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6d Conformal matter

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Abstract: A single M5-brane probing G, an ADE-type singularity, leads to a system which has $G \times G$ global symmetry and can be viewed as "bifundamental" (G, G) matter. For the A_N series, this leads to the usual notion of bifundamental matter. For the other cases it corresponds to a strongly interacting $(1, 0)$ superconformal system in six dimensions. Similarly, an ADE singularity intersecting the Hořava-Witten wall leads to a superconformal matter system with $E_8 \times G$ global symmetry. Using the F-theory realization of these theories, we elucidate the Coulomb/tensor branch of (G, G') conformal matter. This leads to the notion of fractionalization of an M5-brane on an ADE singularity as well as fractionalization of the intersection point of the ADE singularity with the Hořava-Witten wall. Partial Higgsing of these theories leads to new 6d SCFTs in the infrared, which we also characterize. This generalizes the class of $(1, 0)$ theories which can be perturbatively realized by suspended branes in IIA string theory. By reducing on a circle, we arrive at novel duals for 5d affine quiver theories. Introducing many M5-branes leads to large N gravity duals.

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Contents

1 Introduction

One of the remarkable developments in string theory is the close interplay between the geometry of its extra dimensions, and the resulting low energy theories. In the case of geometries with singularities, such methods have led to a host of tools in the construction and study of conformal field theories in diverse dimensions. Of particular significance are conformal field theories in six dimensions, which resist a UV Lagrangian description. The key ingredients of these theories are tensionless strings coupled to dynamical tensor modes.

Notable examples of such theories include the ADE $(2,0)$ theories [1]. For the A-type series, this is realized by a coincident stack of M5-branes [2]. Alternatively, all of the ADE theories can be realized by type IIB strings compactified on an ADE orbifold singularity [1]. Comparatively less is known about $(1,0)$ theories; some examples were found in the past in [3–11]. Recent work [12] gave a complete classification of $(1,0)$ theories without a Higgs branch. Those results also give a systematic starting point for pursuing a full classification of theories which have a Higgs branch.

The CFTs in the classification in [12] do not have a weakly coupled UV Lagrangian. However, one can always go to the Coulomb/tensor branch of these theories, which corresponds to giving vevs to scalars in tensor multiplets. In such cases one can find an effective Lagrangian description for $(1,0)$ theories in terms of a weakly coupled quiver gauge theory, where the scalar in the tensor multiplet controls the coupling constant of the corresponding gauge groups (i.e. the multiplet containing the gauge coupling) and is promoted to a dynamical collection of fields, ending up with a quiver-type theory. Moving to the origin of the tensor branch typically leads to a 6d SCFT with $(1,0)$ supersymmetry. Given the ubiquity of quivers in string theory, it is perhaps not surprising that some of these theories have a straightforward realization in string theory [8, 11, 13].

There are, however, some seemingly obvious quiver gauge theories which do not have a realization in perturbative string theory. For example, the structure of the orientifold projection forbids a bifundamental between $\mathrm{SO}(2p)$ and $\mathrm{SO}(2k)$, but instead leads to bifundamentals between $\mathrm{SO}(2p)$ and $\mathrm{Sp}(k)$. Perhaps even more conspicuous is the absence of E-type gauge theories, let alone an understanding of what a bifundamental between two such nodes would mean.

In this paper we point out that such generalized quivers do exist in string theory, but their matter sector is itself a strongly coupled 6d SCFT. We focus on two primary examples. One of them involves the realization of such 6d SCFTs by treating M5-branes as domain walls in a higher dimensional theory. This case is realized by the theory of M5-branes probing an ADE singularity $\mathbb{C}^2/\Gamma_{ADE}$ in M-theory. The other type involves intersecting an ADE singularity with a Hořava-Witten wall [14, 15].

ADE singularities define a seven-dimensional super Yang-Mills theory; being one dimension lower, M5-branes correspond to domain walls in this theory. Because it cuts the space in two, each such domain wall contributes additional light states to the low energy theory. Each subsequent parallel M5-brane introduces another domain wall, and for each finite segment between adjacent M5-branes we get a dynamical gauge symmetry whose inverse squared coupling constant is proportional to the length of the segment. We thus end up with a linear quiver consisting of gauge groups G, where the "bifundamental" between each adjacent group is interpreted as the associated superconformal matter with $G \times G$ symmetry. For N M5-branes, we therefore get theories $\mathcal{T}(G, N)$.

The question is thus reduced to understanding this matter sector, i.e. the theory living on such a domain wall. To determine this, we use a dual description of conformal matter in F-theory. We take F-theory on a non-compact elliptically fibered Calabi-Yau threefold. In F-theory, the conformal matter degrees of freedom on the domain wall are associated with the collision of two seven-branes, each supporting a gauge group G and wrapping a non-compact curve. Conformal matter is located at the intersection of two such curves, where the associated elliptic fibration can become more singular. In the case of a collision of two A-type gauge groups, there is a single hypermultiplet in the bifundamental of $G_L \times G_R$. For D- and E-type gauge groups, such a collision leads to a theory of tensionless strings which can be studied by introducing a minimal resolution in the base of the F-theory geometry. The existence of additional tensor multiplets suggests that the M5-brane fractionates on a singularity, leading to new gauge symmetries between the fractional M5-branes. We suggest an interpretation of fractional M5-branes as domain walls separating loci of M-theory singularities with different fractional discrete three-form flux of the type proposed in [16].

