FREE GROUP REPRESENTATIONS: DUPLICITY ON THE BOUNDARY

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ABSTRACT. We present a powerful theorem for proving the irreducibility of tempered unitary representations of the free group.

1. INTRODUCTION

Let Γ be a finitely generated non-abelian free group and let $\pi : \Gamma \to \mathcal{U}(\mathcal{H})$ be a unitary representation of Γ. Let $\partial\Gamma$ be the usual boundary of Γ. When π is *tempered*, that is when it is weakly contained in the regular representation, one can sometimes view $\mathcal{H}$ as an L^2-space on $\partial\Gamma$, where the action of $x \in \Gamma$ on an L^2-function is given by a two-stage operation: first, translate the function via x; second, apply some pointwise linear operation. Indeed, one can always view $\mathcal{H}$ as a subspace of an L^2-space with such an action. See Proposition 2.2.

Now suppose that π is irreducible. There are lots of examples where there are precisely two (essentially different) ways in which $\mathcal{H}$ can be identified with an L^2-space on $\partial\Gamma$. Roughly speaking, this is the phenomenon which we call *duplicity*. The single most important result here, the Duplicity Theorem, starts with the hypothesis that we have two such identifications. There is a further technical hypothesis, a Finite Trace Condition, which holds for many interesting examples and fails for many others. Our first main conclusion is that there are no identifications beyond the two we started with.

It is not necessary to suppose that π is irreducible. Alternative hypotheses, much easier to prove, give irreducibility as a second main conclusion. See [PS96] for an application of this technique. The biggest known family of examples where the hypotheses and conclusions of the Duplicity Theorem hold is a certain subfamily of the representations described in [KS04]. We know of no other method to prove their irreducibility in a uniform manner.

A third conclusion, under the same hypotheses, is an analogue of Schur orthogonality. In the formula below A_π is a positive constant, $|x|$ stands for the word-length of $x \in \Gamma$, $v_1, v_2 \in \mathcal{H}$, while v_3 and v_4

1991 *Mathematics Subject Classification.* Primary; 22D10, 43A65. Secondary: 15A48, 22E45, 22E40.

Key words and phrases. free group, crossed-product C^*-algebra, irreducible, tempered, unitary representation.

must be chosen in a certain dense subspace $\mathcal{H}^\infty \subset \mathcal{H}$.

$$\lim_{\epsilon\to 0+} \epsilon \sum_{x\in\Gamma} e^{-\epsilon|x|}\langle v_1, \pi(x)v_3\rangle\overline{\langle v_2, \pi(x)v_4\rangle} = 2A_\pi\langle v_1, v_2\rangle\overline{\langle v_3, v_4\rangle}\ . \tag{1}$$

Theorem 2.4 gives a more elaborate version of this identity which connects limits of this sort with the two identifications of $\mathcal{H}$ with L^2-spaces on $\partial\Gamma$.

Besides duplicity, there are examples of irreducible π illustrating two other phenomena: *monotony*, where there is only one identification between $\mathcal{H}$ and an L^2-space on $\partial\Gamma$ and *oddity*, where there is only one identification, but it has to be with a proper subspace of the L^2-space. This paper has nothing to say about monotony, but there is an Oddity Theorem, closely analogous to the Duplicity Theorem.

2. Definitions and statements of results

So let Γ be a non-abelian free group on a given finite set of free generators. Let $A \subseteq \Gamma$ consist of those generators and their inverses. Let also $\pi : \Gamma \to \mathcal{U}(\mathcal{H})$ be a unitary representation of Γ.

Without going into the matter in any detail, we warn the reader that Γ is *not* a Type I group (see [Dix64]). Among other things, this means that a given unitary representation may be decomposable as a direct integral of irreducibles in more than one way. Also, the unitary dual of Γ, the space of equivalence classes of unitary irreducibles, cannot be parametrized by any standard Borel space (see [Hjo98] and [Gli61]) which means in practice that one cannot hope for a parametrization that one could actually work with. Moreover, the usual machinery of character theory is not applicable.

So what can one do? Many papers construct specific families of representations and prove them irreducible. See for example [Yos51], [PS86], [FTP82], [FTS94], [PS96], [KS96], [Pas01], [KS04], and [BG16]. Some of these papers also prove inequivalence of representations, either within or between families. The first objective of this paper is to explain a powerful indirect method for proving irreducibility and inequivalence.

All the representations which will be of interest here have "realizations" as L^2-spaces on the *boundary* of Γ. Recall that the Cayley graph of Γ with respect to A is a tree, and that this tree has a standard compactification which is obtained by adjoining a boundary, which we denote $\partial\Gamma$. This boundary can be described as the space of ends of the tree; it also coincides with the boundary of Γ considered as a Gromov hyperbolic group. Concretely, if we identify Γ with the set of finite reduced words:

$$\{a_1 a_2 \dots a_n\,;\ a_j \in A, a_j a_{j+1} \neq 1\}$$

then we can identify $\partial\Gamma$ with the set of infinite reduced words

$$\{a_1 a_2 a_3 \dots\,;\ a_j \in A, a_j a_{j+1} \neq 1\}\ .$$

For $x \in \Gamma$ let $\Gamma(x)$ be the set of finite reduced words which start with the reduced word for x; let $\partial\Gamma(x)$ be the set of infinite reduced words which start with the reduced word for x. A basis for the topology on the compactification $\Gamma \sqcup \partial\Gamma$ is given by the singletons $\{x\}$ and the sets $\Gamma(x) \sqcup \partial\Gamma(x)$, as x varies through Γ. The left-action of Γ on Γ extends to a continuous action on the compactification.

Let $C(\partial\Gamma)$ be the commutative C^*-algebra of continuous complex valued functions on $\partial\Gamma$. Likewise for $C(\Gamma \sqcup \partial\Gamma)$. If one wishes to identify an abstract Hilbert space $\mathcal{H}'$ with an L^2-space on $\partial\Gamma$, the essence of the identication is given by the action of $C(\partial\Gamma)$ on $\mathcal{H}'$ corresponding to pointwise multiplication. This action exists no matter what measure on $\partial\Gamma$ is used to construct the L^2-space; also the L^2-space might be vector-valued rather than scalar-valued; indeed the dimension of the vectors might vary in some measurable way from point to point of $\partial\Gamma$. It is the *spectral theorem* for $C(\partial\Gamma)$ (see [Rud73]) which tells us that any $C(\partial\Gamma)$-action on $\mathcal{H}'$ does indeed correspond to an identification of $\mathcal{H}'$ with an L^2-space on $\partial\Gamma$.

Obviously, a Hilbert space with no further structure can be identified with an L^2-space on $\partial\Gamma$ in many, many different ways. Now suppose the Hilbert space carries a unitary representation, $\pi' : \Gamma \to \mathcal{U}(\mathcal{H}')$. We would like to identify $\mathcal{H}'$ with an L^2-space on $\partial\Gamma$ in such a way that the Γ-action on $\mathcal{H}'$ matches up with the Γ-action on $\partial\Gamma$. Specifically, we would like the operator $\pi'(x)$ to be a two-stage operation as described in the introduction: first, translate an L^2-function on $\partial\Gamma$ via x; second, apply some pointwise linear operation.

While it is not hard to make this precise, it is more efficient to express the concept in terms of the compatibility between the two actions on $\mathcal{H}'$: the action of Γ and the action of $C(\partial\Gamma)$. Denote both of these actions by π'. Let $\lambda : \Gamma \to \mathrm{Aut}(C(\partial\Gamma))$ be given by:

$$(\lambda(x)G)(\omega) = G(x^{-1}\omega)$$

i.e. left-translation. The desired compatability is:

$$\pi'(x)\pi'(G)\pi'(x)^{-1} = \pi'(\lambda(x)G) \qquad \text{for } x \in \Gamma \text{ and } G \in C(\partial\Gamma). \tag{2}$$

In fact, a pair of actions which satisfy (2) fit together to give a representation of a certain C^*-algebra, the *crossed-product C^*-algebra*, denoted $\Gamma \ltimes C(\partial\Gamma)$. Vice versa, any $\Gamma \ltimes C(\partial\Gamma)$-representation comes from a pair of actions which fit together as per (2). The definition of $\Gamma \ltimes C(\partial\Gamma)$ is standard, and can be found, for example, in [Dav96], but there is also a short explanation in the following section.

Given a unitary representation $\pi : \Gamma \to \mathcal{U}(\mathcal{H})$, when is it possible to identify $\mathcal{H}$ with the representation space of a $\Gamma \ltimes C(\partial\Gamma)$-representation? Quite often, as it happens, but to get a clean answer we have to modify the question: when is it possible to identify $\mathcal{H}$ with a subspace of the representation space of a $\Gamma \ltimes C(\partial\Gamma)$-representation?

Definition 2.1. Let $\pi : \Gamma \to \mathcal{U}(\mathcal{H})$ be a unitary representation of Γ. A *boundary realization* of π is an isometric Γ-inclusion ι of $\mathcal{H}$ into $\mathcal{H}'$ where

- $\mathcal{H}'$ is the representation space of a $\Gamma \ltimes C(\partial\Gamma)$-representation π',
- and $\iota(\mathcal{H})$ is cyclic for the action of $C(\partial\Gamma)$ on $\mathcal{H}'$.

One thinks of the map ι as an identification of $\mathcal{H}$ with a subspace of an L^2-space on $\partial\Gamma$, where the L^2-space carries a Γ-action compatible with the Γ-action on $\partial\Gamma$. If one omits the second condition in the definition, and if one had a boundary realization as above, then one could replace $\mathcal{H}'$ with $\mathcal{H}' \oplus \mathcal{H}''$ and ι with $\iota \oplus 0$ for any second $\Gamma \ltimes C(\partial\Gamma)$-representation space $\mathcal{H}''$. It is convenient to exclude this second, essentially irrelevant, summand.

Proposition 2.2. *A unitary representation $\pi : \Gamma \to \mathcal{U}(\mathcal{H})$ has a boundary realization if and only if π is weakly contained in the regular representation of Γ.*

Suppose π is *irreducible* and weakly contained in the regular representation. How many different boundary realizations does it have? To make this question precise, we need

Definition 2.3. Let $\pi : \Gamma \to \mathcal{U}(\mathcal{H})$ be a unitary representation of Γ. Two boundary realizations $\iota_j : \mathcal{H} \to \mathcal{H}'_j$ are *equivalent* if there exists a unitary map $J : \mathcal{H}'_1 \to \mathcal{H}'_2$ between the two representation spaces which intertwines both the Γ-actions and the $C(\partial\Gamma)$-actions and such that $J\iota_1 = \iota_2$.

How many inequivalent boundary realizations does π have? There are many known examples where the answer is one; also many known examples where the answer is not one. Call a boundary realization $\iota : \mathcal{H} \to \mathcal{H}'$ a *perfect boundary realization* if ι is a unitary equivalence, i.e. a bijection and not just an injection. In many of the known cases where π has more than one boundary realization, it has exactly two perfect boundary realizations and all other boundary realizations are obtained as combinations of those two. Indeed, given the known examples, one can conjecture that this is the only possibility when π has more than one realization. See the afterword to [KS01] for a more detailed version of this conjecture, and also for indications of how things stand for some known families of representations.

The second main objective of this paper is to present a theorem which (in many cases) allows one to prove, for a representation which has two known perfect boundary realizations, that there are no others.

Duplicity Theorem Using (FTC). *Let $\pi : \Gamma \to \mathcal{U}(\mathcal{H})$ be a unitary representation of Γ. Suppose*

- *$(\pi'_\pm, \mathcal{H}'_\pm)$ are two irreducible $\Gamma \ltimes C(\partial\Gamma)$-representations, inequivalent as $\Gamma \ltimes C(\partial\Gamma)$-representations.*

- *$\iota_\pm : \mathcal{H} \to \mathcal{H}'_\pm$ are two perfect boundary realizations of π.*
- *The following Finite Trace Condition (FTC) holds*

$$\|(\iota_+^* \pi'(\mathbf{1}_{\partial\Gamma(a)})\iota_+)(\iota_-^* \pi'(\mathbf{1}_{\partial\Gamma(b)})\iota_-)\|_{HS} < \infty$$

for $a, b \in A$, $a \neq b$.

Then

- *π is irreducible as a Γ-representation.*
- *Up to equivalence, ι_+ and ι_- are the only perfect boundary realizations of π.*
- *Any imperfect boundary realization of π is equivalent to the map $\sqrt{t_+}\,\iota_+ \oplus \sqrt{t_-}\,\iota_- : \mathcal{H} \to \mathcal{H}'_+ \oplus \mathcal{H}'_-$ for constants $t_+, t_- > 0$ with $t_+ + t_- = 1$.*

A representation π which satisfies the conclusions of this theorem is said to satisfy *duplicity*. Examples where both the hypotheses and the conclusions are valid include the representations of [Yos51] and the non-endpoint representations of [FTP82]. There are other examples of representations where the (FTC) fails but which nonetheless satisfy duplicity. There are further examples without (FTC) where duplicity appears to hold, but for which we have no proof.

The Duplicity Theorem is also a tool for proving the irreducibility of π. To apply it, one must establish the irreducibility and inequivalence of the two $\Gamma \ltimes C(\partial\Gamma)$-representations, $\pi'_\pm$, but proving irreducibility for $\Gamma \ltimes C(\partial\Gamma)$-representations is far easier than proving irreducibility for Γ-representations.

There are lots of examples of irreducible representations π which have only one realization, that realization perfect: *monotony*. Proving this requires different methods than those presented here. See [KS01], [KSS16] and [BG16]. On the other hand, representations which have a single *imperfect* realization can, if an appropriate Finite Trace Condition holds, be attacked with the same methods as in the case of duplicity.

Oddity Theorem Using (FTC). *Let $\pi : \Gamma \to \mathcal{U}(\mathcal{H})$ be a unitary representation of Γ. Suppose*

- *$(\pi', \mathcal{H}')$ is an irreducible $\Gamma \ltimes C(\partial\Gamma)$-representation.*
- *$\iota : \mathcal{H} \to \mathcal{H}'$ is an imperfect realization of π.*
- *The following Finite Trace Condition (FTC) holds*

$$\|P_2 \pi'(\mathbf{1}_{\partial\Gamma(a)}) P_1\|_{HS} < \infty \qquad \text{for each } a \in A$$

where $P_1 : \mathcal{H}' \to \mathcal{H}'$ is the projection onto $\iota(\mathcal{H})$ and $P_2 = \mathrm{Id} - P_1$ is the projection onto the orthogonal complement of $\iota(\mathcal{H})$.

Then

- *π is irreducible as a Γ-representation.*
- *Up to equivalence, ι is the only boundary realization of π.*

Observe that the unitary Γ-action which π' gives on $\mathcal{H}'$ stabilizes $\mathcal{H}_1 = \iota(\mathcal{H})$, so it also stabilizes the orthogonal complement $\mathcal{H}_2 = \mathcal{H} \ominus \mathcal{H}_1$. Let $\pi_2 : \Gamma \to \mathcal{U}(\mathcal{H}_2)$ denote the Γ-action on $\mathcal{H}_2$. One can also conclude:

- *π_2 is irreducible.*
- *π_2 is inequivalent to π.*

A representation π which satisfies the conclusions of this theorem is said to satisfy *oddity*. Among other examples, the non-endpoint representations of [Pas01] satisfy the hypotheses of the theorem, and are examples of oddity. There are also known examples where the (FTC) fails, but oddity holds nonetheless, and yet other known examples where the (FTC) fails, and oddity appears to hold, but is not proved.

The third main objective of this paper is to prove Schur orthogonality relations, like (1). For any $\Gamma \ltimes C(\partial\Gamma)$-representation π' and any function $G \in C(\Gamma \sqcup \partial\Gamma)$, let $\pi'(G) = \pi'\big(G|_{\partial\Gamma}\big)$. Also, for $x \in \Gamma$, let $G^*(x) = \bar{G}(x^{-1})$.

Theorem 2.4. *Let π be a representation of Γ on $\mathcal{H}$ which satisfies the hypotheses of the Duplicity Theorem. There exists a constant $A_\pi > 0$ and a dense subspace $\mathcal{H}^\infty \subset \mathcal{H}$ of* good vectors *of $\mathcal{H}$ so that for any $v_1, v_2 \in \mathcal{H}$, $v_3, v_4 \in \mathcal{H}^\infty$, and $G, \tilde{G} \in C(\Gamma \sqcup \partial\Gamma)$ the following holds*

$$
\begin{aligned}
(3) \quad \lim_{\epsilon \to 0+} \epsilon \sum_{x \in \Gamma} e^{-\epsilon|x|} G(x) \tilde{G}^*(x) \langle v_1, \pi(x) v_3 \rangle \overline{\langle v_2, \pi(x) v_4 \rangle} \\
= A_\pi \Big(\langle \pi'_+(G)\iota_+ v_1, \iota_+ v_2 \rangle \overline{\langle \pi'_-(\tilde{G})\iota_- v_3, \iota_- v_4 \rangle} \\
+ \langle \pi'_-(G)\iota_- v_1, \iota_- v_2 \rangle \overline{\langle \pi'_+(\tilde{G})\iota_+ v_3, \iota_+ v_4 \rangle} \Big) .
\end{aligned}
$$

On the right-hand side of (3) one sees the two actions of $C(\partial\Gamma)$ on $\mathcal{H}$, namely $\iota_\pm^* \pi'_\pm(\cdot)\iota_\pm$, but the sum on the left-hand side is calculated using only the matrix coefficients of the original Γ-representation, π. If one knew a priori what the space of good vectors was, (3) would provide a canonical method for calculating the two boundary realizations starting with π. A slightly simpler formula holds for representations satisfying the conditions of the Oddity Theorem.

