

ON THE EXISTENCE AND THE ENUMERATION OF BIPARTITE REGULAR REPRESENTATIONS OF CAYLEY GRAPHS OVER ABELIAN GROUPS

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ABSTRACT. In this paper we are interested in the asymptotic enumeration of bipartite Cayley digraphs and Cayley graphs over abelian groups. Let A be an abelian group and let ι be the automorphism of A defined by $a^\iota = a^{-1}$, for every $a \in A$. A Cayley graph $\text{Cay}(A, S)$ is said to have an automorphism group as small as possible if $\text{Aut}(\text{Cay}(A, S)) = \langle A, \iota \rangle$. In this paper, we show that, except for two infinite families, almost all bipartite Cayley graphs on abelian groups have automorphism group as small as possible. We also investigate the analogous question for bipartite Cayley digraphs.

These results are used for the asymptotic enumeration of bipartite Cayley digraphs and graphs over abelian groups.

Keywords regular representation, DRR, GRR, bipartite (di)graph, Cayley digraph, automorphism group, Cayley index

1. INTRODUCTION

All digraphs and groups considered in this paper are finite. Let G be a group and let S be a subset of G . The *Cayley digraph* on G with connection set S , denoted $\text{Cay}(G, S)$, is the digraph with vertex-set G and with (g, h) being an arc if and only if $gh^{-1} \in S$. It is easy to see that $\text{Cay}(G, S)$ is a graph if and only if S is inverse-closed (that is, $S^{-1} := \{s^{-1} \mid s \in S\} = S$), in which case it is called a *Cayley graph*. It is also easy to check that G acts regularly as a group of automorphisms on $\text{Cay}(G, S)$ by right multiplication. Therefore, in what follows, we always identify G as a subgroup of the automorphism group $\text{Aut}(\text{Cay}(G, S))$ of $\text{Cay}(G, S)$.

In the extreme case, when G equals $\text{Aut}(\text{Cay}(G, S))$, $\text{Cay}(G, S)$ is called a *DRR* (for *digraphical regular representation*). A DRR which is a graph is called a *GRR* (for *graphical regular representation*). DRRs and GRRs have been widely studied. There are two natural questions on DRRs (and on GRRs):

- which groups admit a DRR (or a GRR)?
- when the size of G tends to infinity, what is the probability that a Cayley digraph (respectively, graph) over G is a DRR (respectively, GRR)?

We do not intend to give here a full account on the study of GRRs, this involved many researchers and papers. Some of the most influential work along the way is due to Babai, Godsil, Hetzel, Imrich, Notwitz and Watkins (to name a few), see [12, 20, 21, 29, 30, 31].

The answer to the first question for DRRs was given by Babai [1]. The analogous answer for GRRs turns out to be considerably harder and was completed by Godsil [8], after a long series of partial results by various authors, see [10, 11, 20] for example. Once this has been established for DRRs and GRRs, it continued to be considered for a large variety of natural Cayley (di)graphs: for instance, oriented regular representations [16, 17, 23], tournament regular representations [2], graphical Frobenius representations [4, 5, 25, 26], and graphical representations of small valency [24, 27, 28].

The second question seems dramatically harder and it has been touched only in a few particular cases and in peculiar situations, see [6, 15]. (There are some recent results on the asymptotic

enumeration of DRRs in [18].) The asymptotic enumeration of vertex-transitive graphs seems also rather difficult, and we refer the interested reader to the seminal work of McKay and Praeger [14] and to [22] for some more recent results on vertex-transitive graphs of fixed valency.

The general aim of this paper is to understand and construct bipartite DRRs and bipartite GRRs. The standard techniques developed in [1, 8, 10] involving a local analysis on the neighborhood of a Cayley (di)graph do not seem to work for bipartite graphs, because the neighborhood of a bipartite graph is the empty graph, which brings little or no information. Therefore, in our paper, we start our investigation by considering bipartite Cayley digraphs and graphs over abelian groups: this allows us to apply the group-theoretic techniques in [6].

Theorem 1.1. *Let A be an abelian group and let B be a subgroup of A having index 2. The number of subsets S of $A \setminus B$ such that $\text{Cay}(A, S)$ is a bipartite DRR is at least $2^{\frac{|A|}{2}} - 3 \cdot 2^{\frac{3|A|}{8} + (\log_2 |A|)^2}$.*

Since $A \setminus B$ has $2^{|A \setminus B|} = 2^{\frac{|A|}{2}}$ subsets, from Theorem 1.1 we immediately obtain the following corollary.

Corollary 1.2. *For every positive real number $\varepsilon > 0$, there exists a natural number n_ε such that, for every abelian group A of order at least n_ε and for every subgroup B of A having index 2, we have*

$$\frac{|\{S \mid S \subseteq A \setminus B, \text{Cay}(A, S) \text{ is a DRR}\}|}{|\{S \mid S \subseteq A \setminus B\}|} \geq 1 - \varepsilon.$$

Broadly speaking Corollary 1.2 says that, when the order of an abelian group A is even and sufficiently large, most bipartite Cayley digraphs over A are DRRs. It worth stressing that Corollary 1.2 says something slightly stronger, that is, if a subgroup B of A of index 2 is given in advance, most Cayley digraphs over A with bipartition $\{B, A \setminus B\}$ are DRRs. The difference seems rather subtle, but it is remarkably important for undirected graphs as we will discuss in detail later, see Theorems 1.8 and 1.10.

Corollary 1.3. *Let A be an abelian group of even order and let $\mathcal{S} := \{S \subseteq A \mid \text{Cay}(A, S) \text{ bipartite}\}$. The proportion of subsets S of \mathcal{S} such that $\text{Cay}(A, S)$ is a bipartite DRR tends to 1 as $|A| \rightarrow \infty$.*

Given a positive integer t , we denote by C_t the cyclic group of order t . Since the estimate in Theorem 1.1 is rather explicit, we also obtain the following corollary.

Corollary 1.4. *Let A be an abelian group and let B be a subgroup of A having index 2. Then, either there exists a subset S of $A \setminus B$ such that $\text{Cay}(A, S)$ is a bipartite DRR or the pair (A, B) is in Table 1.*

A	B	directed bipartite Cayley index
$C_2 \times C_2$	C_2	2
$C_2 \times C_2 \times C_2$	$C_2 \times C_2$	6
$C_2 \times C_2 \times C_2 \times C_2$	$C_2 \times C_2 \times C_2$	24
$C_2 \times C_2 \times C_2 \times C_2 \times C_2$	$C_2 \times C_2 \times C_2 \times C_2$	72
$C_2 \times C_2 \times C_2 \times C_2 \times C_2 \times C_2$	$C_2 \times C_2 \times C_2 \times C_2 \times C_2$	4
$C_3 \times C_6$	$C_3 \times C_3$	2
$C_4 \times C_2 \times C_2 \times C_2$	$C_2 \times C_2 \times C_2 \times C_2$	4
$C_4 \times C_2 \times C_2$	$C_2 \times C_2 \times C_2$	4
$C_4 \times C_2 \times C_2$	$C_4 \times C_2$	2
$C_4 \times C_2$	$C_2 \times C_2$	2

TABLE 1. Abelian groups and their index 2 subgroups not admitting a bipartite DRR

In line with the work of Morris and Tymburski [19] and with the pioneering work of Imrich and Watkins [12], we have included a third column in Table 1, whose meaning we now explain. Let G be a group, let S be a subset of G and let $\Gamma := \text{Cay}(G, S)$ be the Cayley digraph over G with connection set S . The *Cayley index* $c(\Gamma)$ of the digraph Γ is $|\text{Aut}(\Gamma) : G|$. Therefore $c(\Gamma)$ measures the degree of symmetry of a Cayley digraph; intuitively, the larger $c(\Gamma)$ is, the more symmetric Γ is. Moreover, $c(\Gamma)$ is somehow unbiased with respect to the number of vertices of Γ . (We observe here that Morris and Tymburski define and consider the Cayley index only for undirected graphs, see [19, Definition 1.1].) Following the line of research of Imrich and Watkins, we give the following definition.

Definition 1.5. Let G be a group and let B be a subgroup of G having index 2. The *directed bipartite Cayley index* $\vec{c}(G, B)$ of (G, B) is

$$\vec{c}(G, B) := \min_{S \subseteq G \setminus B} |\text{Aut}(\text{Cay}(G, S)) : G|.$$

We also define, for groups admitting an index 2 subgroup, the *global directed bipartite Cayley index*

$$\vec{c}_b(G) := \min_{B \leq G, |G:B|=2} \vec{c}(G, B).$$

In the light of Definition 1.5, Corollary 1.4 says that, except for the ten exceptions in Table 1, $\vec{c}(A, B) = 1$ for every abelian group A and for every subgroup B of A having index 2. In the third column of Table 1, we determine the directed bipartite Cayley index for the ten exceptional pairs.

We also prove the following unlabeled version of Theorem 1.1.

Theorem 1.6. *Let A be an abelian group and let B be a subgroup of A having index 2. Then, the number of bipartite Cayley digraphs (up to graph-isomorphism) over A with bipartition $\{B, A \setminus B\}$ is at least $2^{\frac{|A|}{2} - (\log_2 |A|)^2} - 3 \cdot 2^{\frac{3|A|}{8}}$. Moreover, among all bipartite Cayley digraphs (up to graph-isomorphism) over A with bipartition $\{B, A \setminus B\}$, the proportion that are DRRs tends to 1 as $|A| \rightarrow \infty$.*

In this paper we also consider bipartite Cayley graphs over abelian groups A . We denote by $\iota : A \rightarrow A$ the automorphism of the abelian group A mapping each element to its inverse, that is, $a^\iota = a^{-1}$ for every $a \in A$. Clearly, ι is the identity mapping when A has exponent 2, and ι is an involutory automorphism when A has exponent greater than 2. When A has exponent greater than 2 no Cayley graph is a GRR, because ι is a non-identity graph automorphism. Therefore, we are interested in bipartite Cayley graphs $\text{Cay}(A, S)$ having automorphism group “as small as possible”, that is, $\text{Aut}(\text{Cay}(A, S)) = \langle A, \iota \rangle$. We formalize this idea in the following definition.

Definition 1.7. Let G be a group and let B be a subgroup of G having index 2. The *bipartite Cayley index* $c(G, B)$ of (G, B) is

$$c(G, B) := \min_{\substack{S \subseteq G \setminus B \\ S = S^{-1}}} |\text{Aut}(\text{Cay}(G, S)) : G|.$$

We define, for groups admitting an index 2 subgroup, the *global bipartite Cayley index*

$$c_b(G) := \min_{B \leq G, |G:B|=2} c(G, B).$$

When A is abelian of exponent greater than 2, $c(A, B) \geq 2$ for every subgroup B of A having index 2. Moreover, $c(G, B) \geq c(G)$, where $c(G)$ is the Cayley index of G as defined in [19, Definition 1.1].

