

A note on the entropy of rotating BPS $\text{AdS}_7 \times S^4$ black holes

Seyed Morteza Hosseini,^{a,b,c} Kiril Hristov^d and Alberto Zaffaroni^{b,c}

^a*Kavli IPMU (WPI), UTIAS, The University of Tokyo,
Kashiwa, Chiba, 277-8583 Japan*

^b*Dipartimento di Fisica, Università di Milano-Bicocca,
Piazza della Scienza 3, Milano, I-20126 Italy*

^c*INFN, sezione di Milano-Bicocca,
Piazza della Scienza 3, Milano, I-20126 Italy*

^d*Institute for Nuclear Research and Nuclear Energy, Bulgarian Academy of Sciences,
Tsarigradsko Chaussee 72, Sofia, 1784 Bulgaria*

E-mail: morteza.hosseini@ipmu.jp, khristov@inrne.bas.bg,
alberto.zaffaroni@mib.infn.it

ABSTRACT: In this note we show that the entropy of BPS, rotating, electrically charged $\text{AdS}_7 \times S^4$ black holes can be obtained by an extremization principle involving a particular combination of anomaly coefficients of the six-dimensional $\mathcal{N} = (2, 0)$ theory. This result extends our previous finding for BPS, rotating $\text{AdS}_5 \times S^5$ black holes.

KEYWORDS: AdS-CFT Correspondence, Black Holes in String Theory, Extended Supersymmetry

ARXIV EPRINT: [1803.07568](https://arxiv.org/abs/1803.07568)

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1 Introduction

There has been some recent progress in deriving the entropy of BPS static, asymptotically AdS₄ magnetically charged black holes that can be embedded in string/M-theory [1–6]. The method uses a dual field theory computation based on localization. There are also many examples of BPS, electrically charged, rotating black holes in various dimensions whose entropy cannot be yet explained in this way. The most famous ones are asymptotic to AdS₅ × S⁵ [7–11]. They depend on three electric charges Q_I ($I = 1, 2, 3$), associated with rotations in S⁵, and two angular momenta J_ϕ, J_ψ in AdS₅. Supersymmetry actually requires a constraint among the charges and only four of them are independent. The derivation of their entropy in terms of states of the dual $\mathcal{N} = 4$ SU(N) super Yang-Mills (SYM) theory is still an open problem [12–14]. The natural place where to look for such derivation is the superconformal index [12, 15]

$$\mathcal{I}_{S^3 \times S^1}(\Delta_I, \omega_i) = \text{Tr}_{\mathcal{H}}(-1)^F e^{-2\pi i(\sum_{I=1}^3 \Delta_I r_I - \sum_{i=1}^2 \omega_i h_i)}, \tag{1.1}$$

where h_i are the generators of angular momentum, r_i are the Cartan generators of the SO(6) R-symmetry and ω_i, Δ_I are the conjugate chemical potentials, respectively. (1.1) is defined for $\Delta_1 + \Delta_2 + \Delta_3 + \omega_1 + \omega_2 \in \mathbb{Z}$ since the exponent should commute with the relevant supercharge and chemical potentials are only defined modulo one. The index counts states preserving the same supersymmetries of the black holes and it depends on a number of fugacities equal to the number of conserved charges of the black holes. However, due to a large cancellation between bosonic and fermionic states, the superconformal index

is a quantity of order one for generic values of the fugacities while the entropy scales like N^2 [12]. We recently observed [16] that the Bekenstein-Hawking entropy of these black holes can be obtained as the Legendre transform with respect to ω_i and Δ_I of the quantity

$$E^{\text{SU}(N)} = -i\pi N^2 \frac{\Delta_1 \Delta_2 \Delta_3}{\omega_1 \omega_2}, \tag{1.2}$$

with the determination

$$\sum_{I=1}^3 \Delta_I + \sum_{i=1}^2 \omega_i = 1. \tag{1.3}$$

This constraint among chemical potentials reflects the constraint among charges of the black holes and is compatible with the constraint in the index (1.1). The quantity (1.2) can be expressed in terms of the flavored cubic t'Hooft anomaly coefficients of $\mathcal{N} = 4$ SYM. Indeed it can be obtained by an equivariant integral of the anomaly polynomial of the theory, as shown in [17].¹

In this short note we extend our observation to BPS, electrically charged, rotating black holes in $\text{AdS}_7 \times S^4$. We expect a family of black holes depending on two electric charges Q_I ($I = 1, 2$), associated with rotations in S^4 , and three angular momenta J_i ($i = 1, 2, 3$) in AdS_5 . Supersymmetry again requires a constraint among the charges and only four of them are independent. The dual field theory is the A_{N-1} $\mathcal{N} = (2, 0)$ theory in six dimensions. Inspired by the AdS_5 result, we consider the expression for the equivariant integral of the anomaly polynomial of the theory, which, at large N , is given by [17]

$$E^{(A_{N-1})}(\Delta_I, \omega_i) = i\pi N^3 \frac{(\Delta_1 \Delta_2)^2}{12\omega_1 \omega_2 \omega_3}. \tag{1.4}$$

We will show that the Bekenstein-Hawking entropy of seven-dimensional BPS black holes can be obtained by extremizing the quantity

$$- E^{(A_{N-1})}(\Delta_I, \omega_i) - 2\pi i \sum_{I=1}^2 \Delta_I Q_I - 2\pi i \sum_{i=1}^3 \omega_i J_i, \tag{1.5}$$

with respect to Δ_I, ω_i with the constraint

$$\sum_{I=1}^2 \Delta_I + \sum_{i=1}^3 \omega_i = 1. \tag{1.6}$$

The general black hole solution depending on all four conserved quantities is not available but we will check that (1.5) correctly reproduces the entropy of the existing solutions. A

¹With a different choice of angular determination for the chemical potentials Δ_I and ω_i , the quantity (1.2) becomes the supersymmetric Casimir energy of the theory (see for example eq. (4.50) in [17]). The supersymmetric Casimir energy can be interpreted as the energy of the vacuum [18] and it arises both as a prefactor in the relation that connects the supersymmetric partition function on $S^3 \times S^1$ to the index [19, 20] and also as a term in the high-temperature expansion of the index [21, 22]. All these results seem to have been obtained assuming a particular angular determination of fugacities, which implies, in particular, $\Delta_1 + \Delta_2 + \Delta_3 + \omega_1 + \omega_2 = 0$, instead of (1.3).

two-parameter black hole, with two electric charges and one angular momentum, was found in [23] as the BPS limit of a non-extremal solution [24]. The solution was later extended to a three-parameter one, with three rotations and one electric charge, in [25]. We have explicitly checked the validity of (1.5) in both cases and are thus confident that the result holds in general.

It is remarkable that the equivariant integral of the anomaly polynomial accounts for the entropy of both the $\text{AdS}_5 \times S^5$ and $\text{AdS}_7 \times S^4$ supersymmetric black holes. Moreover, it is noteworthy that, in both cases, the solutions to the extremization equations associated with (1.2) and (1.4) are complex but the value of the Legendre transform at the critical points, the entropy, is a *real* function of the black hole charges. This result still needs a proper field theory interpretation. Here we just make few observations.