Section 3 discusses crossed-product algebras, representations weakly contained in the regular representation, and the proof of Proposition 2.2. Section 4 introduces some general machinery applicable to boundary realizations. Section 5 explains a certain inner product which is used in the main proofs. Section 6 discusses a certain limit closely related to the left-hand side of (3). Section 7 discusses the subspace $\mathcal{H}^\infty \subseteq \mathcal{H}$ of good vectors. Section 8 has the proofs of the Duplicity Theorem and the Oddity Theorem. Section 9 proves (1) and (3).

We follow the convention that a positive constant denoted by C may change its exact value from one line to the next. In general, $\mathbf{1}_S$ is the characteristic function of the set S. However, if $x \in \Gamma$, we abbreviate and write $\mathbf{1}_x$ for the characteristic function of $\partial\Gamma(x)$. Simply $\mathbf{1}$ usually stands for $\mathbf{1}_{\partial\Gamma}$. We assume that all our Hilbert spaces are separable.

3. The crossed product $\Gamma \ltimes C(\partial\Gamma)$ and boundary realizations

The reader who can do without the proof of Proposition 2.2 can skip this section. Or one can read up through the definition of crossed-product C^*-algebras, and skip the rest.

We said already that when a Γ-representation and a $C(\partial\Gamma)$-representation act on the same Hilbert space and satisfy the compatibility condition (2), then this pair of representations can be thought of as a representation of a certain crossed-product C^*-algebra, $\Gamma \ltimes C(\partial\Gamma)$. Here we define crossed-product algebras and clarify the above assertion. After that we give some further definitions. All of this is preparation for the proof of Proposition 3.4 which will give us half of Proposition 2.2.

Assume that Γ acts on a C^*-algebra $\mathcal{A}$ by isometric automorphisms $\lambda : \Gamma \to \mathrm{Aut}(\mathcal{A})$.

Definition 3.1. A *covariant representation* of $(\Gamma, \mathcal{A})$ on a Hilbert space $\mathcal{H}$ is a triple $(\pi, \alpha, \mathcal{H})$ where

- $\pi : \Gamma \to \mathcal{U}(\mathcal{H})$ is a unitary representation of Γ,
- $\alpha : \mathcal{A} \to \mathcal{B}(\mathcal{H})$ is a *-representation of $\mathcal{A}$,
- $\pi(x)\alpha(G)\pi(x)^{-1} = \alpha(\lambda(x)G)$ for every $G \in \mathcal{A}$ and $x \in \Gamma$.

Let $\mathcal{A}[\Gamma]$ denote the space of finitely supported functions from Γ to $\mathcal{A}$:

$$\mathcal{A}[\Gamma] = \left\{ \text{finite sums } \sum_i G_i \delta_{x_i} \,;\; x_i \in \Gamma,\, G_i \in \mathcal{A} \right\}$$

where δ_x denotes the Kroneker function at $x \in \Gamma$. We endow $\mathcal{A}[\Gamma]$ with a C^*-algebra structure as follows: the sum of two elements is defined in the obvious way (as functions on Γ) while for the multiplication and the adjoint we use

$$G_1\delta_x \cdot G_2\delta_y = G_1(\lambda(x)G_2)\delta_{xy}$$
$$(G\delta_x)^* = (\lambda(x^{-1})G^*)\delta_{x^{-1}}$$

and extend by linearity.

For any covariant representation $(\pi, \alpha, \mathcal{H})$ of $(\Gamma, \mathcal{A})$ and, for $\xi = \sum_i G_i \delta_{x_i}$ define

$$(\pi \ltimes \alpha)(\xi) = \sum_i \alpha(G_i)\pi(x_i) \ .$$

Using the covariance relation in Definition 3.1, one sees that $\pi \ltimes \alpha$ defines a $*$-representation of $\mathcal{A}[\Gamma]$. Define a norm on $\mathcal{A}[\Gamma]$ by

$$\|\xi\| = \sup \|\pi \ltimes \alpha(\xi)\|$$

where the supremum is taken over all covariant representations of $(\Gamma, \mathcal{A})$. The *full crossed product C^*-algebra* $\Gamma \ltimes \mathcal{A}$ is defined as the completion of $\mathcal{A}[\Gamma]$ with respect to the above norm.

In this paper we are interested in the following cases:

- $\mathcal{A} = \mathbf{C}$, the complex numbers, with the trivial action of Γ. In this case $\Gamma \ltimes \mathbf{C}$ is $C^*(\Gamma)$, the *full C^*-algebra of* Γ.
- $\mathcal{A} = C(K)$ where K is a second countable compact space on which Γ acts by homeomorphisms.

Remark 3.2. Let $\mathbf{1}_K$ denote the function identically one on the compact space K. The inclusion $\mathbf{C} \to C(K)$ defined by $z \to z\mathbf{1}_K$ induces a map $\psi : C^*(\Gamma) \to \Gamma \ltimes C^*(K)$ defined by

$$\psi(\sum_i c_i \delta_{x_i}) = \sum_i c_i \mathbf{1}_K \delta_{x_i} \ .$$

It is trivial to check that this formula gives a $*$-homomorphism $\mathbf{C}[\Gamma] \to C(K)[\Gamma]$. To pass to the completions, one needs ψ to be norm-decreasing, and this follows because any covariant representation $(\pi, \alpha, \mathcal{H})$ for $\Gamma \ltimes C(K)$ restricts to a covariant representation $(\pi, \alpha|_{\mathbf{C}\mathbf{1}_K}, \mathcal{H})$ for $C^*(\Gamma) = \Gamma \ltimes \mathbf{C}$.

Definition 3.3. A unitary representation $\pi : \Gamma \to \mathcal{U}(\mathcal{H})$ is *weakly contained in the regular representation π_{reg}* if for every $v \in \mathcal{H}$ there exists a sequence $v_n \in \ell^2(\Gamma)$ such that

$$\langle \pi(x)v, v\rangle = \lim_{n\to\infty} \langle \pi_{\mathrm{reg}}(x)v_n, v_n\rangle \quad \text{pointwise.}$$

Based on any Γ-representation π, we define a $C^*(\Gamma)$-representation, also denoted π. Let $\pi : C^*(\Gamma) \to \mathcal{B}(\mathcal{H})$ be the extension of the original π by linearity and continuity. The function $\phi(x) = \langle \pi(x)v, v\rangle$ used in the above definition is known as the *matrix coefficient* associated to v. This also extends by linearity and continuity to a functional $\phi : C^*(\Gamma) \to \mathbf{C}$ given by $\phi(\xi) = \langle \pi(\xi)v, v\rangle$. If we choose v with $\|v\| = 1$, this functional is the *state* corresponding to v.

Expressing Definition 3.3 using $C^*(\Gamma)$-representations and states, one finds that a representation $\pi : C^*(\Gamma) \to \mathcal{B}(\mathcal{H})$ is weakly contained in the regular representation if and only if, for every vector $v \in \mathcal{H}$ with $\|v\| = 1$, the corresponding state $\langle \pi(\cdot)v, v\rangle$ is a limit, in the weak*-topology, of states associated with the regular representation, that is of states of the form $\langle \pi_{\mathrm{reg}}(\cdot)v_n, v_n\rangle$ with $v_n \in \ell^2(\Gamma)$.

The proof of the following proposition depends on basic C^*-algebra theory. Only here do we make use of that theory, or of $C^*(\Gamma)$, or of the fact that $\Gamma \ltimes C(K)$ is a C^*-algebra.

Proposition 3.4. *Let Γ be a discrete countable group acting on a second countable, compact space K. Suppose that the unitary representation $\pi : \Gamma \to \mathcal{U}(\mathcal{H})$ is weakly contained in the regular representation.*

Then there exists an isometric Γ-inclusion $\iota : \mathcal{H} \to \mathcal{H}'$ where $\mathcal{H}'$ is the representation space of a $\Gamma \ltimes C(K)$-representation.

Proof. Fix any $k_0 \in K$ and define a $C(K)$-action on $\ell^2(\Gamma)$ by

$$(\pi'_{\mathrm{reg}}(G)f)(x) = G(xk_0)f(x) \ .$$

Together with π_{reg}, this gives a covariant representation as per Definition 3.1 and so gives a representation $\pi_{\mathrm{reg}} \ltimes \pi'_{\mathrm{reg}}$ of $\Gamma \ltimes C(K)$ on $\ell^2(\Gamma)$.

If the theorem is proved for each summand of a direct sum of representations, it is also proved for the sum. Consequently, we may assume that $\mathcal{H}$ has a unit vector v cyclic for π. Let ϕ be the state of $C^*(\Gamma)$ corresponding to v. By hypothesis, ϕ is the weak*-limit of a sequence $(\phi_n)_n$, where each ϕ_n is the state of $C^*(\Gamma)$ corresponding to a vector $v_n \in \ell^2(\Gamma)$. From the representation $\pi_{\mathrm{reg}} \ltimes \pi'_{\mathrm{reg}}$ and v_n one obtains also a state ϕ'_n of $\Gamma \ltimes C(K)$. The restriction of ϕ'_n to $C^*(\Gamma)$ via the map $C^*(\Gamma) \to \Gamma \ltimes C(K)$ is ϕ_n. The hypotheses on Γ and K guarantee that $\Gamma \ltimes C(K)$ is separable and unital. So, passing to a subsequence, we may assume that ϕ'_n weakly approaches some positive functional ϕ' on $\Gamma \ltimes C(K)$. The restriction of ϕ' to $C^*(\Gamma)$ will be ϕ. Apply the Gelfand–Naimark procedure to ϕ' to generate a representation $\pi' : \Gamma \ltimes C(K) \to \mathcal{B}(\mathcal{H}')$. The state ϕ' is the state of a certain vector $v' \in \mathcal{H}'$, cyclic for π'. If we restrict π' to $C^*(\Gamma)$, the state corresponding to v' is the restriction of ϕ', namely ϕ. This was also the state corresponding to v for π, so there exists a unique Γ-isometry $\iota : \mathcal{H} \to \mathcal{H}'$ with $\iota(v) = v'$. □

Proposition 3.4 proves one implication of Proposition 2.2. The proof of the other implication has been known for some time and we shall give references:

- The action of Γ on $\partial\Gamma$ is *topologically amenable*: see [Ada94] for a general hyperbolic group or the Appendix of [KS96] for the specific case of a free group.
- [Tak03, Chapter X, Theorem 3.8 and 3.15] explains how to realize a $\Gamma \ltimes C(\partial\Gamma)$-representation space as $L^2(\partial\Gamma, d\mu)$.
- Topological amenability implies that every unitary representation of Γ that is realized on $L^2(\partial\Gamma, d\mu)$ is weakly contained in the regular representation. For the precise statement see [Kuh94].

4. Boundary intertwiners ι, maps μ, and vectors F

Throughout this section we will be dealing with a fixed tempered unitary representation $\pi : \Gamma \to \mathcal{U}(\mathcal{H})$.

4.1. **From ι, to μ, to F, and back again.** We are basically interested in the *boundary realizations* of π, that is isometric Γ-maps $\iota : \mathcal{H} \to \mathcal{H}'$ where

- $\mathcal{H}'$ is the representation space of a $\Gamma \ltimes C(\partial\Gamma)$-representation π',

- and $\iota(\mathcal{H})$ is cyclic for the action of $\Gamma \ltimes C(\partial\Gamma)$ on $\mathcal{H}'$.

However it is often convenient to drop the condition that ι be an isometric inclusion, and to consider *all* Γ-maps $\mathcal{H} \to \mathcal{H}'$ satisfying these two conditions. Such maps are called *boundary intertwiners* of π. As is natural, we call two boundary intertwiners $\iota_1 : \mathcal{H} \to \mathcal{H}'_1$ and $\iota_2 : \mathcal{H} \to \mathcal{H}'_2$ *equivalent* if there exists a unitary $\Gamma \ltimes C(\partial\Gamma)$-equivalence $J : \mathcal{H}'_1 \to \mathcal{H}'_2$ such that $\iota_2 = J\iota_1$, and we shall write $\iota_1 \sim \iota_2$.

Remark 4.1. Because of the covariance condition for $\Gamma \ltimes C(\partial\Gamma)$-representations, $\pi'(C(\partial\Gamma))\iota(\mathcal{H})$ is $\pi'(\Gamma)$ invariant. Therefore $\iota(\mathcal{H})$ is cyclic for the action of $\Gamma \ltimes C(\partial\Gamma)$ if and only if it is cyclic for the action of $C(\partial\Gamma)$.

We are now going to establish the correspondence between the set of (equivalence classes of) boundary intertwiners of π and two other sets of objects. To establish a certain parallelism, we will call the boundary intertwiners *iota-intertwiners*.

Definition 4.2. An *iota-intertwiner* of π is none other than a boundary intertwiner of π. A *mu-map* for π is a linear map $\mu : C(\partial\Gamma) \to \mathcal{B}(\mathcal{H})$ which takes non-negative functions to positive semidefinite operators (a positive map) and also satisfies

$$\pi(x)\mu(G)\pi(x)^{-1} = \mu(\lambda(x)G) \qquad \text{for } x \in \Gamma. \tag{4}$$

An *Eff-vector* for π is a vector F of positive semidefinite operators in $\mathcal{B}(\mathcal{H})$, indexed by A, and satisfying $\mathcal{T}F = F$ where

$$(\mathcal{T}F)_a = \sum_{b\in A\,;\, ab\neq 1} \pi(a)F_b\pi(a)^{-1} . \tag{5}$$

We will use several times the following standard elementary lemma.

Lemma 4.3. *If $\mu : C(\partial\Gamma) \to \mathcal{B}(\mathcal{H})$ is a positive map, then there exists a constant $C > 0$ such that $\|\mu(G)\| \le C\|G\|_\infty$.*

Proof. If $G \ge 0$, then $0 \le G \le \|G\|_\infty \mathbf{1}$, hence $0 \le \mu(G) \le \|G\|_\infty \mu(\mathbf{1})$, hence $\|\mu(G)\| \le \|\mu(\mathbf{1})\|\|G\|_\infty$. To treat an arbitrary G, write it as the sum of its positive and negative real and imaginary parts. $\square$

If an iota-intertwiner $\iota : \mathcal{H} \to \mathcal{H}'$ is given, we associate to it the following mu-map:

$$\mu(G) = \iota^*\pi'(G)\iota . \tag{6}$$

To show that this is a mu-map, one checks positivity, which is trivial, and covariance

$$\begin{aligned}\pi(x)\mu(G)\pi(x)^{-1} &= \pi(x)\iota^*\pi'(G)\iota\pi(x)^{-1}\\ &= \iota^*\pi'(x)\pi'(G)\pi'(x)^{-1}\iota = \iota^*\pi'(\lambda(x)G)\iota = \mu(\lambda(x)G) .\end{aligned}$$

Lemma 4.4. *Given iota-intertwiners $(\iota_1, \mathcal{H}'_1)$ and $(\iota_2, \mathcal{H}'_2)$ of π with associated mu-maps μ_1 and μ_2, then $\iota_1 \sim \iota_2$ if and only if $\mu_1 = \mu_2$.*

Proof. Recall that $\iota_1 \sim \iota_2$ if and only if there exists a unitary $\Gamma \ltimes C(\partial\Gamma)$-map $J : \mathcal{H}_1' \to \mathcal{H}_2'$ such that this diagram commutes:

$$\begin{tikzcd} & \mathcal{H}_1' \arrow[dd, "J"] \\ \mathcal{H} \arrow[ur, "\iota_1"] \arrow[dr, "\iota_2"'] & \\ & \mathcal{H}_2' \end{tikzcd}$$

When such a J exists it is obvious that

$$\mu_2(G) = \iota_2^* \pi_2'(G)\iota_2 = \iota_1^* J^* \pi_2'(G) J \iota_1 = \iota_1^* \pi_1'(G)\iota_1 = \mu_1(G) \ .$$

Assume now that $\mu_1 = \mu_2$. We shall first define J on the dense subspace of $\mathcal{H}_1'$ consisting of finite linear combinations $\sum_j \pi_1'(G_j)\iota_1(v_j)$ with $G_j \in C(\partial\Gamma)$ and $v_j \in \mathcal{H}$ by letting

$$J\left(\sum_j \pi_1'(G_j)\iota_1(v_j)\right) = \sum_j \pi_2'(G_j)\iota_2(v_j) \ . \tag{7}$$

Since

$$\begin{aligned} \|\sum_j \pi_1'(G_j)\iota_1(v_j)\|^2 &= \sum_{j,k} \langle \pi_1'(G_j)\iota_1(v_j), \pi_1'(G_k)\iota_1(v_k)\rangle \\ &= \sum_{j,k} \langle \iota_1^*\pi_1'(\overline{G_k}G_j)\iota_1(v_j), v_k\rangle = \sum_{j,k}\langle \mu_1(\overline{G_k}G_j)v_j, v_k\rangle \\ &= \sum_{j,k}\langle \mu_2(\overline{G_k}G_j)v_j, v_k\rangle = \|\sum_j \pi_2'(G_j)\iota_2(v_j)\|^2 \ , \end{aligned}$$

J is well defined and extends to an isometry from $\mathcal{H}_1'$ to $\mathcal{H}_2'$. Likewise we see that J^{-1} exists and is isometric, so J is unitary. It follows from the definition that J intertwines the two actions of $C(\partial\Gamma)$ on the dense set $\sum_j \pi_1'(G_j)\iota_1(v_j)$ and hence everywhere. Finally, to see that J is a Γ-map compute

$$\begin{aligned} &\pi_2'(x)J\left(\sum_j \pi_1'(G_j)\iota_1(v_j)\right) = \pi_2'(x)\left(\sum_j \pi_2'(G_j)\iota_2(v_j)\right) \\ &= \sum_j \pi_2'(x)\pi_2'(G_j)\pi_2'(x)^{-1}\pi_2'(x)\iota_2(v_j) = \sum_j \pi_2'(\lambda(x)G_j)\iota_2(\pi(x)v_j) \\ &= J\left(\sum_j \pi_1'(\lambda(x)G_j)\iota_1(\pi(x)v_j)\right) = J\pi_1'(x)\left(\sum_j \pi_1'(G_j)\iota_1(v_j)\right) \ . \end{aligned}$$

□

Proposition 4.5. *Assume that $\mu : C(\partial\Gamma) \to \mathcal{B}(\mathcal{H})$ is a mu-map for π. Then there exists an iota-intertwiner ι such that μ is the mu-map associated to ι.*

Proof. Let $C(\partial\Gamma)\otimes\mathcal{H}$ be the algebraic tensor product of $C(\partial\Gamma)$ and $\mathcal{H}$. For finite sums $X=\sum_j G_j\otimes v_j$, $Y=\sum_j H_j\otimes w_j$ define

$$\langle \sum_j G_j\otimes v_j, \sum_j H_j\otimes w_j\rangle = \sum_{j,k}\langle \mu(\overline{H_k}G_j)v_j, w_k\rangle \tag{8}$$

where on the right-hand side of (8) we use the inner product of $\mathcal{H}$.