For example, the strongly coupled conformal matter produced by the collision of two $\mathfrak{so}_{2p+8}$ factors has a non-trivial tensor branch, consisting of a single tensor multiplet, an $\mathfrak{sp}_p$ gauge theory, and a half hypermultiplet in the $(\mathbf{2p+8}, \mathbf{2p}, \mathbf{1}) \oplus (\mathbf{1}, \mathbf{2p}, \mathbf{2p+8})$ of $\mathfrak{so}_{2p+8} \times \mathfrak{sp}_p \times \mathfrak{so}_{2p+8}$. In this case, we can view the M5-brane as fractionating to two 1/2 M5-branes between which the gauge symmetry has changed from the $\mathfrak{so}$ type to the $\mathfrak{sp}$ type. As another example, conformal matter between $\mathfrak{e}_8$ and $\mathfrak{e}_8$ leads to a strongly coupled CFT with an eleven-dimensional tensor branch and gauge algebra $(\mathfrak{sp}_1 \times \mathfrak{g}_2)_L \times \mathfrak{f}_4 \times (\mathfrak{g}_2 \times \mathfrak{sp}_1)_R$ with a half hypermultiplet in the $(\mathbf{2}, \mathbf{7}+\mathbf{1})$ for the $(\mathfrak{sp}_1 \times \mathfrak{g}_2)_L$ factor, and a half hypermultiplet in the $(\mathbf{7}+\mathbf{1}, \mathbf{2})$ for the $(\mathfrak{g}_2 \times \mathfrak{sp}_1)_R$ factor. In this case the M5-brane fractionates to 12 fractional M5-branes, and the gauge groups arise from the finite intervals between the fractional M5-branes.[1] Such considerations show that even in the case of a *single* M5-brane, the resulting probe theory of a D- or E-type singularity leads to a non-trivial fixed point which is the reflection of the existence of fractional M5-branes. This is in line with the expectation that additional degrees of freedom enter the low energy theory near the singular point of the moduli space. Upon compactification on a circle, these lead to novel duals of the well studied affine quiver gauge theories.

As the second main example, we consider the M-theory background $\mathbb{R}/\mathbb{Z}_2 \times \mathbb{C}^2/\Gamma_{ADE}$, i.e. ADE singularities intersecting the Hořava-Witten wall. The $\mathbb{Z}_2$ fixed point gives an E_8 nine-brane which wraps $\mathbb{C}^2/\Gamma_{ADE}$. This leads to a conformal system with $E_8 \times G_{ADE}$ global symmetry. In this case we again find the phenomenon of fractionating, but now

[1] Between some fractional M5-brane pairs there are no gauge groups.

the intersection point of the ADE singularity and the wall fractionate. As before, we can also introduce M5-branes along the line of the ADE singularity. In heterotic terms, this is the theory of small E_8 instantons [3–5] probing an ADE singularity. Some aspects of this system have been analyzed using F-theory in [17]. We find G-type gauge symmetries with (G, G) conformal matter system for all of them, except the one adjacent to the wall, which gauges the G symmetry of the (E_8, G) conformal matter system at the wall. We label these theories as $\mathcal{T}(E_8, G, N)$.

These theories also have partial Higgs branches where operators develop vevs which break some of the flavor symmetry, leading to *new* conformal fixed points. By studying the vacua of the 7d SYM theory, or equivalently the vacua of the flavor seven-branes, we show that partial Higgs branches of the $\mathcal{T}(G, N)$ theories are classified by the orbits of nilpotent elements for the flavor symmetry factors. In F-theory, such configurations are examples of T-brane configurations [18–22]. These are non-abelian configurations of intersecting seven-branes which can remain hidden from the complex structure moduli of the Calabi-Yau geometry. We label these theories as $\mathcal{T}(G, \mu_L, \mu_R, N)$, which consists of N M5-branes, and a flavor symmetry $G_L \times G_R$ which can be broken, as dictated by the orbits in $\mathfrak{g}$ of nilpotent elements $\mu_L \in \mathfrak{g}_L$ and $\mu_R \in \mathfrak{g}_R$ for the two Lie algebras.

There are also new conformal theories coming from the partial Higgsing of theories involving M5-brane probes of the ADE singularities intersecting the Hořava-Witten wall. These theories are classified as $\mathcal{T}(E_8, G_R, \gamma_L, \mu_R, N)$: we have a theory of N M5-branes, and flavor symmetry $E_8 \times G_R$ which can be broken, as dictated by a nilpotent element $\mu_R \in \mathfrak{g}$ (and its associated orbit) for the right Lie algebra, as well as a homomorphism $\gamma_L : \Gamma_G \to E_8$ corresponding to the choice of a flat E_8 connection on S^3/Γ_G.

Taking the limit of a large number of M5-branes also leads us to a collection of gravity duals in both M-theory and IIA string theory. An interesting feature of our analysis is that we can see how certain features of IIA duals with a Romans mass show up in our construction.

One can also study, from the perspective of F-theory, the more general case of colliding $G_{ADE} \times G'_{ADE}$ singularities and the associated conformal matter. For completeness, we also include this analysis.

The rest of this paper is organized as follows. To set the stage, we first show in section 2 how to understand conformal matter sectors in F-theory. We use this analysis in section 3 to study M5-branes probing an ADE singularity. In section 4 we show how reduction of these theories on a circle leads to novel 5d dualities. In section 5 we show how to characterize the additional SCFTs generated by moving onto the partial Higgs branches of such theories. Next, in section 6 we turn to the theory of heterotic small instantons probing an ADE singularity, determining both the associated generalized quivers, and their partial Higgs branches. In section 7 we turn to scaling limits of our solutions, and characterize the corresponding holographic dual descriptions. We present our conclusions in section 8. Some additional background, as well as examples of generalized quiver theories in F-theory are presented in a set of appendices.

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2 Conformal matter

One of the aims of our paper will be to show how conformal matter appears in various contexts. In this section we show how to derive properties of these theories via F-theory.

In F-theory, conformal matter arises from the collision of seven-branes where the localized matter is not a weakly coupled hypermultiplet. A convenient way to deduce properties of the matter sector is to formulate the collision of seven-branes in terms of the geometry of an elliptically fibered Calabi-Yau threefold. In minimal Weierstrass form, this is given by:

$$y^2 = x^3 + fx + g, \tag{2.1}$$

where f and g are sections of $\mathcal{O}(-4K_B)$ and $\mathcal{O}(-6K_B)$, with B the base of the elliptic fibration. Seven-branes are associated with irreducible components of the discriminant locus, i.e. the zero set of $4f^3 + 27g^2 = 0$. The corresponding gauge symmetry on such a seven-brane is dictated by the order of vanishing for f and g, which in turn determines the Kodaira-Tate type of the singular fiber. This, in combination with additional geometric data can be used to read off the gauge group on a seven-brane (see e.g. [23]).