According to the standard holographic dictionary, the black hole entropy should account for the 1/16-BPS states in the $\mathcal{N} = (2, 0)$ theory with given electric charge and angular momentum. As in four dimensions, the partition function for such BPS states is too difficult to compute due to the small amount of preserved supersymmetry. The superconformal index of the $\mathcal{N} = (2, 0)$ theory, on the other hand, counts states weighted with signs,²

$$\mathcal{I}_{S^5 \times S^1}(\Delta_I, \omega_i) = \text{Tr}_{\mathcal{H}}(-1)^F e^{-2\pi i(\sum_{I=1}^2 \Delta_I r_I - \sum_{i=1}^3 \omega_i h_i)}, \quad (1.7)$$

where h_i are the generators of angular momentum, r_I are the Cartan generators of the $\text{SO}(5)$ R-symmetry, and $\Delta_1 + \Delta_2 + \omega_1 + \omega_2 + \omega_3 \in \mathbb{Z}$. The index is explicitly computable but, as in four dimensions, is a quantity of order one for generic values of the fugacities. The $S^5 \times S^1$ partition function, on the other hand, is related to the index in the large N limit by [26]

$$\log Z_{S^5 \times S^1}(\Delta_I, \omega_i) = -E_{\text{susy}}(\Delta_I, \omega_i) + \log \mathcal{I}_{S^5 \times S^1}(\Delta_I, \omega_i), \quad (1.8)$$

where E_{susy} is the supersymmetric Casimir energy and scales like $\mathcal{O}(N^3)$. It would be interesting to see if the quantity (1.4) dominates the index or the partition function in some particular regime for the chemical potentials and a choice of angular determination compatible with the constraint (1.6). An expression for E_{susy} has been conjectured in [17] by integrating the anomaly polynomial of the $\mathcal{N} = (2, 0)$ theory, and formally coincides with (1.5). However, the conjecture seems to assume a different angular determination for the chemical potentials, compatible with $\sum_{I=1}^2 \Delta_I + \sum_{i=1}^3 \omega_i = 0$ rather than (1.6).

The $\text{AdS}_5 \times S^5$ and $\text{AdS}_7 \times S^4$ black holes behave quite differently from their magnetically charged relatives in $\text{AdS}_4 \times S^7$ whose entropy has been recently derived [1, 2]. The main difference comes from the magnetic charges that have a dual interpretation as a topological twist [27, 28]. The topologically twisted index [29, 30] has been shown to scale like $N^{3/2}$ in the large N limit [1, 31, 32], suggesting that there is no cancellation between bosonic and fermionic ground states, while the superconformal index is a quantity of order one. It should also be noticed that the derivation of the entropy of $\text{AdS}_4 \times S^7$ black holes is a purely microscopic counting with no reference to the ubiquitous Cardy formula [33]. On the other hand, whatever its field theory interpretation is, the extremization principles

²In our notations, Q_I are eigenstates of r_I and J_i of $-h_i$.

in five and seven dimensions suggests that some sort of Cardy mechanism is at work. First of all, as already said, (1.2) and (1.4) can be written in terms of anomaly coefficients for the R and flavor symmetries of the dual theory. This is simple to see for (1.2). Indeed, by an obvious redefinition of the chemical potentials (see appendix A), (1.2) can be written as the large N limit of³

$$E^{\text{SU}(N)} = \frac{4\pi i}{27} \frac{(\omega_1 + \omega_2 - 1)^3}{\omega_1 \omega_2} a(\hat{\Delta}_I). \tag{1.9}$$

Here

$$a(\hat{\Delta}_I) = \frac{9}{32} \text{Tr} R^3(\hat{\Delta}_I) = \frac{27}{32} (N^2 - 1) \hat{\Delta}_1 \hat{\Delta}_2 \hat{\Delta}_3, \tag{1.10}$$

together with $\hat{\Delta}_1 + \hat{\Delta}_2 + \hat{\Delta}_3 = 2$ is the trial central charge of $\mathcal{N} = 4$ SYM. (1.4) can be written similarly since it arises from an equivariant integration of the eight-form anomaly polynomial of the 6D $\mathcal{N} = (2, 0)$ theory [17]. Moreover, for an extremal BTZ black hole in AdS₃ the relevant quantity to consider is the elliptic genus, whose logarithm in the large N limit goes as c_l/ω where ω is the chemical potential associated with rotation and c_l is the left-moving central charge. The entropy of a black hole of angular momentum j is then correctly reproduced by the Legendre transform with respect to ω , i.e. $S \propto \sqrt{j c_l}$. Obviously, this is nothing else than Cardy formula.

The paper is organized as follows. In section 2 we first review the basic features of the relevant truncation of seven-dimensional maximal gauged supergravity and we later discuss the BPS, rotating black holes of interest. In section 3, we show that the Bekenstein-Hawking entropy of the black holes can be obtained as the Legendre transform of the quantity (1.4). We conclude in section 4 with discussions and future directions. In the appendices we briefly discuss a conjecture to extend our previous result for black holes in AdS₅ × S⁵ to more general compactifications and the dimensional reduction of the seven-dimensional black holes to six dimensions.

2 Supersymmetric rotating AdS₇ black holes

The supersymmetric rotating AdS₇ black holes of interest are solutions of the SO(5) maximal ($\mathcal{N} = 4$) gauged supergravity in seven dimensions [34], obtained by reducing eleven-dimensional supergravity on S⁴ [35, 36]. We will work with a U(1)² consistent truncation [37] of the theory, which consists of the metric, a three-form potential $A_{(3)}$, two Abelian gauge fields $A_{(1)}^I$ ($I = 1, 2$) in the Cartan of SO(5) and two real scalars φ_1 and φ_2 . The bosonic Lagrangian is given by [34]⁴

$$\begin{aligned} \mathcal{L}_7 = & R \star 1 - \frac{1}{2} \sum_{i=1}^2 \star d\varphi_i \wedge d\varphi_i - \frac{1}{2} \sum_{I=1}^2 L_I^{-2} \star F_{(2)}^I \wedge F_{(2)}^I - \frac{1}{2} (L_1 L_2)^2 \star F_{(4)} \wedge F_{(4)} \\ & - 2g^2 [(L_1 L_2)^{-4} - 8L_1 L_2 - 4L_1^{-1} L_2^{-2} - 4L_1^{-2} L_2^{-1}] \star 1 \\ & - g F_{(4)} \wedge A_{(3)} + F_{(2)}^1 \wedge F_{(2)}^2 \wedge A_{(3)}, \end{aligned} \tag{2.1}$$

³Comparing to expressions for the supersymmetric Casimir energy that can be found in [18–20] we have an extra minus one in the numerator which is due to the constraint (1.6).