By Stinespring's Theorem [Sti55] μ is *completely positive* and it follows that (8) defines a semidefinite inner product on $C(\partial\Gamma)\otimes\mathcal{H}$. Let $\mathcal{H}'$ be the quotient-completion of $C(\partial\Gamma)\otimes\mathcal{H}$ with respect to this inner product.

Define

$$\iota:\mathcal{H}\to C(\partial\Gamma)\otimes\mathcal{H} \quad \text{by letting} \quad \iota(v)=\mathbf{1}\otimes v\ . \tag{9}$$

Let $C(\partial\Gamma)$ act on $\sum_j G_j\otimes v_j$ by

$$\pi'(G)\sum_j G_j\otimes v_j=\sum_j GG_j\otimes v_j \tag{10}$$

and check that

$$\begin{aligned}
&\pi'(GH)=\pi'(G)\pi'(H)\\
&\pi'(G+H)=\pi'(G)+\pi'(H)\\
&\langle\pi'(G)X,Y\rangle=\langle X,\pi'(\overline{G})Y\rangle\ .
\end{aligned}$$

In particular, if G is positive one has

$$\pi'(G)=\pi'(\sqrt{G}\sqrt{G})=\pi'(\sqrt{G})(\pi'(\sqrt{G}))^* \tag{11}$$

and hence $\pi'(G)$ is a positive operator. In order to extend $\pi'(G)$ to all $\mathcal{H}'$ we need to show that $|\langle\pi'(G)X,Y\rangle|\le C\|G\|_\infty\cdot\|X\|\cdot\|Y\|$ for some constant C. Assume first that G is positive. Since $0\le G\le\|G\|_\infty\mathbf{1}$, one has

$$\|\pi'(G)\|=\sup_{\|X\|\le1}\langle\pi'(G)X,X\rangle\le\sup_{\|X\|\le1}\langle\pi'(\|G\|_\infty\mathbf{1})X,X\rangle=\|G\|_\infty\ ,$$

and hence $\|\pi'(G)\|\le\|G\|_\infty$. When G is not positive, divide it up into its positive and negative real and imaginary parts.

Make Γ act on $C(\partial\Gamma)\otimes\mathcal{H}$ by

$$\pi'(x)\sum_j G_j\otimes v_j=\sum_j\lambda(x)G_j\otimes\pi(x)v_j\ . \tag{12}$$

It is obvious that π' defines a group action. It is easy to check that $\pi'(x)\iota = \iota\pi(x)$. To see that $\pi'(x)$ is unitary compute

$$\begin{aligned}\langle \pi'(x)\sum_j G_j \otimes v_j, \sum_k H_k \otimes w_k\rangle &= \sum_{j,k}\langle \mu((\lambda(x)G_j)\overline{H_k})\pi(x)v_j, w_k\rangle \\ &= \sum_{j,k}\langle \mu(G_j(\lambda(x^{-1})\overline{H_k}))v_j, \pi(x)^{-1}w_k\rangle \\ &= \langle \sum_j G_j \otimes v_j, \sum_k \lambda(x^{-1})H_k \otimes \pi(x)^{-1}w_k\rangle \\ &= \langle \sum_j G_j \otimes v_j, \pi'(x)^{-1}\sum_k H_k \otimes w_k\rangle \ .\end{aligned}$$

Since $\pi'(x)$ is bounded on $C(\partial\Gamma) \otimes \mathcal{H}$, it extends by continuity to $\mathcal{H}'$.

To see that $\pi' : \Gamma \to \mathcal{U}(\mathcal{H}')$ and $\pi' : C(\partial\Gamma) \to \mathcal{B}(\mathcal{H})$ satisfy (2) and give a $\Gamma \ltimes C(\partial\Gamma)$-representation, compute

$$\begin{aligned}\pi'(x)\pi'(G)\pi'(x^{-1})\sum_j G_j \otimes v_j &= \pi'(x)\sum_j G(\lambda(x^{-1})G_j) \otimes \pi(x)^{-1}v_j \\ &= \sum_j (\lambda(x)G)G_j \otimes v_j = \pi'(\lambda(x)G)\sum_j G_j \otimes v_j \ .\end{aligned}$$

This completes the construction of the iota-intertwiner. Now we check that the associated mu-map is the one we wanted:

$$\langle \iota^*\pi'(G)\iota v_1, v_2\rangle = \langle \pi'(G)\iota v_1, \iota v_2\rangle = \langle G \otimes v_1, \mathbf{1} \otimes v_2\rangle = \langle \mu(G)v_1, v_2\rangle \ .$$

□

Proposition 4.5 is a special case of a considerably more general fact: see Lemma 3.1 of [RSW89].

To any given mu-map $\mu : C(\partial\Gamma) \to \mathcal{B}(\mathcal{H})$, associate an Eff-vector $F(\mu) = F = (F_a)_{a\in A}$ by letting

$$F_a = \mu(\mathbf{1}_a) \ . \tag{13}$$

To check that F is an Eff-vector, observe that each F_a is positive semi-definite and check that $\mathcal{T}F = F$:

$$\begin{aligned}(\mathcal{T}F)_a &= \sum_{b\in A\,;\,ab\neq 1} \pi(a)F_b\pi(a)^{-1} = \sum_{b\in A\,;\,ab\neq 1} \mu(\lambda(a)\mathbf{1}_b) \\ &= \sum_{b\in A\,;\,ab\neq 1} \mu(\mathbf{1}_{ab}) = \mu(\mathbf{1}_a) = F_a \ .\end{aligned}$$

For mu-maps μ_1 and μ_2, we say $\mu_1 \leq \mu_2$ if $\mu_1(G) \leq \mu_2(G)$ as operators for every $G \geq 0$. For Eff-vectors F_1 and F_2, we say that $F_1 \leq F_2$ if $(F_1)_a \leq (F_2)_a$ as operators for each $a \in A$. Indeed we will use this notation whenever F_1 and F_2 are $|A|$-tuples of operators.

Lemma 4.6. *Assume that $\mu_i : C(\partial\Gamma) \to \mathcal{B}(\mathcal{H})$ $(i = 1, 2)$ are mu-maps for π and let $F_i = F(\mu_i)$. Then $F_1 = F_2$ if and only if $\mu_1 = \mu_2$. Furthermore $F_1 \le F_2$ if and only if $\mu_1 \le \mu_2$.*

Proof. Assume that $F_1 = F_2$. Take any $a \in A$ and let $x \in \Gamma$ be such that $|xa| = |x| + 1$. One deduces successively:

$$\begin{aligned} \mu_1(\mathbf{1}_a) &= \mu_2(\mathbf{1}_a) , \\ \pi(x)\mu_1(\mathbf{1}_a)\pi(x)^{-1} &= \pi(x)\mu_2(\mathbf{1}_a)\pi(x)^{-1} , \\ \mu_1(\lambda(x)\mathbf{1}_a) &= \mu_2(\lambda(x)\mathbf{1}_a) , \\ \mu_1(\mathbf{1}_{xa}) &= \mu_2(\mathbf{1}_{xa}) . \end{aligned}$$

Hence $\mu_1(\mathbf{1}_y) = \mu_2(\mathbf{1}_y)$ for all $y \in \Gamma$. By linearity, μ_1 and μ_2 agree on all locally constant functions. By continuity, they agree everywhere. If we replace the equalities in the above calculation with inequalities, we see that $F_1 \le F_2$ implies $\mu_1(\mathbf{1}_y) \le \mu_2(\mathbf{1}_y)$ for all $y \in \Gamma$. Since every positive function of $C(\partial\Gamma)$ can be uniformly approximated by *positive* linear combinations of $\mathbf{1}_y$ $(y \in \Gamma)$ we may conclude that $\mu_1 \le \mu_2$. □

Proposition 4.7. *Let F be any Eff-vector for π. Then there exists a unique mu-map μ for which F is the associated Eff-vector.*

Proof. Uniqueness is the first part of Lemma 4.6 above. Given F, we must construct μ. By (13) we must have $\mu(\mathbf{1}_a) = F_a$, and so by (4)

$$\mu(\mathbf{1}_{xa}) = \mu(\lambda(x)\mathbf{1}_a) = \pi(x)F_a\pi(x)^{-1} \qquad \text{when } |xa| = |x| + 1. \tag{14}$$

Working with (14) we start by defining μ on functions $G(\omega)$ which depend only on the first n letters of ω.

$$\mu(G) = \mu\Big(\sum_{xa\,;\,|xa|=|x|+1=n} G_{xa}\mathbf{1}_{xa}\Big) = \sum_{xa\,;\,|xa|=|x|+1=n} G_{xa}\pi(x)F_a\pi(x)^{-1} .$$

This same function $G(\omega)$ might also be considered as depending on the first $n+1$ letters of ω. This way of looking at G gives a different formula for $\mu(G)$:

$$\mu(G) = \sum_{xab\,;\,|xab|=|x|+2=n+1} G_{xa}\pi(xa)F_b\pi(xa)^{-1}$$

and our definition is consistent only if the two answers agree. They do agree due to the condition $F = \mathcal{T}F$ with $\mathcal{T}$ as in equation (5).

Now $\mu(G)$ is defined for all functions $G(\omega)$ which depend on only finitely many letters of ω. Let $C^\infty(\partial\Gamma)$ denote the subalgebra of all such functions. μ is a positive map on $C^\infty(\partial\Gamma)$ because $F_a \ge 0$ for all $a \in A$. To extend μ to all of $C(\partial\Gamma)$ by continuity we need the inequality $\|\mu(G)\| \le C\|G\|_\infty$ which follows as in the proof of Lemma 4.3.

Now we check for covariance: $\pi(x)\mu(G)\pi(x)^{-1} = \mu(\lambda(x)G)$. By continuity, it is enough to check this when $G \in C^\infty(\partial\Gamma)$. By linearity, it is enough to consider $G = \mathbf{1}_{ya}$ where $|ya| = |y| + 1 > |x|$. Then

$$\begin{aligned}\pi(x)\mu(\mathbf{1}_{ya})\pi(x)^{-1} &= \pi(x)\pi(y)F_a\pi(y)^{-1}\pi(x)^{-1} \\ &= \pi(xy)F_a\pi(xy)^{-1} = \mu(\mathbf{1}_{xya}) = \mu(\lambda(x)\mathbf{1}_{ya})\ .\end{aligned}$$

□

4.2. **Perfect realizations seen in terms of ι, μ, and F.**

Lemma 4.8. *Suppose that ι is an iota-intertwiner for π with associated mu-map μ. Then*

- *ι is a boundary realization if and only of $\mu(\mathbf{1}) = \mathrm{Id}$.*
- *ι is a perfect boundary realization if and only if μ is an algebra homomorphism.*

Proof. Observe that $\mu(\mathbf{1}) = \iota^*\pi'(\mathbf{1})\iota = \iota^*\iota$. Hence ι is an isometry if and only if $\mu(\mathbf{1}) = \mathrm{Id}$. Consider the second statement. If ι is perfect one has $\iota^*\iota = \iota\iota^* = \mathrm{Id}$. Consequently $\mu(G_1G_2) = \iota^*\pi'(G_1G_2)\iota = \iota^*\pi'(G_1)\iota\iota^*\pi'(G_2)\iota = \mu(G_1)\mu(G_2)$. The converse is more delicate. When μ is an algebra homomorphism we can make $\mathcal{H}$ itself into a $\Gamma \ltimes C(\partial\Gamma)$-representation space by defining $\tilde{\pi}'(x) = \pi(x)$ and $\tilde{\pi}'(G) = \mu(G)$ for $x \in \Gamma$ and $G \in C(\partial\Gamma)$. Letting $\tilde{\mathcal{H}}' = \mathcal{H}$ and $\tilde{\iota} = \mathrm{Id}$, we get a perfect boundary realization of $\mathcal{H}$. The mu-map corresponding to this realization is $G \mapsto \tilde{\iota}^*\tilde{\pi}'(G)\tilde{\iota} = \mu(G)$. Since ι and $\tilde{\iota}$ give the same mu-map, Lemma 4.4 says they are equivalent, so ι is also a perfect realization. □

Lemma 4.9. *Suppose that ι is an iota-intertwiner for π with associated Eff-vector (F_a). Then*

- *ι is a boundary realization if and only of $\sum_{a\in A} F_a = \mathrm{Id}$.*
- *ι is a perfect boundary realization if and only if in addition $F_a^2 = F_a$ for each $a \in A$.*

We need this well-known elementary lemma:

Lemma 4.10. *Suppose that $(P_j)_{1\le j\le n}$ are self-adjoint projections such that $\sum_{j=1}^n P_j = \mathrm{Id}$. Then the P_j are mutually orthogonal.*

Proof. Assume that $v = P_k v$. One has

$$\begin{aligned}\langle v, v\rangle = \langle \sum_{j=1}^n P_jP_kv, v\rangle &= \langle P_k^2 v, v\rangle + \sum_{j\neq k}\langle P_jP_kv, v\rangle \\ &= \langle v, v\rangle + \sum_{j\neq k}\langle P_jP_kv, v\rangle\ .\end{aligned}$$

Since $\langle P_jP_kv, v\rangle = \langle P_jP_kv, P_kv\rangle \ge 0$ all such summands in the above sum must be zero. Hence $P_kP_jP_k = P_kP_j(P_kP_j)^* = 0$ for $j \neq k$. □

Proof of Lemma 4.9. ι is a boundary realization if and only if $\mu(\mathbf{1}) = \mathrm{Id}$. Since $\mathbf{1} = \sum_{a\in A} \mathbf{1}_a$ and $F_a = \mu(\mathbf{1}_a)$, $\mu(\mathbf{1}) = \mathrm{Id}$ if and only if $\sum_{a\in A} F_a = \mathrm{Id}$.

If ι is a perfect realization, then μ is an algebra homomorphism, and so each $F_a = \mu(\mathbf{1}_a)$ satisfies $F_a^2 = F_a$. Vice versa, suppose that each F_a is a projection, necessarily orthogonal. Then for any $xa \in \Gamma$ with $|xa| = |x| + 1$, one has that $\mu(\mathbf{1}_{xa}) = \pi(x)\mu(\mathbf{1}_a)\pi(x)^{-1} = \pi(x)F_a\pi(x)^{-1}$ is also an orthogonal projection. For any $n \geq 1$, one has $\mathrm{Id} = \mu(\mathbf{1}) = \sum_{|xa|=|x|+1=n} \mu(\mathbf{1}_{xa})$ and so one may apply Lemma 4.10 and deduce that the projections in this sum are all mutually orthogonal, and the product of any two of them is zero. It follows easily that μ is an algebra homomorphism when restricted to the subalgebra $\sum_{|xa|=|x|+1=n} \mathbf{C}\,\mathbf{1}_{xa}$. This holds for any n, so μ is an algebra homomorphism on $C^\infty(\partial\Gamma)$, hence on all of $C(\partial\Gamma)$. $\square$

4.3. **Direct sums of boundary intertwiners.** Let μ_1 and μ_2 be two mu-maps, and let $(\iota_1, \mathcal{H}_1)$ and $(\iota_2, \mathcal{H}_2)$ be the associated boundary intertwiners. What is the boundary intertwiner associated to $\mu_1 + \mu_2$? Consider $(\iota_1 \oplus \iota_2, \mathcal{H}'_1 \oplus \mathcal{H}'_2)$. It is trivial to check that this has $\mu_1 + \mu_2$ as its mu-map. However it may not satisfy the condition that $(\iota_1 \oplus \iota_2)(\mathcal{H})$ is cyclic in $\mathcal{H}'_1 \oplus \mathcal{H}'_2$. So let $\mathcal{H}'_S$ be the closure of $(\pi'_1 \oplus \pi'_2)(C(\partial\Gamma))(\iota_1 \oplus \iota_2)(\mathcal{H})$, and define ι_S by the following diagram:

$$\begin{tikzcd}
& \mathcal{H}'_S \arrow[d] \\
\mathcal{H} \arrow[ur, "\iota_S"] \arrow[dr, "\iota_1\oplus\iota_2"'] & \\
& \mathcal{H}'_1 \oplus \mathcal{H}'_2
\end{tikzcd}$$

The boundary intertwiner we are looking for is $(\iota_S, \mathcal{H}'_S)$, which we will denote by $(\iota_1, \mathcal{H}'_1) \oplus (\iota_2, \mathcal{H}'_2)$ and call the *direct sum* of the two boundary intertwiners.