Localized matter is associated with the collision of two such irreducible components of the discriminant locus. At these collisions, the Kodaira-Tate singularity type of the elliptic fiber can become more singular, thus leading to the phenomenon of trapped matter. In fact, the fiber can sometimes become so singular that additional blowups in the base B become necessary to understand the resulting matter content. When such blowups are introduced, there are additional exceptional curves in the base. Each such curve can be wrapped by a D3-brane, contributing a string in the six-dimensional effective theory. As these curves shrink to zero size, the tension of this string also tends to zero, yielding a six-dimensional SCFT. The total number of tensor multiplets for such a theory is simply the number of independent curves which simultaneously contract to zero size.

In this section, our primary interest is in the collision of two seven-branes which support the same ADE gauge group, and the corresponding conformal matter. The result of this analysis has been performed in various places, for example in [6, 12, 24, 25]. Rather than launch into a detailed discussion of the necessary blowup structure, we shall use the algorithmic procedure developed and automated in [12], which involves stating some minimal combinatorial data about intersections of curves in the base B.

Using this procedure, we can determine the corresponding degrees of freedom trapped along each collision of singularities. For example, in F-theory, an A-type $\mathfrak{su}_k$ gauge symmetry is realized by a Kodaira-Tate fiber of split I_k type.[2] At the collision of two split I_k and I_p singularities, the singularity becomes I_{k+p}, so we have a Higgsing of $\mathfrak{su}_{k+p}$ down to the product $\mathfrak{su}_k \times \mathfrak{su}_p$, with a corresponding hypermultiplet in the bifundamental $(\mathbf{k}, \overline{\mathbf{p}})$ of $\mathfrak{su}_k \times \mathfrak{su}_p$.

In the remaining cases we consider, the collision of two seven-branes will lead to a strongly coupled conformal sector. Consider next the collision of two D-type singularities. In F-theory, a D-type $\mathfrak{so}_{2p+8}$ gauge symmetry is realized by a Kodaira-Tate fiber of split I_p^* type. The non-split case would realize an $\mathfrak{so}_{2p+7}$ gauge symmetry. At the intersection

[2] Split means there is no monodromy by an outer automorphism of the algebra.

point, the collision of I_k^* and I_p^* leads to an order of vanishing for f and g which does not yield a standard Kodaira-Tate fiber. Thus, a blowup at this point is required. This yields a -1 curve which itself supports a non-split I_{k+p} type fiber [17]. The resulting gauge symmetry from such a non-split singularity is $\mathfrak{sp}_r$ with $r = [(k+p)/2]_+$, that is, the smallest integer obtained from rounding up [6, 25, 26].[3] In addition to this gauge symmetry, we also have a half hypermultiplet in the bifundamental trapped at each collision of our $\mathfrak{sp}_r$ seven-brane with an $\mathfrak{so}_{2k+8}$ and $\mathfrak{so}_{2p+8}$ seven-brane. In the case where $k+p$ is odd, we also have an extra hypermultiplet in the $\mathbf{2r}$ of $\mathfrak{sp}_r$. Now, the key point is that the "matter sector" between our two $\mathfrak{so}$ factors is really a conformal field theory, since the -1 curve is shrunk to zero size in our geometry. This is our first example of conformal matter.

Consider next the collision of two E-type singularities. The Kodaira-Tate fiber for E_6, E_7 and E_8 is respectively a split IV^* fiber, and a III^* and II^* fiber. The pairwise collisions can be conveniently summarized by the minimal Weierstrass models:

$$(E_6, E_6) : y^2 = x^3 + u^4v^4 \tag{2.2}$$
$$(E_7, E_7) : y^2 = x^3 + u^3v^3x \tag{2.3}$$
$$(E_8, E_8) : y^2 = x^3 + u^5v^5, \tag{2.4}$$

with conformal matter located in the base at the point $u = v = 0$. Performing the minimal blowups necessary to get all fibers into Kodaira-Tate form yields an additional configuration of curves, which intersect pairwise at a single point. For details of this resolution algorithm, see reference [12]. Letting a sequence of positive integers denote minus the self-intersection number for these curves, we have the minimal resolutions:

(E_6, E_6) : (2.5)

Gauge Symm:		$\mathfrak{su}_3$	
Curve:	1	3	1

(E_7, E_7) : (2.6)

Gauge Symm:		$\mathfrak{su}_2$		$\mathfrak{so}_7$		$\mathfrak{su}_2$	
Curve:	1	2		3		2	1
Hyper:			$\frac{1}{2}(\mathbf{2},\mathbf{8})$		$\frac{1}{2}(\mathbf{8},\mathbf{2})$		

(E_8, E_8) : (2.7)

Gauge Symm:			$\mathfrak{sp}_1$		$\mathfrak{g}_2$		$\mathfrak{f}_4$		$\mathfrak{g}_2$		$\mathfrak{sp}_1$		
Curve:	1	2	2		3	1	5	1	3		2	2	1
Hyper:				$\frac{1}{2}(\mathbf{2},\mathbf{7}{+}\mathbf{1})$						$\frac{1}{2}(\mathbf{7}{+}\mathbf{1},\mathbf{2})$			

Here, the self-intersection of these curves also dictate the gauge symmetry and matter content for this theory on the resolved branch, as we have indicated. These repeating patterns were noted as basic building blocks of F-theory compactifications in [27, 28]. For earlier work where these building blocks were also identified see [6].

