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Dynamical phase diagrams of a love capacity constrained prey–predator model

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Abstract. One interesting question in love relationships is: finally, what and when is the end of this love relationship? Using a prey–predator Verhulst–Lotka–Volterra (VLV) model we imply cooperation and competition tendency between people in order to describe a "love dilemma game". We select the most simple but immediately most complex case for studying the set of nonlinear differential equations, i.e. that implying three persons, being at the same time prey and predator. We describe four different scenarios in such a love game containing either a one-way love or a love triangle. Our results show that it is hard to love more than one person simultaneously. Moreover, to love several people simultaneously is an unstable state. We find some condition in which persons tend to have a friendly relationship and love someone in spite of their antagonistic interaction. We demonstrate the dynamics by displaying flow diagrams.

1 Introduction

Empirical studies and theoretical modeling of interacting agents have been the subject of a large body of recent research in statistical physics and applied mathematics [1]. No need to recall (an exhaustive list of references would be impossible neither to print nor read) the flurry of papers on the "prisoner's dilemma game" since it was proposed by Flood and Dresher in 1950 [2,3]. Nowadays, agents are considered to behave on networks rather than on lattices. Their "opinion evolution" can be imagined to occur according to complex algorithms, but also basing their evolutive behavior on nonlinear differential equations. Let it be recalled that the Lotka–Volterra differential equation set for preys–predators [4] has been used in various ways to model many types of complex systems, with competitive or cooperation aspects. For instance, outside demography or biology, effects of competition on growth, pertinent in economy and opinion formation [5,6] taking into account a "market capacity" [7,8] has been studied in [9–11].

As recently pointed out [12], dynamic system theory is useful for describing complex behavior of humans. The usefulness of nonlinear differential equations is hereby used for outlining possible "love story" scenarios. The classical Lotka–Volterra equations [4] for prey–predator population evolution is taken as a model for one of the most common human behavior: love. The psychological conditions of a triad of potential lovers is described through an adjacency matrix. This 3-agent pattern is justified to be the basic unit in social networks as already discussed in [13,14], where the ensemble of configurations pertinent to a weighted-directed network are discussed from eigenvectors and eigenvalues points of view. The mathematical analysis follows ideas presented elsewhere for economic-financial markets [9–11], introducing the notion of maximum love capacity, as in Verhulst logistic equation [7,8], that is, limited resources for a population in a country.

The purpose of the described work here below touches upon a classical aspect of human nature, described by many but rarely touched upon in terms of complex system theories, yet quite useful in quantitative social sciences, including psychology and cognition [12]. Nevertheless, it is fair to point to a pioneering work on the matter by Strogatz [15]. He considers the evolution of love in a couple. We add that such an evolution depends in an

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endogenous way of the possible other partners. Moreover, it is reasonable to consider an exogenous constraint: a "love capacity" concept can also be introduced.

In Section 2, we present and develop a generalized Verhulst–Lotka–Volterra model in order to describe the "love states" of three interacting persons. In Section 3, we analyze the four possible interaction schemes, searching for steady states through a fixed point and stability analysis. We present the nontrivial dynamical phase diagrams, depending on initial love conditions of the partners. In Section 4, we point to the qualitative "love story" aspects mimicked by our quantifying nonlinear dynamics model, and suggest extensions.

2 Method

The classical Lotka–Volterra equations or the predator–prey model equations are used to describe the dynamics of a population made of two species which interact, one as a predator and the other as a prey. The size s_i of populations ($i = 1$ or 2) changes according to the set of equations

$$\dot{s}_i = \lambda_i s_i - \zeta_i s_i s_j, \qquad (1)$$

where λ's and ζ's are system parameters.

On the other hand, the simple logistic growth equation also known as the Verhulst equation describes the time evolution of a single species. Interestingly, and realistically, this equation considers the lack of resources in the growth of the population of a species according to the equation

$$\dot{s}_i = s_i(a - b s_i), \qquad (2)$$

where the parameter b is related to the available resources for a species.

One can combine these two aspects of a population growth of species to find a generalized model which describes the evolution of the population of several species, on a "market" with finite resources. The combination of these equations leads to the generalized Verhulst–Lotka–Volterra model [9–11]

$$\dot{s}_i = \alpha_i s_i(\beta_i - s_i) - \sum_{i \neq j} \gamma(s_i, s_j) s_i s_j, \quad i = 1, \ldots, n. \quad (3)$$

In the present case, s_i is the "size of the love" of the agent i, such that $0 < s_i < 1$; α_i is the growth rate for the love when there is no interaction; β_i is the maximum capacity of the love for the agent i. According to psychological aspects, the maximum amount of "love capacity" β is supposed to be such that

$$\beta_i = \beta - \sum_{j \neq i} s_j,$$

similar to market constraints in economy and finance [9–11]. For simplicity, we may assume that all the lovers have the same growth rate $\alpha_i = 1$ and also assume that the love capacity is $\beta = 1$.

The interaction function in equation (3) is defined by

$$\gamma(s_i, s_j) \equiv \Gamma_{i,j} \exp\left[-\left(\frac{s_i - s_j}{\sigma}\right)^2\right]. \qquad (4)$$

This Gaussian term expresses the effect of the difference between the strength of the love between two agents. This term notices some life reality. The interaction function $\gamma(s_i, s_j)$ is the largest if the strength of the lover i and that of j are similar, while $\gamma(s_i, s_j) \approx 0$ if the strengths are quite different, though not neglected.

The parameter σ is positive and scales the intensity distribution of the "lover strengths". For simplicity, we assume $\sigma = 1$.

The parameter Γ_{ij} can be considered as an element of a matrix Γ which specifies various scenarios on the types of lover's interactions.

Here we define four different Γ-matrices each of which characterizes different love scenarios among three lovers A_1, A_2 and A_3. The possible Γ-matrices for describing one-way love are

$$\Gamma_1 = \begin{pmatrix} 0 & -1 & -1 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad \Gamma_2 = \begin{pmatrix} 0 & 1 & 1 \\ -1 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}. \quad (5)$$

Γ_1 represent a situation that A_1 is not interested in A_2 and A_3, but A_2 and A_3 care about A_1 and wish A_1 as a lover. This can be imagined as a situation in which two individuals fall in love with a (famous) person, who a priori does not care.

In contrast, Γ_2 expresses that A_1 falls in love with both A_2 and A_3, but A_2 and A_3 do not care about A_1.

We also describe a famous situation in relationships called "love triangle". Two main forms of love triangle have to be distinguished: there is the rivalrous triangle, where the lover is competing with a rival for the love of the desired, and the split-object triangle, where a lover has split some attention between two "love objects".

Thus, we consider A_1 is interested in both A_2 and A_3, and also A_2 and A_3 care about A_1 and wish love; A_2 and A_3 do not "know" each other, such that there is no interaction between A_2 and A_3. The possible Γ-matrix for expressing such a love triangle game is

$$\Gamma_3 = \begin{pmatrix} 0 & -1 & -1 \\ -1 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}. \qquad (6)$$

In contrast, one may consider a situation in which A_1 does not like A_2 and A_3 and also A_2 and A_3 distaste A_1. (An analogy for this scenario is the case where a country has a conflict with two other countries which they are neither allied nor hostile.) The possible Γ-matrix for expressing this kind of interaction is

$$\Gamma_4 = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}. \qquad (7)$$

2.1 Fixed point analysis and stability

To access to all possible states and systems? Evolving states, it is useful to find the phase space of the parameters in the dynamical equations. The phase space trajectory represents the set of states compatible with starting from any initial condition. The dynamics equations could be represented as a vector field in the space s_1, s_2, s_3; the points of these vector space are:

$$(\dot{s}_1(s_1, s_2, s_3), \dot{s}_2(s_1, s_2, s_3), \dot{s}_3(s_1, s_2, s_3)). \tag{8}$$

Each point $r(s_1, s_2, s_3)$ of space evolves in time from t_0 to t_n.

$$r(t_0) \rightarrow r(t_1) \rightarrow r(t_2) \rightarrow \cdots \rightarrow r(t_n). \tag{9}$$

This evolution path draws a trajectory in the space s_1, s_2, s_3; each vector shows the "speed" and the evolving direction in this space.

In order to investigate the dynamics of the system, a fixed point analysis can be fruitful. A fixed point of a function is a point which is mapped to itself by the function. In other words, the evolution of the equations stops at the fixed points, thus the time derivative in the evolution equations is equal to zero. At the fixed point (s_1^*, s_2^*, s_3^*), each component of the vector field vanishes. The fixed points are found by solving equation (10):

$$\dot{s}_1(s_1^*, s_2^*, s_3^*) = 0, \quad \dot{s}_2(s_1^*, s_2^*, s_3^*) = 0,$$
$$\dot{s}_3(s_1^*, s_2^*, s_3^*) = 0. \tag{10}$$

A fixed point is stable if the trajectory of all points in the vicinity of the fixed point tends to evolve toward the fixed point. These points are called attractors or sinks. A fixed point is called repeller or source if all the points in the vicinity of the fixed point flow away or diverge from the fixed point; it is called a saddle point if the vicinity of the fixed point converges to the fixed point along some directions and diverges along other directions [15].