Lemma 4.11. *Let $(\iota_j, \mathcal{H}'_j)_{j=1,2}$ be boundary intertwiners mapping $\mathcal{H}$ to* irreducible and inequivalent *$\Gamma \ltimes C(\partial\Gamma)$-spaces $(\mathcal{H}'_j)_{j=1,2}$. Then their direct sum is just $(\iota_1 \oplus \iota_2, \mathcal{H}'_1 \oplus \mathcal{H}'_2)$.*

Proof. By construction, $\mathcal{H}'_S$ is a $\Gamma \ltimes C(\partial\Gamma)$-subspace of $\mathcal{H}'_1 \oplus \mathcal{H}'_2$. Because $\mathcal{H}'_1$ and $\mathcal{H}'_2$ are irreducible and inequivalent, the only possibilities for $\mathcal{H}'_S$ are 0, $\mathcal{H}'_1$, $\mathcal{H}'_2$, and $\mathcal{H}'_1 \oplus \mathcal{H}'_2$. Since $\iota_1(\mathcal{H}) \oplus \iota_2(\mathcal{H}) \subseteq \mathcal{H}'_S$ we see that only $\mathcal{H}'_S = \mathcal{H}'_1 \oplus \mathcal{H}'_2$ is possible. $\square$

4.4. **Scalar multiples of boundary intertwiners.** Let μ be a mu-map and let $(\iota, \mathcal{H}')$ be the associated boundary intertwiner. For $t > 0$, $t\mu$ is another mu-map, whose associated boundary intertwiner is $(\sqrt{t}\,\iota, \mathcal{H}')$. For $t = 0$, we have $t\mu = 0$, and the associated boundary intertwiner is the zero map to the zero $\Gamma \ltimes C(\partial\Gamma)$-space.

4.5. What does $\mu \le \mu_1$ mean?

Proposition 4.12. *Assume that μ and μ_1 are mu-maps for π with associated boundary intertwiners $(\iota, \mathcal{H}')$ and $(\iota_1, \mathcal{H}_1')$. If $\mu \le \mu_1$ then there exists a $\Gamma \ltimes C(\partial\Gamma)$-map $\phi : \mathcal{H}_1' \to \mathcal{H}'$ so that $\phi\iota_1 = \iota$. The image of ϕ is dense and $\|\phi\| \le 1$.*

Proof. By Lemma 4.4, two intertwiners corresponding to the same mu-map are equivalent. So we may assume that $(\iota_1, \mathcal{H}_1')$ and, $(\iota, \mathcal{H}')$ are as constructed in the proof of Proposition 4.5 starting from μ_1 and μ respectively. Let $X = \sum_j G_j \otimes v_j$. Using the definition of the norms one has

$$\|X\|^2_{\mathcal{H}'} = \sum_{j,k} \langle \mu(\overline{G_k} G_j) v_j, v_k\rangle \le \sum_{j,k} \langle \mu_1(\overline{G_k} G_j) v_j, v_k\rangle = \|X\|^2_{\mathcal{H}_1{}'} \tag{15}$$

where the inequality holds because, according to Stinespring's Theorem, $\mu_1 - \mu$ is completely positive. Hence the identity map extends to a continuous $\Gamma \ltimes C(\partial\Gamma)$-map $\phi : \mathcal{H}_1{}' \to \mathcal{H}'$. Clearly the image is dense, $\|\phi\| \le 1$ and

$$\phi\iota_1(v) = \phi(\mathbf{1} \otimes v) = \mathbf{1} \otimes v = \iota(v) \in \mathcal{H}' \ .$$

$\square$

Corollary 4.13. *Let μ, μ_1, $(\iota, \mathcal{H}')$, $(\iota_1, \mathcal{H}_1')$ be as in Proposition 4.12. If $\mathcal{H}_1'$ is an irreducible $\Gamma \ltimes C(\partial\Gamma)$-representation, then $\mu = t\mu_1$ for some nonnegative constant $t \le 1$.*

Proof. Let ϕ be as in 4.12 and set $T = \phi^*\phi : \mathcal{H}_1' \to \mathcal{H}_1'$. Then T intertwines π_1' to itself, hence $T = t\,\mathrm{Id}$ for some $t \ge 0$. Then $t \le 1$ follows from $\|\phi\| \le 1$. The following diagram commutes:

$$\begin{tikzcd} & \mathcal{H}_1' \arrow[d, "\phi"] \\ \mathcal{H} \arrow[ur, "\iota_1"] \arrow[dr, "\iota"'] & \\ & \mathcal{H}' \end{tikzcd}$$

Using $\iota = \phi\iota_1$ and $\pi'(G)\phi = \phi\pi_1'(G)$ we have

$$\mu(G) = \iota^*\pi'(G)\iota = \iota_1^*\phi^*\pi'(G)\phi\iota_1 = \iota_1^*\phi^*\phi\pi_1'(G)\iota_1 = t\mu_1(G) \ .$$

$\square$

Corollary 4.14. *Let $(\iota_j)_{j=1,2}$ be boundary intertwiners mapping $\mathcal{H}$ to* irreducible and inequivalent *$\Gamma \ltimes C(\partial\Gamma)$-spaces $(\mathcal{H}_j')_j$. Let $(\mu_j)_{j=1,2}$ be the corresponding mu-maps. If μ is some other mu-map satisfying $\mu \le C(\mu_1 + \mu_2)$ for some $C > 0$, then $\mu = t_1\mu_1 + t_2\mu_2$ for some pair (t_1, t_2) of nonnegative coefficients.*

Proof. By scaling we may assume that $C = 1$. Let $(\iota, \mathcal{H}')$ be the boundary intertwiner corresponding to μ. According to Lemma 4.11

the boundary intertwiner associated to $\mu_1 + \mu_2$ is $(\iota_1 \oplus \iota_2, \mathcal{H}'_1 \oplus \mathcal{H}'_2)$. Let $\phi : \mathcal{H}'_1 \oplus \mathcal{H}'_2 \to \mathcal{H}'$ be as in Proposition 4.12. We have

$$\begin{tikzcd} & \mathcal{H}'_1 \oplus \mathcal{H}'_2 \arrow[d, "\phi"] \\ \mathcal{H} \arrow[ur, "\iota_1 \oplus \iota_2"] \arrow[r, "\iota"'] & \mathcal{H}' \end{tikzcd}$$

Since $\mathcal{H}'_1$ and $\mathcal{H}'_2$ are irreducible and inequivalent and since $\phi^*\phi$ is a $\Gamma \ltimes C(\partial\Gamma)$-intertwiner, it must be given by a block matrix of the form

$$\phi^*\phi = \begin{pmatrix} t_1 \operatorname{Id}_{\mathcal{H}'_1} & 0 \\ 0 & t_2 \operatorname{Id}_{\mathcal{H}'_2} \end{pmatrix} \qquad \text{with } t_1, t_2 \geq 0.$$

$$\begin{aligned} \mu(G) &= \iota^*\pi'(G)\iota = (\iota_1 \oplus \iota_2)^*\phi^*\pi'(G)\phi(\iota_1 \oplus \iota_2) \\ &= (\iota_1 \oplus \iota_2)^*\phi^*\phi(\pi'_1(G) \oplus \pi'_2(G))(\iota_1 \oplus \iota_2) \\ &= \iota_1^*(t_1 \operatorname{Id}_{\mathcal{H}'_1})\pi'_1(G)\iota_1 + \iota_2^*(t_2 \operatorname{Id}_{\mathcal{H}'_2})\pi'_2(G)\iota_2 = t_1\mu_1(G) + t_2\mu_2(G) . \end{aligned}$$

□

5. The trace inner-product

5.1. $\mathrm{TR}(T_1, T_2)$ **for** T_1 **and** T_2 **positive semidefinite.** We denote by $\mathcal{B}^+(\mathcal{H}) \subseteq \mathcal{B}(\mathcal{H})$ the subset of positive semidefinite operators. For $T \in \mathcal{B}^+(\mathcal{H})$ we recall the definition of the *trace*:

$$\operatorname{tr}(T) = \sum_{i=1}^{\infty} \langle Te_i, e_i \rangle$$

for some fixed orthonormal basis $\{e_i\}_{i=0}^{\infty}$. Let $S, T \in \mathcal{B}^+(\mathcal{H})$. It is well known (see for example [Dix81] Section 1.6.6) that

- $\operatorname{tr}(T) \in [0, +\infty]$.
- $\operatorname{tr}(\alpha T + \beta S) = \alpha \operatorname{tr}(T) + \beta \operatorname{tr}(S)$ for $\alpha, \beta \in \mathbf{R}^+$.
- $\operatorname{tr}(TT^*) = \operatorname{tr}(T^*T) = \|T\|_{HS}^2$ where $\|\cdot\|_{HS}$ denotes the Hilbert–Schmidt norm.
- If $U \in \mathcal{B}(H)$ is unitary, then $\operatorname{tr}(UTU^{-1}) = \operatorname{tr}(T)$.
- $\operatorname{tr}(T)$ is independent of the choice of basis.

Definition 5.1. For $S, T \in \mathcal{B}^+(\mathcal{H})$ we define

$$\mathrm{TR}(S, T) = \operatorname{tr}(\sqrt{S}T\sqrt{S}) . \tag{16}$$

The following properties are easily deduced from the above-mentioned properties of the trace:

- $\mathrm{TR}(S, T) \in [0, +\infty]$.
- $\mathrm{TR}(S, T) = \|\sqrt{S}\sqrt{T}\|_{HS}^2$.
- $\mathrm{TR}(S, T) = \mathrm{TR}(T, S)$.
- $\mathrm{TR}(S, T)$ is bilinear.
- If $T, S \in \mathcal{B}^+(H)$ and $\mathrm{TR}(S, T) = 0$, then $ST = TS = 0$.

- If U is unitary $\mathrm{TR}(USU^{-1}, UTU^{-1}) = \mathrm{TR}(S,T)$.
- If $T \leq A$ and $S \leq B$ then $\mathrm{TR}(T,S) \leq \mathrm{TR}(A,B)$.

Proof. The first statement is obvious. Let us turn to the others:

$$\begin{aligned}\mathrm{TR}(S,T) &= \mathrm{tr}(\sqrt{S}T\sqrt{S}) = \mathrm{tr}(\sqrt{S}\sqrt{T}\sqrt{T}\sqrt{S}) \\ &= \mathrm{tr}((\sqrt{S}\sqrt{T})(\sqrt{S}\sqrt{T})^*) = \|\sqrt{S}\sqrt{T}\|_{HS}^2 \\ &= \mathrm{tr}((\sqrt{S}\sqrt{T})^*(\sqrt{S}\sqrt{T})) = \mathrm{TR}(T,S)\ .\end{aligned}$$

Bilinearity is now obvious. Suppose that $\mathrm{TR}(S,T) = 0$. By the second statement $\sqrt{S}\sqrt{T} = 0$. Multiply on the left by $\sqrt{S}$ and on the right by $\sqrt{T}$ to get $ST = 0$. The next to the last statement follows from the fact that $\sqrt{USU^{-1}} = U\sqrt{S}U^{-1}$ when U is unitary. Finally, assume that $T \leq A$ and $S \leq B$. Choose an orthonormal basis e_i and compute

$$\begin{aligned}\mathrm{TR}(T,S) = \mathrm{tr}(\sqrt{T}S\sqrt{T}) &= \sum_i \langle S\sqrt{T}e_i, \sqrt{T}e_i\rangle \\ &\leq \sum_i \langle B\sqrt{T}e_i, \sqrt{T}e_i\rangle = \mathrm{TR}(T,B)\ .\end{aligned}$$

Use now that $\mathrm{TR}(T,B) = \mathrm{TR}(B,T)$ to get $\mathrm{TR}(T,S) \leq \mathrm{TR}(B,A) = \mathrm{TR}(A,B)$. □

We will use the following versions of Fatou's Lemma and of the Bounded Convergence Theorem. They are immediate consequences of the usual Fatou's Lemma and Bounded Convergence Theorem for counting measure and we prove them together.

Proposition 5.2. *Suppose that T is positive semidefinite and that the sequence $(T_j)_{j\geq 0}$ is made up of positive semidefinite operators, is increasing, and has a weak limit T_∞. Then*

$$\mathrm{TR}(T_\infty, T) \leq \liminf_{j\to\infty} \mathrm{TR}(T_j, T)\ .$$

Proposition 5.3. *Suppose that T and T_B are positive semidefinite and satisfy $\mathrm{TR}(T_B, T) < \infty$. Suppose that the sequence $(T_j)_{j\geq 0}$ is made up of positive semidefinite operators, all bounded above by T_B, and has a weak limit T_∞. Then*

$$\mathrm{TR}(T_\infty, T) = \lim_{j\to\infty} \mathrm{TR}(T_j, T)\ .$$

Proof. Choose an orthonormal basis $\{e_k\}$ for $\mathcal{H}$ and consider the quantities $\langle T_j\sqrt{T}e_k, \sqrt{T}e_k\rangle = \varphi_j(k)$. Since $(T_j)_{j\geq 0}$ has a weak limit we know that φ_j is pointwise convergent for each k.

To get Proposition 5.2 apply Fatou's Lemma to φ_j with respect to counting measure. To get Proposition 5.3 observe that the φ_j are all bounded above by $\varphi_B(k) = \langle T_B\sqrt{T}e_k, \sqrt{T}e_k\rangle$ and apply the Bounded Convergence Theorem. □

5.2. (F_1, F_2) **for Eff-vectors** F_1 **and** F_2**.**

Definition 5.4. If $F = (F_a)_{a\in A}$ and $\tilde{F} = (\tilde{F}_a)_{a\in A}$ are $|A|$-tuples of positive semidefinite operators, define

$$(F, \tilde{F}) = \sum_{a\neq b} \mathrm{TR}(F_a, \tilde{F}_b) \ . \tag{17}$$

Given this definition, the following Corollaries follow immediately from Propositions 5.2 and 5.3.

Corollary 5.5. *Suppose that F is an $|A|$-tuple of positive semidefinite operators, that the sequence $(F_j)_{j\geq 0}$ is made up of similar tuples, that the sequence is increasing componentwise, and has a componentwise weak limit F_∞. Then*

$$(F_\infty, F) \leq \liminf_{j\to\infty} (F_j, F) \ .$$

Corollary 5.6. *Suppose that F and F_B are $|A|$-tuples of positive semidefinite operators satisfying $(F_B, F) < \infty$. Suppose that the sequence $(F_j)_{j\geq 0}$ is likewise made up of $|A|$-tuples of positive semidefinite operators, all bounded above componentwise by F_B, and suppose that the sequence has a componentwise weak limit F_∞. Then*

$$(F_\infty, F) = \lim_{j\to\infty} (F_j, F) \ .$$

Proposition 5.7. *Let $(\iota, \mathcal{H}')$ be a boundary realization with associated mu-map μ and associated Eff-vector $F = (F_a)$ as per equations (6) and (13). Then $(\iota, \mathcal{H}')$ is perfect if and only if $(F, F) = 0$.*

Proof. If $(\iota, \mathcal{H}')$ is perfect then μ is an algebra homomorphism. In particular one has $F_aF_b = \mu(\mathbf{1}_a)\mu(\mathbf{1}_b) = \mu(\mathbf{1}_a\mathbf{1}_b) = \mu(0) = 0$ if $a \neq b$ and hence $(F, F) = 0$. Conversely, assume that $(F, F) = \sum_{a\neq b} \mathrm{TR}(F_a, F_b) = 0$. Since each F_a is positive semidefinite one has $F_aF_b = 0 = F_bF_a$ when $a \neq b$. Moreover, since ι is an isometry, one has $\mathrm{Id} = \mu(\mathbf{1}) = \sum_{a\in A} \mu(\mathbf{1}_a)$. Multiply both sides by F_b :

$$F_b = \sum_{a\in A} F_bF_a = F_b^2 \ .$$

Now apply Lemma 4.9. □

Proposition 5.8. *Let $(\iota, \mathcal{H}')$, $(\tilde{\iota}, \tilde{\mathcal{H}}')$ be two boundary realizations with associated mu-maps μ and $\tilde{\mu}$ and associated Eff-vectors $F = (F_a)$ and $\tilde{F} = (\tilde{F}_a)$. Then $(\iota, \mathcal{H}')$ and $(\tilde{\iota}, \tilde{\mathcal{H}}')$ are perfect and equivalent if and only if $(F, \tilde{F}) = 0$.*

Proof. If $(\iota, \mathcal{H}')$ and $(\tilde{\iota}, \tilde{\mathcal{H}}')$ are equivalent the corresponding Eff-vectors F and $\tilde{F}$ are equal, so that the statement follows from Proposition 5.7.

Conversely, assume that $(F, \tilde{F}) = 0$ or, equivalently, that $F_a\tilde{F}_b = \tilde{F}_b F_a = 0$ for all $a \neq b$. Since both μ and $\tilde{\mu}$ are isometries one has

$$\sum_{a\in A} F_a = \sum_{a\in A} \tilde{F}_a = \mathrm{Id} \ . \tag{18}$$

Fix $b \in A$ and multiply the left-hand side of (18) by $\tilde{F}_b$ and the right-hand side by F_b to get

$$\tilde{F}_b = \sum_{a\in A} \tilde{F}_b F_a = \tilde{F}_b F_b \ , \qquad F_b = \sum_{a\in A} \tilde{F}_a F_b = \tilde{F}_b F_b \ ,$$

and conclude that the two realizations are equivalent by Lemmas 4.4 and 4.6 and perfect by Proposition 5.7. □

Definition 5.9. Let $(\iota, \mathcal{H}')$ be a boundary intertwiner with associated mu-map μ and associated Eff-vector $F = (F_a)$. We say that $(\iota, \mathcal{H}')$ satisfies the *finite trace condition* or briefly (FTC) if

$$(F, F) < \infty \ . \tag{FTC}$$

Remark 5.10. Since $(F, F) = 0$ for any perfect boundary realization, (FTC) is of interest mostly for imperfect boundary realizations. When $(\iota, \mathcal{H})$ is the direct sum of two perfect boundary realizations, $(\iota_1, \mathcal{H}'_1)$ and $(\iota_2, \mathcal{H}'_2)$, the corresponding Eff-vector F is the sum of the Eff-vectors corresponding to ι_1 and ι_2 and the (FTC) for F becomes

$$(F, F) = 2(F_1, F_2) < \infty \ .$$

Remark 5.11. The following straightforward property of $(\cdot, \cdot)$ is crucial in the next section. Here $a, b, c, d \in A$.

$$\begin{aligned}
(\mathcal{T}F, \tilde{F}) &= \sum_{a\neq b} \mathrm{TR}((\mathcal{T}F)_a, \tilde{F}_b) \\
&= \sum_{a\neq b} \sum_{c\neq a^{-1}} \mathrm{TR}(\pi(a) F_c \pi(a)^{-1}, \tilde{F}_b) \\
&= \sum_{a\neq b} \sum_{c\neq a^{-1}} \mathrm{TR}(F_c, \pi(a)^{-1} \tilde{F}_b \pi(a)) \\
&= \sum_{b\neq d^{-1}} \sum_{c\neq d} \mathrm{TR}(F_c, \pi(d) \tilde{F}_b \pi(d)^{-1}) = (F, \mathcal{T}\tilde{F}) \ .
\end{aligned} \tag{19}$$

6. A weak limit

Recall that when F_1 and F_2 are $|A|$-tuples of operators, we write $F_1 \leq F_2$ to mean $(F_1)_a \leq (F_2)_a$ for every $a \in A$.