[3] In the case where $k+p$ is even, this can be understood by quotienting by the outer automorphism of $\mathfrak{su}_{k+p}$, thus producing an $\mathfrak{sp}_r$ algebra. In the case where $k+p$ is odd, the analysis of roots in the associated resolution of the fiber is more subtle, and only an $\mathfrak{sp}_{r-1}$ algebra can be identified geometrically [26]. However, the structure of 6d anomaly cancelation and consistent Higgsing patterns in the field theory is such that the only self-consistent way to get a gauge symmetry is to have $\mathfrak{sp}_r$ gauge symmetry, with some additional matter fields attached to the $\mathfrak{sp}$ factor [6].

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Thus, what the F-theory realization gives us is a direct description of the tensor branch of the conformal matter sector. By following a similar procedure, other collisions with different singularity types $G \times G'$ lead to canonical notions of conformal matter. We give a list of such conformal matter sectors in appendix B.

2.1 Higgsing and brane recombination

A hallmark of bifundamental matter is that activating a vev breaks some of the symmetries of the system. Even in our non-Lagrangian systems, this characterization still carries over. As a warmup, consider again the collision of two A_{k-1}-type singularities, with a bifundamental hypermultiplet in the $(\mathbf{k}, \overline{\mathbf{k}})$ of $\mathrm{SU}(k)_L \times \mathrm{SU}(k)_R$. The corresponding geometric singularity is locally given by:

$$y^2 = x^2 + u^k v^k. \tag{2.8}$$

Activating a bifundamental corresponds to a brane recombination operation. As explained in [29], this can be viewed as the deformation $uv \longmapsto uv + a$. So in other words, the flavor symmetry is broken to an $\mathrm{SU}(k)_{\mathrm{diag}}$ stack supported at $uv + a = 0$:

$$y^2 = x^2 + (uv + a)^k\,. \tag{2.9}$$

Similar considerations hold for the strongly coupled conformal matter. For example, in the collision of two E_8 singularities, we have the breaking pattern:

$$y^2 = x^3 + u^5 v^5 \longmapsto x^3 + (uv + a)^5\,, \tag{2.10}$$

that is, we break to the diagonal of $E_8 \times E_8$.

Following up on our discussion of collision of singularities given earlier, we can see that a similar characterization holds for all of the other collisions. In other words, if we have flavor symmetry G_L supported on $u = 0$ and G_R supported on $v = 0$, then the brane recombination $uv \longmapsto uv + a$ breaks this to the diagonal subgroup.[4]

3 CFTs from domain walls

In this section we introduce our first class of examples of 6d SCFTs with conformal matter. In M-theory, these will be realized by M5-branes probing an ADE singularity. In field theory terms, we introduce a class of $(1, 0)$ superconformal field theories which are realized as domain wall solutions in seven-dimensional gauge theory. This leads to theories where the flavor symmetry of the CFT is a product $G_L \times G_R$ with $G_L \simeq G_R$ an ADE group. The problem naturally reduces to the study of a single M5-brane probing the singularity, leading to the conformal matter, from which one can deduce the quiver theory associated with N parallel M5-branes. We shall therefore label these theories as $\mathcal{T}(G, N)$, in the obvious notation.

This section is organized as follows. First, we begin with some general considerations in both M- and F-theory. Next, we consider in turn each type of orbifold singularity. In the case of the A- and D-series, we also provide realizations in IIA string theory.

[4]For further details on these brane recombination operators see [30].

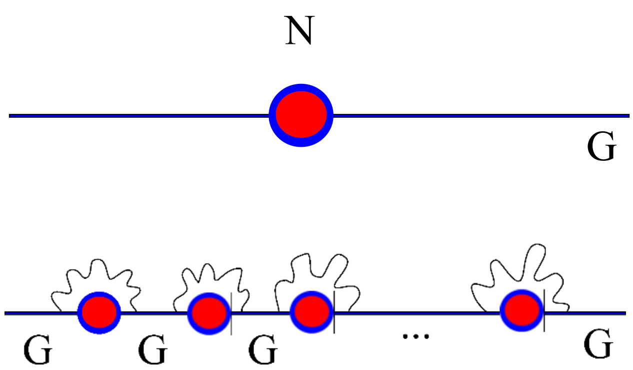


Figure 1. UP: N M5-branes at an orbifold singularity (SCFT point). DOWN: the Coulomb/tensor branch deformation where the N M5-branes are separated along the singularity locus and associated conformal matter which are symbolically represented by wavy lines between adjacent gauge factors.

3.1 M5-brane probes of ADE singularities

To begin, we recall that seven-dimensional super Yang-Mills theory with 16 supercharges is realized by the M-theory background $\mathbb{R}^{6,1} \times \mathbb{C}^2/\Gamma_G$, where Γ_G is an ADE discrete subgroup of SU(2). For additional properties of the group theory and associated geometry of these singularities, see appendix C. The bosonic field content of this theory consists of a seven-dimensional gauge field, and three real adjoint-valued scalars.

Domain wall solutions of the M-theory realization correspond to M5-brane probes which fill six spacetime dimensions and sit at the orbifold fixed point. In more detail, the domain wall fills $\mathbb{R}^{5,1}$ and sits at a point of the real line factor of $\mathbb{R} \subset \mathbb{R} \times \mathbb{C}^2/\Gamma_G$. We are interested here precisely in the 6d theory living on the worldvolume of this domain wall; see figure 1. Being half-BPS, this 6d system has $(1,0)$ supersymmetry. Moreover, being a domain wall for the 7d theory, the 7d gauge theory degrees of freedom serve as flavor symmetry currents in the 6d system. Thus the system has a flavor group $G_L \times G_R$. The 7d gauge symmetry may be (partially) broken by suitable choices of boundary conditions for the 7d fields at the domain wall [13, 31].