⁴Here we use the conventions of [23].

where

$$\begin{aligned} F_{(2)}^I &= dA_{(1)}^I, & F_{(4)} &= dA_{(3)}, \\ L_1 &= e^{-\frac{1}{\sqrt{2}}\varphi_1 - \frac{1}{\sqrt{10}}\varphi_2}, & L_2 &= e^{\frac{1}{\sqrt{2}}\varphi_1 - \frac{1}{\sqrt{10}}\varphi_2}, \end{aligned} \tag{2.2}$$

and g is the gauge coupling constant. There is a ‘‘first-order self-duality’’ condition for the four-form field strength that has to be imposed after the variation of the Lagrangian and that can be conveniently written by including a two-form potential $A_{(2)}$, and defining

$$F_{(3)} = dA_{(2)} - \frac{1}{2}A_{(1)}^1 \wedge dA_{(1)}^2 - \frac{1}{2}A_{(1)}^2 \wedge dA_{(1)}^1. \tag{2.3}$$

The self-duality equation then reads

$$(L_1 L_2)^2 \star F_{(4)} = -2gA_{(3)} - F_{(3)}. \tag{2.4}$$

We are interested in supersymmetric black holes with electric charges Q_I ($I = 1, 2$) under the $U(1)^2$ and angular momenta J_i ($i = 1, 2, 3$) in AdS_7 . We expect supersymmetry to impose a constraint among the five charges, leaving four independent ones. The most general family of such black holes has not been written yet. A two-parameter black hole, with two electric charges and one angular momentum with a constraint among them, was found in [23]. A three-parameter family of black holes, with three rotations and one electric charge with a constraint, was later found in [25]. Note that in both cases the near-horizon geometry is a warped product of AdS_2 and a squashed S^5 . We now write explicitly these solutions and their thermodynamic quantities. We have corrected few misprints in [23].

2.1 Single-rotation two-charge black holes

The solution can be written as [23]

$$\begin{aligned} ds^2 &= (H_1 H_2)^{1/5} \left(-\frac{V}{H_1 H_2 B} r^2 dt^2 + B(\sigma + f dt)^2 + \frac{dr^2}{V} + r^2 ds_{\mathbb{C}P^2}^2 \right), \\ A_{(1)}^I &= \frac{2ms_I}{\rho^4 \Xi H_I} (\alpha_I \Xi_- dt + \beta_I \sigma), \\ A_{(2)} &= \frac{mas_1 s_2}{\rho^4 \Xi_-} \left(\frac{1}{H_1} + \frac{1}{H_2} \right) dt \wedge \sigma, & A_{(3)} &= \frac{2mas_1 s_2}{\rho^2 \Xi \Xi_-} \sigma \wedge J, \\ L_I &= (H_1 H_2)^{2/5} H_I^{-1}, & H_I &= 1 + \frac{2ms_I^2}{\rho^4}, & \rho &= \sqrt{\Xi} r, \\ \alpha_1 &= c_1 - \frac{1}{2}(1 - \Xi_+^2)(c_1 - c_2), & \alpha_2 &= c_2 + \frac{1}{2}(1 - \Xi_+^2)(c_1 - c_2), \\ \beta_1 &= -a\alpha_2, & \beta_2 &= -a\alpha_1, & \Xi_{\pm} &= 1 \pm ag, & \Xi &= 1 - a^2 g^2, \\ s_I &\equiv \sinh \delta_I, & c_I &\equiv \cosh \delta_I. \end{aligned} \tag{2.5}$$

The metric functions V , B and f depends on the radial coordinate r and are given by

$$V = \frac{Y}{\Xi \rho^6}, \quad B = \frac{f_1}{H_1 H_2 \Xi^2 \rho^4}, \quad f = -\frac{2f_2 \Xi_-}{f_1}, \tag{2.6}$$

where

$$\begin{aligned}
f_1 &= \Xi\rho^6 H_1 H_2 - \frac{4\Xi_+^2 m^2 a^2 s_1^2 s_2^2}{\rho^4} + \frac{1}{2} m a^2 [4\Xi_+^2 - 2c_1 c_2 (\Xi_+^4 - 1) + (c_1^2 + c_2^2) (\Xi_+^2 - 1)^2] , \\
f_2 &= -\frac{1}{2} g \Xi_+ \rho^6 H_1 H_2 + \frac{1}{4} m a [2c_1 c_2 (\Xi_+^4 + 1) - (c_1^2 + c_2^2) (\Xi_+^4 - 1)] , \\
Y &= g^2 \rho^8 H_1 H_2 + \Xi \rho^6 + \frac{1}{2} m a^2 [4\Xi_+^2 - 2c_1 c_2 (\Xi_+^4 - 1) + (c_1^2 + c_2^2) (\Xi_+^2 - 1)^2] \\
&\quad - \frac{1}{2} m \rho^2 [4\Xi + 2c_1 c_2 a^2 g^2 (3a^2 g^2 + 8ag + 6) - (c_1^2 + c_2^2) a^2 g^2 (ag + 2)(3ag + 2)] .
\end{aligned} \tag{2.7}$$

Only two parameters are independent due to the constraints⁵

$$\begin{aligned}
e^{\delta_1 + \delta_2} &= 1 - \frac{2}{3ag} , \\
m &= \frac{128e^{\delta_1 + \delta_2} (3e^{\delta_1 + \delta_2} - 1)^3}{729g^4 (e^{2\delta_1} - 1)(e^{2\delta_2} - 1)(e^{\delta_1 + \delta_2} + 1)^2 (e^{\delta_1 + \delta_2} - 1)^4} .
\end{aligned} \tag{2.8}$$

The former comes from the BPS condition and the latter is required in order to avoid naked closed timelike curves (CTCs). With these choices, the function V becomes

$$V = \frac{g^2 (r^2 - r_0^2)^2}{r^2} \left(1 + \frac{9e^{2(\delta_1 + \delta_2)} - 6e^{\delta_1 + \delta_2} + 17}{3(e^{\delta_1 + \delta_2} + 1)(3e^{\delta_1 + \delta_2} - 5)g^2 r^2} + \frac{h}{g^4 r^4} \right) , \tag{2.9}$$

where

$$\begin{aligned}
h &= \left[32(-2d_1^2 - 2d_2^2 + 9d_1 d_2 + 9d_1^5 d_2^5 - 3d_1^3 d_2^3 (d_1 + d_2)^2 + 2d_1^2 d_2^2 (2d_1^2 - 3d_1 d_2 + 2d_2^2) \right. \\
&\quad \left. - d_1 d_2 (3d_1^2 - 2d_1 d_2 + 3d_2^2)) \right] / \left[9d_1 d_2 (d_1^2 - 1)(d_2^2 - 1)(d_1 d_2 + 1)(3d_1 d_2 - 5)^2 \right] ,
\end{aligned} \tag{2.10}$$

and we defined $d_I \equiv e^{\delta_I}$ ($I = 1, 2$). The black hole has an event horizon at $V(r_0) = 0$ which reads

$$r_0^2 = \frac{16}{3g^2 (e^{\delta_1 + \delta_2} + 1)(3e^{\delta_1 + \delta_2} - 5)} . \tag{2.11}$$