Proposition 6.1. *Let F be an Eff-vector. Let F_0 be any $|A|$-tuple of positive semidefinite operators satisfying $(\mathcal{T}^N F_0)_a \leq C F_a$ for some* *fixed integer $N \geq 0$, some fixed $C > 0$, and all $a \in A$. Then there exists a sequence $(\epsilon_j)_j \to 0+$ such that the componentwise weak limit*

$$F_L = \operatorname*{wk\text{-}lim}_{j\to\infty} \epsilon_j \sum_{n\geq 0} e^{-\epsilon_j n} \mathcal{T}^n F_0$$

exists. Moreover

- *$F_L \leq CF$;*
- *$\mathcal{T} F_L = F_L$;*
- *if F_1 is any Eff-vector satisfying $(F, F_1) < +\infty$, then $(F_L, F_1) = (F_0, F_1)$.*

One can choose $(\epsilon_j)_j$ to be a subsequence of any given sequence decreasing to 0.

Proof. From equation (5), which defines $\mathcal{T}$, it follows that $\mathcal{T}F' \leq \mathcal{T}F''$ componentwise whenever $F' \leq F''$ componentwise. Since $\mathcal{T}F = F$, our hypotheses imply

$$(\mathcal{T}^n F_0)_a \leq CF_a \quad \text{for all } n \geq N$$

hence

$$(\epsilon \sum_{n\geq N} e^{-\epsilon n} \mathcal{T}^n F_0)_a \leq \frac{\epsilon}{1 - e^{-\epsilon}} CF_a \,. \tag{20}$$

From (20) deduce first that the series $\sum_{n\geq 0} e^{-\epsilon n} \mathcal{T}^n F_0$ converges componentwise in the norm topology, and then that the quantities

$$\|(\epsilon \sum_{n\geq 0} e^{-\epsilon n} \mathcal{T}^n F_0)_a\|$$

are uniformly bounded for $0 < \epsilon \leq 1$. Since the unit ball of $\mathcal{B}(\mathcal{H})$ is compact and metrizable in the weak operator topology, we conclude that there exists a sequence $\epsilon_j \to 0+$ such that

$$\operatorname*{wk\text{-}lim}_{j\to\infty} \epsilon_j \sum_{n\geq 0} e^{-\epsilon_j n} \mathcal{T}^n F_0 = F_L$$

exists componentwise. One gets $F_L \leq CF$ from (20).

From the definition of $\mathcal{T}$ it follows easily that $\mathcal{T}$ commutes with componentwise weak limits. Thus

$$\begin{aligned}\mathcal{T} F_L &= \mathcal{T}(\operatorname*{wk\text{-}lim}_{j\to\infty} \epsilon_j \sum_{n\geq 0} e^{-\epsilon_j n} \mathcal{T}^n F_0) = \operatorname*{wk\text{-}lim}_{j\to\infty} \mathcal{T} \epsilon_j \sum_{n\geq 0} e^{-\epsilon_j n} \mathcal{T}^n F_0 \\ &= \operatorname*{wk\text{-}lim}_{j\to\infty} \epsilon_j \sum_{n\geq 0} e^{-\epsilon_j n} \mathcal{T}^{n+1} F_0 = \operatorname*{wk\text{-}lim}_{j\to\infty} e^{\epsilon_j} \epsilon_j \sum_{n\geq 1} e^{-\epsilon_j n} \mathcal{T}^n F_0 = F_L \,.\end{aligned}$$

Finally, assume that F_1 is another Eff-vector and that $(F, F_1) < \infty$. Use (20) for boundedness and apply Corollary 5.6.

$$\begin{aligned}(F_L, F_1) &= (\lim_{j\to\infty} \epsilon_j \sum_{n\geq N} e^{-\epsilon_j n}\mathcal{T}^n F_0, F_1) = \lim_{j\to\infty} (\epsilon_j \sum_{n\geq N} e^{-\epsilon_j n}\mathcal{T}^n F_0, F_1) \\ &= \lim_{j\to\infty} \epsilon_j \sum_{n\geq N} e^{-\epsilon_j n}(\mathcal{T}^n F_0, F_1) = \lim_{j\to\infty} \epsilon_j \sum_{n\geq N} e^{-\epsilon_j n}(F_0, \mathcal{T}^n F_1) \\ &= \lim_{j\to\infty} \epsilon_j \sum_{n\geq N} e^{-\epsilon_j n}(F_0, F_1) = (F_0, F_1)\end{aligned}$$

since $\mathcal{T} F_1 = F_1$. □

7. Good vectors

Throughout this section we consider a fixed representation π of Γ on $\mathcal{H}$ and a fixed boundary realization $\iota : \mathcal{H} \to \mathcal{H}'$ of π. The concepts of *good vector* and *special good vector* are relative to this π and this ι. As usual, let

$$\mu(G) = \iota^*\pi'(G)\iota \qquad\qquad F_a = \iota^*\pi'(\mathbf{1}_a)\iota$$

be the corresponding mu-map and Eff-vector.

For v_1 and $v_2 \in \mathcal{H}$, recall that $v_1\otimes\bar{v}_2$ stands for the rank one operator given by

$$(v_1 \otimes \bar{v}_2)(v) = \langle v, v_2\rangle v_1$$

and that

$$\pi(x)(v_1 \otimes \bar{v}_2)\pi(x)^{-1} = (\pi(x)v_1) \otimes \overline{(\pi(x)v_2)}\ .$$

Definition 7.1. Say that a vector $v \in \mathcal{H}$ is a *good vector* with respect to ι if there exist $C > 0$, $N \geq 0$ so that

$$\mathcal{T}^N E \leq CF$$

where $E_a = v \otimes \bar{v}$ for every $a \in A$. Say that v is a *special good vector* with respect to ι if for some $z \in \Gamma$ and some $C > 0$ we have

$$v \otimes \bar{v} \leq C\mu(\mathbf{1}_z) \qquad\qquad v \otimes \bar{v} \leq C\mu(\mathbf{1} - \mathbf{1}_z)\ . \tag{21}$$

Remark 7.2. If v is a good vector, arguing as in the proof of Proposition 6.1, we see that

$$\mathcal{T}^n E \leq CF \qquad \text{for all } n \geq N\ .$$

Remark 7.3. If our realization is perfect, then $\mu(\mathbf{1}_z)$ and $\mu(\mathbf{1}-\mathbf{1}_z)$ are disjoint projections. From that it follows easily that the only special good vector is the null vector.

Remark 7.4. If (21) holds with $z = e$, then the second inequality gives $v = 0$. Therefore the definition of special good vector would be equivalent if we considered only nontrivial z.

Lemma 7.5. *The good vectors make up a linear subspace of $\mathcal{H}$.*

Proof. $(u+v)\otimes\overline{(u+v)} \le 2(u\otimes\bar{u} + v\otimes\bar{v})$. □

Recall that $\Gamma(x)\subseteq\Gamma$ is the subset of reduced words which start with the reduced word for x. Likewise, let $\tilde{\Gamma}(x)$ be the subset of reduced words which end with the reduced word for x.

Lemma 7.6. *Let $L=(L)_a$ be an $|A|$-tuple of operators in $\mathcal{B}(\mathcal{H})$ and let $n\ge 1$. Then*

$$(\mathcal{T}^n L)_a = \sum_b \sum_{\substack{x\in\Gamma\,;\ |x|=n \\ x\in\Gamma(a),\ x\notin\tilde{\Gamma}(b^{-1})}} \pi(x)L_b\pi(x^{-1})\ . \tag{22}$$

Proof. For $B\in\mathcal{B}(\mathcal{H})$ let $P(x)B=\pi(x)B\pi(x)^{-1}$. Using this notation one has

$$(\mathcal{T}L)_a = \sum_b \mathcal{T}_{a,b}L_b = \sum_{b\ne a^{-1}} P(a)L_b\ .$$

Now use induction. For $n=1$ one has

$$(\mathcal{T}^1)_{a,b} = (\mathcal{T})_{a,b} = \left\{\begin{array}{rl} P(a) & \text{if } b\ne a^{-1} \\ 0 & \text{if } b=a^{-1}\end{array}\right\} = \sum_{\substack{|x|=1 \\ x\in\Gamma(a),\ x\notin\tilde{\Gamma}(b^{-1})}} P(x)\ .$$

For $n>1$

$$\begin{aligned}(\mathcal{T}^n)_{a,b} &= \sum_c \mathcal{T}_{a,c}\mathcal{T}^{n-1}_{c,b} = \sum_{c\ne a^{-1}} P(a)\mathcal{T}^{n-1}_{c,b} \\ &= \sum_{c\ne a^{-1}} \sum_{\substack{|x|=n-1 \\ x\in\Gamma(c),\ x\notin\tilde{\Gamma}(b^{-1})}} P(a)P(x) = \sum_{\substack{|y|=n \\ y\in\Gamma(a),\ y\notin\tilde{\Gamma}(b^{-1})}} P(y)\ .\end{aligned}$$

Thus

$$\begin{aligned}(\mathcal{T}^n L)_a &= \sum_b \mathcal{T}^n_{a,b}L_b \\ &= \sum_b \sum_{\substack{|y|=n \\ y\in\Gamma(a),\ y\notin\tilde{\Gamma}(b^{-1})}} P(y)L_b = \sum_b \sum_{\substack{|y|=n \\ y\in\Gamma(a),\ y\notin\tilde{\Gamma}(b^{-1})}} \pi(y)L_b\pi(y)^{-1}\ .\end{aligned}$$

□

Lemma 7.7. *The set of good vectors is stable under $\pi(\Gamma)$.*

Proof. Let E be the vector with $E_a = v\otimes\bar{v}$ and E' the vector with $E'_a = \pi(y)v\otimes\overline{\pi(y)v}$. If v is good, then we have $\mathcal{T}^N E\le CF$ for some N, hence $(\mathcal{T}^n E)_a\le CF_a$ for all $a\in A$, and all $n\ge N$. According to (22) this implies that $\pi(x)v\otimes\overline{\pi(x)v}\le CF_a$ whenever $x\in\Gamma(a)$ and $|x|\ge N$. Consequently $\pi(x)\pi(y)v\otimes\overline{\pi(x)\pi(y)v}\le CF_a$ whenever $x\in\Gamma(a)$ and $|x|\ge N+|y|$. Use (22) again to deduce that $(\mathcal{T}^{N+|y|}E')_a\le C'F_a$. □

Lemma 7.8. *If v is a good vector, then there exists $C = C(v) > 0$, independent of n so that*

$$\sum_{|x|=n} \pi(x)v \otimes \overline{\pi(x)v} \leq C \,\mathrm{Id} \ .$$

Proof. By Remark 7.2 there exist $C > 0$, $N > 0$ so that $CF \geq \mathcal{T}^n E$ for every $n \geq N$ where $E_a = v \otimes \bar{v}$ for every a. The finite number of values of n with $n < N$ create no difficulty, so assume $n \geq N$ and $n \geq 1$. Sum over a the inequalities $CF_a \geq (\mathcal{T}^n E)_a$ and apply Lemma 7.6 to get

$$\begin{aligned} C \,\mathrm{Id} = C \sum_{a \in A} F_a &\geq \sum_{a,b \in A} \sum_{\substack{|x|=n \\ x \in \Gamma(a),\, x \notin \tilde{\Gamma}(b^{-1})}} \pi(x)(v \otimes \bar{v})\pi(x)^{-1} \\ &= \sum_{a,b \in A} \sum_{\substack{x \in \Gamma\,;\, |x|=n \\ x \in \Gamma(a),\, x \notin \tilde{\Gamma}(b^{-1})}} \pi(x)v \otimes \overline{\pi(x)v} = q \sum_{|x|=n} \pi(x)v \otimes \overline{\pi(x)v} \end{aligned}$$

where $q + 1 = |A|$. □

Corollary 7.9. *If v is a good vector, then there exists $C = C(v) > 0$, independent of n and w so that*

$$\sum_{|x|=n} |\langle w, \pi(x)v \rangle|^2 \leq C \|w\|^2 \ .$$

What about the existence of good vectors?

Lemma 7.10. *If v is a special good vector then*

- *There exist $C > 0$, $N > 0$ so that if $a \in A$, $x \in \Gamma$,$|ax| = 1+|x|$, and $|x| \geq N$, then $\pi(ax)(v \otimes \bar{v})\pi(ax)^{-1} \leq C\mu(\mathbf{1}_a) = CF_a$.*
- *v is a good vector.*

Proof. Choose C and z so that (21) holds. Choose $N = |z|$. For the first assertion, note that if $x \notin \tilde{\Gamma}(z^{-1})$, then

$$\begin{aligned} \pi(ax)(v \otimes \bar{v})\pi(ax)^{-1} &\leq C\pi(ax)\mu(\mathbf{1}_z)\pi(ax)^{-1} \\ &= C\mu(\lambda(ax)\mathbf{1}_z) \leq C\mu(\mathbf{1}_a) = CF_a \end{aligned}$$

and contrariwise if $x \in \tilde{\Gamma}(z^{-1})$, then

$$\begin{aligned} \pi(ax)(v \otimes \bar{v})\pi(ax)^{-1} &\leq C\pi(ax)\mu(\mathbf{1} - \mathbf{1}_z)\pi(ax)^{-1} \\ &= C\mu(\lambda(ax)(\mathbf{1} - \mathbf{1}_z)) \leq C\mu(\mathbf{1}_a) = CF_a \ . \end{aligned}$$

The second assertion now follows from Lemma 7.6 since, putting $E_a = v \otimes \bar{v}$ for each $a \in A$,

$$(\mathcal{T}^{N+1}E)_a = \sum_{\substack{x \in \Gamma,\, b \in A \\ |ax| = 1 + |x| = N+1,\, x \notin \tilde{\Gamma}(b^{-1})}} \pi(ax)(v \otimes \bar{v})\pi(ax)^{-1} \leq CF_a$$

for a new value of C. □

Lemma 7.11. *The set of special good vectors is stable under $\pi(\Gamma)$.*

Proof. If v satisfies (21) then $\pi(x)v$ satisfies

$$\begin{aligned}(\pi(x)v)\otimes\overline{(\pi(x)v)} &= \pi(x)(v\otimes\bar{v})\pi(x^{-1})\\ &\le C\pi(x)\mu(\mathbf{1}_z)\pi(x^{-1}) = C\mu(\lambda(x)\mathbf{1}_z)\ ,\\ (\pi(x)v)\otimes\overline{(\pi(x)v)} &= \pi(x)(v\otimes\bar{v})\pi(x^{-1})\\ &\le C\pi(x)\mu(\mathbf{1}-\mathbf{1}_z)\pi(x^{-1}) = C\mu(\lambda(x)(\mathbf{1}-\mathbf{1}_z))\ .\end{aligned}$$

Now observe that the pair $\{\mathbf{1}_z, \mathbf{1}-\mathbf{1}_z\}$ is translated by $\lambda(x)$ to another such pair. If $x \notin \tilde{\Gamma}(z^{-1})$ then the translated pair is $\{\mathbf{1}_{xz}, \mathbf{1}-\mathbf{1}_{xz}\}$; if $x \in \tilde{\Gamma}(z^{-1})$ and if $z = wa$ with $|z| = |w|+1$, then the translated pair is $\{\mathbf{1}-\mathbf{1}_{xw}, \mathbf{1}_{xw}\}$. This is easiest to understand by drawing diagrams of the tree which is the Cayley graph of Γ. ∎

Lemma 7.12. *Let $Q \in \mathcal{B}(\mathcal{H})$ be a nonnegative operator and let $u \in \mathcal{H}$. Then*

$$(Q^{1/2}u)\otimes\overline{(Q^{1/2}u)} \le \|u\|^2 Q\ .$$

Proof. Let $w \in \mathcal{H}$. Then

$$\begin{aligned}\langle (Q^{1/2}u)\otimes\overline{(Q^{1/2}u)}w, w\rangle &= |\langle Q^{1/2}u, w\rangle|^2\\ &= |\langle u, Q^{1/2}w\rangle|^2 \le \|u\|^2\|Q^{1/2}w\|^2 = \|u\|^2\langle Qw, w\rangle\ .\end{aligned}$$

∎

Lemma 7.13. *For any $u \in \mathcal{H}$ and $z \in \Gamma$,*

$$v = (\mu(\mathbf{1}_z) - \mu(\mathbf{1}_z)^2)^{1/2}u$$

is a special good vector.