By a similar token, we can introduce multiple domain walls and partition up the real line factor in $\mathbb{R} \times \mathbb{C}^2/\Gamma_G$ into finite size segments such that the the leftmost and rightmost segments are still non-compact. Each such segment on the real line specifies a six-dimensional gauge theory with gauge group G. The value of the gauge coupling is in turn specified by the length of the interval. As an interval segment becomes large, the corresponding gauge theory factor becomes weakly coupled. In particular, we see that the leftmost and rightmost intervals are non-compact and thus support flavor symmetries.

Summarizing then, we have arrived at a six-dimensional theory with $(1,0)$ supersymmetry, i.e. eight real supercharges. For each segment of the real line, we have a corresponding gauge group:

$$G_{\text{quiver}} = G_1 \times \ldots \times G_{N-1} \tag{3.1}$$

where we have partitioned up the real line into $N-1$ finite segments, and $G_i \simeq G$ for all i. The 6d gauge coupling of each segment is proportional to the length of the segment:

$$\frac{1}{g_i^2} \sim L_i, \tag{3.2}$$

where L_i is the length of the interval. Hence, the leftmost and rightmost segments define flavor symmetries, while finite size intervals contribute dynamical degrees of freedom. An additional feature of this construction is that the length of each line segment is itself a dynamical mode, i.e. the scalar of a tensor multiplet.

We reach a conformal fixed point by shrinking the distance between the domain walls to zero size, that is, by passing to a strongly coupled fixed point of the gauge theory. In other words, our description in terms of domain walls partitioning up the real line characterizes the tensor branch of a six-dimensional theory.

Each domain wall contributes additional degrees of freedom trapped along its worldvolume. From this characterization, we see that we have a generalized notion of a quiver gauge theory: we have a set of gauge groups and matter sectors which act as links between them. Clearly then, it is important to know what are the additional degrees of freedom living on each domain wall, i.e., the conformal matter. The conformal matter sector corresponds to the special case of $N = 1$ given by a single M5-brane which in F-theory corresponds to the case of two non-compact G-type seven-branes intersecting at a single point. The more general situation with N M5-branes, translates in F-theory to the case where the seven-branes intersect at the $\mathbb{Z}_N$ fixed point of the A_{N-1} singularity. The conformal matter will automatically have $G \times G$ symmetry as discussed before. In the F-theory setup this simply comes from the fact that the non-compact seven-branes play the role of global symmetries.

As for what this conformal matter is, we know the answer, as it follows directly from the results reviewed in section 2. To see this, observe that in M-theory, the A_{N-1} $(2,0)$ theory is realized by N coincident M5-branes, while in F-theory, it is realized by the geometry $\mathbb{C}^2/\mathbb{Z}_N \times T^2$. In the resolution of the base, we have $N-1$ $\mathbb{P}^1$'s, each with self-intersection -2, with neighboring intersections dictated by the A_{N-1} Dynkin diagram. Moreover, the volumes of the $\mathbb{P}^1$'s control the relative positions of the M5-branes on the tensor branch. We are interested in the case where there are some additional flavors, so we can also introduce two stacks of non-compact seven-branes, with respective gauge groups G_L and G_R.[5] In the resolution of the $\mathbb{C}^2/\mathbb{Z}_N$ singularity, G_L intersects the leftmost $\mathbb{P}^1$ while G_R intersects the rightmost $\mathbb{P}^1$. Following the analysis of [12], we can see that the minimal singularity type over each of the -2 curves enhances to an algebra $\mathfrak{g}$. So in other words, we have a configuration of curves:

$$[G_L] \;\; \overset{\mathfrak{g}}{2} \;\; \overset{\mathfrak{g}}{2} \;\; \overset{\cdots}{\cdots} \;\; \overset{\mathfrak{g}}{2} \;\; \overset{\mathfrak{g}}{2} \;\; [G_R], \tag{3.3}$$

that is, each -2 curve is wrapped by a seven-brane with gauge symmetry $\mathfrak{g}$. At each intersection of a divisor in the base, we get a conformal matter sector. for which the G-symmetries are gauged by the adjacent G on the -2 curve. In the following sections we

[5] In fact, sometimes such a flavor symmetry is *required* in order to satisfy the condition that an elliptic fibration exists. In field theory, it is required to satisfy 6d gauge anomaly cancelation.

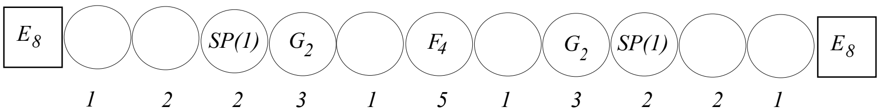
discuss the E case first, which has no type IIA realization, and then turn to the A and D cases which do have IIA realizations.

3.2 M5-branes probing E-type singularity

As discussed above, the problem reduces to finding the conformal matter which arises when two E_i singularities collide. This was already discussed in section 2. For example, in the case $G = E_8$, the relevant conformal matter is given by line (2.7); thus we end up with a theory with gauge symmetries:

$$\mathfrak{sp}_1 \times \mathfrak{g}_2 \times \mathfrak{f}_4 \times \mathfrak{g}_2 \times \mathfrak{sp}_1 \,, \tag{3.4}$$

with half hypermultiplets in the $(\mathbf{2}, \mathbf{7}+\mathbf{1})$ of each $\mathfrak{sp}_1 \times \mathfrak{g}_2$ factor (and in the $(\mathbf{7}+\mathbf{1}, \mathbf{2})$ of each $\mathfrak{g}_2 \times \mathfrak{sp}_1$ factor), and with flavor group $E_8 \times E_8$. More precisely we have a *generalized* quiver theory of the form:



(3.5)

where the notation above denotes one tensor multiplet per each circle, the number below the circle denotes the (negative of) self intersection number of the corresponding cycle in the F-theory geometry. The two systems

(3.6)

denote, respectively, the theory of a single small E_8 instanton which has a global E_8 symmetry (consistent with the gauge and flavor symmetries attached to it as in line (3.5)), and the $(2,0)$ theory of 2 parallel M5-branes (i.e. the A_1 (2,0) system). The configuration of a -1 curve next to a -2 curve corresponds to the theory of two small E_8 instantons. In the present case, the -2 curve touches a curve with $\mathfrak{sp}_1$ gauge symmetry, which is obtained by gauging a subalgebra of the $\mathfrak{so}_4$ global symmetry.[6]

3.2.1 Fractional M5-branes

This picture suggests that the single M5-brane on the line of E_8 singularity has split to 12 points on it, leading to 11 finite segments whose lengths are controlled by the scalars of the associated tensor multiplet (see figure 2). In other words, we have branes with *fractional*

[6] For further discussion of the anomaly polynomial for multiple small E_8 instantons, see [35]. The physical interpretation of the $\mathfrak{sp}_1$ gauge symmetry was presented in reference [36], after the present work first appeared.