The line element $ds_{\mathbb{CP}^2}^2$ in (2.5) is the standard Fubini-Study metric on \mathbb{CP}^2 :

$$ds_{\mathbb{CP}^2}^2 = d\xi^2 + \frac{1}{4} \sin^2 \xi (\sigma_1^2 + \sigma_2^2 + \cos^2 \xi \sigma_3^2) , \tag{2.12}$$

where σ_i ($i = 1, 2, 3$) are left-invariant one-forms on $SU(2)$, satisfying $d\sigma_i = -\frac{1}{2}\epsilon_{ijk}\sigma_j \wedge \sigma_k$. Note that, the Kähler form on \mathbb{CP}^2 is $J = \frac{1}{2}d\mathcal{B}$ with $\mathcal{B} = \frac{1}{2}\sin^2 \xi \sigma_3$ being the connection of the $U(1)$ bundle over \mathbb{CP}^2 whose total space is the unit S^5 . We also have $\sigma = d\psi + \mathcal{B}$ and $0 \leq \psi \leq 2\pi$ is the coordinate along the $U(1)$ fiber of S^5 . The thermodynamic quantities

⁵We correct a misprint in [23] here, i.e. $m_{\text{here}} = (3e^{\delta_1 + \delta_2} - 1)m_{\text{there}}$.

are given by⁶

$$\begin{aligned}
E &= \frac{m\pi^2}{32G_N\Xi^4} \left[12\Xi_+^2(\Xi_+^2 - 2) - 2c_1c_2a^2g^2(21\Xi_+^4 - 20\Xi_+^3 - 15\Xi_+^2 - 10\Xi_+ - 6) \right. \\
&\quad \left. + (c_1^2 + c_2^2)(21\Xi_+^6 - 62\Xi_+^5 + 40\Xi_+^4 + 13\Xi_+^2 - 2\Xi_+ + 6) \right], \\
S &= \frac{\pi^3}{4G_N} [B(r_0)H_1(r_0)H_2(r_0)]^{1/2}r_0^4, \\
J &= \frac{ma\pi^2}{16G_N\Xi^4} \left[4ag\Xi_+^2 - 2c_1c_2(2\Xi_+^5 - 3\Xi_+^4 - 1) + ag(c_1^2 + c_2^2)(\Xi_+ + 1)(2\Xi_+^3 - 3\Xi_+^2 - 1) \right], \\
Q_I &= \frac{m\pi^2s_I}{4G_N\Xi^3} \left[a^2g^2\frac{c_1c_2}{c_I}(2\Xi_+ + 1) - c_I(2\Xi_+^3 - 3\Xi_+^2 - 1) \right], \\
T &= 0, \quad \Omega = -g, \quad \Phi_I = -1.
\end{aligned} \tag{2.13}$$

The charges satisfy the BPS condition $E + 3gJ - \sum_{I=1}^2 Q_I = 0$.

2.2 Three-rotation single-charge black holes

The solution reads [25]

$$\begin{aligned}
ds^2 &= H^{2/5} \left\{ \frac{(r^2 + y^2)(r^2 + z^2)}{R} dr^2 + \frac{(r^2 + y^2)(y^2 - z^2)}{Y} dy^2 + \frac{(r^2 + y^2)(z^2 - y^2)}{Z} dz^2 \right. \\
&\quad - \frac{R}{H^2(r^2 + y^2)(r^2 + z^2)} \mathcal{A}^2 \\
&\quad + \frac{Y}{(r^2 + y^2)(y^2 - z^2)} \left[dt + (z^2 - r^2)d\psi_1 - r^2z^2d\psi_2 - \frac{q}{H(r^2 + y^2)(r^2 + z^2)} \mathcal{A} \right]^2 \\
&\quad + \frac{Z}{(r^2 + y^2)(z^2 - y^2)} \left[dt + (y^2 - r^2)d\psi_1 - r^2y^2d\psi_2 - \frac{q}{H(r^2 + y^2)(r^2 + z^2)} \mathcal{A} \right]^2 \\
&\quad + \frac{a_1^2a_2^2a_3^2}{r^2y^2z^2} \left[dt + (y^2 + z^2 - r^2)d\psi_1 + (y^2z^2 - r^2y^2 - r^2z^2)d\psi_2 - r^2y^2z^2d\psi_3 \right. \\
&\quad \left. - \frac{q}{H(r^2 + y^2)(r^2 + z^2)} \left(1 + \frac{gy^2z^2}{a_1a_2a_3} \right) \mathcal{A} \right]^2 \Big\}, \\
L &= H^{-1/5}, \quad A_{(1)} = -\frac{q(1 - a_1g - a_2g - a_3g)}{H(r^2 + y^2)(r^2 + z^2)} \mathcal{A}, \\
A_{(2)} &= \frac{q}{H(r^2 + y^2)(r^2 + z^2)} \mathcal{A} \\
&\quad \wedge \left\{ dt + \sum_{i=1}^3 a_i^2(g^2dt + d\psi_i) + \sum_{i<j} a_i^2a_j^2(g^2d\psi_i + d\psi_j) + a_1^2a_2^2a_3^2(g^2d\psi_2 + d\psi_3) \right. \\
&\quad \left. - g^2(y^2 + z^2)dt - g^2y^2z^2d\psi_1 + a_1a_2a_3g [d\psi_1 + (y^2 + z^2)d\psi_2 + y^2z^2d\psi_3] \right\},
\end{aligned}$$

⁶We correct a misprint in [23] here, i.e. $S_{\text{here}} = \frac{1}{4}S_{\text{there}}$.

$$\begin{aligned}
 A_{(3)} &= qa_1a_2a_3 [d\psi_1 + (y^2 + z^2)d\psi_2 + y^2z^2d\psi_3] \\
 &\wedge \left[\frac{1}{(r^2 + y^2)z} dz \wedge (d\psi_1 + y^2d\psi_2) + \frac{1}{(r^2 + z^2)y} dy \wedge (d\psi_1 + z^2d\psi_2) \right] \\
 &- qg\mathcal{A} \wedge \left[\frac{z}{r^2 + y^2} dz \wedge (d\psi_1 + y^2d\psi_2) + \frac{y}{r^2 + z^2} dy \wedge (d\psi_1 + z^2d\psi_2) \right],
 \end{aligned} \tag{2.14}$$

where

$$\begin{aligned}
 R &= \frac{(r^2 - r_0^2)^2}{r^2} \left\{ g^2r^4 + [1 + (a_1^2 + a_2^2 + a_3^2)g^2 + 2g^2r_0^2] r^2 + \frac{(a_1a_2a_3 - qg)^2}{r_0^4} \right\}, \\
 Y &= \frac{1 - g^2y^2}{y^2} \prod_{i=1}^3 (a_i^2 - y^2), \quad Z = \frac{1 - g^2z^2}{z^2} \prod_{i=1}^3 (a_i^2 - z^2), \\
 \mathcal{A} &= dt + (y^2 + z^2)d\psi_1 + y^2z^2d\psi_2, \\
 H &= 1 + \frac{q}{(r^2 + y^2)(r^2 + z^2)}.
 \end{aligned} \tag{2.15}$$

The black hole has an event horizon at $r = r_0$:

$$r_0^2 = \frac{a_1a_2 + a_2a_3 + a_3a_1 - a_1a_2a_3g}{1 - a_1g - a_2g - a_3g}. \tag{2.16}$$