Proof. Note that $\mu(\mathbf{1}_z) + \mu(\mathbf{1}-\mathbf{1}_z) = \mu(\mathbf{1}) = \mathrm{Id}$, hence

$$0 \le \mu(\mathbf{1}_z) \le \mathrm{Id}\ , \qquad 0 \le \mu(\mathbf{1}-\mathbf{1}_z) \le \mathrm{Id}\ .$$

By Lemma 7.12 we have $v\otimes\bar{v} \le \|u\|^2(\mu(\mathbf{1}_z) - \mu(\mathbf{1}_z)^2)$. Now use

$$\begin{aligned}\mu(\mathbf{1}_z) - \mu(\mathbf{1}_z)^2 &\le \mu(\mathbf{1}_z)\ ,\\ \mu(\mathbf{1}_z) - \mu(\mathbf{1}_z)^2 &= (\mathrm{Id} - \mu(\mathbf{1}_z)) - (\mathrm{Id} - \mu(\mathbf{1}_z))^2\\ &= \mu(\mathbf{1}-\mathbf{1}_z) - \mu(\mathbf{1}-\mathbf{1}_z)^2 \le \mu(\mathbf{1}-\mathbf{1}_z)\ .\end{aligned}$$

∎

Definition 7.14. Let $\mathcal{H}_B \subseteq \mathcal{H}$ consist of those vectors which are orthogonal to all special good vectors and let $\mathcal{H}_G = \mathcal{H}\ominus\mathcal{H}_B$. Thus $\mathcal{H}_G$ is the closure of the linear span of the special good vectors.

Proposition 7.15. *$\mathcal{H}_G$ contains a dense linear subspace made up of good vectors.*

Proof. This follows from Lemmas 7.10 and 7.5. ∎

Proposition 7.16.
(1) *$\mathcal{H}_B$ is a closed linear subspace.*
(2) *$\mathcal{H}_B$ is invariant under $\pi(\Gamma)$.*
(3) *$w \in \mathcal{H}$ belongs to $\mathcal{H}_B$ if and only if $\mu(\mathbf{1}_z)w = \mu(\mathbf{1}_z)^2 w$ for all $z \in \Gamma$.*
(4) *For $w \in \mathcal{H}_B$, $\mu(\mathbf{1}_y)\mu(\mathbf{1}_z)w = 0$ whenever $\partial\Gamma(y)$ and $\partial\Gamma(z)$ are disjoint.*
(5) *For $w \in \mathcal{H}_B$, $\mu(G_1)\mu(G_2)w = \mu(G_1 G_2)w$ for G_1, $G_2 \in C(\partial\Gamma)$.*
(6) *$\mathcal{H}_B$ is stable under the action of $\mu(C(\partial\Gamma))$.*

Proof. The first assertion is trivial. The second assertion follows from Lemma 7.11. For the third assertion, first suppose that $w \in \mathcal{H}_B$. Then by Lemma 7.13 $\langle w, (\mu(\mathbf{1}_z) - \mu(\mathbf{1}_z)^2)^{1/2} u\rangle = 0$ for any $u \in \mathcal{H}$, hence $(\mu(\mathbf{1}_z) - \mu(\mathbf{1}_z)^2)^{1/2} w = 0$, hence $(\mu(\mathbf{1}_z) - \mu(\mathbf{1}_z)^2)w = 0$.

Conversely, suppose that $\mu(\mathbf{1}_z)w = \mu(\mathbf{1}_z)^2 w$ for all $z \in \Gamma$ and suppose that v is a special good vector satisfying (21) for some particular $z \in \Gamma$. Then $\langle v, w\rangle = \langle v, \mu(\mathbf{1}_z)w\rangle + \langle v, \mu(\mathbf{1} - \mathbf{1}_z)w\rangle$ and

$$\begin{aligned}
|\langle v, \mu(\mathbf{1}_z)w\rangle|^2 &= \langle (v \otimes \bar{v})\mu(\mathbf{1}_z)w, \mu(\mathbf{1}_z)w\rangle \\
&\leq C\langle \mu(\mathbf{1} - \mathbf{1}_z)\mu(\mathbf{1}_z)w, \mu(\mathbf{1}_z)w\rangle \ , \\
|\langle v, \mu(\mathbf{1} - \mathbf{1}_z)w\rangle|^2 &= \langle (v \otimes \bar{v})\mu(\mathbf{1} - \mathbf{1}_z)w, \mu(\mathbf{1} - \mathbf{1}_z)w\rangle \\
&\leq C\langle \mu(\mathbf{1}_z)\mu(\mathbf{1} - \mathbf{1}_z)w, \mu(\mathbf{1} - \mathbf{1}_z)w\rangle \ .
\end{aligned}$$

Now use $\mu(\mathbf{1} - \mathbf{1}_z)\mu(\mathbf{1}_z)w = (\mu(\mathbf{1}_z) - \mu(\mathbf{1}_z)^2)w = 0$ in both terms.

In the fourth assertion, we assume that $\partial\Gamma(y)$ and $\partial\Gamma(z)$ are disjoint, hence that $\mathbf{1}_y \leq \mathbf{1} - \mathbf{1}_z$.

$$\begin{aligned}
\langle \mu(\mathbf{1}_y)^{1/2}\mu(\mathbf{1}_z)w, \mu(\mathbf{1}_y)^{1/2}\mu(\mathbf{1}_z)w\rangle &= \langle \mu(\mathbf{1}_y)\mu(\mathbf{1}_z)w, \mu(\mathbf{1}_z)w\rangle \\
&\leq \langle \mu(\mathbf{1} - \mathbf{1}_z)\mu(\mathbf{1}_z)w, \mu(\mathbf{1}_z)w\rangle = 0 \ .
\end{aligned}$$

Hence $\mu(\mathbf{1}_y)^{1/2}\mu(\mathbf{1}_z)w = 0$, hence $\mu(\mathbf{1}_y)\mu(\mathbf{1}_z)w = 0$.

Let $w \in \mathcal{H}_B$. Suppose that for some $n > 0$ each of G_1, $G_2 \in C(\partial\Gamma)$ is of the form $\sum_{|z|=n} c_z \mathbf{1}_z$. Then $\mu(G_1)\mu(G_2)w = \mu(G_1 G_2)w$ follows from the third and fourth assertions and linearity. Taking limits, we see that this formula is valid for arbitrary G_1, $G_2 \in C(\partial\Gamma)$.

Finally, let $w \in \mathcal{H}_B$ and $G \in C(\partial\Gamma)$. For any $z \in \Gamma$

$$\mu(\mathbf{1}_z)^2\mu(G)w = \mu(\mathbf{1}_z)\mu(\mathbf{1}_z G)w = \mu(\mathbf{1}_z G)w = \mu(\mathbf{1}_z)\mu(G)w$$

and according to the criterion from the third assertion, this shows that $\mu(G)w \in \mathcal{H}_B$. □

Definition 7.17. Let $\mathcal{H}'_B$ and $\mathcal{H}'_G$ be the closures of $\pi'(C(\partial\Gamma))\iota(\mathcal{H}_B)$ and $\pi'(C(\partial\Gamma))\iota(\mathcal{H}_G))$ in $\mathcal{H}'$. By Proposition 7.16 $\mathcal{H}_B$ and $\mathcal{H}_G$ are stable under $\pi(\Gamma)$ and consequently each of $\mathcal{H}'_B$ and $\mathcal{H}'_G$ is stable under $\pi'(\Gamma \ltimes C(\partial\Gamma))$. Let π_B denote the restriction of π to $\mathcal{H}_B$ and π_G the restriction to $\mathcal{H}_G$. Likewise let π'_B be the restriction of π' to $\mathcal{H}'_B$ and π'_G the restriction to $\mathcal{H}'_G$. Let $\iota_B : \mathcal{H}_B \to \mathcal{H}'_B$ and $\iota_G : \mathcal{H}_G \to \mathcal{H}'_G$ be the respective restrictions of ι. They are boundary realizations of $\mathcal{H}_B$ and $\mathcal{H}_G$ respectively.

Corollary 7.18. $\mathcal{H}' = \mathcal{H}'_B \oplus \mathcal{H}'_G$.

Proof.

$$\begin{aligned}&\langle \pi'(C(\partial\Gamma))\iota(\mathcal{H}_B), \pi'(C(\partial\Gamma))\iota(\mathcal{H}_G)\rangle \\ &\quad = \langle \iota^*\pi'(C(\partial\Gamma))\iota\mathcal{H}_B, \mathcal{H}_G\rangle = \langle \mu(C(\partial\Gamma))\mathcal{H}_B, \mathcal{H}_G\rangle = \langle \mathcal{H}_B, \mathcal{H}_G\rangle = 0\end{aligned}$$

using the last assertion of Proposition 7.16. This shows that $\mathcal{H}'_B$ and $\mathcal{H}'_G$ are perpendicular. Consequently, their direct sum is a closed subspace of $\mathcal{H}'$. That sum contains $\pi'(C(\partial\Gamma))\iota(\mathcal{H})$ and is consequently total. □

Corollary 7.19. *$(\iota_B, \mathcal{H}'_B)$ is a perfect realization of $\mathcal{H}_B$. Consequently $\mathcal{H}'_B = \iota_B(\mathcal{H}_B)$.*

Proof. Use the fifth assertion of Proposition 7.16 and Lemma 4.8. □

We never use this last lemma, but it rounds out the picture.

Lemma 7.20. *All good vectors lie in $\mathcal{H}_G$.*

Sketch of proof. Suppose that $v = v_B + v_G \in \mathcal{H}$ with $v_B \in \mathcal{H}_B$ and $v_G \in \mathcal{H}_G$. Let E be the vector with

$$E_a = v \otimes \bar{v} = (v_B + v_G) \otimes \overline{(v_B + v_G)} = \begin{pmatrix} v_B \otimes \bar{v}_B & v_B \otimes \bar{v}_G \\ v_G \otimes \bar{v}_B & v_G \otimes \bar{v}_G \end{pmatrix} .$$

Suppose v is good for ι. This translates to $\mathcal{T}^n E \leq CF$. Calculate both sides as block matrices. Looking at the upper left hand block shows that v_B is good for ι_B. Corollary 7.19 says that ι_B is perfect, and so Remark 7.3 says that $v_B = 0$. □

8. Main proofs

Lemma 8.1. *Assume that $\mu : C(\partial\Gamma) \to \mathcal{B}(\mathcal{H})$ is a *-map satisfying $\pi(x)\mu(G)\pi(x^{-1}) = \mu(\lambda(x)G)$. Suppose also that $\|\mu(G)\| \leq C\|G\|_\infty$. If μ is* not *a positive map, then there is some $a \in A$ such that $\mu(\mathbf{1} - \mathbf{1}_a)$ is* not *positive semidefinite.*

Proof. Observe first that a norm-continuous *-map $\mu : C(\partial\Gamma) \to \mathcal{B}(\mathcal{H})$ is positive if and only if $\mu(\mathbf{1}_z)$ is positive for each $z \in \Gamma$. Hence, if μ is not positive, there exists $z \in \Gamma$ and $v \in \mathcal{H}$ such that $\langle \mu(\mathbf{1}_z)v, v\rangle < 0$. Denote by c the last letter of z and let $w = \pi(z^{-1})v$. One has

$$\begin{aligned}\langle \mu(\mathbf{1}_z)v, v\rangle = \langle \pi(z^{-1})\mu(\mathbf{1}_z)\pi(z)w, w\rangle = \\ \langle \mu(\lambda(z^{-1})\mathbf{1}_z)w, w\rangle = \langle \mu(\mathbf{1} - \mathbf{1}_{c^{-1}})w, w\rangle < 0\end{aligned}$$

implying that $\mu(\mathbf{1} - \mathbf{1}_{c^{-1}})$ is not positive semidefinite. □

8.1. **Oddity.** In this subsection, the reigning hypotheses are as follows:

- $\pi : \Gamma \to \mathcal{U}(\mathcal{H})$ is a unitary representation of Γ.
- $\pi' : \Gamma \ltimes C(\partial\Gamma) \to \mathcal{B}(\mathcal{H}')$ is an irreducible representation of $\Gamma \ltimes C(\partial\Gamma)$.
- $\iota : \mathcal{H} \to \mathcal{H}'$ is an imperfect boundary realization of π, i.e. ι is a Γ-map which is an isometry but is *not* a unitary isomorphism.
- As per Section 4, μ and F are associated to ι.
- The (FTC) holds for F.

Let $\mathcal{H}_1 = \mathcal{H}$ and $\mathcal{H}_2 = \mathcal{H}' \ominus \iota(\mathcal{H})$; let $\iota_1 = \iota$ and let $\iota_2 : \mathcal{H}_2 \to \mathcal{H}'$ be the inclusion map. This sets up the natural symmetry between $\mathcal{H} = \mathcal{H}_1$ and $\mathcal{H}_2$, with $\mathcal{H}' = \iota_1(\mathcal{H}_1) \oplus \iota_2(\mathcal{H}_2)$. Note that $\mathcal{H}_2$ is not stable under $\Gamma \ltimes C(\partial\Gamma)$, but it is stable under Γ; set $\pi_1 = \pi : \Gamma \to \mathcal{U}(\mathcal{H}_1)$ and let $\pi_2 : \Gamma \to \mathcal{U}(\mathcal{H}_2)$ be the Γ-action on $\mathcal{H}_2$ obtained from the $\Gamma \ltimes C(\partial\Gamma)$ representation. The only difference between the two direct summands is that we have assumed (FTC) for ι_1. This asymmetry is only apparent.

Proposition 8.2. *Let $F_1 = F$ and F_2 be associated with ι_1 and ι_2 respectively. Suppose that the block matrix for $\pi'(\mathbf{1}_a)$ is given by*

$$\begin{pmatrix} \pi'_{11} & \pi'_{12} \\ \pi'_{21} & \pi'_{22} \end{pmatrix} (\mathbf{1}_a) \ .$$

Then

$$(F_1, F_1) = \sum_a \|\pi'_{21}(\mathbf{1}_a)\|^2_{HS} = \sum_a \|\pi'_{12}(\mathbf{1}_a)\|^2_{HS} = (F_2, F_2) \ .$$

Besides establishing the (FTC) for ι_2, this shows that the (FTC), which we are assuming here, is equivalent to the corresponding condition in the statement of the Oddity Theorem in Section 2.

Proof. The middle equality is trivial because $\pi'_{12}(\mathbf{1}_a) = {\pi'}_{21}(\mathbf{1}_a)^*$, which holds because $\pi'(\mathbf{1}_a)$ is self adjoint. By definition

$$(F_1, F_1) = \sum_{a,d\,;\, a \neq d} \mathrm{TR}(\pi'_{11}(\mathbf{1}_a), \pi'_{11}(\mathbf{1}_d)) = \sum_a \mathrm{TR}(\pi'_{11}(\mathbf{1}_a), \pi'_{11}(\mathbf{1} - \mathbf{1}_a)) \ .$$

Since $\pi'_{11}(\mathbf{1}) = \mathrm{Id}$, $\pi'_{11}(\mathbf{1}_a)$ and $\pi'_{11}(\mathbf{1} - \mathbf{1}_a)$ commute, so

$$\mathrm{TR}(\pi'_{11}(\mathbf{1}_a), \pi'_{11}(\mathbf{1} - \mathbf{1}_a)) = \mathrm{tr}(\pi'_{11}(\mathbf{1}_a)\pi'_{11}(\mathbf{1} - \mathbf{1}_a)) \ .$$

One calculates

$$0 = (\pi'(\mathbf{1}_a)\pi'(\mathbf{1} - \mathbf{1}_a))_{11} = \pi'_{11}(\mathbf{1}_a)\pi'_{11}(\mathbf{1} - \mathbf{1}_a) + \pi'_{12}(\mathbf{1}_a)\pi'_{21}(\mathbf{1} - \mathbf{1}_a) \ .$$

Since $\pi'(\mathbf{1}) = \mathrm{Id}$ we have $\pi'_{21}(\mathbf{1}) = 0$, whence

$$\pi'_{11}(\mathbf{1}_a)\pi'_{11}(\mathbf{1} - \mathbf{1}_a) = \pi'_{12}(\mathbf{1}_a)\pi'_{21}(\mathbf{1}_a) \ .$$

Take traces of both sides and use $\pi'_{12}(\mathbf{1}_a) = {\pi'}_{21}(\mathbf{1}_a)^*$ to get $(F_1, F_1) = \sum_a \|\pi'_{21}(\mathbf{1}_a)\|^2_{HS}$. The formula for (F_2, F_2) follows in exactly the same way. $\square$

Proposition 8.3. *Assume the reigning hypotheses of this subsection. Let F_0 be an $|A|$-tuple of positive semidefinite operators satisfying $(\mathcal{T}^N F_0)_a \le CF_a$ for some fixed integer $N \ge 0$, some fixed $C > 0$, and for all $a \in A$. Then*

$$\operatorname*{wk-lim}_{\epsilon\to 0+} \epsilon \sum_{n\ge 0} e^{-\epsilon n}\mathcal{T}^n F_0 = \frac{(F_0, F)}{(F,F)} F \; . \tag{23}$$

Proof. Note that $(F_0, F) = (F_0, \mathcal{T}^N F) = (\mathcal{T}^N F_0, F) \le C(F,F) < \infty$. Use Proposition 6.1 to see that the limit exists for some subsequence $\epsilon_j \to 0+$. Use Corollary 4.13 to see that the limit must be of the form tF. Again by Proposition 6.1 the value of t must be as shown. Since any subsequence such that the limit exists gives the same limit, that limit must be valid for $\epsilon \to 0+$. □

Remark 8.4. If the hypotheses of Proposition 8.3 hold, except that instead of the (FTC) one has $(F, F) = \infty$, and if $(F_0, F) < \infty$, then a similar argument shows that the limit is zero.

Theorem 8.5. *Assume the reigning hypotheses of this subsection. Let $\pi'_N : \Gamma \ltimes C(\partial\Gamma) \to \mathcal{B}(\mathcal{H}'_N)$ be a boundary representation and $\iota_N : \mathcal{H} \to \mathcal{H}'_N$ a boundary intertwiner. Let μ_N and F_N be associated to ι_N, as in Section 4. Then μ_N is a scalar multiple of μ.*

Proof. Any mention of "good vectors" in this proof means good vectors in $\mathcal{H}$ relative to the boundary realization ι; we never consider good vectors relative to ι_N or to any other boundary intertwiner. Suppose the good vectors weren't dense in $\mathcal{H}$. According to Corollary 7.19 this would mean there was a nonzero Γ-invariant subspace $\mathcal{H}_B \subsetneq \mathcal{H}$ such that $\iota|_{\mathcal{H}_B}$ was a perfect realization; hence $\iota(\mathcal{H}_B)$ would be a $\Gamma \ltimes C(\partial\Gamma)$-subspace of $\mathcal{H}'$. Since $\mathcal{H}'$ is irreducible, this would imply that $\iota(\mathcal{H}_B)$ was all of $\mathcal{H}'$, a contradiction since ι isn't surjective.