If the derivative of the vector field along a specified direction at the fixed point is positive, the fixed point is an attractor along this direction; if it is negative, the fixed point is a repeller along that direction. Numerically or analytically,

$$f(s_1, s_2, s_3) \equiv \dot{s}_i(s_1, s_2, s_3), \quad i = 1, 2, 3, \tag{11}$$

$$f(s_1, s_2, s_3) \approx f(s_1^*, s_2^*, s_3^*) + \left(\frac{\partial f}{\partial s_1}\right)_{s_1^*} (s_1 - s_1^*)$$
$$+ \left(\frac{\partial f}{\partial s_2}\right)_{s_2^*} (s_2 - s_2^*)$$
$$+ \left(\frac{\partial f}{\partial s_3}\right)_{s_3^*} (s_3 - s_3^*). \tag{12}$$

Using the first order coefficient of the expansion, the Jacobian matrix is

$$J \equiv \begin{pmatrix} \left(\frac{\partial \dot{s}_1}{\partial s_1}\right)_{s_1^*} & \left(\frac{\partial \dot{s}_1}{\partial s_2}\right)_{s_2^*} & \left(\frac{\partial \dot{s}_1}{\partial s_3}\right)_{s_3^*} \\ \left(\frac{\partial \dot{s}_2}{\partial s_1}\right)_{s_1^*} & \left(\frac{\partial \dot{s}_2}{\partial s_2}\right)_{s_2^*} & \left(\frac{\partial \dot{s}_2}{\partial s_3}\right)_{s_3^*} \\ \left(\frac{\partial \dot{s}_3}{\partial s_1}\right)_{s_1^*} & \left(\frac{\partial \dot{s}_3}{\partial s_2}\right)_{s_2^*} & \left(\frac{\partial \dot{s}_3}{\partial s_3}\right)_{s_3^*} \end{pmatrix}. \tag{13}$$

One can probe the stability of a fixed point by evaluating the eigenvalues and eigenvectors of the Jacobian matrix computed at each corresponding fixed point.

$$J \vec{u}_i = \lambda_i \vec{u}_i. \tag{14}$$

In the above equation, $\vec{u}_i$ is the ith eigenvector and λ_i is the corresponding eigenvalue. The fixed point is an attractor along an eigenvector if the corresponding eigenvalue is negative; it is a repeller if the eigenvalue is positive; when all the eigenvalues are negative, the fixed point is an attractor and vice versa; if there are both negative and positive eigenvalues, the fixed point is a saddle point [15].

3 Results

The dynamics equations are usefully rewritten as

$$\dot{s}_1 = s_1(1 - s_1 - s_2 - s_3) - \Gamma_{12} s_1 s_2 e^{-\left(\frac{s_1 - s_2}{\sigma}\right)^2}$$
$$- \Gamma_{13} s_1 s_3 e^{-\left(\frac{s_1 - s_3}{\sigma}\right)^2}, \tag{15}$$

$$\dot{s}_2 = s_2(1 - s_1 - s_2 - s_3) - \Gamma_{12} s_1 s_2 e^{-\left(\frac{s_1 - s_2}{\sigma}\right)^2}, \tag{16}$$

$$\dot{s}_3 = s_3(1 - s_1 - s_2 - s_3) - \Gamma_{13} s_1 s_3 e^{-\left(\frac{s_1 - s_3}{\sigma}\right)^2}. \tag{17}$$

In the following, we solve the equations for each Γ.

3.1 One-way love Γ_1 type

$$\Gamma_1 = \begin{pmatrix} 0 & -1 & -1 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}. \tag{18}$$

– $r_0 = (0, 0, 0)$ is a trivial fixed point of the system. It is unstable, the eigenvalues of the Jacobian matrix are positive $\lambda_1 = 1$, $\lambda_2 = 1$, $\lambda_3 = 1$.
– $r_1 = (1, 0, 0)$ is an attractor fixed point; its eigenvalues are $\lambda_1 = -1$, $\lambda_2 = -1/e \equiv -0.3679$, $\lambda_3 = -0.3679$. The corresponding eigenvectors are $\vec{u}_1 = (1, 0, 0)$, $\vec{u}_2 = (-1, 1, 0)$ and $\vec{u}_3 = (-1, 0, 1)$.
– $r_2 = (0, 1, 0)$ is a saddle point with the eigenvalues $\lambda_1 = -1$, $\lambda_2 = 0.3679$, $\lambda_3 = 0$; the corresponding eigenvectors are $\vec{u}_1 = (0, 1, 0)$, $\vec{u}_2 = (1, -1, 0)$ and $\vec{u}_3 = (0, -1, 1)$. It is an attractor along vector $\vec{u}_1 = (0, 1, 0)$ and repeller along two other directions u_2
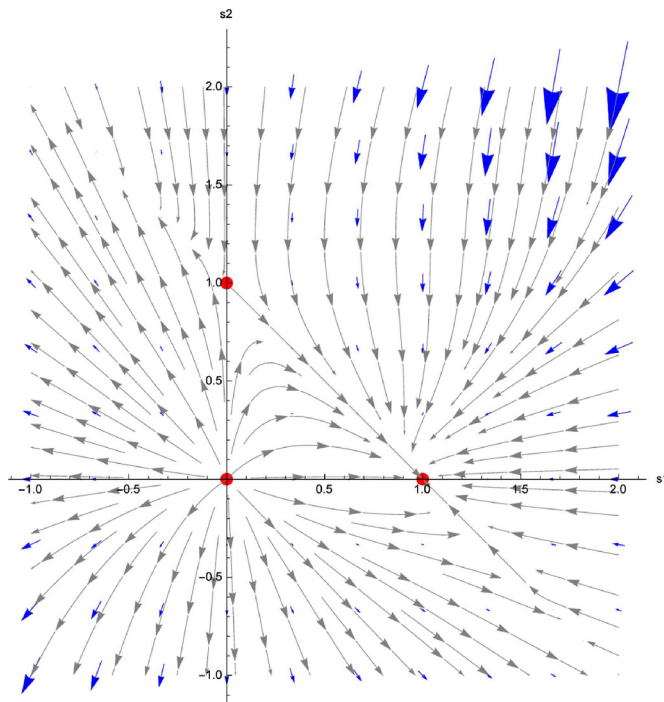
Fig. 1. Vector field in the plane $s_3 = 0$ for Γ_1. This plane contains a stable fixed point $r_1 = (1, 0, 0)$ and two unstable fixed points $r_0 = (0, 0, 0)$, $r_2 = (0, 1, 0)$.

and u_3, so all the points on the axis s_2 end up to this point; see Figure 1.

– $r_3 = (0, 0, 1)$ is similar to the point r_2 due to the symmetry of the evolution equations; it is a saddle point; its eigenvalues are $\lambda_1 = -1$, $\lambda_2 = 0.3679$, $\lambda_3 = 0$; the eigenvectors are $\overrightarrow{u_1} = (0, 0, 1)$, $\overrightarrow{u_2} = (1, 0, -1)$ and $\overrightarrow{u_3} = (0, 1, -1)$. All the points along the direction u_1 converge to r_3, whence the basin of attraction for this fixed point is the axis s_3; see Figure 2.

– There is a line of fixed points in the plane $s_1 = 0$. All the points on the line $L = (0, s_2, 1 - s_2)$ are fixed point and the eigenvalues are:

$$\lambda_1 = 0, \quad \lambda_2 = -1,$$
$$\lambda_3 = e^{-(1-s_2)^2 - s_2^2}$$
$$\times (e^{s_2^2} + e^{(1-s_2)^2} s_2 - e^{s_2^2} s_2). \quad (19)$$

The eigenvalue λ_3 is greater than 0 in the region $0 < s_1, s_2, s_3 < 1$, as shown on Figure 3; whence these points are unstable and saddle points. The eigenvector corresponding to the eigenvalue $\lambda_2 = -1$ is $\overrightarrow{u_2} = (0, -\frac{s_2}{s_2-1}, 1)$ and therefore all the points in the plane $s_1 = 0$ converge to this line along the direction u_2; Figure 4.

The results are summarized in Table 1.

In this case, the vector field is perpendicular to the normal vector of the surface $s_2 = s_3$; thus, the field is tangent to this surface and the trajectory remains on this surface forever and converges to the fixed point $r_1 = (1, 0, 0)$ for all the initial values. This means that if the two lovers A_2 and A_3 were initially and equally interested in A_1, their amount of love remains equal

for ever. On the other hand, if none of A_2 and A_3 are interested in A_1, their amount of initial love, decreases and converges to zero; this is in contrast to the love of A_1 which increases and goes to the full capacity. When the strength of the love interaction between A_1 and the two lovers A_2, A_3 is equal, no one can win the love competition.

The point $r_1 = (1, 0, 0)$ is the only stable fixed point; all the points in the space except those on the plane $s_1 = 0$ converge to this point. This result seems intuitive. As the two lovers are not interested in A_1 at all, A_1 cannot impress them but as A_1 is fascinated by both A_2 and A_3 the amount of its love increases; it will end up to be deeply in love with both A_2 and A_3.

We also find an interesting situation when the lover A_1 starts with a little love at the beginning of the game and when the lovers A_2 and A_3 get somehow interested in A_1; both demand more attention from the lover A_1; therefore their love increases at first but later they become less interested and their love decreases, ending up to zero; see Figure 5. This case occurs when the initial s_1 is small and for $1 - s_1 - s_2 - s_3 > 0$; in such a case, $\dot{s}_2 > 0$ and $\dot{s}_3 > 0$. One can numerically find a region in which s_2 and s_3 increase at first; see Figure 6.