We also denote $\Xi_i = 1 - a_i^2g^2$ and $\Xi_{i\pm} = 1 \pm a_i g$, which are positive to have a correct signature. The parameters q and a_i have to satisfy

$$q = -\frac{\Xi_{1-}\Xi_{2-}\Xi_{3-}(a_1 + a_2)(a_2 + a_3)(a_1 + a_3)}{(1 - a_1g - a_2g - a_3g)^2g}, \tag{2.17}$$

in order for the solution to be free from naked CTCs. The thermodynamic quantities are given by

$$\begin{aligned}
 E &= -\frac{\pi^2}{8G_N} \frac{\prod_{k<l}(a_k + a_l) \left[\sum_i \Xi_i + \sum_{i<j} \Xi_i \Xi_j - (1 + a_1a_2a_3g^3) \left(2 + \sum_i a_i g + \sum_{i<j} a_i a_j g^2 \right) \right]}{\Xi_{1+}\Xi_{2+}\Xi_{3+}(1 - a_1g - a_2g - a_3g)^2gr_0}, \\
 S &= -\frac{\pi^3}{4G_N} \frac{(a_1 + a_2)(a_2 + a_3)(a_1 + a_3)(a_1a_2 + a_2a_3 + a_1a_3 - a_1a_2a_3g)}{\Xi_{1+}\Xi_{2+}\Xi_{3+}(1 - a_1g - a_2g - a_3g)^2gr_0}, \\
 J_i &= -\frac{\pi^2}{8G_N} \frac{(a_1 + a_2)(a_2 + a_3)(a_1 + a_3) \left[a_i - \left(a_i^2 + 2a_i \sum_{j \neq i} a_j + \prod_{j \neq i} a_j \right) g + a_1a_2a_3g^2 \right]}{\Xi_{1+}\Xi_{2+}\Xi_{3+}\Xi_{i+}(1 - a_1g - a_2g - a_3g)^2g}, \\
 Q &= -\frac{\pi^2}{4G_N} \frac{(a_1 + a_2)(a_2 + a_3)(a_1 + a_3)}{\Xi_{1+}\Xi_{2+}\Xi_{3+}(1 - a_1g - a_2g - a_3g)g}, \\
 T &= 0, \quad \Omega_i = -g, \quad \Phi = -1.
 \end{aligned} \tag{2.18}$$

Finally, the charges satisfy the BPS condition

$$E + g \sum_{i=1}^3 J_i - 2Q = 0. \tag{2.19}$$

3 An extremization principle for the entropy

In this section we will show that the Bekenstein-Hawking entropy of the BPS black holes (2.5) and (2.14) can be obtained as a Legendre transform of a combination of anomaly coefficients of the dual $\mathcal{N} = (2, 0)$ theory in six dimensions. The result is a natural generalization of the analogous one for $\text{AdS}_5 \times S^5$ black holes [16].

3.1 Anomaly polynomials for 6D $\mathcal{N} = (2, 0)$ field theories

Our analysis involves a quantity formally equal to the supersymmetric Casimir energy of the theory, which we now briefly review.

The supersymmetric Casimir energy, $E^{(\mathfrak{g})}$, for an $\mathcal{N} = (2, 0)$ theory with algebra \mathfrak{g} arises in the regularization of the $S^5 \times S^1$ partition function [26, 38–43] and is related to the superconformal index defined in (1.7) by

$$\log Z_{S^5 \times S^1}(\Delta_I, \omega_i) = -E^{(\mathfrak{g})}(\Delta_I, \omega_i) + \log \mathcal{I}_{S^5 \times S^1}(\Delta_I, \omega_i). \quad (3.1)$$

It is the leading contribution to $\log Z_{S^5 \times S^1}$ for $\beta \rightarrow \infty$, where β is the radius of S^1 when the chemical potentials are rescaled as $\Delta_I = \beta \hat{\Delta}_I$ and $\omega_i = \beta \hat{\omega}_i$.⁷ Since the superconformal index is a quantity of order one for generic values of the fugacities, the supersymmetric Casimir energy, which scales as N^3 , is also the leading contribution to the $S^5 \times S^1$ partition function in the large N limit. The supersymmetric Casimir energy of the $\mathcal{N} = (2, 0)$ theory has been conjectured to be equal to an equivariant integral of the eight-form anomaly polynomial and it reads [17]

$$E^{(\mathfrak{g})}(\Delta_I, \omega_i) = r_{\mathfrak{g}} E^{(1)}(\Delta_I, \omega_i) + \frac{i\pi}{12} d_{\mathfrak{g}} h_{\mathfrak{g}}^{\vee} \frac{(\Delta_1 \Delta_2)^2}{\omega_1 \omega_2 \omega_3}, \quad (3.2)$$

where $r_{\mathfrak{g}}$, $d_{\mathfrak{g}}$ and $h_{\mathfrak{g}}^{\vee}$ are the rank, dimension and dual Coxeter number of the simply laced Lie algebra \mathfrak{g} , respectively; $E^{(1)}$ is the supersymmetric Casimir energy of the Abelian tensor multiplet theory:

$$E^{(1)}(\Delta_I, \omega_i) = \frac{i\pi}{24\omega_1\omega_2\omega_3} \left[(\Delta_1 \Delta_2)^2 - \sum_{i < j} (\omega_i \omega_j)^2 + \frac{1}{4} \left(\sum_{i=1}^3 \omega_i^2 - \Delta_1^2 - \Delta_2^2 \right) \right]. \quad (3.3)$$

Here Δ_I ($I = 1, 2$) are the chemical potentials conjugate to the R-symmetry generators r_I and ω_i ($i = 1, 2, 3$) are the chemical potentials conjugate to the Cartan generators of rotations h_i in three orthogonal planes in \mathbb{R}^6 .

Superconformal indices are defined in general as

$$\mathcal{I}(\mu_a) = \text{Tr}_{\mathcal{H}}(-1)^F e^{-\beta\{Q, Q^\dagger\}} e^{-\sum_a \mu_a R_a}, \quad (3.4)$$

for a choice of supercharge Q , and R_a is the set of all R and flavor symmetries that commute with Q . For the superconformal index (1.7), the linear combination $\sum_{i=1}^3 h_i + \sum_{I=1}^2 r_I$ does

⁷We have reabsorbed a standard factor of β in the definition of $E^{(\mathfrak{g})}$ for convenience. (3.1) is usually written as $\log Z_{S^5 \times S^1} = -\beta E^{(\mathfrak{g})} + \log \mathcal{I}_{S^5 \times S^1}$.

not commute with Q . This translates into a linear constraint among the chemical potentials

$$\sum_{I=1}^2 \Delta_I + \sum_{i=1}^3 \omega_i = n, \quad n \in \mathbb{Z}. \quad (3.5)$$

Notice that, since r_I and J_i have integer eigenvalues, the chemical potentials are only defined modulo one. For this reason the right-hand side of (3.5) is not required to vanish but it must be an integer. In four dimensions, where an analogous constraint appears in the definition of the four-dimensional superconformal index, the statement (3.1) has been derived under the assumption $n = 0$. In six dimensions, things are less clear.

