\cdot\nabla W\, dz = \int_{\partial^+ B_r^+} t^{1-2s}(A\nabla W\cdot\nu)\, W\, dS + \kappa_s\int_{\Gamma_r^-}\tilde h|\mathrm{Tr}W|^2 dy.$$

Remark 2.11. The term $\int_{S_r^-}\tilde h|\mathrm{Tr}W|^2 dS'$ is understood for a.e. $r \in (0, R_0)$ as the L^1-function given by the weak derivative of the $W^{1,1}(0,R_0)$-function $r \mapsto \int_{\Gamma_r^-}\tilde h|\mathrm{Tr}W|^2 dy$. Likewise, the two terms

$$\int_{\partial^+ B_r^+} t^{1-2s}A\nabla W\cdot\nabla W\, dS \quad\text{and}\quad \int_{\partial^+ B_r^+} t^{1-2s}\frac{|A\nabla W\cdot\nu|^2}{\mu}\, dS$$

are understood for a.e. $r \in (0, R_0)$ as the L^1-functions given by the weak derivative of the $W^{1,1}(0,R_0)$-functions $r \mapsto \int_{B_r^+} A\nabla W\cdot\nabla W\, dz$ and $r \mapsto \int_{B_r^+} t^{1-2s}\frac{|A\nabla W\cdot\nu|^2}{\mu}\, dz$ respectively.

Proof. Since $U_n \in H^2(O_{r,n,\delta})$, the following Rellich-Nečas identity holds in a distributional sense in $O_{r,n,\delta}$:

$$\begin{aligned}(2.66)\quad \mathrm{div}\Big((\tilde A\nabla U_n\cdot\nabla U_n)\beta - 2(\beta\cdot\nabla U_n)\tilde A\nabla U_n\Big) &= (\tilde A\nabla U_n\cdot\nabla U_n)\,\mathrm{div}\,\beta - 2(\beta\cdot\nabla U_n)\,\mathrm{div}(\tilde A\nabla U_n)\\ &\quad + (d\tilde A\nabla U_n\nabla U_n)\cdot\beta - 2J_\beta(\tilde A\nabla U_n)\cdot\nabla U_n,\end{aligned}$$

where $\tilde A(z) = t^{1-2s}A(y)$ and β has been defined in (2.21). Since $U_n \in H^2(O_{r,n,\delta})$ and $\tilde A$ and β have Lipschitz components, we have that

$$(\tilde A\nabla U_n\cdot\nabla U_n)\beta - 2(\beta\cdot\nabla U_n)\tilde A\nabla U_n \in W^{1,1}(O_{r,n,\delta})$$

so that we can use the integration by parts formula for Sobolev functions on the Lipschitz domain $O_{r,n,\delta}$ and obtain, in view of (2.66) and (2.52),

$$\begin{aligned}
& r\int_{\tau_{r,n,\delta}} t^{1-2s} A\nabla U_n\cdot\nabla U_n\,dS - 2r\int_{\tau_{r,n,\delta}} t^{1-2s}\frac{|A\nabla U_n\cdot\nu|^2}{\mu}\,dS \\
&\quad - \int_{\gamma_{r,n,\delta}} \frac{t^{1-2s}}{\mu}|\partial_\nu U_n|^2(A\nu\cdot\nu)(Az\cdot\nu)\,dS - \int_{\sigma_{r,n,\delta}} \delta^{2-2s}\frac{\alpha}{\mu}A\nabla U_n\cdot\nabla U_n\,dy \\
&\quad + 2\int_{\sigma_{r,n,\delta}} \delta^{1-2s}\frac{\alpha}{\mu}\partial_t U_n(D\nabla_y U_n\cdot y)\,dy + 2\int_{\sigma_{r,n,\delta}} \delta^{2-2s}\frac{\alpha^2}{\mu}|\partial_t U_n|^2\,dy \\
&= \int_{O_{r,n,\delta}} t^{1-2s}A\nabla U_n\cdot\nabla U_n\ \mathrm{div}\beta\,dz - 2\int_{O_{r,n,\delta}} t^{1-2s}J_\beta(A\nabla U_n)\cdot\nabla U_n\,dz \\
&\quad + \int_{O_{r,n,\delta}} t^{1-2s}(dA\nabla U_n\nabla U_n)\cdot\beta\,dz + (1-2s)\int_{O_{r,n,\delta}} t^{1-2s}\frac{\alpha}{\mu}A\nabla U_n\cdot\nabla U_n\,dz.
\end{aligned}$$

In the previous computation we have used the following facts: on $\tau_{r,n,\delta}$ the outward unit normal vector is $\nu=\frac{z}{r}$, on $\gamma_{r,n,\delta}$ we have that $\nabla U_n=\pm|\nabla U_n|\nu$ due to vanishing of U_n on $\gamma_{r,n,\delta}$, and on $\sigma_{r,n,\delta}$ one has $\nu=(0,0,\dots,0,-1)$, $A\nabla U_n\cdot\nu=-\alpha\partial_t U_n$ and

$$A\nabla U_n\cdot z=(D\nabla_y U_n,\alpha\partial_t U_n)\cdot z=D\nabla_y U_n\cdot y+\alpha\delta\partial_t U_n.$$

From Lemma 2.8 and uniform ellipticity of A it follows that

$$\int_{\gamma_{r,n,\delta}} \frac{t^{1-2s}}{\mu}|\partial_\nu U_n|^2(A\nu\cdot\nu)(Az\cdot\nu)\geq 0.$$

Hence, we get the following inequality

$$\begin{aligned}
& r\int_{\tau_{r,n,\delta}} t^{1-2s}A\nabla U_n\cdot\nabla U_n\,dS - 2r\int_{\tau_{r,n,\delta}} t^{1-2s}\frac{|A\nabla U_n\cdot\nu|^2}{\mu}\,dS - \int_{\sigma_{r,n,\delta}} \delta^{2-2s}\frac{\alpha}{\mu}A\nabla U_n\cdot\nabla U_n\,dy \\
&\quad + 2\int_{\sigma_{r,n,\delta}} \delta^{1-2s}\frac{\alpha}{\mu}\partial_t U_n(D\nabla_y U_n\cdot y)\,dy + 2\int_{\sigma_{r,n,\delta}} \delta^{2-2s}\frac{\alpha^2}{\mu}|\partial_t U_n|^2\,dy \\
&\geq \int_{O_{r,n,\delta}} t^{1-2s}A\nabla U_n\cdot\nabla U_n\ \mathrm{div}\beta\,dz - 2\int_{O_{r,n,\delta}} t^{1-2s}J_\beta(A\nabla U_n)\cdot\nabla U_n\,dz \\
&\quad + \int_{O_{r,n,\delta}} t^{1-2s}(dA\nabla U_n\nabla U_n)\cdot\beta\,dz + (1-2s)\int_{O_{r,n,\delta}} t^{1-2s}\frac{\alpha}{\mu}A\nabla U_n\cdot\nabla U_n\,dz.
\end{aligned}\tag{2.67}$$

We are going to pass to the limit as $\delta\to 0$. We denote as $O_{r,n}$ the limit domain with its boundary $\partial O_{r,n}=\sigma_{r,n}\cup\gamma_{r,n}\cup\tau_{r,n}$, i.e.

$$\begin{aligned}
& O_{r,n}=\mathcal{U}_n\cap B_r, \quad \tau_{r,n}=\{(y',y_N,t)\in\partial B_r : y_N<f_n(t),\ t\geq 0\}, \\
& \gamma_{r,n}=\left\{(y',y_N,t)\in\overline{B_r^+} : y_N=f_n(t)\right\}, \quad \sigma_{r,n}=\left\{(y',y_N)\in B_r' : y_N<n^{-1/8}\right\}.
\end{aligned}$$

We claim that, for every fixed $r\in(0,R_0)$ and $n>r^{-8}$, there exists a sequence $\delta_k\to 0^+$ such that

$$-\int_{\sigma_{r,n,\delta_k}} \delta_k^{2-2s}\frac{\alpha}{\mu}A\nabla U_n\cdot\nabla U_n\,dy + 2\int_{\sigma_{r,n,\delta_k}} \delta_k^{2-2s}\frac{\alpha^2}{\mu}|\partial_t U_n|^2\,dy\to 0 \quad \text{as } k\to\infty.$$

Since α is bounded, $\mu\geq 1/4$ in B_{R_0}, and A has bounded coefficients, it is enough to prove that there exists a sequence $\delta_k\to 0^+$ such that $\lim_{k\to\infty}\int_{\sigma_{r,n,\delta_k}}\delta_k^{2-2s}|\nabla U_n|^2\,dy=0$. To prove this, we argue by contradiction and assume that there exist a positive constant $c>0$ and $\delta_0>0$ such that, for any $\delta\in(0,\delta_0)$,

$$\frac{c}{\delta}\leq\int_{\sigma_{r,n,\delta}}\delta^{1-2s}|\nabla U_n(y,\delta)|^2\,dy,$$

which, after integration in $(0,\delta_0)$, gives the contradiction

$$\int_0^{\delta_0}\frac{c}{\delta}\,d\delta\leq\int_0^{\delta_0}\delta^{1-2s}\left(\int_{\sigma_{r,n,\delta}}|\nabla U_n(y,\delta)|^2\,dy\right)d\delta\leq\|U_n\|^2_{H^1(B^+_{R_0},t^{1-2s}\,dz)},$$

since the first integral diverges.

In order to prove the convergence

$$2\int_{\sigma_{r,n,\delta}} \delta^{1-2s}\frac{\alpha}{\mu}\partial_t U_n(D\nabla_y U_n\cdot y)\,dy \underset{\delta\to 0}{\longrightarrow} -2\kappa_s\int_{\sigma_{r,n}}\frac{1}{\mu}\eta_n h_n \mathrm{Tr}U_n(D\nabla_y \mathrm{Tr}U_n\cdot y)\,dy,$$

we exploit a continuity result for $t^{1-2s}\partial_t U_n$ and $\nabla_y U_n$ over $\overline{\mathcal{U}_n\cap B_r}$, which allows us to pass to the limit by the Dominated Convergence Theorem. More precisely we claim that, for all $r\in(0,R_0)$ and $n>r^{-8}$,

$$t^{1-2s}\partial_t U_n\in C^0(\overline{\mathcal{U}_n\cap B_r}),\quad \nabla_y U_n\in C^0(\overline{\mathcal{U}_n\cap B_r}). \tag{2.68}$$

The continuity of $t^{1-2s}\partial_t U_n$ and $\nabla_y U_n$ away from $\{t=0\}$ easily follows from classical elliptic regularity theory, since U_n is solution of an uniformly elliptic equation. Nevertheless, Lemma 3.3 in [13] allows us to prove continuity of $t^{1-2s}\partial_t U_n$ and $\nabla_y U_n$ up to $\{t=0\}$ when we stay away from the corner between $\sigma_{r,n}$ and $\gamma_{r,n}$, i.e. away from the edge $\{(y',y_N,t)\in\overline{B_r}: t=0 \text{ and } y_N=n^{-1/8}\}$: to this aim it is enough to apply [13, Lemma 3.3] to the function $U_n\circ F^{-1}$. Eventually, we can deduce continuity of $t^{1-2s}\partial_t U_n$ and $\nabla_y U_n$ also in the set $\{(y',y_N,t)\in\overline{B_r}: t\in[0,\frac{1}{2n}] \text{ and } y_N\in[0,n^{-1/8}]\}$ as a consequence of the regularity result given in Lemma A.1 applied to the function $U_n\circ F^{-1}$.

Using the fact that, for all $r\in(0,R_0)$ and $n>r^{-8}$, the terms integrated over $\tau_{r,n,\delta}$ belong to $L^1(\tau_{r,n})$ in view of (2.68) and the terms integrated over $O_{r,n,\delta}$ belong to $L^1(\mathcal{U}_n\cap B_r)$ since $U_n\in H^1(\mathcal{U}_n,t^{1-2s}\,dz)$, then by absolute continuity of the Lebesgue integral, we can pass to the limit in (2.67) along $\delta=\delta_k$ as $k\to\infty$, thus ending up with the inequality

$$\begin{aligned} r\int_{\tau_{r,n}} & t^{1-2s}A\nabla U_n\cdot\nabla U_n\,dS-2r\int_{\tau_{r,n}}t^{1-2s}\frac{|A\nabla U_n\cdot\nu|^2}{\mu}\,dS-2\kappa_s\int_{\sigma_{r,n}}\frac{1}{\mu}\eta_n h_n\mathrm{Tr}U_n(D\nabla_y\mathrm{Tr}U_n\cdot y)\,dy\\ &\geq\int_{\mathcal{U}_n\cap B_r}t^{1-2s}A\nabla U_n\cdot\nabla U_n\mathrm{div}\beta\,dz-2\int_{\mathcal{U}_n\cap B_r}t^{1-2s}J_\beta(A\nabla U_n)\cdot\nabla U_n\,dz\\ &\quad+\int_{\mathcal{U}_n\cap B_r}t^{1-2s}(dA\nabla U_n\nabla U_n)\cdot\beta\,dz+(1-2s)\int_{\mathcal{U}_n\cap B_r}t^{1-2s}\frac{\alpha}{\mu}A\nabla U_n\cdot\nabla U_n\,dz, \end{aligned}\tag{2.69}$$

for all $r\in(0,R_0)$ and $n>r^{-8}$. For $r\in(0,R_0)$ fixed, we are going to pass to the limit in (2.69) as $n\to+\infty$. We extend the functions U_n to be zero in $B_r^+\setminus\mathcal{U}_n$. By the strong convergence $U_n\to W$ in $H^1(B_{R_0}^+,t^{1-2s}dz)$ (see Proposition 2.9), it follows that

$$\int_0^{R_0}\left(\int_{\partial^+B_r^+}t^{1-2s}\left(|\nabla(U_n-W)|^2+|U_n-W|^2\right)dS\right)dr\to 0,$$

i.e. the sequence of functions $u_n(r)=\int_{\partial^+B_r^+}t^{1-2s}\left(|\nabla(U_n-W)|^2+|U_n-W|^2\right)dS$ converges to 0 in $L^1(0,R_0)$ and hence a.e. along a subsequence u_{n_k}. In particular we have that

$$U_{n_k}\to W\quad\text{as } k\to\infty \text{ in } H^1(\partial^+B_r^+,t^{1-2s}\,dS) \text{ for a.e. } r\in(0,R_0), \tag{2.70}$$

where $H^1(\partial^+B_r^+,t^{1-2s}\,dS)$ is the completion of $C^\infty(\overline{\partial^+B_r^+})$ with respect to the norm

$$\|\psi\|_{H^1(\partial^+B_r^+,t^{1-2s}\,dS)}=\left(\int_{\partial^+B_r^+}t^{1-2s}\left(|\nabla\psi|^2+\psi^2\right)dS\right)^{1/2}.$$

Let us now discuss the behavior of the term $\int_{\sigma_{r,n}}\frac{1}{\mu}\eta_n h_n\mathrm{Tr}U_n(D\nabla_y\mathrm{Tr}U_n\cdot y)\,dy$ as $n\to\infty$. Since $\eta_n(y',y_N)=0$ if $y_N>-\frac{1}{n}$, by the Divergence Theorem we have that

$$\begin{aligned} \int_{\sigma_{r,n}} & \frac{1}{\mu}\eta_n h_n\mathrm{Tr}U_n(D\nabla_y\mathrm{Tr}U_n\cdot y)\,dy=\int_{\Gamma_r^-}\frac{1}{\mu}\eta_n h_n\mathrm{Tr}U_n(D\nabla_y\mathrm{Tr}U_n\cdot y)\,dy\\ &=\frac12\int_{\Gamma_r^-}\mathrm{div}_y\left(\frac{1}{\mu}\eta_n h_n|\mathrm{Tr}U_n|^2Dy\right)dy-\frac12\int_{\Gamma_r^-}|\mathrm{Tr}U_n|^2\mathrm{div}_y\left(\frac{1}{\mu}\eta_n h_n Dy\right)dy\\ &=\frac12\int_{S_r^-}\frac{1}{\mu}\eta_n h_n|\mathrm{Tr}U_n|^2Dy\cdot\nu\,dS'-\frac12\int_{\Gamma_r^-}|\mathrm{Tr}U_n|^2\mathrm{div}_y\left(\eta_n h_n\beta'\right)dy\\ &=\frac{r}{2}\int_{S_r^-}\eta_n h_n|\mathrm{Tr}U_n|^2\,dS'-\frac12\int_{\Gamma_r^-}|\mathrm{Tr}U_n|^2\left(\eta_n\nabla_y h_n\cdot\beta'+\eta_n h_n\mathrm{div}_y\beta'\right)dy\\ &\quad-\frac12\int_{\Gamma_r^-}|\mathrm{Tr}U_n|^2h_n\nabla_y\eta_n\cdot\beta'\,dy, \end{aligned}\tag{2.71}$$

where β' has been defined in (2.21). From the strong convergence $U_n \to W$ in $H^1(B^+_{R_0}, t^{1-2s}dz)$ proved in Proposition 2.9 and (2.1), it follows that

$$\int_0^{R_0} \left(\int_{S_r^-} \left(\eta_n h_n |\mathrm{Tr} U_n|^2 - \tilde{h} |\mathrm{Tr} W|^2 \right) dS' \right) dr \to 0,$$

i.e. the sequence of functions $r \mapsto \int_{S_r^-} \left(\eta_n h_n |\mathrm{Tr} U_n|^2 - \tilde{h}|\mathrm{Tr} W|^2 \right) dS'$ converges to 0 in $L^1(0, R_0)$ and hence a.e. along a further subsequence, which we still index by n_k. In particular we have that

$$\int_{S_r^-} \eta_{n_k} h_{n_k} |\mathrm{Tr} U_{n_k}|^2 \, dS' \to \int_{S_r^-} \tilde{h} |\mathrm{Tr} W|^2 \, dS' \quad \text{as } k \to \infty \text{ for a.e. } r \in (0, R_0). \tag{2.72}$$

The strong convergence of U_n to W in $H^1(B^+_{R_0}, t^{1-2s}dz)$, which implies that $\mathrm{Tr} U_n \to \mathrm{Tr} W$ in $L^{2^*(s)}(B'_{R_0})$ by (2.1), the strong convergence (2.43) of h_n to $\tilde{h}$ in $W^{1,p}(\Gamma^-_{R_0})$, and the fact that $\eta_n \to 1$ a.e. in $\Gamma^-_{R_0}$ imply that

$$\lim_{n\to\infty} \int_{\Gamma_r^-} |\mathrm{Tr} U_n|^2 \left(\eta_n \nabla_y h_n \cdot \beta' + \eta_n h_n \mathrm{div}_y \beta' \right) dy = \int_{\Gamma_r^-} |\mathrm{Tr} W|^2 \left(\nabla_y \tilde{h} \cdot \beta' + \tilde{h} \mathrm{div}_y \beta' \right) dy \tag{2.73}$$

for all $r \in (0, R_0)$. Finally, we have that, by (2.50) and (2.21),

$$\nabla_y \eta_n \cdot \beta' = \frac{1}{\mu} \frac{n}{2} \eta'\big(-\tfrac{n y_N}{2} \big) (D(y)y)_N.$$