On the surface $s_1 = 0$, there is a line of fixed points; the basin of attraction is the surface $s_1 = 0$; this means that if the lover A_1 wishes neither A_2 nor A_3 at first it will remain uninterested, but there will be a competition between A_2 and A_3 to obtain A_1's attention; the initial ratio of their love $\frac{s_2}{s_3}$ remains unchanged and their interest reaches a balance state on the line $L = (0, s_2, 1 - s_2)$; see Figure 4 again.
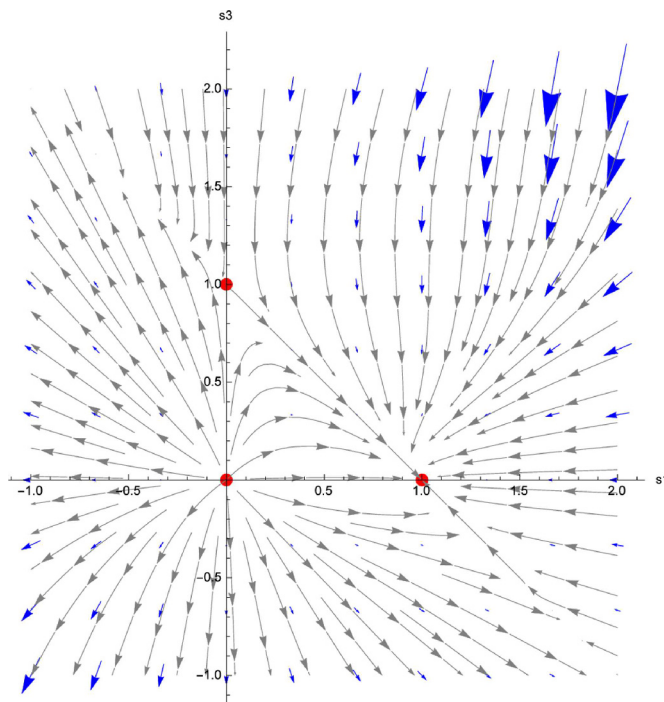
Fig. 2. Vector field in the plane $s_2 = 0$ for Γ_1. This plane contains a stable fixed point $r_1 = (1, 0, 0)$ and two unstable fixed points $r_0 = (0, 0, 0)$, $r_3 = (0, 0, 1)$.

Table 1. Fixed points, stability regime, basin of attraction in the Γ_1 case.

Fixed point	Stability	Basin of attraction
$r_0 = (0, 0, 0)$	Repeller	–
$r_1 = (1, 0, 0)$	Attractor	Global
$r_2 = (0, 1, 0)$	Saddle	Axis s_2
$r_3 = (0, 0, 1)$	Saddle	Axis s_3
The line $L = (0, s_2, 1 - s_2)$	Saddle	The surface $s_1 = 0$
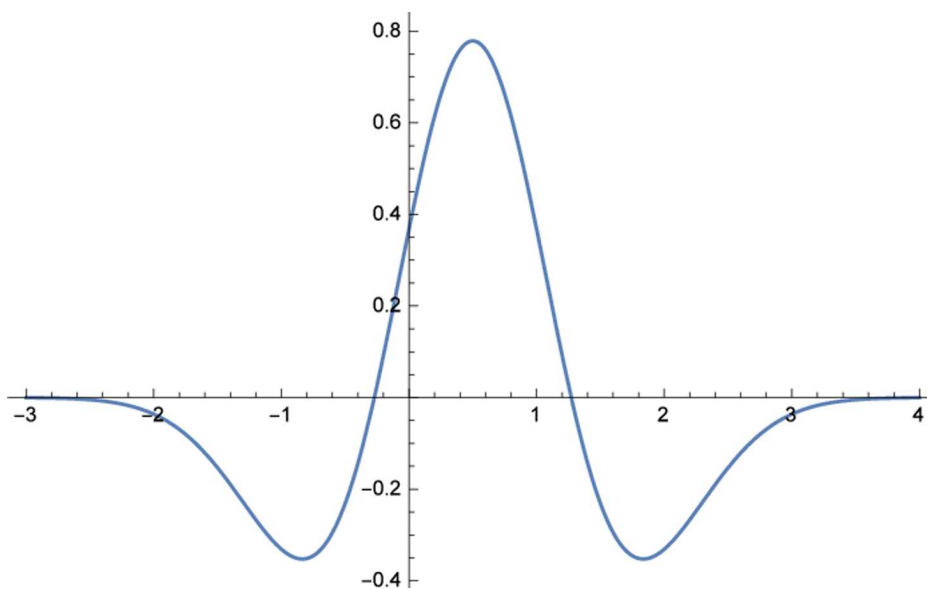
Fig. 3. Eigenvalue λ_3 for the Γ_1 case.
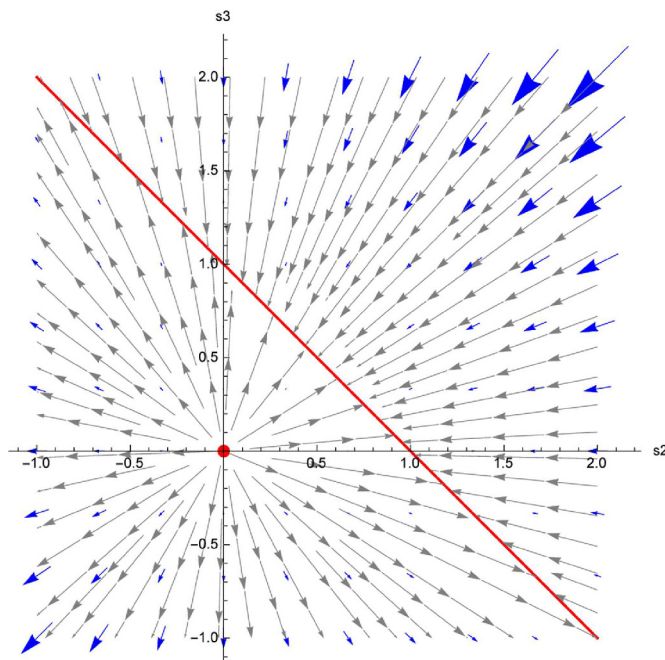
Fig. 4. Vector field in the plane $s_1 = 0$ for Γ_1.

Table 2. Fixed points, stability regime, basin of attraction in the Γ_2 case.

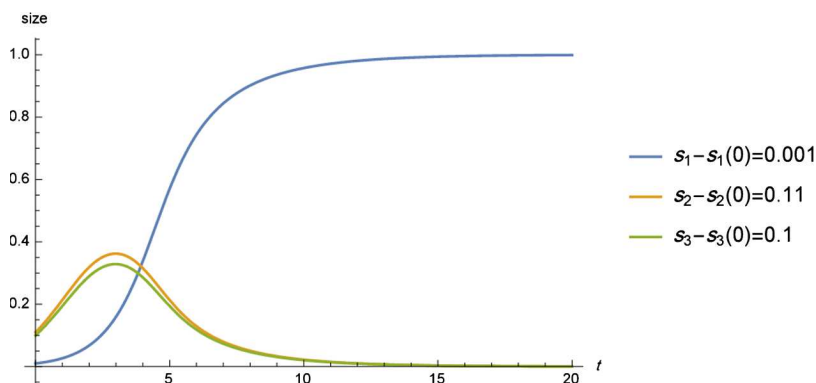| Fixed point	Stability	Basin of attraction
$r_0 = (0, 0, 0)$	Repeller	–
$r_1 = (1, 0, 0)$	Saddle	Axis s_2
$r_2 = (0, 1, 0)$	Stable	The surface $s_3 = 0$
$r_3 = (0, 0, 1)$	Stable	The surface $s_2 = 0$
The line $L = (0, s_2, 1 - s_2)$	Stable	Global

Fig. 5. Time evolution of the size of love for the case Γ_1.

3.2 One-way love Γ_2 type

$$\Gamma_2 = \begin{pmatrix} 0 & 1 & 1 \\ -1 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}. \quad (20)$$

The fixed points, stability regimes, and basins of attraction are summarized in Table 2.