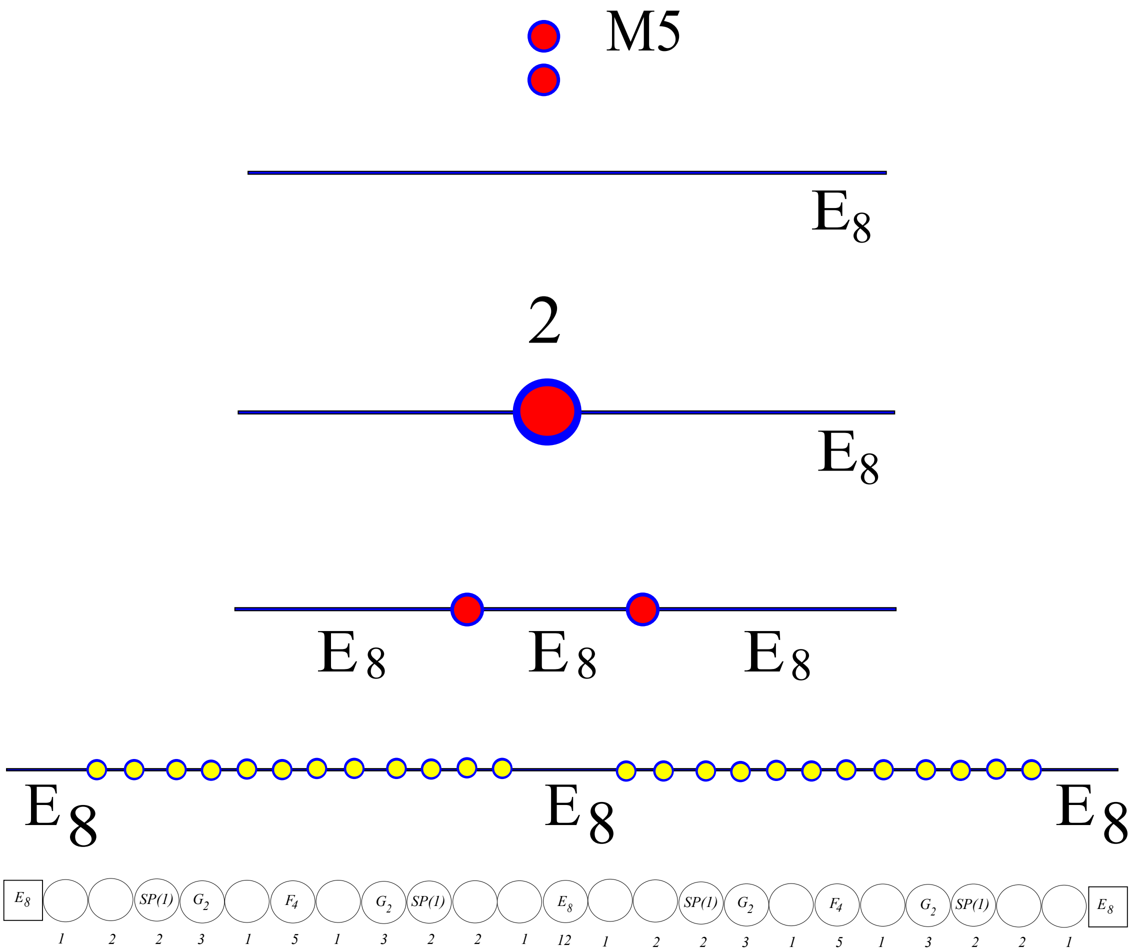


Figure 2. Example of two M5-branes probing an E_8 singularity. Moving onto the tensor branch gives rise to (E_8, E_8) conformal matter that are SCFTs themselves with their own tensor branches, as described by the generalized quiver of line (3.5). This suggests that each M5-brane on E_8 has fractionated to 12 pieces.

M5 brane charge. Looking at the list of colliding E-singularities, we discover in this way that the fractionalization of M5-branes for the ADE series are given by

	E_8	E_7	E_6	D_p	A_k
# of M5 Fractions	12	6	4	2	1

(3.7)

where as we will discuss in the context of D-type singularities, an M5-brane on it can fractionate to 2, while in the case of M5-branes on an A-type singularity no fractionation occurs.

This raises the question of why the gauge group factor on each interval is *not* the E_8 gauge symmetry, but rather the list above. We propose an answer to this question:[7] It

[7]This proposal was motivated by a question posed by E. Witten at Strings 2014.

has already been suggested that by choices of discrete three-form fluxes stuck at seven-dimensional M-theory singularities, the type of the gauge symmetry can change [16]. We thus propose that fractional M5-branes change the discrete flux from one value to the next, changing the gauge group in the process. Moreover we propose that each fractional M5-brane changes the three-form flux fraction by equal amounts. So for example in the E_8 case, each fractional M5-brane will change the fractional three-form flux by $1/12$. Our description of fractional M5-branes also matches to the list of groups (up to what we hope is a typo for the E_6 entry) listed in table 14 of reference [16].