Hence, since (2.19) implies that $(D(y)y)_N = O(y_N)$ as $y_N \to 0$ and (2.38) yields that $\eta'\big(-\frac{n y_N}{2}\big) \neq 0$ only for $y_N \in \big(-\frac{2}{n}, -\frac{1}{n} \big)$, we conclude that

$$\nabla_y \eta_n \cdot \beta' \quad \text{is bounded in } \Gamma_r^- \text{ uniformly with respect to } n.$$

Therefore, by Hölder's inequality,

$$\left| \int_{\Gamma_r^-} |\mathrm{Tr} U_n|^2 h_n \nabla_y \eta_n \cdot \beta' \, dy \right| \\ \leq \mathrm{const} \, \|\mathrm{Tr} U_n\|^2_{L^{2^*(s)}(\Gamma_r^-)} \|h_n\|_{L^p(\Gamma_r^-)} \left| \{ (y', y_N) \in \Gamma_r^- : -\tfrac{2}{n} < y_N < -\tfrac{1}{n} \} \right|_N^{\frac{2s}{pN}(p - \frac{N}{2s})}$$

where $|\cdot|_N$ stands for the N-dimensional Lebesgue measure; hence, by boundedness of $\{\mathrm{Tr} U_n\}$ in $L^{2^*(s)}(\Gamma_r^-)$, ensured by (2.1), and boundedness of $\{h_n\}$ in $L^p(\Gamma_r^-)$, we conclude that

$$\lim_{n\to\infty} \int_{\Gamma_r^-} |\mathrm{Tr} U_n|^2 h_n \nabla_y \eta_n \cdot \beta' \, dy = 0 \tag{2.74}$$

Combining (2.72), (2.73), and (2.74), we can pass to the limit in (2.71) along the subsequence, obtaining that

$$\lim_{k\to\infty} \int_{\sigma_{r,n_k}} \frac{1}{\mu} \eta_{n_k} h_{n_k} \mathrm{Tr} U_{n_k} (D \nabla_y \mathrm{Tr} U_{n_k} \cdot y) \, dy \\ = \frac{r}{2} \int_{S_r^-} \tilde{h} |\mathrm{Tr} W|^2 \, dS' - \frac{1}{2} \int_{\Gamma_r^-} |\mathrm{Tr} W|^2 \left(\nabla_y \tilde{h} \cdot \beta' + \tilde{h} \mathrm{div}_y \beta' \right) dy. \tag{2.75}$$

In view of (2.70), (2.75), and the strong convergence of U_n to W in $H^1(B^+_{R_0}, t^{1-2s}dz)$, we can pass to the limit as $n = n_k \to \infty$ in (2.69) obtaining the desired Pohozaev-type inequality (2.64) for the solution W.

Finally, to prove (2.65), we first multiply equation (2.51) by U_n and integrate by parts over $O_{r,n,\delta}$; then we pass to the limit as $\delta \to 0^+$ using (2.68) and as $n = n_k \to \infty$, taking into account (2.70). □

3. Almgren type frequency function

In this section we analyze the properties of the Almgren frequency function $\mathcal{N}(r)$ associated to (2.11), see (3.9). To perform a blow-up analysis, the boundedness of the frequency will be crucial; to this aim we are going to prove that $\mathcal{N}$ possesses a nonnegative finite limit as $r \to 0^+$.

Let $W \in H^1_{\Gamma^+_{R_1}}(B^+_{R_1}, t^{1-2s}\, dz)$ be a nontrivial weak solution of (2.11). For all $r \in (0, R_1)$ we define

$$E(r) = \frac{1}{r^{N-2s}} \left(\int_{B_r^+} t^{1-2s} A \nabla W \cdot \nabla W \, dz - \kappa_s \int_{\Gamma_r^-} \tilde{h} |\, \mathrm{Tr}\, W|^2 \, dy \right) \tag{3.1}$$

and

$$H(r) = \frac{1}{r^{N+1-2s}} \int_{\partial^+ B_r^+} t^{1-2s}\mu(z)W^2(z)\, dS. \tag{3.2}$$

Let us first estimate the derivative of H.

Lemma 3.1. *Let E and H be the functions defined as in* (3.1) *and* (3.2)*. Then $H \in W^{1,1}_{\rm loc}(0, R_1)$ and*

$$H'(r) = \frac{2}{r^{N+1-2s}} \int_{\partial^+ B_r^+} t^{1-2s}\mu W \frac{\partial W}{\partial \nu}\, dS + H(r)O(1) \quad \text{as } r \to 0^+ \tag{3.3}$$

in a distributional sense and for a.e. $r \in (0, R_1)$, where $\nu = \nu(z) = \frac{z}{|z|}$ denotes the unit outer normal vector to $\partial^+ B_r^+$. Moreover

$$H'(r) = \frac{2}{r^{N+1-2s}} \int_{\partial^+ B_r^+} t^{1-2s}(A\nabla W \cdot \nu)W\, dS + H(r)O(1) \tag{3.4}$$

and

$$H'(r) = \frac{2}{r}E(r) + H(r)O(1) \tag{3.5}$$

as $r \to 0^+$.

Proof. We observe that $H \in L^1_{\rm loc}(0, R_1)$ by definition and it can be rewritten as

$$H(r) = \int_{\mathbb{S}^N_+} \theta_{N+1}^{1-2s}\mu(r\theta)W^2(r\theta)\, dS.$$

Thus, for all test functions $\varphi \in C^\infty_c(0, R_1)$, we have that

$$\begin{aligned}
-\int_0^{R_1} H(r)\varphi'(r)dr &= -\int_0^{R_1} \left(\int_{\mathbb{S}^N_+} \theta_{N+1}^{1-2s}\mu(r\theta)W^2(r\theta)\, dS\right)\varphi'(r)dr \\
&= -\int_{B^+_{R_1}} t^{1-2s}\frac{\mu(z)W^2(z)}{|z|^{N+2-2s}}\nabla\tilde\varphi(z)\cdot z\, dz = \int_{B^+_{R_1}} \operatorname{div}\left(\frac{t^{1-2s}\mu(z)W^2(z)z}{|z|^{N+2-2s}}\right)\tilde\varphi(z)dz \\
&= \int_{B^+_{R_1}} t^{1-2s}\left(\frac{2\mu(z)W(z)\nabla W(z) + W^2(z)\nabla\mu(z)}{|z|^{N+2-2s}}\cdot z\right)\tilde\varphi(z)dz \\
&= \int_0^{R_1}\left(\int_{\mathbb{S}^N_+} \theta_{N+1}^{1-2s}\big[2\mu(r\theta)W(r\theta)\nabla W(r\theta)\cdot\theta + W^2(r\theta)\nabla\mu(r\theta)\cdot\theta\big]dS\right)\varphi(r)\, dr,
\end{aligned}$$

where $\tilde\varphi(z) := \varphi(|z|)$, so that $\varphi'(|z|) = \nabla\tilde\varphi(z)\cdot\frac{z}{|z|}$. Hence the distributional derivative of H in $(0, R_1)$ is given by

$$H'(r) = \frac{2}{r^{N+1-2s}} \int_{\partial^+ B_r^+} t^{1-2s}\mu W\frac{\partial W}{\partial\nu}\, dS + \frac{1}{r^{N+1-2s}}\int_{\partial^+ B_r^+} t^{1-2s}W^2\nabla\mu\cdot\nu\, dS. \tag{3.6}$$

Since $W, \nabla W \in L^2(B^+_{R_1}, t^{1-2s}dz)$, from (2.25) and (2.26) we easily infer that $H \in W^{1,1}_{\rm loc}(0, R_1)$ and (3.6) also holds for a.e. $r \in (0, R_1)$. Moreover, combining (2.25), (2.26), (3.2) and (3.6), we obtain (3.3).

In order to prove (3.4), we define

$$\gamma(z) = \frac{\mu(z)(\beta(z) - z)}{|z|}.$$

Observing that $\gamma(z)\cdot z = 0$ by definition, we deduce that, for a.e. $r \in (0, R_1)$,

$$\begin{aligned}
\int_{\partial^+ B_r^+} t^{1-2s}(A\nabla W\cdot\nu)W\, dS &= \int_{\partial^+ B_r^+} t^{1-2s}\mu W\frac{\partial W}{\partial\nu}dS + \frac{1}{2}\int_{\partial^+ B_r^+} t^{1-2s}\gamma\cdot\nabla(W^2)\, dS \\
&= \int_{\partial^+ B_r^+} t^{1-2s}\mu W\frac{\partial W}{\partial\nu}dS - \frac{1}{2}\int_{\partial^+ B_r^+} \operatorname{div}(t^{1-2s}\gamma)W^2\, dS \\
&= \int_{\partial^+ B_r^+} t^{1-2s}\mu W\frac{\partial W}{\partial\nu}dS + H(r)O(r^{N+1-2s}),
\end{aligned} \tag{3.7}$$

where we used that

$$\operatorname{div}(t^{1-2s}\gamma) = t^{1-2s}\operatorname{div}\gamma + (1-2s)\gamma_{N+1}t^{-2s}$$

and

$$\gamma_{N+1}(z) = t\,O(1) \quad \text{as } |z| \to 0^+,$$

$$\mathrm{div}\gamma = \left(\frac{\nabla\mu(z)}{|z|} - \frac{\mu(z)z}{|z|^3}\right)(\beta(z) - z) + \frac{\mu(z)}{|z|}(\mathrm{div}\beta - (N+1)) = O(1) \quad \text{as } |z| \to 0^+,$$

as a consequence of (2.21), (2.7), (2.25), (2.26), (2.27), (2.29). Hence, from (3.3) and (3.7) it follows (3.4). From (3.1), (2.65) and (3.4) we infer that

$$r^{N-2s}E(r) = \int_{\partial^+ B_r^+} t^{1-2s}(A\nabla W \cdot \nu)W\, dS = \frac{r^{N+1-2s}}{2}H'(r) + H(r)O(r^{N+1-2s}),$$

as $r \to 0^+$, which gives (3.5), thus proving the lemma. □

Lemma 3.2. *The function H defined as in* (3.2) *is strictly positive for every $0 < r \le R_0$, with R_0 being defined in* (2.45)*.*

Proof. We prove the statement by contradiction. To this aim, we suppose that there exists $\overline{R} \le R_0$ such that $H(\overline{R}) = 0$. Then, since for every $r \le R_0$ it holds that $\mu \ge 1/4$ in B_r, $\int_{\partial^+ B_{\overline{R}}^+} t^{1-2s}W^2 = 0$ and hence $W \equiv 0$ on $\partial^+ B_{\overline{R}}^+$. From (3.5) it follows that H is differentiable in a classical sense in $\overline{R}$ and $H'(\overline{R}) = 2\overline{R}^{-1}E(\overline{R})$; on the other hand, $H(r) \ge 0 = H(\overline{R})$ implies that $0 = H'(\overline{R}) = 2\overline{R}^{-1}E(\overline{R})$ and hence $E(\overline{R}) = 0$. Then from (2.31) it follows that

$$0 = \int_{B_{\overline{R}}^+} t^{1-2s}A\nabla W \cdot \nabla W\, dz - \kappa_s \int_{\Gamma_{\overline{R}}^-} \tilde{h}|\operatorname{Tr} W|^2\, dy \ge \tilde{C}_{N,s}\int_{B_{\overline{R}}^+} t^{1-2s}|\nabla W|^2\, dz. \tag{3.8}$$

By (3.8) and Lemma 2.3 we can conclude that $W \equiv 0$ in $B_{\overline{R}}^+$, which in turn leads to $W \equiv 0$ in $B_{R_1}^+ \cap \{t > \delta\}$ from classical unique continuation principles for second order elliptic equations with Lipschitz coefficients (see [23]). Since $\delta > 0$ can be taken arbitrarily small, we end up with $W \equiv 0$ in $B_{R_1}^+$, which is a contradiction. □

As a consequence of Lemma 3.2, the *Almgren type frequency* function

$$\mathcal{N}(r) = \frac{E(r)}{H(r)} \tag{3.9}$$

is well defined in $(0, R_0]$, with R_0 as in (2.45).

In the following lemma we provide an estimate for the derivative of the function E.

Lemma 3.3. *Let E be the function defined as in* (3.1)*. Then $E \in W^{1,1}_{\mathrm{loc}}((0, R_0])$ and*

$$E'(r) \ge \frac{2}{r^{N-2s}}\int_{\partial^+ B_r^+} t^{1-2s}\frac{|A\nabla W \cdot \nu|^2}{\mu} + O(r^{-1+\bar{\delta}})\left[E(r) + \frac{N-2s}{2}H(r)\right] \quad \text{as } r \to 0^+ \tag{3.10}$$

for a.e. $r \in (0, R_0)$, where

$$\bar{\delta} = \min\{\bar{\varepsilon}, 1\} \in (0, 1] \tag{3.11}$$

and $\bar{\varepsilon}$ is defined as in (2.35)*.*

Proof. From (3.1) we deduce that $E \in L^1_{\mathrm{loc}}(0, R_0)$. From coarea formula $E \in W^{1,1}_{\mathrm{loc}}((0, R_0])$ and

$$\begin{aligned} E'(r) = &\frac{2s-N}{r^{N+1-2s}}\left(\int_{B_r^+} t^{1-2s}A\nabla W \cdot \nabla W\, dz - \kappa_s\int_{\Gamma_r^-} \tilde{h}|\operatorname{Tr} W|^2\, dy\right) \\ &+ \frac{1}{r^{N-2s}}\left(\int_{\partial^+ B_r^+} t^{1-2s}A\nabla W \cdot \nabla W\, dS - \kappa_s\int_{S_r^-} \tilde{h}|\operatorname{Tr} W|^2\, dS'\right) \end{aligned} \tag{3.12}$$

in a distributional sense and a.e. in $(0, R_0)$. Using (2.64), Lemma 2.1 and Lemma 2.2, we obtain that

$$\begin{aligned} E'(r) \ge &\frac{2}{r^{N-2s}}\int_{\partial^+ B_r^+} t^{1-2s}\frac{|A\nabla W \cdot \nu|^2}{\mu}\, dS + \frac{O(r)}{r^{N+1-2s}}\int_{B_r^+} t^{1-2s}A\nabla W \cdot \nabla W\, dz \\ &+ \frac{O(1)}{r^{N+1-2s}}\int_{\Gamma_r^-} (|\tilde{h}| + |\nabla_y \tilde{h}|)|\operatorname{Tr} W|^2\, dy \end{aligned} \tag{3.13}$$

as $r \to 0^+$, for a.e. $r \in (0, R_0)$. One can estimate the last two terms of the right hand side in (3.13) using (2.31), thus obtaining

$$\frac{O(r)}{r^{N+1-2s}} \int_{B_r^+} t^{1-2s} A\nabla W \cdot \nabla W \, dz = O(1)\left[E(r) + \frac{N-2s}{2} H(r)\right] \tag{3.14}$$

and

$$\begin{aligned}\frac{O(1)}{r^{N+1-2s}} \int_{\Gamma_r^-} (|\tilde{h}| + |\nabla_y \tilde{h}|) W^2 \, dy &= \frac{O(r^{\bar{\varepsilon}})}{r^{N+1-2s}} \left(\int_{\Gamma_r^-} |\operatorname{Tr} W|^{2^*(s)} \, dy\right)^{2/2^*(s)} \\ &= O(r^{-1+\bar{\varepsilon}})\left[E(r) + \frac{N-2s}{2} H(r)\right],\end{aligned} \tag{3.15}$$

where in (3.15) we have taken into account (2.34) as well. Estimate (3.10) follows from (3.13), (3.14), and (3.15). □

Lemma 3.4. *Let $\mathcal{N}$ be the function defined in* (3.9)*. Then, for every $0 < r \le R_0$,*

$$\mathcal{N}(r) > -\frac{N-2s}{2} \tag{3.16}$$

and

$$\liminf_{r \to 0^+} \mathcal{N}(r) \ge 0. \tag{3.17}$$

Proof. We deduce (3.16) from (2.31). By (3.1), (3.2), (2.37) and (2.32), we obtain that for all $0 < r \le R_0$

$$\begin{aligned} r^{N-2s} E(r) &= \int_{B_r^+} t^{1-2s} A\nabla W \cdot \nabla W \, dz - \kappa_s \int_{\Gamma_r^-} \tilde{h} |\operatorname{Tr} W|^2 \, dy \\ &\ge \frac{3}{8} \int_{B_r^+} t^{1-2s} |\nabla W|^2 \, dz - \kappa_s \tilde{S}_{N,s} \tilde{c}_{N,s,p} r^{\bar{\varepsilon}} \bar{\alpha}_0 \frac{2(N-2s)}{r} \int_{\partial^+ B_r^+} t^{1-2s} \mu W^2 \, dS \ge -\tilde{C} r^{\bar{\varepsilon}+N-2s} H(r), \end{aligned}$$

with $\bar{\alpha}_0$ as in (2.44) and $\tilde{C} := 2(N-2s)\kappa_s \tilde{S}_{N,s} \tilde{c}_{N,s,p} \bar{\alpha}_0 > 0$. From this and (3.9) it follows that, for every $0 < r \le R_0$,

$$\mathcal{N}(r) \ge -\tilde{C} r^{\bar{\varepsilon}},$$

which leads to (3.17). □

Lemma 3.5. *Let $\mathcal{N}$ be the function defined in* (3.9)*. Then $\mathcal{N} \in W^{1,1}_{\mathrm{loc}}((0, R_0])$ and*

$$\mathcal{N}'(r) \ge V_1(r) + V_2(r) \tag{3.18}$$

for almost every $r \in (0, R_0)$, where

$$V_1(r) = \frac{2r\left[\left(\int_{\partial^+ B_r^+} t^{1-2s} \frac{|A\nabla W \cdot \nu|^2}{\mu} \, dS\right)\left(\int_{\partial^+ B_r^+} t^{1-2s} \mu W^2 \, dS\right) - \left(\int_{\partial^+ B_r^+} t^{1-2s} (A\nabla W \cdot \nu) W \, dS\right)^2\right]}{\left(\int_{\partial^+ B_r^+} t^{1-2s} \mu W^2\right)^2}$$

and

$$V_2(r) = O(r^{-1+\bar{\delta}})\left(\mathcal{N}(r) + \frac{N-2s}{2}\right) \quad \text{as } r \to 0^+,$$

with $\bar{\delta}$ as in (3.11)*.*

Proof. The fact that $\mathcal{N} \in W^{1,1}_{\mathrm{loc}}((0, R_0])$ follows from Lemmas 3.1, 3.2, and 3.3. Moreover, exploiting (3.4), (3.5) and (3.10), we obtain that

$$\begin{aligned} \mathcal{N}'(r) &= \frac{E'(r)H(r) - H'(r)E(r)}{H^2(r)} = \frac{E'(r)H(r)}{H^2(r)} - \frac{H'(r)}{H^2(r)}\left(\frac{r}{2} H'(r) + H(r) O(r)\right) \\ &\ge \frac{2r\left[\left(\int_{\partial^+ B_r^+} t^{1-2s} \frac{|A\nabla W \cdot \nu|^2}{\mu} dS\right)\left(\int_{\partial^+ B_r^+} t^{1-2s} \mu W^2 dS\right) - \left(\int_{\partial^+ B_r^+} t^{1-2s} (A\nabla W \cdot \nu) W dS\right)^2\right]}{\left(\int_{\partial^+ B_r^+} t^{1-2s} \mu W^2 dS\right)^2} \\ &\quad + O(r^{-1+\bar{\delta}})\left(\mathcal{N}(r) + \frac{N-2s}{2}\right) + O(r) + O(1)\frac{1}{H(r)} \frac{1}{r^{N-2s}} \int_{\partial^+ B_r^+} t^{1-2s} (A\nabla W \cdot \nu) W \, dS. \end{aligned} \tag{3.19}$$