Let $t = \max\{t \ge 0\,;\ \mu_N - t\mu \text{ is a positive map}\}$ and let $\mu_t = \mu_N - t\mu$. As per Section 4, we find ι_t and F_t associated to μ_t. After several steps, we shall show that $\mu_t = 0$, and it will follow that $\mu_N = t\mu$, proving the theorem.

Fix any $\delta > 0$. From the definition of t it follows that $\mu_t - \delta\mu$ is *not* a positive map. As per Lemma 8.1 choose $a \in A$ so that $(\mu_t - \delta\mu)(\mathbf{1} - \mathbf{1}_a)$ is not positive semidefinite, and a good vector $u \in \mathcal{H}$ so that

$$\langle(\mu_t - \delta\mu)(\mathbf{1} - \mathbf{1}_a)u, u\rangle < 0 \; . \tag{24}$$

Define F_0 by

$$(F_0)_a = u \otimes \bar{u} \qquad\qquad (F_0)_c = 0 \qquad \text{for } c \ne a.$$

Then it follows from (24) that

$$\begin{aligned}(F_0, F_t) &= \mathrm{TR}(u \otimes \bar{u}, \mu_t(\mathbf{1} - \mathbf{1}_a)) \\ &= \langle\mu_t(\mathbf{1} - \mathbf{1}_a)u, u\rangle < \langle\delta\mu(\mathbf{1} - \mathbf{1}_a)u, u\rangle = \delta(F_0, F) \; .\end{aligned} \tag{25}$$

According to Proposition 8.3

$$F_L = \operatorname*{wk-lim}_{\epsilon \to 0+} \epsilon \sum_{n \geq 0} e^{-\epsilon n} \mathcal{T}^n F_0 = \frac{(F_0, F)}{(F, F)} F$$

and this is a nonzero multiple of F because of (25). Multiplying u by a scalar, we may assume the limit F_L is F itself.

Now using Proposition 6.1, Corollary 5.5, the identities $(\mathcal{T}F_1, F_2) = (F_1, \mathcal{T}F_2)$, $\mathcal{T}F_t = F_t$, and $\mathcal{T}F = F$, equation (25) and again Proposition 6.1, we find:

$$\begin{aligned}
(F, F_t) &= (F_L, F_t) = (\operatorname*{wk-lim}_{\epsilon \to 0+} \epsilon \sum_{n \geq 0} e^{-\epsilon n} \mathcal{T}^n F_0, F_t) \\
&\leq \liminf_{\epsilon \to 0+} (\epsilon \sum_{n \geq 0} e^{-\epsilon n} \mathcal{T}^n F_0, F_t) \\
&= \liminf_{\epsilon \to 0+} \epsilon \, (F_0, F_t)/(1 - e^{-\epsilon}) = (F_0, F_t) < \delta (F_0, F) \\
&= \delta \lim_{\epsilon \to 0+} \epsilon (F_0, F)/(1 - e^{-\epsilon}) = \delta (\operatorname*{wk-lim}_{\epsilon \to 0+} \epsilon \sum_{n \geq 0} e^{-\epsilon n} \mathcal{T}^n F_0, F) \\
&= \delta (F_L, F) = \delta (F, F) \; .
\end{aligned}$$

Send $\delta \to 0+$ to conclude that $(F, F_t) = 0$.

Unless $\mu_t = 0$, you may apply Proposition 8.1 to $-\mu_t$ to find $b \in A$ such that $-\mu_t(\mathbf{1} - \mathbf{1}_b)$ is not positive definite. Then choose a good vector $v \in \mathcal{H}$ so that $\langle \mu_t(\mathbf{1} - \mathbf{1}_b)v, v) \rangle > 0$, and define F_1 by

$$(F_1)_b = v \otimes \bar{v} \qquad\qquad (F_1)_c = 0 \qquad \text{for } c \neq b.$$

One may then calculate $(F_1, F_t) = \langle \mu_t(\mathbf{1} - \mathbf{1}_b)v, v \rangle > 0$. Since v is good, one knows that for n large enough $\mathcal{T}^n F_1 \leq CF$. This leads to the contradiction

$$0 < (F_1, F_t) = (F_1, \mathcal{T}^n F_t) = (\mathcal{T}^n F_1, F_t) \leq C(F, F_t) = 0 \; .$$

□

Corollary 8.6. *Under the above hypotheses, $\pi = \pi_1$ and π_2 are irreducible Γ-representations.*

Proof. By Proposition 8.2 π_2 satisfies the same hypothesis as π_1, hence it is enough to prove the assertion for π_1. Assume, by contradiction, that π_1 is reducible. Split $\mathcal{H} = \mathcal{H}_0 \oplus \mathcal{H}_0^\perp$ into the direct sum of π_1 invariant subspaces and let P_0 be the projection onto $\mathcal{H}_0$. Let $\iota' = \iota P_0$: then ι' is another boundary intertwiner for π_1, and it is clearly not equivalent to any scalar multiple of ι. □

Corollary 8.7. *Under the above hypotheses, $\pi = \pi_1$ and π_2 are inequivalent as Γ-representations.*

Proof. Assume, by contradiction, that $U\pi_1 = \pi_2 U$ for some unitary $U : \mathcal{H}_1 \to \mathcal{H}_2$. Let $\iota' = \iota_2 U$. Then ι' is another boundary realization of π_1

which must be equivalent to ι, that is $\iota' = J\iota$ where J intertwines π' to itself. Since π' is irreducible J must be a scalar, which is impossible. □

Corollary 8.8. *Under the above hypotheses, for $j = 1$ or 2, any boundary realization of π_j is equivalent to ι_j.*

Proof. By Proposition 8.2 π_2 satisfies the same hypothesis as π_1. □

8.2. **Duplicity.** The proof of the Oddity Theorem, in the previous subsection, and the proof of the Duplicity Theorem, in this subsection, are closely parallel. In this subsection the reigning hypotheses are as follows:

- $\pi : \Gamma \to \mathcal{U}(\mathcal{H})$ is a unitary representation of Γ.
- We have two irreducible representations $\pi'_\pm : \Gamma \ltimes C(\partial\Gamma) \to \mathcal{B}(\mathcal{H}'_\pm)$, which are inequivalent as representations of $\Gamma \ltimes C(\partial\Gamma)$.
- We have two perfect boundary realizations of π, $\iota_\pm : \mathcal{H} \to \mathcal{H}'_\pm$.
- Let $\iota : \mathcal{H} \to \mathcal{H}_+ \oplus \mathcal{H}_-$ be defined by $\iota = \frac{1}{\sqrt{2}}(\iota_+ \oplus \iota_-)$. As per Section 4, μ and F are associated to ι.
- The (FTC) holds for F.

By hypothesis, $\iota_\pm$ are perfect realizations, but clearly ι is not. Does $\iota(\mathcal{H})$ lie in some proper $\Gamma \ltimes C(\partial\Gamma)$-subspace of $\mathcal{H}_1 \oplus \mathcal{H}_2$? According to Lemma 4.11 it does not. Let $\mu_\pm$ and $F_\pm$ be associated with $\iota_\pm$. As explained in subsection 4.3, $\mu = \frac{1}{2}(\mu_+ + \mu_-)$ and $F = \frac{1}{2}(F_+ + F_-)$. The (FTC) for F is $(F, F) < \infty$. Proposition 5.7 says that $(F_+, F_+) = (F_-, F_-) = 0$. Thus, the (FTC) is equivalent to $(F_+, F_-) < \infty$. Since ι is not perfect, Proposition 5.7 says that $(F, F) > 0$, i.e. $(F_+, F_-) > 0$.

Proposition 8.9. *The (FTC) holds for F if and only if*

$$\|(F_+)_a(F_-)_b\|_{HS} < \infty$$

whenever $a, b \in A$, $a \neq b$.

Proof. By definition of the inner product (F_+, F_-), the (FTC) means that $\mathrm{TR}((F_+)_a, (F_-)_b) < \infty$ whenever $a \neq b$. Because $\iota_\pm$ are perfect realizations, the components of $F_\pm$ are projections. For a pair of projections one has

$$\begin{aligned}\mathrm{TR}((F_+)_a, (F_-)_b) &= \mathrm{tr}((F_+)_a(F_-)_b(F_+)_a)\\ &= \mathrm{tr}((F_+)_a(F_-)_b((F_+)_a(F_-)_b)^*) = \|(F_+)_a(F_-)_b\|^2_{HS}.\end{aligned}$$

□

This shows that the (FTC) which we are assuming here is equivalent to the finiteness condition in the statement of the Duplicity Theorem in Section 2.

Proposition 8.10. *Assume the reigning hypotheses of this subsection. Let F_0 be an $|A|$-tuple of positive operators satisfying $(\mathcal{T}^N F_0)_a \leq CF_a$*

for some fixed integer $N \geq 0$, some fixed $C > 0$, and for all $a \in A$. Then

$$F_L = \operatorname*{wk-lim}_{\epsilon \to 0+} \epsilon \sum_{n \geq 0} e^{-\epsilon n} \mathcal{T}^n F_0 = \frac{(F_0, F_-)F_+ + (F_0, F_+)F_-}{(F_+, F_-)} . \tag{26}$$

Proof. Proposition 6.1, Corollary 4.14, and Corollary 5.6 together with the finiteness of (F_+, F_-). $\square$

Remark 8.11. If the hypotheses of Proposition 8.10 hold, except that instead of the (FTC) one has $(F_+, F_-) = \infty$, and if $(F_0, F) < \infty$, then a similar argument shows that the limit is zero.

Theorem 8.12. *Assume the reigning hypotheses of this subsection. Let $\pi'_N : \Gamma \ltimes C(\partial\Gamma) \to \mathcal{B}(\mathcal{H}'_N)$ be any boundary representation and suppose that $\iota_N : \mathcal{H} \to \mathcal{H}'_N$ is a boundary intertwiner (a Γ-map). Let μ_N and F_N be associated to ι_N, as per Section 4. Then $\mu_N = t_+\mu_+ + t_-\mu_-$ for nonnegative coefficients $t_\pm$.*

Proof. Any mention of "good vectors" in this proof means good vectors in $\mathcal{H}$ relative to the boundary realization ι; we never consider good vectors relative to ι_N, to $\iota_\pm$, or to any other boundary intertwiner.

Suppose the good vectors weren't dense in $\mathcal{H}$. According to Corollary 7.19 this would mean there was a nonzero Γ-invariant subspace $\mathcal{H}_B \subsetneq \mathcal{H}$ such that $\iota|_{\mathcal{H}_B}$ was a perfect realization; hence $\iota(\mathcal{H}_B)$ would be a $\Gamma \ltimes C(\partial\Gamma)$-subspace of $\mathcal{H}'$. Since $\mathcal{H}'_\pm$ are irreducible and inequivalent, the only possibilities for that subspace would be 0, $\mathcal{H}'_+$, $\mathcal{H}'_-$, and $\mathcal{H}'_+ \oplus \mathcal{H}'_-$. Since $\iota(v) = \frac{1}{\sqrt{2}}(\iota_+(v), \iota_-(v))$, no nonzero vector of $\iota(\mathcal{H})$ lies in any of the three proper $\Gamma \ltimes C(\partial\Gamma)$-subspaces. Moreover, $\iota(\mathcal{H})$ isn't all of $\mathcal{H}'_+ \oplus \mathcal{H}'_-$, so its subspace $\iota(\mathcal{H}_B)$ can't be either. One concludes that the good vectors are dense in $\mathcal{H}$.

Let $t_+ = \max\{t \geq 0\,;\ \mu_N - t\mu_+ \text{ is a positive map}\}$ and then let $t_- = \max\{t \geq 0\,;\ \mu_N - t_+\mu_+ - t\mu_- \text{ is a positive map}\}$. Let $\mu_R = \mu_N - t_+\mu_+ - t_-\mu_-$. Note that for any $\delta > 0$, neither $\mu_R - \delta\mu_+$ nor $\mu_R - \delta\mu_-$ is a positive map. As per Section 4, we find ι_R and F_R associated to μ_R. After several steps, we shall show that $\mu_R = 0$, and it will follow that $\mu_N = t_+\mu_+ + t_-\mu_-$, proving the theorem.

Fix any $\delta > 0$. Using Lemma 8.1 choose $a \in A$ so that $(\mu_R - \delta\mu_+)(\mathbf{1} - \mathbf{1}_a)$ is not positive semidefinite, and a good vector $u \in \mathcal{H}$ so that

$$\langle (\mu_R - \delta\mu_+)(\mathbf{1} - \mathbf{1}_a)u, u\rangle < 0 . \tag{27}$$

Define F_0 by

$$(F_0)_a = u \otimes \bar{u} \qquad\qquad (F_0)_c = 0 \qquad \text{for } c \neq a.$$

Then it follows from (27) that

$$\begin{aligned} (F_0, F_R) &= \mathrm{TR}(u \otimes \bar{u}, \mu_R(\mathbf{1} - \mathbf{1}_a)) \\ &= \langle \mu_R(\mathbf{1} - \mathbf{1}_a)u, u\rangle < \langle \delta\mu_+(\mathbf{1} - \mathbf{1}_a)u, u\rangle = \delta(F_0, F_+) . \end{aligned} \tag{28}$$

In particular this shows that $(F_0, F_+) > 0$.

Now using Corollary 5.5, the identities $(\mathcal{T} F_1, F_2) = (F_1, \mathcal{T} F_2)$, $\mathcal{T} F_R = F_R$, and $\mathcal{T} F = F$, equation (28) and Corollary 5.6, we find:

$$\begin{aligned}
\frac{(F_0,F_+)(F_-,F_R)}{(F_+,F_-)} &\le \left(\frac{(F_0,F_-)F_+ + (F_0,F_+)F_-}{(F_+,F_-)}, F_R\right) = (F_L, F_R) \\
&= (\lim_{\epsilon\to 0+} \epsilon \sum_{n\ge 0} e^{-\epsilon n}\mathcal{T}^n F_0, F_R) \le \liminf_{\epsilon\to 0+}(\epsilon \sum_{n\ge 0} e^{-\epsilon n}\mathcal{T}^n F_0, F_R) \\
&= \liminf_{\epsilon\to 0+} \epsilon\,(F_0,F_R)/(1-e^{-\epsilon}) = (F_0,F_R) < \delta(F_0,F_+) \\
&= \delta \lim_{\epsilon\to 0+} \epsilon(F_0,F_+)/(1-e^{-\epsilon}) = \delta(\lim_{\epsilon\to 0+} \epsilon \sum_{n\ge 0} e^{-\epsilon n}\mathcal{T}^n F_0, F_+) \\
&= \delta(F_L,F_+) = \left(\frac{(F_0,F_-)F_+ + (F_0,F_+)F_-}{(F_+,F_-)}, F_+\right) = \delta(F_0,F_+)\,.
\end{aligned}$$

Cancel the factor of (F_0, F_+) and send $\delta \to 0+$ to conclude that $(F_-, F_R) = 0$. An identical argument shows that $(F_+, F_R) = 0$. Consequently $(F, F_R) = 0$.

The following final step is identical to the corresponding step in the previous subsection. Unless $\mu_R = 0$, you may apply Proposition 8.1 to $-\mu_R$ to find $b \in A$ such that $-\mu_R(\mathbf{1}-\mathbf{1}_b)$ is not positive semidefinite. Then choose a good vector $v \in \mathcal{H}$ so that $\langle \mu_R(\mathbf{1}-\mathbf{1}_b)v, v\rangle > 0$, and define F_1 by

$$(F_1)_b = v \otimes \bar{v} \qquad\qquad (F_1)_c = 0 \qquad \text{for } c \ne b.$$

One may then calculate $(F_1, F_R) = \langle \mu_R(\mathbf{1}-\mathbf{1}_b)v, v\rangle > 0$. Since v is good, it follows that for n large enough $\mathcal{T}^n F_1 \le CF$. This leads to the contradiction

$$0 < (F_1, F_R) = (F_1, \mathcal{T}^n F_R) = (\mathcal{T}^n F_1, F_R) \le C(F, F_R) = 0\,.$$

□

Corollary 8.13. *Under the above hypotheses, π is an irreducible Γ-representation.*

Proof. Argue as in the proof of Corollary 8.6. □

Corollary 8.14. *Under the above hypotheses, any nonzero boundary intertwiner of π is equivalent to $s_+\iota_+$ for some $s_+ > 0$, to $s_-\iota_-$ for some $s_- > 0$, or to $s_+\iota_+ \oplus s_-\iota_-$ for some $s_\pm > 0$.*

In the last case, the boundary intertwiner will be a boundary realization (an isometry) if $s_+^2 + s_-^2 = 1$.

9. "Schur orthogonality"

The aim of this Section is to show how Proposition 8.10 can be specialized using *good vectors* to get formulae for limits of sums of products of matrix coefficents. The version of Schur orthogonality discussed here involves matrix coefficients of one fixed representation π. Similar Schur orthogonality for coefficients of two different representations is worth looking into, but we do not know how to proceed.

9.1. **Duplicity.** We consider the case of duplicity first. The reigning hypotheses are:

- $\pi : \Gamma \to \mathcal{U}(\mathcal{H})$ is a unitary representation of Γ.
- We have two irreducible representations $\pi'_\pm : \Gamma \ltimes C(\partial\Gamma) \to \mathcal{B}(\mathcal{H}'_\pm)$, which are inequivalent as representations of $\Gamma \ltimes C(\partial\Gamma)$.
- We have two perfect boundary realizations of π, $\iota_\pm : \mathcal{H} \to \mathcal{H}'_\pm$.
- Let $\iota : \mathcal{H} \to \mathcal{H}_+ \oplus \mathcal{H}_-$ be defined by $\iota = \frac{1}{\sqrt{2}}(\iota_+ \oplus \iota_-)$. As per Section 4, associate μ and F to ι.
- The (FTC) holds for F.

