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Multiplicity of solutions of a zero mass nonlinear equation on a Riemannian manifold

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Abstract

The relation between the number of solutions of a nonlinear equation on a Riemannian manifold and the topology of the manifold itself is studied. The technique is based on Ljusternik–Schnirelmann category and Morse theory.

Keywords: Elliptic equations on manifolds; Multiplicity of solutions; Ljusternik–Schnirelmann category; Morse theory

1. Introduction

In this paper we are interested in the relation between the number of solutions of a nonlinear equation on a Riemannian manifold and the topology of the manifold itself.

Let (M, g) be a compact, connected, orientable, boundary-less Riemannian manifold of class C^∞ with Riemannian metric g. Let $\dim(M) = n \geqslant 3$.

We consider the problem

$$-\epsilon^2 \Delta u = f'(u) \tag{1.1}$$

with $u \in H_1^2(M)$.

As it has been pointed out in [9] problem (1.1) admits solutions on $\mathbb{R}^n$ if $f'(0) < 0$, while there are no solutions if $f'(0) > 0$. The limiting case $f'(0) = 0$, i.e. the "zero mass" case, depends on

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the structure of f. Berestycki and Lions proved the existence of ground state solutions if $f(u)$ behaves as $|u|^p$ for u large and $|u|^q$ for u small, with p and q respectively sub- and super-critical. In [8] they proved also the existence of infinitely many bound state solutions.

Problem (1.1) has been studied also in [7], where existence and non-existence results have been given on an exterior domain in $\mathbb{R}^n$.

The problem of the multiplicity of solutions of a nonlinear elliptic equation on a Riemannian manifold has been studied in [3], where the authors consider an equation with sub-critical growth.

The effect of the domain shape on the number of positive solutions of some semilinear elliptic problems has been widely studied. Here we only mention [1,5,6,10] and [4].

Let $f:\mathbb{R}\to\mathbb{R}$ be an even function such that:

(f1) $0<\mu f(s)\leqslant f'(s)s<f''(s)s^2$ for any $s\neq 0$ and for some $\mu>2$;
(f2) $f(0)=f'(0)=f''(0)=0$ and there exist positive constants c_0, c_1, p, q with $2<p<2^*<q$ such that

$$f(s)\geqslant\begin{cases} c_0|s|^p & \text{for } |s|\geqslant 1,\\ c_0|s|^q & \text{for } |s|\leqslant 1,\end{cases} \tag{1.2}$$

$$f''(s)\leqslant\begin{cases} c_1|s|^{p-2} & \text{for } |s|\geqslant 1,\\ c_1|s|^{q-2} & \text{for } |s|\leqslant 1.\end{cases} \tag{1.3}$$

We denote by $\operatorname{cat}(M)$ the Ljusternik–Schnirelmann category of M and by $\mathcal{P}_t(M)$ the Poincaré polynomial of M.

Our main results are the following:

Theorem 1.1. *For $\epsilon>0$ sufficiently small, Eq.* (1.1) *has at least* $\operatorname{cat}(M)+1$ *solutions in* $H_1^2(M)$.

Theorem 1.2. *If for $\epsilon>0$ sufficiently small the solutions of Eq.* (1.1) *are non-degenerate, then there are at least* $2\mathcal{P}_1(M)-1$ *solutions.*

2. Notation and preliminary results

We denote by $B(0,R)$ the ball in $\mathbb{R}^n$ of centre 0 and radius R and by $B_g(x,R)$ the ball in M of centre x and radius R.

We define a smooth real function χ_R on $\mathbb{R}^+$ such that

$$\chi_R(t)=\begin{cases} 1 & \text{if } 0\leqslant t\leqslant \frac{R}{2},\\ 0 & \text{if } t\geqslant R,\end{cases} \tag{2.1}$$

and $|\chi_R'(t)|\leqslant\frac{\chi_0}{R}$, with χ_0 positive constant.

We recall some definitions and results about compact connected Riemannian manifolds of class C^∞ (see for example [12]).

Remark 2.1. On the tangent bundle TM of M the exponential map $\exp: TM\to M$ is defined. This map has the following properties:

(i) exp is of class C^∞;
(ii) there exists a constant $R > 0$ such that

$$\exp_x|_{B(0,R)} : B(0,R) \to B_g(x,R)$$

is a diffeomorphism for all $x \in M$.

It is possible to choose an atlas $\mathcal{C}$ on M, whose charts are given by the exponential map (normal coordinates). We denote by $\{\psi_C\}_{C\in\mathcal{C}}$ a partition of unity subordinate to the atlas $\mathcal{C}$. Let g_{x_0} be the Riemannian metric in the normal coordinates of the map $\exp_{x_0}$.

For any $u \in H_1^2(M)$ we have that

$$\begin{aligned}\int_M \left|\nabla u(x)\right|_g^2 d\mu_g &= \sum_{C\in\mathcal{C}} \int_C \psi_C(x)\left|\nabla u(x)\right|_g^2 d\mu_g \\ &= \sum_{C\in\mathcal{C}} \int_{B(0,R)} \psi_C\big(\exp_{x_C}(z)\big) g_{x_C}^{ij}(z) \frac{\partial u(\exp_{x_C}(z))}{\partial z_i} \frac{\partial u(\exp_{x_C}(z))}{\partial z_j} \left|g_{x_C}(z)\right|^{\frac{1}{2}} dz,\end{aligned}$$

where Einstein notation is adopted, that is

$$g^{ij} z_i z_j = \sum_{i,j=1}^n g^{ij} z_i z_j,$$

$(g_{x_0}^{ij}(z))$ is the inverse matrix of $g_{x_0}(z)$ and $|g_{x_0}(z)| = \det(g_{x_0}(z))$. In particular we have that $g_{x_0}(0) = \mathrm{Id}$. A similar relation holds for the integration of $|u(x)|^p$. For convenience we will also write for all $x_0 \in M$ and $z, \xi \in T_{x_0}M$

$$|\xi|_{g_{x_0}(z)}^2 = g_{x_0}^{ij}(z)\xi_i\xi_j. \tag{2.2}$$

Remark 2.2. Since M is compact, there are two strictly positive constants h and H such that for all $x \in M$ and all $z \in T_xM$

$$h|z|^2 \leqslant g_x(z,z) \leqslant H|z|^2,$$

where $|\cdot|$ is the standard metric in $\mathbb{R}^n$. Hence there holds

$$h^n \leqslant \left|g_x(z)\right| \leqslant H^n.$$

We are going to find the solutions of (1.1) as critical points of the functional $J_\epsilon : H_1^2(M) \to \mathbb{R}$, defined by

$$J_\epsilon(u) = \frac{\epsilon^2}{2\epsilon^n} \int_M \left|\nabla u(x)\right|_g^2 d\mu_g - \frac{1}{\epsilon^n} \int_M f\big(u(x)\big)\, d\mu_g, \tag{2.3}$$

constrained on the Nehari manifold

$$\mathcal{N}_\epsilon = \left\{ u \in H_1^2(M) \;\middle|\; u \neq 0 \text{ and } \int_M \epsilon^2 |\nabla u|_g^2 \, d\mu_g = \int_M f'(u) u \, d\mu_g \right\}. \tag{2.4}$$

Let $\mathcal{D}^{1,2}(\mathbb{R}^n)$ be the completion of $C_0^\infty(\mathbb{R}^n)$ with respect to the norm

$$\|v\|^2_{\mathcal{D}^{1,2}(\mathbb{R}^n)} = \int_{\mathbb{R}^n} \left|\nabla v(z)\right|^2 dz.$$

We consider also the following functional $J : \mathcal{D}^{1,2}(\mathbb{R}^n) \to \mathbb{R}$ defined by

$$J(v) := \int_{\mathbb{R}^n} \left(\frac{1}{2} \left|\nabla v(x)\right|^2 - f\big(v(x)\big) \right) dx \tag{2.5}$$

and the associated Nehari manifold

$$\mathcal{N} = \left\{ v \in \mathcal{D}^{1,2}\big(\mathbb{R}^n\big) \;\middle|\; v \neq 0 \text{ and } \int_{\mathbb{R}^n} \left|\nabla v(x)\right|^2 dx = \int_{\mathbb{R}^n} f'(u) u \, dx \right\}. \tag{2.6}$$

The functionals J_ϵ and J are C^2 respectively on $H_1^2(M)$ and on $\mathcal{D}^{1,2}(\mathbb{R}^n)$. In fact, we have

Lemma 2.3. (i) *The functional $F_{\epsilon,M} : L^p(M) \to \mathbb{R}$, defined by*

$$F_{\epsilon,M}(u) := \frac{1}{\epsilon^n} \int_M f\big(u(x)\big) \, d\mu_g \tag{2.7}$$

is of class C^2 and

$$F'_{\epsilon,M}(u_0)u_1 = \frac{1}{\epsilon^n} \int_M f'\big(u_0(x)\big) u_1(x) \, d\mu_g,$$

$$F''_{\epsilon,M}(u_0)u_1 u_2 = \frac{1}{\epsilon^n} \int_M f''\big(u_0(x)\big) u_1(x) u_2(x) \, d\mu_g.$$

(ii) *The functional $F : L^{2^*}(\mathbb{R}^n) \to \mathbb{R}$ defined by*

$$F(v) := \int_{\mathbb{R}^n} f\big(v(z)\big) \, dz \tag{2.8}$$

is of class C^2 and

$$F'(v_0)v_1 = \int_{\mathbb{R}^n} f'\big(v_0(z)\big)v_1(z)\,dz,$$

$$F''(v_0)v_1v_2 = \int_{\mathbb{R}^n} f''\big(v_0(z)\big)v_1(z)v_2(z)\,dz.$$

The proof of this lemma is analogous to the proof of Lemma 2.7 in [7].
We also have the following lemma:

Lemma 2.4. *The functionals $\widetilde{F}_{\epsilon,M} : L^p(M) \to \mathbb{R}$, defined by*

$$\widetilde{F}_{\epsilon,M}(u) := \frac{1}{\epsilon^n}\int_M \left[\frac{1}{2}f'\big(u(x)\big)u(x) - f\big(u(x)\big)\right]d\mu_g \tag{2.9}$$

and $\widetilde{F}_\Omega : L^{2^*}(\Omega) \to \mathbb{R}$ defined by

$$\widetilde{F}_\Omega(v) := \int_\Omega \left[\frac{1}{2}f'\big(v(z)\big)v(z) - f\big(v(z)\big)\right]dz \tag{2.10}$$

are strongly continuous.

We write

$$m(J) := \inf\big\{J(v) \bigm| v \in \mathcal{N}\big\}. \tag{2.11}$$

There exists a positive, spherically symmetric and decreasing with $|z|$ solution $U \in \mathcal{D}^{1,2}(\mathbb{R}^n)$ of

$$-\Delta U = f'(U) \quad \text{in } \mathbb{R}^n, \tag{2.12}$$

such that $J(U) = m(J)$ (see [9] and [7]).
The function $U_\epsilon(z) = U(\frac{z}{\epsilon})$ is solution of

$$-\epsilon^2 \Delta U_\epsilon = f'(U_\epsilon).$$

For any $\delta > 0$ we consider the subset of $\mathcal{N}_\epsilon$

$$\Sigma_{\epsilon,\delta} := \big\{u \in \mathcal{N}_\epsilon \bigm| J_\epsilon(u) < m(J) + \delta\big\}. \tag{2.13}$$

We recall now the definition of Palais–Smale condition:

Definition 2.5. Let J be a C^1 functional on a Banach space X. A sequence $\{u_m\}$ in X is a Palais–Smale sequence for J if $|J(u_m)| \leqslant c$, uniformly in m, while $J'(u_m) \to 0$ strongly, as $m \to \infty$. We say that J satisfies the Palais–Smale condition ((PS) condition) if any Palais–Smale sequence has a convergent subsequence.

3. Ideas of the proof for the category theory result

We recall the definition of Ljusternik–Schnirelmann category (see [13]).

Definition 3.1. Let M be a topological space and consider a closed subset $A \subset M$. We say that A has category k relative to M ($\mathrm{cat}_M(A) = k$) if A is covered by k closed sets A_j, $1 \leqslant j \leqslant k$, which are contractible in M and if k is minimal with this property. If no such finite covering exists, we let $\mathrm{cat}_M(A) = \infty$. If $A = M$, we write $\mathrm{cat}_M(M) = \mathrm{cat}(M)$.

Remark 3.2. Let M_1 and M_2 be topological spaces. If $g_1 : M_1 \to M_2$ and $g_2 : M_2 \to M_1$ are continuous operators such that $g_2 \circ g_1$ is homotopic to the identity on M_1, then $\mathrm{cat}(M_1) \leqslant \mathrm{cat}(M_2)$ (see [5]).

Using the notation in the previous section, Theorem 1.1 can be stated more precisely like this:

Theorem 3.3. *There exists $\delta_0 \in (0, m(J))$ such that for any $\delta \in (0, \delta_0)$ there exists $\epsilon_0 = \epsilon_0(\delta) > 0$ and for any $\epsilon \in (0, \epsilon_0)$ the functional J_ϵ has at least* $\mathrm{cat}(M)$ *critical points $u \in H_2^1(M)$ satisfying $J_\epsilon(u) < m(J) + \delta$ and at least one critical point with $J_\epsilon(u) > m(J) + \delta$.*

This theorem is a consequence of the following classical result (see for example [6]):

Theorem 3.4. *Let J be a C^1 real functional on a complete $C^{1,1}$ submanifold N of a Banach space. If J is bounded below and satisfies the* (PS) *condition then it has at least* $\mathrm{cat}(J^d)$ *critical points in J^d, where $J^d := \{u \in N:\ J(u) < d\}$, and at least one critical point $u \notin J^d$.*

More precisely, Theorem 3.3 follows from the previous theorem, Remark 3.2 and the following proposition:

Proposition 3.5. *There exists $\delta_0 \in (0, m(J))$ such that for any $\delta \in (0, \delta_0)$ there exists $\epsilon_0 = \epsilon_0(\delta) > 0$ and for any $\epsilon \in (0, \epsilon_0)$ we have*

$$\mathrm{cat}(M) \leqslant \mathrm{cat}(\Sigma_{\epsilon,\delta}).$$

In order to prove this we will present two suitable functions g_1 and g_2.
By the embedding theorem, we assume that M is embedded in $\mathbb{R}^N$, with $N \geqslant 2n$.

Definition 3.6. We define the radius of topological invariance $r(M)$ of M as

$$r(M) := \sup\bigl\{\rho > 0 \bigm| \mathrm{cat}(M_\rho) = \mathrm{cat}(M)\bigr\},$$

where $M_\rho := \{z \in \mathbb{R}^N \mid d(z, M) < \rho\}$.

We can now show a function $\phi_\epsilon : M \to \Sigma_{\epsilon,\delta}$ and a function $\beta : \Sigma_{\epsilon,\delta} \to M_r$, with $0 < r < r(M)$ such that

$$I_\epsilon := \beta \circ \phi_\epsilon : M \to M_r \tag{3.1}$$

is well defined and homotopic to the identity on M.

4. The function ϕ_ϵ

Next lemma presents some properties of the Nehari manifold.

Lemma 4.1. (i) *The set $\mathcal{N}_\epsilon$ (respectively $\mathcal{N}$) is a C^1 manifold.*

(ii) *For all not constant $u \in H_1^2(M)$ (for all $v \in \mathcal{D}^{1,2}(\mathbb{R}^n)$, $v \not\equiv 0$), there exists a unique $t_\epsilon(u) > 0$ ($t(v) > 0$) such that $t_\epsilon(u)u \in \mathcal{N}_\epsilon$ ($t(v)v \in \mathcal{N}$) and $J_\epsilon(t_\epsilon(u)u)$ ($J(t(v)v)$) is the maximum value of $J_\epsilon(tu)$ ($J(tv)$) for $t \geqslant 0$.*

(iii) *The dependence of $t_\epsilon(u)$ on u (of $t(v)$ on v) is C^1.*

For the proof see Lemma 3.1 in [7].

Let U be the function defined in Section 2. We write

$$\widetilde{U}_{\frac{R}{\epsilon}} = U(z) \quad \text{with } z \in \mathbb{R}^n \text{ such that } |z| = \frac{R}{\epsilon}.$$

For any $x_0 \in M$ and $\epsilon > 0$, we consider the function on M

$$W_{x_0,\epsilon}(x) := \begin{cases} U_\epsilon(\exp_{x_0}^{-1}(x)) - \widetilde{U}_{\frac{R}{\epsilon}} & \text{if } x \in B_g(x_0, R), \\ 0 & \text{otherwise,} \end{cases} \tag{4.1}$$

where R is chosen as in Remark 2.1(ii).

The function $W_{x_0,\epsilon}$ is in $H_1^2(M)$ and is not identically zero. Then, by the previous lemma, we can define

$$\begin{aligned} \phi_\epsilon : M &\longrightarrow \mathcal{N}_\epsilon, \\ x_0 &\longmapsto t_\epsilon\big(W_{x_0,\epsilon}(x)\big) W_{x_0,\epsilon}(x). \end{aligned} \tag{4.2}$$

The choice of the function ϕ_ϵ different from the one in [3] has been made for the function U can be not in $L^2(\mathbb{R}^n)$.

Proposition 4.2. *For any $\epsilon > 0$ the map $\phi_\epsilon : M \to \mathcal{N}_\epsilon$ is continuous. For any $\delta > 0$ there exists $\epsilon_0 > 0$ such that if $\epsilon < \epsilon_0$*

$$\phi_\epsilon(x_0) \in \Sigma_{\epsilon,\delta}$$

for all $x_0 \in M$.

Proof. (I) *The map $\phi_\epsilon : M \to \mathcal{N}_\epsilon$ is continuous.*

By Lemma 4.1(iii), it is enough to prove that

$$\lim_{k\to\infty} \|W_{x_k,\epsilon} - W_{\hat{x},\epsilon}\|_{H_2^1(M)} = 0$$

for any sequence $\{x_k\}$ in M, converging to $\hat{x}$.

We choose a finite atlas $\mathcal{C}$ for M, which contains the chart $C = B_g(\hat{x}, R)$. The functions $W_{x_k,\epsilon}$ and $W_{\hat{x},\epsilon}$ have support respectively on $B_g(x_k, R)$ and on $B_g(\hat{x}, R)$. Since $x_k \to \hat{x}$ the set

$Z_k = [B_g(x_k, R) \setminus B_g(\hat{x}, R)] \cup [B_g(\hat{x}, R) \setminus B_g(x_k, R)]$ is such that $\mu_g(Z_k) \to 0$ as $k \to \infty$. Then we have

$$\int_{Z_k} \left|\nabla\left(W_{x_k,\epsilon}(x) - W_{\hat{x},\epsilon}(x)\right)\right|_g^2 d\mu_g \to 0 \quad \text{as } k \to \infty.$$

We still have to check the integral on $B_g(x_k, R) \cap B_g(\hat{x}, R)$. We write $A_k = \exp_{\hat{x}}^{-1}(B_g(x_k, R) \cap B_g(\hat{x}, R))$ and $\eta_k(z) = \exp_{x_k}^{-1}(\exp_{\hat{x}}(z))$,

$$\begin{aligned}\int_{\exp_{\hat{x}}(A_k)} \left|\nabla\left[W_{x_k,\epsilon}(x) - W_{\hat{x},\epsilon}(x)\right]\right|_g^2 d\mu_g &= \int_{A_k} \left|\nabla\left[U_\epsilon\left(\eta_k(z)\right) - U_\epsilon(z)\right]\right|_{g_{\hat{x}}(z)}^2 \left|g_{\hat{x}}(z)\right|^{\frac{1}{2}} dz \\ &\leqslant \frac{H^{\frac{n}{2}}}{h} \int_{A_k} \left|\nabla\left[U_\epsilon\left(\eta_k(z)\right) - U_\epsilon(z)\right]\right|^2 dz.\end{aligned}$$

Since $\eta_k(z)$ tends point-wise to z and ∇U_ϵ is continuous, $|\nabla[U_\epsilon(\eta_k(z)) - U_\epsilon(z)]|^2$ tends point-wise to zero. Applying the Lebesgue theorem, we obtain that

$$\int_M \left|\nabla\left[W_{x_k,\epsilon}(x) - W_{\hat{x},\epsilon}(x)\right]\right|_g^2 d\mu_g \to 0.$$

In an analogous way we have that $\|W_{x_k,\epsilon} - W_{\hat{x},\epsilon}\|_{L^2(M)}^2$ tends to zero.