Theorem 1.8. *If $A \cong C_4 \times C_2^\ell$ for some $\ell \geq 1$ and $B \cong C_2^{\ell+1}$, or $A \cong C_4^2 \times C_2^\ell$ for some $\ell \geq 0$ and $B \cong C_4 \times C_2^{\ell+1}$, then there exists no subset $S \subseteq A \setminus B$ such that $\text{Cay}(A, S)$ is a bipartite Cayley graph with bipartite Cayley index 2.*

In particular Theorem 1.8 shows that there exist two infinite families with $c(A, B) > 2$. This behavior is a novelty compared with the statement of Theorem 1.1.

Problem 1.9. Determine the bipartite Cayley index for the two exception families in Theorem 1.8. That is, determine $c(A, B)$, where $A \cong C_4 \times C_2^\ell$ for some $\ell \geq 1$ and $B \cong C_2^{\ell+1}$, or $A \cong C_4^2 \times C_2^\ell$ for some $\ell \geq 0$ and $B \cong C_4 \times C_2^{\ell+1}$.

Observe that Theorem 1.8 does not make any claim on the index 2 subgroups B' of A different from the subgroup B in the statement. Among other things, this point is cleared in the next result. (Given an abelian group A and $a \in A$, we denote by $o(a)$ the order of a and by A_2 the subgroup $A_2 := \{a \in A \mid a^2 = 1\}$.)

Theorem 1.10. *Let A be an abelian group and let B be a subgroup of A having index 2. Let $c := 1$ when A has exponent 2 and let $c := 2$ when A has exponent greater than 2. Then A contains $2^{\frac{|A|}{4} + \frac{|A_2 \setminus B|}{2}}$ inverse-closed subsets S with $S \subseteq A \setminus B$. Moreover, one of the following holds:*

- (1) *the number of inverse-closed subsets S of $A \setminus B$ such that $\text{Cay}(A, S)$ has bipartite Cayley index c is at least $2^{\frac{|A|}{4} + \frac{|A_2 \setminus B|}{2}} - 2^{\frac{11|A|}{48} + \frac{|A_2 \setminus B|}{2} + (\log_2 |A|)^2 + 2}$,*
- (2) *$A \cong C_4 \times C_2^\ell$ for some $\ell \geq 1$ and $B \cong C_2^{\ell+1}$, or $A \cong C_4^2 \times C_2^\ell$ for some $\ell \geq 0$ and $B \cong C_4 \times C_2^{\ell+1}$.*

Theorem 1.10 shows that the pairs in the statement of Theorem 1.8 are the only exceptional pairs and, more importantly, for any other possible pair (A, B) , the number of ‘‘highly symmetric’’ subsets (that is, inverse-closed subsets $S \subseteq A \setminus B$ with $\text{Cay}(A, S)$ not having Cayley index 2) is bounded above by a relatively slow growing function.

From Theorem 1.10 we immediately obtain the following analogue of Corollary 1.2.

Corollary 1.11. *For every positive real number $\varepsilon > 0$, there exists a natural number n_ε such that, for every abelian group A of order at least n_ε and for every subgroup B of A having index 2 and with (A, B) not one of the pairs in Theorem 1.8 (or in Theorem 1.10 (2)), we have*

$$\frac{|\{S \mid S \subseteq A \setminus B, S = S^{-1}, \text{Cay}(A, S) \text{ has Cayley index } c\}|}{|\{S \mid S \subseteq A \setminus B, S = S^{-1}\}|} \geq 1 - \varepsilon,$$

where $c := 1$ when A has exponent 2 and $c := 2$ when A has exponent greater than 2.

Exactly as for Corollary 1.2, Corollary 1.11 says that, when the order of an abelian group A is even and sufficiently large, most bipartite Cayley graphs over A have bipartite Cayley index 2, aside from the two exceptional pairs described in Theorem 1.8.

Since the estimate in Theorem 1.10 is rather explicit, we also obtain the following corollary.

Corollary 1.12. *Let A be an abelian group and let B be a subgroup of A having index 2. Let $c := 1$ when A has exponent 2 and let $c := 2$ when A has exponent greater than 2. Then, either there exists an inverse-closed subset S of $A \setminus B$ such that $\text{Cay}(A, S)$ has Cayley index c or the pair (A, B) is in Table 2.*

In the third column of Table 2, we have computed $c(A, B)$, except for the two infinite pairs arising from Theorem 1.8.

We also prove the following unlabelled version of Theorem 1.10.

Theorem 1.13. *Let A be an abelian group and let B be a subgroup of A having index 2. Suppose that (A, B) is not one of the pairs in Theorem 1.8. Then, the number of bipartite Cayley graphs (up to graph-isomorphism) over A with bipartition $\{B, A \setminus B\}$ is $2^{\frac{|A|}{4} + \frac{|A_2 \setminus B|}{2} + o(|A|)}$.*

Based on the work in this paper and on some computer computations, we dare to make the following two conjectures.

Conjecture 1.14. *There exists a positive integer n such that, if G is a group of order at least n and B is a subgroup of G having index 2, then $\check{c}(G, B) = 1$.*

A	B	bipartite Cayley index
$C_4 \times C_2^\ell$	$C_2^{\ell+1}$	not known for $\ell \geq 4$
$C_4 \times C_4 \times C_2^\ell$	$C_4 \times C_2^{\ell+1}$	not known for $\ell \geq 2$
$C_2 \times C_2 \times C_2$	$C_2 \times C_2$	6
$C_2 \times C_2 \times C_2 \times C_2$	$C_2 \times C_2 \times C_2$	24
$C_2 \times C_2 \times C_2 \times C_2 \times C_2$	$C_2 \times C_2 \times C_2 \times C_2$	72
$C_2 \times C_2 \times C_2 \times C_2 \times C_2 \times C_2$	$C_2 \times C_2 \times C_2 \times C_2 \times C_2$	4
$C_2 \times C_4$	C_4	6
$C_2 \times C_4$	$C_2 \times C_2$	16
$C_2 \times C_8$	$C_2 \times C_4$	16
$C_4 \times C_4$	$C_4 \times C_2$	24
$C_4 \times C_2 \times C_2$	$C_2 \times C_2 \times C_2$	768
$C_4 \times C_2 \times C_2$	$C_4 \times C_2$	24
$C_3 \times C_6$	$C_3 \times C_3$	8
$C_2 \times C_{12}$	$C_2 \times C_6$	4
$C_2 \times C_2 \times C_6$	$C_2 \times C_6$	4
$C_4 \times C_8$	$C_4 \times C_4$	4
$C_4 \times C_8$	$C_2 \times C_8$	4
$C_2 \times C_2 \times C_8$	$C_2 \times C_2 \times C_4$	12
$C_2 \times C_4 \times C_4$	$C_4 \times C_4$	12
$C_2 \times C_4 \times C_4$	$C_2 \times C_2 \times C_4$	128
$C_2 \times C_2 \times C_2 \times C_4$	$C_2 \times C_2 \times C_2 \times C_2$	786 432
$C_2 \times C_2 \times C_2 \times C_4$	$C_2 \times C_2 \times C_4$	72
$C_3 \times C_{12}$	$C_3 \times C_6$	4
$C_2 \times C_2 \times C_{12}$	$C_2 \times C_2 \times C_6$	4
$C_3 \times C_3 \times C_6$	$C_3 \times C_3 \times C_3$	12
$C_2 \times C_2 \times C_2 \times C_8$	$C_2 \times C_2 \times C_2 \times C_4$	8
$C_4 \times C_4 \times C_4$	$C_2 \times C_4 \times C_4$	4
$C_2 \times C_2 \times C_2 \times C_4$	$C_2 \times C_2 \times C_2 \times C_4$	4

TABLE 2. Abelian groups and their index 2 subgroups not admitting a bipartite Cayley graph with Cayley index 2

Conjecture 1.15. There exists a positive integer n such that, if G is a group of order at least n and B is a subgroup of G having index 2, then either

- $c(G, B) = 1$, or
- there exists $\alpha \in \text{Aut}(G)$ with $\alpha \neq 1$, $B^\alpha = B$ and $G = B \cup \{g \in G \mid g^\alpha = g\} \cup \{g \in G \mid g^\alpha = g^{-1}\}$.

Clearly, the groups satisfying the second condition in Conjecture 1.15 include all abelian groups. At the time of this writing, we are not sure if the groups satisfying the second condition in Conjecture 1.15 might have a meaningful and useful classification. However, we observe that the work of Fitzpatrick, Hegarty, Liebeck and MacHale [7, 9, 13] on groups admitting automorphisms inverting many elements seems to be relevant.

2. PRELIMINARY FACTS

In what follows we use repeatedly the following facts.

- (1) Let X be a finite group. Since a chain of subgroups of X has length at most $\lfloor \log_2 |X| \rfloor$, X has a generating set of cardinality at most $\lfloor \log_2 |X| \rfloor \leq \log_2 |X|$.

- (2) Any automorphism of X is uniquely determined by the images of the elements of a generating set for X . Therefore $|\text{Aut}(X)| \leq |X|^{\lfloor \log_2 |X| \rfloor} \leq 2^{(\log_2 |X|)^2}$.
- (3) Any subgroup Y of X is determined by a generating set, which has cardinality at most $\lfloor \log_2 |Y| \rfloor \leq \lfloor \log_2 |X| \rfloor$. Therefore X has at most $|X|^{\lfloor \log_2 |X| \rfloor} \leq 2^{(\log_2 |X|)^2}$ subgroups.
- (4) Let A be an abelian group. Then A has at most $|A|$ subgroups H with $|H|$ a prime number. Similarly, A has at most $|A|$ subgroups K with $|A : K|$ a prime number.
- (5) Let X be a finite group, let Y be a subgroup of X of index 2 and let Z be a proper subgroup of X . If $Z \leq Y$, then $|Z \setminus Y| = 0$. If $Z \not\leq Y$, then $|Z \setminus Y| = |Z|/2 \leq (|X|/2)/2 = |X|/4$. Therefore, in either case, $|Z \setminus Y| \leq |X|/4$.