– $r_0 = (0, 0, 0)$ is an unstable fixed point and its eigenvalues are $\lambda_1 = 1$, $\lambda_2 = 1$, $\lambda_3 = 1$.
– $r_1 = (1, 0, 0)$ is a saddle point with eigenvalues $\lambda_1 = -1$, $\lambda_2 = 0.3679$, $\lambda_3 = -0.3679$. The eigenvector corresponding to the eigenvalue $\lambda_1 = -1$ is $\overrightarrow{u_1} = (1, 0, 0)$; thus, it is an attractor along the axis s_1.
– $r_2 = (0, 1, 0)$ is a stable fixed point: the eigenvalues are $\lambda_1 = -1$, $\lambda_2 = -0.3679$, $\lambda_3 = 0$; the corresponding eigenvectors are $\overrightarrow{u_1} = (0, 1, 0)$, $\overrightarrow{u_2} = (-1, 1, 0)$ and $\overrightarrow{u_3} = (0, -1, 1)$; see Figure 7.
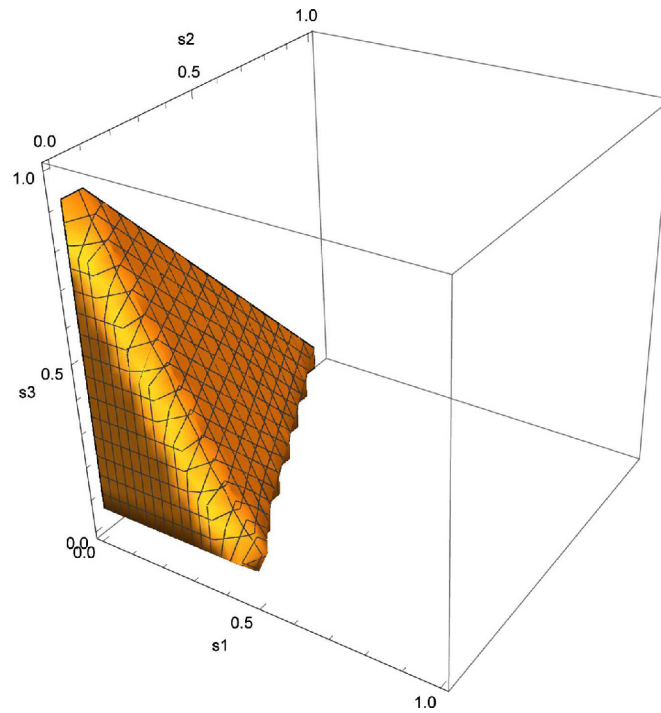
Fig. 6. Phase space region of weak attraction between the 3 lovers for the Γ_1 case.
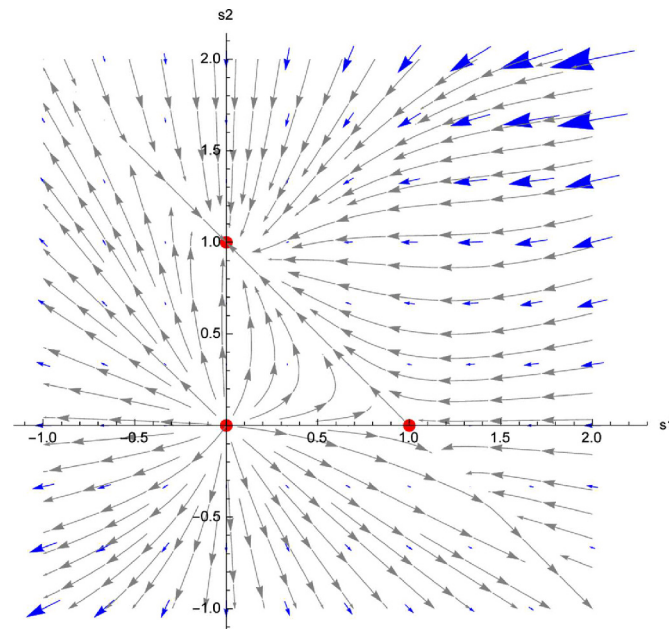
Fig. 7. Vector field in the plane $s_3 = 0$ for Γ_2. This plane contains a stable point $r_2 = (0,1,0)$ and two unstable points $r_0 = (0,0,0)$, $r_1 = (1,0,0)$.

- $r_3 = (0,0,1)$ is similar to the point r_2; thus, it is a stable fixed point; he eigenvalues are $\lambda_1 = -1$, $\lambda_2 = -0.3679$, $\lambda_3 = 0$; the corresponding eigenvectors are $\overrightarrow{u_1} = (0,0,1)$, $\overrightarrow{u_2} = (1,0,-1)$ and $\overrightarrow{u_3} = (0,1,-1)$.
- There is a line of fixed points in the plane $s_1 = 0$. All the points on the line $L = (0, s_2, 1 - s_2)$ are

fixed points; their eigenvalues are:

$$\lambda_1 = 0, \quad \lambda_2 = -1,$$
$$\lambda_3 = e^{-(1-s_2)^2 - s_2^2}$$
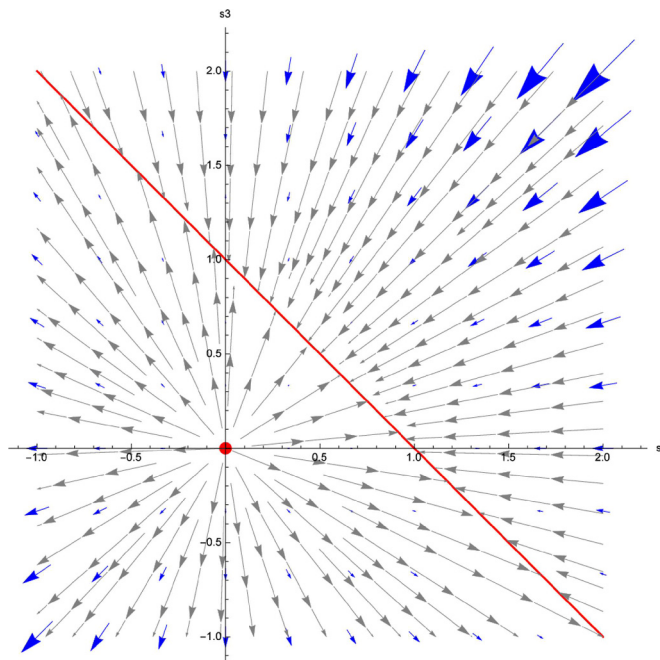$$\times(-e^{s_2^2} - e^{(1-s_2)^2} s_2 + e^{s_2^2} s_2). \quad (21)$$

Fig. 8. Vector field in the plane $s_1 = 0$ for the Γ_2 case.
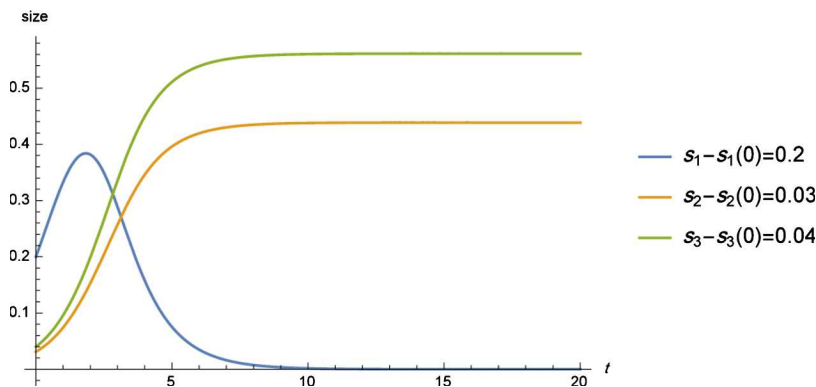

Fig. 9. Example of time evolution of the size of love for the case Γ_2.

The eigenvalue λ_3 is negative in the region $0 < s_1, s_2, s_3 < 1$, so that these points are stable and attractors. The eigenvector corresponding to the eigenvalue $\lambda_2 = -1$ is $\overrightarrow{u_2} = \left(0, -\frac{s_2}{s_2-1}, 1\right)$; therefore all the points in the plane $s_1 = 0$ converge to this line along the direction u_2; see Figure 8.

All the points in the space except the axis s_1 converge to this line. This result seems intuitive, as A_1 is not interested in A_2 and A_3 at all, A_1 cannot be impressed by the lovers A_2 and A_3. But as A_2 and A_3 are fascinated by A_1 the amount of their love increases up to be deeply in love with A_1 forever.

When the initial love of the lovers A_2 and A_3 is small and under the condition $1 - s_1 - s_2 - s_3 > 0$ with $\dot{s}_1 > 0$, the lover A_1 gets somehow interested in A_2 and A_3 and demands more attention from these lovers; whence its love increases at first. As an example consider a movie star at the beginning of its work when he or she is not a celebrated person and is known by a few people, the movie star is interested in getting fans and greedily demands their attention. But later on, A_1 looses to be interested; thereafter s_1 decreases and ends up at zero; see Figure 9. One can numerically find a region in which s_1 increases at first; Figure 10.
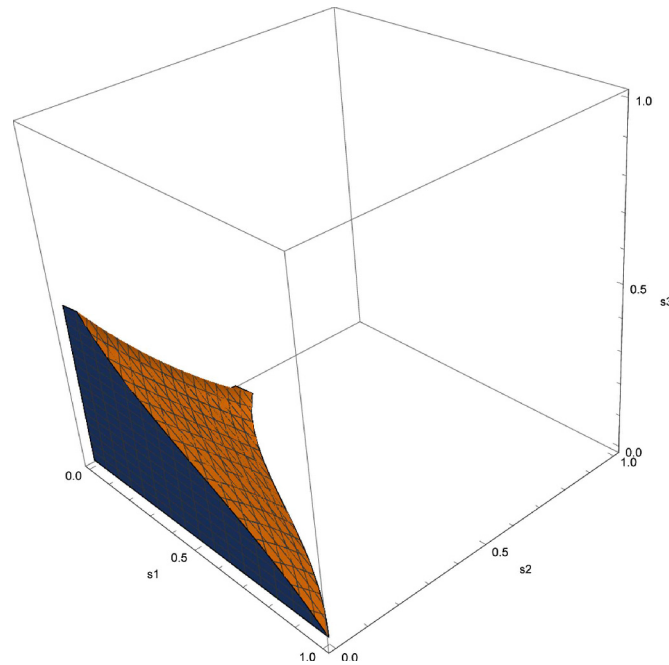
Just like in the previous section, one finds the vector field to be tangent to the surface $s_2 = s_3$; all the trajectories remain on this surface forever and converge to the fixed point $r_1 = (0, 0.5, 0.5)$.
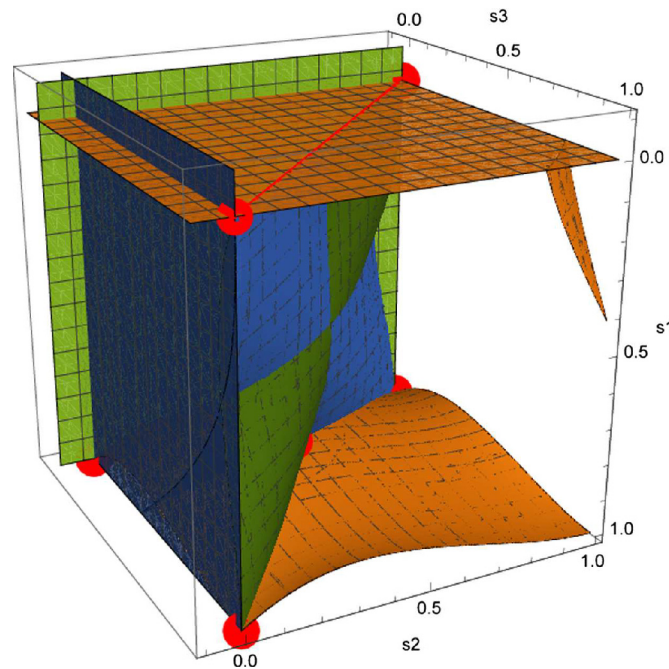
3.3 Love triangle Γ_3 type

For the Γ_3 case, the three evolution surfaces $s_1 = 0$, $s_2 = 0$ and $s_3 = 0$ can be displayed in the phase space as on Figure 11.

$$\Gamma_3 = \begin{pmatrix} 0 & -1 & -1 \\ -1 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}. \tag{22}$$

Fig. 10. Numerical calculation and display of the region in which s_1 increases at first, in the case Γ_2.