(II) *The limit of* $\frac{\epsilon^2}{\epsilon^n}\int_M |\nabla W_{x_0,\epsilon}(x)|_g^2\, d\mu_g$ *is* $\|U\|_{\mathcal{D}^{1,2}(\mathbb{R}^n)}^2$.

To prove the second statement of this proposition, first we show that

$$\lim_{\epsilon\to 0} \frac{\epsilon^2}{\epsilon^n} \int_M \left|\nabla W_{x_0,\epsilon}(x)\right|_g^2 d\mu_g = \|U\|_{\mathcal{D}^{1,2}(\mathbb{R}^n)}^2 \tag{4.3}$$

uniformly with respect to $x_0 \in M$.

We evaluate the following:

$$\begin{aligned}\left|\frac{\epsilon^2}{\epsilon^n}\int_M |\nabla W_{x_0,\epsilon}|_g^2\, d\mu_g - \int_{\mathbb{R}^n} |\nabla U|^2\, dz\right| &= \left|\frac{\epsilon^2}{\epsilon^n} \int_{B_g(x_0,R)} \left|\nabla\left[U_\epsilon\left(\exp_{x_0}^{-1}(x)\right)\right]\right|_g^2 d\mu_g - \int_{\mathbb{R}^n} |\nabla U|^2\, dz\right| \\ &= \left|\frac{\epsilon^2}{\epsilon^n} \int_{B(0,R)} \left|\nabla U_\epsilon(z)\right|_{g_{x_0}(z)}^2 \left|g_{x_0}(z)\right|^{\frac{1}{2}} dz - \int_{\mathbb{R}^n} |\nabla U|^2\, dz\right|.\end{aligned}$$

Changing variables, we obtain

$$\left|\int_{\mathbb{R}^n} \left(\chi_{B(0,\frac{R}{\epsilon})}(z) g_{x_0}^{ij}(\epsilon z) \left|g_{x_0}(\epsilon z)\right|^{\frac{1}{2}} - \delta^{ij}\right) \frac{\partial U}{\partial z_i} \frac{\partial U}{\partial z_j}\, dz\right|,$$

where $\chi_{B(0,\frac{R}{\epsilon})}(z)$ denotes the characteristic function of the set $B(0,\frac{R}{\epsilon})$ and where δ^{ij} is the Kronecker delta (it takes value 0 for $i \neq j$ and 1 for $i = j$). The previous integral is bounded from above by the following sum

$$\left| \int\limits_{B(0,T)} \left(\chi_{B(0,\frac{R}{\epsilon})}(z) g_{x_0}^{ij}(\epsilon z) \left|g_{x_0}(\epsilon z)\right|^{\frac{1}{2}} - \delta^{ij}\right) \frac{\partial U}{\partial z_i} \frac{\partial U}{\partial z_j} \, dz \right| \\ + \left| \int\limits_{\mathbb{R}^n \setminus B(0,T)} \left(\chi_{B(0,\frac{R}{\epsilon})}(z) g_{x_0}^{ij}(\epsilon z) \left|g_{x_0}(\epsilon z)\right|^{\frac{1}{2}} - \delta^{ij}\right) \frac{\partial U}{\partial z_i} \frac{\partial U}{\partial z_j} \, dz \right|$$

with $T > 0$. It is easy to see that the second addendum vanishes as $T \to \infty$. As regards the first addendum, fixed T, by compactness of the manifold M and regularity of the Riemannian metric g the limit

$$\lim_{\epsilon \to 0} \left| \chi_{B(0,\frac{R}{\epsilon})}(z) g_{x_0}^{ij}(\epsilon z) \left|g_{x_0}(\epsilon z)\right|^{\frac{1}{2}} - \delta^{ij} \right| = 0$$

holds true uniformly with respect to $x_0 \in M$ and $z \in B(0,T)$ and (4.3) is proved.

(III) *There exists $t_1 > 0$ such that $t_\epsilon(W_{x_0,\epsilon}) \geqslant t_1$ for any $\epsilon \in (0,1]$ and $x_0 \in M$.*

Let $g_{\epsilon,u}(t) = J_\epsilon(tu)$. By Lemma 4.1(ii), it is enough to find $t_1 > 0$ such that for all $t \in [0,t_1]$ $g'_{\epsilon,W_{x_0,\epsilon}}(t) > 0$ for all $\epsilon \leqslant 1$ and for all $x_0 \in M$. Then we look for a lower bound of $g'_{\epsilon,W_{x_0,\epsilon}}(t)$:

$$\begin{aligned} g'_{\epsilon,W_{x_0,\epsilon}}(t) &= \frac{\epsilon^2 t}{\epsilon^n} \int\limits_M |\nabla W_{x_0,\epsilon}|_g^2 \, d\mu_g - \frac{1}{\epsilon^n} \int\limits_M f'(tW_{x_0,\epsilon}) W_{x_0,\epsilon} \, d\mu_g \\ &= \frac{1}{\epsilon^n} \int\limits_{B(0,R)} \left[\epsilon^2 t \left|\nabla U_\epsilon(z)\right|^2_{g_{x_0}(z)} - f'\left(tU_\epsilon(z) - t\widetilde{U}_{\frac{R}{\epsilon}}\right)\left(U_\epsilon(z) - \widetilde{U}_{\frac{R}{\epsilon}}\right)\right] \left|g_{x_0}(z)\right|^{\frac{1}{2}} dz \\ &= \int\limits_{B(0,\frac{R}{\epsilon})} \left[t \left|\nabla U(z)\right|^2_{g_{x_0}(\epsilon z)} - f'\left(tU(z) - t\widetilde{U}_{\frac{R}{\epsilon}}\right)\left(U(z) - \widetilde{U}_{\frac{R}{\epsilon}}\right)\right] \left|g_{x_0}(\epsilon z)\right|^{\frac{1}{2}} dz. \end{aligned}$$

Using Remark 2.2, the fact that $\epsilon \leqslant 1$ and the properties of f (f1) and (f2), we obtain the following inequality:

$$\begin{aligned} g'_{\epsilon,W_{x_0,\epsilon}}(t) > \frac{h^{\frac{n}{2}} t}{H} \int\limits_{B(0,R)} \left|\nabla U(z)\right|^2 dz - c_1 H^{\frac{n}{2}} \int\limits_{G_{t,\epsilon}} t^{p-1} \left|U(z) - \widetilde{U}_{\frac{R}{\epsilon}}\right|^p dz \\ - c_1 H^{\frac{n}{2}} \int\limits_{L_{t,\epsilon}} t^{q-1} \left|U(z) - \widetilde{U}_{\frac{R}{\epsilon}}\right|^q dz, \end{aligned}$$

where $G_{t,\epsilon} = \{z \in B(0,\frac{R}{\epsilon}) \mid t(U(z) - \widetilde{U}_{\frac{R}{\epsilon}}) \geqslant 1\}$ and $L_{t,\epsilon} = \{z \in B(0,\frac{R}{\epsilon}) \mid t(U(z) - \widetilde{U}_{\frac{R}{\epsilon}}) \leqslant 1\}$. If $t \leqslant 1$, the following inclusions hold:

$$G_{t,\epsilon} \subset \left\{ z \in B\left(0, \frac{R}{\epsilon}\right) \,\middle|\, U(z) - \widetilde{U}_{\frac{R}{\epsilon}} \geqslant 1 \right\}$$
$$\subset \left\{ z \in B\left(0, \frac{R}{\epsilon}\right) \,\middle|\, U(z) \geqslant 1 \right\} \subset \left\{ z \in \mathbb{R}^n \,\middle|\, U(z) \geqslant 1 \right\} = G.$$

By these inclusions and the fact that $|U(z) - \widetilde{U}_{\frac{R}{\epsilon}}| \leqslant |U(z)|$,

$$\int_{G_{t,\epsilon}} t^{p-1} \left| U(z) - \widetilde{U}_{\frac{R}{\epsilon}} \right|^p dz \leqslant \int_G t^{p-1} \left| U(z) \right|^p dz.$$

Let $L = \{z \in \mathbb{R}^n \mid U(z) \leqslant 1\}$. We have

$$\begin{aligned}
\int_{L_{t,\epsilon}} t^{q-1} \left| U(z) - \widetilde{U}_{\frac{R}{\epsilon}} \right|^q dz &= \int_{L \cap B(0, \frac{R}{\epsilon})} t^{q-1} \left| U(z) - \widetilde{U}_{\frac{R}{\epsilon}} \right|^q dz + \int_{L_{t,\epsilon} \setminus L} t^{q-1} \left| U(z) - \widetilde{U}_{\frac{R}{\epsilon}} \right|^q dz \\
&\leqslant \int_L t^{q-1} \left| U(z) \right|^q dz + \int_{L_{t,\epsilon} \setminus L} t^{p-1} \left| U(z) - \widetilde{U}_{\frac{R}{\epsilon}} \right|^p dz \\
&\leqslant \int_L t^{q-1} \left| U(z) \right|^q dz + \int_G t^{p-1} \left| U(z) \right|^p dz.
\end{aligned}$$

We conclude that

$$g'_{\epsilon, W_{x_0,\epsilon}}(t) > \gamma_1 t - \gamma_2 t^{p-1} - \gamma_3 t^{q-1}$$

with γ_1, γ_3 positive constants and γ_2 nonnegative constant. This proves the existence of t_1.

(IV) *There exists $t_2 > 0$ such that $t_\epsilon(W_{x_0,\epsilon}) \leqslant t_2$ for any $\epsilon \in (0, 1]$ and $x_0 \in M$.*

If u is a function in the Nehari manifold $\mathcal{N}_\epsilon$, we have that $J_\epsilon(u) = \widetilde{F}_{\epsilon,M}(u)$, as defined in (2.9). Then by property (f1) $J_\epsilon(u)$ is positive. By Lemma 4.1(ii), it is enough to find $t_2 > 0$ such that for all $t \geqslant t_2$ $J_\epsilon(t W_{x_0,\epsilon}) < 0$ for all $\epsilon \leqslant 1$ and for all $x_0 \in M$. Then we look for an upper bound of $J_\epsilon(t W_{x_0,\epsilon})$:

$$\begin{aligned}
J_\epsilon(t W_{x_0,\epsilon}) &= \frac{\epsilon^2 t^2}{2\epsilon^n} \int_M |\nabla W_{x_0,\epsilon}|_g^2 \, d\mu_g - \frac{1}{\epsilon^n} \int_M f(t W_{x_0,\epsilon}) \, d\mu_g \\
&= \frac{1}{\epsilon^n} \int_{B(0,R)} \left[\frac{\epsilon^2 t^2}{2} \left| \nabla U_\epsilon(z) \right|^2_{g_{x_0}(z)} - f\left(t U_\epsilon(z) - t \widetilde{U}_{\frac{R}{\epsilon}}\right) \right] \left| g_{x_0}(z) \right|^{\frac{1}{2}} dz \\
&= \int_{B(0, \frac{R}{\epsilon})} \left[\frac{t^2}{2} \left| \nabla U(z) \right|^2_{g_{x_0}(\epsilon z)} - f\left(t U(z) - t \widetilde{U}_{\frac{R}{\epsilon}}\right) \right] \left| g_{x_0}(\epsilon z) \right|^{\frac{1}{2}} dz
\end{aligned}$$

$$\leqslant \frac{H^{\frac{n}{2}}t^2}{2h}\|U\|^2_{\mathcal{D}^{1,2}(\mathbb{R}^n)} - c_0 h^{\frac{n}{2}} \int\limits_{G_{t,\epsilon}} t^p\big|U(z)-\widetilde{U}_{\frac{R}{\epsilon}}\big|^p\,dz$$

$$- c_0 h^{\frac{n}{2}} \int\limits_{L_{t,\epsilon}} t^q\big|U(z)-\widetilde{U}_{\frac{R}{\epsilon}}\big|^q\,dz.$$

If we consider $t \geqslant 1$ and $\widetilde{U}_R = U(z)$ with $z \in \mathbb{R}^n$ such that $|z| = R$, there holds

$$\begin{aligned}
&\int\limits_{G_{t,\epsilon}} t^p\big|U(z)-\widetilde{U}_{\frac{R}{\epsilon}}\big|^p\,dz + \int\limits_{L_{t,\epsilon}} t^q\big|U(z)-\widetilde{U}_{\frac{R}{\epsilon}}\big|^q\,dz \\
&\geqslant t^p\Bigg[\int\limits_{G_{1,\epsilon}} \big|U(z)-\widetilde{U}_{\frac{R}{\epsilon}}\big|^p\,dz + \int\limits_{G_{t,\epsilon}\setminus G_{1,\epsilon}} \big|U(z)-\widetilde{U}_{\frac{R}{\epsilon}}\big|^p\,dz \\
&\quad + \int\limits_{L_{1,\epsilon}} \big|U(z)-\widetilde{U}_{\frac{R}{\epsilon}}\big|^q\,dz - \int\limits_{L_{1,\epsilon}\setminus L_{t,\epsilon}} \big|U(z)-\widetilde{U}_{\frac{R}{\epsilon}}\big|^q\,dz\Bigg] \\
&\geqslant t^p\Bigg[\int\limits_{G_{1,\epsilon}} \big|U(z)-\widetilde{U}_{\frac{R}{\epsilon}}\big|^p\,dz + \int\limits_{L_{1,\epsilon}} \big|U(z)-\widetilde{U}_{\frac{R}{\epsilon}}\big|^q\,dz\Bigg] \\
&\geqslant t^p\Bigg[\int\limits_{G_{1,\epsilon}\cap B(0,R)} \big|U(z)-\widetilde{U}_{\frac{R}{\epsilon}}\big|^p\,dz + \int\limits_{L_{1,\epsilon}\cap B(0,R)} \big|U(z)-\widetilde{U}_{\frac{R}{\epsilon}}\big|^q\,dz\Bigg] \\
&\geqslant t^p\Bigg[\int\limits_{G_{1,\epsilon}\cap B(0,R)} \big|U(z)-\widetilde{U}_{R}\big|^p\,dz + \int\limits_{L_{1,\epsilon}\cap B(0,R)} \big|U(z)-\widetilde{U}_{R}\big|^q\,dz\Bigg] \\
&= t^p\Bigg[\int\limits_{G_{1,1}} \big|U(z)-\widetilde{U}_{R}\big|^p\,dz + \int\limits_{G_{1,\epsilon}\cap B(0,R)\setminus G_{1,1}} \big|U(z)-\widetilde{U}_{R}\big|^p\,dz \\
&\quad + \int\limits_{L_{1,1}} \big|U(z)-\widetilde{U}_{R}\big|^q\,dz - \int\limits_{L_{1,1}\setminus L_{1,\epsilon}} \big|U(z)-\widetilde{U}_{R}\big|^q\,dz\Bigg] \\
&\geqslant t^p\Bigg[\int\limits_{G_{1,1}} \big|U(z)-\widetilde{U}_{R}\big|^p\,dz + \int\limits_{L_{1,1}} \big|U(z)-\widetilde{U}_{R}\big|^q\,dz\Bigg].
\end{aligned}$$

So $J_\epsilon(tW_{x_0,\epsilon}) \leqslant \gamma_4 t^2 - \gamma_5 t^p$ with γ_4, γ_5 positive constants and for t big enough it is negative.

(V) *The parameter $t_\epsilon(W_{x_0,\epsilon})$ tends to* 1 *for ϵ tending to zero uniformly with respect to $x_0 \in M$.*

By the previous steps $t_\epsilon(W_{x_0,\epsilon}) \in [t_1, t_2]$ for any $\epsilon \in (0,1]$ and $x_0 \in M$. Let us write $t_{x_0,\epsilon} = t_\epsilon(W_{x_0,\epsilon})$. Then there exists a sequence $\epsilon_k \to 0$ for $k \to \infty$ such that t_{x_0,ϵ_k} converges to $t^*_{x_0}$. By step (II) we have $\lim_{k\to\infty} \frac{\epsilon_k^2}{\epsilon_k^n}\int_M |t_{x_0,\epsilon_k}\nabla W_{x_0,\epsilon_k}(x)|_g^2\,d\mu_g = \|t^*_{x_0}U\|^2_{\mathcal{D}^{1,2}(\mathbb{R}^n)}$. By definition we have

$$\begin{aligned}
&\frac{1}{\epsilon_k^n}\int\limits_M f'(t_{x_0,\epsilon_k}W_{x_0,\epsilon_k})t_{x_0,\epsilon_k}W_{x_0,\epsilon_k}\,d\mu_g \\
&\quad=\frac{1}{\epsilon_k^n}\int\limits_{B(0,R)} f'\big(t_{x_0,\epsilon_k}\big(U_{\epsilon_k}(z)-\widetilde{U}_{\frac{R}{\epsilon_k}}\big)\big)t_{x_0,\epsilon_k}\big(U_{\epsilon_k}(z)-\widetilde{U}_{\frac{R}{\epsilon_k}}\big)\big|g_{x_0}(z)\big|^{\frac{1}{2}}\,dz \\
&\quad=\int\limits_{B(0,\frac{R}{\epsilon_k})} f'\big(t_{x_0,\epsilon_k}\big(U(z)-\widetilde{U}_{\frac{R}{\epsilon_k}}\big)\big)t_{x_0,\epsilon_k}\big(U(z)-\widetilde{U}_{\frac{R}{\epsilon_k}}\big)\big|g_{x_0}(\epsilon_k z)\big|^{\frac{1}{2}}\,dz \\
&\quad=\int\limits_{\mathbb{R}^n} \chi_{B(0,\frac{R}{\epsilon_k})}(z) f'\big(t_{x_0,\epsilon_k}\big(U(z)-\widetilde{U}_{\frac{R}{\epsilon_k}}\big)\big)t_{x_0,\epsilon_k}\big(U(z)-\widetilde{U}_{\frac{R}{\epsilon_k}}\big)\big|g_{x_0}(\epsilon_k z)\big|^{\frac{1}{2}}\,dz.
\end{aligned}$$