3. EXISTENCE AND ASYMPTOTIC ENUMERATION OF BIPARTITE CAYLEY DIGRAPHS

Lemma 3.1. *Let A be an abelian group and let B be a subgroup of A having index 2. The number of subsets S of $A \setminus B$ with $\langle S \rangle$ a proper subgroup of A is at most $2^{\frac{|A|}{4} + \log_2 |A|}$.*

Proof. Set $N := |\{S \subseteq A \setminus B \mid \langle S \rangle < A\}|$. Clearly,

$$\{S \subseteq A \setminus B \mid \langle S \rangle < A\} = \bigcup_{\substack{C < A \\ |A:C| \text{ prime}}} \{S \subseteq A \setminus B \mid \langle S \rangle \leq C\}.$$

Since $\{S \subseteq A \setminus B \mid \langle S \rangle \leq C\} = \{S \mid S \subseteq C \setminus (C \cap B)\}$, we have

$$N \leq \sum_{\substack{C < A \\ |A:C| \text{ prime}}} |\{S \mid S \subseteq C \setminus (C \cap B)\}| \leq \sum_{\substack{C < A \\ |A:C| \text{ prime}}} 2^{\frac{|A|}{4}} \leq 2^{\frac{|A|}{4} + \log_2 |A|},$$

where in the second and in the third inequality we used the facts listed in Section 2. \square

Lemma 3.2. *Let A be a group, let B be a subgroup of A having index 2 and let α be a non-identity automorphism of A with $B^\alpha = B$. The number of subsets S of $A \setminus B$ with $S^\alpha = S$ is at most $2^{\frac{3|A|}{8}}$.*

Proof. Since B is α -invariant, so is $A \setminus B$. Let O_1, \dots, O_ℓ be the orbits of $\langle \alpha \rangle$ on $A \setminus B$. If $S \subseteq A \setminus B$ is α -invariant, then S is a union of some of O_1, \dots, O_ℓ and hence

$$(3.1) \quad |\{S \subseteq A \setminus B \mid S^\alpha = S\}| = 2^\ell.$$

The orbits of $\langle \alpha \rangle$ on A of cardinality one correspond exactly to the elements of $\mathbf{C}_A(\alpha) := \{a \in A \mid a^\alpha = a\}$, whereas the orbits of $\langle \alpha \rangle$ on $A \setminus \mathbf{C}_A(\alpha)$ have cardinality at least 2. Now, observing that $|\mathbf{C}_A(\alpha)| \leq |A|/2$ and that

$$|\mathbf{C}_A(\alpha) \cap (A \setminus B)| = \begin{cases} 0 & \text{when } \mathbf{C}_A(\alpha) \leq B, \\ |\mathbf{C}_A(\alpha) \cap B| = |\mathbf{C}_A(\alpha)|/2 & \text{when } \mathbf{C}_A(\alpha) \not\leq B, \end{cases}$$

we get

$$(3.2) \quad \begin{aligned} \ell &\leq |\mathbf{C}_A(\alpha) \cap (A \setminus B)| + \frac{|(A \setminus B) \setminus (\mathbf{C}_A(\alpha) \cap (A \setminus B))|}{2} = \frac{|\mathbf{C}_A(\alpha) \cap (A \setminus B)|}{2} + \frac{|A \setminus B|}{2} \\ &= \frac{|\mathbf{C}_A(\alpha) \cap (A \setminus B)|}{2} + \frac{|A|}{4} \leq \frac{|\mathbf{C}_A(\alpha)|}{4} + \frac{|A|}{4} \leq \frac{|A|/2}{4} + \frac{|A|}{4} = \frac{3}{8}|A|. \end{aligned}$$

The proof now follows from (3.1) and (3.2). \square

Lemma 3.3. *Let A be a group, let B be a subgroup of A having index 2 and let H and K be subgroups of A with $1 < H \leq K < A$ and $H \leq B$. The number of subsets S of $A \setminus B$ such that $S \setminus K$ is a union of H -cosets is at most $2^{\frac{3|A|}{8}}$.*

Proof. Observe that $A \setminus (K \cup B)$ is a union of H -cosets because $H \leq K \cap B$. Set

$$\mathcal{N} := \{S \subseteq A \setminus B \mid S \setminus K \text{ is a union of } H\text{-cosets}\}.$$

If $S \in \mathcal{N}$, then $S \cap K$ is an arbitrary subset of $K \setminus B$ and hence we have $2^{|K \setminus B|}$ choices for $S \cap K$. From this it follows

$$\begin{aligned} |\mathcal{N}| &= 2^{|K \setminus B| + \frac{|(A \setminus B) \setminus (K \setminus B)|}{|H|}} \leq 2^{|K \setminus B| + \frac{|(A \setminus B) \setminus (K \setminus B)|}{2}} = 2^{|K \setminus B| + \frac{|A \setminus B|}{2} - \frac{|K \setminus B|}{2}} \\ &\leq 2^{\frac{|K \setminus B|}{2} + \frac{|A \setminus B|}{2}} \leq 2^{\frac{|K|}{4} + \frac{|A|}{4}} \leq 2^{\frac{|A|}{8} + \frac{|A|}{4}} = 2^{\frac{3|A|}{8}}. \end{aligned} \quad \square$$

Proof of Theorem 1.1. We partition the set $2^{A \setminus B} := \{S \mid S \subseteq A \setminus B\}$ in (not necessarily disjoint) subsets:

$$\begin{aligned} \mathcal{A}_1 &:= \{S \in 2^{A \setminus B} \mid \langle S \rangle < A\}, \\ \mathcal{A}_2 &:= \{S \in 2^{A \setminus B} \mid \text{there exists } \alpha \in \text{Aut}(A) \text{ with } \alpha \neq 1, S^\alpha = S \text{ and } B^\alpha = B\}, \\ \mathcal{A}_3 &:= \{S \in 2^{A \setminus B} \mid \text{there exist two subgroups } H \text{ and } K \text{ with } 1 < H \leq K < A, H \leq B, \\ &\quad |A : K| \text{ and } |H| \text{ both prime numbers, and } S \setminus K \text{ is a union of } H\text{-cosets}\}, \\ \mathcal{A}_4 &:= 2^{A \setminus B} \setminus (\mathcal{A}_1 \cup \mathcal{A}_2 \cup \mathcal{A}_3). \end{aligned}$$

From Lemma 3.1,

$$(3.3) \quad |\mathcal{A}_1| \leq 2^{\frac{|A|}{4} + \log_2 |A|}.$$

Observe now that, if $S \in 2^{A \setminus B} \setminus \mathcal{A}_1$, then $\text{Cay}(A, S)$ is connected and hence $\{B, A \setminus B\}$ is the only bipartition of $\text{Cay}(A, S)$. In particular, every automorphism of $\text{Cay}(A, S)$ must preserve the bipartition $\{B, A \setminus B\}$.

From Lemma 3.2,

$$(3.4) \quad |\mathcal{A}_2| \leq 2^{\frac{3|A|}{8}} (|\text{Aut}(A)| - 1) \leq 2^{\frac{3|A|}{8} + (\log_2 |A|)^2}.$$

Since A contains at most $|A|^2$ subgroups H and K with $|H|$ and $|A : K|$ both primes, Lemma 3.3 yields

$$(3.5) \quad |\mathcal{A}_3| \leq 2^{\frac{3|A|}{8} + 2 \log_2 |A|}.$$

CLAIM: For every $S \in \mathcal{A}_4$, $\text{Cay}(A, S)$ is a bipartite DRR with bipartition $\{B, A \setminus B\}$.

Let $S \in \mathcal{A}_4$, let $\Gamma := \text{Cay}(A, S)$ and let $G := \text{Aut}(\Gamma)$. As $S \notin \mathcal{A}_1$, Γ is connected, bipartite and $\{B, A \setminus B\}$ is the only bipartition of Γ .

Since Γ is a Cayley digraph over A , the group A is embedded in G via its right regular representation. Thus we may identify A as a subgroup of G , and we do so. Let G_1 be the stabilizer of the vertex 1 of Γ . Since $1 \in B$, the group G_1 fixes setwise the two parts B and $A \setminus B$ of the bipartition of Γ , that is, $B^\alpha = B$ for each $\alpha \in G_1$.

Let $N := \mathbf{N}_{G_1}(A)$. Given $\alpha \in N$, we see that α acts as an automorphism on A ; moreover, $S^\alpha = S$ and $B^\alpha = B$ because α is an automorphism of Γ fixing 1. Thus $N \leq \{\alpha \in \text{Aut}(A) \mid S^\alpha = S, B^\alpha = B\}$. Since $S \notin \mathcal{A}_2$, we deduce $N = 1$. Thus A is self-normalizing in G , that is, $A = \mathbf{N}_G(A)$. Therefore, since A is abelian, we are in the position to apply [6, Theorem 4.2], see also [6, Definition 4.1] for some terminology. We deduce that either $G = A$ and Γ is a DRR, or there exist two subgroups H' and K' of A with $1 < H' \leq K' < A$ and with $S \setminus K'$ a union of H' -cosets. We show that the latter possibility yields $S \in \mathcal{A}_3$, contradicting our choice of S . Thus, arguing by contradiction, let H' and K' be subgroups of A with $1 < H' \leq K' \leq A$ and with $S \setminus K'$ a union of H' -cosets. Let $H \leq H'$ and let $K \geq K'$ with $|A : K|$ and $|H|$ both prime numbers. Observe that, since $H \leq H'$ and $K \geq K'$, the set $S \setminus K$ is a union of H -cosets. Now, to deduce that $S \in \mathcal{A}_3$, it suffices to show that $H \leq B$. Since Γ is connected, $S \not\subseteq K$ and hence there exists $s \in S \setminus K$. Since $s \in S \setminus K$, we have $sH \subseteq S$; therefore, for every $h \in H$, we have $sh \in A \setminus B$. As $s \in A \setminus B$ and

$|A : B| = 2$, this implies $h \in B$, for every $h \in H$, that is, $H \leq B$. Therefore $S \in \mathcal{A}_3$, a contradiction. Our claim is now proven. ■

Now, the proof follows immediately from the previous claim, (3.3), (3.4) and (3.5), and a computation. □

Proof of Corollary 1.3. Let

$$\begin{aligned} \mathcal{B} &:= \{B \mid B \leq A, |A : B| = 2\}, \\ \mathcal{S} &:= \{S \subseteq A \mid \text{Cay}(A, S) \text{ is a bipartite Cayley digraph}\}, \\ \mathcal{D} &:= \{S \subseteq A \mid \text{Cay}(A, S) \text{ is a bipartite DRR}\}, \\ \mathcal{S}_B &:= \{S \subseteq A \setminus B \mid \text{Cay}(A, S) \text{ is a bipartite Cayley digraph with bipartition } \{B, A \setminus B\}\}, \\ \mathcal{D}_B &:= \{S \subseteq A \setminus B \mid \text{Cay}(A, S) \text{ is a bipartite DRR with bipartition } \{B, A \setminus B\}\}. \end{aligned}$$

We aim to prove that $|\mathcal{D}|/|\mathcal{S}| \rightarrow 1$ as $|A| \rightarrow \infty$. Observe that

$$\mathcal{S} = \bigcup_{B \in \mathcal{B}} \mathcal{S}_B, \quad \mathcal{D} = \bigcup_{B \in \mathcal{B}} \mathcal{D}_B.$$

For $B \in \mathcal{B}$, we have $|\mathcal{S}| \leq |\mathcal{B}||\mathcal{S}_B| = |\mathcal{B}|2^{\frac{|A|}{2}}$. Moreover, using the inclusion-exclusion principle, we have

$$|\mathcal{D}| \geq \sum_{B \in \mathcal{B}} |\mathcal{D}_B| - \frac{1}{2} \sum_{\substack{B_1, B_2 \in \mathcal{B} \\ B_1 \neq B_2}} |\mathcal{D}_{B_1} \cap \mathcal{D}_{B_2}|.$$

If $S \in \mathcal{D}_{B_1} \cap \mathcal{D}_{B_2}$, then $S \subseteq (A \setminus B_1) \cap (A \setminus B_2) = A \setminus (B_1 \cup B_2)$ and hence we have at most $2^{\frac{|A|}{4}}$ possibilities for S . Therefore $|\mathcal{D}_{B_1} \cap \mathcal{D}_{B_2}| \leq 2^{\frac{|A|}{4}}$. Using Theorem 1.1, we get

$$|\mathcal{D}| \geq |\mathcal{B}|(2^{\frac{|A|}{2}} - 3 \cdot 2^{\frac{3|A|}{8} + (\log_2 |A|)^2}) - \frac{(|\mathcal{B}| - 1)|\mathcal{B}|}{2} 2^{\frac{|A|}{4}}.$$

Thus

$$\frac{|\mathcal{D}|}{|\mathcal{S}|} \geq \frac{|\mathcal{D}|}{|\mathcal{B}|2^{\frac{|A|}{2}}} \geq 1 - 3 \cdot 2^{-\frac{|A|}{8} + (\log_2 |A|)^2} - \frac{(|\mathcal{B}| - 1)}{2} 2^{-\frac{|A|}{4}} \rightarrow 1,$$

as $|A| \rightarrow \infty$. □

Proof of Corollary 1.4. Let A be an abelian group and let B be a subgroup of A having index 2. If $|A| \geq 744$, then a computation shows that $|A|/2 > 3|A|/8 + (\log_2 |A|)^2 + 2$ and hence, by Theorem 1.1, there exists a subset $S \subseteq A \setminus B$ with $\text{Cay}(A, S)$ a DRR.