Fig. 11. For the Γ_3 case, the three evolution surfaces $\dot{s}_1 = 0$, $\dot{s}_2 = 0$ and $\dot{s}_3 = 0$ can be displayed in the phase space as: the blue surface corresponds to $\dot{s}_1 = 0$, the green one is for $\dot{s}_2 = 0$, and the $\dot{s}_3 = 0$ surface is illustrated in orange. The fixed points are indicated by the red points.

– $r_0 = (0,0,0)$ is a trivial fixed point of the system. It is unstable, the eigenvalues of the Jacobian matrix are positive $\lambda_1 = 1$, $\lambda_2 = 1$, $\lambda_3 = 1$.
– There are three fixed points on the surface $s_2 = 0$. $r_1 = (1,0,1)$, $r_2 = (1,0,0)$, $r_3 = (0,0,1)$; r_1 is an attractor fixed point; its eigenvalues are $\lambda_1 = -1$, $\lambda_2 = -1$, $\lambda_3 = -1 + 1/e \equiv -0.632$.

The eigenvalues of the point r_2 are $\lambda_1 = -1$, $\lambda_2 = 0.3679$, $\lambda_3 = 0.3679$; thus it is a saddle point. The eigenvector corresponding to $\lambda_1 = -1$ is $\overrightarrow{u_1} = (1,0,0)$. Therefore, it is an attractor along the axis s_1. On the other hand, it is a repeller along the two directions $\overrightarrow{u_1} = (-0.42, 0.91, 0)$ and $\overrightarrow{u_1} = (-0.42, 0, 0.91)$; Figure 12.
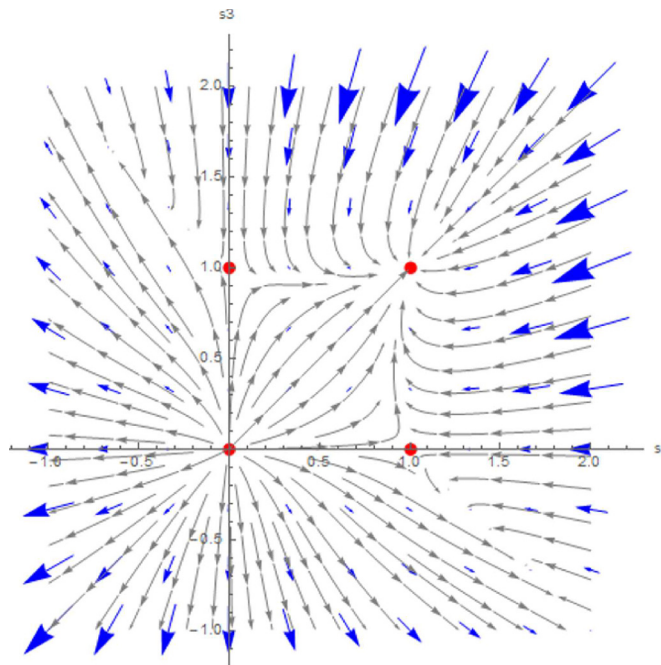
Fig. 12. The vector field in the plane $s_2 = 0$ for Γ_3. This plane contains a stable fixed point $r_1 = (1, 0, 1)$ and three unstable fixed points $r_0 = (0, 0, 0)$, $r_2 = (1, 0, 0)$, $r_3 = (0, 0, 1)$.

Similarly, the point r_3 is a saddle point with the eigenvalues $\lambda_1 = -1$, $\lambda_2 = 0.3679$, $\lambda_3 = 0$. The eigenvector corresponding to $\lambda_1 = -1$ is $\overrightarrow{u_1} = (0, 0, 1)$. Therefore, it is an attractor along the axis s_3.

- There are two other fixed points $r_4 = (1, 1, 0)$, $r_5 = (0, 1, 0)$ on the $s_3 = 0$ plane.

  The fixed point r_4 is similar to the fixed point r_1 and its eigenvalues are the same $\lambda_1 = -1, \lambda_2 = -1, \lambda_3 = -0.632$; thus, it is an attractor fixed point.

  The eigenvalues of the point r_5 are $\lambda_1 = -1, \lambda_2 = 0.3679, \lambda_3 = 0$. The eigenvector corresponding to $\lambda_1 = -1$ is $\overrightarrow{u_1} = (0, 1, 0)$. So this point is a saddle point which attracts along the axis s_2; see Figure 13.

- There exists a line of fixed points in the plane $s_1 = 0$. In other words, all the points on the line $L = (0, s_2, 1 - s_2)$ are fixed points. These points are saddle points and their eigenvalues are:

$$\lambda_1 = 0, \quad \lambda_2 = -1,$$
$$\lambda_3 = e^{-(1-s_2)^2 - s_2^2}$$
$$\times (e^{s_2^2} + e^{(1-s_2)^2} s_2 - e^{s_2^2} s_2). \quad (23)$$

Since $0 < s_1, s_2, s_3 < 1$, the eigenvalue λ_3 is greater than 0 in this interval, so these points are saddle points. The eigenvector corresponding to $\lambda_2 = -1$ is $\overrightarrow{u_2} = (0, -\frac{s_2}{-1+s_1}, 1)$ and is located in the plane $s_1 = 0$; thus, all the points in this plane converge to the line of fixed points; see Figure 14.

- The r_6 fixed point $\approx (0.8565, 0.4282, 0.4282)$ is an unstable saddle point. Indeed, the eigenvalues are

$\lambda_1 \approx -1.2615, \lambda_2 \approx -0.7130, \lambda_3 \approx 0.2615$; the corresponding eigenvectors are:

$$\overrightarrow{u_1} = (-0.8165, -0.4082, -0.4082), \quad (24)$$

$$\overrightarrow{u_2} = (0.2429, 0.6859, 0.6859), \quad (25)$$

$$\overrightarrow{u_3} = (0, -0.7071, 0.7071). \quad (26)$$

This point is a repeller along the third eigenvector $\overrightarrow{u_3}$ with eigenvalue $\lambda_1 \approx -1.2615$. Thus, all the trajectories perpendicular to this vector at this point will not be repelled. The surface $s_2 = s_3$ is perpendicular to the vector $\overrightarrow{u_3}$; the inner product of vector field $\overrightarrow{S} = (\dot{s_1}, \dot{s_1}, \dot{s_1})$ lies on the surface $s_2 = s_3$; the normal vector to the surface $\overrightarrow{u_3}$ is zero:

$$\overrightarrow{S} \cdot \overrightarrow{u_3} = 0. \quad (27)$$

Therefore, all the trajectories on the surface $s_2 = s_3$ remain on this surface and converge to the fixed point r_6. Thus, the surface $s_2 = s_3$ is the basin of attraction of this point and all the trajectories on this surface converge to the fixed point r_6; Figure 15.

The trajectories which are not on the surface $s_2 = s_3$ diverge from the surface. The points that are located under this surface $s_3 < s_2$ converge to the stable fixed point $r_4 = (1, 1, 0)$ (Fig. 16); the points that are located above this surface $s_3 > s_2$ converge to the fixed point $r_1 = (1, 0, 1)$; see Figure 17.