The integrand point-wise tends to $f'(t^*_{x_0}U(z))t^*_{x_0}U(z)$ for k tending to infinity and is bounded from above by a function in $L^1(\mathbb{R}^n)$ as follows:

$$\begin{aligned}
&\chi_{B(0,\frac{R}{\epsilon_k})}(z) f'\big(t_{x_0,\epsilon_k}\big(U(z)-\widetilde{U}_{\frac{R}{\epsilon_k}}\big)\big)t_{x_0,\epsilon_k}\big(U(z)-\widetilde{U}_{\frac{R}{\epsilon_k}}\big)\big|g_{x_0}(\epsilon_k z)\big|^{\frac{1}{2}} \\
&\quad\leqslant H^{\frac{n}{2}}\chi_{B(0,\frac{R}{\epsilon_k})}(z) f'\big(t_2\big(U(z)-\widetilde{U}_{\frac{R}{\epsilon_k}}\big)\big)t_2\big(U(z)-\widetilde{U}_{\frac{R}{\epsilon_k}}\big) \\
&\quad\leqslant \begin{cases} c_1 H^{\frac{n}{2}} t_2^p(U(z)-\widetilde{U}_{\frac{R}{\epsilon_k}})^p & \text{if } t_2(U(z)-\widetilde{U}_{\frac{R}{\epsilon_k}})\geqslant 1 \text{ and } |z|\leqslant \frac{R}{\epsilon_k}, \\ c_1 H^{\frac{n}{2}} t_2^q(U(z)-\widetilde{U}_{\frac{R}{\epsilon_k}})^q & \text{if } t_2(U(z)-\widetilde{U}_{\frac{R}{\epsilon_k}})\leqslant 1 \text{ and } |z|\leqslant \frac{R}{\epsilon_k}, \\ 0 & \text{otherwise} \end{cases} \\
&\quad\leqslant \begin{cases} c_1 H^{\frac{n}{2}} t_2^p(U(z))^p & \text{if } t_2(U(z)-\widetilde{U}_{\frac{R}{\epsilon_k}})\geqslant 1,\ U(z)\geqslant 1 \text{ and } |z|\leqslant \frac{R}{\epsilon_k}, \\ c_1 H^{\frac{n}{2}} t_2^q(U(z)-\widetilde{U}_{\frac{R}{\epsilon_k}})^q & \text{if } t_2(U(z)-\widetilde{U}_{\frac{R}{\epsilon_k}})\geqslant 1,\ U(z)< 1 \text{ and } |z|\leqslant \frac{R}{\epsilon_k}, \\ c_1 H^{\frac{n}{2}} t_2^p(U(z)-\widetilde{U}_{\frac{R}{\epsilon_k}})^p & \text{if } t_2(U(z)-\widetilde{U}_{\frac{R}{\epsilon_k}})< 1,\ U(z)\geqslant 1 \text{ and } |z|\leqslant \frac{R}{\epsilon_k}, \\ c_1 H^{\frac{n}{2}} t_2^q(U(z))^q & \text{if } t_2(U(z)-\widetilde{U}_{\frac{R}{\epsilon_k}})< 1,\ U(z)< 1 \text{ and } |z|\leqslant \frac{R}{\epsilon_k}, \\ 0 & \text{otherwise} \end{cases} \\
&\quad\leqslant \begin{cases} c_1 H^{\frac{n}{2}} t_2^p(U(z))^p & \text{if } U(z)\geqslant 1, \\ c_1 H^{\frac{n}{2}} t_2^q(U(z))^q & \text{if } U(z)< 1 \end{cases} \\
&\quad\leqslant \frac{c_1 H^{\frac{n}{2}} t_2^q}{c_0} f\big(U(z)\big).
\end{aligned}$$

Then by the Lebesgue theorem $\lim_{k\to\infty}\frac{1}{\epsilon_k^n}\int_M f'(t_{x_0,\epsilon_k}W_{x_0,\epsilon_k})t_{x_0,\epsilon_k}W_{x_0,\epsilon_k}\,d\mu_g = \int_{\mathbb{R}^n} f'(t^*_{x_0}U(z))t^*_{x_0}U(z)\,dz$. By the fact that $U\in\mathcal{N}$ and $\|t^*_{x_0}U\|^2_{\mathcal{D}^{1,2}(\mathbb{R}^n)} = \int_{\mathbb{R}^n} f'(t^*_{x_0}U(z))t^*_{x_0}\times U(z)\,dz$, we conclude that $t^*_{x_0}=1$.

To prove that the convergence is uniform with respect to $x_0\in M$, we show that $\lim_{\epsilon\to 0}\sup_{x\in M}|t_{x,\epsilon}-1|=0$. For any ϵ there exists $x(\epsilon)\in M$ such that $\sup_{x\in M}|t_{x,\epsilon}-1| = |t_{x(\epsilon),\epsilon}-1|$. By compactness there exists a sequence $\epsilon_k\to 0$ for $k\to\infty$ such that $x(\epsilon_k)$ tends to $x_*\in M$. Let us fix $\eta>0$. There exists k_0 such that for all $k\geqslant k_0$ $|t_{x_*,\epsilon_k}-1|<\frac{\eta}{3}$. Possibly increasing k_0 we also have that for all $k \geqslant k_0$ and $h > k$ $|t_{x(\epsilon_k),\epsilon_k} - t_{x(\epsilon_h),\epsilon_k}| < \frac{\eta}{3}$. Finally there exists h_0 such that for all $h \geqslant h_0$ $|t_{x(\epsilon_h),\epsilon_k} - t_{x_*,\epsilon_k}| < \frac{\eta}{3}$. Summing the three terms one has that $|t_{x(\epsilon_k),\epsilon_k} - 1| < \eta$ for all $k \geqslant k_0$.

(VI) *The limit of* $\frac{1}{\epsilon^n}\int_M f(t_{x_0,\epsilon}W_{x_0,\epsilon})\,d\mu_g$ *is* $\int_{\mathbb{R}^n} f(U)\,dz$.

Changing variables and using the mean value theorem, we have

$$\frac{1}{\epsilon^n}\int\limits_M f(t_{x_0,\epsilon}W_{x_0,\epsilon})\,d\mu_g$$
$$= \int\limits_{B(0,\frac{R}{\epsilon})} \big[f\big(U(z)-\widetilde{U}_{\frac{R}{\epsilon}}\big) + (t_{x_0,\epsilon}-1)f'\big(\Theta_{x_0,\epsilon}(z)\big(U(z)-\widetilde{U}_{\frac{R}{\epsilon}}\big)\big)\big(U(z)-\widetilde{U}_{\frac{R}{\epsilon}}\big)\big]\big|g_{x_0}(\epsilon z)\big|^{\frac{1}{2}}\,dz,$$

where $\Theta_{x_0,\epsilon}(z) = (\theta_{x_0,\epsilon}(z)t_{x_0,\epsilon} + 1 - \theta_{x_0,\epsilon}(z))$ with a suitable $0 < \theta_{x_0,\epsilon}(z) < 1$. We want to prove that

$$\int\limits_{B(0,\frac{R}{\epsilon})} f\big(U(z)-\widetilde{U}_{\frac{R}{\epsilon}}\big)\big|g_{x_0}(\epsilon z)\big|^{\frac{1}{2}}\,dz \xrightarrow{\epsilon\to 0} \int\limits_{\mathbb{R}^n} f(U)\,dz,$$

$$\int\limits_{B(0,\frac{R}{\epsilon})} (t_{x_0,\epsilon}-1)f'\big(\Theta_{x_0,\epsilon}(z)\big(U(z)-\widetilde{U}_{\frac{R}{\epsilon}}\big)\big)\big(U(z)-\widetilde{U}_{\frac{R}{\epsilon}}\big)\big|g_{x_0}(\epsilon z)\big|^{\frac{1}{2}}\,dz \xrightarrow{\epsilon\to 0} 0 \tag{4.4}$$

uniformly with respect to $x_0 \in M$.

It is easy to see that

$$\int\limits_{B(0,\frac{R}{\epsilon})} f\big(U(z)-\widetilde{U}_{\frac{R}{\epsilon}}\big)\big|\big|g_{x_0}(\epsilon z)\big|^{\frac{1}{2}} - 1\big|\,dz \xrightarrow{\epsilon\to 0} 0$$

uniformly with respect to $x_0 \in M$. The function $\chi_{B(0,\frac{R}{\epsilon})}(z)f(U(z)-\widetilde{U}_{\frac{R}{\epsilon}})$ tends point-wise to $f(U(z))$ for any $z \in \mathbb{R}^n$. Moreover

$$\begin{aligned}
\chi_{B(0,\frac{R}{\epsilon})}(z)f\big(U(z)-\widetilde{U}_{\frac{R}{\epsilon}}\big) &\leqslant \begin{cases} \frac{c_1}{\mu}(U(z)-\widetilde{U}_{\frac{R}{\epsilon}})^p & \text{if } U(z)-\widetilde{U}_{\frac{R}{\epsilon}} \geqslant 1,\ |z| \leqslant \frac{R}{\epsilon},\\ \frac{c_1}{\mu}(U(z)-\widetilde{U}_{\frac{R}{\epsilon}})^q & \text{if } U(z)-\widetilde{U}_{\frac{R}{\epsilon}} \leqslant 1,\ |z| \leqslant \frac{R}{\epsilon},\\ 0 & \text{otherwise}\end{cases}\\
&\leqslant \begin{cases} \frac{c_1}{\mu}(U(z)-\widetilde{U}_{\frac{R}{\epsilon}})^p & \text{if } U(z) \geqslant 1,\ |z| \leqslant \frac{R}{\epsilon},\\ \frac{c_1}{\mu}(U(z)-\widetilde{U}_{\frac{R}{\epsilon}})^q & \text{if } U(z) < 1,\ |z| \leqslant \frac{R}{\epsilon},\\ 0 & \text{otherwise}\end{cases}\\
&\leqslant \begin{cases} \frac{c_1}{\mu}(U(z))^p & \text{if } U(z) \geqslant 1,\\ \frac{c_1}{\mu}(U(z))^q & \text{if } U(z) \leqslant 1\end{cases}\\
&\leqslant \frac{c_1}{c_0\mu}f\big(U(z)\big)
\end{aligned}$$

and by Lebesgue's theorem we obtain the first limit in (4.4). The function of t $f'(tu)u$ is increasing in t, since its derivative is $f''(tu)u^2 > 0$. Then we have

$$\int\limits_{B(0,\frac{R}{\epsilon})} f'\big(\Theta_{x_0,\epsilon}(z)\big(U(z)-\widetilde{U}_{\frac{R}{\epsilon}}\big)\big)\big(U(z)-\widetilde{U}_{\frac{R}{\epsilon}}\big)\big|g_{x_0}(\epsilon z)\big|^{\frac{1}{2}}\,dz$$

$$< H^{\frac{n}{2}} \int\limits_{B(0,\frac{R}{\epsilon})} f'\big((t_2+1)\big(U(z)-\widetilde{U}_{\frac{R}{\epsilon}}\big)\big)\big(U(z)-\widetilde{U}_{\frac{R}{\epsilon}}\big)\,dz.$$

By the usual standard inequalities, the previous integral is bounded from above by $\frac{c_1 H^{\frac{n}{2}}}{c_0(t_2+1)}\int_{\mathbb{R}^n} f((t_2+1)U(z))\,dz$ and the second limit in (4.4) is proved, because of (V).

(VII) *Conclusion.*

By (II), (V) and (VI) we obtain that $J_\epsilon(\phi_\epsilon(x_0))$ tends to $J(U)=m(J)$ for ϵ tending to zero uniformly with respect to x_0. This completes the proof. □

Remark 4.3. By the previous proposition, in particular we know that, given $\delta > 0$, for any positive ϵ sufficiently small $\Sigma_{\epsilon,\delta}$ is not empty.

5. The function β

Given a function $u \in L^p(M)$, $u \not\equiv 0$, it is possible to define its centre of mass $\beta(u) \in \mathbb{R}^N$ by

$$\beta(u) = \frac{\int_M x\Phi(u)\,d\mu_g}{\int_M \Phi(u)\,d\mu_g}, \tag{5.1}$$

where

$$\Phi(u) = \frac{1}{2}f'(u)u - f(u). \tag{5.2}$$

By the properties of f, $\Phi(s) > 0$ for all $s \neq 0$. To prove that $\beta : \Sigma_{\epsilon,\delta} \to M_{r(M)}$ (see Section 3 and Definition 3.6), we use the fact that the functions in $\Sigma_{\epsilon,\delta}$ concentrate for ϵ and δ tending to zero.

First of all we find a positive inferior bound for the functional J_ϵ on the Nehari manifold. Let us denote

$$m_\epsilon = \inf_{u\in\mathcal{N}_\epsilon} J_\epsilon(u). \tag{5.3}$$

It is easy to see that

$$\inf_{u\in\mathcal{N}_\epsilon} \|u\|_{H_2^1(M)} > 0$$

(the proof is analogous to Lemma 3.2 of [7]) and, since the manifold M is compact, that the infimum m_ϵ is achieved.

Lemma 5.1. *There exist positive constants α and ϵ_0 such that for any $0<\epsilon<\epsilon_0$ the inequality $m_\epsilon \geqslant \alpha$ holds.*

To prove this lemma we need the following technical lemma (for the proof see Appendix A).

Lemma 5.2. *For any $r \in (0, r(M))$, there exist constants $k_1, k_2, k_3, k_4 > 0$ such that for any $u \in H_2^1(M)$ there exists $v \in \mathcal{D}^{1,2}(M_r)$ such that $v|_M \equiv u$ and*

$$\|v\|^2_{\mathcal{D}^{1,2}(M_r)} \leqslant k_1 \int_M |\nabla u|_g^2 \, d\mu_g, \tag{5.4}$$

$$\int_{M_r} f\big(v(z)\big)\, dz \geqslant k_2 \int_M f\big(u(x)\big)\, d\mu_g, \tag{5.5}$$

$$\int_{M_r} f\big(v(z)\big)\, dz \leqslant k_3 \int_M f\big(u(x)\big)\, d\mu_g, \tag{5.6}$$

$$\|v\|^2_{L^2(M_r)} \geqslant k_4 \|u\|^2_{L^2(M)}. \tag{5.7}$$

Proof of Lemma 5.1. By definition m_ϵ is the infimum of $J_\epsilon(u)$ on the Nehari manifold $\mathcal{N}_\epsilon$. If $u \in \mathcal{N}_\epsilon$ we have

$$J_\epsilon(u) \geqslant \left(\frac{1}{2} - \frac{1}{\mu}\right)\frac{\epsilon^2}{\epsilon^n} \int_M |\nabla u|_g^2 \, d\mu_g.$$

Rescaling u, it is easy to see that m_ϵ is greater than or equal to the infimum of the functional $(\frac{1}{2} - \frac{1}{\mu})\frac{\epsilon^2}{\epsilon^n} t_\epsilon^2 \int_M |\nabla w|_g^2 \, d\mu_g$ on the set of the functions $w \in H_2^1(M)$ such that $\frac{1}{\epsilon^n}\int_M f(w)\, d\mu_g = 1$ and where $t_\epsilon = t_\epsilon(w)$ is as in (ii), Lemma 4.1. First of all, we check that there exists a constant $\tilde{\alpha} > 0$ and for such functions w it holds

$$\frac{\epsilon^2}{\epsilon^n} \int_M |\nabla w|_g^2 \, d\mu_g \geqslant \tilde{\alpha}.$$

By Lemma 5.2, for any function w there exists a function $v \in \mathcal{D}^{1,2}(M_r)$ such that (5.4) and (5.5) hold. We consider $\tilde{v} \in \mathcal{D}^{1,2}(\mathbb{R}^N)$, defined as $\tilde{v}(y) = v(y)$ for all $y \in M_r$ and $\tilde{v}(y) = 0$ for all $y \in \mathbb{R}^N \setminus M_r$. We can now consider the following rescalement $V(y) = \tilde{v}(\epsilon^\sigma y)$ with $\sigma = \frac{2n-(n-2)p}{2N-(N-2)p}$. In case the denominator is equal to 0, we can choose a bigger N. We have

$$\|V\|^2_{\mathcal{D}^{1,2}(\mathbb{R}^N)} = \frac{\epsilon^{2\sigma}}{\epsilon^{N\sigma}} \|v\|^2_{\mathcal{D}^{1,2}(M_r)} \quad \text{and} \quad \int_{\mathbb{R}^N} f\big(V(y)\big)\, dy = \frac{1}{\epsilon^{N\sigma}} \int_{M_r} f\big(v(y)\big)\, dy.$$

By these equalities, (5.4) and (5.5), we have

$$\frac{\epsilon^2}{\epsilon^n}\int\limits_M |\nabla w|_g^2\,d\mu_g = \frac{\frac{\epsilon^2}{\epsilon^n}\int_M |\nabla w|_g^2\,d\mu_g}{(\frac{1}{\epsilon^n}\int_M f(w)\,d\mu_g)^{\frac{2}{p}}} \geqslant \frac{k_2^{\frac{2}{p}}}{k_1}\frac{\frac{\epsilon^2}{\epsilon^n}\|v\|^2_{\mathcal{D}^{1,2}(M_r)}}{(\frac{1}{\epsilon^n}\int_{M_r} f(v)\,dy)^{\frac{2}{p}}}$$
$$= \frac{k_2^{\frac{2}{p}}}{k_1}\frac{\frac{\epsilon^{(N-2)\sigma}}{\epsilon^{n-2}}\|V\|^2_{\mathcal{D}^{1,2}(\mathbb{R}^N)}}{(\frac{\epsilon^{N\sigma}}{\epsilon^n}\int_{\mathbb{R}^N} f(V)\,dy)^{\frac{2}{p}}} = \frac{k_2^{\frac{2}{p}}}{k_1}\frac{\|V\|^2_{\mathcal{D}^{1,2}(\mathbb{R}^N)}}{(\int_{\mathbb{R}^N} f(V)\,dy)^{\frac{2}{p}}}. \tag{5.8}$$

We show now that for ϵ sufficiently small we have $\int_{\mathbb{R}^N} f(V)\,dy < 1$. In fact, by (5.6) there holds

$$\int\limits_{\mathbb{R}^N} f(V)\,dy = \frac{1}{\epsilon^{N\sigma}}\int\limits_{M_r} f\big(v(y)\big)\,dy \leqslant \frac{k_3}{\epsilon^{N\sigma}}\int\limits_M f(w)\,d\mu_g = \frac{k_3\epsilon^n}{\epsilon^{N\sigma}}.$$

By definition of σ $\lim_{N\to\infty} N\sigma = \frac{2n-(n-2)p}{2-p} < 0$ and so there exists N sufficiently big such that $n - N\sigma > 0$.