In this paper we shall consider the quantity (3.2), arising from the equivariant integral of the eight-form anomaly polynomial, for a general choice of angular ambiguities in (3.5). For $\mathfrak{g} = A_{N-1}$, the equivariant integral at large N reads

$$E^{(A_{N-1})}(\Delta_I, \omega_i) = i\pi N^3 \frac{(\Delta_1 \Delta_2)^2}{12\omega_1 \omega_2 \omega_3}. \quad (3.6)$$

3.2 Reproducing the entropy

We now show that the Bekenstein-Hawking entropy of the BPS black holes (2.5) and (2.14) can be obtained by extremizing the quantity

$$\mathcal{S}(\Delta_I, \omega_i) \equiv -E^{(A_{N-1})}(\Delta_I, \omega_i) - 2\pi i \sum_{I=1}^2 \Delta_I Q_I - 2\pi i \sum_{i=1}^3 \omega_i J_i, \quad (3.7)$$

where $E^{(A_{N-1})}$ is given in (3.6), with respect to Δ_I , ω_i and subject to the constraint

$$\sum_{I=1}^2 \Delta_I + \sum_{i=1}^3 \omega_i = 1. \quad (3.8)$$

In order to check it, it is convenient to work with the following parameterization of the chemical potentials:

$$\begin{aligned} \omega_1 &= \frac{1}{1 + z_1 + z_2 + z_3 + z_4}, & \Delta_1 &= \frac{z_1}{1 + z_1 + z_2 + z_3 + z_4}, \\ \Delta_2 &= \frac{z_2}{1 + z_1 + z_2 + z_3 + z_4}, & \omega_2 &= \frac{z_3}{1 + z_1 + z_2 + z_3 + z_4}, \\ \omega_3 &= \frac{z_4}{1 + z_1 + z_2 + z_3 + z_4}. \end{aligned} \quad (3.9)$$

Note that $\sum_{I=1}^2 \Delta_I + \sum_{i=1}^3 \omega_i = 1$. Then the extremization equations become

$$\begin{aligned} \frac{2^7 g^5 G_N}{\pi^2} (q_a - q_0) &= -\frac{z_1^2 z_2^2}{z_3 z_4} (1 + 2/z_a), & \text{for } a = 1, 2, \\ \frac{2^7 g^5 G_N}{\pi^2} (q_b - q_0) &= -\frac{z_1^2 z_2^2}{z_3 z_4} (1 - 1/z_b), & \text{for } b = 3, 4, \end{aligned} \quad (3.10)$$

where we have relabeled the black hole charges as

$$J_1 = q_0, \quad Q_1 = q_1, \quad Q_2 = q_2, \quad J_2 = q_3, \quad J_3 = q_4. \quad (3.11)$$

The value of \mathcal{S} at the critical point \tilde{z}_i , as a function of the charges, is given by

$$\mathcal{S}(Q_I, J_i) = \frac{i\pi^3}{2^6 g^5 G_N} \frac{\tilde{z}_1^2 \tilde{z}_2^2}{\tilde{z}_3 \tilde{z}_4} - 2\pi i J_1. \tag{3.12}$$

By an explicit computation one can check that the solution to the extremization equations is complex; however, quite remarkably, \mathcal{S} at the critical point is a *real* function of the black hole charges. Moreover, by equating two electric charges or three angular momenta, one can check that it precisely coincides with the entropy of the black holes (2.5) and (2.14)

$$\mathcal{S}|_{\text{crit}}(Q_I, J_i) = S_{\text{BH}}(Q_I, J_i). \tag{3.13}$$

In order to compare the field theory inspired result with the gravity ones in (3.13) we made use of the relation between field theory and gravitational parameters in the large N limit, which is given by

$$N^3 = \frac{3\pi^2}{16g^5 G_N}. \tag{3.14}$$

It is quite remarkable that the entropy of the black holes is reproduced as a Legendre transform of the integrated anomaly polynomial with the correct field theory normalization.

4 Discussion and conclusions

In this note we have extended our previous observation [16] that the entropy of BPS, rotating $\text{AdS}_5 \times S^5$ black holes can be written as the Legendre transform of a combination of anomaly coefficients for R and flavor symmetries of the dual theory to the case of $\text{AdS}_7 \times S^4$ black holes. It would be interesting to see if the same results hold only for maximally supersymmetric dual theories or it can be also extended to rotating black holes asymptotic to $\text{AdS}_5 \times Y_5$, where Y_5 is a five-dimensional Sasaki-Einstein manifold. For such black holes there is a natural conjecture that we discuss in appendix A.

An important rôle in our analysis is played by the angular ambiguities (3.5) in the definition of chemical potentials, which affect both the partition function on $S^5 \times S^1$ and the index. A choice of determination for the chemical potentials should be made when performing limits, for example low- and high-temperature, or modular transformations of the integrand of the corresponding matrix models, since these operations typically involve multi-valued functions. Examples in the analogous four-dimensional case can be found in [19–22, 44]. It is then interesting to ask whether there exists a limit in the fugacities, subject to the constraint (1.6), where the quantity (1.4) dominates the partition function or the index. We notice that, also for static magnetically charged black holes in AdS_4 , the ambiguities played a crucial rôle. It was shown indeed in [1, 31, 32] that, if we assume that all the real parts of chemical potentials live in the interval $[0, 2\pi]$, one can find a consistent saddle point for the topologically twisted index and reproduce the entropy of the black holes, only if the sum of all chemical potentials is a very specific multiple of 2π .⁸

⁸More precisely, with the determination $\text{Re } \Delta_I \in [0, 2\pi]$, one finds a saddle point, up to discrete symmetries, only if the sum of all Δ_I appearing in each superpotential term is 2π . The interval $[0, 2\pi]$ in [1, 31, 32] is analogous to the interval $[0, 1]$ in this paper.

It would be also very interesting to compute, using supersymmetric holographic renormalization, the on-shell action of the black holes in $\text{AdS}_5 \times S^5$ and $\text{AdS}_7 \times S^4$. It has been shown in [45–47] that, for a class of BPS static AdS_4 black holes, the on-shell action indeed reproduces the entropy of the black holes and, in the grand canonical picture, the large N limit of the twisted index. It would be interesting to see if we can reproduce (1.4) via a holographic computation. Notice also that anomalies seem to affect the field theory and holographic computation [48, 49]. These anomalies could also be responsible for the choice of determination (1.6).

Finally, we noticed in [16] that the extremization for $\text{AdS}_5 \times S^5$ black holes with equal rotations has a nice interpretation in terms of an attractor mechanism for static black holes in four-dimensional gauged supergravity upon dimensional reduction of the squashed S^3 horizon geometry along the Hopf fiber. It would be interesting to show that a similar mechanism is at work here, using $\mathcal{N} = (1, 1)$ six-dimensional gauged supergravity for the solution that is obtained by dimensional reduction of the squashed S^5 horizon geometry of the black hole with equal angular momenta along the Hopf fiber. We briefly discuss the physical interpretation of this reduction in appendix B.

Acknowledgments

We would like to thank Francesco Benini and Paolo Milan for useful discussions and especially Achilleas Passias for numerous comments and collaboration on a related project. The work of SMH was supported by World Premier International Research Center Initiative (WPI Initiative), MEXT, Japan and in part by the INFN. KH is supported in part by the Bulgarian NSF grant DN08/3 and the bilateral grant STC/Bulgaria-France 01/6. AZ is partially supported by the INFN and ERC-STG grant 637844-HBQFTNCER. SMH would like to thank the Bulgarian Academy of Sciences in Sofia and the String Theory group at University of Padova for their kind hospitality during his visit, where part of this work was done.