Suppose then $|A| < 744$. In this case the proof follows with the invaluable help of the computer algebra system `magma` [3]. Except in the case $A = C_2^6$ all the computations are straightforward. We give some details of these computations. Except for the pairs listed in Table 1, we generate at random 10 000 subsets S of $A \setminus B$ and we check whether $\text{Cay}(A, S)$ is a DRR: in all cases, we find a DRR among our digraphs. When (A, B) is one of the pairs in Table 1 with $A \neq C_2^6$, we construct all subsets S of $A \setminus B$ and we compute $|\text{Aut}(\text{Cay}(A, S)) : A|$; therefore we compute $\bar{c}(A, B)$ by brute force. When $A = C_2^6$ and B has index 2 in A this naive approach does not work because we have $2^{|A \setminus B|} = 2^{32}$ subsets to check.

Suppose $A = C_2^6$ and B has index 2 in A . We aim to prove that $\bar{c}(A, B) = 4$. We identify A with the 6-dimensional vector space \mathbb{F}_2^6 of column vectors over the field \mathbb{F}_2 of size 2 and we identify B with the hyperplane of A with equation $x_1 + x_2 + x_3 + x_4 + x_5 + x_6 = 0$. Let S be a subset of $A \setminus B$ with $\bar{c}(A, B) = |\text{Aut}(\text{Cay}(A, S)) : A|$. Write $\Gamma := \text{Cay}(A, S)$, $s := |S|$, $G := \text{Aut}(A)$ and $H := \{\alpha \in G \mid B^\alpha = B\}$. Clearly, $G \cong \text{GL}_6(2)$ and $H \cong \text{AGL}_5(2)$.

Replacing S with $(A \setminus B) \setminus S$ if necessary, we may assume that $s = |S| \leq |A \setminus B|/2 = 16$. If $\langle S \rangle < A$, then Γ is disconnected and we can apply Table 1 to the elementary abelian 2-group $\langle S \rangle \cong C_2^\ell$ with

$\ell \leq 5$. For instance, if $\langle S \rangle \cong C_2^5$, then $|\text{Aut}(\text{Cay}(A, S)) : A| \geq |\text{Aut}(\text{Cay}(C_2^5, S)) \text{ wr } C_2|/2^6 \geq (72 \cdot 2^5)^2 \cdot 2/2^6 = 2! \cdot 72^2 \cdot 2^4$. Thus, we obtain

$$|\text{Aut}(\text{Cay}(A, S)) : A| \geq \min\{2! \cdot 72^2 \cdot 2^4, 4! \cdot 24^4 \cdot 2^{10}, 8! \cdot 6^8 \cdot 2^{18}, 16! \cdot 2^{16} \cdot 2^{26}\} = 165\,888.$$

It remains to consider the case that $A = \langle S \rangle$, that is, S contains an \mathbb{F}_2 -basis of A .

A computation shows that $H = \text{AGL}_5(2)$ acts transitively on the \mathbb{F}_2 -basis of A contained in $A \setminus B$. Therefore, we may assume that S contains the six canonical vectors $e_1, e_2, e_3, e_4, e_5, e_6$. Write

$$\mathcal{B} := \{e_1, e_2, e_3, e_4, e_5, e_6\} \quad \text{and} \quad K := \{h \in H \mid \mathcal{B}^h = \mathcal{B}\}.$$

Observe that $K \cong \text{Sym}(6)$ is the group of monomial matrices.

Now, we may write $S = \mathcal{B} \cup T$, for some subset T of $A \setminus (B \cup \mathcal{B})$ of cardinality at most $16 - 6 = 10$. If $T = \emptyset$, then $|\text{Aut}(\text{Cay}(A, S)) : A| \geq |K| = 6! = 720$, because the group of monomial matrices K fixes setwise S and hence is a group of automorphisms of $\text{Cay}(A, S)$. Suppose that $T \neq \emptyset$. A computation reveals that K has two orbits on the vectors in $A \setminus (B \cup \mathcal{B})$, with representatives $e_1 + e_2 + e_3$ and $e_1 + e_2 + e_3 + e_4 + e_5$. Write $\mathcal{B}_1 := \mathcal{B} \cup \{e_1 + e_2 + e_3\}$ and $\mathcal{B}_2 := \mathcal{B} \cup \{e_1 + e_2 + e_3 + e_4 + e_5\}$. Replacing S by a suitable K -conjugate if necessary, we may assume that S contains either \mathcal{B}_1 or \mathcal{B}_2 . Therefore, we have two cases to consider

- $S = \mathcal{B}_1 \cup T_1$, for some subset T_1 of $A \setminus (B \cup \mathcal{B}_1)$ of cardinality at most $16 - 7 = 9$,
- $S = \mathcal{B}_2 \cup T_2$, for some subset T_2 of $A \setminus (B \cup \mathcal{B}_2)$ of cardinality at most $16 - 7 = 9$.

Since $|A \setminus (B \cup \mathcal{B}_i)| = 25$, the number of possibilities for S is

$$2 \left(\binom{25}{0} + \binom{25}{1} + \binom{25}{2} + \binom{25}{3} + \binom{25}{4} + \binom{25}{5} + \binom{25}{6} + \binom{25}{7} + \binom{25}{8} + \binom{25}{9} \right) = 7\,701\,512.$$

This number is within computational reach, therefore we have generated all possible subsets S as above and we have checked that the minimum for $|\text{Aut}(\text{Cay}(A, S)) : A|$ is 4. \square

An *unlabeled* digraph is simply an equivalence class of digraphs under the relation “being digraph-isomorphic to”. In the proof of Theorem 1.6, we identify a representative with its class.

Proof of Theorem 1.6. For the proof, we let $\text{DRR}(A, B)$ denote the set of unlabelled bipartite DRRs over A with bipartition $\{B, A \setminus B\}$ and let $2_{\text{DRR}}^{A \setminus B}$ be the collection of the subsets S of $A \setminus B$ with $\text{Cay}(A, S)$ a DRR.

Let S_1 and S_2 be in $2_{\text{DRR}}^{A \setminus B}$ and let $\Gamma_1 := \text{Cay}(A, S_1)$ and $\Gamma_2 := \text{Cay}(A, S_2)$. Suppose that $\Gamma_1 \cong \Gamma_2$ and let φ be a digraph isomorphism from Γ_1 to Γ_2 . Without loss of generality, we may assume that $1^\varphi = 1$. Note that φ induces a group automorphism from $\text{Aut}(\Gamma_1) = A$ to $\text{Aut}(\Gamma_2) = A$. In particular, $\varphi \in \text{Aut}(A)$ and S_1 and S_2 are conjugate via an element of $\text{Aut}(A)$. This shows that

$$|\text{DRR}(A, B)| \geq \frac{|2_{\text{DRR}}^{A \setminus B}|}{|\text{Aut}(A)|}.$$

By Theorem 1.1, we have

$$|2_{\text{DRR}}^{A \setminus B}| \geq 2^{\frac{|A|}{2}} - 3 \cdot 2^{\frac{3|A|}{|S|} + (\log_2 |A|)^2}.$$

Since $|\text{Aut}(A)| \leq 2^{(\log_2 |A|)^2}$, it follows that

$$|\text{DRR}(A, B)| \geq 2^{\frac{|A|}{2} - (\log_2 |A|)^2} - 3 \cdot 2^{\frac{3|A|}{|S|}}.$$

In particular, this proves the first part of the theorem.

Let $\text{UCD}(A, B)$ denote the set of unlabeled bipartite Cayley digraphs on A with bipartition $\{B, A \setminus B\}$ that are not DRRs. Clearly, $|\text{DRR}(A, B)| + |\text{UCD}(A, B)|$ is the number of unlabelled bipartite Cayley graphs on A with bipartition $\{B, A \setminus B\}$. Note that

$$\frac{|\text{DRR}(A, B)|}{|\text{DRR}(A, B)| + |\text{UCD}(A, B)|} = 1 - \frac{|\text{UCD}(A, B)|}{|\text{DRR}(A, B)| + |\text{UCD}(A, B)|} \geq 1 - \frac{|\text{UCD}(A, B)|}{|\text{DRR}(A, B)|}.$$

By Theorem 1.1, we have $|\text{UCD}(A, B)| \leq 2^{\frac{3|A|}{8} + o(|A|)}$ and thus

$$\frac{|\text{UCD}(A, B)|}{|\text{DRR}(A, B)|} \rightarrow 0,$$

as $|A| \rightarrow \infty$. This completes the proof of the second part of the theorem. \square

4. EXISTENCE AND ASYMPTOTIC ENUMERATION OF BIPARTITE CAYLEY GRAPHS

Given a group A , we write A_2 for the subset $\{a \in A \mid o(a) \leq 2\}$. When A is abelian, A_2 is a subgroup of A .