Since $\int_{\mathbb{R}^N} f(tV(y))\,dy$ is an increasing function of t for positive t, there exists $t_* > 1$ such that $\int_{M_r} f(t_*V(y))\,dy = 1$. Let $V_*(y) = t_*V(y)$ for any $y \in \mathbb{R}^N$. With the usual computation we obtain

$$\begin{aligned}
\int\limits_{\mathbb{R}^N} f\big(V(y)\big)\,dy &= \int\limits_{\mathbb{R}^N} f\left(\frac{1}{t_*}V^*(y)\right)dy \\
&< \frac{c_1}{\mu}\Bigg(\int\limits_{\{y\in\mathbb{R}^N \,|\, |V_*(y)|\geqslant t_*\}} \frac{1}{t_*^p}\big|V_*(y)\big|^p\,dy + \int\limits_{\{y\in\mathbb{R}^N \,|\, |V_*(y)|\leqslant t_*\}} \frac{1}{t_*^q}\big|V_*(y)\big|^q\,dy\Bigg) \\
&\leqslant \frac{c_1}{\mu}\Bigg(\int\limits_{\{y\in\mathbb{R}^N \,|\, |V_*(y)|\geqslant 1\}} \frac{1}{t_*^p}\big|V_*(y)\big|^p\,dy + \int\limits_{\{y\in\mathbb{R}^N \,|\, |V_*(y)|\leqslant 1\}} \frac{1}{t_*^q}\big|V_*(y)\big|^q\,dy\Bigg) \\
&\leqslant \frac{c_1}{c_0\mu t_*^p}\int\limits_{\mathbb{R}^N} f\big(V_*(y)\big)\,dy = \frac{c_1}{c_0\mu t_*^p}.
\end{aligned}$$

Concluding we have that the last term in (5.8) is equal to

$$\frac{k_2^{\frac{2}{p}}}{k_1}\frac{\frac{1}{t_*^2}\|V_*\|^2_{\mathcal{D}^{1,2}(\mathbb{R}^N)}}{(\int_{\mathbb{R}^N} f(\frac{1}{t_*}V_*)\,dy)^{\frac{2}{p}}} \geqslant \frac{k_2^{\frac{2}{p}}}{k_1}\left(\frac{c_0\mu}{c_1}\right)^{\frac{2}{p}}\|V_*\|^2_{\mathcal{D}^{1,2}(\mathbb{R}^N)},$$

which is bounded from below because (see [9])

$$\inf_{\substack{V\in\mathcal{D}^{1,2}(\mathbb{R}^N)\\ \int_{\mathbb{R}^N} f(V)\,dy=1}} \|V\|^2_{\mathcal{D}^{1,2}(\mathbb{R}^N)} = \hat{\alpha} > 0.$$

We still have to show that t_ϵ is bounded from below by a positive constant. By the properties (f1) and (f2) we have

$$\frac{1}{\epsilon^n}\int_M f'(t_\epsilon w)t_\epsilon w\,d\mu_g < \frac{c_1}{\epsilon^n}\left[\int_{\{x\in M\,||t_\epsilon w(x)|\geqslant 1\}} \left|t_\epsilon w(x)\right|^p d\mu_g + \int_{\{x\in M\,||t_\epsilon w(x)|\leqslant 1\}} \left|t_\epsilon w(x)\right|^q d\mu_g\right]$$
$$\leqslant \frac{c_1}{\epsilon^n}\left[\int_{\{x\in M\,||w(x)|\geqslant 1\}} \left|t_\epsilon w(x)\right|^p d\mu_g + \int_{\{x\in M\,||w(x)|\leqslant 1\}} \left|t_\epsilon w(x)\right|^q d\mu_g\right]$$
$$\leqslant \frac{c_1 t_\epsilon^p}{c_0\epsilon^n}\int_M f\big(w(x)\big)\,d\mu_g = \frac{c_1 t_\epsilon^p}{c_0},$$

where the last equality is due the property of the functions w. Since $t_\epsilon w \in \mathcal{N}_\epsilon$, $\frac{1}{\epsilon^n}\int_M f'(t_\epsilon w)t_\epsilon w\,d\mu_g = \frac{\epsilon^2 t_\epsilon^2}{\epsilon^n}\int_M |\nabla w|_g^2\,d\mu_g$ and by the previous inequalities we have

$$t_\epsilon^{p-2} \geqslant \frac{c_0}{c_1}\frac{\epsilon^2}{\epsilon^n}\int_M |\nabla w|_g^2\,d\mu_g \geqslant \frac{c_0}{c_1}\tilde{\alpha}$$

and this completes the proof. □

In the following lemma for every function $u \in \mathcal{N}_\epsilon$ it is stated the existence of a point in the manifold where u in some sense concentrates.

Lemma 5.3. *Let $\mathcal{C}$ be an atlas for M with open cover given by $B_g(x_i, R)$, $i = 1, \ldots, A$, and partition of unity $\{\psi_i\}_{i=1,\ldots,A}$. There exists a constant $\gamma > 0$ such that for any $0 < \epsilon < \epsilon_0$, where ϵ_0 is defined in Lemma* 5.1*, if $u \in \mathcal{N}_\epsilon$ there exists $i = i(u)$ such that*

$$\frac{1}{\epsilon^n}\int_{B_g(x_i,\frac{R}{2})}\left[\frac{1}{2}f'(u)u - f(u)\right]d\mu_g \geqslant \gamma,$$
$$\frac{\epsilon^2}{2\epsilon^n}\int_{B_g(x_i,\frac{R}{2})} |\nabla u|_g^2\,d\mu_g - \frac{1}{\epsilon^n}\int_{B_g(x_i,\frac{R}{2})} f(u)\,d\mu_g \geqslant \gamma. \tag{5.9}$$

Proof. Let u be in $\mathcal{N}_\epsilon$. We assume that $\widetilde{\mathcal{C}} = \{B_g(x_i, \frac{R}{2})\}_{i=1,\ldots,A}$ is still an open cover (otherwise we complete $\mathcal{C}$). Let $\{\tilde{\psi}_i\}_{i=1,\ldots,A}$ be a partition of unity subordinate to the atlas $\widetilde{\mathcal{C}}$. If $\widetilde{F}_{\epsilon,M}(u)$ is as in (2.9), it is possible to write

$$J_\epsilon(u) = \big(\widetilde{F}_{\epsilon,M}(u)\big)^{\frac{1}{2}}\big(J_\epsilon(u)\big)^{\frac{1}{2}}$$
$$= \left(\frac{1}{\epsilon^n}\sum_{i=1}^{A}\int_{B_g(x_i,\frac{R}{2})} \tilde{\psi}_i(x)\left[\frac{1}{2}f'\big(u(x)\big)u(x) - f\big(u(x)\big)\right]d\mu_g\right)^{\frac{1}{2}}\big(J_\epsilon(u)\big)^{\frac{1}{2}}$$
$$\leqslant \sqrt{A}\max_{1\leqslant i\leqslant A}\big(\widetilde{F}_{\epsilon,B_g(x_i,\frac{R}{2})}(u)\big)^{\frac{1}{2}}\big(J_\epsilon(u)\big)^{\frac{1}{2}}.$$

By this inequality and Lemma 5.1 we conclude that

$$\max_{1\leqslant i\leqslant A} \widetilde{F}_{\epsilon, B_g(x_i, \frac{R}{2})}(u) \geqslant \frac{1}{A} J_\epsilon(u) \geqslant \frac{\alpha}{A}.$$

The second equation in (5.9) is proved analogously. □

In the following proposition the concentration property is better specified.

Proposition 5.4. *For any $\eta \in (0,1)$ there exists $\delta_0 < m(J)$ such that, for any $\delta \in (0,\delta_0)$ there exists $\epsilon_0 = \epsilon_0(\delta) > 0$ and for any $\epsilon \in (0,\epsilon_0)$ with every function $u \in \Sigma_{\epsilon,\delta}$ it is associated a point $x_0 = x_0(u)$ in M with the property*

$$\widetilde{F}_{\epsilon, B_g(x_0, \frac{r(M)}{2})}(u) > \eta m(J).$$

The proof of this proposition needs the following lemmas. The first lemma we need is the splitting lemma proved in [7, Lemma 4.1]:

Lemma 5.5. *Let $\{v_k\}_{k\in\mathbb{N}} \subset \mathcal{N}$ be a sequence such that*

$$\begin{aligned} &J(v_k) \to m(J) && \text{as } k \to \infty,\\ &J'(v_k) \to 0 && \text{in } \mathcal{D}^{1,2}(\mathbb{R}^n) \text{ as } k \to \infty. \end{aligned}$$

Then

- *either v_k converges strongly in $\mathcal{D}^{1,2}(\mathbb{R}^n)$ to a ground state solution of (2.12), or*
- *there exist a sequence of points $\{y_k\}_{k\in\mathbb{N}} \subset \mathbb{R}^n$ with $|y_k| \to \infty$ as $k \to \infty$, a ground state solution U of (2.12) and a sequence of functions $\{v_k^0\}_{k\in\mathbb{N}}$ such that, up to a subsequence:*
  - (i) *$v_k(z) = v_k^0(z) + U(z - y_k)$ for all $z \in \mathbb{R}^n$;*
  - (ii) *$v_k^0 \to 0$ as $k \to \infty$ in $\mathcal{D}^{1,2}(\mathbb{R}^n)$.*

Lemma 5.6. *Let ϵ_k and δ_k be two positive sequences tending to zero for k tending to infinity. For any $k \in \mathbb{N}$ let u_k be a function in $\Sigma_{\epsilon_k,\delta_k}$ such that for any $u \in H_2^1(M)$*

$$\left|J'_{\epsilon_k}(u_k)(u)\right| = o\left(\frac{\epsilon_k}{\epsilon_k^{\frac{n}{2}}}\|u\|_{H_2^1(M)}\right).$$

There exist a sequence $\{x_k\}_{k\in\mathbb{N}}$ of points in M and a sequence of functions w_k on $\mathbb{R}^n$, defined as

$$w_k(z) = u_k\big(\exp_{x_k}(\epsilon_k z)\big)\chi_{\frac{R}{\epsilon_k}}\big(|z|\big), \tag{5.10}$$

such that the following properties hold:

(i) *There exists $w \in \mathcal{D}^{1,2}(\mathbb{R}^n)$ such that, up to a subsequence, w_k tends to w weakly in $\mathcal{D}^{1,2}(\mathbb{R}^n)$ and strongly in $L^p_{\mathrm{loc}}(\mathbb{R}^n)$.*

(ii) *The function w is a weak solution of $-\Delta w = f'(w)$ on $\mathbb{R}^n$.*
(iii) *The function w is a ground state solution.*
(iv) *The following equality holds*

$$\lim_{k\to\infty} J_{\epsilon_k}(u_k) = m(J).$$

Proof. To get started we consider x_k to be the points in M such that u_k has the property (5.9). We will be more precise in point (iii).

(i) It is sufficient to prove that the sequence w_k is bounded in $\mathcal{D}^{1,2}(\mathbb{R}^n)$. We write

$$\begin{aligned}
\|w_k\|^2_{\mathcal{D}^{1,2}(\mathbb{R}^n)} &= \int_{B(0,\frac{R}{\epsilon_k})} \left|\nabla w_k(z)\right|^2 dz \\
&\leqslant 2 \int_{B(0,\frac{R}{\epsilon_k})} \left|\nabla\left[u_k\left(\exp_{x_k}(\epsilon_k z)\right)\right]\right|^2 \left[\chi_{\frac{R}{\epsilon_k}}\left(|z|\right)\right]^2 dz \\
&\quad + 2 \int_{B(0,\frac{R}{\epsilon_k})} \left[\chi'_{\frac{R}{\epsilon_k}}\left(|z|\right)\right]^2 \left[u_k\left(\exp_{x_k}(\epsilon_k z)\right)\right]^2 dz = I_1 + I_2.
\end{aligned}$$

We consider the following inequality:

$$\begin{aligned}
\frac{\epsilon_k^2}{\epsilon_k^n} \int_M |\nabla u_k|_g^2 \, d\mu_g &\geqslant \frac{\epsilon_k^2}{\epsilon_k^n} \int_{B_g(x_k,R)} |\nabla u_k|_g^2 \, d\mu_g \\
&= \frac{\epsilon_k^2}{\epsilon_k^n} \int_{B(0,R)} \left|\nabla u_k\left(\exp_{x_k}(z)\right)\right|^2_{g_{x_k}(z)} \left|g_{x_k}(z)\right|^{\frac{1}{2}} dz \\
&= \int_{B(0,\frac{R}{\epsilon_k})} \left|\nabla u_k\left(\exp_{x_k}(\epsilon_k z)\right)\right|^2_{g_{x_k}(\epsilon_k z)} \left|g_{x_k}(\epsilon_k z)\right|^{\frac{1}{2}} dz \\
&\geqslant \frac{h^{\frac{n}{2}}}{H} \int_{B(0,\frac{R}{\epsilon_k})} \left|\nabla u_k\left(\exp_{x_k}(\epsilon_k z)\right)\right|^2 dz \geqslant \frac{h^{\frac{n}{2}}}{2H} I_1.
\end{aligned} \tag{5.11}$$

Moreover the following inequality holds

$$\begin{aligned}
I_2 &\leqslant \frac{2\chi_0^2\epsilon_k^2}{R^2} \int_{B(0,\frac{R}{\epsilon_k})} \left[u_k\left(\exp_{x_k}(\epsilon_k z)\right)\right]^2 dz \\
&= \frac{2\chi_0^2\epsilon_k^2}{R^2\epsilon_k^n} \int_{B(0,R)} \left[u_k\left(\exp_{x_k}(z)\right)\right]^2 dz \\
&\leqslant \frac{2\chi_0^2\epsilon_k^2}{h^{\frac{n}{2}} R^2\epsilon_k^n} \int_{B_g(x_k,R)} \left(u_k(x)\right)^2 d\mu_g.
\end{aligned} \tag{5.12}$$

By (5.11) and (5.12), we have that the sum $I_1 + I_2$ is bounded by a constant times $\frac{\epsilon_k^2}{\epsilon_k^n}\|u_k\|^2_{H_2^1(M)}$. We show then that this quantity must be bounded. Since $u_k \in \Sigma_{\epsilon_k,\delta_k}$ and

$$J_{\epsilon_k}(u_k) \geqslant \left(\frac{1}{2} - \frac{1}{\mu}\right)\frac{\epsilon_k^2}{\epsilon_k^n}\int_M |\nabla u_k|_g^2\, d\mu_g,$$

the right-hand side of the preceding inequality must be bounded. We still have to check that $\frac{\epsilon_k^2}{\epsilon_k^n}\|u_k\|^2_{L^2(M)}$ is bounded too. In fact, by (5.7) in Lemma 5.2 we have a sequence v_k of functions in $\mathcal{D}^{1,2}(M_r)$ and

$$\frac{\epsilon_k^2}{\epsilon_k^n}\|u_k\|^2_{L^2(M)} \leqslant \frac{\epsilon_k^2}{k_4\epsilon_k^n}\|v_k\|^2_{L^2(M_r)} \leqslant \frac{C\epsilon_k^2}{k_4\epsilon_k^n}\|v_k\|^2_{\mathcal{D}^{1,2}(M_r)} \leqslant \frac{Ck_1\epsilon_k^2}{k_4\epsilon_k^n}\int_M |\nabla u_k|_g^2\, d\mu_g,$$

where C is the constant in the Poincaré inequality and we have used (5.4) in the last inequality.

(ii) First of all we prove that for any $\xi \in C_0^\infty(\mathbb{R}^n)$ $J'(w_k)(\xi)$ tends to zero for k tending to infinity:

$$\begin{aligned}
J'(w_k)(\xi) &= \int_{\mathbb{R}^n} \nabla w_k(z)\cdot\nabla\xi(z)\,dz - \int_{\mathbb{R}^n} f'\big(w_k(z)\big)\xi(z)\,dz \\
&= \int_{\mathbb{R}^n} \Big[\nabla\Big[u_k\big(\exp_{x_k}(\epsilon_k z)\big)\chi_{\frac{R}{\epsilon_k}}\big(|z|\big)\Big]\cdot\nabla\xi(z) - f'\Big(u_k\big(\exp_{x_k}(\epsilon_k z)\big)\chi_{\frac{R}{\epsilon_k}}\big(|z|\big)\Big)\xi(z)\Big]\,dz \\
&= \int_{\mathbb{R}^n} \Big[\nabla\Big[u_k\big(\exp_{x_k}(\epsilon_k z)\big)\Big]\cdot\nabla\xi(z) - f'\Big(u_k\big(\exp_{x_k}(\epsilon_k z)\big)\Big)\xi(z)\Big]\,dz,
\end{aligned}$$

where in the last equality we have used the fact that for k sufficiently large for any z in the support of ξ $\chi_{\frac{R}{\epsilon_k}}(|z|) = 1$. Now we define the function $\tilde{\xi}_k$ in $H_2^1(M)$ as follows:

$$\tilde{\xi}_k(x) = \begin{cases} \xi\big(\frac{\exp_{x_k}^{-1}(x)}{\epsilon_k}\big) & \forall x \in B_g(x_k, R), \\ 0 & \text{otherwise.} \end{cases}$$

Then we want to write

$$J'(w_k)(\xi) = \frac{\epsilon_k^2}{\epsilon_k^n}\int_M g_{x_k}\big(\nabla u_k(x), \nabla\tilde{\xi}_k(x)\big)\,d\mu_g - \frac{1}{\epsilon_k^n}\int_M f'\big(u_k(x)\big)\tilde{\xi}_k(x)\,d\mu_g + E_k,$$

where E_k is an error. By hypothesis

$$\left|\int_M \left[\frac{\epsilon_k^2}{\epsilon_k^n} g_{x_k}\big(\nabla u_k(x), \nabla\tilde{\xi}_k(x)\big) - \frac{1}{\epsilon_k^n} f'\big(u_k(x)\big)\tilde{\xi}_k(x)\right] d\mu_g\right|$$

$$= \left|J'_{\epsilon_k}(u_k)(\tilde{\xi}_k)\right| = o\left(\frac{\epsilon_k}{\epsilon_k^{\frac{n}{2}}}\|\tilde{\xi}\|_{H_2^1(M)}\right) = o\big(\|\xi\|_{H_2^1(\mathbb{R}^n)}\big).$$

Now we have to check the error:

$$\begin{aligned}
|E_k| &= \Bigg|\int_{\mathbb{R}^n} \big[\nabla\big[u_k\big(\exp_{x_k}(\epsilon_k z)\big)\big]\cdot\nabla\xi(z) - f'\big(u_k\big(\exp_{x_k}(\epsilon_k z)\big)\big)\xi(z)\big]\,dz \\
&\quad - \frac{\epsilon_k^2}{\epsilon_k^n}\int_M g_{x_k}\big(\nabla u_k(x), \nabla\tilde{\xi}_k(x)\big)\,d\mu_g - \frac{1}{\epsilon_k^n}\int_M f'\big(u_k(x)\big)\tilde{\xi}_k(x)\,d\mu_g\Bigg| \\
&\leqslant \left|\int_{\mathbb{R}^n} \nabla\big[u_k\big(\exp_{x_k}(\epsilon_k z)\big)\big]\cdot\nabla\xi(z)\,dz - \frac{\epsilon_k^2}{\epsilon_k^n}\int_M g_{x_k}\big(\nabla u_k(x), \nabla\tilde{\xi}_k(x)\big)\,d\mu_g\right| \\
&\quad + \left|\int_{\mathbb{R}^n} f'\big(u_k\big(\exp_{x_k}(\epsilon_k z)\big)\big)\xi(z)\,dz - \frac{1}{\epsilon_k^n}\int_M f'\big(u_k(x)\big)\tilde{\xi}_k(x)\,d\mu_g\right| \\
&= |E_{1,k}| + |E_{2,k}|.
\end{aligned}$$

For the first term we have

$$|E_{1,k}| \leqslant \int_\Xi \left|\big(\delta^{ij} - g_{x_k}^{ij}(\epsilon_k z)\big|g_{x_k}(\epsilon_k z)\big|^{\frac{1}{2}}\big)\frac{\partial[u_k(\exp_{x_k}(\epsilon_k z))]}{\partial z_i}\frac{\partial\xi(z)}{\partial z_j}\right| dz,$$

where Ξ denotes the compact support of ξ. The limit

$$\lim_{k\to\infty}\big|\delta^{ij} - g_{x_k}^{ij}(\epsilon_k z)\big|g_{x_k}(\epsilon_k z)\big|^{\frac{1}{2}}\big| = 0$$

is uniform with respect to $z \in \Xi$. Since

$$\int_\Xi \left|\frac{\partial[u_k(\exp_{x_k}(\epsilon_k z))]}{\partial z_i}\frac{\partial\xi(z)}{\partial z_j}\right| dz \leqslant \big\|u_k\big(\exp_{x_k}(\epsilon_k z)\big)\big\|_{\mathcal{D}^{1,2}(\Xi)}\|\xi\|_{\mathcal{D}^{1,2}(\mathbb{R}^n)}$$

and for k sufficiently large

$$\begin{aligned}
\int_\Xi \big|\nabla\big[u_k\big(\exp_{x_k}(\epsilon_k z)\big)\big]\big|^2\,dz &\leqslant \frac{H}{h^{\frac{n}{2}}}\frac{\epsilon_k^2}{\epsilon_k^n}\int_M |\nabla u_k|_g^2\,d\mu_g \\
&\leqslant \frac{2\mu H}{(\mu-2)h^{\frac{n}{2}}} J_{\epsilon_k}(u_k) \leqslant \frac{4\mu H m(J)}{(\mu-2)h^{\frac{n}{2}}},
\end{aligned}$$

we conclude that $|E_{1,k}|$ tends to zero. For the second term we have

$$|E_{2,k}| = \left| \int\limits_{\Xi} \left(1 - \left|g_{x_k}(\epsilon_k z)\right|^{\frac{1}{2}}\right) f'\left(u_k\left(\exp_{x_k}(\epsilon_k z)\right)\right)\xi(z)\, dz \right|.$$