In fact, the fraction of discrete flux matches *where* we find the corresponding group in our repeated pattern of exceptional curves! For example, reading from left to right in the configuration $1, 2, 2, 3, 1, 5, 1, 3, 2, 2, 1$ for the conformal (E_8, E_8) matter, we find that there is an $\mathfrak{sp}_1$ gauge symmetry on the third curve, which would give $3/12 = 1/4$, associated with a $\mathbb{Z}_4$ flux. Further, on the fourth curve, we have $4/12 = 1/3$ flux giving $\mathfrak{g}_2$ ($\mathbb{Z}_3$ flux), the sixth curve yields $6/12 = 1/2$, giving $\mathfrak{f}_4$ ($\mathbb{Z}_2$ flux). Similar considerations hold for E_7 and E_6 (and D type) conformal matter. One subtlety, however, is that the labels which correspond to trivial gauge group are different from [16]. For example, for E_8 there is no $\mathbb{Z}_5$. But in addition there are all the other fractions of $1/12$ which lead to no gauge factors.

3.3 IIA realization of $\mathcal{T}(\mathrm{SU}(k), N)$ theories

Let us now consider more closely the theory of N M5-branes probing an A_{k-1}-type singularity $\mathbb{C}^2/\mathbb{Z}_k$. A convenient description of the domain wall discussed above is obtained via a standard duality with Type IIA. The A_{k-1}-type singularity can be thought of as an infinite radius limit of the charge k Taub-NUT space, or A_{k-1} ALF space. More precisely, the TN_k space has a canonical fibration as a circle of radius R, S^1_R, over $\mathbb{R}^3$: in the limit $R \to \infty$ one obtains the A_{k-1} ALE space. M-theory on the TN_k geometry describes a system of k Kaluza-Klein monopoles that dualize to a system of k parallel infinite D6-branes on the Type IIA side, once one identifies S^1_R with the M-theory circle [32]. In the limit in which the KK monopoles coincide, one obtains an enhanced $\mathrm{SU}(k)$ gauge symmetry; these are the degrees of freedom of the 7d gauge theory we discussed above. Instead, in Type IIA the D6-branes fill the $X^0, X^1, \ldots, X^6$ directions and the enhanced symmetry comes from the gauge theory living on the worldvolume of the stack of k coincident D6-branes. Under such a duality the N M5 probes turn into N NS5 probes of the stack of k D6-branes. On the worldvolume of the N coincident NS5s lives a well-known but still quite mysterious 6d $(1, 0)$ SCFT with tensionless strings [11]. Let us briefly discuss the field content of this theory. Each NS5 contributes to the worldvolume a $(2, 0)$ tensor multiplet. As usual, the center of mass degrees of freedom decouple and we are left with a system of $N - 1$ tensor multiplets at the superconformal point. Each $(2, 0)$ multiplet decomposes into a $(1, 0)$ tensor plus two $(1, 0)$ hypers. The scalars of the multiplets arise from quantizing the motion transverse to the NS5s in the whole geometry, including the M-theory circle. In particular the vev of the scalars in the tensor multiplets parameterize the relative distance between the NS5s along the X^6 direction. Separating the NS5s we break superconformal invariance and move onto the tensor branch of the system, eventually landing on a quiver gauge theory (see figure 3)

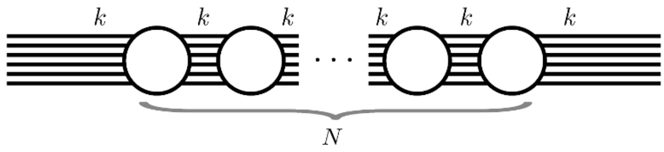
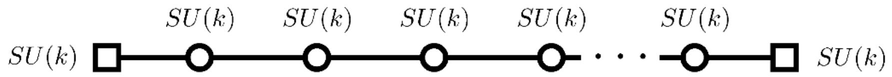


Figure 3. Type IIA description of the system of N M5-branes probing the $\mathbb{C}^2/\mathbb{Z}_k$ singularity: here we are drawing schematically the X^6 direction for the tensor branch. One obtains N NS5-branes sitting on top of k infinite D6 branes.



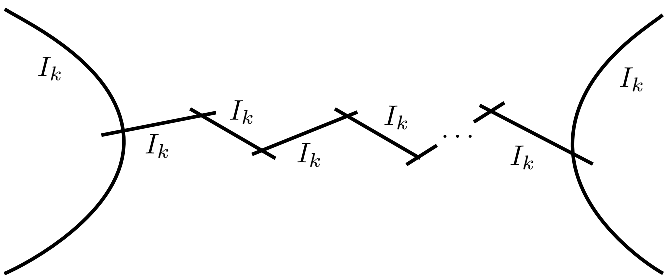
(3.8)

where edges stands for bifundamental hypers, the $N-1$ round nodes for gauge groups, and the two square nodes for flavor groups as usual. Notice that naively the brane system in figure 3 should correspond to a U(k) quiver gauge theory. In this context, however, the U(1)'s are anomalous because of the nonzero term $F_{\mathrm{U}(1)} \wedge \mathrm{Tr}(F^3_{\mathrm{SU(k)}})$ in the anomaly polynomial. To cure this pathology, one couples the compact scalar corresponding to the M-theory circle to the abelian gauge field, making it massive [11, 33]. This is the reason why in 6d one obtains SU(k) gauge groups on the tensor branch.

3.3.1 IIB/F-theory description

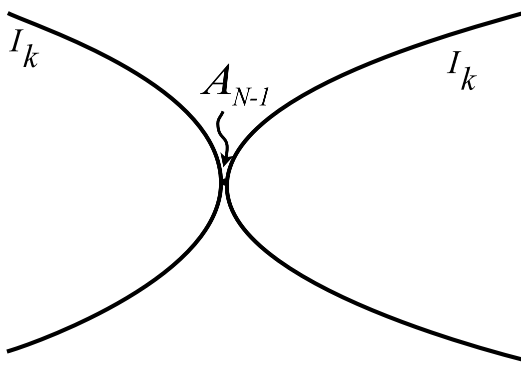
The $\mathcal{T}(\mathrm{SU}(k), N)$ theories also have a straightforward realization in type IIB string theory. To obtain it, we recall that if we T-dualize the S^1 at the boundary of an ALE space, we reach a collection of coincident NS5-branes. In such a configuration, we can next consider a stack of D7-branes which pass through the singular locus of this geometry. Upon T-dualizing this circle, we see that a D7-brane wrapped over a collapsing $\mathbb{P}^1$ of the geometry $\mathbb{C}^2/\mathbb{Z}_N$ will become a D6-brane in the dual description. Putting these elements together, we see that for each such $\mathbb{P}^1$, we get a stack of D6-branes suspended between NS5-branes.