Lemma 4.1. *Let A be an abelian group and let B be a subgroup of A having index 2. Then*

$$|\{S \subseteq A \setminus B \mid S = S^{-1}\}| = 2^{\frac{|A|}{4} + \frac{|A_2 \setminus B|}{2}} = \begin{cases} 2^{\frac{|A|}{4}} & \text{if } A_2 \leq B, \\ 2^{\frac{|A|}{4} + \frac{|A_2|}{4}} & \text{if } A_2 \not\leq B. \end{cases}$$

Proof. We may partition the inverse-closed subsets S of $A \setminus B$ in two parts $S_1 := S \cap A_2$ and $S_2 := S \setminus A_2$. Clearly, S_1 is an arbitrary subset of $A_2 \setminus B$, whereas since none of the elements in $A \setminus A_2$ is an involution, the elements in S_2 come in pairs: each element paired up to its inverse. Therefore, for S_1 we have $2^{|A_2 \setminus B|}$ choices and for S_2 we have $2^{\frac{|A \setminus (A_2 \cup B)|}{2}} = 2^{\frac{|(A \setminus B) \setminus (A_2 \setminus B)|}{2}} = 2^{\frac{|A \setminus B|}{2} - \frac{|A_2 \setminus B|}{2}}$ choices. Now, the proof follows. \square

Lemma 4.2. *Let A be an abelian group and let B be a subgroup of A having index 2. The number of inverse-closed subsets S of $A \setminus B$ with $\langle S \rangle$ a proper subgroup of A is at most $2^{\frac{|A|}{8} + \frac{|A_2 \setminus B|}{2} + \log_2 |A|}$.*

Proof. Set $N := |\{S \subseteq A \setminus B \mid S = S^{-1}, \langle S \rangle < A\}|$. Clearly,

$$\{S \subseteq A \setminus B \mid S = S^{-1}, \langle S \rangle < A\} = \bigcup_{\substack{C < A \\ |A:C| \text{ prime}}} \{S \subseteq A \setminus B \mid S = S^{-1}, \langle S \rangle \leq C\}.$$

Since $\{S \subseteq A \setminus B \mid S = S^{-1}, \langle S \rangle \leq C\} \subseteq \{S \mid S = S^{-1}, S \subseteq C \setminus (C \cap B)\}$, using Lemma 4.1 we have

$$\begin{aligned} N &\leq \sum_{\substack{C < A \\ |A:C| \text{ prime}}} |\{S \mid S = S^{-1}, S \subseteq C \setminus (C \cap B)\}| = \sum_{\substack{C < A \\ |A:C| \text{ prime}}} 2^{\frac{|C|}{4} + \frac{|C_2 \setminus B|}{2}} \\ &\leq \sum_{\substack{C < A \\ |A:C| \text{ prime}}} 2^{\frac{|A|}{8} + \frac{|A_2 \setminus B|}{2}} \leq 2^{\frac{|A|}{8} + \frac{|A_2 \setminus B|}{2} + \log_2 |A|}. \end{aligned} \quad \square$$

Example 4.3. Let ℓ be a positive integer with $\ell \geq 1$, let $A := \langle x \rangle \times \langle y_1 \rangle \times \langle y_2 \rangle \times \cdots \times \langle y_\ell \rangle$ with $o(x) = 4$ and $o(y_i) = 2$, for each $i \in \{1, \dots, \ell\}$, and let $B := \langle x^2, y_1, y_2, \dots, y_\ell \rangle$. Here we show that no bipartite Cayley graph over A with bipartition $\{B, A \setminus B\}$ has Cayley index 2, that is, $c(A, B) > 2$.

Let $\alpha : A \rightarrow A$ be the automorphism of A defined on the generators by

$$\begin{aligned} x^\alpha &= x^{-1}, \\ y_1^\alpha &= x^2 y_1, \text{ and} \\ y_i^\alpha &= y_i, \text{ for each } i \in \{2, \dots, \ell\}. \end{aligned}$$

Observe that α is a non-identity automorphism and $\alpha \neq \iota$. Moreover,

$$(xy_1)^\alpha = x^\alpha y_1^\alpha = x^{-1} x^2 y_1 = xy_1.$$

Therefore α fixes each element in the subgroup

$$T_1 := \langle xy_1, y_2, \dots, y_\ell \rangle$$

and α inverts each element in the subgroup

$$T_{-1} := \langle x, y_2, \dots, y_\ell \rangle.$$

Clearly,

$$T_1 \cup T_{-1} \supseteq A \setminus B$$

and hence $\{a, a^{-1}\}^\alpha = \{a, a^{-1}\}$, for every $a \in A \setminus B$. In particular, for every inverse-closed subset $S \subseteq A \setminus B$, α is a non-identity graph automorphism of $\text{Cay}(A, S)$. Thus $\text{Aut}(\text{Cay}(A, S)) \geq \langle A, \iota, \alpha \rangle$ and $c(A, B) \geq 4$.

Example 4.4. Let ℓ be a non-negative integer, let $A := \langle x_1 \rangle \times \langle x_2 \rangle \times \langle y_1 \rangle \times \langle y_2 \rangle \times \dots \times \langle y_\ell \rangle$ with $o(x_1) = o(x_2) = 4$ and $o(y_i) = 2$, for each $i \in \{1, \dots, \ell\}$, and let $B := \langle x_1^2, x_2, y_1, \dots, y_\ell \rangle$. Here we show that no bipartite Cayley graph over A with bipartition $\{B, A \setminus B\}$ has Cayley index 2, that is, $c(A, B) > 2$.

Let $\alpha : A \rightarrow A$ be the automorphism of A defined on the generators by

$$\begin{aligned} x_1^\alpha &= x_1, \\ x_2^\alpha &= x_1^2 x_2^{-1}, \text{ and} \\ y_i^\alpha &= y_i, \text{ for each } i \in \{1, \dots, \ell\}. \end{aligned}$$

Observe that α is a non-identity automorphism and $\alpha \neq \iota$. Moreover,

$$(x_1 x_2)^\alpha = x_1^\alpha x_2^\alpha = x_1 x_1^2 x_2^{-1} = x_1^{-1} x_2^{-1} = (x_1 x_2)^{-1}.$$

Therefore α fixes each element in the subgroup

$$T_1 := \langle x_1, x_2^2, y_1, y_2, \dots, y_\ell \rangle$$

and α inverts each element in the subgroup

$$T_{-1} := \langle x_1^2, x_1 x_2, y_1, \dots, y_\ell \rangle.$$

Clearly,

$$T_1 \cup T_{-1} \supseteq A \setminus B$$

and hence $\{a, a^{-1}\}^\alpha = \{a, a^{-1}\}$, for every $a \in A \setminus B$. In particular, for every inverse-closed subset $S \subseteq A \setminus B$, α is a non-identity graph automorphism of $\text{Cay}(A, S)$. Thus $\text{Aut}(\text{Cay}(A, S)) \geq \langle A, \iota, \alpha \rangle$ and $c(A, B) \geq 4$.

Proof of Theorem 1.8. Let ℓ be a positive integer with $\ell \geq 1$, let $A := \langle x \rangle \times \langle y_1 \rangle \times \langle y_2 \rangle \times \dots \times \langle y_\ell \rangle$ with $o(x) = 4$ and $o(y_i) = 2$, for each $i \in \{1, \dots, \ell\}$. Thus $A \cong C_4 \times C_2^\ell$. Let B be a subgroup of A with $B \cong C_2^{\ell+1}$. It is an easy computation to see that $\text{Aut}(A)$ has only two orbits in its action on the subgroups of A having index 2; moreover, for the two orbits we may take representatives $\langle x^2, y_1, y_2, \dots, y_\ell \rangle \cong C_2^{\ell+1}$ and $\langle x, y_1, y_2, \dots, y_{\ell-1} \rangle \cong C_4 \times C_2^{\ell-1}$. Therefore, $B = \langle x^2, y_1, y_2, \dots, y_\ell \rangle$ and, in this case, the proof follows from Example 4.3.

Let ℓ be a non-negative integer, let $A := \langle x_1 \rangle \times \langle x_2 \rangle \times \langle y_1 \rangle \times \langle y_2 \rangle \times \dots \times \langle y_\ell \rangle$ with $o(x_1) = o(x_2) = 4$ and $o(y_i) = 2$, for each $i \in \{1, \dots, \ell\}$. Thus $A \cong C_4^2 \times C_2^\ell$. Let B be a subgroup of A with $B \cong C_4 \times C_2^{\ell+1}$. When $\ell \geq 1$, the group $\text{Aut}(A)$ has only two orbits in its action on the subgroups of A having index 2; moreover, for the two orbits we may take representatives $\langle x_1^2, x_2, y_1, y_2, \dots, y_\ell \rangle \cong C_4 \times C_2^{\ell+1}$ and $\langle x_1, x_2, y_1, y_2, \dots, y_{\ell-1} \rangle \cong C_4^2 \times C_2^{\ell-1}$. Therefore, replacing B by a suitable $\text{Aut}(A)$ -conjugate, we may assume that $B = \langle x_1^2, x_2, y_1, y_2, \dots, y_\ell \rangle$. When $\ell = 0$, every subgroup of A having index 2 is isomorphic to $C_4 \times C_2$ and hence, again, we may take $B = \langle x_1^2, x_2 \rangle$. Now, the proof in this case follows from Example 4.4. \square

Lemma 4.5. *Let A be an abelian group not having cardinality a power of 2, let B be a subgroup of A having index 2, and let H and K be subgroups of A with $1 < H \leq K < A$ and $H \leq B$. The number of inverse-closed subsets S of $A \setminus B$ such that $S \setminus K$ is a union of H -cosets is at most $2^{\frac{11|A|}{48} + \frac{|A_2 \setminus B|}{2}}$.*

Proof. We subdivide the proof in various cases. For simplicity, we write

$$\mathcal{S} := \{S \subseteq A \setminus B \mid S = S^{-1}, S \setminus K \text{ is a union of } H\text{-cosets}\} \quad \text{and} \quad s := |\mathcal{S}|.$$

CASE 1: $K \leq B$ and $|H| > 2$.

If $S \in \mathcal{S}$, then $S \setminus K = S$ and hence the whole of S is a union of H -cosets. Since $A \setminus B$ has $2^{\frac{|A \setminus B|}{|H|}}$ subsets that are union of H -cosets (regardless of whether they are inverse-closed or not) and since $2^{\frac{|A \setminus B|}{|H|}} = 2^{\frac{|A|}{2|H|}} \leq 2^{\frac{|A|}{6}}$, we have

$$(4.1) \quad s \leq 2^{\frac{|A|}{6}}.$$

CASE 2: $K \not\leq B$ and $|H| > 2$.

Let $S \in \mathcal{S}$. We may partition S into two parts $S_1 := S \cap K$ and $S_2 := S \setminus K$. Observe that $K \cap B$ has index 2 in K because $K \not\leq B$. Therefore, by Lemma 4.1 applied to the group K and to the index 2 subgroup $B \cap K$, the number of choices for S_1 is exactly

$$2^{\frac{|K|}{4} + \frac{|K_2 \setminus B|}{2}}.$$

Similarly to Case 1, to obtain an upper bound for the number of choices of S_2 , we simply count the number of subsets S_2 of $(A \setminus B) \setminus (K \setminus B)$ that are union of H -cosets, regardless of whether S_2 is inverse-closed or not. Thus the number of choices for S_2 is at most

$$2^{\frac{|(A \setminus B) \setminus (K \setminus B)|}{|H|}} = 2^{\frac{|A|}{2|H|} - \frac{|K|}{2|H|}}.$$

Combining the upper bounds for S_1 and S_2 , we have

$$s \leq 2^{\frac{|A|}{2|H|} - \frac{|K|}{2|H|} + \frac{|K|}{4} + \frac{|K_2 \setminus B|}{2}} = 2^{\frac{|A|}{2|H|} + \frac{|K|}{4} \left(1 - \frac{2}{|H|}\right) + \frac{|K_2 \setminus B|}{2}} \leq 2^{\frac{|A|}{2|H|} + \frac{|K|}{4} \left(1 - \frac{2}{|H|}\right) + \frac{|A_2 \setminus B|}{2}}.$$

Using $|H| > 2$, we deduce $1/2|H| \leq 1/6$ and $1 - 2/|H| \leq 1/3$. Thus

$$s \leq 2^{\frac{|A|}{6} + \frac{|K|}{12} + \frac{|A_2 \setminus B|}{2}}.$$

Since $|K| \leq |A|/2$, we have

$$(4.2) \quad s \leq 2^{\frac{5|A|}{24} + \frac{|A_2 \setminus B|}{2}}.$$

For the rest of the proof we may assume that $|H| = 2$. Write $H := \langle h \rangle$. We start with a preliminary observation. Let $a \in A$. If $\{a, a^{-1}\}h = \{a, a^{-1}\}$, then $ah = a^{-1}$ and $a^{-1}h = a$, that is, $a^2 = h$. Therefore, under the action of right multiplication by h , the only pairs $\{a, a^{-1}\}$ that are fixed by h satisfy $a^2 = h$.