As before, $\lim_{k\to\infty} |g_{x_k}(\epsilon_k z)|^{\frac{1}{2}}$ is 1 uniformly with respect to $z \in \Xi$ and

$$\begin{aligned}
&\int\limits_{\Xi} \left|f'\left(u_k\left(\exp_{x_k}(\epsilon_k z)\right)\right)\xi(z)\right| dz \\
&\quad \leqslant \left(\int\limits_{\{z\in\Xi \mid |u_k(\exp_{x_k}(\epsilon_k z))|\geqslant 1\}} \left|f'\left(u_k\left(\exp_{x_k}(\epsilon_k z)\right)\right)\right|^{\frac{p}{p-1}} dz \right)^{\frac{p-1}{p}} \|\xi\|_{L^p(\mathbb{R}^n)} \\
&\qquad + \left(\int\limits_{\{z\in\Xi \mid |u_k(\exp_{x_k}(\epsilon_k z))|\leqslant 1\}} \left|f'\left(u_k\left(\exp_{x_k}(\epsilon_k z)\right)\right)\right|^{\frac{q}{q-1}} dz \right)^{\frac{q-1}{q}} \|\xi\|_{L^q(\mathbb{R}^n)}.
\end{aligned}$$

It is easy to see that there exists a positive constant C such that the right side is bounded from above by

$$\begin{aligned}
&C\left[\left(\frac{1}{\epsilon_k^n} \int\limits_M f'(u_k)u_k\, d\mu_g \right)^{\frac{p-1}{p}} \|\xi\|_{L^p(\mathbb{R}^n)} + \left(\frac{1}{\epsilon_k^n} \int\limits_M f'(u_k)u_k\, d\mu_g \right)^{\frac{q-1}{q}} \|\xi\|_{L^q(\mathbb{R}^n)} \right] \\
&\quad \leqslant C\left[\left(\frac{2\mu}{\mu-2}\left(m(J)+1\right) \right)^{\frac{p-1}{p}} \|\xi\|_{L^p(\mathbb{R}^n)} + \left(\frac{2\mu}{\mu-2}\left(m(J)+1\right) \right)^{\frac{q-1}{q}} \|\xi\|_{L^q(\mathbb{R}^n)} \right]
\end{aligned}$$

and this proves that $|E_{2,k}|$ tends to zero. Our second and last step is to prove that for any $\xi \in C_0^\infty(\mathbb{R}^n)$ $J'(w_k)(\xi)$ tends to $J'(w)(\xi)$ for k tending to infinity. It is immediate that $\int_{\mathbb{R}^n} \nabla w_k \cdot \nabla\xi\, dz$ tends to $\int_{\mathbb{R}^n} \nabla w \cdot \nabla\xi\, dz$. By the mean value theorem there exists a function $\theta(z)$ with values in (0, 1) such that

$$\begin{aligned}
&\int\limits_{\mathbb{R}^n} \left|f'\left(w_k(z)\right) - f'\left(w(z)\right)\right|\left|\xi(z)\right| dz \\
&\quad = \int\limits_{\mathbb{R}^n} \left|f''\left(\theta(z)w_k(z) + \left(1-\theta(z)\right)w(z)\right)\right|\left|w_k(z) - w(z)\right|\left|\xi(z)\right| dz.
\end{aligned}$$

By Hölder inequality the right-hand side is bounded from above by

$$\|w_k - w\|_{L^p(\Xi)} \|\xi\|_{L^p(\Xi)} \left(\int\limits_{\mathbb{R}^n} \left|f''\left(\theta(z)w_k(z) + \left(1-\theta(z)\right)w(z)\right)\right|^{\frac{p}{p-2}} dz \right)^{\frac{p-2}{p}},$$

where $\|w_k - w\|_{L^p(\Xi)}$ tends to zero by (i). Besides, we have

$$\int_{\mathbb{R}^n} \left| f''\big(\theta(z)w_k(z) + \big(1-\theta(z)\big)w(z)\big)\right|^{\frac{p}{p-2}} dz$$
$$\leqslant c_1 \int_{\{z\in\Xi \,|\, |\theta(z)w_k(z)+(1-\theta(z))w(z)|\geqslant 1\}} \left|\theta(z)w_k(z) + \big(1-\theta(z)\big)w(z)\right|^p dz + c_1 \operatorname{vol}(\Xi)$$
$$\leqslant c_1 2^{p-1}\big(\|w_k\|^p_{L^p(\Xi)} + \|w\|^p_{L^p(\Xi)}\big) + c_1 \operatorname{vol}(\Xi)$$

and this quantity is bounded by a constant.

(iii) Let $t_k = t(w_k)$ be the multiplier defined in (ii), Lemma 4.1. First of all we prove that there exist $0 < t_1 \leqslant 1 \leqslant t_2$ such that for all k $t_1 \leqslant t_k \leqslant t_2$. Let $g_w(t) = J(tw)$. By Lemma 4.1(ii), it is enough to find $t_1 > 0$ such that for all $t \in [0, t_1]$ $g'_{w_k}(t) > 0$ for all $k \in \mathbb{N}$. There holds

$$g'_{w_k}(t) = t\int_{\mathbb{R}^n} \left|\nabla w_k(z)\right|^2 dz - \int_{\mathbb{R}^n} f'\big(tw_k(z)\big)w_k(z)\,dz$$
$$> t\int_{\mathbb{R}^n} \left|\nabla w_k(z)\right|^2 dz - \frac{c_1 t^{p-1}}{c_0}\int_{\mathbb{R}^n} f\big(w_k(z)\big)\,dz.$$

Since we have

$$\int_{\mathbb{R}^n} \left|\nabla w_k(z)\right|^2 dz \geqslant \frac{h\epsilon_k^2}{H^{\frac{n}{2}}\epsilon_k^n}\int_{B_g(x_k,\frac{R}{2})} |\nabla u_k|_g^2\,d\mu_g$$
$$\geqslant \frac{2h}{H^{\frac{n}{2}}}\left(\frac{\epsilon_k^2}{2\epsilon_k^n}\int_{B_g(x_k,\frac{R}{2})} |\nabla u_k|_g^2\,d\mu_g - \frac{1}{\epsilon_k^n}\int_{B_g(x_k,\frac{R}{2})} f(u_k)\,d\mu_g\right) \geqslant \frac{2h}{H^{\frac{n}{2}}}\gamma,$$

where we have used the second equation of (5.9), and

$$\int_{\mathbb{R}^n} f\big(w_k(z)\big)\,dz \leqslant \frac{1}{h^{\frac{n}{2}}\epsilon_k^n}\int_{B_g(x_k,\frac{R}{2})} f(u_k)\,d\mu_g$$
$$\leqslant \frac{2}{h^{\frac{n}{2}}(\mu-2)\epsilon_k^n}\int_{B_g(x_k,\frac{R}{2})} \left[\frac{1}{2}f'(u_k)u_k - f(u_k)\right]d\mu_g \leqslant \frac{2(m(J)+1)}{h^{\frac{n}{2}}(\mu-2)},$$

then there exist $C_1, C_2 > 0$ such that $g'_{w_k}(t) > C_1 t - C_2 t^{p-1}$. So we consider $t_1 = (\frac{C_1}{C_2})^{\frac{1}{p-2}}$.

If v is a function in the Nehari manifold $\mathcal{N}$, $J(v) = \widetilde{F}_{\mathbb{R}^n}(v)$, as defined in (2.10). Then by property (f1) $J(v)$ is positive. By Lemma 4.1(ii), it is enough to find $t_2 > 0$ such that for all $t \geqslant t_2$ $J(tw_k) < 0$ for all $k \in \mathbb{N}$. Since

$$J(tw_k) = \frac{t^2}{2}\int_{\mathbb{R}^n} \left|\nabla w_k(z)\right|^2 dz - \int_{\mathbb{R}^n} f\big(tw_k(z)\big)\,dz$$

and we already proved that $\{w_k\}_{k\in\mathbb{N}}$ is bounded in $\mathcal{D}^{1,2}(\mathbb{R}^n)$, we still have to bound the second part for $t \geqslant 1$

$$\begin{aligned}\int_{\mathbb{R}^n} f\big(tw_k(z)\big)\,dz &\geqslant c_0 t^p\bigg(\int_{\{z\in\mathbb{R}^n\,||w_k(z)|\geqslant 1\}} \big|w_k(z)\big|^p\,dz + \int_{\{z\in\mathbb{R}^n\,||w_k(z)|\leqslant 1\}} \big|w_k(z)\big|^q\,dz\bigg)\\ &> \frac{c_0 t^p}{c_1}\int_{\mathbb{R}^n} f''\big(w_k(z)\big)\big(w_k(z)\big)^2\,dz > \frac{2c_0 t^p}{c_1-2c_0}\widetilde{F}_{\mathbb{R}^n}(w_k)\\ &\geqslant \frac{2c_0 t^p}{(c_1-2c_0)H^{\frac{n}{2}}}\widetilde{F}_{\epsilon_k, B_g(x_k,\frac{R}{2})}(u_k) \geqslant \frac{2c_0\gamma t^p}{(c_1-2c_0)H^{\frac{n}{2}}},\end{aligned}$$

where we have used (5.9). So there exist $C_3, C_4 > 0$ such that $J(tw_k) < C_3 t^2 - C_4 t^p$ and $t_2 = (\frac{C_3}{C_4})^{\frac{1}{p-2}}$.

By the boundedness of t_k we conclude that up to subsequences t_k converges to $\bar{t}$ for k tending to infinity.

We apply the splitting lemma (Lemma 5.5) to the sequence $t_k w_k$. Then in the first case we have that $t_k w_k$ converges strongly in $\mathcal{D}^{1,2}(\mathbb{R}^n)$ to a ground state solution $\bar{w}$. It is easy to see that $t_k w_k$ weakly converges to $\bar{t}w$, in fact for any $\xi \in C_0^\infty(\mathbb{R}^n)$ there holds

$$\begin{aligned}\bigg|\int_{\mathbb{R}^n} \nabla(t_k w_k - \bar{t}w)\cdot\nabla\xi\bigg| &= \bigg|\int_{\mathbb{R}^n} \nabla(t_k w_k - \bar{t}w_k)\cdot\nabla\xi + \int_{\mathbb{R}^n}\nabla(\bar{t}w_k - \bar{t}w)\cdot\nabla\xi\bigg|\\ &\leqslant |t_k - \bar{t}|\|\xi\|_{\mathcal{D}^{1,2}(\mathbb{R}^n)}\|w_k\|_{\mathcal{D}^{1,2}(\mathbb{R}^n)} + o(1) = o(1).\end{aligned}$$

We can conclude that $\bar{w} = \bar{t}w$. In particular $w \not\equiv 0$ and by the fact that both $\bar{w}$ and w are in $\mathcal{N}$, $\bar{t} = 1$ and we have finished.

Otherwise, there exist a sequence of points $\{y_k\}_{k\in\mathbb{N}}$ tending to infinity, a ground state solution U and a sequence of functions $\{w_k^0\}_{k\in\mathbb{N}}$ such that, up to a subsequence, $t_k w_k(z) = w_k^0(z) + U(z - y_k)$ for all $z \in \mathbb{R}^n$ and w_k^0 tends strongly to zero. We consider three different cases: $\lim_{k\to\infty}|y_k| - \frac{R}{\epsilon_k} = 2T > 0$, $\lim_{k\to\infty}|y_k| - \frac{R}{\epsilon_k} = 0$ and $\lim_{k\to\infty}\frac{R}{\epsilon_k} - |y_k| = 2T > 0$. In the first case, since by definition $w_k \equiv 0$ in $\mathbb{R}^n \setminus B(0, \frac{R}{\epsilon_k})$, $w_k^0(z) = -U(z - y_k)$. Then we have

$$\begin{aligned}\int_{\mathbb{R}^n\setminus B(0,\frac{R}{\epsilon_k})} \big|\nabla w_k^0(z)\big|^2\,dz &= \int_{\mathbb{R}^n\setminus B(0,\frac{R}{\epsilon_k})} \big|\nabla U(z-y_k)\big|^2\,dz\\ &\geqslant \int_{B(y_k,T)} \big|\nabla U(z-y_k)\big|^2\,dz = \int_{B(0,T)}\big|\nabla U(z)\big|^2\,dz > 0\end{aligned}$$

and this is in contradiction with the fact that w_k^0 tends strongly to zero. If $\lim_{k\to\infty}|y_k| - \frac{R}{\epsilon_k} = 0$, let $\pi(y_k)$ denote the projection of y_k onto the sphere centred in the origin with radius $\frac{R}{\epsilon_k}$ and $T > 0$. Then

$$\int\limits_{\{z\in B(\pi(y_k),T)\,|\,|z|\geqslant \frac{R}{\epsilon_k}\}} \left|\nabla U\big(z-\pi(y_k)\big)\right|^2 dz = \int\limits_{\{z\in B(0,T)\,|\,|z+\pi(y_k)|\geqslant \frac{R}{\epsilon_k}\}} \left|\nabla U(z)\right|^2 dz$$
$$\geqslant \min_{\zeta\in S^n} \int\limits_{\{z\in B(0,T)\,|\,z\cdot\zeta\geqslant 0\}} \left|\nabla U(z)\right|^2 dz = C>0,$$

where S^n is the unit sphere in $\mathbb{R}^n$ and $z\cdot\zeta$ is the scalar product in $\mathbb{R}^n$. Similarly to the first case we have

$$\int\limits_{\mathbb{R}^n\setminus B(0,\frac{R}{\epsilon_k})} \left|\nabla w_k^0(z)\right|^2 dz = \int\limits_{\mathbb{R}^n\setminus B(0,\frac{R}{\epsilon_k})} \left|\nabla U(z-y_k)\right|^2 dz$$
$$\geqslant \int\limits_{\{z\in B(y_k,T)\,|\,|z|\geqslant \frac{R}{\epsilon_k}\}} \left|\nabla U(z-y_k)\right|^2 dz$$
$$= \int\limits_{\{z\in B(\pi(y_k),T)\,|\,|z|\geqslant \frac{R}{\epsilon_k}\}} \left|\nabla U\big(z-\pi(y_k)\big)\right|^2 dz + o(1)$$

and this is greater than $\frac{C}{2}$ for k sufficiently large, which is a contradiction. Finally, if $\lim_{k\to\infty}\frac{R}{\epsilon_k}-|y_k|=2T>0$, for k sufficiently large $B(y_k,T)$ is contained in $B(0,\frac{R}{\epsilon_k})$. There holds

$$\int\limits_{B(y_k,T)} \left[\frac{1}{2}f'\big(U(z-y_k)\big)U(z-y_k)-f\big(U(z-y_k)\big)\right]dz$$
$$= \int\limits_{B(0,T)} \left[\frac{1}{2}f'\big(U(z)\big)U(z)-f\big(U(z)\big)\right]dz=\gamma_0>0.$$

We consider the new sequence of points

$$\tilde{x}_k=\exp_{x_k}(\epsilon_k y_k)\in B_g(x_k,R).$$

For any k sufficiently large, let $U(\tilde{x}_k)$ be the neighborhood of $\tilde{x}_k$ defined as $\exp_{x_k}(\epsilon_k B(y_k,T))$, then

$$\frac{1}{\epsilon_k^n}\int\limits_{U(\tilde{x}_k)} \left[\frac{1}{2}f'(u_k)u_k-f(u_k)\right]d\mu_g$$
$$=\frac{1}{\epsilon_k^n}\int\limits_{\epsilon_k B(y_k,T)} \left[\frac{1}{2}f'\big(u_k\big(\exp_{x_k}(z)\big)\big)u_k\big(\exp_{x_k}(z)\big)-f\big(u_k\big(\exp_{x_k}(z)\big)\big)\right]\left|g_{x_k}(z)\right|^{\frac{1}{2}}dz$$
$$\geqslant h^{\frac{n}{2}}\int\limits_{B(y_k,T)} \left[\frac{1}{2}f'\big(w_k(z)\big)w_k(z)-f\big(w_k(z)\big)\right]dz.$$

Since $t_k \in (t_1, t_2)$ and using the properties of the function f we obtain

$$\begin{aligned}
&\int\limits_{B(y_k,T)} \left[\frac{1}{2} f'\big(w_k(z)\big) w_k(z) - f\big(w_k(z)\big)\right] dz \\
&\quad \geqslant \int\limits_{B(y_k,T)} \left[\frac{1}{2} f'\left(\frac{t_k}{t_2} w_k(z)\right) \frac{t_k}{t_2} w_k(z) - f\left(\frac{t_k}{t_2} w_k(z)\right)\right] dz \\
&\quad > \frac{(\mu-2)c_0}{(c_1-2c_0)t_2^q} \int\limits_{B(y_k,T)} \left[\frac{1}{2} f'\big(t_k w_k(z)\big) t_k w_k(z) - f\big(t_k w_k(z)\big)\right] dz.
\end{aligned}$$

By the splitting lemma we have

$$\begin{aligned}
&\int\limits_{B(y_k,T)} \left[\frac{1}{2} f'\big(t_k w_k(z)\big) t_k w_k(z) - f\big(t_k w_k(z)\big)\right] dz \\
&= \int\limits_{B(y_k,T)} \left[\frac{1}{2} f'\big(w_k^0(z) + U(z-y_k)\big)\big(w_k^0(z) + U(z-y_k)\big) - f\big(w_k^0(z) + U(z-y_k)\big)\right] dz \\
&= \int\limits_{B(y_k,T)} \left[\frac{1}{2} f'\big(U(z-y_k)\big)\big(U(z-y_k)\big) - f\big(U(z-y_k)\big)\right] dz + o(1) \\
&= \gamma_0 + o(1).
\end{aligned}$$

So we have proved that for any k sufficiently large

$$\frac{1}{\epsilon_k^n} \int\limits_{U(\tilde{x}_k)} \left[\frac{1}{2} f'(u_k) u_k - f(u_k)\right] d\mu_g > \tilde{\gamma}_0 > 0. \tag{5.13}$$

By definition, for k big enough $U(\tilde{x}_k)$ is contained in $B_g(\tilde{x}_k, R)$ and so we can substitute x_k by $\tilde{x}_k$ and w_k by $\tilde{w}_k$, defined as in (5.10) with the new choice of points. Steps (i) and (ii) are independent of x_k (provided w_k is not identically zero) and so $\tilde{w}_k$ tends weakly to a weak solution $\tilde{w}$. It is possible to see that there exists $\widetilde{T} > 0$ such that for any k $U(\tilde{x}_k) \subset B_g(\tilde{x}_k, \epsilon_k \widetilde{T})$. Then we have

$$\begin{aligned}
&\int\limits_{B(0,\widetilde{T})} \left[\frac{1}{2} f'\big(\tilde{w}_k(z)\big) \tilde{w}_k(z) - f\big(\tilde{w}_k(z)\big)\right] dz \\
&\quad \geqslant \frac{1}{H^{\frac{n}{2}} \epsilon_k^n} \int\limits_{B_g(\tilde{x}_k, \epsilon_k \widetilde{T})} \left[\frac{1}{2} f'\big(u_k(x)\big) u_k(x) - f\big(u_k(x)\big)\right] d\mu_g \\
&\quad \geqslant \frac{1}{H^{\frac{n}{2}} \epsilon_k^n} \int\limits_{U(\tilde{x}_k)} \left[\frac{1}{2} f'\big(u_k(x)\big) u_k(x) - f\big(u_k(x)\big)\right] d\mu_g.
\end{aligned}$$

By (5.13) and by the strong convergence of $\tilde{w}_k$ to $\tilde{w}$ in $L^p(B(0,\widetilde{T}))$, we conclude that

$$\int_{B(0,\widetilde{T})} \left[\frac{1}{2} f'\big(\tilde{w}(z)\big)\tilde{w}(z) - f\big(\tilde{w}(z)\big)\right] dz \geqslant \frac{\tilde{\gamma}_0}{H^{\frac{n}{2}}}$$

and so $\tilde{w} \not\equiv 0$ and $\tilde{w} \in \mathcal{N}$.