The results are summarized in Table 3.
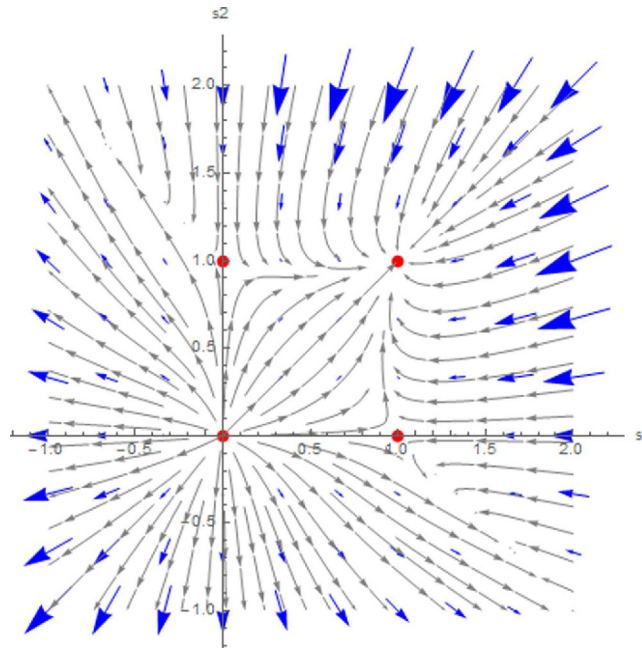
Fig. 13. Vector field in the plane $s_3 = 0$ for Γ_3, it is similar to the vector field in the plane $s_2 = 0$ due to the symmetry in dynamical equations. This plane contains a stable fixed point $r_4 = (1, 1, 0)$ and three unstable fixed points $r_0 = (0, 0, 0)$, $r_2 = (1, 0, 0)$, $r_5 = (0, 1, 0)$.
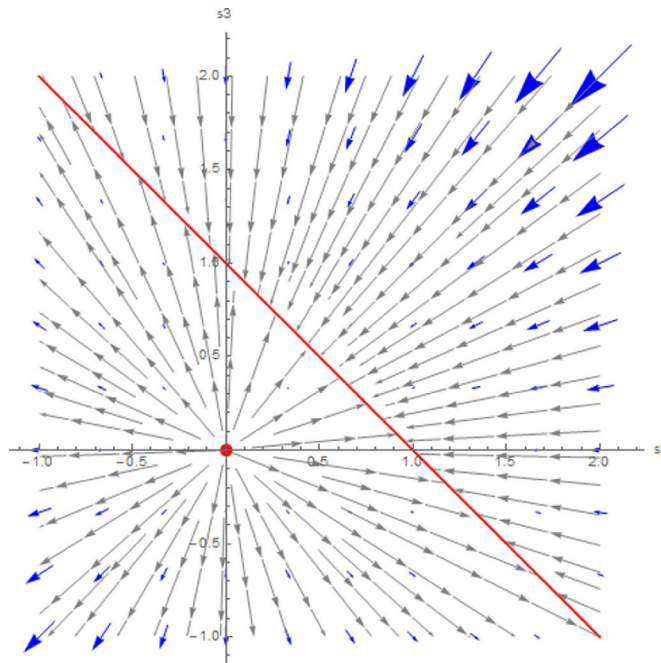

Fig. 14. Vector field in the plane $s_1 = 0$ for the Γ_3 case.

So, two fixed points $r_1 = (1, 0, 1)$ and $r_4 = (1, 1, 0)$ correspond to a situation in which one of the lovers wins the love game in a love triangle and can get all the attention of the lover A_1; if one of the lovers A_2 and A_3 is initially more interested in A_1, it will capture the full attention of the lover A_1.

But if the two lovers love A_1 equally at first, no one wins the game and both share the love and attention of A_1, and reach the point r_6. As this fixed point is strongly unstable and if the condition changes a little bit, the love path diverges from the point r_6. For example, if the love of A_2 regresses and gets lower than A_3,
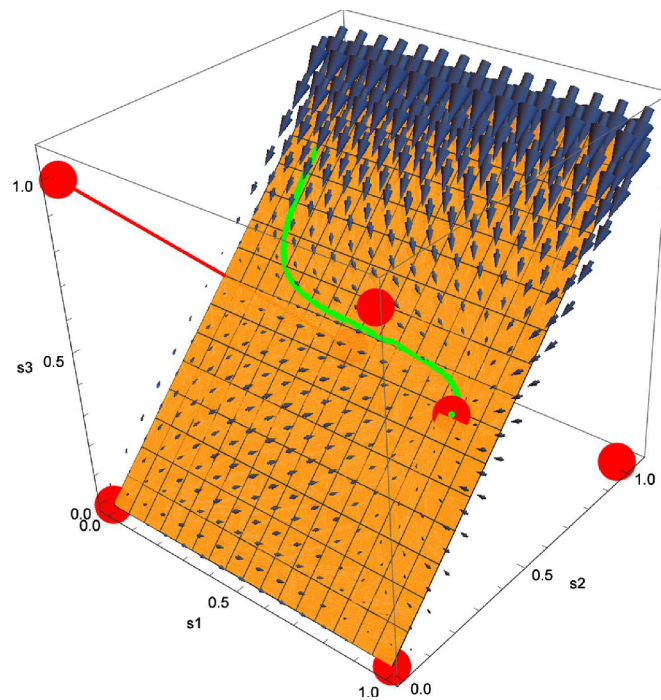
Fig. 15. Vector field in the plane $s_2 = s_3$ for Γ_3; fixed points indicated by red dots.
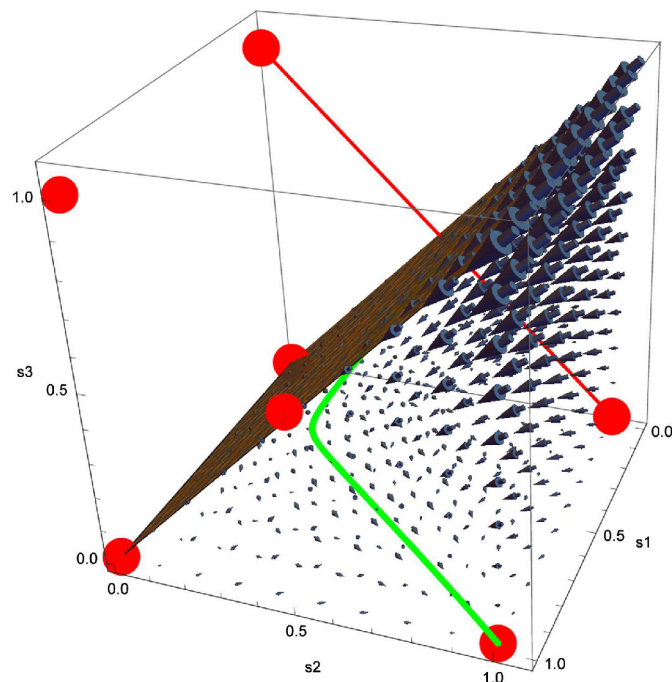
Fig. 16. The vector field in the region $s_3 < s_2$ for Γ_3. The trajectories of an arbitrary initial point in the region $s_3 < s_2$ converge to $r_4 = (1, 1, 0)$; fixed points indicated by red dots.
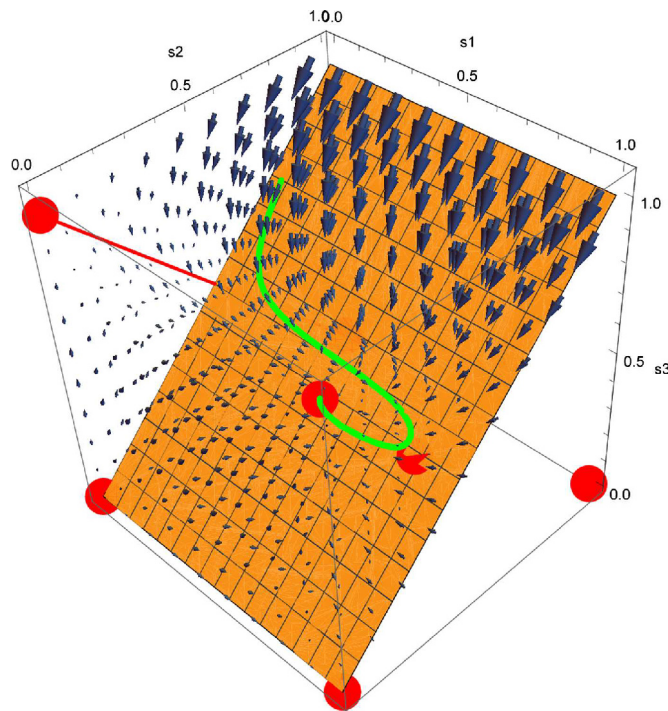
Fig. 17. The vector field in the region $s_3 > s_2$ for Γ_3. The trajectory of an arbitrary initial point in the region $s_3 > s_2$ converges to $r_1 = (1, 0, 1)$.

Table 3. Fixed points, stability regimes, basins of attraction in the Γ_3 case.

Fixed point	Stability	Basin of attraction
$r_0 = (0, 0, 0)$	Repeller	–
$r_1 = (1, 0, 1)$	Attractor	$s_3 > s_2$
$r_2 = (1, 0, 0)$	Saddle	Axis s_1
$r_3 = (0, 0, 1)$	Saddle	Axis s_3
$r_4 = (1, 1, 0)$	Attractor	$s_3 < s_2$
$r_5 = (0, 1, 0)$	Saddle	Axis s_2
$r_6 \approx (0.8565, 0.4282, 0.4282)$	Saddle	The surface $s_2 = s_3$
The line $L = (0, s_2, 1 - s_2)$	Saddle	The surface $s_1 = 0$

then A_2 will loose the game and the love path ends up in $r_1 = (1, 0, 1)$.

3.4 Love triangle Γ_4 type

Now consider:

$$\Gamma_4 = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}. \tag{28}$$

In this case the fixed points and their stability quite differ from the previous case. The three surfaces $\dot{s_1} = 0$, $\dot{s_2} = 0$ and $\dot{s_3} = 0$ are illustrated in Figure 18.