In order to estimate the last term in (3.19), we observe that

$$O(1)\frac{1}{H(r)}\frac{1}{r^{N-2s}}\int_{\partial^+B_r^+} t^{1-2s}(A\nabla W\cdot\nu)W\,dS = \frac{H'(r)}{H(r)}O(r)+O(r) = \mathcal{N}(r)O(1)+O(r), \tag{3.20}$$

where we used (3.4) and (3.5). Combining (3.19) and (3.20) we obtain that

$$\mathcal{N}'(r) \geq \frac{2r\left[\left(\int_{\partial^+B_r^+} t^{1-2s}\frac{|A\nabla W\cdot\nu|^2}{\mu}dS\right)\left(\int_{\partial^+B_r^+} t^{1-2s}\mu W^2 dS\right) - \left(\int_{\partial^+B_r^+} t^{1-2s}(A\nabla W\cdot\nu)W dS\right)^2\right]}{\left(\int_{\partial^+B_r^+} t^{1-2s}\mu W^2 dS\right)^2}$$
$$+\mathcal{N}(r)O(1)+O(r)+O(r^{-1+\bar\delta})\left(\mathcal{N}(r)+\frac{N-2s}{2}\right) \quad \text{as } r\to 0^+,$$

which yields (3.18) in view of (3.17). □

Proposition 3.6. *Let $\mathcal{N}$ be the function defined in* (3.9). *Then there exists $C_1>0$ such that, for every $r\in(0,R_0]$,*

$$\mathcal{N}(r)\leq C_1. \tag{3.21}$$

Moreover the limit

$$\gamma := \lim_{r\to 0^+}\mathcal{N}(r) \tag{3.22}$$

exists, is finite and nonnegative.

Proof. From Lemma 3.5, we deduce that $\mathcal{N}'(r)\geq V_2(r)$ a.e. in $(0,R_0)$, since $V_1(r)\geq 0$ as a consequence of Schwarz's inequality. Hence there exist $0<\hat R<R_0$ and $C_2>0$ such that

$$\mathcal{N}'(r)\geq -C_2 r^{-1+\bar\delta}\left(\mathcal{N}(r)+\frac{N-2s}{2}\right), \tag{3.23}$$

for a.e. $r\in(0,\hat R)$. Then

$$\frac{d}{dr}\left[\log\left(\mathcal{N}(r)+\frac{N-2s}{2}\right)\right]\geq -C_2 r^{-1+\bar\delta} \quad \text{a.e. in } (0,\hat R),$$

and, integrating the above inequality between $(r,\hat R)$ with $r<\hat R$, we obtain the upper bound

$$\mathcal{N}(r)\leq\left(\mathcal{N}(\hat R)+\frac{N-2s}{2}\right)e^{C_2\frac{\hat R^{\bar\delta}}{\bar\delta}}-\frac{N-2s}{2} \quad \text{for all } r\in(0,\hat R),$$

which yields (3.21), in view of the continuity of $\mathcal{N}$ in $(0,R_0]$. From (3.23), we derive that

$$\frac{d}{dr}\left[e^{C_2\frac{r^{\bar\delta}}{\bar\delta}}\left(\mathcal{N}(r)+\frac{N-2s}{2}\right)\right]\geq 0 \quad \text{a.e. in } (0,\hat R),$$

hence

$$r\mapsto e^{C_2\frac{r^{\bar\delta}}{\bar\delta}}\left(\mathcal{N}(r)+\frac{N-2s}{2}\right)$$

is a monotonically increasing function in $(0,\hat R)$, thus its limit as $r\to 0^+$ does exist, and the same holds true for the limit of the function $\mathcal{N}$. From Lemma 3.4 and (3.21), we can conclude that the limit $\gamma:=\lim_{r\to 0^+}\mathcal{N}(r)$ is finite and nonnegative. □

Lemma 3.7. *Let $\gamma=\lim_{r\to 0^+}\mathcal{N}(r)$. Then:*

(i) *there exists $k_1>0$ such that, for all $r\in(0,R_0)$,*

$$H(r)\leq k_1 r^{2\gamma}; \tag{3.24}$$

(ii) *for any $\sigma>0$, there exists $k_2(\sigma)>0$ such that, for all $r\in(0,R_0)$,*

$$H(r)\geq k_2(\sigma)r^{2\gamma+\sigma}.$$

Proof. (i) By (3.21), (3.23), and (3.22) $\mathcal{N}' \in L^1(0, R_0)$, then using (3.23) and (3.21)

$$\mathcal{N}(r) - \gamma = \int_0^r \mathcal{N}'(s)ds \geq -C_2 \int_0^r s^{-1+\bar{\delta}} \left(\mathcal{N}(s) + \frac{N-2s}{2}\right) ds$$
$$\geq -C_2 \left(C_1 + \tfrac{N-2s}{2}\right) \int_0^r s^{-1+\bar{\delta}} ds = -C_2 \left(C_1 + \tfrac{N-2s}{2}\right) \frac{r^{\bar{\delta}}}{\bar{\delta}} \quad \text{for all } r \in (0, R_0).$$

Using (3.5), one has

$$\frac{H'(r)}{H(r)} = \frac{2\mathcal{N}(r)}{r} + O(1) \geq \frac{2\gamma}{r} - 2C_2 \left(C_1 + \tfrac{N-2s}{2}\right) \frac{r^{-1+\bar{\delta}}}{\bar{\delta}} + O(1) \quad \text{as } r \to 0^+.$$

Integrating the above estimate and taking into account that H is continuous on $(0, R_0]$, we obtain (3.24).

(ii) Since $\gamma = \lim_{r\to 0^+} \mathcal{N}(r)$, for any $\sigma > 0$ there exists $r_\sigma > 0$ such that, for any $r \in (0, r_\sigma)$,

$$\mathcal{N}(r) < \gamma + \frac{\sigma}{2},$$

and hence

$$\frac{H'(r)}{H(r)} = \frac{2\mathcal{N}(r)}{r} + O(1) < \frac{2\gamma + \sigma}{r} + c,$$

for some positive constant c. Integrating over the interval (r, r_σ) and taking into account that H is continuous and positive in $(0, R_0]$, we prove the second statement. □

4. Blow-up analysis and local asymptotics

4.1. **Blow-up analysis.** As in Section 3, let $W \in H^1_{\Gamma^+_{R_1}}(B^+_{R_1}, t^{1-2s}\, dz)$ be a nontrivial weak solution of (2.11). For every $\lambda \in (0, R_0)$, with R_0 being as in (2.45), let us define

$$w^\lambda(z) = \frac{W(\lambda z)}{\sqrt{H(\lambda)}}. \tag{4.1}$$

We have that w^λ is a weak solution to

$$\begin{cases} -\mathrm{div}\left(t^{1-2s} A(\lambda\,\cdot)\nabla w^\lambda\right) = 0 & \text{in } B^+_{R_1/\lambda}, \\ \lim_{t\to 0^+}\left(t^{1-2s} A(\lambda\,\cdot)\nabla w^\lambda \cdot \nu\right) = \kappa_s \lambda^{2s} \tilde{h}(\lambda\cdot)\,\mathrm{Tr}\, w^\lambda & \text{on } \Gamma^-_{R_1/\lambda}, \\ w^\lambda = 0 & \text{on } \Gamma^+_{R_1/\lambda}. \end{cases} \tag{4.2}$$

Moreover we have that

$$\int_{\mathbb{S}^N_+} \theta_{N+1}^{1-2s} \mu(\lambda\theta) |w^\lambda(\theta)|^2\, dS = 1. \tag{4.3}$$

Lemma 4.1. *The family of functions* $\{w^\lambda\}_{\lambda\in(0,R_0)}$ *is bounded in* $H^1(B_1^+, t^{1-2s}dz)$.

Proof. By (3.9) and using (2.32), (2.36) and (2.37), we obtain that, for every $\lambda \in (0, R_0)$,

$$\mathcal{N}(\lambda) = \frac{\lambda^{2s-N}}{H(\lambda)} \left(\int_{B^+_\lambda} t^{1-2s} A\nabla W \cdot \nabla W\, dz - \kappa_s \int_{\Gamma^-_\lambda} \tilde{h} |\,\mathrm{Tr}\, W|^2\, dy \right)$$
$$\geq \frac{\lambda^{2s-N}}{H(\lambda)} \left[\frac{3}{8} \int_{B^+_\lambda} t^{1-2s} |\nabla W|^2\, dz - \kappa_s \tilde{S}_{N,s} \tilde{c}_{N,s,p} \lambda^{\bar{\varepsilon}} \bar{\alpha}_0 \frac{2(N-2s)}{\lambda} \int_{\partial^+ B^+_\lambda} t^{1-2s} \mu W^2 dS \right]$$
$$\geq \frac{3}{8} \int_{B_1^+} t^{1-2s} |\nabla w^\lambda|^2\, dz - 2(N-2s)\kappa_s \tilde{S}_{N,s} \tilde{c}_{N,s,p} \lambda^{\bar{\varepsilon}} \bar{\alpha}_0$$
$$\geq \frac{3}{8} \int_{B_1^+} t^{1-2s} |\nabla w^\lambda|^2\, dz - \frac{N-2s}{4},$$

which together with (3.21) implies that $\left\{ \|\nabla w^\lambda\|_{L^2(B_1^+, t^{1-2s}dz)} \right\}_{\lambda\in(0,R_0)}$ is bounded. From this and (4.3), the boundedness of $\{w^\lambda\}_{\lambda\in(0,R_0)}$ in $H^1(B_1^+, t^{1-2s}dz)$ follows by Lemma 2.3. □

We are going to prove strong convergence in $H^1(B_1^+, t^{1-2s}dz)$ of $\{w^\lambda\}$ along a proper vanishing sequence of λ's; to this aim, we first need to establish the following doubling properties.

Lemma 4.2. *There exists $C_3 > 0$ such that*

$$\frac{1}{C_3} H(\lambda) \le H(R\lambda) \le C_3 H(\lambda), \tag{4.4}$$

$$\int_{B_R^+} t^{1-2s} |\nabla w^\lambda|^2 dz \le C_3 2^{N-2s} \int_{B_1^+} t^{1-2s} |\nabla w^{R\lambda}|^2 dz, \tag{4.5}$$

and

$$\int_{B_R^+} t^{1-2s} |w^\lambda|^2 dz \le C_3 2^{N+2-2s} \int_{B_1^+} t^{1-2s} |w^{R\lambda}|^2 dz. \tag{4.6}$$

for any $\lambda < R_0/2$ and $R \in [1,2]$.

Proof. From (3.5) we deduce that, for a.e. $r \in (0, R_0)$,

$$\frac{H'(r)}{H(r)} = \frac{2\mathcal{N}(r)}{r} + O(1) \quad \text{as } r \to 0^+.$$

Hence there exist a positive constant $C > 0$ and $\tilde{R}_0 < R_0$ such that, for all $r \in (0, \tilde{R}_0)$,

$$-C - \frac{N-2s}{r} \le \frac{H'(r)}{H(r)} \le C + \frac{2C_1}{r},$$

where we used (3.16) and (3.21). Integrating the above inequalities over the interval $(\lambda, R\lambda)$, with $R \in (1,2]$ and $\lambda < \tilde{R}_0/R$, we obtain that

$$2^{-(N-2s)} e^{-C\frac{\tilde{R}_0}{R}(R-1)} \le \frac{H(R\lambda)}{H(\lambda)} \le 4^{C_1} e^{C\frac{\tilde{R}_0}{R}(R-1)}. \tag{4.7}$$

The above chain of inequalities trivially extends to the case $R = 1$. Estimate (4.4) follows from (4.7) and the fact that H is continuous and strictly positive in $(0, R_0]$ (Lemmas 3.1 and 3.2). By scaling and (4.4), we easily deduce (4.5) and (4.6) (see [9] for details in a similar situation). □

Lemma 4.3. *Let w^λ be as in (4.1), with $\lambda \in (0, R_0)$. Then there exist $M > 0$ and $\lambda_0 > 0$ such that, for any $\lambda \in (0, \lambda_0)$, there exists $R_\lambda \in [1,2]$ such that*

$$\int_{\partial^+ B_{R_\lambda}^+} t^{1-2s} |\nabla w^\lambda|^2 \, dS \le M \int_{B_{R_\lambda}^+} t^{1-2s} |\nabla w^\lambda|^2 \, dz.$$

Proof. We recall that, by Lemma 4.1, the set $\{w^\lambda\}_{\lambda \in (0,R_0)}$ is bounded in $H^1(B_1^+, t^{1-2s} dz)$ and

$$w^\lambda \in H^1_{\Gamma_1^+}(B_1^+, t^{1-2s}\, dz) \quad \text{for all } \lambda \in (0, R_0). \tag{4.8}$$

Moreover, by Lemma 4.2, we have that $\{w^\lambda\}_{\lambda \in (0,R_0/2)}$ is bounded in $H^1(B_2^+, t^{1-2s} dz)$, hence

$$\limsup_{\lambda \to 0^+} \int_{B_2^+} t^{1-2s} |\nabla w^\lambda|^2 \, dz < +\infty. \tag{4.9}$$

For every $\lambda \in (0, R_0/2)$, let

$$f_\lambda(r) = \int_{B_r^+} t^{1-2s} |\nabla w^\lambda|^2 \, dz.$$

Then f_λ is absolutely continuous in $[0,2]$ with distributional derivative given by

$$f'_\lambda(r) = \int_{\partial^+ B_r^+} t^{1-2s} |\nabla w^\lambda|^2 \, dS \quad \text{for almost every } r \in (0,2).$$

Let us suppose by contradiction that for any $M > 0$ there exists a sequence $\lambda_n \to 0^+$ such that

$$\int_{\partial^+ B_r^+} t^{1-2s} |\nabla w^{\lambda_n}|^2 \, dS > M \int_{B_r^+} t^{1-2s} |\nabla w^{\lambda_n}|^2 \, dz$$

for all $r \in [1,2]$ and $n \in \mathbb{N}$, i.e.

$$f'_{\lambda_n}(r) > M f_{\lambda_n}(r) \tag{4.10}$$

for a.e. $r \in (1,2)$ and any $n \in \mathbb{N}$. Integrating (4.10) over $[1,2]$, we obtain that, for any $n \in \mathbb{N}$, $f_{\lambda_n}(2) > e^M f_{\lambda_n}(1)$, and hence

$$\liminf_{n \to +\infty} f_{\lambda_n}(1) \le \limsup_{n \to +\infty} f_{\lambda_n}(1) \le e^{-M} \limsup_{n \to +\infty} f_{\lambda_n}(2),$$

which implies that

$$\liminf_{\lambda\to 0^+} f_\lambda(1) \le e^{-M} \limsup_{\lambda\to 0^+} f_\lambda(2), \tag{4.11}$$

for all $M > 0$. From (4.11) and (4.9), letting $M \to +\infty$ we deduce that $\liminf_{\lambda\to 0^+} f_\lambda(1) = 0$. Then there exist a sequence $\tilde\lambda_n \to 0^+$ and some $w \in H^1(B_1^+, t^{1-2s}dz)$ such that $w^{\tilde\lambda_n} \rightharpoonup w$ in $H^1(B_1^+, t^{1-2s}dz)$ with

$$\lim_{n\to+\infty} \int_{B_1^+} t^{1-2s}|\nabla w^{\tilde\lambda_n}|^2 dz = 0.$$

However, by compactness of trace map $H^1(B_1^+, t^{1-2s}dz) \hookrightarrow\hookrightarrow L^2(\partial^+ B_1^+, t^{1-2s}dS)$, (4.3), (2.25), and weak lower semicontinuity of norms, we necessarily have that

$$\int_{B_1^+} t^{1-2s}|\nabla w|^2 dz = 0 \qquad \text{and} \qquad \int_{\partial^+ B_1^+} t^{1-2s} w^2 dS = 1.$$

Hence there exists $c \in \mathbb{R}$ such that $w \equiv c$ in B_1^+ and $c \neq 0$. Since $H^1_{\Gamma_1^+}(B_1^+, t^{1-2s}\,dz)$ is weakly closed in $H^1(B_1^+, t^{1-2s}\,dz)$, from (4.8) we deduce that $w \equiv c \in H^1_{\Gamma_1^+}(B_1^+, t^{1-2s}\,dz)$, so that $0 = \operatorname{Tr} w\big|_{\Gamma_1^+} = c$, a contradiction. □

Lemma 4.4. *Let w^λ and R_λ be as in the statement of Lemma 4.3. Then there exists $\overline{M} > 0$ such that*

$$\int_{\partial^+ B_1^+} t^{1-2s}|\nabla w^{\lambda R_\lambda}|^2\, dz \le \overline{M}$$

for any $\lambda \in (0, \min\{\lambda_0, R_0/2\})$.