From now on we will write as before w_k instead of $\tilde{w}_k$, x_k instead of $\tilde{x}_k$ and w instead of $\tilde{w}$. The last step is to verify that $J(w) = m(J)$. Let us consider the following inequalities

$$\begin{aligned} m(J) + \delta_k \geqslant J_{\epsilon_k}(u_k) &= \frac{1}{\epsilon_k^n} \int_M \left[\frac{1}{2} f'(u_k)u_k - f(u_k)\right] d\mu_g \\ &\geqslant \int_{\mathbb{R}^n} \left[\frac{1}{2} f'(w_k)w_k - f(w_k)\right] \left|g_{x_k}(\epsilon_k z)\right|^{\frac{1}{2}} dz. \end{aligned} \tag{5.14}$$

We define the sequence of functions in $L^2(\mathbb{R}^n)$:

$$F_k(z) = \left[\frac{1}{2} f'\big(w_k(z)\big)w_k(z) - f\big(w_k(z)\big)\right]^{\frac{1}{2}} \left|g_{x_k}(\epsilon_k z)\right|^{\frac{1}{4}}.$$

By (5.14) this sequence is bounded in $L^2(\mathbb{R}^n)$ and there exists a weak limit $F \in L^2(\mathbb{R}^n)$. We prove that

$$F(z) = \left[\frac{1}{2} f'\big(w(z)\big)w(z) - f\big(w(z)\big)\right]^{\frac{1}{2}}. \tag{5.15}$$

Let ξ be in $C_0^\infty(\mathbb{R}^n)$. On Ξ, the support of ξ, w_k strongly converges to w in $L^p(\Xi)$. So up to a subsequence $w_k(z)$ converges to $w(z)$ almost everywhere. Then point-wise

$$F_k(z)\xi(z) \to \left[\frac{1}{2} f'\big(w(z)\big)w(z) - f\big(w(z)\big)\right]^{\frac{1}{2}} \xi(z)$$

for almost every $z \in \Xi$. We can now apply Lebesgue's theorem. In fact, there holds

$$\begin{aligned} \left|F_k(z)\right|\left|\xi(z)\right| &< \begin{cases} H^{\frac{n}{4}}(\frac{c_1}{2} - c_0)^{\frac{1}{2}} |w_k(z)|^{\frac{p}{2}} |\xi(z)| & \text{if } |w_k(z)| \geqslant 1, \\ H^{\frac{n}{4}}(\frac{c_1}{2} - c_0)^{\frac{1}{2}} |w_k(z)|^{\frac{q}{2}} |\xi(z)| & \text{if } |w_k(z)| \leqslant 1 \end{cases} \\ &\leqslant H^{\frac{n}{4}} \left(\frac{c_1}{2} - c_0\right)^{\frac{1}{2}} \left(1 + \left|w_k(z)\right|^{\frac{p}{2}}\right)\left|\xi(z)\right| \end{aligned}$$

and, since w_k converges strongly to w in $L^p(\Xi)$, there exists $W \in L^p(\Xi)$ such that for all k $|w_k(z)| \leqslant W(z)$ almost everywhere and $|F_k(z)||\xi(z)| \leqslant H^{\frac{n}{4}}(\frac{c_1}{2} - c_0)^{\frac{1}{2}}(1 + (W(z))^{\frac{p}{2}})|\xi(z)| \in L^2(\Xi)$. So (5.15) is proved. By weak lower semicontinuity of the norm

$$\|F\|^2_{L^2(\mathbb{R}^n)} \leqslant \liminf_{k\to\infty} \|F_k\|^2_{L^2(\mathbb{R}^n)},$$

that is

$$\int_{\mathbb{R}^n} \left[\frac{1}{2} f'(w)w - f(w)\right] dz \leqslant \liminf_{k\to\infty} \int_{\mathbb{R}^n} \left[\frac{1}{2} f'(w_k)w_k - f(w_k)\right] \left|g_{x_k}(\epsilon_k z)\right|^{\frac{1}{2}} dz.$$

By this inequality and (5.14) we conclude that

$$\begin{aligned} m(J) &= \lim_{k\to\infty} m(J) + \delta_k \geqslant \lim_{k\to\infty} J_{\epsilon_k}(u_k) \\ &\geqslant \liminf_{k\to\infty} \int_{\mathbb{R}^n} \left[\frac{1}{2} f'(w_k)w_k - f(w_k)\right] \left|g_{x_k}(\epsilon_k z)\right|^{\frac{1}{2}} dz \\ &\geqslant \int_{\mathbb{R}^n} \left[\frac{1}{2} f'(w)w - f(w)\right] dz \geqslant m(J). \end{aligned}$$

(iv) The equality is immediate from (5.14). □

We recall here Ekeland principle (see for instance [11]).

Definition 5.7. Let X be a complete metric space and $\Psi : X \to \mathbb{R} \cup \{+\infty\}$ be a lower semi-continuous function on X, bounded from below. Given $\eta > 0$ and $\bar{u} \in X$ such that

$$\Psi(\bar{u}) < \inf_{u\in X} \Psi(u) + \frac{\eta}{2},$$

for all $\lambda > 0$ there exists $u_\lambda \in X$ such that

$$\Psi(u_\lambda) < \Psi(\bar{u}), \qquad d(u_\lambda, \bar{u}) < \lambda$$

and for all $u \neq u_\lambda$ it holds

$$\Psi(u_\lambda) < \Psi(u) + \frac{\eta}{\lambda} d(u_\lambda, u).$$

Remark 5.8. 1. We apply Lemma 5.6 when u_k is a minimum solution $u_k \in \mathcal{N}_{\epsilon_k}$, $J_{\epsilon_k}(u_k) = m_{\epsilon_k}$. By (iv) we have $\lim_{k\to\infty} m_{\epsilon_k} = m(J)$. In particular for any $\delta > 0$ there exists $\epsilon_0 = \epsilon_0(\delta) > 0$ sufficiently small such that for all $\epsilon \leqslant \epsilon_0$ $|m_\epsilon - m(J)| < \delta$.

2. Applying Ekeland principle for $X = \Sigma_{\epsilon,\delta}$, with $\epsilon \leqslant \epsilon_0(\delta)$ as in 1, we obtain that for all $\bar{u} \in \Sigma_{\epsilon,\delta}$ there exists $u_\delta \in \Sigma_{\epsilon,\delta}$ such that

$$J_\epsilon(u_\delta) < J_\epsilon(\bar{u}), \qquad \frac{\epsilon}{\epsilon^{\frac{n}{2}}} \|u_\delta - \bar{u}\|_{H^1_2(M)} < 4\sqrt{\delta}$$

and for all $u \in T\Sigma_{\epsilon,\delta}$

$$\left|J'_\epsilon(u_\delta)(u)\right| < \frac{\sqrt{\delta}\epsilon}{\epsilon^{\frac{n}{2}}} \|u\|_{H^1_2(M)}. \tag{5.16}$$

Proof of Proposition 5.4. We choose $\epsilon_0(\delta)$ as in point 1 of Remark 5.8. We also assume that $\epsilon_0(\delta_0)$ is less than ϵ_0 in Lemma 5.1.

By contradiction, we assume that there is $\eta_0 \in (0,1)$ such that there exist two positive sequences $\{\delta_k\}_{k\in\mathbb{N}}$, $\{\epsilon_k\}_{k\in\mathbb{N}}$ tending to zero as k tends to infinity and a sequence of functions $\{u_k\}_{k\in\mathbb{N}}$, with $u_k \in \Sigma_{\epsilon_k,\delta_k}$, and for any $x \in M$

$$\widetilde{F}_{\epsilon_k, B_g(x,\frac{r(M)}{2})}(u_k) \leqslant \eta_0 m(J). \tag{5.17}$$

By Ekeland principle for any k we can consider $\tilde{u}_k$ as in 2 of Remark 5.8. Property (5.17) becomes

$$\widetilde{F}_{\epsilon_k, B_g(x,\frac{r(M)}{2})}(\tilde{u}_k) \leqslant \eta_1 m(J) \tag{5.18}$$

with η_1 still in $(0,1)$. To prove this we have to evaluate the difference

$$\frac{1}{\epsilon_k^n} \int\limits_{B_g(x,\frac{r(M)}{2})} \left| \frac{1}{2} f'(\tilde{u}_k)\tilde{u}_k - f(\tilde{u}_k) - \frac{1}{2} f'(u_k)u_k + f(u_k) \right| d\mu_g,$$

which by the mean value theorem can be written

$$\frac{1}{2\epsilon_k^n} \int\limits_B \left| f''\left(u_k^*\right)u_k^* - f'\left(u_k^*\right)\right| |\tilde{u}_k - u_k| \, d\mu_g, \tag{5.19}$$

where B is $B_g(x, \frac{r(M)}{2})$ and $u_k^*(x) = \theta(x)\tilde{u}_k(x) + (1-\theta(x))u_k(x)$ for a suitable function $\theta(x)$ with values in $(0,1)$. By Hölder's inequality (5.19) is bounded from above by

$$\frac{1}{2}\left(\frac{1}{\epsilon_k^n} \int\limits_B \left| f''\left(u_k^*\right)u_k^* - f'\left(u_k^*\right)\right|^{\frac{2n}{n+2}} d\mu_g \right)^{\frac{n+2}{2n}} \left(\frac{1}{\epsilon_k^n} \int\limits_B |\tilde{u}_k - u_k|^{\frac{2n}{n-2}} d\mu_g \right)^{\frac{n-2}{2n}}.$$

We prove that the first factor is bounded and the second one is infinitesimal. In fact, we have

$$\begin{aligned} \left(\frac{1}{\epsilon_k^n} \int\limits_B |\tilde{u}_k - u_k|^{\frac{2n}{n-2}} d\mu_g \right)^{\frac{n-2}{2n}} &= \frac{\epsilon_k}{\epsilon_k^{\frac{n}{2}}} \|\tilde{u}_k - u_k\|_{L^{\frac{2n}{n-2}}(B)} \\ &\leqslant C \frac{\epsilon_k}{\epsilon_k^{\frac{n}{2}}} \|\tilde{u}_k - u_k\|_{H_2^1(M)} < 4C\sqrt{\delta}. \end{aligned}$$

The proof of the bound

$$\frac{1}{\epsilon_k^n} \int\limits_B \left| f''\left(u_k^*\right)u_k^* - f'\left(u_k^*\right)\right|^{\frac{2n}{n+2}} d\mu_g \leqslant C \tag{5.20}$$

for a positive constant C can be found in Appendix A.

We apply Lemma 5.6 to the sequences $\{\delta_k\}_{k\in\mathbb{N}}$, $\{\epsilon_k\}_{k\in\mathbb{N}}$ and $\{\tilde{u}_k\}_{k\in\mathbb{N}}$, obtaining a sequence of functions on $\mathbb{R}^n$ $\{w_k\}_{k\in\mathbb{N}}$ (it is easy to see that (5.16) holds for any $u \in H_2^1(M)$). Let w be the weak limit in $\mathcal{D}^{1,2}(\mathbb{R}^n)$ of w_k. Let η_2 be a constant in $(0,1)$ such that $\eta_2 > \frac{1+\eta_1}{2}$. Since $J(w) = m(J)$, there exists $T > 0$ such that

$$\int\limits_{B(0,T)} \left[\frac{1}{2} f'\bigl(w(z)\bigr)w(z) - f\bigl(w(z)\bigr)\right] dz \geqslant \eta_2 m(J). \tag{5.21}$$

On the other hand, up to a subsequence, we have

$$\begin{aligned}
&\int\limits_{B(0,T)} \left[\frac{1}{2} f'(w)w - f(w)\right] dz \\
&\quad = \lim_{k\to\infty} \int\limits_{B(0,T)} \left[\frac{1}{2} f'(w_k)w_k - f(w_k)\right] dz \\
&\quad = \lim_{k\to\infty} \frac{1}{\epsilon_k} \int\limits_{B(0,\epsilon_k T)} \left[\frac{1}{2} f'(\tilde{u}_k \circ \exp_{x_k})\tilde{u}_k \circ \exp_{x_k} - f(\tilde{u}_k \circ \exp_{x_k})\right] dz.
\end{aligned} \tag{5.22}$$

By compactness the sequence x_k converges (up to a subsequence) to $\bar{x}$ and for any $z \in B(0,T)$ the limit of $|g_{x_k}(\epsilon_k z)|^{\frac{1}{2}}$ for k tending to infinity is $|g_{\bar{x}}(0)|^{\frac{1}{2}} = 1$. Since $\frac{2\eta_1}{1+\eta_1} \in (0,1)$, for k sufficiently big for any $z \in B(0,\epsilon_k T)$ we have $|g_{x_k}(z)|^{\frac{1}{2}} > \frac{2\eta_1}{1+\eta_1}$. So the last limit in (5.22) is less than

$$\begin{aligned}
&\frac{1+\eta_1}{2\eta_1} \lim_{k\to\infty} \frac{1}{\epsilon_k} \int\limits_{B(0,\epsilon_k T)} \left[\frac{1}{2} f'(\tilde{u}_k \circ \exp_{x_k})\tilde{u}_k \circ \exp_{x_k} - f(\tilde{u}_k \circ \exp_{x_k})\right] \bigl|g_{x_k}(z)\bigr|^{\frac{1}{2}}\, dz \\
&\quad = \frac{1+\eta_1}{2\eta_1} \lim_{k\to\infty} \frac{1}{\epsilon_k} \int\limits_{B(x_k,\epsilon_k T)} \left[\frac{1}{2} f'(\tilde{u}_k)\tilde{u}_k - f(\tilde{u}_k)\right] d\mu_g \leqslant \frac{1+\eta_1}{2} m(J),
\end{aligned}$$

where we have used property (5.18). By this inequality together with (5.22) and (5.21) we get $\eta_2 \leqslant \frac{1+\eta_1}{2}$ which is in contradiction with the choice of η_2. $\square$

It is now possible to prove the following proposition:

Proposition 5.9. *There exists $\delta_0 \in (0, m(J))$ such that for any $\delta \in (0,\delta_0)$ there exists $\epsilon_0 = \epsilon_0(\delta) > 0$ and for any $\epsilon \in (0,\epsilon_0)$ and $u \in \Sigma_{\epsilon,\delta}$ the barycentre $\beta(u)$ is in $M_{r(M)}$.*

Proof. By Proposition 5.4, for any $\eta \in (0,1)$ and for any $u \in \Sigma_{\epsilon,\delta}$ with ϵ and δ sufficiently small there exists a point x_0 such that

$$\widetilde{F}_{\epsilon, B_g(x_0, \frac{r(M)}{2})}(u) > \eta m(J).$$

Since $u \in \Sigma_{\epsilon,\delta}$ we also have

$$\widetilde{F}_{\epsilon,M}(u) \leqslant m(J) + \delta.$$

We define

$$\rho\big(u(x)\big) = \frac{\frac{1}{2}f'(u(x))u(x) - f(u(x))}{\int_M [\frac{1}{2}f'(u(x))u(x) - f(u(x))]\, d\mu_g}.$$

By the previous inequalities we have then

$$\int\limits_{B_g(x_0, \frac{r(M)}{2})} \rho\big(u(x)\big)\, d\mu_g > \frac{\eta}{1 + \frac{\delta}{m(J)}}.$$

We can now esteem

$$\begin{aligned}
\big|\beta(u) - x_0\big| &= \bigg|\int\limits_M (x - x_0)\rho\big(u(x)\big)\, d\mu_g\bigg| \\
&\leqslant \bigg|\int\limits_{B_g(x_0, \frac{r(M)}{2})} (x - x_0)\rho\big(u(x)\big)\, d\mu_g\bigg| + \bigg|\int\limits_{M \setminus B_g(x_0, \frac{r(M)}{2})} (x - x_0)\rho\big(u(x)\big)\, d\mu_g\bigg| \\
&< \frac{r(M)}{2} + D\bigg(1 - \frac{\eta}{1 + \frac{\delta}{m(J)}}\bigg),
\end{aligned}$$

where D is the diameter of the manifold M. For η near to 1 and δ sufficiently small we obtain $\beta(u) \in M_{r(M)}$. $\square$

6. The function I_ϵ

We prove now that the composition I_ϵ of ϕ_ϵ and β is well defined and homotopic to the identity on M:

Proposition 6.1. *There exists $\epsilon_0 > 0$ such that for any $\epsilon \in (0, \epsilon_0)$ the composition*

$$I_\epsilon = \beta \circ \phi_\epsilon : M \to M_{r(M)}$$

is well defined and homotopic to the identity on M.