CASE 3: $K \leq B$ and $|H| = 2$.

As in Case 1, all of S is a union of H -cosets. Write

$$T := \{a \in A \setminus B \mid a^2 = h\}.$$

Observe that T contains only elements having order 4 and hence $T \cap A_2 = \emptyset$. Let $S \in \mathcal{S}$. The elements in $S \cap T$ come in pairs: each element x paired up to x^{-1} . The elements in $S \cap (A_2 \setminus B)$ also come in pairs: each element x paired up with xh . The elements in $S \setminus (T \cup (A_2 \setminus B))$ come in fours: each element x comes along with x, x^{-1}, xh and $x^{-1}h$. Thus, from our preliminary observation, we have

$$(4.3) \quad \begin{aligned} s &= 2^{\frac{|A_2 \setminus B|}{2} + \frac{|T|}{2} + \frac{|(A \setminus B) \setminus (T \cup (A_2 \setminus B))|}{4}} = 2^{\frac{|A_2 \setminus B|}{2} + \frac{|T|}{2} + \frac{|A \setminus B|}{4} - \frac{|A_2 \setminus B|}{4} - \frac{|T|}{4} + \frac{|T \cap A_2|}{4}} \\ &= 2^{\frac{|A \setminus B|}{4} + \left(\frac{|T|}{2} - \frac{|T|}{4}\right) + \left(\frac{|A_2 \setminus B|}{2} - \frac{|A_2 \setminus B|}{4}\right)} = 2^{\frac{|A|}{8} + \frac{|T|}{4} + \frac{|A_2 \setminus B|}{4}}. \end{aligned}$$

If $T = \emptyset$, then (4.3) gives

$$s = 2^{\frac{|A|}{8} + \frac{|A_2 \setminus B|}{4}}.$$

Suppose $T \neq \emptyset$ and let $a_0 \in T$. An easy computation yields

$$T = a_0(A_2 \cap B) = \{a_0c \mid c \in A_2 \cap B\}.$$

Thus, $|T| = |A_2 \cap B|$. Since A is not a 2-group and $|A : B| = 2$, we have $|A : A_2 \cap B| = |A : B||B : A_2 \cap B| = 2|B : A_2 \cap B| \geq 6$ and hence (4.3) gives

$$s = 2^{\frac{|A|}{8} + \frac{|A_2 \cap B|}{4} + \frac{|A_2 \setminus B|}{4}} \leq 2^{\frac{|A|}{8} + \frac{|A|}{24} + \frac{|A_2 \setminus B|}{4}} \leq 2^{\frac{|A|}{6} + \frac{|A_2 \setminus B|}{4}}.$$

Summing up, we have shown

$$(4.4) \quad s \leq 2^{\frac{|A|}{6} + \frac{|A_2 \setminus B|}{4}}.$$

CASE 4: $K \not\leq B$ and $|H| = 2$.

We use the ideas in Cases 2 and 3. Write

$$T := \{a \in (A \setminus B) \setminus (K \setminus B) \mid a^2 = h\}.$$

Observe that T contains only elements having order 4 and hence $T \cap A_2 = \emptyset$. Write also

$$R := (A_2 \setminus B) \setminus (K \setminus B).$$

The sets T , R and $K \setminus B$ are mutually disjoint and $R \cup (K_2 \setminus B) = A_2 \setminus B$.

Let $S \in \mathcal{S}$. By Lemma 4.1 applied to the group K and to the index 2 subgroup $B \cap K$, the number of choices for $S \cap K$ is exactly $2^{\frac{|K|}{4} + \frac{|K_2 \setminus B|}{2}}$. The elements in $S \setminus K$ can be partitioned in three subsets

$$S_1 := (S \setminus K) \cap T, \quad S_2 := (S \setminus K) \cap R \quad \text{and} \quad S_3 := (S \setminus K) \setminus (S_1 \cup S_2).$$

As R , T and $S \setminus K$ are inverse-closed and unions of H -cosets, so are S_1 , S_2 and S_3 . Therefore the elements in S_1 and in S_2 come in pairs (the element x in S_1 paired up to x^{-1} and the element x in S_2 paired up to xh), whereas the elements in S_3 come in fours. Thus

$$(4.5) \quad \begin{aligned} s &= 2^{\frac{|(A \setminus B) \setminus (R \cup T \cup (K \setminus B))|}{4} + \frac{|T|}{2} + \frac{|R|}{2} + \frac{|K|}{4} + \frac{|K_2 \setminus B|}{2}} = 2^{\frac{|A \setminus B|}{4} - \frac{|R|}{4} - \frac{|T|}{4} - \frac{|K \setminus B|}{4} + \frac{|T|}{2} + \frac{|R|}{2} + \frac{|K|}{4} + \frac{|K_2 \setminus B|}{2}} \\ &= 2^{\frac{|A|}{8} + \left(\frac{|R|}{2} - \frac{|R|}{4}\right) + \left(\frac{|T|}{2} - \frac{|T|}{4}\right) - \frac{|K|}{8} + \frac{|K|}{4} + \frac{|K_2 \setminus B|}{2}} \\ &= 2^{\frac{|A|}{8} + \frac{|R|}{4} + \frac{|T|}{4} + \frac{|K|}{8} + \frac{|K_2 \setminus B|}{2}} \leq 2^{\frac{|A|}{8} + \frac{|T|}{4} + \frac{|K|}{8} + \frac{|A_2 \setminus B|}{2}}. \end{aligned}$$

If $T = \emptyset$, then

$$s \leq 2^{\frac{|A|}{8} + \frac{|K|}{8} + \frac{|A_2 \setminus B|}{2}} \leq 2^{\frac{|A|}{8} + \frac{|A|}{16} + \frac{|A_2 \setminus B|}{2}} = 2^{\frac{3|A|}{16} + \frac{|A_2 \setminus B|}{2}}.$$

Suppose $T \neq \emptyset$ and let $a_0 \in T$. An easy computation gives

$$T \subseteq a_0(A_2 \cap B) = \{a_0c \mid c \in A_2 \cap B\}.$$

Thus, $|T| \leq |A_2 \cap B|$. Since A is not a 2-group and $|A : B| = 2$, we have $|A_2 \cap B| \leq |A|/6$ and hence $|T| \leq |A|/6$. From (4.5), we deduce

$$(4.6) \quad s \leq 2^{\frac{|A|}{8} + \frac{|A|}{24} + \frac{|A|}{16} + \frac{|A_2 \setminus B|}{2}} = 2^{\frac{11|A|}{48} + \frac{|A_2 \setminus B|}{2}}.$$

Now the proof follows from (4.1), (4.2), (4.4) and (4.6). \square

Lemma 4.6. *Let A be an abelian group of exponent greater than 2, let B be a subgroup of A having index 2 and let α be a non-identity automorphism of A with $B^\alpha = B$ and $\alpha \neq \iota$. Suppose that (A, B) is not one of the pairs in the statement of Theorem 1.8. Then, the number of inverse-closed subsets S of $A \setminus B$ with $S^\alpha = S$ is at most $2^{\frac{11|A|}{48} + \frac{|A_2 \setminus B|}{2}}$.*

Proof. Set

$$\begin{aligned} T_1 &:= \{a \in A \mid a^\alpha = a\}, \\ T_{-1} &:= \{a \in A \mid a^\alpha = a^{-1}\}, \\ \mathcal{S} &:= \{S \subseteq A \setminus B \mid S = S^{-1}, S^\alpha = S\}, \\ s &:= |\mathcal{S}|. \end{aligned}$$

Observe that $T_1 \cap T_{-1} \leq A_2$.

We start with a preliminary remark: given a subset $S \subseteq A \setminus B$, S satisfies $S = S^{-1}$ and $S^\alpha = S$ if and only if S is invariant by the subgroup $\langle \iota, \alpha \rangle$ of $\text{Aut}(A)$. Now, we divide the proof in two cases, and each case in various subcases.

Case 1: $A_2 \leq B$.

CASE 1A: $T_1 \cup T_{-1} \subseteq B$.

The conditions in Case 1a guarantee that each orbit of $\langle \iota, \alpha \rangle$ on $A \setminus B$ has cardinality at least 4. Therefore,

$$s \leq 2 \frac{|A \setminus B|}{4} = 2 \frac{|A|}{8} < 2 \frac{11|A|}{48} + \frac{|A_2 \setminus B|}{2}.$$

CASE 1B: $T_1 \leq B$ and $T_{-1} \not\leq B$, or $T_1 \not\leq B$ and $T_{-1} \leq B$.

We only deal with the case $T_1 \leq B$, $T_{-1} \not\leq B$ and $A_2 \leq B$, the other case is similar. As $A_2 \leq B$, $\langle \alpha, \iota \rangle$ has orbits of size at least 2 on $(A \setminus B) \cap (T_1 \cup T_{-1}) = T_{-1} \setminus B$ and of size at least 4 on $(A \setminus B) \setminus (T_1 \cup T_{-1})$. Therefore,

$$\begin{aligned} s &\leq 2 \frac{|(A \setminus B) \cap (T_1 \cup T_{-1})|}{2} + \frac{|(A \setminus B) \setminus (T_1 \cup T_{-1})|}{4} = 2 \frac{|A \setminus B|}{4} + \frac{|(A \setminus B) \cap (T_1 \cup T_{-1})|}{4} \\ &= 2 \frac{|A|}{8} + \frac{|(A \setminus B) \cap (T_1 \cup T_{-1})|}{4} = 2 \frac{|A|}{8} + \frac{|T_{-1} \setminus B|}{4} = 2 \frac{|A|}{8} + \frac{|T_{-1}|}{8} \leq 2 \frac{|A|}{8} + \frac{|A|}{16} \\ &= 2 \frac{3|A|}{16} < 2 \frac{11|A|}{48} + \frac{|A_2 \setminus B|}{2}. \end{aligned}$$

CASE 1C: $T_1 \not\leq B$ and $T_{-1} \not\leq B$.