Proof. We observe that, by scaling and (4.1),

$$\int_{\partial^+ B_1^+} t^{1-2s}|\nabla w^{\lambda R_\lambda}|^2\, dS = \frac{R_\lambda^{1-N+2s} H(\lambda)}{H(\lambda R_\lambda)} \int_{\partial^+ B_{R_\lambda}^+} t^{1-2s}|\nabla w^\lambda|^2\, dS,$$

so that, in view of Lemmas 4.2, 4.3, and 4.1, we have that

$$\begin{aligned}\int_{\partial^+ B_1^+} t^{1-2s}|\nabla w^{\lambda R_\lambda}|^2\, dS &\le 2C_3 M \int_{B_{R_\lambda}^+} t^{1-2s}|\nabla w^\lambda|^2\, dz \\ &\le 2^{1+N-2s} M C_3^2 \int_{B_1^+} t^{1-2s}|\nabla w^{\lambda R_\lambda}|^2\, dz \le \overline{M} < +\infty.\end{aligned}$$

The proof is thereby complete. □

Proposition 4.5. *Let $W \in H^1_{\Gamma_{R_1}^+}(B_{R_1}^+, t^{1-2s}\,dz)$, $W \not\equiv 0$, be a nontrivial weak solution of (2.11). Let γ be as in Proposition 3.6. Then*

(i) *there exists $k_0 \in \mathbb{N}$ such that $\gamma = s + k_0$;*

(ii) *for any sequence $\lambda_n \to 0^+$, there exist a subsequence $\{\lambda_{n_k}\}$ and an eigenfunction ψ of problem (1.11) associated to the eigenvalue $\mu_{k_0} = (k_0+s)(k_0+N-s)$ such that $\|\psi\|_{L^2(\mathbb{S}^N_+, \theta_{N+1}^{1-2s} dS)} = 1$ and*

$$w^{\lambda_{n_k}}(z) = \frac{W(\lambda_{n_k} z)}{\sqrt{H(\lambda_{n_k})}} \to |z|^\gamma \psi\left(\frac{z}{|z|}\right)$$

strongly in $H^1(B_1^+, t^{1-2s}dz)$.

Proof. Let $w^\lambda \in H^1_{\Gamma_1^+}(B_1^+, t^{1-2s}\,dz)$ be as in (4.1) and R_λ as in Lemma 4.3. From Lemma 4.1 we deduce that the set $\{w^{\lambda R_\lambda}\}_{\lambda \in (0, \min\{\lambda_0, R_0/2\})}$ is bounded in $H^1(B_1^+, t^{1-2s}dz)$. Let us consider a sequence $\lambda_n \to 0^+$. Then there exist a subsequence $\{\lambda_{n_k}\}$ and $w \in H^1_{\Gamma_1^+}(B_1^+, t^{1-2s}\,dz)$ such that $w^{\lambda_{n_k} R_{\lambda_{n_k}}} \rightharpoonup w$ weakly in $H^1(B_1^+, t^{1-2s}dz)$. Moreover we have that

$$\int_{\partial^+ B_1^+} t^{1-2s} w^2 dS = 1 \tag{4.12}$$

by compactness of trace map $H^1(B_1^+, t^{1-2s}dz) \hookrightarrow\hookrightarrow L^2(\partial^+ B_1^+, t^{1-2s}dS)$, (4.3), and (2.25). This allows us to conclude that w is non-trivial.

We now claim strong convergence

$$w^{\lambda_{n_k} R_{\lambda_{n_k}}} \to w \quad \text{in } H^1(B_1^+, t^{1-2s}dz). \tag{4.13}$$

We note that $w^{\lambda_{n_k} R_{\lambda_{n_k}}}$ weakly solves (4.2) with $\lambda = \lambda_{n_k} R_{\lambda_{n_k}}$. Since $B_1^+ \subset B^+_{R_{1/(\lambda_{n_k} R_{\lambda_{n_k}})}}$ for sufficiently large k, we then have that

$$\begin{aligned}\int_{B_1^+} & t^{1-2s} A(\lambda_{n_k} R_{\lambda_{n_k}} y) \nabla w^{\lambda_{n_k} R_{\lambda_{n_k}}}(z) \cdot \nabla \Phi(z)\, dz \\ &= \kappa_s (\lambda_{n_k} R_{\lambda_{n_k}})^{2s} \int_{\Gamma_1^-} \tilde{h}(\lambda_{n_k} R_{\lambda_{n_k}} y) \operatorname{Tr} w^{\lambda_{n_k} R_{\lambda_{n_k}}}(y) \operatorname{Tr} \Phi(y)\, dy \\ &\quad + \int_{\partial^+ B_1^+} \big(t^{1-2s} A(\lambda_{n_k} R_{\lambda_{n_k}} y) \nabla w^{\lambda_{n_k} R_{\lambda_{n_k}}}(z) \cdot \nu\big) \Phi(z)\, dS\end{aligned} \tag{4.14}$$

for sufficiently large k and for every $\Phi \in C_c^\infty(\overline{B_1^+} \setminus \Gamma_1^+)$, hence by density for every $\Phi \in H^1_{\Gamma_1^+}(B_1^+, t^{1-2s}dz)$. We are going to pass to the limit in (4.14). To this aim, we observe that (2.20) implies that

$$\begin{aligned}&\left| \int_{B_1^+} t^{1-2s} (A(\lambda y) \nabla w^\lambda(z) - \nabla w(z)) \cdot \nabla \Phi(z)\, dx \right| \\ &\leq \left| \int_{B_1^+} t^{1-2s} \nabla (w^\lambda - w) \cdot \nabla \Phi\, dz \right| + C\lambda \int_{B_1^+} t^{1-2s} |\nabla w^\lambda| |\nabla \Phi|\, dz \\ &\leq \left| \int_{B_1^+} t^{1-2s} \nabla (w^\lambda - w) \cdot \nabla \Phi\, dz \right| + C\lambda \left(\int_{B_1^+} t^{1-2s} |\nabla w^\lambda|^2\, dz \right)^{1/2} \left(\int_{B_1^+} t^{1-2s} |\nabla \Phi|^2\, dz \right)^{1/2}\end{aligned} \tag{4.15}$$

for some $C > 0$ and for sufficiently small λ, and

$$\begin{aligned}&\lambda^{2s} \left| \int_{\Gamma_1^-} \tilde{h}(\lambda y) \operatorname{Tr} w^\lambda(y) \operatorname{Tr} \Phi(y)\, dy \right| \\ &\leq \lambda^{2s} \left(\int_{B_1'} |\operatorname{Tr} w^\lambda(y)|^{2^*(s)}\, dy \right)^{\frac{1}{2^*(s)}} \left(\int_{B_1'} |\operatorname{Tr} \Phi(y)|^{2^*(s)}\, dy \right)^{\frac{1}{2^*(s)}} \left(\int_{\Gamma_1^-} |\tilde{h}(\lambda y)|^{\frac{N}{2s}}\, dy \right)^{\frac{2s}{N}} \\ &= O(1) \left(\int_{\Gamma_\lambda^-} |\tilde{h}(y)|^{\frac{N}{2s}} dy \right)^{\frac{2s}{N}} = o(1) \quad \text{as } \lambda \to 0^+,\end{aligned} \tag{4.16}$$

from Hölder's inequality, Lemmas 2.4, 4.1, and (4.3), since $\mu(\lambda y) \geq 1/4$ for all $\lambda \leq R_0$. Taking $\lambda = \lambda_{n_k} R_{\lambda_{n_k}}$ in (4.15) and (4.16), letting $k \to \infty$, and recalling that $w^{\lambda_{n_k} R_{\lambda_{n_k}}} \rightharpoonup w$ weakly in $H^1(B_1^+, t^{1-2s}dz)$, we obtain that

$$\lim_{k \to \infty} \int_{B_1^+} t^{1-2s} A(\lambda_{n_k} R_{\lambda_{n_k}} y) \nabla w^{\lambda_{n_k} R_{\lambda_{n_k}}}(z) \cdot \nabla \Phi(z)\, dz = \int_{B_1^+} t^{1-2s} \nabla w \cdot \nabla \Phi\, dz \tag{4.17}$$

and

$$\lim_{k \to \infty} (\lambda_{n_k} R_{\lambda_{n_k}})^{2s} \int_{\Gamma_1^-} \tilde{h}(\lambda_{n_k} R_{\lambda_{n_k}} y) \operatorname{Tr} w^{\lambda_{n_k} R_{\lambda_{n_k}}}(y) \operatorname{Tr} \Phi(y)\, dy = 0. \tag{4.18}$$

Thanks to (2.20), we have that

$$\begin{aligned}&\int_{\partial^+ B_1^+} \big(t^{1-2s} A(\lambda y) \nabla w^\lambda(z) \cdot \nu\big) \Phi(z)\, dS \\ &= \int_{\partial^+ B_1^+} t^{1-2s} \frac{\partial w^\lambda}{\partial \nu} \Phi\, dS + \int_{\partial^+ B_1^+} t^{1-2s} (A(\lambda y) - \mathrm{Id}_N) \nabla w^\lambda(z) \cdot \nu\, \Phi(z)\, dS \\ &= \int_{\partial^+ B_1^+} t^{1-2s} \frac{\partial w^\lambda}{\partial \nu} \Phi\, dS + O(\lambda) \left(\int_{\partial^+ B_1^+} t^{1-2s} |\nabla w^\lambda|^2\, dS \right)^{1/2} \left(\int_{\partial^+ B_1^+} t^{1-2s} \Phi^2\, dS \right)^{1/2}.\end{aligned} \tag{4.19}$$

Moreover, from Lemma 4.4, up to a further subsequence, we have that

$$\frac{\partial w^{\lambda_{n_k} R_{\lambda_{n_k}}}}{\partial \nu} \rightharpoonup f \quad \text{weakly in } L^2(\partial^+ B_1^+, t^{1-2s}dS) \tag{4.20}$$

for some $f \in L^2(\partial^+ B_1^+, t^{1-2s} dS)$. Then, taking $\lambda = \lambda_{n_k} R_{\lambda_{n_k}}$ in (4.19), letting $k \to \infty$, and taking into account Lemma 4.4, we obtain that

$$\lim_{k\to\infty} \int_{\partial^+ B_1^+} \big(t^{1-2s} A(\lambda_{n_k} R_{\lambda_{n_k}} y) \nabla w^{\lambda_{n_k} R_{\lambda_{n_k}}}(z) \cdot \nu\big) \Phi(z)\, dS = \int_{\partial^+ B_1^+} t^{1-2s} f \Phi \, dS. \tag{4.21}$$

Hence, passing to the limit as $k \to \infty$ in (4.14) and combining (4.17), (4.18), and (4.21), we find that

$$\int_{B_1^+} t^{1-2s} \nabla w \cdot \nabla \Phi \, dz = \int_{\partial^+ B_1^+} t^{1-2s} f \Phi \, dS \quad \text{for any } \Phi \in H^1_{\Gamma_1^+}(B_1^+, t^{1-2s} dz). \tag{4.22}$$

On the other hand, if we take $\Phi = w^{\lambda_{n_k} R_{\lambda_{n_k}}}$ in (4.14), we have that

$$\begin{aligned} &\int_{B_1^+} t^{1-2s} A(\lambda_{n_k} R_{\lambda_{n_k}} y) \nabla w^{\lambda_{n_k} R_{\lambda_{n_k}}}(z) \cdot \nabla w^{\lambda_{n_k} R_{\lambda_{n_k}}}(z)\, dz \\ &\qquad = \kappa_s (\lambda_{n_k} R_{\lambda_{n_k}})^{2s} \int_{\Gamma_1^-} \tilde{h}(\lambda_{n_k} R_{\lambda_{n_k}} y) |\operatorname{Tr} w^{\lambda_{n_k} R_{\lambda_{n_k}}}(y)|^2 \, dy \\ &\qquad\quad + \int_{\partial^+ B_1^+} \big(t^{1-2s} A(\lambda_{n_k} R_{\lambda_{n_k}} z) \nabla w^{\lambda_{n_k} R_{\lambda_{n_k}}}(z) \cdot \nu\big) w^{\lambda_{n_k} R_{\lambda_{n_k}}}(z)\, dS, \end{aligned}$$

hence, by (2.20), arguing as in (4.19) and (4.16) and using Lemma 4.4 and (4.20), we obtain that

$$\begin{aligned} \lim_{k\to\infty} \int_{B_1^+} |\nabla w^{\lambda_{n_k} R_{\lambda_{n_k}}}|^2 &= \lim_{k\to\infty} \int_{B_1^+} t^{1-2s} A(\lambda_{n_k} R_{\lambda_{n_k}} y) \nabla w^{\lambda_{n_k} R_{\lambda_{n_k}}}(z) \cdot \nabla w^{\lambda_{n_k} R_{\lambda_{n_k}}}(z)\, dz \\ &= \lim_{k\to\infty} \int_{\partial^+ B_1^+} t^{1-2s} \frac{\partial w^{\lambda_{n_k} R_{\lambda_{n_k}}}}{\partial \nu} w^{\lambda_{n_k} R_{\lambda_{n_k}}} \, dS \\ &= \int_{\partial^+ B_1^+} t^{1-2s} f w \, dS = \int_{B_1^+} t^{1-2s} |\nabla w|^2, \end{aligned} \tag{4.23}$$

where we used the compactness of the trace operator $H^1(B_1^+, t^{1-2s} dz) \hookrightarrow\hookrightarrow L^2(\partial^+ B_1^+, t^{1-2s} dS)$ and (4.22) with $\Phi = w$. The weak convergence $w^{\lambda_{n_k} R_{\lambda_{n_k}}} \rightharpoonup w$ in $H^1(B_1^+, t^{1-2s} dz)$ together with (4.23) imply (4.13).

For every $k \in \mathbb{N}$ and $r \in (0,1]$, let us define

$$\begin{aligned} E_k(r) = \frac{1}{r^{N-2s}} \Bigg[&\int_{B_r^+} t^{1-2s} A(\lambda_{n_k} R_{\lambda_{n_k}} y) \nabla w^{\lambda_{n_k} R_{\lambda_{n_k}}} \cdot \nabla w^{\lambda_{n_k} R_{\lambda_{n_k}}} \, dz \\ &- \kappa_s \lambda_{n_k}^{2s} R_{\lambda_{n_k}}^{2s} \int_{\Gamma_r^-} \tilde{h}(\lambda_{n_k} R_{\lambda_{n_k}} y) |\operatorname{Tr} w^{\lambda_{n_k} R_{\lambda_{n_k}}}|^2 dy \Bigg] \end{aligned}$$

and

$$H_k(r) = \frac{1}{r^{N+1-2s}} \int_{\partial^+ B_r^+} t^{1-2s} \mu(\lambda_{n_k} R_{\lambda_{n_k}} z) |w^{\lambda_{n_k} R_{\lambda_{n_k}}}(z)|^2 dS.$$

We also define, for any $r \in (0,1]$,

$$E_w(r) = \frac{1}{r^{N-2s}} \int_{B_r^+} t^{1-2s} |\nabla w(z)|^2 dz \tag{4.24}$$

and

$$H_w(r) = \frac{1}{r^{N+1-2s}} \int_{\partial^+ B_r^+} t^{1-2s} w^2(z)\, dS. \tag{4.25}$$

By scaling, one can easily verify that

$$\mathcal{N}_k(r) := \frac{E_k(r)}{H_k(r)} = \frac{E(\lambda_{n_k} R_{\lambda_{n_k}} r)}{H(\lambda_{n_k} R_{\lambda_{n_k}} r)} = \mathcal{N}(\lambda_{n_k} R_{\lambda_{n_k}} r) \quad \text{for all } r \in (0,1]. \tag{4.26}$$

From (4.13), (2.20), and (4.16), it follows that, for any fixed $r \in (0,1]$,

$$E_k(r) \to E_w(r). \tag{4.27}$$

On the other hand, by compactness of the trace operator and (2.25), we also have, for any fixed $r \in (0,1]$,

$$H_k(r) \to H_w(r). \tag{4.28}$$

In order to prove that H_w is strictly positive, we argue by contradiction and assume that there exists $r \in (0,1]$ such that $H_w(r) = 0$; then r is a minimum point for H_w and hence, arguing as in Lemma 3.1, we obtain that necessarily $0 = H_w'(r) = 2r^{2s-N-1} \int_{B_r^+} t^{1-2s} |\nabla w(z)|^2 dz$ and hence w is constant in B_r^+. From Lemma 2.3 we conclude that $w \equiv 0$ in B_r^+, which implies that $w \equiv 0$ in B_1^+ from classical unique continuation principles for second order elliptic equations, thus contradicting (4.12).

Hence $H_w(r) > 0$ for all $r \in (0,1]$, thus the function

$$\mathcal{N}_w : (0,1] \to \mathbb{R}, \quad \mathcal{N}_w(r) := \frac{E_w(r)}{H_w(r)}$$

is well defined and one can easily prove that it belongs to $W^{1,1}_{\rm loc}((0,1])$. From (4.26), (4.27), (4.28), and Proposition 3.6, we deduce that

$$\mathcal{N}_w(r) = \lim_{k\to\infty} \mathcal{N}(\lambda_{n_k} R_{\lambda_{n_k}} r) = \gamma \tag{4.29}$$

for all $r \in (0,1]$. Therefore $\mathcal{N}_w$ is constant in $(0,1]$ and hence

$$\mathcal{N}_w'(r) = 0 \quad \text{for any } r \in (0,1). \tag{4.30}$$

Recalling the equation satisfied by w, i.e. (4.22), and arguing as in Lemma 3.5 with $A = \mathrm{Id}_N$ and $\tilde{h} \equiv 0$, we can prove that, for a.e. $r \in (0,1)$,

$$\mathcal{N}_w'(r) \geq \frac{2r\left[\left(\int_{\partial^+ B_r^+} t^{1-2s} |\partial_\nu w|^2 dS\right)\left(\int_{\partial^+ B_r^+} t^{1-2s} w^2 dS\right) - \left(\int_{\partial^+ B_r^+} t^{1-2s} \partial_\nu w \, w \, dS\right)^2\right]}{\left(\int_{\partial^+ B_r^+} t^{1-2s} w^2 dS\right)^2}. \tag{4.31}$$

Combining (4.30) and (4.31) with Schwarz's inequality, we obtain that, for a.e. $r \in (0,1)$,

$$\left(\int_{\partial^+ B_r^+} t^{1-2s} |\partial_\nu w|^2 dS\right)\left(\int_{\partial^+ B_r^+} t^{1-2s} w^2 dS\right) - \left(\int_{\partial^+ B_r^+} t^{1-2s} \partial_\nu w \, w \, dS\right)^2 = 0.$$