Proof. Let us consider the function $H : [0, 1] \times M \to M_{r(M)}$, defined by $H(t, x) = tI_\epsilon(x) + (1 - t)x$. This function is a homotopy if for any $t \in [0, 1]$ $H(t, x) \in M_{r(M)}$. It is enough to prove that for any $x_0 \in M$ $|I_\epsilon(x_0) - x_0| < r(M)$. Since the support of $\phi_\epsilon(x_0)$ is contained in $B_g(x_0, R)$

$$I_\epsilon(x_0) - x_0 = \int_M (x - x_0)\rho\big(\phi_\epsilon(x_0)(x)\big)\,d\mu_g = \int_{B_g(x_0,R)} (x - x_0)\rho\big(\phi_\epsilon(x_0)(x)\big)\,d\mu_g$$

$$= \frac{\int_{B(0,R)} z\Phi(t_\epsilon(W_{x_0,\epsilon})W_{x_0,\epsilon}(\exp_{x_0}(z)))|g_{x_0}(z)|^{\frac{1}{2}}\,dz}{\int_{B(0,R)} \Phi(t_\epsilon(W_{x_0,\epsilon})W_{x_0,\epsilon}(\exp_{x_0}(z)))|g_{x_0}(z)|^{\frac{1}{2}}\,dz}$$

$$= \frac{\epsilon\int_{B(0,\frac{R}{\epsilon})} z\Phi(t_\epsilon(W_{x_0,\epsilon})W_{x_0,\epsilon}(\exp_{x_0}(\epsilon z)))|g_{x_0}(\epsilon z)|^{\frac{1}{2}}\,dz}{\int_{B(0,\frac{R}{\epsilon})} \Phi(t_\epsilon(W_{x_0,\epsilon})W_{x_0,\epsilon}(\exp_{x_0}(\epsilon z)))|g_{x_0}(\epsilon z)|^{\frac{1}{2}}\,dz},$$

where Φ is defined in (5.2). We recall that for any $\epsilon \in (0,1]$ and $x_0 \in M$ $t_1 \leqslant t_\epsilon(W_{x_0,\epsilon}) \leqslant t_2$. By definition of ϕ_ϵ, we have

$$\int_{B(0,\frac{R}{\epsilon})} \Phi\big(t_\epsilon(W_{x_0,\epsilon})W_{x_0,\epsilon}\big(\exp_{x_0}(\epsilon z)\big)\big)\big|g_{x_0}(\epsilon z)\big|^{\frac{1}{2}}\,dz \geqslant h^{\frac{n}{2}} \int_{B(0,R)} \Phi\big(t_1\big(U(z) - \widetilde{U}_R\big)\big)\,dz > 0,$$

where $\widetilde{U}_R$ is the value $U(z)$ for any $z \in \mathbb{R}^n$ such that $|z| = R$. Furthermore, we have

$$\begin{aligned}
&\epsilon \int_{B(0,\frac{R}{\epsilon})} |z|\Phi\big(t_\epsilon(W_{x_0,\epsilon})W_{x_0,\epsilon}\big(\exp_{x_0}(\epsilon z)\big)\big)\big|g_{x_0}(\epsilon z)\big|^{\frac{1}{2}}\,dz \\
&\quad \leqslant \epsilon H^{\frac{n}{2}} \int_{B(0,\frac{R}{\epsilon})} |z|\Phi\big(t_2 U(z)\big)\,dz \\
&\quad < \frac{(c_1 - 2c_0)H^{\frac{n}{2}}\epsilon}{2}\Bigg[\int_{\{z\in B(0,\frac{R}{\epsilon}) | t_2U(z)\geqslant 1\}} |z|t_2^p\big(U(z)\big)^p\,dz + \int_{\{z\in B(0,\frac{R}{\epsilon}) | t_2U(z)\leqslant 1\}} |z|t_2^q\big(U(z)\big)^q\,dz\Bigg].
\end{aligned}$$

Since U is spherically symmetric and decreasing, there exists $\rho_0 > 0$ such that the last quantity is equal to

$$\frac{(c_1 - 2c_0)H^{\frac{n}{2}}\epsilon}{2}\Bigg[\int_{B(0,\rho_0)} |z|t_2^p\big(U(z)\big)^p\,dz + \int_{B(0,\frac{R}{\epsilon})\setminus B(0,\rho_0)} |z|t_2^q\big(U(z)\big)^q\,dz\Bigg]. \tag{6.1}$$

Obviously, the integral

$$\int_{B(0,\rho_0)} |z|t_2^p\big(U(z)\big)^p\,dz \leqslant t_2^p\rho_0 \int_{B(0,\rho_0)} \big(U(z)\big)^p\,dz$$

is bounded. For the second integral in (6.1), we use the well-known inequality by Strauss (see [15]):

$$\epsilon \int\limits_{B(0,\frac{R}{\epsilon})\setminus B(0,\rho_0)} |z|\big(U(z)\big)^q\,dz \leqslant C_n \|U\|^q_{\mathcal{D}^{1,2}(\mathbb{R}^n)}\epsilon \int\limits_{B(0,\frac{R}{\epsilon})\setminus B(0,\rho_0)} \frac{|z|}{|z|^{\frac{(n-2)q}{2}}}\,dz,$$

where C_n is a positive constant. Then we conclude that there exist two positive constants C_1, C_2 such that (6.1) is bounded from above by $C_1\epsilon + C_2\epsilon^{\frac{(n-2)q-2n}{2}}$, where the second exponent is positive and so $|I_\epsilon(x_0) - x_0|$ tends to zero as ϵ tends to zero. □

Finally, by standard arguments it is easy to see that the Palais–Smale condition holds for J_ϵ constrained on $\mathcal{N}_\epsilon$.

7. The Morse theory result

For an introduction to Morse theory we refer the reader to [14], while for the applications to problems of functional analysis we mention [2].

Let (X, Y) be a couple of topological spaces, with $Y \subset X$, and $H_k(X, Y)$ be the kth homology group with coefficients in some field. We recall the following definition:

Definition 7.1. The Poincaré polynomial of (X, Y) is the formal power series

$$\mathcal{P}_t(X, Y) = \sum_{k=0}^{\infty} \dim\big[H_k(X, Y)\big]t^k.$$

The Poincaré polynomial of X is defined as $\mathcal{P}_t(X) = \mathcal{P}_t(X, \emptyset)$.

If X is a compact n-dimensional manifold $\dim[H_k(X)]$ is finite for any k and $\dim[H_k(X)] = 0$ for any $k > n$. In particular $\mathcal{P}_t(X)$ is a polynomial and not a formal series.

We define now the Morse index.

Definition 7.2. Let J be a C^2 functional on a Banach space X and let u be an isolated critical point of J with $J(u) = c$. The (polynomial) Morse index of u is defined as

$$i_t(u) = \sum_{k=0}^{\infty} \dim\big[H_k\big(J^c, J^c \setminus \{u\}\big)\big]t^k,$$

where $J^c = \{v \in X \mid J(v) \leqslant c\}$. If u is a non-degenerate critical point then $i_t(u) = t^{\mu(u)}$, where $\mu(u)$ is the (numerical) Morse index of u and represents the dimension of the maximal subspace on which the bilinear form $J''(u)[\cdot,\cdot]$ is negative definite.

It is now possible to state Theorem 1.2 more precisely:

Theorem 7.3. *There exists $\epsilon_0 > 0$ such that for any $\epsilon \in (0, \epsilon_0)$, if the set K_ϵ of solutions of Eq.* (1.1) *is discrete, then*

$$\sum_{u \in K_\epsilon} i_t(u) = t\mathcal{P}_t(M) + t^2\big[\mathcal{P}_t(M) - 1\big] + t(1+t)\mathcal{Q}_\epsilon(t),$$

where $\mathcal{Q}_\epsilon(t)$ is a polynomial with nonnegative integer coefficients.

In the non-degenerate case, the above theorem becomes:

Corollary 7.4. *There exists $\epsilon_0 > 0$ such that for any $\epsilon \in (0, \epsilon_0)$, if the set K_ϵ of solutions of Eq.* (1.1) *is discrete and the solutions are non-degenerate, then*

$$\sum_{u \in K_\epsilon} t^{\mu(u)} = t\mathcal{P}_t(M) + t^2\big[\mathcal{P}_t(M) - 1\big] + t(1+t)\mathcal{Q}_\epsilon(t),$$

where $\mathcal{Q}_\epsilon(t)$ is a polynomial with nonnegative integer coefficients.

Since we have proved that the composition I_ϵ of ϕ_ϵ and β from M to $M_{r(M)}$ for ϵ sufficiently small is homotopic to the identity on M, the following equation holds (see [4]):

$$\mathcal{P}_t(\Sigma_{\epsilon,\delta}) = \mathcal{P}_t(M) + \mathcal{Z}(t), \tag{7.1}$$

where $\mathcal{Z}(t)$ is a polynomial with nonnegative integer coefficients (here ϵ and δ are chosen as in Proposition 5.9).

Let α and ϵ be as in Lemma 5.1, $\delta > 0$, then

$$\begin{aligned} \mathcal{P}_t\big(J_\epsilon^{m(J)+\delta}, J_\epsilon^{\frac{\alpha}{2}}\big) &= t\mathcal{P}_t(\Sigma_{\epsilon,\delta}), \\ \mathcal{P}_t\big(H_2^1(M), J_\epsilon^{m(J)+\delta}\big) &= t\big[\mathcal{P}_t\big(J_\epsilon^{m(J)+\delta}, J_\epsilon^{\frac{\alpha}{2}}\big) - t\big]. \end{aligned} \tag{7.2}$$

By Morse theory we have

$$\sum_{u \in K_\epsilon} i_t(u) = \mathcal{P}_t\big(H_2^1(M), J_\epsilon^{m(J)+\delta}\big) + \mathcal{P}_t\big(J_\epsilon^{m(J)+\delta}, J_\epsilon^{\frac{\alpha}{2}}\big) + (1+t)\mathcal{Q}_\epsilon(t),$$

where $\mathcal{Q}_\epsilon(t)$ is a polynomial with nonnegative coefficients. Using this relation with (7.1) and (7.2), we obtain Theorem 7.3 and Corollary 7.4. Theorem 1.2 easily follows by evaluating the power series in $t = 1$.

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Appendix A

Proof of Lemma 5.2. Given any $0 < r < r(M)$, we can choose $\rho < r$ small enough so that there exists a finite open cover of M_ρ $\{C_\alpha\}_{\alpha=1,\dots,k}$ of subsets of $\mathbb{R}^N$ with smooth charts $\xi_\alpha : D_\alpha \subset \mathbb{R}^N \to C_\alpha$ induced on M_ρ by the manifold structure of M. We assume that $D_\alpha = Z_\alpha \times T_\alpha$, with Z_α a subset of $\mathbb{R}^n$ star-shaped centred in the origin and T_α the ball of $\mathbb{R}^{N-n}$ with centre the origin and radius ρ. For any α and any $(z, 0) \in Z_\alpha \times T_\alpha$, let $\xi_\alpha(z, 0) \in \widetilde{C}_\alpha = C_\alpha \cap M$. Vice versa for any $x \in \widetilde{C}_\alpha$, let $\xi_\alpha^{-1}(x) = (z, 0)$.

We denote by $\{\psi_\alpha(y)\}_{\alpha=1,\dots,k}$ a partition of unity subordinate to the cover $\{C_\alpha\}_{\alpha=1,\dots,k}$. For all $y \in M_\rho$ we write $\xi_\alpha^{-1}(y) = (z_\alpha(y), t_\alpha(y))$.

Given a function $u \in H_2^1(M)$, we define a function $v \in \mathcal{D}^{1,2}(M_r)$ by $v(y) \equiv 0$ for all $y \in M_r \setminus M_\rho$ and

$$v(y) = \sum_{\alpha=1}^{k} \psi_\alpha(y) u\big(\xi_\alpha\big(z_\alpha(y), 0\big)\big) \chi_\rho\big(\big|t_\alpha(y)\big|\big)$$

for all $y \in M_\rho$, where χ_ρ is defined in (2.1).

Inequality (5.4). Let us write

$$C_0 = \Big[\sup_{i,j=1,\dots,N} \sup_{\alpha=1,\dots,k} \sup_{y\in C_\alpha} \big(D_y\big(\xi_\alpha\big(z_\alpha(y),0\big)\big)\big)_{ij}\Big]^2,$$

$$C_1 = \Big[\sup_{\substack{i=1,\dots,N\\ j=1,\dots,N-n}} \sup_{\alpha=1,\dots,k} \sup_{y\in C_\alpha} \big(D\big(t_\alpha(y)\big)\big)_{ij}\Big]^2,$$

$$C_2 = \sup_{\alpha=1,\dots,k} \sup_{y\in C_\alpha} \big(\big|\nabla\psi_\alpha(y)\big|^2 + 1\big),$$

$$C_3 = \sup_{\alpha=1,\dots,k} \sup_{(z,t)\in D_\alpha} \big|\det D\big(\xi_\alpha(z,t)\big)\big|,$$

$$C_4 = \int_{\mathbb{R}^{N-n}} \big[\big(\chi_\rho\big(|t|\big)\big)^2 + \big(\chi_\rho'\big(|t|\big)\big)^2\big]\,dt.$$

Then we can estimate

$$\begin{aligned}
\int_{M_r} \big|\nabla v(y)\big|^2\,dy \leqslant{}& 2\sum_{\alpha=1}^{k} \int_{C_\alpha} \big[\big|\nabla\psi_\alpha(y)\big|^2 \big(u\big(\xi_\alpha\big(z_\alpha(y),0\big)\big)\chi_\rho\big(\big|t_\alpha(y)\big|\big)\big)^2 \\
&+ \big|\nabla_y\big(u\big(\xi_\alpha\big(z_\alpha(y),0\big)\big)\big)\big|^2 \big(\psi_\alpha(y)\chi_\rho\big(\big|t_\alpha(y)\big|\big)\big)^2 \\
&+ \big|\nabla_y\big(\chi_\rho\big(\big|t_\alpha(y)\big|\big)\big)\big|^2 \big(\psi_\alpha(y)u\big(\xi_\alpha\big(z_\alpha(y),0\big)\big)\big)^2\big]\,dy \\
\leqslant{}& 2\sum_{\alpha=1}^{k} \int_{C_\alpha} \big[\big|\nabla\psi_\alpha(y)\big|^2 \big(u\big(\xi_\alpha\big(z_\alpha(y),0\big)\big)\chi_\rho\big(\big|t_\alpha(y)\big|\big)\big)^2 \\
&+ C_0\big|\nabla u\big(\xi_\alpha\big(z_\alpha(y),0\big)\big)\big|^2 \big(\psi_\alpha(y)\chi_\rho\big(\big|t_\alpha(y)\big|\big)\big)^2 \\
&+ C_1\big(\chi_\rho'\big(\big|t_\alpha(y)\big|\big)\big)^2 \big(\psi_\alpha(y)u\big(\xi_\alpha\big(z_\alpha(y),0\big)\big)\big)^2\big]\,dy \\
\leqslant{}& \sum_{\alpha=1}^{k} \int_{C_\alpha} \big[2C_0\big|\nabla u\big(\xi_\alpha\big(z_\alpha(y),0\big)\big)\big|^2 \big(\chi_\rho\big(\big|t_\alpha(y)\big|\big)\big)^2 \\
&+ 2(1+C_1)C_2\big(u\big(\xi_\alpha\big(z_\alpha(y),0\big)\big)\big)^2\big[\big(\chi_\rho\big(\big|t_\alpha(y)\big|\big)\big)^2 + \big(\chi_\rho'\big(\big|t_\alpha(y)\big|\big)\big)^2\big]\,dy\big]
\end{aligned}$$

$$\leqslant 2C_0C_3\sum_{\alpha=1}^{k}\int\limits_{D_\alpha}\left|\nabla u\big(\xi_\alpha(z,0)\big)\right|^2\big(\chi_\rho\big(|t|\big)\big)^2\,dz\,dt$$

$$+\,2(1+C_1)C_2C_3\sum_{\alpha=1}^{k}\int\limits_{D_\alpha}\big(u\big(\xi_\alpha(z,0)\big)\big)^2\big[\big(\chi_\rho\big(|t|\big)\big)^2+\big(\chi_\rho'\big(|t|\big)\big)^2\big]\,dz\,dt$$

$$\leqslant 2C_3\big(C_0+(1+C_1)C_2\big)\sum_{\alpha=1}^{k}\bigg[\int\limits_{T_\alpha}\big(\chi_\rho\big(|t|\big)\big)^2\,dt\int\limits_{Z_\alpha}\left|\nabla u\big(\xi_\alpha(z,0)\big)\right|^2\,dz$$

$$+\int\limits_{T_\alpha}\big[\big(\chi_\rho\big(|t|\big)\big)^2+\big(\chi_\rho'\big(|t|\big)\big)^2\big]\,dt\int\limits_{Z_\alpha}\big(u\big(\xi_\alpha(z,0)\big)\big)^2\,dz\bigg]$$

$$\leqslant 2C_3\big(C_0+(1+C_1)C_2\big)C_4\sum_{\alpha=1}^{k}\int\limits_{Z_\alpha}\big[\left|\nabla u\big(\xi_\alpha(z,0)\big)\right|^2+\big(u\big(\xi_\alpha(z,0)\big)\big)^2\big]\,dz$$

$$\leqslant 2C_3\big(C_0+(1+C_1)C_2\big)C_4\frac{H}{h^{\frac{n}{2}}}\sum_{\alpha=1}^{k}\int\limits_{\widetilde{C}_\alpha}\big[\left|\nabla u(x)\right|_g^2+\big(u(x)\big)^2\big]\,d\mu_g.$$

One can easily see that there exists a constant $C_5>0$, depending only on the charts ξ_α and on the partition of unity ψ_α, such that

$$\sum_{\alpha=1}^{k}\int\limits_{\widetilde{C}_\alpha}\big[\left|\nabla u(x)\right|_g^2+\big(u(x)\big)^2\big]\,d\mu_g\leqslant C_5\|u\|^2_{H_2^1(M)}$$

and by the Sobolev embedding of $H_2^1(M)$ in $L^2(M)$ (5.4) is proved.

Inequality (5.5). We show that for any $s,t\in\mathbb{R}$, $s+t\neq 0$,

$$f(s+t)>\frac{c_0\mu}{c_1}\big[f(s)+f(t)\big].$$

Let us consider first the case $|s+t|\geqslant 1$, $|s|\geqslant 1$ and $|t|\geqslant 1$:

$$f(s+t)\geqslant c_0|s+t|^p\geqslant c_0\big(|s|^p+|t|^p\big)\geqslant\frac{c_0}{c_1}\big(f''(s)s^2+f''(t)t^2\big)>\frac{c_0\mu}{c_1}\big(f(s)+f(t)\big).$$

If $|s+t|\geqslant 1$, $|s|\geqslant 1$ and $|t|<1$, we have

$$f(s+t)\geqslant c_0\big(|s|^p+|t|^p\big)\geqslant c_0\big(|s|^p+|t|^q\big)>\frac{c_0\mu}{c_1}\big(f(s)+f(t)\big).$$

The same kind of inequalities holds true in the other cases.