– The point $r_0 = (0, 0, 0)$ is an unstable fixed point with eigenvalues $\lambda_1 = 1$, $\lambda_2 = 1$, $\lambda_3 = 1$.
– There are three fixed points on the plane $s_2 = 0$. The point $r_1 = (0.3333, 0, 0.3333)$ with the eigenvalues $\lambda_1 = -1$, $\lambda_2 = 0.3333$, $\lambda_3 = 0.3505$ is an unstable saddle point. It is attractor along $\overrightarrow{u_1} = (1, 0, 1)$ with

the eigenvalue $\lambda_1 = -1$, as shown in Figure 19. The point $r_2 = (0, 0, 1)$ is an attractor fixed point and its eigenvalues are $\lambda_1 = -1$, $\lambda_2 = -0.3679$, $\lambda_3 = 1 - 0$. The point $r_3 = (1, 0, 0)$ with eigenvalues $\lambda_1 = -1$, $\lambda_2 = -0.3679$, $\lambda_3 = -0.3679$ is an attractor.

– There are two other fixed points on the plane $s_3 = 0$. The point $r_4 = (0.3333, 0.3333, 0)$ with eigenvalues $\lambda_1 = -1$, $\lambda_2 = 0.3333$, $\lambda_3 = 0.3505$ is an unstable saddle point. All the points along the eigenvector $\overrightarrow{u_1} = (1, 1, 0)$ converge to this point; see Figure 20. The point $r_5 = (0, 1, 0)$ with eigenvalues $\lambda_1 = -1$, $\lambda_2 = -0.3679$, $\lambda_3 = 0$ is an attractor.

– There is a line of fixed points in the plane $s_1 = 0$. All the points on the line $L = (0, s_2, 1 - s_2)$ are fixed points with eigenvalues:
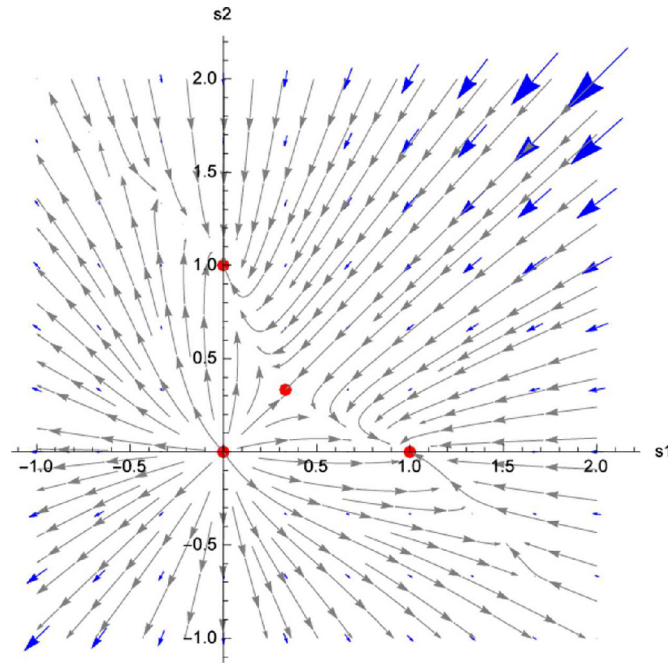
$$\lambda_1 = 0, \quad \lambda_2 = -1,$$
$$\lambda_3 = e^{-(1-s_2)^2 - s_2^2}$$
$$\times \left(-e^{s_2^2} - e^{(1-s_2)^2} s_2 + e^{s_2^2} s_2 \right). \tag{29}$$
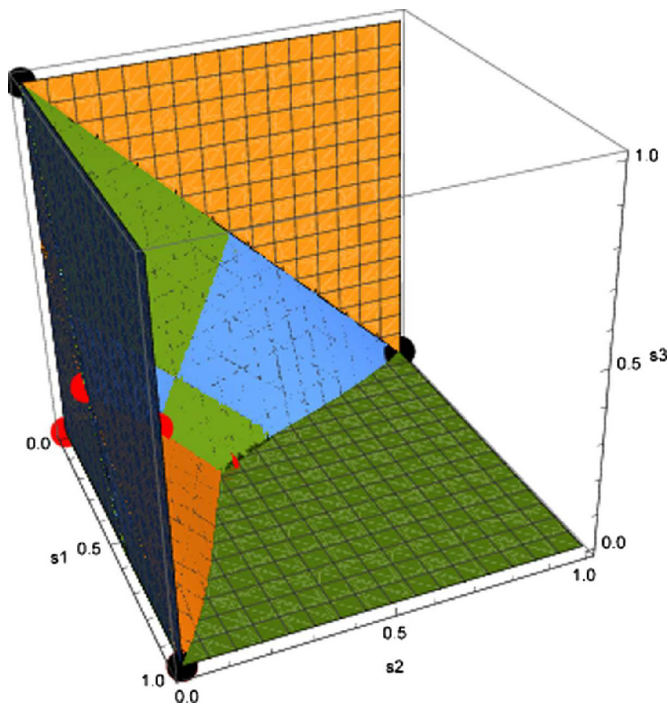
Fig. 18. The three evolution surfaces $\dot{s}_1 = 0$, $\dot{s}_2 = 0$ and $\dot{s}_3 = 0$ for the Γ_4 case. The orange one is the surface $\dot{s}_1 = 0$, the blue one is $\dot{s}_2 = 0$ and the surface $\dot{s}_3 = 0$ is illustrated in green. The stable fixed points are the the black points and the unstable fixed points are the red points.
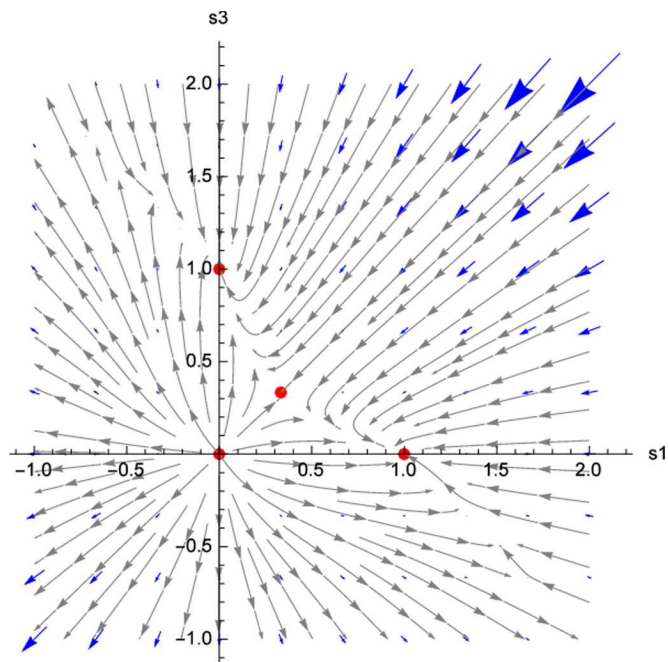
Fig. 19. Vector field on the plane $s_2 = 0$, for the Γ_4 case.

Fig. 20. Vector field on the plane $s_3 = 0$ for the Γ_4 case.

The third eigenvector λ_3 is negative in the region $0 < s_2 < 1$. So, unlike in the previous case, all the fixed points on this line are stable; Figure 21.

– In the region $0 < s_1, s_2, s_3 < 1$, there is a fixed point $r_6 \approx (0.3365, 0.1682, 0.1682)$ which is an unstable saddle point; its eigenvalues are $\lambda_1 \approx -0.9815$, $\lambda_2 \approx 0.3271$, $\lambda_3 \approx -0.0185$. Consider the surface $-s_1 + s_2 + s_3 = 0$. The normal vector of this surface is $\overrightarrow{n} = (-1, 1, 1)$ The vector field on this surface is:

$$\dot{s_1} = (s_2 + s_3)(1 + (-2 - e^{-s_3^2})s_2 + (-2 - e^{-s_2^2})s_3), \tag{30}$$

$$\dot{s_2} = s_2 \left(1 - 2\, s_2 - 2\, s_3 - e^{-s_3^2}(s_2 + s_3)\right), \tag{31}$$

$$\dot{s_3} = s_3 \left(1 - 2\, s_2 - 2\, s_3 - e^{-s_2^2}(s_2 + s_3)\right). \tag{32}$$

The inner product of vector field $\overrightarrow{S} = (\dot{s_1}, \dot{s_1}, \dot{s_1})$ on the surface $-s_1 + s_2 + s_3 = 0$ and the normal vector $\overrightarrow{n}$ to the surface, $\overrightarrow{n} = (-1, 1, 1)$, is zero.

$$\overrightarrow{S} \cdot \overrightarrow{n} = 0. \tag{33}$$

Therefore, the vector field is perpendicular to $\overrightarrow{n}$; thus, the vector field is tangent to the surface and all the trajectories remain on this surface. These trajectories converge to $r_6 \approx (0.3365, 0.1682, 0.1682)$. The trajectories initially located under the $s_2 + s_3 < s_1$ surface converge to the stable fixed point $r_3 = (1, 0, 0)$ (see Fig. 22); those located above the $s_2 + s_3 > s_1$ surface converge to the line $L = (0, s_2, 1 - s_2)$; see Figure 23.

As an example for this case of interaction, consider three countries A_1, A_2 and A_3. There is a hostility between the country A_1 and the two countries A_2 and A_3 but there is no any hostility or alliance between the countries A_2 and A_3.

The point $r_3 = (1, 0, 0)$ corresponds to a situation in which A_1 is interested in the two others and demands a good relationship, but A_2 and A_3 are not interested in A_1 and tend to boycott it.

As mentioned before there is a fixed point on the surface $s_1 = s + 2 + s_3$ for which the basin of attraction is $s_1 = s_2 + s_3$. This means that under this special condition, there is an unusual fixed point in which all the countries/lovers are interested in each other and seek a mutually peaceful relationship, like a "menage a 3". However, this balanced state is extremely unstable; any small deviation (perturbation) from the surface, gives rise to a divergent trajectory from the fixed point, either leading to the fixed point r_3 or to the line L.