Hence, for a.e. $r \in (0,1)$, w and $\partial_\nu w$ have the same direction as vectors in $L^2(\partial^+ B_r^+, t^{1-2s} dS)$, so that there exists a function $\eta = \eta(r)$, defined a.e. in $(0,1)$, such that $\partial_\nu w(r\theta) = \eta(r) w(r\theta)$ for a.e. $r \in (0,1)$ and for all $\theta \in \mathbb{S}^N_+$. It is easy to verify that $\eta(r) = \frac{H_w'(r)}{2H_w(r)}$ for a.e $r \in (0,1)$, so that $\eta \in L^1_{\rm loc}((0,1])$. After integration we obtain that

$$w(r\theta) = e^{\int_1^r \eta(s) ds} w(\theta) = g(r)\psi(\theta), \quad r \in (0,1), \ \theta \in \mathbb{S}^N_+, \tag{4.32}$$

where $g(r) = e^{\int_1^r \eta(s) ds}$ and $\psi = w\big|_{\mathbb{S}^N_+}$. We observe that (4.12) implies that

$$\|\psi\|_{L^2(\mathbb{S}^N_+, \theta_{N+1}^{1-2s} dS)} = 1. \tag{4.33}$$

From the fact that $w \in H^1_{\Gamma_1^+}(B_1^+, t^{1-2s}\, dz)$ it follows that $\psi \in \mathcal{H}_0$; moreover, plugging (4.32) into (4.22) we obtain that ψ satisfies (1.12) for some $\mu \in \mathbb{R}$, so that ψ is an eigenfunction of (1.11). Recalling (1.13) and letting $k_0 \in \mathbb{N}$ be such that $\mu = \mu_{k_0} = (k_0 + s)(k_0 + N - s)$, we can rewrite the equation $-\mathrm{div}\left(t^{1-2s}\nabla w\right) = 0$ in polar coordinates exploiting [13, Lemma 2.1], thus obtaining, for all $r \in (0,1)$ and $\theta \in \mathbb{S}^N_+$,

$$\begin{aligned} 0 &= \frac{1}{r^N}(r^{N+1-2s} g')' \theta_{N+1}^{1-2s} \psi(\theta) + r^{-1-2s} g(r) \mathrm{div}_{\mathbb{S}^N}(\theta_{N+1}^{1-2s} \nabla_{\mathbb{S}^N} \psi(\theta)) \\ &= \frac{1}{r^N}(r^{N+1-2s} g')' \theta_{N+1}^{1-2s} \psi(\theta) - r^{-1-2s} g(r) \theta_{N+1}^{1-2s} \mu_{k_0} \psi(\theta). \end{aligned}$$

Then $g(r)$ solves the equation

$$-\frac{1}{r^N}(r^{N+1-2s} g')' + \mu_{k_0} r^{-1-2s} g(r) = 0 \quad \text{in } (0,1)$$

i.e.

$$-g''(r) - \frac{N+1-2s}{r} g'(r) + \frac{\mu_{k_0}}{r^2} g(r) = 0 \quad \text{in } (0,1).$$

Hence $g(r)$ is of the form

$$g(r) = c_1 r^{k_0+s} + c_2 r^{s-N-k_0}$$

for some $c_1, c_2 \in \mathbb{R}$. Since $w \in H^1(B_1^+, t^{1-2s}\,dz)$ and the function $|z|^{-1}|z|^{s-N-k_0}\psi\big(\frac{z}{|z|}\big) \notin L^2(B_1^+, t^{1-2s}\,dz)$, from Lemma 2.3 we deduce that necessarily $c_2 = 0$ and $g(r) = c_1 r^{k_0+s}$. Moreover, from $g(1) = 1$, we obtain that $c_1 = 1$ and then

$$w(r\theta) = r^{k_0+s}\psi(\theta), \quad \text{for all } r \in (0,1) \text{ and } \theta \in \mathbb{S}^N_+. \tag{4.34}$$

Let us now consider the sequence $\{w^{\lambda_{n_k}}\}$. Up to a further subsequence still denoted by $\{w^{\lambda_{n_k}}\}$, we may suppose that $w^{\lambda_{n_k}} \rightharpoonup \overline{w}$ weakly in $H^1(B_1^+, t^{1-2s}\,dz)$ for some $\overline{w} \in H^1(B_1^+, t^{1-2s}\,dz)$ and that $R_{\lambda_{n_k}} \to \overline{R}$ for some $\overline{R} \in [1,2]$.

Strong convergence of $w^{\lambda_{n_k} R_{\lambda_{n_k}}}$ in $H^1(B_1^+, t^{1-2s}dz)$ implies that, up to a subsequence, both $w^{\lambda_{n_k} R_{\lambda_{n_k}}}$ and $|\nabla w^{\lambda_{n_k} R_{\lambda_{n_k}}}|$ are a.e. dominated by a $L^2(B_1^+, t^{1-2s}dz)$-function uniformly with respect to k. Moreover, by (4.4), up to a further subsequence, we may assume that the limit

$$\ell := \lim_{k\to+\infty} \frac{H(\lambda_{n_k} R_{\lambda_{n_k}})}{H(\lambda_{n_k})}$$

exists and is finite, with $\ell > 0$. Then, by Dominated Convergence Theorem, we have

$$\begin{aligned}
\lim_{k\to+\infty} \int_{B_1^+} t^{1-2s} w^{\lambda_{n_k}}(z) v(z) dz &= \lim_{k\to+\infty} R_{\lambda_{n_k}}^{N+2-2s} \int_{B^+_{1/R_{\lambda_{n_k}}}} t^{1-2s} w^{\lambda_{n_k}}(R_{\lambda_{n_k}} z) v(R_{\lambda_{n_k}} z) dz \\
&= \lim_{k\to+\infty} R_{\lambda_{n_k}}^{N+2-2s} \sqrt{\frac{H(\lambda_{n_k} R_{\lambda_{n_k}})}{H(\lambda_{n_k})}} \int_{B_1^+} t^{1-2s} \chi_{B^+_{1/R_{\lambda_{n_k}}}}(z)\, w^{\lambda_{n_k} R_{\lambda_{n_k}}}(z) v(R_{\lambda_{n_k}} z) dz \\
&= \overline{R}^{N+2-2s} \sqrt{\ell} \int_{B_1^+} t^{1-2s} \chi_{B^+_{1/\overline{R}}}(z) w(z) v(\overline{R} z) dz \\
&= \overline{R}^{N+2-2s} \sqrt{\ell} \int_{B^+_{1/\overline{R}}} t^{1-2s} w(z) v(\overline{R} z) dz = \sqrt{\ell} \int_{B_1^+} t^{1-2s} w(z/\overline{R}) v(z) dz
\end{aligned}$$

for any $v \in C^\infty(\overline{B_1^+})$. By density, the above convergence actually holds for all $v \in L^2(B_1^+, t^{1-2s}dz)$. This proves that $w^{\lambda_{n_k}} \rightharpoonup \sqrt{\ell} w(\cdot/\overline{R})$ weakly in $L^2(B_1^+, t^{1-2s}dz)$. Since we know that $w^{\lambda_{n_k}} \rightharpoonup \overline{w}$ weakly in $H^1(B_1^+, t^{1-2s}\,dz)$, we conclude that $\overline{w} = \sqrt{\ell} w(\cdot/\overline{R})$ and then $w^{\lambda_{n_k}} \rightharpoonup \sqrt{\ell} w(\cdot/\overline{R})$. Moreover

$$\begin{aligned}
\lim_{k\to+\infty} \int_{B_1^+} t^{1-2s} |\nabla w^{\lambda_{n_k}}(z)|^2 dz &= \lim_{k\to+\infty} R_{\lambda_{n_k}}^{N+2-2s} \int_{B^+_{1/R_{\lambda_{n_k}}}} t^{1-2s} |\nabla w^{\lambda_{n_k}}(R_{\lambda_{n_k}} z)|^2 dz \\
&= \lim_{k\to+\infty} R_{\lambda_{n_k}}^{N-2s} \frac{H(\lambda_{n_k} R_{\lambda_{n_k}})}{H(\lambda_{n_k})} \int_{B_1^+} t^{1-2s} \chi_{B^+_{1/R_{\lambda_{n_k}}}} |\nabla w^{\lambda_{n_k} R_{\lambda_{n_k}}}(z)|^2 dz \\
&= \overline{R}^{N-2s} \ell \int_{B_1^+} t^{1-2s} \chi_{B^+_{1/\overline{R}}}(z) |\nabla w(z)|^2 dz = \overline{R}^{N-2s} \ell \int_{B^+_{1/\overline{R}}} t^{1-2s} |\nabla w(z)|^2 dz \\
&= \int_{B_1^+} t^{1-2s} \Big|\sqrt{\ell} \nabla \Big(w\Big(\frac{z}{\overline{R}}\Big)\Big)\Big|^2 dz.
\end{aligned}$$

This shows that $w^{\lambda_{n_k}} \to \overline{w} = \sqrt{\ell} w(\cdot/\overline{R})$ strongly in $H^1(B_1^+, t^{1-2s}dz)$.

By (4.34) w is homogeneous of degree $k_0 + s$, hence $\overline{w} = \sqrt{\ell}\, \overline{R}^{-k_0-s} w$. Furthermore (4.3) and the strong convergence $w^{\lambda_{n_k}} \to \overline{w}$ in $L^2(\partial^+ B_1^+, t^{1-2s}dS)$ imply that

$$1 = \int_{\partial^+ B_1^+} t^{1-2s} \overline{w}^2\, dS = \ell\, \overline{R}^{-2k_0-2s} \int_{\partial^+ B_1^+} t^{1-2s} w^2\, dS = \ell\, \overline{R}^{-2k_0-2s}$$

in view of (4.12), thus implying that $\overline{w} = w$.

It remains to prove part (i). By (4.34), (4.33) and the fact that ψ is an eigenfunction of (1.11) with associated eigenvalue $\mu_{k_0} = (k_0+s)(k_0+N-s)$, we have that

$$\int_{B_r^+} t^{1-2s} |\nabla w(z)|^2 dz = \frac{r^{N+2k_0}}{N+2k_0} \big((k_0+s)^2 + \mu_{k_0}\big) = (k_0+s) r^{N+2k_0}$$

and

$$\int_{\partial^+ B_r^+} t^{1-2s} w^2 dS = r^{N+1-2s} \int_{\mathbb{S}^N_+} \theta_{N+1}^{1-2s} w^2(r\theta)\, dS = r^{N+2k_0+1}.$$

Therefore, by (4.24), (4.25) and (4.29), it follows that

$$\gamma = \mathcal{N}_w(r) = \frac{E_w(r)}{H_w(r)} = \frac{r\int_{B_r^+} t^{1-2s}|\nabla w(z)|^2 dz}{\int_{\partial^+ B_r^+} t^{1-2s} w^2 dS} = k_0 + s.$$

This completes the proof. □

To complete the blow-up analysis and detect the sharp asymptotic behaviour of W at 0, it remains to describe the behavior of $H(\lambda)$ as $\lambda \to 0^+$.

Lemma 4.6. *Let $\gamma = \lim_{r\to 0}\mathcal{N}(r)$ be as in Proposition 3.6. Then the limit $\lim_{r\to 0^+} r^{-2\gamma}H(r)$ exists and is finite.*

Proof. Thanks to (3.24), it is enough to show that the limit exists. From (3.5) we deduce that, a.e. in $(0, R_0)$,

$$\frac{d}{dr}\frac{H(r)}{r^{2\gamma}} = \frac{H'(r)}{r^{2\gamma}} - 2\gamma\frac{H(r)}{r^{2\gamma+1}} = \frac{2}{r^{2\gamma+1}}\left(E(r) + H(r)O(r) - \gamma H(r)\right) = \frac{2H(r)}{r^{2\gamma+1}}\left(\mathcal{N}(r) - \gamma + O(r)\right) = \frac{2H(r)}{r^{2\gamma+1}}\left(\int_0^r \mathcal{N}'(s)\, ds + O(r)\right) \tag{4.35}$$

as $r \to 0^+$. Using the notation of Lemma 3.6, we can write $\mathcal{N}' = \alpha_1 + \alpha_2$ in $(0, \hat{R})$, with

$$\alpha_1(r) = \mathcal{N}'(r) + C_2 r^{-1+\bar{\delta}}\left(C_1 + \frac{N-2s}{2}\right) \quad \text{and} \quad \alpha_2(r) = -C_4 r^{-1+\bar{\delta}},$$

where $\bar{\delta} \in (0, 1]$ has been defined in (3.11) and $C_4 = C_2\left(C_1 + \frac{N-2s}{2}\right)$. Integrating (4.35) between $(r, \hat{R})$, we obtain that

$$\frac{H(\hat{R})}{\hat{R}^{2\gamma}} - \frac{H(r)}{r^{2\gamma}} = \int_r^{\hat{R}} \tfrac{2H(\rho)}{\rho^{2\gamma+1}}\left(\int_0^\rho \alpha_1(\tau)d\tau\right)d\rho + \int_r^{\hat{R}} \tfrac{2H(\rho)}{\rho^{2\gamma+1}}\left(\int_0^\rho \alpha_2(\tau)d\tau\right)d\rho + \int_r^{\hat{R}} \tfrac{H(\rho)}{\rho^{2\gamma}}O(1)d\rho$$
$$= \int_r^{\hat{R}} \tfrac{2H(\rho)}{\rho^{2\gamma+1}}\left(\int_0^\rho \alpha_1(\tau)d\tau\right)d\rho - \int_r^{\hat{R}} \frac{H(\rho)}{\rho^{2\gamma}}\left(-\frac{2C_4}{\bar{\delta}}\rho^{-1+\bar{\delta}} + O(1)\right)d\rho.$$

Since $\alpha_1 \geq 0$ by (3.21) and (3.23), we have that $\lim_{r\to 0^+}\int_r^{\hat{R}} \frac{2H(\rho)}{\rho^{2\gamma+1}}\left(\int_0^\rho \alpha_1(\tau)d\tau\right)d\rho$ exists. On the other hand, estimate (3.24) ensures that $\rho \mapsto \frac{H(\rho)}{\rho^{2\gamma}}\left(-\frac{2C_4}{\bar{\delta}}\rho^{-1+\bar{\delta}} + O(1)\right) \in L^1(0, \hat{R})$, so that the limit $\lim_{r\to 0^+}\int_r^{\hat{R}} \frac{H(\rho)}{\rho^{2\gamma}}\left(-\frac{2C_4}{\bar{\delta}}\rho^{-1+\bar{\delta}} + O(1)\right)d\rho$ exists and is finite. The lemma is thereby proved. □

The next step is the proof that the limit $\lim_{r\to 0^+} r^{-2\gamma}H(r)$ is actually strictly positive. To this aim, we first define the Fourier coefficients associated with W, with respect to the orthonormal basis (1.14) of $L^2(\mathbb{S}^N_+, \theta_{N+1}^{1-2s}dS)$, as

$$\varphi_{k,m}(\lambda) = \int_{\mathbb{S}^N_+} \theta_{N+1}^{1-2s} W(\lambda\theta) Y_{k,m}(\theta)dS, \quad \lambda \in (0, R_1],\ k \in \mathbb{N},\ m = 1, \dots, M_k. \tag{4.36}$$

We also define

$$\Upsilon_{k,m}(\lambda) = -\int_{B_\lambda^+} t^{1-2s}(A - \mathrm{Id}_{N+1})\nabla W(z)\cdot\frac{\nabla_{\mathbb{S}^N} Y_{k,m}(z/|z|)}{|z|}dz + \kappa_s\int_{\Gamma_\lambda^-}\tilde{h}(y)\mathrm{Tr}W(y)\mathrm{Tr}Y_{k,m}\Big(\frac{y}{|y|}\Big)dy + \int_{\partial^+ B_\lambda^+} t^{1-2s}(A - \mathrm{Id}_{N+1})\nabla W\cdot\frac{z}{|z|}Y_{k,m}\Big(\frac{z}{|z|}\Big)dS, \tag{4.37}$$

for a.e. $\lambda \in (0, R_1]$, $k \in \mathbb{N}$ and $m \in \{1, 2, ..., M_k\}$.

Lemma 4.7. *Let k_0 be as in Proposition 4.5. Then, for all $m \in \{1, 2, \dots, M_{k_0}\}$ and $R \in (0, R_0]$,*

$$\varphi_{k_0,m}(\lambda) = \lambda^{k_0+s}\Bigg(R^{-k_0-s}\varphi_{k_0,m}(R) + \frac{(k_0+s)R^{-N-2k_0}}{N+2k_0}\int_0^R \rho^{k_0+s-1}\Upsilon_{k_0,m}(\rho)\,d\rho + \frac{N-s+k_0}{N+2k_0}\int_\lambda^R \rho^{-N-1+s-k_0}\Upsilon_{k_0,m}(\rho)\,d\rho\Bigg) + O(\lambda^{k_0+s+\bar{\delta}}) \quad \text{as } \lambda \to 0^+, \tag{4.38}$$

where $\bar{\delta}$ is defined in (3.11).

Proof. Let $k \in \mathbb{N}$ and $m \in \{1, 2, ..., M_k\}$. Testing (2.11) with $\phi = \omega(|z|)|z|^{-N-1+2s} Y_{k,m}(z/|z|)$ for any test function $\omega \in C_c^\infty(0, R_1)$ and using (1.12), we can easily verify that $\varphi_{k,m}$ solves the following second order differential equation

$$-\varphi''_{k,m}(\lambda) - \frac{N+1-2s}{\lambda}\varphi'_{k,m}(\lambda) + \frac{\mu_k}{\lambda^2}\varphi_{k,m}(\lambda) = \zeta_{k,m}(\lambda) \qquad \text{in } (0, R_1) \tag{4.39}$$

in a distributional sense, with μ_k as in (1.13), where the distribution $\zeta_{k,m} \in \mathcal{D}'(0, R_1)$ is defined by

$$\begin{aligned} {}_{\mathcal{D}'(0,R_1)}\langle \zeta_{k,m}, \omega\rangle_{\mathcal{D}(0,R_1)} = \kappa_s \int_0^{R_1} \frac{\omega(\lambda)}{\lambda^{2-2s}} \left(\int_{\mathbb{S}^{N-1}_-} \tilde{h}(\lambda\theta') \operatorname{Tr} W(\lambda\theta') Y_{k,m}(\theta', 0) dS' \right) d\lambda \\ - \int_{B^+_{R_1}} t^{1-2s}(A - \mathrm{Id}_{N+1})\nabla W \cdot \nabla \left(\omega(|z|)|z|^{-N-1+2s} Y_{k,m}(z/|z|)\right) dz \end{aligned}$$

for any $\omega \in C_c^\infty(0, R_1)$, where $\mathbb{S}^{N-1}_- = \{(\theta_1, \dots, \theta_N) \in \mathbb{S}^{N-1} : \theta_N \le 0\}$. Letting $\Upsilon_{k,m}$ be as in (4.37), by direct calculations we have that $\Upsilon_{k,m} \in L^1(0, R_1)$ and

$$\Upsilon'_{k,m}(\lambda) = \lambda^{N+1-2s}\zeta_{k,m}(\lambda) \qquad \text{in } \mathcal{D}'(0, R_1). \tag{4.40}$$

In view of (4.40) and (1.13), we have that (4.39) is equivalent to

$$-\left(\lambda^{N+1+2k}\left(\lambda^{-k-s}\varphi_{k,m}\right)'\right)' = \lambda^{k+s}\Upsilon'_{k,m} \quad \text{in } \mathcal{D}'(0, R_1).$$

Integrating the above equation, we obtain that, for every $R \in (0, R_1]$, $k \in \mathbb{N}$ and $m \in \{1, 2, \dots, M_k\}$, there exists a real number $c_{k,m}(R)$ (depending also on R) such that

$$\begin{aligned} \left(\lambda^{-k-s}\varphi_{k,m}(\lambda)\right)' = -\lambda^{-N-1+s-k}\Upsilon_{k,m}(\lambda) \\ - (k+s)\lambda^{-N-1-2k}\left(c_{k,m}(R) + \int_\lambda^R \rho^{k+s-1}\Upsilon_{k,m}(\rho)\, d\rho\right), \end{aligned} \tag{4.41}$$

in the sense of distributions in $(0, R_1)$. From (4.41) we infer that $\varphi_{k,m} \in W^{1,1}_{\rm loc}((0, R_1])$, thus a new integration leads to

$$\begin{aligned} \varphi_{k,m}(\lambda) = \lambda^{k+s}\left(\frac{\varphi_{k,m}(R)}{R^{k+s}} - \frac{(k+s)c_{k,m}(R)}{(N+2k)R^{N+2k}} + \frac{N+k-s}{N+2k}\int_\lambda^R \rho^{-N-k+s-1}\Upsilon_{k,m}(\rho)\, d\rho\right) \\ + \frac{(k+s)\lambda^{-N-k+s}}{N+2k}\left(c_{k,m}(R) + \int_\lambda^R \rho^{k+s-1}\Upsilon_{k,m}(\rho)\, d\rho\right) \end{aligned} \tag{4.42}$$

for all $\lambda \in (0, R_1]$.