Observe that $T_1 \cap T_{-1} \leq A_2 \leq B$. In particular, $T_1 \neq T_{-1}$. We argue as in the case above but slightly refining the argument. As $A_2 \leq B$, $\langle \alpha, \iota \rangle$ has orbits of size at least 2 on $(A \setminus B) \cap (T_1 \cup T_{-1})$ and of size at least 4 on $(A \setminus B) \setminus (T_1 \cup T_{-1})$. Therefore,

$$\begin{aligned} s &\leq 2 \frac{|(A \setminus B) \cap (T_1 \cup T_{-1})|}{2} + \frac{|(A \setminus B) \setminus (T_1 \cup T_{-1})|}{4} = 2 \frac{|A|}{8} + \frac{|(A \setminus B) \cap (T_1 \cup T_{-1})|}{4} = 2 \frac{|A|}{8} + \frac{|T_1 \setminus B|}{4} + \frac{|T_{-1} \setminus B|}{4} \\ &= 2 \frac{|A|}{8} + \frac{|T_1|}{8} + \frac{|T_{-1}|}{8}. \end{aligned}$$

If $|A : T_1| \geq 3$ or $|A : T_{-1}| \geq 3$, then

$$s \leq 2 \frac{|A|}{8} + \frac{|A|}{16} + \frac{|A|}{24} = 2 \frac{11|A|}{48} = 2 \frac{11|A|}{48} + \frac{|A_2 \setminus B|}{2}.$$

Finally, suppose that $|A : T_1| = |A : T_{-1}| = 2$. In particular, as $T_1 \cap T_{-1} \leq A_2$, we deduce A_2 has index either 2 or 4 in A .

Assume first that $|A : A_2| = 4$. Then $A_2 = T_1 \cap T_{-1}$. Moreover, since T_1/A_2 and T_{-1}/A_2 are two distinct subgroups of A/A_2 of order 2, we deduce $A/A_2 \cong C_2 \times C_2$. Then $A \cong C_4 \times C_4 \times C_2^\ell$ for some $\ell \geq 0$. Since B contains A_2 , we have $B \cong C_4 \times C_2^{\ell+1}$. Therefore (A, B) is one of the pairs in the statement of Theorem 1.8, contradicting one of the hypotheses of this lemma.

Assume that $|A : A_2| = 2$. Therefore $A \cong C_4 \times C_2^\ell$, for some $\ell \geq 1$. (If $\ell = 0$, then $A \cong C_4$. However, C_4 has a unique subgroup of index 2, forcing $T_1 = T_{-1} = A_2$.) As $A_2 \leq B$ and $|B| = |A_2|$, we must have $B = A_2 \cong C_2^{\ell+1}$. Therefore (A, B) is one of the pairs in the statement of Theorem 1.8, contradicting one of the hypotheses of this lemma.

This concludes the proof of **Case 1**.

Case 2: $A_2 \not\leq B$.

CASE 2A: $T_1 \cap A_2 \not\leq B$.

Since $T_1 \cap A_2 \not\leq B$, we may write $A = B \times \langle a \rangle$, with $a \in T_1 \cap A_2$. In particular, $a^\alpha = a = a^{-1}$. Consider $\beta := \alpha|_B$, the restriction of α to B . As α is not the identity or ι but fixes the involution a , β is neither the identity nor the inverse automorphism of B . Now every subset $S \subseteq A \setminus B$ satisfying $S = S^{-1}$ and $S^\alpha = S$ is of the form $S = aT$, where T is a subset of B satisfying $T = T^{-1}$ and $T^\beta = T$. Observe that since $A = B \times \langle a \rangle$ and A has exponent greater than 2, the group B has exponent greater than 2. In particular we are in the position to apply Lemma 5.5 in [6] to the group B . From [6, Lemma 5.5], the number of choices for T is at most $2^{\frac{11|B|}{24} + \frac{|B_2|}{2}}$. (Strictly speaking, [6, Lemma 5.5] only says that the number of choices for T is at most $2^{\frac{11|B|}{24} + \frac{|B_2|}{2} + (\log_2 |B|)^2}$, in our application here we may delete the extra factor $2^{(\log_2 |B|)^2}$ because the automorphism β of B has been fixed.) Therefore

$$s \leq 2^{\frac{11|B|}{24} + \frac{|B_2|}{2}} = 2^{\frac{11|A|}{48} + \frac{|A_2 \setminus B|}{2}}.$$

CASE 2B: $T_1 \cap A_2 \leq B$.

Since $T_1 \cap A_2 = T_{-1} \cap A_2$ and $T_1 \cap T_{-1} \leq A_2$, the sets $T_1 \setminus B$, $T_{-1} \setminus B$ and $A_2 \setminus B$ are pairwise disjoint. For simplicity, we write

$$\bar{T}_1 := T_1 \setminus B, \quad \bar{T}_{-1} := T_{-1} \setminus B \quad \text{and} \quad \bar{A}_2 := A_2 \setminus B.$$

Observe that $\langle \alpha, \iota \rangle$ has orbits of size at least 2 on $\bar{T}_1 \cup \bar{T}_{-1} \cup \bar{A}_2$ and of size at least 4 on $(A \setminus B) \setminus (\bar{T}_1 \cup \bar{T}_{-1} \cup \bar{A}_2)$. Therefore,

$$\begin{aligned} s &\leq 2^{\frac{|(A \setminus B) \setminus (\bar{T}_1 \cup \bar{T}_{-1} \cup \bar{A}_2)|}{4} + \frac{|\bar{T}_1|}{2} + \frac{|\bar{T}_{-1}|}{2} + \frac{|\bar{A}_2|}{2}} \\ &= 2^{\frac{|(A \setminus B)|}{4} + \frac{|\bar{T}_1|}{4} + \frac{|\bar{T}_{-1}|}{4} + \frac{|\bar{A}_2|}{4}} \\ &= 2^{\frac{|A|}{8} + \frac{|\bar{T}_1|}{4} + \frac{|\bar{T}_{-1}|}{4} + \frac{|A_2 \setminus B|}{4}}. \end{aligned}$$

For $i \in \{1, -1\}$, write $a_i := |A : T_i|$ if $T_i \not\leq B$ and $a_i := \infty$ otherwise. Now, a simple computation shows that either

(a):

$$\left(\frac{1}{8} + \frac{1}{8a_1} + \frac{1}{8a_{-1}} \right) \leq \frac{11}{48}, \quad \text{or}$$

(b): $|A : T_1| = |A : T_{-1}| = 2$, $T_1 \not\leq B$ and $T_{-1} \not\leq B$.

In (a), we have

$$\frac{|A|}{8} + \frac{|\bar{T}_1|}{4} + \frac{|\bar{T}_{-1}|}{4} = |A| \left(\frac{1}{8} + \frac{1}{8a_1} + \frac{1}{8a_{-1}} \right) \leq \frac{11|A|}{48}$$

and the proof of this lemma follows in this case.

Suppose (b) holds, that is,

$$|A : T_1| = |A : T_{-1}| = 2, \quad T_1 \not\leq B \quad \text{and} \quad T_{-1} \not\leq B.$$

As $T_1 \cap A_2 \leq B$, T_1 , T_{-1} and B are three distinct subgroups of A having index 2 and containing the index four subgroup $T_1 \cap T_{-1} \leq A_2$.

Assume $|A : A_2| = 4$. Moreover, since T_1/A_2 and T_{-1}/A_2 are two distinct subgroups of A/A_2 of order 2, we deduce $A/A_2 \cong C_2 \times C_2$. Then $A_2 = T_1 \cap T_{-1}$ and $A \cong C_4 \times C_4 \times C_2^\ell$ for some $\ell \geq 0$. Since B contains A_2 , we have $B \cong C_4 \times C_2^{\ell+1}$. Therefore (A, B) is one of the pairs in the statement of Theorem 1.8, contradicting one of the hypotheses of this lemma.

Assume that $|A : A_2| = 2$. Therefore $A \cong C_4 \times C_2^\ell$, for some $\ell \geq 1$. (If $\ell = 0$, then $A \cong C_4$. However, C_4 has a unique subgroup of index 2, forcing $T_1 = T_{-1} = A_2$.) Now, T_1 , T_{-1} and A_2 are three distinct subgroups of A having index 2 and containing $T_1 \cap T_{-1} = T_1 \cap A_2 = T_{-1} \cap A_2$. As

$A/(T_1 \cap T_{-1}) \cong C_2 \times C_2$, the subgroup equals one of T_1, T_{-1}, A_2 , contradicting one of the previous paragraphs. Therefore, this case does not arise. \square

Lemma 4.7. *Let A be an abelian group and let B be a subgroup of A having index 2. The number of triples (C, Z, S) with*

- $A = C \times Z$,
- C a cyclic subgroup of A of order $t \geq 4$,
- Z an elementary abelian 2-subgroup of A and
- $S \subseteq A \setminus B$ such that $S = S' \times S''$, for some $S' \in \{C, \emptyset, \{1\}, C \setminus \{1\}\}$ and for some $S'' \subseteq Z$,

is at most $2^{\frac{|A|}{8} + 2 \log_2 |A| - 1}$.

Proof. Clearly, we may assume that $A = \langle \lambda \rangle \times Z'$, for some elementary abelian 2-subgroup Z' and some cyclic subgroup $\langle \lambda \rangle$ of order $t \geq 4$, otherwise we have no triple. If t is odd, then this decomposition is unique. If t is even, then the number of choices for C is $|Z'|$ because the subgroup C equals $\langle \lambda k \rangle$, for some $k \in Z'$; while the number of choices for Z is at most the number of subgroups of index 2 in $\langle \lambda^{|Z|/2} \rangle \times Z'$, which is at most $2|Z'|$. Assume now that C and Z are fixed.

We have 4 choices for S' . Moreover, if $S' = \emptyset$, then $S = S' \times S'' = \emptyset$, for every subset S'' of Z . Therefore, when $S' = \emptyset$, we have only one choice for S . Similarly, when $S'' = \emptyset$, we have only one choice for S . Therefore, let $S' \in \{C, \{1\}, C \setminus \{1\}\}$ and let $S'' \subseteq Z$ with $S'' \neq \emptyset$ such that $S := S' \times S'' \subseteq A \setminus B$. As $S'' \neq \emptyset$, Z is not contained in B . Then $B \cap Z$ has index 2 in Z and hence we have $2^{|Z:Z \cap B|} = 2^{\frac{|Z|}{2}}$ choices for S'' . Since S' equals $C, \{1\}$ or $C \setminus \{1\}$, we have $S' \cap B \neq \emptyset$ and hence there exists $s_1 \in S' \cap B$. Since $s_1 S'' \subseteq A \setminus B$ and $s_1 \in B$, we deduce $S'' \subseteq Z \setminus B$. Therefore, when S' and S'' are both non-empty, we have at most $3 \cdot 2^{\frac{|Z|}{2}}$ choices for S .

It follows that there are most

$$|Z'| \cdot 2|Z'| \cdot (1 + 3 \cdot 2^{\frac{|A|}{8}}) \leq 2^{2 \log_2 |Z'| + 1 + \frac{|A|}{8} + 2} \leq 2^{2 \log_2 (|A|/4) + 1 + \frac{|A|}{8} + 2} \leq 2^{\frac{|A|}{8} + 2 \log_2 |A| - 1}$$

triples. \square

Proof of Theorem 1.10. The group A contains $2^{\frac{|A|}{4} + \frac{|A_2 \setminus B|}{2}}$ inverse-closed subsets S with $S \subseteq A \setminus B$ by Lemma 4.1.