(A "symmetric solution" is possible: the line $L = (0, s_2, 1 - s_2)$ turns out to be obviously the opposite of this previously discussed situation.)

The results are summarized in Table 4.

4 Conclusions

In this paper, we have tried to describe one of the most mysterious aspects of a human life, love, in terms of complex system theories and nonlinear dynamics. We have used the Verhulst–Lotka–Volterra prey–predator model in order to take into account competitive or cooperative attitudes between "agents". We have considered 3 agents ("lovers") for establishing the minimal complexity set of nonlinear differential equations, and have introduced
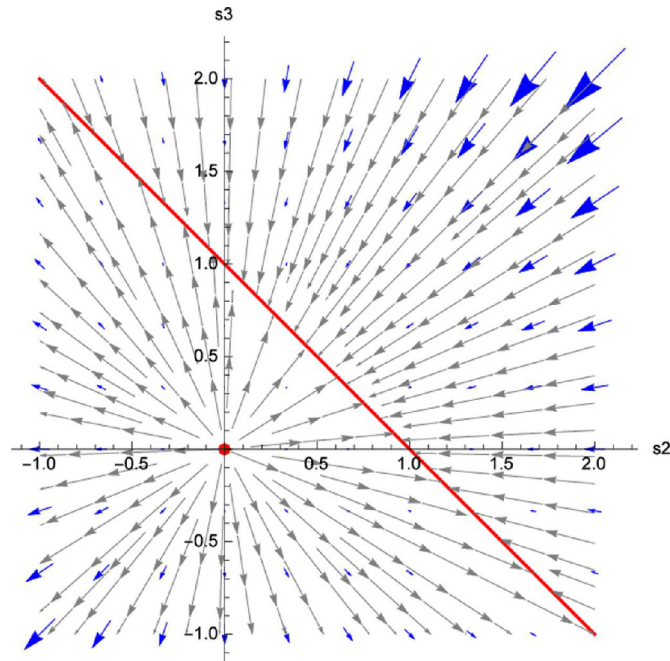
**Fig. 21.** Vector field in the plane $s_1 = 0$, for the Γ_4 case.
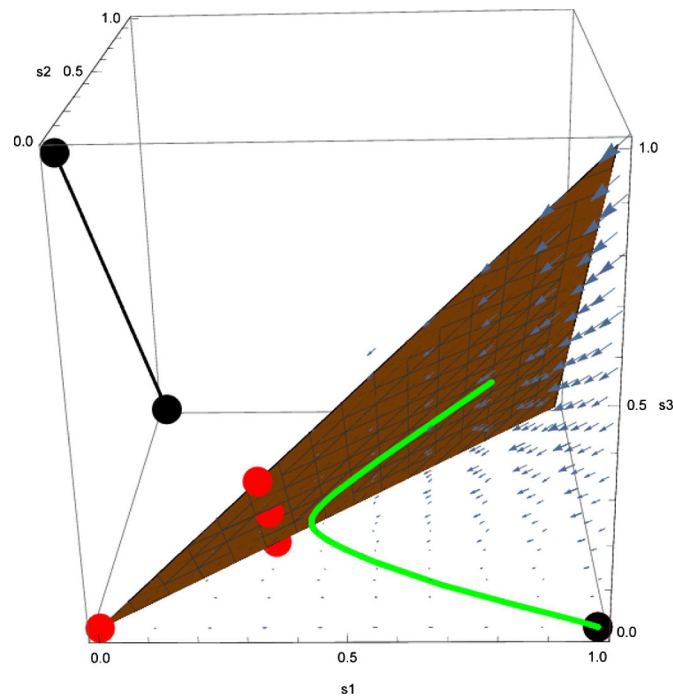
Fig. 22. Illustration of the phase space in the Γ_4 case, when trajectories initially located under the $s_2 + s_3 < s_1$ surface converge to the stable fixed point $r_3 = (1, 0, 0)$; the opposite case is shown in Figure 23.

different interaction matrices in order to mimick some (well-known) situations in a 3 partner love story game. We have tried to answer some questions, like: What is the end of a love story in different scenarios? Under which condition will occur different situations? Which ones of these final situations are stable and under which conditions?

In this love triangle, we have further reduced the number of scenarios, only considering that there is no direct interaction (love or hate) between the lovers A_2 and A_3, yet keeping A_1 in relation with both neighboring partners. Therefore, most of the time one of the rivals wins the love game and the other one will be completely ignored.
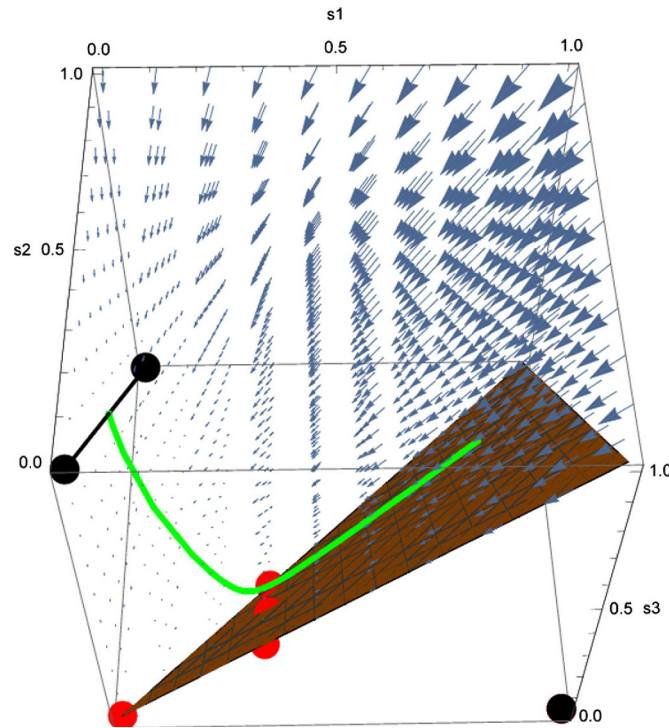
Fig. 23. Illustration of the phase space in the Γ_4 case, when trajectories initially located above the $s_2 + s_3 > s_1$ surface converge to the line $L = (0, s_2, 1 - s_2)$; the opposite case is shown in Figure 22.

Table 4. Fixed points, stability regimes, basins of attraction in the Γ_4 case.

Fixed point	Stability	Basin of attraction
$r_0 = (0, 0, 0)$	Repeller	$-$
$r_1 = (0.3333, 0, 0.3333)$	Saddle	$s_2 = 0, s_1 = s_3$
$r_2 = (0, 0, 1)$	Attractor	$s_2 = 0, s_1 < s_3$
$r_3 = (1, 0, 0)$	Attractor	$s_2 + s_3 < s_1$
$r_4 = (0.3333, 0.3333, 0)$	Saddle	$s_3 = 0, s_1 = s_2$
$r_5 = (0, 1, 0)$	Attractor	$s_3 = 0, s_1 < s_2$
$r_6 \approx (0.3365, 0.1682, 0.1682)$	Saddle	The surface $s_1 = s_2 + s_3$
The line $L = (0, s_2, 1 - s_2)$	Attractor	$s_2 + s_3 > s_1$

However under some conditions, as outlined in the main text, the game does not necessarily have any "winner"; there is a possible stable trio of lovers. Interestingly, this situation is very sensitive and is not stable. This means that it is hard to love several persons simultaneously, in this kind of model. It is fair to stress that this difficulty is found at the end of the game. Indeed, displayed curves show that two "love sizes" can be equal. We stress that, in this case, the love size flows have opposite signs. Then, there is necessarily a divergence: one loves grows, the other decays.

We have also considered the opposite of the previous love triangle case, that is A_1 is not interested in A_2 and A_3. As an example for this case consider the political relationship among three countries which are hostile together. We argued that in this case, in spite of the unfriendly direct interaction between the agents, there is always some agent looking for a friendly relationship; under special conditions, described in the previous sections, everybody demands friendly relationship with the others. But this situation is again not stable. We have discussed two different scenarios in this so called "one-way love". In the first scenario, the central lover A_1 is interested by the two agents A_2 and A_3 but these do not love (care about) A_1. The second scenario is the opposite: the two lovers A_2 and A_3 love A_1, but the agent A_1 is not interested in them. The analogy with "public stars" or "celebrity" has been mentioned in the main text.

In the first scenario, A_1 will eventually be deeply in love with the two others, but these two will not be fond of A_1. Under special conditions in which A_1 starts the love game from a low amount of interest, the two agents A_2 and A_3 get somehow interested in the central lover A_1; it seems they are appealing for more love and attention from the lover A_1 at the beginning of their love journey.

In the second scenario of one-way love, A_1 will not be in love with the two other lovers, but these two lovers reach a balance point due to their competition in loving A_1. Just like in the first scenario, if the lovers A_2 and A_3 start the love game from a low amount of interest, the central agent

will be interested in them and be looking for more love and attention, right from the beginning of the love journey.

In our analysis, we have supposed that the strength of the interaction is a binary variable, whatever the couples, and is time modulated in a self-organized, symmetric, and preferential way through some Gaussian strength weight; we have also supposed that the interaction term between A_2 and A_3 can be neglected. In order to achieve a deeper understanding it can be fruitful to reconsider these constraints in future research. It seems that it would be useful also to think about the evolution of love under "external fields", and about the dynamics of the system over time, allowing for memory effects [16–18].

Author contribution statement

Authors contributed in various complex ways to this paper. Most of the numerical work and the figures are due to P. Toranj Simin skill.

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