From now on, we fix k_0 as in Proposition 4.5, R_0 as in (2.45), and $m \in \{1, 2, \dots, M_{k_0}\}$. We prove that

$$\int_0^{R_0} \rho^{-N-k_0+s-1}|\Upsilon_{k_0,m}(\rho)|\, d\rho < +\infty. \tag{4.43}$$

To this purpose, exploiting (2.20) and using Hölder's inequality, one can estimate the first term in (4.37) for all $\rho \in (0, R_0)$ as follows

$$\begin{aligned} &\left|\int_{B^+_\rho} t^{1-2s}(A - \mathrm{I}_{N+1})\nabla W(z) \cdot \frac{\nabla_{\mathbb{S}^N} Y_{k_0,m}(z/|z|)}{|z|} dz\right| \\ &\le \text{const} \sqrt{\int_{B^+_\rho} t^{1-2s}|\nabla W|^2 dz} \cdot \sqrt{\int_{B^+_\rho} t^{1-2s}|\nabla_{\mathbb{S}^N} Y_{k_0,m}(z/|z|)|^2 dz} \\ &=: \text{const}\, I_1(\rho) \cdot I_2(\rho), \end{aligned} \tag{4.44}$$

where

$$\begin{aligned} I_1(\rho) = \sqrt{\rho^{N+2-2s}\int_{B_1^+} t^{1-2s}|\nabla W(\rho z)|^2 dz} = \rho^{\frac{N-2s}{2}}\sqrt{H(\rho)}\sqrt{\int_{B_1^+} t^{1-2s}|\nabla w^\rho(z)|^2 dz} \\ \le \text{const}\, \rho^{\frac{N-2s}{2}}\sqrt{H(\rho)}, \end{aligned} \tag{4.45}$$

as a consequence of Lemma 4.1, and

$$I_2(\rho) = \sqrt{\int_0^\rho \tau^{N+1-2s}\left(\int_{\mathbb{S}^N_+} \theta_{N+1}^{1-2s}|\nabla_{\mathbb{S}^N} Y_{k_0,m}(\theta)|^2 dS\right) d\tau} = \frac{\sqrt{\mu_{k_0}}}{\sqrt{N+2-2s}}\rho^{\frac{N+2-2s}{2}}, \tag{4.46}$$

due to (1.12). Combining (4.44), (4.45), (4.46), and (3.24) we obtain that, for every $R \in (0, R_0]$,

$$\int_0^R \rho^{-N-1+s-k_0}\left|\int_{B_\rho^+} t^{1-2s}(A - \mathrm{I}_{N+1})\nabla W(z)\cdot \frac{\nabla_{\mathbb{S}^N} Y_{k_0,m}(z/|z|)}{|z|}dz\right| d\rho \tag{4.47}$$
$$\leq \mathrm{const} \int_0^R \rho^{-s-k_0}\sqrt{H(\rho)}\, ds \leq \mathrm{const}\, R.$$

Moreover, Hölder's inequality implies that

$$\left|\int_{\Gamma_\lambda^-} \tilde{h}(y)\,\mathrm{Tr}\, W(y)\,\mathrm{Tr}\, Y_{k_0,m}\big(\tfrac{y}{|y|}\big)\, dy\right| \leq \sqrt{\int_{\Gamma_\lambda^-} |\tilde{h}||\,\mathrm{Tr}\, W|^2 dy}\cdot\sqrt{\int_{\Gamma_\lambda^-} |\tilde{h}(y)||\,\mathrm{Tr}\, Y_{k_0,m}\big(\tfrac{y}{|y|}\big)|^2 dy}. \tag{4.48}$$

From (2.34) and homogeneity of the function $Y_{k_0,m}(y/|y|)$ it follows that, for all $\rho \in (0, R_0]$,

$$\sqrt{\int_{\Gamma_\rho^-} |\tilde{h}(y)||\,\mathrm{Tr}\, Y_{k_0,m}\big(\tfrac{y}{|y|}\big)|^2 dy} \leq \sqrt{\tilde{c}_{N,s,p}}\, \|\tilde{h}\,\|^{1/2}_{L^p(\Gamma^-_{R_1})}\rho^{\bar{\varepsilon}/2}\left(\int_{\Gamma_\rho^-} |\,\mathrm{Tr}\, Y_{k_0,m}\big(\tfrac{y}{|y|}\big)|^{2^*(s)}\, dy\right)^{\frac{1}{2^*(s)}}$$
$$= \sqrt{\tilde{c}_{N,s,p}}\, \|\tilde{h}\,\|^{1/2}_{L^p(\Gamma^-_{R_1})}\rho^{\frac{\bar{\varepsilon}+N-2s}{2}}\left(\int_{\Gamma_1^-} |\,\mathrm{Tr}\, Y_{k_0,m}\big(\tfrac{y}{|y|}\big)|^{2^*(s)}\, dy\right)^{\frac{1}{2^*(s)}}.$$

Using (2.34), (2.31), and (3.21), we obtain that, for all $\rho \in (0, R_0]$,

$$\sqrt{\int_{\Gamma_\rho^-} |\tilde{h}||\,\mathrm{Tr}\, W|^2 dy} \leq \sqrt{\frac{\tilde{c}_{N,s,p}}{\tilde{C}_{N,s}}\|\tilde{h}\,\|_{L^p(\Gamma^-_{R_1})}\rho^{\bar{\varepsilon}+N-2s}H(\rho)\left(\mathcal{N}(\rho)+\frac{N-2s}{2}\right)}$$
$$\leq \mathrm{const}\,\rho^{\frac{N-2s+\bar{\varepsilon}}{2}}\sqrt{H(\rho)}.$$

Putting the above estimates together and recalling (3.24), we conclude that, for every $R \in (0, R_0]$,

$$\int_0^R \rho^{-N-k_0+s-1}\left|\int_{\Gamma_\rho^-} \tilde{h}(y)\mathrm{Tr}W(y)\mathrm{Tr}Y_{k_0,m}\big(\tfrac{y}{|y|}\big)\, dy\right| d\rho \tag{4.49}$$
$$\leq \mathrm{const}\int_0^R \rho^{-1+\bar{\varepsilon}-k_0-s}\sqrt{H(\rho)}\, d\rho \leq \mathrm{const}\, R^{\bar{\varepsilon}}.$$

In order to estimate the last term, we observe that, since $\|Y_{k_0,m}\|_{L^2(\mathbb{S}^N_+,\theta_{N+1}^{1-2s}dS)} = 1$,

$$\int_{B_\lambda^+} t^{1-2s}\big|Y_{k_0,m}\big(\tfrac{z}{|z|}\big)\big|^2 dz = \int_0^\lambda \tau^{N+1-2s}\left(\int_{\mathbb{S}^N_+} \theta_{N+1}^{1-2s}|Y_{k_0,m}(\theta)|^2 dS\right) d\tau = \frac{\lambda^{N+2-2s}}{N+2-2s}. \tag{4.50}$$

Hence, integrating by parts, we have that, for every $R \in (0, R_0]$,

$$\int_0^R \rho^{-N+s-1-k_0}\left|\int_{\partial^+ B_\rho^+} t^{1-2s}(A - \mathrm{Id}_{N+1})\nabla W\cdot \frac{z}{|z|}Y_{k_0,m}\big(\tfrac{z}{|z|}\big)dS\right| d\rho \tag{4.51}$$
$$\leq \mathrm{const}\int_0^R \rho^{-N+s-k_0}\left(\int_{\partial^+ B_\rho^+} t^{1-2s}|\nabla W||Y_{k_0,m}\big(\tfrac{z}{|z|}\big)|\, dS\right) d\rho$$
$$= \mathrm{const}\left(R^{-N+s-k_0}\int_{B_R^+} t^{1-2s}|\nabla W|\big|Y_{k_0,m}\big(\tfrac{z}{|z|}\big)\big|\, dz\right.$$
$$\left. + (N+k_0-s)\int_0^R \rho^{-N+s-1-k_0}\left(\int_{B_\rho^+} t^{1-2s}|\nabla W|\big|Y_{k_0,m}\big(\tfrac{z}{|z|}\big)\big|\, dz\right) d\rho\right)$$
$$\leq \mathrm{const}\left(R^{1-s-k_0}\sqrt{H(R)} + \int_0^R \rho^{-s-k_0}\sqrt{H(\rho)}\, d\rho\right) \leq \mathrm{const}\, R,$$

thanks to (2.20), Hölder inequality, (4.45), (4.50), and (3.24). From (4.37), (4.47), (4.49), and (4.51) it follows that, for every $R \in (0, R_0]$,

$$\int_0^R \rho^{-N-k_0+s-1} |\Upsilon_{k_0,m}(\rho)|\, d\rho \le \text{const}\, R^{\bar\delta} \tag{4.52}$$

for some const > 0 independent of R. From (4.52), we derive immediately (4.43).

From (4.43) it follows that, for every $R \in (0, R_0]$,

$$\lambda^{k_0+s}\left(\frac{\varphi_{k_0,m}(R)}{R^{k_0+s}} - \frac{(k_0+s)c_{k_0,m}(R)}{(N+2k_0)R^{N+2k_0}} + \frac{N+k_0-s}{N+2k_0}\int_\lambda^R \rho^{-N-k_0+s-1}\Upsilon_{k_0,m}(\rho)\, d\rho\right) = O(\lambda^{k_0+s}) = o(\lambda^{-N-k_0+s}) \quad \text{as } \lambda \to 0^+. \tag{4.53}$$

Now we prove that, for every $R \in (0, R_0]$,

$$c_{k_0,m}(R) + \int_0^R \rho^{k_0+s-1}\Upsilon_{k_0,m}(\rho)\, d\rho = 0. \tag{4.54}$$

In order to do this, first we observe that

$$\int_0^{R_0} \rho^{k_0+s-1}|\Upsilon_{k_0,m}(\rho)| d\rho < +\infty, \tag{4.55}$$

as a direct consequence of (4.43), since $k_0 + s - 1 > -N - k_0 + s - 1$. Suppose by contradiction that (4.54) does not hold true for some $R \in (0, R_0]$; then from (4.42), (4.53) and (4.55), we should have that

$$\varphi_{k_0,m}(\lambda) \sim \frac{(k_0+s)\lambda^{-N-k_0+s}}{N+2k_0}\left(c_{k_0,m}(R) + \int_0^R \rho^{k_0+s-1}\Upsilon_{k_0,m}(\rho)\, d\rho\right) \quad \text{as } \lambda \to 0^+,$$

and hence

$$\int_0^{R_0} \lambda^{N-1-2s}|\varphi_{k_0,m}(\lambda)|^2 d\lambda = +\infty.$$

On the other hand, by (4.36), we have that

$$\int_0^{R_0} \lambda^{N-1-2s}|\varphi_{k_0,m}(\lambda)|^2 d\lambda \le \int_0^{R_0} \lambda^{N-1-2s}\left(\int_{\mathbb{S}^N_+} \theta_{N+1}^{1-2s}|W(\lambda\theta)|^2 dS\right) d\lambda = \int_{B^+_{R_0}} t^{1-2s}\frac{W^2(z)}{|z|^2} dz < \infty,$$

as a consequence of Lemma 2.3, giving rise to a contradiction. Hence (4.54) holds true. From (4.54) and (4.52) we deduce that, for every $R \in (0, R_0]$,

$$\begin{aligned}
\left|\lambda^{-N-k_0+s}\left(c_{k_0,m}(R) + \int_\lambda^R \rho^{k_0+s-1}\Upsilon_{k_0,m}(\rho)\, d\rho\right)\right| &= \lambda^{-N+s-k_0}\left|\int_0^\lambda \rho^{k_0+s-1}\Upsilon_{k_0,m}(\rho)\, d\rho\right| \\
&\le \lambda^{-N+s-k_0}\int_0^\lambda \rho^{N+2k_0}|\rho^{-N-1+s-k_0}\Upsilon_{k_0,m}(\rho)|\, d\rho \\
&\le \lambda^{k_0+s}\int_0^\lambda \rho^{-N-1+s-k_0}|\Upsilon_{k_0,m}(\rho)|\, d\rho = O(\lambda^{k_0+s+\bar\delta}) \quad \text{as } \lambda \to 0^+.
\end{aligned}$$

Combining this last information with (4.54) and (4.42), we finally obtain (4.38). □

Using Lemma 4.7, we now prove that $\lim_{r\to 0^+} r^{-2\gamma}H(r) = \lim_{r\to 0^+} r^{-2(k_0+s)}H(r) > 0$.

Lemma 4.8. *Let $\gamma = \lim_{r\to 0}\mathcal{N}(r)$ be as in Proposition 3.6. Then*

$$\lim_{r\to 0^+} r^{-2\gamma}H(r) > 0.$$

Proof. By (2.25) and using the Parseval identity we have that

$$\begin{aligned}
H(\lambda) &= \int_{\mathbb{S}^N_+} \theta_{N+1}^{1-2s}\mu(\lambda\theta)|W(\lambda\theta)|^2 dS \\
&= (1+O(\lambda))\int_{\mathbb{S}^N_+} \theta_{N+1}^{1-2s}|W(\lambda\theta)|^2 dS = (1+O(\lambda))\sum_{k=0}^\infty\sum_{m=1}^{M_k}|\varphi_{k,m}(\lambda)|^2.
\end{aligned} \tag{4.56}$$

Let $k_0 \in \mathbb{N}$ be as in Proposition 4.5, thus $\gamma = k_0 + s$. We argue by contradiction, assuming that

$$\lim_{\lambda\to 0^+} \lambda^{-2\gamma} H(\lambda) = 0. \tag{4.57}$$

Hence from (4.56) it follows that $\lim_{\lambda\to 0^+} \lambda^{-(k_0+s)}\varphi_{k_0,m}(\lambda) = 0$ for any $m \in \{1, 2, \ldots, M_{k_0}\}$. This and Lemma 4.7 lead to

$$R^{-k_0-s}\varphi_{k_0,m}(R) + \frac{(k_0+s)R^{-N-2k_0}}{N+2k_0}\int_0^R \rho^{k_0+s-1}\Upsilon_{k_0,m}(\rho)\,d\rho + \frac{N-s+k_0}{N+2k_0}\int_0^R \rho^{-N-1+s-k_0}\Upsilon_{k_0,m}(\rho)\,d\rho = 0, \tag{4.58}$$

for all $m \in \{1, 2, \ldots, M_{k_0}\}$ and for every $R \in (0, R_0]$. From (4.58), (4.38), and (4.52) it follows that

$$\varphi_{k_0,m}(\lambda) = -\lambda^{k_0+s}\frac{N-s+k_0}{N+2k_0}\int_0^\lambda \rho^{-N-1+s-k_0}\Upsilon_{k_0,m}(\rho)\,d\rho + O(\lambda^{k_0+s+\bar\delta}) = O(\lambda^{k_0+s+\bar\delta})$$

as $\lambda \to 0^+$ for all $m \in \{1, 2, \ldots, M_{k_0}\}$. Hence

$$\sqrt{H(\lambda)}\,(w^\lambda, \psi)_{L^2(\mathbb{S}^N_+,\theta_{N+1}^{1-2s}dS)} = O(\lambda^{k_0+s+\bar\delta}) \quad \text{as } \lambda \to 0^+ \tag{4.59}$$

for every $\psi \in \operatorname{span}\{Y_{k_0,m} : m = 1, \ldots, M_{k_0}\}$. From Lemma 3.7-(ii), $\sqrt{H(\lambda)} \geq \sqrt{k_2(\bar\delta)}\lambda^{k_0+s+\frac{\bar\delta}{2}}$ for λ small, so that (4.59) yields

$$(w^\lambda, \psi)_{L^2(\mathbb{S}^N_+,\theta_{N+1}^{1-2s}dS)} = O(\lambda^{\bar\delta/2}) \quad \text{as } \lambda \to 0^+ \tag{4.60}$$

for every $\psi \in \operatorname{span}\{Y_{k_0,m} : m = 1, \ldots, M_{k_0}\}$. On the other hand, by Proposition 4.5 and continuity of the trace map from $H^1(B_1^+, t^{1-2s}dz)$ into $L^2(\partial^+B_1^+, t^{1-2s}dS)$, for any sequence $\lambda_n \to 0^+$, there exist a subsequence $\{\lambda_{n_k}\}$ and $\psi_0 \in \operatorname{span}\{Y_{k_0,m} : m = 1, \ldots, M_{k_0}\}$ such that

$$\|\psi_0\|_{L^2(\mathbb{S}^N_+,\theta_{N+1}^{1-2s}dS)} = 1 \quad \text{and} \quad w^{\lambda_{n_k}} \to \psi_0 \quad \text{in } L^2(\mathbb{S}^N_+, \theta_{N+1}^{1-2s}dS). \tag{4.61}$$

From (4.60) and (4.61) we deduce that

$$0 = \lim_{k\to\infty}(w^{\lambda_{n_k}}, \psi_0)_{L^2(\mathbb{S}^N_+,\theta_{N+1}^{1-2s}dS)} = \|\psi_0\|^2_{L^2(\mathbb{S}^N_+,\theta_{N+1}^{1-2s}dS)} = 1,$$

thus reaching a contradiction. □

Theorem 4.9. *Let $k_0 \in \mathbb{N}$ be as in Proposition 4.5. Let $M_{k_0} \in \mathbb{N}\setminus\{0\}$ be the multiplicity of the eigenvalue $\mu_{k_0} = (k_0+s)(k_0+N-s)$ and let $\{Y_{k_0,m}\}_{m=1,\ldots,M_{k_0}}$ be a $L^2(\mathbb{S}^N_+, \theta_{N+1}^{1-2s}dS)$-orthonormal basis of the eigenspace of* (1.11) *associated to μ_{k_0}. Then, for every $m \in \{1, 2, \ldots, M_{k_0}\}$, there exists $\beta_m \in \mathbb{R}$ such that $(\beta_1, \beta_2, \ldots, \beta_{M_{k_0}}) \neq (0, 0, \ldots, 0)$,*

$$\frac{W(\lambda z)}{\lambda^{k_0+s}} \to |z|^{k_0+s}\sum_{m=1}^{M_{k_0}} \beta_m Y_{k_0,m}(z/|z|) \quad \text{in } H^1(B_1^+, t^{1-2s}dz) \text{ as } \lambda \to 0^+,$$

and

$$\beta_m = R^{-(k_0+s)}\varphi_{k_0,m}(R) + \frac{(k_0+s)R^{-N-2k_0}}{N+2k_0}\int_0^R \rho^{k_0+s-1}\Upsilon_{k_0,m}(\rho)\,d\rho + \frac{N-s+k_0}{N+2k_0}\int_0^R \rho^{-N-1+s-k_0}\Upsilon_{k_0,m}(\rho)\,d\rho \quad \text{for all } R \in (0, R_0], \tag{4.62}$$

with $\varphi_{k_0,m}$ and $\Upsilon_{k_0,m}$ given by (4.36) *and* (4.37) *respectively.*