A AdS_5 black holes entropy and anomalies in four dimensions

In this appendix we make some remarks on a possible generalization of the relation between the entropy of BPS, rotating AdS_5 black holes and the anomaly polynomials of their field theory duals originally presented in [16]. In particular, we consider supersymmetric black hole solutions asymptotic to $\text{AdS}_5 \times Y_5$, where the internal space Y_5 is a Sasaki-Einstein manifold.

Consider five-dimensional gauged supergravity with n_V massless vector multiplets and Fayet-Iliopoulos (FI) gauging. The Lagrangian is completely determined by the symmetric coefficients C_{IJK} , $I, J, K \in \{1, \dots, n_V\}$ which can be read off from the Chern-Simons terms in the Lagrangian,

$$\mathcal{L}_5 = eR_5 + \dots - \frac{1}{6}C_{IJK}F^I \wedge F^J \wedge A^K + \dots, \tag{A.1}$$

and the FI parameters ξ_I , that specify the linear combination $\xi_I A^I$ used for electrically gauging the R-symmetry. Here, A^I are the U(1) gauge fields and F^I their corresponding field strengths. See for example [50] and references therein for a comprehensive description of five-dimensional gauged supergravity.

The general supersymmetric rotating black holes in the above class of five-dimensional gauged supergravity were written down in [11] after the seminal paper [7] and further developments. The black hole solutions, apart from explicitly depending on the numbers C_{IJK} and ξ_I , depend on the set of asymptotic charges given by n_V electric charges Q_I and two angular momenta J_{\pm} . Due to the requirement of supersymmetry and the existence of a smooth black hole horizon, there is one additional constraint among the set of asymptotic charges. It is particularly useful to consider a Scherk-Schwarz dimensional reduction down to four dimensions as it was done in [16], where one can explicitly write down the black hole attractor mechanism. The resulting four-dimensional supergravity has $(n_V + 1)$ U(1) vector fields (the new Kaluza-Klein gauge field is labeled by A^0) and is uniquely specified by a the holomorphic prepotential,

$$\mathcal{F}(X^\Lambda) = -\frac{1}{6} \frac{C_{IJK} X^I X^J X^K}{X^0}. \tag{A.2}$$

The prepotential uniquely determines the scalar manifold given by the holomorphic sections $X^\Lambda, \Lambda \in \{0, I\}$, in turn defining all kinetic terms in the four-dimensional Lagrangian. The R-symmetry in four dimensions is again gauged by the linear combination $\xi_\Lambda A^\Lambda$ where the new gauge field A^0 is included with a weight $\xi_0 = 1$. From a four-dimensional perspective the same black holes can be described by $(n_V + 1)$ electric charges $(q_0, q_I) = G_N^{(5)}(J_+/2, -Q_I)/\pi$, an angular momentum $j = G_N^{(5)} J_-/2\pi$, and the KK magnetic charge $p^0 = 1$. Note that $G_N^{(5)} = 4\pi G_N^{(4)}$ and the black hole entropy remains the same upon reduction to four dimensions. The static limit ($J_- = 0$) is particularly useful since we can write down the black hole entropy, in terms of four-dimensional variables, in a compact form [51, 52]⁹

$$\mathcal{S}(X^\Lambda) = -\frac{i\pi}{2G_N^{(4)}} \left(q_\Lambda X^\Lambda - p^\Lambda \frac{\partial \mathcal{F}(X^\Lambda)}{\partial X^\Lambda} \right) = -\frac{i\pi}{2G_N^{(4)}} \left(q_0 X^0 + q_I X^I - \frac{1}{6} \frac{C_{IJK} X^I X^J X^K}{(X^0)^2} \right), \tag{A.3}$$

under the constraint

$$\xi_\Lambda X^\Lambda = X^0 + \xi_I X^I = 1. \tag{A.4}$$

Upon extremizing $\mathcal{S}(X^\Lambda)$ as given above, one fixes the scalar fields in terms of the conserved charges and recovers the correct Bekenstein-Hawking entropy at the extremum.

Introducing an extra parameter X^- (being conjugate to J_-), in [16] we showed that the function \mathcal{S} can be extended to include also the last remaining charge J_- for the *stu* model. In this case the only nonvanishing triple intersection numbers are $C_{123} = 1$ (and cyclic permutation) and $\xi_I = 1$. It would be interesting to similarly generalize also the ($J_- \propto j \neq 0$) case to arbitrary parameters C_{IJK} , but at the moment we are lacking proper understanding of the four-dimensional rotating attractor mechanism.

⁹Here we correct a sign mistake in (4.23), (4.24) and (4.30) in [16].

However, we can try to apply these arguments to the case of BPS, rotating black holes in $\text{AdS}_5 \times Y_5$. The five-dimensional effective theory contains n_V massless vector multiplets, corresponding to the R- and global symmetries of the dual field theory. Generically the reduction on Y_5 leads to other matter multiplets in five-dimensional supergravity, such as hypermultiplets and massive vector multiplets. These, however, do not carry additional $U(1)$ gauge symmetries and we will work under the assumptions that they decouple in the description of the black hole near-horizon geometry. With this working assumption, we could expect that the entropy is given by the minimum of (A.3) in the case of equal angular momenta and by its natural extension for $J_- \neq 0$. This is particularly intriguing because the coefficients C_{IJK} in a compactification on $\text{AdS}_5 \times Y_5$ are proportional to the anomaly coefficients $\text{Tr } \mathcal{Q}_I \mathcal{Q}_J \mathcal{Q}_K$ for the n_V symmetries \mathcal{Q}_I associated with the gauge fields A^I in the bulk five-dimensional theory [53, 54]. As a consequence, it is tempting to speculate that the entropy of a black hole with electric charges Q_I and angular momenta J_i should be obtained as a Legendre transform of¹⁰

$$E(\Delta_I, \omega_i) = -i\pi N^2 \sum_{I,J,K=1}^{n_V} \frac{C_{IJK}}{6} \frac{\Delta_I \Delta_J \Delta_K}{\omega_1 \omega_2}, \tag{A.5}$$

with respect to Δ_I and ω_i with the constraint

$$\omega_1 + \omega_2 + \sum_{I=1}^{n_V} \Delta_I = 1. \tag{A.6}$$

The expression (A.5) is fully determined by anomalies. By setting $\Delta_I = (1 - \omega_1 - \omega_2) \cdot \hat{\Delta}_I/2$, it can be written as

$$E(\Delta_I, \omega_i) = \frac{4\pi i (\omega_1 + \omega_2 - 1)^3}{27 \omega_1 \omega_2} a(\hat{\Delta}_I), \tag{A.7}$$

where

$$a(\hat{\Delta}_I) = \frac{9N^2}{64} \sum_{I,J,K=1}^{n_V} C_{IJK} \hat{\Delta}_I \hat{\Delta}_J \hat{\Delta}_K, \tag{A.8}$$

subject to $\sum_{I=1}^{n_V} \hat{\Delta}_I = 2$ is the trial R-charge of the conformal field theory in the large N limit [53, 54]. The expression (A.7) has a strong resemblance with the refined supersymmetric Casimir energy for the Hopf surface $\mathcal{H}_{p,q} \simeq S^3 \times S^1$ in the large N limit [18–20]. Indeed it differs from it only by the -1 in the numerator.¹¹ (A.5) reduces exactly to the supersymmetric Casimir energy if we impose $\omega_1 + \omega_2 + \sum_{I=1}^{n_V} \Delta_I = 0$ instead of (A.6), corresponding to a different choice of angular determinations for the chemical potentials.