Now, we assume that Part (2) does not hold, that is, (A, B) is not one of the pairs in Theorem 1.8; we show that Part (1) holds. If A has exponent 2, then $A_2 = A$ and $|A_2 \setminus B| = |A|/2$. Thus the result follows from Theorem 1.1 because every Cayley digraph over an elementary abelian 2-group is undirected. For the rest of the proof we assume that A has exponent at least 3, and hence the mapping $\iota : A \rightarrow A$ defined by $a^\iota = a^{-1}$, for every $a \in A$, is a non-identity automorphism of A .

We partition the set $\mathcal{S} := \{S \mid S \subseteq A \setminus B, S = S^{-1}\}$ in five (not necessarily disjoint) subsets:

$$\mathcal{A}_1 := \{S \in \mathcal{S} \mid \langle S \rangle < A\},$$

$$\mathcal{A}_2 := \{S \in \mathcal{S} \mid \text{there exists } \alpha \in \text{Aut}(A) \text{ with } \alpha \neq 1, \alpha \neq \iota, S^\alpha = S \text{ and } B^\alpha = B\},$$

$$\mathcal{A}_3 := \{S \in \mathcal{S} \mid \text{there exist two subgroups } H \text{ and } K \text{ with } 1 < H \leq K < A, H \leq B,$$

$$|H| \text{ and } |A : K| \text{ both prime numbers, and } S \setminus K \text{ is a union of } H\text{-cosets}\}$$

when $|A|$ is not a 2-group, and

$$\mathcal{A}_3 := \emptyset \text{ when } A \text{ is a 2-group,}$$

$$\mathcal{A}_4 := \{S \in \mathcal{S} \mid \text{there exist a cyclic subgroup } C \text{ of order at least 4 and an elementary abelian 2-subgroup } Z \text{ with } A = C \times Z, \text{ there exist}$$

$$S' \in \{\emptyset, \{1\}, C, C \setminus \{1\}\} \text{ and } S'' \subseteq Z \text{ with } S = S' \times S''\},$$

$$\mathcal{A}_5 := \mathcal{S} \setminus (\mathcal{A}_1 \cup \mathcal{A}_2 \cup \mathcal{A}_3 \cup \mathcal{A}_4).$$

From Lemma 4.2,

$$(4.7) \quad |\mathcal{A}_1| \leq 2^{\frac{|A|}{8} + \frac{|A_2 \setminus B|}{2} + \log_2 |A|}.$$

Observe that, if $S \in \mathcal{S} \setminus \mathcal{A}_1$, then $\text{Cay}(A, S)$ is connected and hence $\{B, A \setminus B\}$ is the only bipartition of $\text{Cay}(A, S)$. In particular, every automorphism of $\text{Cay}(A, S)$ must preserve the bipartition $\{B, A \setminus B\}$.

From Lemma 4.6,

$$(4.8) \quad |\mathcal{A}_2| \leq 2^{\frac{11|A|}{48} + \frac{|A_2 \setminus B|}{2}} (|\text{Aut}(A)| - 1) \leq 2^{\frac{11|A|}{48} + \frac{|A_2 \setminus B|}{2} + (\log_2 |A|)^2}.$$

Since A contains at most $|A|^2$ subgroups H and K with $|H|$ and $|A : K|$ both prime numbers, Lemma 4.5 yields

$$(4.9) \quad |\mathcal{A}_3| \leq 2^{\frac{11|A|}{48} + \frac{|A_2 \setminus B|}{2} + 2 \log_2 |A|}.$$

From Lemma 4.7, we have

$$(4.10) \quad |\mathcal{A}_4| \leq 2^{\frac{|A|}{8} + 2 \log_2 |A| - 1}.$$

CLAIM: For every $S \in \mathcal{A}_5$, $\text{Cay}(A, S)$ is a bipartite Cayley graph on A with $c(A, B) = 2$.

Let $S \in \mathcal{A}_5$, let $\Gamma := \text{Cay}(A, S)$ and let $G := \text{Aut}(\Gamma)$. As $S \notin \mathcal{A}_1$, Γ is connected, bipartite and $\{B, A \setminus B\}$ is the only bipartition of Γ .

Since Γ is a Cayley graph over A , the group A is embedded in G via its right regular representation. Thus we may identify A as a subgroup of G , and we do so. Let G_1 be the stabilizer of the vertex 1 of Γ . Since $1 \in B$, the group G_1 fixes the two parts B and $A \setminus B$ of the bipartition of Γ , that is, $B^\alpha = B$ for each $\alpha \in G_1$.

Let $N := \mathbf{N}_{G_1}(A)$. Clearly, $N = \{\alpha \in \text{Aut}(A) \mid S^\alpha = S\} = \langle \iota \rangle$, because $S \notin \mathcal{A}_2$. Therefore, $\mathbf{N}_G(A)$ is a generalized dihedral group over the abelian group A , that is, $\langle A, \iota \rangle = \mathbf{N}_G(A)$. Therefore, we are in the position to apply [6, Theorem 4.3] to the group G , see also [6, Definition 4.1].

Part (1) of Theorem 4.3 in [6] does not hold for G because $S \notin \mathcal{A}_3$ (observe here that arguing as in the proof of Theorem 1.1 we may assume that $|H|$ and $|A : K|$ are both prime numbers in the statement of [6, Theorem 4.3]). Similarly, part (2) of Theorem 4.3 in [6] also does not hold because $S \notin \mathcal{A}_4$. Therefore, from Theorem 4.3 in [6], we deduce $G = \mathbf{N}_G(A) = \langle A, \iota \rangle$. Thus $|\text{Aut}(\text{Cay}(A, S)) : A| = 2$ and so $c(A, B) = 2$. ■

Now, the proof follows immediately from the previous claim, together with a computation using (4.7), (4.8), (4.9) and (4.10). \square

Proof of Corollary 1.12. Let A be an abelian group and let B be a subgroup of A having index 2. If A has exponent 2, then the proof follows from Corollary 1.4. Suppose that A has exponent at least 3. If $|A| \geq 8214$, then a computation shows that $|A|/4 > 11|A|/48 + (\log_2 |A|)^2 + 2$ and hence, by Theorem 1.10, there exists an inverse-closed subset $S \subseteq A \setminus B$ with $\text{Cay}(A, S)$ having Cayley index 2, that is, $c(A, B) = 2$.

Suppose first $|A| \leq 4096$. In this case the proof follows with the invaluable help of the computer algebra system `magma` [3]. All the computations are straightforward and use the same method explained in the proof of Corollary 1.4. Except for the pairs listed in Table 2, we generate at random 10 000 inverse-closed subsets S of $A \setminus B$ and we check whether $\text{Cay}(A, S)$ has Cayley index 2: in all cases, we have shown that $c(A, B) = 2$. When (A, B) is one of the pairs in Table 2, we construct all inverse-closed subsets S of $A \setminus B$ and we compute $c(A, B)$.

Suppose then $|A| > 4096$ and $|A| < 8214$. Following the argument in the proof of Theorem 1.10, Eqs. (4.7), (4.8), (4.9) and (4.10), we see that there exists a subset $S \subseteq A \setminus B$ with $S = S^{-1}$ and with $\text{Cay}(A, S)$ having Cayley index 2 as long as

$$2^{\frac{|A|}{4} + \frac{|A_2 \setminus B|}{2}} > 2^{\frac{|A|}{8} + \frac{|A_2 \setminus B|}{2} + \log_2 |A|} + 2^{\frac{11|A|}{48} + \frac{|A_2 \setminus B|}{2}} |\text{Aut}(A)| + 2^{\frac{11|A|}{48} + \frac{|A_2 \setminus B|}{2} + 2 \log_2 |A|} + 2^{\frac{|A|}{8} + 2 \log_2 |A| - 1}.$$

With `magma`, we have checked this inequality computing explicitly $|\text{Aut}(A)|$; every abelian group A with $4096 < |A| < 8214$ satisfies this inequality. \square

Proof of Theorem 1.13. Let $c := 1$ when A has exponent 2 and let $c := 2$ when A has exponent greater than 2. Let $\iota : A \rightarrow A$ be the automorphism of A with $a^\iota = a^{-1}$, for every $a \in A$. For the proof, we let $\text{GRR}(A, B)$ denote the set of unlabelled bipartite Cayley graphs over A with bipartition $\{B, A \setminus B\}$ and having Cayley index c . Also, we let $2_{\text{GRR}}^{A \setminus B}$ be the collection of the subsets S of $A \setminus B$ with $\text{Cay}(A, S)$ having Cayley index 2.

Let S_1 and S_2 be in $2_{\text{GRR}}^{A \setminus B}$ and let $\Gamma_1 := \text{Cay}(A, S_1)$ and $\Gamma_2 := \text{Cay}(A, S_2)$. Suppose that $\Gamma_1 \cong \Gamma_2$ and let φ be a graph isomorphism from Γ_1 to Γ_2 . Without loss of generality, we may assume that $1^\varphi = 1$. Note that φ induces a group automorphism from $\text{Aut}(\Gamma_1) = \langle A, \iota \rangle$ to $\text{Aut}(\Gamma_2) = \langle A, \iota \rangle$. Using the fact that the pair (A, B) is not one of the exceptional pairs described in Theorem 1.8, we deduce $A^\varphi = A$. In particular, $\varphi \in \text{Aut}(A)$ and S_1 and S_2 are conjugate via an element of $\text{Aut}(A)$. This shows that

$$|\text{GRR}(A, B)| \geq \frac{|2_{\text{GRR}}^{A \setminus B}|}{|\text{Aut}(A)|}.$$

By Theorem 1.10, we have

$$|2_{\text{GRR}}^{A \setminus B}| \geq 2^{\frac{|A|}{4} + \frac{|A_2 \setminus B|}{2}} - 2^{\frac{11|A|}{48} + \frac{|A_2 \setminus B|}{2} + (\log_2 |A|)^2 + 2}.$$

Since $|\text{Aut}(A)| \leq 2^{(\log_2 |A|)^2}$, it follows that

$$|\text{GRR}(A, B)| \geq 2^{\frac{|A|}{4} + \frac{|A_2 \setminus B|}{2} - (\log_2 |A|)^2} - 2^{\frac{11|A|}{48} + \frac{|A_2 \setminus B|}{2} + 2} \geq 2^{\frac{|A|}{4} + \frac{|A_2 \setminus B|}{2}} (2^{-(\log_2 |A|)^2} - 2^{-\frac{|A|}{48} + 2}).$$

From Lemma 4.1, the number of inverse-closed subsets of A contained in $A \setminus B$ is $2^{\frac{|A|}{4} + \frac{|A_2 \setminus B|}{2}}$. Therefore

$$2^{\frac{|A|}{4} + \frac{|A_2 \setminus B|}{2}} \geq |\text{GRR}(A, B)|.$$

This completes the proof. \square

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