Our aim is to prove Theorem 2.4. In the statement of that theorem appears a dense linear subspace $\mathcal{H}^\infty \subseteq \mathcal{H}$. Let $\mathcal{H}^\infty$ be the subspace of good vectors with respect to ι. $\mathcal{H}^\infty$ is dense (see the proof of Theorem 8.12), closed under linear combinations (Lemma 7.5) and Γ-invariant (Lemma 7.7). Also in Theorem 2.4 appears a positive constant A_π. Let $A_\pi = 1/(F_+, F_-)$.

It is clear that the restriction map from $C(\Gamma \sqcup \partial\Gamma)$ to $C(\partial\Gamma)$ is a map of C^*-algebras. Abusing notation we define, for any G in $C(\Gamma \sqcup \partial\Gamma)$,

$$\pi'_\pm(G) = \pi'_\pm(G|_{\partial\Gamma}) \qquad \text{and} \qquad \mu_\pm(G) = \mu_\pm(G|_{\partial\Gamma}) \ .$$

For $G \in C(\Gamma \sqcup \partial\Gamma)$ and $x \in \Gamma$ set $G^*(x) = \overline{G(x^{-1})}$. We must prove

$$\begin{aligned}(29)\quad & \lim_{\epsilon\to 0+} \epsilon \sum_{x\in\Gamma} e^{-\epsilon|x|} G(x)\tilde{G}^*(x) \langle v_1, \pi(x)v_3\rangle \overline{\langle v_2, \pi(x)v_4\rangle} \\ &= \frac{1}{(F_+, F_-)} \Big(\langle \pi'_+(G)\iota_+ v_1, \iota_+ v_2\rangle \overline{\langle \pi'_-(\tilde{G})\iota_- v_3, \iota_- v_4\rangle} \\ &\qquad + \langle \pi'_-(G)\iota_- v_1, \iota_- v_2\rangle \overline{\langle \pi'_+(\tilde{G})\iota_+ v_3, \iota_+ v_4\rangle} \Big) \ .\end{aligned}$$

for any $G, \tilde{G} \in C(\Gamma \sqcup \partial\Gamma)$, $v_1, v_2 \in \mathcal{H}$, and $v_3, v_4 \in \mathcal{H}^\infty$. We need only prove (29) with $v_1 = v_2 = w \in \mathcal{H}$ and $v_3 = v_4 = v \in \mathcal{H}^\infty$. Once this is done, polarization, first with respect to v_1 and v_2, then with respect to v_3 and v_4 will give (29) in generality.

Taking into account the definition of $\mu_\pm$ we see that with $v_1 = v_2 = w$ and $v_3 = v_4 = v$ formula (29) becomes:

$$\begin{aligned}(30)\quad & \lim_{\epsilon\to 0+} \epsilon \sum_{x\in\Gamma} e^{-\epsilon|x|} G(x)\tilde{G}^*(x) \, |\langle w, \pi(x)v\rangle|^2 \\ &= \frac{1}{(F_+, F_-)} \Big(\langle \mu_+(G)w, w\rangle \overline{\langle \mu_-(\tilde{G})v, v\rangle} + \langle \mu_-(G)w, w\rangle \overline{\langle \mu_+(\tilde{G})v, v\rangle} \Big) \ .\end{aligned}$$

Remark 9.1. The limit on the left hand side of (30) remains the same if we omit any finite number of values of x. For instance, we can restrict the sum to $x \in \Gamma$ with $|x| \geq N$.

Lemma 9.2. *Let $v \in \mathcal{H}^\infty$ and $w \in \mathcal{H}$. Then there exists a constant $C(v, w)$ depending only on v and w, such that*

$$\epsilon \sum_{x\in\Gamma} e^{-\epsilon|x|} \left|\langle w, \pi(x)v\rangle\right|^2 \leq C(v, w) \tag{31}$$

whenever $0 < \epsilon \leq 1$.

Proof. By Corollary 7.9 there exists a constant C depending only on v such that

$$\sum_{|x|=n} \left|\langle w, \pi(x)v\rangle\right|^2 \leq C(v)\|w\|^2 .$$

Multiply the above inequality by $e^{-\epsilon n}$ and add up the geometric series to get the result. □

Corollary 9.3. *Let $v \in \mathcal{H}^\infty$ and $w \in \mathcal{H}$. Let H be any function in $C(\Gamma \sqcup \partial\Gamma)$ such that $\|H\|_\infty \leq \delta$. Then there exists a constant $C = C(v, w)$ such that*

$$\limsup_{\epsilon\to 0+} \left| \epsilon \sum_{x\in\Gamma} e^{-\epsilon|x|} H(x) \left|\langle w, \pi(x)v\rangle\right|^2 \right| \leq C\delta .$$

For $z \in \Gamma$ we defined $\lambda(z)$ as left-translation acting on $C(\partial\Gamma)$. Here we will use the same notation for left-translation acting on $C(\Gamma \sqcup \partial\Gamma)$.

Lemma 9.4. *Fix $G, \tilde{G} \in C(\Gamma \sqcup \partial\Gamma)$. Suppose that (30) holds for that G and $\tilde{G}$ together with any $w \in \mathcal{H}$ and $v \in \mathcal{H}^\infty$. Let $z \in \Gamma$. Then (30) also holds if we replace G by $\lambda(z)G$ or if we replace $\tilde{G}$ with $\lambda(z)\tilde{G}$.*

Proof. When we replace G with $\lambda(z)G$ the left hand side of (30) becomes

$$\begin{aligned}
\text{LHS} &= \lim_{\epsilon\to 0+} \epsilon \sum_{x\in\Gamma} e^{-\epsilon|x|} G(z^{-1}x)\tilde{G}^*(x) \left|\langle w, \pi(x)v\rangle\right|^2 \\
&= \lim_{\epsilon\to 0+} \epsilon \sum_{x\in\Gamma} e^{-\epsilon|zx|} G(x)\tilde{G}^*(zx) \left|\langle w, \pi(zx)v\rangle\right|^2 \\
&= \lim_{\epsilon\to 0+} \epsilon \sum_{x\in\Gamma} e^{-\epsilon|zx|} G(x)\tilde{G}^*(zx) \left|\langle \pi(z^{-1})w, \pi(x)v\rangle\right|^2 \\
&= \lim_{\epsilon\to 0+} \epsilon \sum_{x\in\Gamma} e^{-\epsilon|x|} G(x)\tilde{G}^*(x) \left|\langle \pi(z^{-1})w, \pi(x)v\rangle\right|^2 + \text{“vanishing term”} \\
&= \frac{1}{(F_+, F_-)} \Big(\langle \mu_+(G)\pi(z^{-1})w, \pi(z^{-1})w\rangle \overline{\langle \mu_-(\tilde{G})v, v\rangle} \\
&\qquad\qquad + \langle \mu_-(G)\pi(z^{-1})w, \pi(z^{-1})w\rangle \overline{\langle \mu_+(\tilde{G})v, v\rangle} \Big) \\
&= \frac{1}{(F_+, F_-)} \Big(\langle \mu_+(\lambda(z)G)w, w\rangle \overline{\langle \mu_-(\tilde{G})v, v\rangle} \\
&\qquad\qquad + \langle \mu_-(\lambda(z)G)w, w\rangle \overline{\langle \mu_+(\tilde{G})v, v\rangle} \Big) = \text{RHS}.
\end{aligned}$$

It remains to show that the "vanishing term" vanishes. Write it as:

$$\lim_{\epsilon\to 0+} \epsilon \sum_{x\in\Gamma} e^{-\epsilon|x|} G(x)(e^{\epsilon(|x|-|zx|)}\tilde{G}^*(zx) - \tilde{G}^*(x))\, |\langle \pi(z^{-1})w, \pi(x)v\rangle|^2 \,.$$

Note that $e^{-\epsilon|z|} \le e^{\epsilon(|x|-|zx|)} \le e^{\epsilon|z|}$, so $|e^{\epsilon(|x|-|zx|)} - 1| \le e^{\epsilon|z|} - 1$. Fix any $\delta > 0$. Since $\tilde{G}$ is continuous on $\Gamma \sqcup \partial\Gamma$ we can choose $N > 0$ so that $|\tilde{G}^*(zx) - \tilde{G}^*(x)| = |\tilde{G}(x^{-1}z^{-1}) - \tilde{G}(x^{-1})| \le \delta$ when $|x| \ge N$. According to Remark 9.1 we can omit all terms where $|x| < N$. Then

$$|G(x)(e^{\epsilon(|x|-|zx|)}\tilde{G}^*(zx) - \tilde{G}^*(x))| \le \|G\|_\infty\big((e^{\epsilon|z|} - 1)\|\tilde{G}\|_\infty + \delta\big) \,.$$

Hence Corollary 9.3 says

$$\limsup_{\epsilon\to 0+} \left| \epsilon \sum_{x\in\Gamma} e^{-|x|} G(x)(e^{\epsilon(|x|-|zx|)}\tilde{G}^*(zx) - \tilde{G}^*(x))\, |\langle \pi(z^{-1})w, \pi(x)v\rangle|^2 \right| \le C(v,w)\, \|G\|_\infty\, \delta \,.$$

As $\delta > 0$ is arbitrary the term does indeed vanish.

A strictly analogous calculation takes care of the case when we replace $\tilde{G}$ with $\lambda(z)\tilde{G}$. Note that in this second case one uses Lemma 7.7, the Γ-invariance of $\mathcal{H}^\infty$. □

Definition 9.5. Recall that $\mathbf{1}_x$ was defined as $\mathbf{1}_{\partial\Gamma(x)} \in C(\partial\Gamma)$. Abusing notation, we will also let $\mathbf{1}_x = \mathbf{1}_{\Gamma(x)\sqcup\partial\Gamma(x)} \in C(\Gamma \sqcup \partial\Gamma)$. Thus

$$\mathbf{1}_x(\text{reduced word}) = \begin{cases} 1 & \text{if it starts with the reduced word for } x \\ 0 & \text{otherwise} \end{cases}$$

for both finite and infinite reduced words.

We proceed to prove (30) for $G = \mathbf{1}_a$ and $\tilde{G} = \mathbf{1} - \mathbf{1}_b$:

Proposition 9.6. *For $v \in \mathcal{H}^\infty$ and $w \in \mathcal{H}$ one has*

$$\begin{aligned} \lim_{\epsilon\to 0+} \epsilon \sum_{x\in\Gamma} e^{-\epsilon|x|} \mathbf{1}_a(x)(\mathbf{1} - \mathbf{1}_b)(x^{-1})\, |\langle w, \pi(x)v\rangle|^2 \\ = \frac{1}{(F_+, F_-)} \big(\langle \mu_+(\mathbf{1}_a)w, w\rangle \langle \mu_-(\mathbf{1} - \mathbf{1}_b)v, v\rangle \\ + \langle \mu_-(\mathbf{1}_a)w, w\rangle \langle \mu_+(\mathbf{1} - \mathbf{1}_b)v, v\rangle\big) \,. \end{aligned} \tag{32}$$

Proof. Observe that the left hand side of (32) is equivalent to:

$$\lim_{\epsilon\to 0+} \epsilon \sum_{x\in\Gamma(a),\, x\notin\tilde{\Gamma}(b^{-1})} e^{-\epsilon|x|}\, |\langle w, \pi(x)v\rangle|^2 \,. \tag{33}$$

As in the proof of Theorem 8.12 define

$$(F_0)_b = v \otimes \bar{v} \qquad\qquad (F_0)_c = 0 \qquad \text{for } c \ne b.$$

As per Proposition 8.10 one has

$$\operatorname*{wk-lim}_{\epsilon\to 0+} \epsilon \sum_{n\geq 0} e^{-\epsilon n}\mathcal{T}^n F_0 = \frac{(F_0,F_-)F_+ + (F_0,F_+)F_-}{(F_+,F_-)} \ . \tag{34}$$

Dropping the $n=0$ term from the sum on the left has no effect on the limit. Consider the left-hand side of (34). For $n\geq 1$ Lemma 7.6 gives

$$\begin{aligned}(\mathcal{T}^n F_0)_a = \sum_c (\mathcal{T}^n)_{a,c}(F_0)_c &= \sum_{\substack{x\in\Gamma\,;\,|x|=n\\ x\in\Gamma(a)\,;\,x\notin\tilde\Gamma(b^{-1})}} P(x)(v\otimes\bar v)\\ &= \sum_{\substack{x\in\Gamma\,;\,|x|=n\\ x\in\Gamma(a)\,;\,x\notin\tilde\Gamma(b^{-1})}} \pi(x)v\otimes\overline{\pi(x)v}\ .\end{aligned}$$

On applying the a-th component of the left-hand side of (34) to the vector w and then calculating the inner product with w, one obtains

$$\begin{aligned}&\lim_{\epsilon\to 0+} \quad \epsilon\,\langle\left(\sum_{n\geq 1} e^{-\epsilon n}\mathcal{T}^n F_0\right)_a w, w\rangle\\ &= \lim_{\epsilon\to 0+} \quad \epsilon \sum_{n\geq 1} e^{-\epsilon n}\left(\sum_{\substack{x\in\Gamma\,;\,|x|=n\\ x\in\Gamma(a)\,;\,x\notin\tilde\Gamma(b^{-1})}} \langle w,\pi(x)v\rangle\overline{\langle w,\pi(x)v\rangle}\right)\end{aligned}$$

which is equal to (33).

Now let us compute the right-hand side of (34):

$$\begin{aligned}(F_0,F_\pm) = \sum_{c,d\,;\,c\neq d}\mathrm{TR}((F_0)_c,(F_\pm)_d) &= \sum_{d\,;\,d\neq b}\mathrm{TR}(v\otimes\bar v,(F_\pm)_d)\\ &= \sum_{d\,;\,d\neq b}\langle\mu_\pm(\mathbf{1}_d)v,v\rangle = \langle\mu_\pm(\mathbf{1}-\mathbf{1}_b)v,v\rangle\ .\end{aligned}$$

Hence the a-th component of the right-hand side of (34) is given by

$$\frac{\langle\mu_-(\mathbf{1}-\mathbf{1}_b)v,v\rangle\mu_+(\mathbf{1}_a) + \langle\mu_+(\mathbf{1}-\mathbf{1}_b)v,v\rangle\mu_-(\mathbf{1}_a)}{(F_+,F_-)}\ . \tag{35}$$

On applying this operator to the vector w and then taking the inner product with w, one gets the right-hand side of (32). ☐

Corollary 9.7. *For every $v\in\mathcal{H}^\infty$, $w\in\mathcal{H}$ and $y,z\in\Gamma$ one has*

$$\begin{aligned}&\lim_{\epsilon\to 0+}\epsilon\sum_{x\in\Gamma}\mathbf{1}_z(x)\mathbf{1}^*_y(x)\,|\langle w,\pi(x)v\rangle|^2 \qquad (36)\\ &= \frac{1}{(F_+,F_-)}\left(\langle\mu_+(\mathbf{1}_z)w,w\rangle\overline{\langle\mu_-(\mathbf{1}_y)v,v\rangle} + \langle\mu_-(\mathbf{1}_z)w,w\rangle\overline{\langle\mu_+(\mathbf{1}_z)v,v\rangle}\right)\ .\end{aligned}$$

Proof. Since $\lambda(b^{-1})(\mathbf{1} - \mathbf{1}_b) = \mathbf{1}_{b^{-1}}$ (36) follows from Proposition 9.6 and Lemma 9.4 for $|z| = |y| = 1$. The general case follows again from Lemma 9.4: write $z = wc$ for the reduced word for z and apply $\lambda(w)$ to $\mathbf{1}_c$, analogously for $\mathbf{1}_y$. □

Now we finish the proof of (30), and so of (29) and of Theorem 2.4. The linear span of the functions $\{\mathbf{1}_x\}_{x\in\Gamma}$ together with all finitely supported functions on Γ is dense in $C(\Gamma \sqcup \partial\Gamma)$. From Corollary 9.7 one deduces that (30) holds for G and $\tilde{G}$ in this dense subset. Finally, Corollary 9.3 allows us to pass from the dense subset to all of $C(\Gamma\sqcup\partial\Gamma)$.

9.2. **Oddity.** Here the reigning hypotheses are as follows:

- $\pi : \Gamma \to \mathcal{U}(\mathcal{H})$ is a unitary representation of Γ.
- $\pi' : \Gamma \ltimes C(\partial\Gamma) \to \mathcal{B}(\mathcal{H}')$ is an irreducible representation of $\Gamma \ltimes C(\partial\Gamma)$.
- $\iota : \mathcal{H} \to \mathcal{H}'$ is an imperfect boundary realization of π.
- As per Section 4, μ and F are associated to ι.
- The (FTC) holds for F.

Let $\mathcal{H}^\infty \subset \mathcal{H}$ be the dense subset of good vectors relative to ι.

Theorem 9.8. *Let $G, \tilde{G} \in C(\Gamma \sqcup \partial\Gamma)$; $v_1, v_2 \in \mathcal{H}$; $v_3, v_4 \in \mathcal{H}^\infty$. Then*

$$\lim_{\epsilon\to 0+} \epsilon \sum_{x\in\Gamma} e^{-\epsilon|x|} G(x)\tilde{G}^*(x)\langle v_1, \pi(x)v_3\rangle\overline{\langle v_2, \pi(x)v_4\rangle} = \frac{1}{(F,F)}\left(\langle \pi'(G)\iota v_1, \iota v_2\rangle\overline{\langle \pi'(\tilde{G})\iota v_3, \iota v_4\rangle}\right) . \tag{37}$$

The proof is analogous to the proof in the previous subsection and we omit it.

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