Hereafter, for all $y \in M_r$ we denote $v_\alpha(y) = \psi_\alpha(y) u(\xi_\alpha(z_\alpha(y), 0)) \chi_\rho(|t_\alpha(y)|)$. The following integrals are always meant on the intersection with the support of v:

$$\begin{aligned}\int_{M_r} f\big(v(y)\big)\, dy &= \int_{M_r} f\left(\sum_{\alpha=1}^{k} v_\alpha(y)\right) dy > \frac{c_0\mu}{c_1} \sum_{\alpha=1}^{k} \int_{C_\alpha} f\big(v_\alpha(y)\big)\, dy \\ &\geqslant \frac{c_0^2\mu}{c_1} \sum_{\alpha=1}^{k} \Bigg[\int_{\{y\in C_\alpha \mid |v_\alpha(y)| \geqslant 1\}} \big|v_\alpha(y)\big|^p\, dy + \int_{\{y\in C_\alpha \mid |v_\alpha(y)| \leqslant 1\}} \big|v_\alpha(y)\big|^q\, dy \Bigg].\end{aligned}$$

For all $\alpha = 1, \dots, k$ it is possible to choose $C'_\alpha \subset C_\alpha$ such that on this subset $\psi_\alpha(y) \geqslant \frac{1}{k}$. Then the previous chain of inequalities is bounded from below by

$$\begin{aligned}&\frac{c_0^2\mu}{c_1 k^q} \sum_{\alpha=1}^{k} \Bigg[\int_{\{y\in C'_\alpha \mid |v_\alpha(y)| \geqslant 1\}} \Big|u\big(\xi_\alpha\big(z_\alpha(y), 0\big)\big) \chi_\rho\big(\big|t_\alpha(y)\big|\big)\Big|^p\, dy \\ &\quad + \int_{\{y\in C'_\alpha \mid |v_\alpha(y)| \leqslant 1\}} \Big|u\big(\xi_\alpha\big(z_\alpha(y), 0\big)\big) \chi_\rho\big(\big|t_\alpha(y)\big|\big)\Big|^q\, dy \Bigg].\end{aligned} \tag{A.1}$$

Let D'_α be the set $\xi_\alpha^{-1}(C'_\alpha)$. We consider the following constants:

$$\begin{aligned}C_6 &= \inf_{\alpha=1,\dots,k} \inf_{(z,t)\in D_\alpha} \big|\det D\big(\xi_\alpha(z,t)\big)\big|, \\ C_7 &= \int_{\mathbb{R}^{N-n}} \big(\chi_\rho\big(|t|\big)\big)^q\, dt, \\ C_8 &= \inf_{\alpha=1,\dots,k} \inf_{x\in \widetilde{C}_\alpha} \big|\det D\big(z_\alpha(x)\big)\big|.\end{aligned}$$

The inequality (A.1) is bounded from below by

$$\begin{aligned}&\frac{c_0^2\mu C_6}{c_1 k^q} \sum_{\alpha=1}^{k} \Bigg[\int_{\{(z,t)\in D'_\alpha \mid |v_\alpha(\xi_\alpha(z,t))| \geqslant 1\}} \Big|u\big(\xi_\alpha(z,0)\big) \chi_\rho\big(|t|\big)\Big|^p\, dz\, dt \\ &\quad + \int_{\{(z,t)\in D'_\alpha \mid |v_\alpha(\xi_\alpha(z,t))| \leqslant 1\}} \Big|u\big(\xi_\alpha(z,0)\big) \chi_\rho\big(|t|\big)\Big|^q\, dz\, dt \Bigg] \\ &\geqslant \frac{c_0^2\mu C_6}{c_1 k^q} \sum_{\alpha=1}^{k} \Bigg[\int_{\{(z,t)\in D'_\alpha \mid |u(\xi_\alpha(z,0))| \geqslant 1\}} \big|u\big(\xi_\alpha(z,0)\big)\big|^p \big(\chi_\rho\big(|t|\big)\big)^q\, dz\, dt \\ &\quad + \int_{\{(z,t)\in D'_\alpha \mid |u(\xi_\alpha(z,0))| \leqslant 1\}} \big|u\big(\xi_\alpha(z,0)\big)\big|^q \big(\chi_\rho\big(|t|\big)\big)^q\, dz\, dt\end{aligned}$$

$$
\begin{aligned}
&- \int\limits_{\{(z,t)\in D'_\alpha \mid |v_\alpha(\xi_\alpha(z,t))|\leqslant 1,\ |u(\xi_\alpha(z,0))|\geqslant 1\}} \big|u\big(\xi_\alpha(z,0)\big)\big|^p \big(\chi_\rho\big(|t|\big)\big)^q \,dz\,dt \\
&+ \int\limits_{\{(z,t)\in D'_\alpha \mid |v_\alpha(\xi_\alpha(z,t))|\leqslant 1,\ |u(\xi_\alpha(z,0))|\geqslant 1\}} \big|u\big(\xi_\alpha(z,0)\big)\big|^q \big(\chi_\rho\big(|t|\big)\big)^q \,dz\,dt \Bigg] \\
&= \frac{c_0^2 \mu C_6 C_7}{c_1 k^q} \sum_{\alpha=1}^k \Bigg[\int\limits_{\{(z,0)\in D'_\alpha \mid |u(\xi_\alpha(z,0))|\geqslant 1\}} \big|u\big(\xi_\alpha(z,0)\big)\big|^p \,dz \\
&+ \int\limits_{\{(z,0)\in D'_\alpha \mid |u(\xi_\alpha(z,0))|\leqslant 1\}} \big|u\big(\xi_\alpha(z,0)\big)\big|^q \,dz \Bigg] \\
&\geqslant \frac{c_0^2 \mu C_6 C_7 C_8}{c_1 k^q} \sum_{\alpha=1}^k \Bigg[\int\limits_{\{x\in \widetilde{C}_\alpha \mid x\in C'_\alpha,\ |u(x)|\geqslant 1\}} \big|u(x)\big|^p \,dx \\
&+ \int\limits_{\{x\in \widetilde{C}_\alpha \mid x\in C'_\alpha,\ |u(x)|\leqslant 1\}} \big|u(x)\big|^q \,dx \Bigg].
\end{aligned}
$$

Since for all $x \in M$ the sum of the $\psi_\alpha(x)$ is one, there exists $\hat{\alpha}$ such that $x \in C'_\alpha$. Then for any $u \in L^1(M)$

$$
\begin{aligned}
\sum_{\alpha=1}^k \int\limits_{C'_\alpha \cap M} \big|u(x)\big| \,dx &= \sum_{\alpha=1}^k \int\limits_M \chi_{C'_\alpha}(x) \big|u(x)\big| \,dx = \int\limits_M \left(\sum_{\alpha=1}^k \chi_{C'_\alpha}(x) \right) \big|u(x)\big| \,dx \\
&\geqslant \int\limits_M \big|u(x)\big| \,dx.
\end{aligned}
$$

This means that

$$
\begin{aligned}
&\sum_{\alpha=1}^k \Bigg[\int\limits_{\{x\in \widetilde{C}_\alpha \mid x\in C'_\alpha,\ |u(x)|\geqslant 1\}} \big|u(x)\big|^p \,dx + \int\limits_{\{x\in \widetilde{C}_\alpha \mid x\in C'_\alpha,\ |u(x)|\leqslant 1\}} \big|u(x)\big|^q \,dx \Bigg] \\
&\geqslant \int\limits_{\{x\in M \mid |u(x)|\geqslant 1\}} \big|u(x)\big|^p \,dx + \int\limits_{\{x\in M \mid |u(x)|\leqslant 1\}} \big|u(x)\big|^q \,dx \\
&\geqslant \frac{1}{c_1} \int\limits_M f''\big(u(x)\big)\big(u(x)\big)^2 \,dx > \frac{\mu}{c_1} \int\limits_M f\big(u(x)\big) \,dx \geqslant \frac{\mu}{c_1 H^{\frac{n}{2}}} \int\limits_M f\big(u(x)\big) \,d\mu_g.
\end{aligned}
$$

Inequality (5.6). For $s > 0$ $f(s)$ is increasing. Then we have

$$\begin{aligned}
\int_{M_r} f\big(v(y)\big)\,dy &< \frac{c_1}{c_0\mu}\int_{M_r} f\big(\big|v(y)\big|\big)\,dy \leqslant \frac{c_1}{c_0\mu}\int_{M_r} f\left(\sum_{\alpha=1}^{k}\big|v_\alpha(y)\big|\right)dy \\
&\leqslant \frac{c_1}{c_0\mu}\int_{M_r} f\left(\sum_{\alpha=1}^{k}\big|\psi_\alpha(y)u\big(\xi_\alpha\big(z_\alpha(y),0\big)\big)\big|\right)dy \\
&= \frac{c_1}{c_0\mu}\sum_{\beta=1}^{k}\int_{C_\beta}\psi_\beta(y) f\left(\sum_{\alpha=1}^{k}\big|\psi_\alpha(y)u\big(\xi_\alpha\big(z_\alpha(y),0\big)\big)\big|\right)dy \\
&\leqslant \frac{c_1C_3}{c_0\mu}\sum_{\beta=1}^{k}\int_{D_\beta} f\left(\sum_{\alpha=1}^{k}\big|\chi_{D_\alpha}(z,t)u\big(\xi_\alpha(z,0)\big)\big|\right)dz\,dt \\
&\leqslant \frac{c_1C_3C_9}{c_0\mu}\sum_{\beta=1}^{k}\int_{Z_\beta} f\left(\sum_{\alpha=1}^{k}\big|\chi_{Z_\alpha}(z)u\big(\xi_\alpha(z,0)\big)\big|\right)dz,
\end{aligned}$$

where C_9 is the volume of the ball of radius ρ in $\mathbb{R}^{N-n}$. Proceeding with the chain of inequalities we obtain

$$\begin{aligned}
&\sum_{\beta=1}^{k}\int_{Z_\beta} f\left(\sum_{\alpha=1}^{k}\big|\chi_{Z_\alpha}(z)u\big(\xi_\alpha(z,0)\big)\big|\right)dz \\
&\quad= \sum_{\beta=1}^{k}\int_{\widetilde{C}_\beta} f\left(\sum_{\alpha=1}^{k}\big|\chi_{\widetilde{C}_\alpha}(x)u(x)\big|\right)dx \\
&\quad\leqslant k\int_{M} f\big(k\big|u(x)\big|\big)\,dx \\
&\quad< \frac{kc_1}{\mu}\left[\int_{\{x\in M|k|u(x)|\geqslant 1\}} k^p\big|u(x)\big|^p\,dx + \int_{\{x\in M|k|u(x)|\leqslant 1\}} k^q\big|u(x)\big|^q\,dx\right] \\
&\quad= \frac{kc_1}{\mu}\left[\int_{\{x\in M||u(x)|\geqslant 1\}} k^p\big|u(x)\big|^p\,dx + \int_{\{x\in M||u(x)|\leqslant 1\}} k^q\big|u(x)\big|^q\,dx\right. \\
&\qquad\left. + \int_{\{x\in M||u(x)|\leqslant 1,\ k|u(x)|\geqslant 1\}} k^p\big|u(x)\big|^p\,dx - \int_{\{x\in M||u(x)|\leqslant 1,\ k|u(x)|\geqslant 1\}} k^q\big|u(x)\big|^q\,dx\right]
\end{aligned}$$

$$\leqslant \frac{kc_1}{\mu}\Bigg[\int\limits_{\{x\in M \mid |u(x)|\geqslant 1\}} k^p\big|u(x)\big|^p\,dx + \int\limits_{\{x\in M \mid |u(x)|\leqslant 1\}} k^q\big|u(x)\big|^q\,dx\Bigg]$$

$$\leqslant \frac{k^{q+1}c_1}{c_0\mu}\int\limits_M f\big(u(x)\big)\,dx \leqslant \frac{k^{q+1}c_1}{c_0\mu h^{\frac{n}{2}}}\int\limits_M f\big(u(x)\big)\,d\mu_g.$$

Inequality (5.7). The proof is analogous to the proof of (5.5). $\square$

We complete now the proof of Proposition 5.4.

Proof of Eq. (5.20). The following inequalities hold:

$$\frac{1}{\epsilon_k^n}\int\limits_B \big|f''\big(u_k^*\big)u_k^* - f'\big(u_k^*\big)\big|^{\frac{2n}{n+2}}\,d\mu_g$$

$$\leqslant \frac{2^{\frac{2n}{n+2}}}{\epsilon_k^n}\int\limits_B \big(\big|f''\big(u_k^*\big)u_k^*\big|^{\frac{2n}{n+2}} + \big|f'\big(u_k^*\big)\big|^{\frac{2n}{n+2}}\big)\,d\mu_g$$

$$< \frac{2(2c_1)^{\frac{2n}{n+2}}}{\epsilon_k^n}\Bigg(\int\limits_{\{x\in B \mid |u_k^*(x)|\geqslant 1\}} \big|u_k^*(x)\big|^{\frac{(p-1)2n}{n+2}}\,d\mu_g + \int\limits_{\{x\in B \mid |u_k^*(x)|\leqslant 1\}} \big|u_k^*(x)\big|^{\frac{(q-1)2n}{n+2}}\,d\mu_g\Bigg)$$

$$\leqslant \frac{2(2c_1)^{\frac{2n}{n+2}}}{\epsilon_k^n}\Bigg(\int\limits_{\{x\in B \mid |u_k^*(x)|\geqslant 1\}} \big|u_k^*(x)\big|^p\,d\mu_g + \int\limits_{\{x\in B \mid |u_k^*(x)|\leqslant 1\}} \big|u_k^*(x)\big|^q\,d\mu_g\Bigg),$$

where in the last inequality we have used the fact that $\frac{(p-1)2n}{n+2} < p$ and $\frac{(q-1)2n}{n+2} > q$. We can write

$$\int\limits_{\{x\in B \mid |u_k^*(x)|\geqslant 1\}} \big|u_k^*(x)\big|^p\,d\mu_g + \int\limits_{\{x\in B \mid |u_k^*(x)|\leqslant 1\}} \big|u_k^*(x)\big|^q\,d\mu_g$$

$$= \int\limits_{\{x\in B \mid |u_k^*(x)|\geqslant 1,\ |\tilde{u}_k(x)|\geqslant 1,\ |u_k(x)|\geqslant 1\}} \big|u_k^*(x)\big|^p\,d\mu_g$$

$$+ \int\limits_{\{x\in B \mid |u_k^*(x)|\geqslant 1,\ |\tilde{u}_k(x)|\leqslant 1,\ |u_k(x)|\leqslant 1\}} \big|u_k^*(x)\big|^p\,d\mu_g$$

$$+ \int\limits_{\{x\in B \mid |u_k^*(x)|\geqslant 1,\ |\tilde{u}_k(x)|\geqslant 1,\ |u_k(x)|\leqslant 1\}} \big|u_k^*(x)\big|^p\,d\mu_g$$

$$+ \int\limits_{\{x\in B \mid |u_k^*(x)|\geqslant 1,\ |\tilde{u}_k(x)|\leqslant 1,\ |u_k(x)|\geqslant 1\}} \big|u_k^*(x)\big|^p\,d\mu_g$$

$$+ \int\limits_{\{x\in B \mid |u_k^*(x)|\leqslant 1,\ |\tilde{u}_k(x)|\geqslant 1,\ |u_k(x)|\geqslant 1\}} \left|u_k^*(x)\right|^q d\mu_g$$

$$+ \int\limits_{\{x\in B \mid |u_k^*(x)|\leqslant 1,\ |\tilde{u}_k(x)|\leqslant 1,\ |u_k(x)|\leqslant 1\}} \left|u_k^*(x)\right|^q d\mu_g$$

$$+ \int\limits_{\{x\in B \mid |u_k^*(x)|\leqslant 1,\ |\tilde{u}_k(x)|\geqslant 1,\ |u_k(x)|\leqslant 1\}} \left|u_k^*(x)\right|^q d\mu_g$$

$$+ \int\limits_{\{x\in B \mid |u_k^*(x)|\leqslant 1,\ |\tilde{u}_k(x)|\leqslant 1,\ |u_k(x)|\geqslant 1\}} \left|u_k^*(x)\right|^q d\mu_g$$

$$\leqslant \int\limits_{\{x\in B \mid |u_k^*(x)|\geqslant 1,\ |\tilde{u}_k(x)|\geqslant 1,\ |u_k(x)|\geqslant 1\}} \left|u_k^*(x)\right|^p d\mu_g$$

$$+ \int\limits_{\{x\in B \mid |u_k^*(x)|\geqslant 1,\ |\tilde{u}_k(x)|\leqslant 1,\ |u_k(x)|\leqslant 1\}} \left|u_k^*(x)\right|^q d\mu_g$$

$$+ \int\limits_{\{x\in B \mid |u_k^*(x)|\geqslant 1,\ |\tilde{u}_k(x)|\geqslant 1,\ |u_k(x)|\leqslant 1\}} \left|u_k^*(x)\right|^p d\mu_g$$

$$+ \int\limits_{\{x\in B \mid |u_k^*(x)|\geqslant 1,\ |\tilde{u}_k(x)|\leqslant 1,\ |u_k(x)|\geqslant 1\}} \left|u_k^*(x)\right|^p d\mu_g$$

$$+ \int\limits_{\{x\in B \mid |u_k^*(x)|\leqslant 1,\ |\tilde{u}_k(x)|\geqslant 1,\ |u_k(x)|\geqslant 1\}} \left|u_k^*(x)\right|^p d\mu_g$$

$$+ \int\limits_{\{x\in B \mid |u_k^*(x)|\leqslant 1,\ |\tilde{u}_k(x)|\leqslant 1,\ |u_k(x)|\leqslant 1\}} \left|u_k^*(x)\right|^q d\mu_g$$

$$+ \int\limits_{\{x\in B \mid |u_k^*(x)|\leqslant 1,\ |\tilde{u}_k(x)|\geqslant 1,\ |u_k(x)|\leqslant 1\}} \left|u_k^*(x)\right|^p d\mu_g$$

$$+ \int\limits_{\{x\in B \mid |u_k^*(x)|\leqslant 1,\ |\tilde{u}_k(x)|\leqslant 1,\ |u_k(x)|\geqslant 1\}} \left|u_k^*(x)\right|^p d\mu_g$$

$$\leqslant \int\limits_{\{x\in B \mid |u_k^*(x)|\geqslant 1,\ |\tilde{u}_k(x)|\geqslant 1,\ |u_k(x)|\geqslant 1\}} 2^p\left(\left|\tilde{u}_k(x)\right|^p + \left|u_k(x)\right|^p\right) d\mu_g$$

$$+ \int\limits_{\{x\in B \mid |u_k^*(x)|\geqslant 1,\ |\tilde{u}_k(x)|\leqslant 1,\ |u_k(x)|\leqslant 1\}} 2^q\left(\left|\tilde{u}_k(x)\right|^q + \left|u_k(x)\right|^q\right) d\mu_g$$

$$
\begin{aligned}
&+ \int\limits_{\{x\in B\,||u_k^*(x)|\geqslant 1,\ |\tilde u_k(x)|\geqslant 1,\ |u_k(x)|\leqslant 1\}} 2^p \left|\tilde u_k(x)\right|^p d\mu_g \\
&+ \int\limits_{\{x\in B\,||u_k^*(x)|\geqslant 1,\ |\tilde u_k(x)|\leqslant 1,\ |u_k(x)|\geqslant 1\}} 2^p \left|u_k(x)\right|^p d\mu_g \\
&+ \int\limits_{\{x\in B\,||u_k^*(x)|\leqslant 1,\ |\tilde u_k(x)|\geqslant 1,\ |u_k(x)|\geqslant 1\}} 2^p \left(\left|\tilde u_k(x)\right|^p + \left|u_k(x)\right|^p\right) d\mu_g \\
&+ \int\limits_{\{x\in B\,||u_k^*(x)|\leqslant 1,\ |\tilde u_k(x)|\leqslant 1,\ |u_k(x)|\leqslant 1\}} 2^q \left(\left|\tilde u_k(x)\right|^q + \left|u_k(x)\right|^q\right) d\mu_g \\
&+ \int\limits_{\{x\in B\,||u_k^*(x)|\leqslant 1,\ |\tilde u_k(x)|\geqslant 1,\ |u_k(x)|\leqslant 1\}} 2^p \left|\tilde u_k(x)\right|^p d\mu_g \\
&+ \int\limits_{\{x\in B\,||u_k^*(x)|\leqslant 1,\ |\tilde u_k(x)|\leqslant 1,\ |u_k(x)|\geqslant 1\}} 2^p \left|u_k(x)\right|^p d\mu_g \\
\leqslant\ & \int\limits_{\{x\in B\,||\tilde u_k(x)|\geqslant 1\}} 2^p \left|\tilde u_k(x)\right|^p d\mu_g + \int\limits_{\{x\in B\,||\tilde u_k(x)|\leqslant 1\}} 2^q \left|\tilde u_k(x)\right|^q d\mu_g \\
&+ \int\limits_{\{x\in B\,||u_k(x)|\geqslant 1\}} 2^p \left|u_k(x)\right|^p d\mu_g + \int\limits_{\{x\in B\,||u_k(x)|\leqslant 1\}} 2^q \left|u_k(x)\right|^q d\mu_g \\
\leqslant\ & \frac{2^q}{c_0} \int\limits_M \left[f(\tilde u_k) + f(u_k)\right] d\mu_g.
\end{aligned}
$$

Concluding there exists a constant $C > 0$ such that

$$
\begin{aligned}
\frac{1}{\epsilon_k^n} \int\limits_B \left|f''\left(u_k^*\right)u_k^* - f'\left(u_k^*\right)\right|^{\frac{2n}{n+2}} d\mu_g &< \frac{C}{\epsilon_k^n} \int\limits_M \left[f(\tilde u_k) + f(u_k)\right] d\mu_g \\
&\leqslant \frac{2C}{(\mu-2)}\left[J_{\epsilon_k}(\tilde u_k) + J_{\epsilon_k}(\tilde u_k)\right] \leqslant \frac{8Cm(J)}{(\mu-2)}
\end{aligned}
$$

and this completes the proof of (5.20). □

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