B Dimensional reduction and topological twist on \mathbb{CP}^2

For the case of asymptotically AdS_5 BPS rotating black holes that we considered previously in [16], the reduction of the solutions from five to four dimensions gave us an additional

¹⁰To compare with (A.3), we set $X^I = \Delta_I$, $X^0 = \omega_1 + \omega_2$, $X^- = \omega_1 - \omega_2$ and $J^\pm = J_1 \pm J_2$.

¹¹See, for example, eq. (C.3) in [16] with $\omega_i = -i|b_i|$ where $p = e^{-2\pi|b_1|}$, $q = e^{-2\pi|b_2|}$. Recall also that in the large N limit $a = c$.

physical understanding. The rotating black holes with equal angular momenta reduce to static domain-wall solutions in four dimensions with near horizon geometry $\text{AdS}_2 \times S^2$. We showed that from four-dimensional perspective supersymmetry is preserved by the $U(1)_R$ gauge field canceling the spin connection on the internal S^2 manifold via a topological twist [27, 28]. Without going into so much details, now we would like to argue that a similar dimensional reduction gives analogous understanding of the asymptotically AdS_7 black holes from six-dimensional point of view.

We focus on the single-rotation class of solutions described in section 2.1. We can dimensionally reduce the metric and all other fields along the $U(1)$ fiber of S^5 , with the remaining \mathbb{CP}^2 retaining all its symmetries and obtaining a static solution. We consider the usual Kaluza-Klein (KK) ansatz for the metric,

$$ds_7^2 = e^{\phi^{\text{KK}}} ds_6^2 + e^{-4\phi^{\text{KK}}} (d\psi + A^{\text{KK}})^2, \tag{B.1}$$

where ds_6^2 is the resulting 6D line element, ϕ^{KK} is the KK scalar field, and A^{KK} the $U(1)$ KK vector field. We see that the resulting six-dimensional line-element has the usual time and radial directions, as well as internal space \mathbb{CP}^2 , whose metric we repeat again here,

$$ds_{\mathbb{CP}^2}^2 = d\xi^2 + \frac{1}{4} \sin^2 \xi (\sigma_1^2 + \sigma_2^2 + \cos^2 \xi \sigma_3^2), \tag{B.2}$$

where σ_i ($i = 1, 2, 3$) are left-invariant one-forms on $SU(2)$, satisfying $d\sigma_i = -\frac{1}{2}\epsilon_{ijk}\sigma_j \wedge \sigma_k$. The Kähler form on \mathbb{CP}^2 is $J = \frac{1}{2}d\mathcal{B}$ with $\mathcal{B} = \frac{1}{2} \sin^2 \xi \sigma_3$. Comparing to the explicit solution in (2.5), we see that the KK vector field has a leg along the time direction carrying an electric charge in 6D (which corresponds to the angular momentum in 7D) but also has a leg along the internal \mathbb{CP}^2 manifold,

$$A_{\mathbb{CP}^2}^{\text{KK}} = \mathcal{B} = \frac{1}{2} \sin^2 \xi \sigma_3. \tag{B.3}$$

The reduction of the remaining fields give rise to additional electric charges for the other six-dimensional vector fields.

The only vector field along the internal manifold is the KK vector (B.3), therefore we are interested in seeing how the Killing spinor covariant derivative depends on it. Via a general Scherk-Schwarz ansatz for the reduction of fermions along a $U(1)$ isometry, see e.g. [55], the Killing spinor covariant derivative looks like

$$D_\mu \epsilon = \partial_\mu \epsilon + \frac{1}{4} \left(\omega_\mu^{ab} \gamma_{ab} + 2g A_\mu^{\text{KK}} \Gamma_R \right) \epsilon + \dots, \tag{B.4}$$

where the ellipsis denotes additional connections that will not be important below. The coupling constant g is left arbitrary, the γ_{ab} are spatial gamma matrices, and the matrix Γ_R allows for some internal structure of the spinors as typically the $U(1)$ KK gauge field becomes part of a bigger R-symmetry mixing the fermions.

An arbitrary four-manifold has an $SO(4) = SU(2)_l \times SU(2)_r$ holonomy, but for Kähler manifolds such as \mathbb{CP}^2 we have a further simplification and one of the two $SU(2)$ factors, say $SU(2)_r$ becomes $U(1)_r$. Explicitly, in the coordinates we already introduced, the

nonvanishing components of the spin connection read

$$\begin{aligned} \omega_{\sigma_1}{}^{14} = \omega_{\sigma_1}{}^{32} &= \frac{1}{2} \cos \xi, & \omega_{\sigma_2}{}^{13} = \omega_{\sigma_2}{}^{24} &= \frac{1}{2} \cos \xi, \\ \omega_{\sigma_3}{}^{12} &= \frac{1}{2} \cos^2 \xi - 1, & \omega_{\sigma_3}{}^{34} &= \frac{1}{2} (\cos^2 \xi - \sin^2 \xi). \end{aligned} \tag{B.5}$$

It is easy to see that splitting the spin connection into a self-dual and antiself-dual part ω^\pm is equivalent to splitting it into a $U(1)_r$ and an $SU(2)_l$ factor, respectively. In order for the supersymmetric twist to be performed, we need to cancel completely both ω^+ and ω^- in the Killing spinor covariant derivative, (B.4). The $SU(2)_l$ part drops out automatically if we impose the projection

$$\gamma^{1234} \epsilon = -\epsilon \Rightarrow \omega_\mu^{-ab} \gamma_{ab}^- \epsilon = 0, \tag{B.6}$$

where $2\gamma_{ab}^- \equiv \gamma_{ab} - \varepsilon_{abcd} \gamma^{cd}$. The $U(1)_r$ part of the spin connection is then

$$\omega^{+12} = -\frac{3}{2} \sin^2 \xi \sigma_3 = -3\mathcal{B} = -3A_{\mathbb{CP}^2}^{\text{KK}}. \tag{B.7}$$

We therefore see that the spin connection is precisely canceled and the supersymmetric twist is completed upon imposing

$$\gamma^{12} \epsilon = \Gamma_R \epsilon, \quad g = \frac{3}{2}, \tag{B.8}$$

in (B.4).

Note that supersymmetric flows in six dimensions with a $U(1)$ twist have already been explicitly found in [56] and further studied in [57], in the absence of electric charges, two-form field and additional scalars. Here we have shown that the 7D rotating black holes we consider, upon reduction to six dimensions, fit in the same category of solutions in [56, 57] with additional conserved charges.

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