Proof. If we consider any sequence of strictly positive real numbers $\lambda_n \to 0^+$, then from Proposition 4.5 and Lemmas 4.6 and 4.8, we deduce that there exist a subsequence $\{\lambda_{n_k}\}$ and real numbers $\beta_1, \beta_2, \ldots \beta_{M_{k_0}}$ not all equal to 0 such that

$$\frac{W(\lambda_{n_k} z)}{\lambda_{n_k}^{k_0+s}} \to |z|^{k_0+s}\sum_{m=1}^{M_{k_0}} \beta_m Y_{k_0,m}(z/|z|) \quad \text{in } H^1(B_1^+, t^{1-2s}dz) \text{ as } k \to \infty. \tag{4.63}$$

We claim that the coefficients β_m depend neither on the sequence $\{\lambda_n\}$, nor on its subsequence $\{\lambda_{n_k}\}$. To this aim, we observe that (4.36), (4.63), and the continuity of the trace map from $H^1(B_1^+, t^{1-2s}dz)$ into $L^2(\partial^+ B_1^+, t^{1-2s}dS)$ imply that, for all $m \in \{1, 2, \dots, M_{k_0}\}$,

$$\lim_{k\to\infty} \lambda_{n_k}^{-(k_0+s)} \varphi_{k_0,m}(\lambda_{n_k}) = \lim_{k\to+\infty} \int_{\mathbb{S}^N_+} \theta_{N+1}^{1-2s} \lambda_{n_k}^{-\gamma} W(\lambda_{n_k}\theta) Y_{k_0,m}(\theta)\, dS$$
$$= \sum_{i=1}^{M_{k_0}} \beta_i \int_{\mathbb{S}^N_+} \theta_{N+1}^{1-2s} Y_{k_0,i}(\theta) Y_{k_0,m}(\theta)\, dS = \beta_m,$$

for all $m \in \{1, 2, \dots, M_{k_0}\}$. At the same time, after fixing $R < R_0$, by (4.38) we have that

$$\lim_{k\to\infty} \lambda_{n_k}^{-(k_0+s)} \varphi_{k_0,m}(\lambda_{n_k}) = R^{-(k_0+s)} \varphi_{k_0,m}(R) + \frac{(k_0+s)R^{-N-2k_0}}{N+2k_0} \int_0^R \rho^{k_0+s-1} \Upsilon_{k_0,m}(\rho)\, d\rho$$
$$+ \frac{N-s+k_0}{N+2k_0} \int_0^R \rho^{-N-1+s-k_0} \Upsilon_{k_0,m}(\rho)\, d\rho,$$

hence, by uniqueness of the limit, we can deduce that, for all $m \in \{1, 2, \dots, M_{k_0}\}$,

$$\beta_m = R^{-(k_0+s)} \varphi_{k_0,m}(R) + \frac{(k_0+s)R^{-N-2k_0}}{N+2k_0} \int_0^R \rho^{k_0+s-1} \Upsilon_{k_0,m}(\rho)\, d\rho$$
$$+ \frac{N-s+k_0}{N+2k_0} \int_0^R \rho^{-N-1+s-k_0} \Upsilon_{k_0,m}(\rho)\, d\rho.$$

This is enough to conclude that the coefficients β_m depend neither on the sequence $\{\lambda_n\}$, nor on its subsequence $\{\lambda_{n_k}\}$. Urysohn's Subsequence Principle allows us to conclude that the convergence in (4.63) holds as $\lambda \to 0^+$, thus completing the proof. □

We are now in position to prove Theorem 1.3.

Proof of Theorem 1.3. Up to a translation, we can assume that $x_0 = 0$. If U is as in the assumptions of Theorem 1.3, then, letting F as in Section 2.1, $W = U \circ F \in H^1_{\Gamma^+_{R_1}}(B^+_{R_1}, t^{1-2s}\, dz)$ is a nontrivial weak solution of (2.11). We notice that the nontriviality of U in any neighbourhood of 0, and consequently of W in $B^+_{R_1}$, can be easily deduced from nontriviality of U in $\mathbb{R}^{N+1}_+$ and classical unique continuation principles for second order elliptic equations with Lipschitz coefficients [23].

Then, by Proposition 4.5 and Theorem 4.9, there exist $k_0 \in \mathbb{N}$ and an eigenfunction Y of problem (1.11) associated to the eigenvalue $\mu_{k_0} = (k_0+s)(k_0+N-s)$ such that

$$\frac{W(\lambda z)}{\lambda^{k_0+s}} \to |z|^{k_0+s} Y(z/|z|) \quad \text{in } H^1(B_1^+, t^{1-2s}dz) \text{ as } \lambda \to 0^+. \tag{4.64}$$

We observe that

$$\frac{U(\lambda z)}{\lambda^{k_0+s}} = \frac{W(\lambda G_\lambda(z))}{\lambda^{k_0+s}}, \quad \nabla\left(\frac{U(\lambda \cdot)}{\lambda^{k_0+s}}\right)(z) = \nabla\left(\frac{W(\lambda \cdot)}{\lambda^{k_0+s}}\right)(G_\lambda(z))\, J_{G_\lambda}(z), \tag{4.65}$$

where

$$G_\lambda(z) = \frac{1}{\lambda} F^{-1}(\lambda z).$$

From (2.9) we have that

$$G_\lambda(z) = z + O(\lambda) \quad \text{and} \quad J_{G_\lambda}(z) = \mathrm{Id}_{N+1} + O(\lambda) \tag{4.66}$$

as $\lambda \to 0^+$ uniformly with respect to $z \in B_1^+$. From (4.66) one can easily deduce that, if $f_\lambda \to f$ in $L^2(B_1^+, t^{1-2s}dz)$, then $f_\lambda \circ G_\lambda \to f$ in $L^2(B_1^+, t^{1-2s}dz)$. In view of (4.64) and (4.65), this yields the conclusion. □

As a direct consequence of Theorem 1.3 and of the equivalent formulation of problem (1.1) given in (1.9), we obtain Theorem 1.2

Proof of Theorem 1.2. If $u \in \mathcal{D}^{s,2}(\mathbb{R}^N)$, $u \not\equiv 0$, is a nontrivial weak solution to (1.1), then its extension $U = \mathcal{H}(u) \in \mathcal{D}^{1,2}(\mathbb{R}^{N+1}_+, t^{1-2s}\, dz)$ weakly solves (1.9) in the weak sense specified in (1.10), see [6] and Section 1. Then the conclusion follows from Theorem 1.3 applied to U and the continuity of the trace map from $H^1(B_1^+, t^{1-2s}dz)$ into $H^s(B_1')$, see e.g. [25, Proposition 2.1]. □

Appendix A. Some boundary regularity results at edges of cylinders

Let us consider the following local problem: $\Omega \subset \mathbb{R}^N$ is a $C^{1,1}$ domain, $x_0 \in \partial\Omega$, $R, T > 0$ and U is a weak solution to

$$\begin{cases} \operatorname{div}\left(t^{1-2s}\nabla U\right) = 0 & \text{in } C_{R,T}(x_0), \\ U = 0 & \text{in } D_{R,T}(x_0), \\ \lim_{t\to 0} t^{1-2s}\partial_t U = 0 & \text{in } \sigma_{R,T}(x_0), \end{cases} \tag{A.1}$$

where we denoted

$$C_{R,T}(x_0) := (B'_R(x_0) \cap \Omega) \times (0,T), \quad D_{R,T}(x_0) := (B'_R(x_0) \cap \partial\Omega) \times (0,T),$$
$$\sigma_{R,T}(x_0) = (B'_R(x_0) \cap \Omega) \times \{0\};$$

i.e. U belongs to the space $\mathcal{H}$ defined as the closure of the set

$$\{v \in C^\infty(\overline{C_{R,T}(x_0)}) : v = 0 \text{ in a neighbourhood of } D_{R,T}(x_0)\}$$

in $H^1(C_{R,T}(x_0), t^{1-2s}\,dz)$, and

$$\int_{C_{R,T}(x_0)} t^{1-2s}\nabla U \cdot \nabla\Phi \, dz = 0 \quad \text{for all } \Phi \in C_c^\infty(C_{R,T}(x_0) \cup \sigma_{R,T}(x_0)).$$

The following regularity result holds true.

Lemma A.1. *Let $\alpha \in (0,1)$, $\beta \in (0,1) \cap (0, 2-2s)$, $r < R$, and $\tau < T$. Then there exists a positive constant C such that, for every weak solution U to* (A.1),

$$\|U\|_{C^{1,\alpha}(C_{r,\tau}(x_0))} + \|t^{1-2s}\partial_t U\|_{C^{0,\beta}(C_{r,\tau}(x_0))} \le C\|U\|_{L^2(C_{R,T}(x_0), t^{1-2s}dz)}.$$

Proof. Denoting the total variable $z = (x,t) \in \mathbb{R}^N \times (0,+\infty)$, with $x = (x', x_N) \in \mathbb{R}^{N-1} \times \mathbb{R}$, let us consider $g \in C^{1,1}(\mathbb{R}^{N-1})$ such that $B'_R(x_0) \cap \Omega = \{x = (x', x_N) \in B'_R(x_0) : x_N < g(x')\}$. Without loss of generality we can assume that $x_0 = 0$, $g(0) = 0$ and $\nabla g(0) = 0$. Starting from this function g, we can argue as in Section 2.1 and construct a function F as in (2.5), which turns out to be a diffeomorphism in a neighbourhood of 0. Hence there exist positive constants $r_0 < R$ and $\tau_0 < T$ such that the composition $W = U \circ F$ weakly solves the following straightened problem

$$\begin{cases} \operatorname{div}\left(t^{1-2s}A\nabla W\right) = 0 & \text{in } \Gamma^-_{r_0} \times (0,\tau_0), \\ W = 0 & \text{in } (B'_{r_0} \cap \{y_N = 0\}) \times (0,\tau_0), \\ \lim_{t\to 0} t^{1-2s}A\nabla W \cdot \nu = 0 & \text{in } \Gamma^-_{r_0}, \end{cases}$$

with $A = A(y)$ being as in (2.12); in particular the matrix $A(y)$ does not depend on the vertical variable t, is symmetric, uniformly elliptic, and possesses $C^{0,1}$ coefficients.

Let us consider the odd reflection of W (which we still denote as W) through the hyperplane $\{y_N = 0\}$ in $B'_{r_0} \times (0,\tau_0)$, i.e. we set $W(y', y_N, t) = -W(y', -y_N, t)$ for $y_N < 0$; it is easy to verify that W weakly satisfies

$$\begin{cases} \operatorname{div}\left(t^{1-2s}\widetilde{A}\nabla W\right) = 0 & \text{in } B'_{r_0} \times (0,\tau_0), \\ \lim_{t\to 0} t^{1-2s}\widetilde{A}\nabla W \cdot \nu = 0 & \text{in } B'_{r_0}, \end{cases}$$

where

$$\widetilde{A}(y) = \widetilde{A}(y', y_N) := \begin{cases} A(y', y_N), & \text{if } y_N \le 0, \\ SA(y', -y_N)S, & \text{if } y_N > 0, \end{cases}$$

with

$$S := \left(\begin{array}{c|c|c} \mathrm{Id}_{N-1} & \mathbf{0} & \mathbf{0} \\ \hline \mathbf{0}^T & -1 & 0 \\ \hline \mathbf{0}^T & 0 & 1 \end{array}\right),$$

We observe that no discontinuities appear in the coefficients of the matrix $\widetilde{A}$ since, denoting as (a_{ij}) the entries of the matrix A, $a_{i,N}(y', 0, t) = 0$ for all $i < N$ thanks to (2.18) and (2.19). Then the matrix $\widetilde{A}$ has Lipschitz continuous coefficients. Let us then consider the even reflection of W (which we still denote as W) through the hyperplane $\{t = 0\}$ in $B'_{r_0} \times (-\tau_0, \tau_0)$, i.e. we set $W(y', y_N, t) = W(y', y_N, -t)$ for

$t < 0$; due to the homogeneous Neumann type boundary condition satisfied by W on B'_{r_0} and the fact that the matrix A is independent of t, we obtain that such even reflection through $\{t = 0\}$ weakly solves

$$\operatorname{div}\left(|t|^{1-2s}\widetilde{A}\nabla W\right) = 0 \quad \text{in } B'_{r_0} \times (-\tau_0, \tau_0).$$

From [36, Lemma 7.1] it follows that $V = |t|^{1-2s}\partial_t W \in H^1_{\rm loc}(B'_{r_0} \times (-\tau_0, \tau_0), |t|^{2s-1}\,dz)$ is a weak solution to

$$\operatorname{div}\left(|t|^{2s-1}\widetilde{A}\nabla V\right) = 0 \quad \text{in } B'_{r_0} \times (-\tau_0, \tau_0)$$

which is odd with respect to $\{t = 0\}$, i.e. $V(y', y_N, -t) = -V(y', y_N, -t)$.

From [36, Theorem 1.2] it follows that, for all $r \in (0, r_0)$ and $\tau \in (0, \tau_0)$, $W \in C^{1,\alpha}(B'_r \times (-\tau, \tau))$ and $\|W\|_{C^{1,\alpha}(B'_r\times(-\tau,\tau))} \leq \text{const}\|W\|_{L^2(B'_{r_0}\times(-\tau_0,\tau_0),|t|^{1-2s}\,dz)}$ for some const > 0 (independent of W). Furthermore, [12] ensures that V is locally Hölder continous. More precisely, [37, Proposition 2.10] yields that the function $\Phi(x,t) = \frac{V(x,t)}{t|t|^{1-2s}}$, which is even in the variable t, belongs to the weighted Sobolev space $H^1_{\rm loc}(B'_{r_0} \times (-\tau_0, \tau_0), |t|^{3-2s}\,dz)$, and weakly solves

$$\operatorname{div}\left(|t|^{3-2s}\widetilde{A}\nabla \Phi\right) = 0 \quad \text{in } B'_{r_0} \times (-\tau_0, \tau_0),$$

thanks to the fact that the matrix $\widetilde{A}$ is independent of t.

From [36, Theorem 1.2] we have that $\Phi \in C^{0,\gamma}(B'_r \times (-\tau, \tau))$ for all $\gamma \in (0, 1)$, $r \in (0, r_0)$ and $\tau \in (0, \tau_0)$, and

$$\|\Phi\|_{C^{0,\gamma}(B'_r\times(-\tau,\tau))} = \left\|\frac{V}{t|t|^{1-2s}}\right\|_{C^{0,\gamma}(B'_r\times(-\tau,\tau))} \leq \text{const}\,\|V\|_{L^2(B'_{r_0}\times(-\tau_0,\tau_0),|t|^{2s-1}\,dz)}$$

for some const > 0 (independent of V). Therefore $V \in C^{0,\delta}(B'_r \times (-\tau, \tau))$ with $\delta = \min\{2 - 2s, \gamma\}$ and $\|V\|_{C^{0,\delta}(B'_r\times(-\tau,\tau))} \leq \text{const}\,\|V\|_{L^2(B'_{r_0}\times(-\tau_0,\tau_0),|t|^{2s-1}\,dz)}$.

The conclusion follows by recalling that $U = W \circ F^{-1}$ with F^{-1} being of class $C^{1,1}$ and taking into account the particular form of the matrix in (2.6). □

Appendix B. Homogeneity degrees and eigenvalues of the spherical problem

In this appendix, we derive an explicit formula for the eigenvalues of problem (1.11), which follows from a complete classification of possible homogeneity degrees of homogeneous weak solutions to the problem

$$\text{(B.1)}\qquad \begin{cases} -\operatorname{div}\left(t^{1-2s}\nabla\Psi\right) = 0 & \text{in } \mathbb{R}^{N+1}_+, \\ \lim_{t\to0^+}\left(t^{1-2s}\nabla\Psi\cdot\nu\right) = 0 & \text{in } \Gamma^-, \\ \Psi = 0 & \text{in } \Gamma^+, \end{cases}$$

where $\Gamma^- := \{(y', y_N, 0) \in \mathbb{R}^N \times \{0\} : y_N < 0\}$ and $\Gamma^+ := \{(y', y_N, 0) \in \mathbb{R}^N \times \{0\} : y_N \geq 0\}$.

Proposition B.1. *Let $\Psi \in \cap_{r>0} H^1_{\Gamma^+_r}(B^+_r, t^{1-2s}\,dz)$ be a weak solution to* (B.1), *i.e.*

$$\int_{\mathbb{R}^{N+1}_+} t^{1-2s}\nabla\Psi\cdot\nabla\Phi\,dz = 0, \quad \textit{for all } \Phi \in C^\infty_c(\overline{\mathbb{R}^{N+1}_+} \setminus \Gamma^+).$$

If, for some $\gamma \geq 0$, $\Psi(z) = |z|^\gamma\Psi(\frac{z}{|z|})$, then there exists $j \in \mathbb{N}$ such that $\gamma = j + s$.

The proof of Proposition B.1 requires a polynomial Liouville type theorem for even solutions to degenerate equations with a weight which is possibly out of the A_2-Muckenhoupt class. To this aim, Lemma B.2 below provides a generalization of Lemma 2.7 in [7]. For all $a \in (-1, +\infty)$ and $r > 0$, we define $H^1(B_r, |t|^a\,dz)$ as the completion of $C^\infty(\overline{B_r})$ with respect to the norm $\sqrt{\int_{B_r}|t|^a\left(|\Psi|^2 + |\nabla\Psi|^2\right)dz}$ and $H^{1,a}_{\rm loc}(\mathbb{R}^{N+1})$ as

$$H^{1,a}_{\rm loc}(\mathbb{R}^{N+1}) = \{\Psi \in L^2_{\rm loc}(\mathbb{R}^{N+1}, |t|^a\,dz) : \Psi \in H^1(B_r, |t|^a\,dz) \text{ for all } r > 0\}.$$

We also define

$$H^{1,a}_{\rm loc}(\overline{\mathbb{R}^{N+1}_+}) = \{\Psi \in L^2_{\rm loc}(\overline{\mathbb{R}^{N+1}_+}, t^a\,dz) : \Psi \in H^1(B^+_r, t^a\,dz) \text{ for all } r > 0\}.$$

Lemma B.2. *Let $a \in (-1,+\infty)$ and $v \in H^{1,a}_{\rm loc}(\mathbb{R}^{N+1})$ be a weak solution to*

$$\operatorname{div}(|t|^a \nabla v) = 0 \quad \text{in } \mathbb{R}^{N+1} \tag{B.2}$$

which is even in t, i.e.

$$v(x,-t) = v(x,t) \quad \text{a.e. in } \mathbb{R}^{N+1}.$$

If there exist $k \in \mathbb{N}$ and $c > 0$ such that

$$|v(z)| \le c(1+|z|^k) \qquad \text{for all } z \in \mathbb{R}^{N+1},$$

then v is a polynomial.

Proof. Let $a > -1$ and $v \in H^{1,a}_{\rm loc}(\mathbb{R}^{N+1})$ be a weak solution to (B.2) even in t. For $\alpha \in (0,1)$ and $k \in \mathbb{N}$, let $D_x^{\beta_k} v$ be a partial derivative in the variables $x = (x_1,...,x_N)$ of order $k = |\beta_k|$, with $\beta_k \in \mathbb{N}^N$ multiindex. Then, there exists a positive constant C depending only on N, α, a, k such that

$$\sup_{B_{r/2}} |D_x^{\beta_k} v| \le \frac{C}{r^k} \sup_{B_r} |v| \tag{B.3}$$

and

$$[D_x^{\beta_k} v]_{C^{0,\alpha}(B_{r/2})} \le \frac{C}{r^{k+\alpha}} \sup_{B_r} |v|, \tag{B.4}$$

where $[w]_{C^{0,\alpha}(\Lambda)} := \sup_{z,z'\in\Lambda} |z-z'|^{-\alpha}|w(z)-w(z')|$. In order to prove the previous inequalities we apply some local regularity estimates for even solutions contained in [36]. If $k=0$, then the inequalities follow by scaling

$$\|v\|_{C^{0,\alpha}(B_{1/2})} \le C\|v\|_{L^\infty(B_1)}$$

proved in [36, Theorem 1.2 part *i*)]. If $k \ge 1$, we remark that any partial derivation in variables x_i for $i \in \{1,...,N\}$ commutes with the operator $\operatorname{div}(|t|^a\nabla\cdot)$ and $D_x^{\beta_k} v$ are actually even solutions to the same equation, (see [36, Section 7] for details). Hence, inequalities (B.3) and (B.4) follow by scaling and iterating the estimate

$$\|v\|_{C^{1,\alpha}(B_{1/2})} \le C\|v\|_{L^\infty(B_1)}$$

proved in [36, Theorem 1.2 part *ii*)]. Indeed, fixed a multiindex β_k, we can choose

$$r_k = 1/2 < r_{k-1} < ... < r_0 = 1,$$

then

$$\begin{aligned}\|D_x^{\beta_k} v\|_{C^{0,\alpha}(B_{1/2})} &\le C_{k-1} \sup_{B_{r_{k-1}}} |D_x^{\beta_{k-1}} v| \le C_{k-1}C_{k-2} \sup_{B_{r_{k-2}}} |D_x^{\beta_{k-2}} v| \\ &\le ... \le \left(\prod_{i=0}^{k-1} C_i\right) \sup_{B_1} |v|.\end{aligned}$$

Once we have (B.3) and (B.4), we can proceed exactly as in proof of [7, Lemma 2.7]. We have only to remark that for any $a \in (-1,+\infty)$, given an even solution to (B.2) v, then $\partial^2_{tt} v + \frac{a}{t}\partial_t v = -\Delta_x v$ is also an even solution to (B.3). □

Now we are able to prove Proposition B.1.

Proof of Proposition B.1. Let $\Psi \in H^{1,1-2s}_{\rm loc}(\overline{\mathbb{R}^{N+1}_+})$ be a weak solution to (B.1), such that

$$\Psi(z) = |z|^\gamma \Psi\left(\frac{z}{|z|}\right) \quad \text{in } \mathbb{R}^{N+1}_+,$$

for some $\gamma \ge 0$. The homogeneity condition trivially implies a polynomial global bound on the growth of Ψ. The same bound is inherited by the trace $\phi = \operatorname{Tr}\Psi$ on $\mathbb{R}^N = \partial\mathbb{R}^{N+1}_+$, which is also γ-homogeneous. Moreover, $\phi \in C^\infty(\Gamma^-)$ by [36, Theorem 1.1] and $\phi \in C^0(\mathbb{R}^N)$ by [27, Proposition 5.3]. With these premises, we can define the extension V of ϕ in the sense of [1, Lemma 3.3]. Actually, we introduce a minor change in the definition of the extension given in [1]; that is, for every $R > 0$ we define

$$\phi_R = \phi\eta_R \tag{B.5}$$

(instead of $\phi_R = \phi\chi_{B'_R}$), where $\eta_R \in C^\infty_c(B'_{2R})$ is a radially decreasing cut-off function with $|\eta_R| \le 1$ and $\eta_R \equiv 1$ in B'_R. We remark that the adjusted family of functions ϕ_R convoluted with the usual Poisson kernel of the upper half-space converge in a suitable way to the same extension V obtained by

Abatangelo and Ros-Oton in [1]. Moreover, defining the extension starting from (B.5), we can easily ensure that $V \in H^{1,1-2s}_{\rm loc}(\overline{\mathbb{R}^{N+1}_+})$ and that it is weak solution to (B.1). Nevertheless, also V inherits from ϕ an at most polynomial growth. Let us consider $W = V - \Psi \in H^{1,1-2s}_{\rm loc}(\overline{\mathbb{R}^{N+1}_+})$, which weakly solves

$$\begin{cases} \operatorname{div}(t^{1-2s}\nabla W) = 0 & \text{in } \mathbb{R}^{N+1}_+, \\ \operatorname{Tr} W = 0 & \text{on } \mathbb{R}^N = \partial\mathbb{R}^{N+1}_+. \end{cases}$$

Then, denoting as $\widetilde{W}$ the odd reflection of W through $\mathbb{R}^N = \partial\mathbb{R}^{N+1}_+$, by [37, Proposition 2.10]

$$\overline{W} = \frac{\widetilde{W}}{t|t|^{2s-1}} \in H^{1,1+2s}_{\rm loc}(\mathbb{R}^{N+1})$$

is an even entire weak solution to (B.2) with $a = 1+2s \in (1,3)$. We have that $\overline{W}$ satisfies the assumptions of Lemma B.2, being a polynomial bound on its growth ensured by the polynomial bounds of Ψ and V. From Lemma B.2 we can promptly conclude that $\overline{W}$ is a polynomial. We also have that

$$t^{1-2s}\partial_t V = t^{1-2s}\partial_t \Psi + t^{1-2s}\partial_t(t^{2s}\overline{W}) = t^{1-2s}\partial_t\Psi + P_k$$

for some polynomial P_k of degree $k \in \mathbb{N}$. Hence, passing to the trace of the weighted derivative above, by [1, Lemma 3.3] it follows that

$$(-\Delta)^s\phi \overset{k+1}{=} 0 \quad \text{in } \Gamma^-$$

and $\phi = 0$ in Γ^+, where the above identity is meant in the sense of the notion of "fractional Laplacian modulus polynomials of degree at most k" given in [1, Definition 3.1], see also [11]. Hence, by [1, Theorem 3.10], we have that

$$\phi(x) = p(x)(x_N)^s_-,$$

for some polynomial p. By homogeneity of ϕ, this implies that necessary there exists $j \in \mathbb{N}$ such that $\gamma = j + s$. □

We are now going to derive from Proposition B.1 the explicit formula (1.13) for the eigenvalues of problem (1.11). We first observe that, if μ is an eigenvalue of (1.11) with an associated eigenfunction ψ, then the function $\Psi(\rho\theta) = \rho^\sigma\psi(\theta)$ with $\sigma = -\frac{N-2s}{2} + \sqrt{\left(\frac{N-2s}{2}\right)^2 + \mu}$ belongs to $H^{1,1-2s}_{\rm loc}(\overline{\mathbb{R}^{N+1}_+})$ and is a weak solution to (B.1). From Proposition B.1 we then deduce that there exists $j \in \mathbb{N}$ such that $\sigma = j + s$ and hence

$$\mu = (j+s)(j+N-s).$$

Viceversa, we prove now that all numbers of the form $\mu = (j+s)(j+N-s)$ with $j \in \mathbb{N}$ are eigenvalues of (1.11). For any fixed $j \in \mathbb{N}$, we consider the function Ψ defined, in cylindrical coordinates, as

$$\Psi(x', r\cos\tau, r\sin\tau) = r^{s+j}\left|\sin\left(\frac{\tau}{2}\right)\right|^{2s} {}_2F_1\left(-j, j+1; 1-s; \frac{1+\cos\tau}{2}\right), \; r \ge 0, \; \tau \in [0, 2\pi],$$

where ${}_2F_1$ is the hypergeometric function. From [29] we have that $\Psi \in H^{1,1-2s}_{\rm loc}(\overline{\mathbb{R}^{N+1}_+})$ is a weak solution to (B.1). Furthermore Ψ is homogeneous of degree $s + j$ and therefore the function $\psi := \Psi\big|_{\mathbb{S}^N_+}$ belongs to $\mathcal{H}_0$, $\psi \not\equiv 0$, and

$$\Psi(\rho\theta) = \rho^{s+j}\psi(\theta), \quad \rho \ge 0, \; \theta \in \mathbb{S}^N_+.$$

Plugging the above characterization of Ψ into (B.1), we obtain that

$$\rho^{j-1-s}\Big((j+s)(j+N-s)\theta_{N+1}^{1-2s}\psi(\theta) + \operatorname{div}_{\mathbb{S}^N}\left(\theta_{N+1}^{1-2s}\nabla_{\mathbb{S}^N}\psi\right)\Big) = 0, \quad \rho > 0, \; \theta \in \mathbb{S}^N_+,$$

so that $(j+s)(j+N-s)$ is an eigenvalue of (1.11).

We then conclude that the set of all eigenvalues of problem (1.11) is $\{(j+s)(j+N-s) : j \in \mathbb{N}\}$.

Acknowledgments

We thank Giorgio Tortone for kindly providing us with the reference [1].

References

[1] Abatangelo, N., and Ros-Oton, X. Obstacle problems for integro-differential operators: higher regularity of free boundaries. *Adv. Math. 360* (2020), 106931, 61.

[2] Adolfsson, V., and Escauriaza, L. $C^{1,\alpha}$ domains and unique continuation at the boundary. *Comm. Pure Appl. Math. 50*, 10 (1997), 935–969.

[3] Adolfsson, V., Escauriaza, L., and Kenig, C. Convex domains and unique continuation at the boundary. *Rev. Mat. Iberoamericana 11*, 3 (1995), 513–525.

[4] Bonforte, M., Figalli, A., and Vázquez, J. L. Sharp boundary behaviour of solutions to semilinear nonlocal elliptic equations. *Calc. Var. Partial Differential Equations 57*, 2 (2018), Paper No. 57, 34.

[5] Brändle, C., Colorado, E., de Pablo, A., and Sánchez, U. A concave-convex elliptic problem involving the fractional Laplacian. *Proc. Roy. Soc. Edinburgh Sect. A 143*, 1 (2013), 39–71.

[6] Caffarelli, L., and Silvestre, L. An extension problem related to the fractional Laplacian. *Comm. Partial Differential Equations 32*, 7-9 (2007), 1245–1260.

[7] Caffarelli, L. A., Salsa, S., and Silvestre, L. Regularity estimates for the solution and the free boundary of the obstacle problem for the fractional Laplacian. *Invent. Math. 171*, 2 (2008), 425–461.

[8] Cotsiolis, A., and Tavoularis, N. K. Best constants for Sobolev inequalities for higher order fractional derivatives. *J. Math. Anal. Appl. 295*, 1 (2004), 225–236.

[9] De Luca, A., and Felli, V. Unique continuation from the edge of a crack. *Math. Eng. 3*, 3 (2021), Paper No. 023, 40.

[10] Dipierro, S., Felli, V., and Valdinoci, E. Unique continuation principles in cones under nonzero Neumann boundary conditions. *Ann. Inst. H. Poincaré Anal. Non Linéaire 37*, 4 (2020), 785–815.

[11] Dipierro, S., Savin, O., and Valdinoci, E. Definition of fractional Laplacian for functions with polynomial growth. *Rev. Mat. Iberoam. 35*, 4 (2019), 1079–1122.

[12] Fabes, E. B., Kenig, C. E., and Serapioni, R. P. The local regularity of solutions of degenerate elliptic equations. *Comm. Partial Differential Equations 7*, 1 (1982), 77–116.

[13] Fall, M. M., and Felli, V. Unique continuation property and local asymptotics of solutions to fractional elliptic equations. *Comm. Partial Differential Equations 39*, 2 (2014), 354–397.

[14] Fall, M. M., and Felli, V. Unique continuation properties for relativistic Schrödinger operators with a singular potential. *Discrete Contin. Dyn. Syst. 35*, 12 (2015), 5827–5867.

[15] Fall, M. M., Felli, V., Ferrero, A., and Niang, A. Asymptotic expansions and unique continuation at Dirichlet-Neumann boundary junctions for planar elliptic equations. *Math. Eng. 1*, 1 (2019), 84–117.

[16] Felli, V., and Ferrero, A. Almgren-type monotonicity methods for the classification of behaviour at corners of solutions to semilinear elliptic equations. *Proc. Roy. Soc. Edinburgh Sect. A 143*, 5 (2013), 957–1019.

[17] Felli, V., and Ferrero, A. Unique continuation and classification of blow-up profiles for elliptic systems with Neumann boundary coupling and applications to higher order fractional equations. *Nonlinear Anal. 196* (2020), 111826, 28.

[18] Felli, V., and Ferrero, A. Unique continuation principles for a higher order fractional Laplace equation. *Nonlinearity 33*, 8 (2020), 4133–4190.

[19] Felli, V., Ferrero, A., and Terracini, S. Asymptotic behavior of solutions to Schrödinger equations near an isolated singularity of the electromagnetic potential. *J. Eur. Math. Soc. (JEMS) 13*, 1 (2011), 119–174.

[20] Felli, V., Ferrero, A., and Terracini, S. On the behavior at collisions of solutions to Schrödinger equations with many-particle and cylindrical potentials. *Discrete Contin. Dyn. Syst. 32*, 11 (2012), 3895–3956.

[21] Fernández-Real, X., and Ros-Oton, X. Stable cones in the thin one-phase problem. *Preprint,* `arXiv:2009.11626` (2020).

[22] García-Ferrero, M. A., and Rüland, A. Strong unique continuation for the higher order fractional Laplacian. *Math. Eng. 1*, 4 (2019), 715–774.

[23] Garofalo, N., and Lin, F.-H. Monotonicity properties of variational integrals, A_p weights and unique continuation. *Indiana Univ. Math. J. 35*, 2 (1986), 245–268.

[24] Grisvard, P. *Elliptic problems in nonsmooth domains*, vol. 24 of *Monographs and Studies in Mathematics.* Pitman (Advanced Publishing Program), Boston, MA, 1985.

[25] Jin, T., Li, Y., and Xiong, J. On a fractional Nirenberg problem, part I: blow up analysis and compactness of solutions. *J. Eur. Math. Soc. (JEMS) 16*, 6 (2014), 1111–1171.

[26] Kukavica, I., and Nyström, K. Unique continuation on the boundary for Dini domains. *Proc. Amer. Math. Soc. 126*, 2 (1998), 441–446.

[27] Niang, A. Boundary regularity for a degenerate elliptic equation with mixed boundary conditions. *Commun. Pure Appl. Anal. 18*, 1 (2019), 107–128.

[28] Ros-Oton, X., and Serra, J. The Dirichlet problem for the fractional Laplacian: regularity up to the boundary. *J. Math. Pures Appl. (9) 101*, 3 (2014), 275–302.

[29] Ros-Oton, X., and Serra, J. Boundary regularity for fully nonlinear integro-differential equations. *Duke Math. J. 165*, 11 (2016), 2079–2154.

[30] Ros-Oton, X., and Serra, J. Boundary regularity estimates for nonlocal elliptic equations in C^1 and $C^{1,\alpha}$ domains. *Ann. Mat. Pura Appl. (4) 196*, 5 (2017), 1637–1668.

[31] Rüland, A. Unique continuation for fractional Schrödinger equations with rough potentials. *Comm. Partial Differential Equations 40*, 1 (2015), 77–114.

[32] Seo, I. On unique continuation for Schrödinger operators of fractional and higher orders. *Math. Nachr. 287*, 5-6 (2014), 699–703.

[33] Seo, I. Carleman inequalities for fractional Laplacians and unique continuation. *Taiwanese J. Math. 19*, 5 (2015), 1533–1540.
[34] Seo, I. Unique continuation for fractional Schrödinger operators in three and higher dimensions. *Proc. Amer. Math. Soc. 143*, 4 (2015), 1661–1664.
[35] Sire, Y., Terracini, S., and Tortone, G. On the nodal set of solutions to degenerate or singular elliptic equations with an application to s-harmonic functions. *J. Math. Pures Appl. (9) 143* (2020), 376–441.
[36] Sire, Y., Terracini, S., and Vita, S. Liouville type theorems and regularity of solutions to degenerate or singular problems part I: even solutions. *Comm. Partial Differential Equations 46*, 2 (2021), 310–361.
[37] Sire, Y., Terracini, S., and Vita, S. Liouville type theorems and regularity of solutions to degenerate or singular problems part II: odd solutions. *Math. Eng. 3*, 1 (2021), Paper No. 5, 50.
[38] Tao, X., and Zhang, S. Boundary unique continuation theorems under zero Neumann boundary conditions. *Bull. Austral. Math. Soc. 72*, 1 (2005), 67–85.
[39] Tao, X., and Zhang, S. Weighted doubling properties and unique continuation theorems for the degenerate Schrödinger equations with singular potentials. *J. Math. Anal. Appl. 339*, 1 (2008), 70–84.
[40] Yang, R. On higher order extensions for the fractional Laplacian. *Preprint,* arXiv:1302.4413 (2013).
[41] Yu, H. Unique continuation for fractional orders of elliptic equations. *Ann. PDE 3*, 2 (2017), Paper No. 16, 21.

(A. De Luca, V. Felli) Dipartimento di Matematica e Applicazioni
Università degli Studi di Milano - Bicocca
Via Cozzi 55, 20125, Milano, Italy
E-mail address: a.deluca41@campus.unimib.it, veronica.felli@unimib.it

(S. Vita) Dipartimento di Matematica "Giuseppe Peano"
Università degli Studi di Torino
Via Carlo Alberto 10, 10123, Torino, Italy
E-mail address: stefano.vita@unito.it