In fact, we can also lift this IIB description back to F-theory. The tensor branch of the 6d $(1,0)$ theory engineered with F-theory is given by a local system of -2 curves in the base of an elliptically fibered Calabi-Yau threefold. In the case of the system of N M5-branes probing the $\mathbb{C}^2/\mathbb{Z}_k$ singularity, each such curve supports a Kodaira type I_k singular fiber, corresponding to enhanced gauge symmetry of type SU(k). Flavor groups in the F-theory description correspond to non-compact divisors of the base of the elliptically fibered Calabi-Yau threefold. Since we know that the system carries an SU(k) $\times$ SU(k) flavor group, we need two non-compact divisors in the base. Moreover, the fact that the system we are engineering corresponds to the tensor branch of an SCFT translates to the requirement that all compact -2 curves in the base can be shrunk simultaneously to zero. Since bifundamental hypers are trapped at the intersections in between the various $\mathbb{P}^1$'s in the base, starting from the tensor branch description of line (3.8), the resulting configuration of -2 curves for the F-theory description is:



(3.9)

the leftmost and rightmost -2 curves supporting I_k type Kodaira fibers are non-compact. Taking the conformal limit of such a system would correspond to shrinking the compact -2 curves to zero size: at the SCFT point the two non-compact curves corresponding to the flavor symmetries touch at an A_{N-1} singularity of the base:



(3.10)

This local configuration of curves corresponds to the SCFT describing the worldvolume of N M5-branes probing an A_{k-1} singularity in M-theory.

3.4 IIA Realization of $\mathcal{T}(\mathrm{SO}(2p), N)$ theories

Let us now turn to the theory of N M5-branes probing a D-type singularity. First of all, a D_{p+4} singularity gives rise to 7d SYM theory with gauge group $\mathrm{SO}(2p+8)$ for $p \geq 0$. Following the same reasoning of the previous section, we seek a 6d $(1,0)$ system with $\mathrm{SO}(2p+8) \times \mathrm{SO}(2p+8)$ flavor group.

Following [32, 34] we can obtain a Type IIA description of the system by replacing the D_{p+4} singularity with the corresponding D_{p+4} ALF space. Eventually, one obtains a stack of $p+4$ parallel D6-branes on top of an $\mathrm{O6}_-$ plane, together with $p+4$ mirror images of the D6s below. The resulting 7d theory has $\mathrm{SO}(2p+8)$ gauge symmetry. Consider now introducing domain walls. By construction, the theory living on the wall has $\mathrm{SO}(2p+8) \times \mathrm{SO}(2p+8)$ flavor symmetry. However, when an $\mathrm{O6}_\pm$ plane meets an NS5-brane, it turns into an $\mathrm{O6}_\mp$ plane, with a net shift of eight units of D6 charge. Now, a system of $p+4$ D6-branes parallel to an $\mathrm{O6}_+$ plane gives rise to an $\mathfrak{sp}_p$ gauge theory. Therefore, we do not just get a set of N NS5-branes sitting on top of the $p+4$ D6s in the presence of an $\mathrm{O6}_-$ plane. Indeed, for N odd such a system would have the wrong flavor symmetry, i.e. $\mathrm{SO}(2p+8) \times \mathrm{Sp}(p)$, and this is impossible. The only way out from this paradox is to conclude that there are $2N$ NS5-branes, and Sp factors in between our SO factors, for a total of $2N-1$ tensor multiplets.

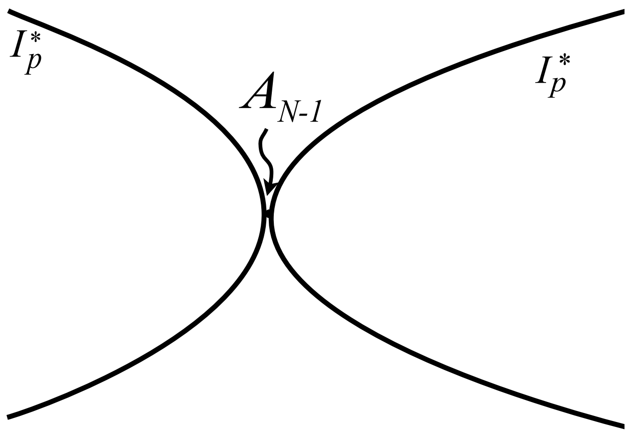
We have just found that in contrast with M5-brane probes of A-type singularities, for D-type singularities we find fractional M5-branes. Since there are only two varieties of these branes for the D-type singularity, we shall refer to them as 1/2-M5-branes.

This notion of fractionalization of M5-brane for the D-type singularity further supports the picture we proposed for fractionalization of M5-branes probing the E-type singularities. Indeed, as noted in [37], the singularity with $\mathbb{Z}_2$ flux is a simple lift of a D6-O6 system of Type IIA.

3.4.1 IIB/F-theory description

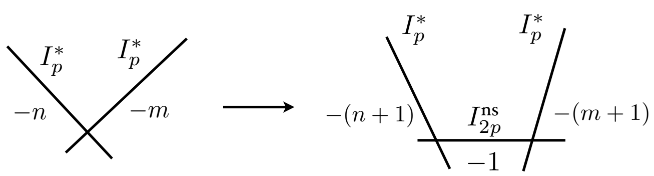
Let us now turn to the IIB / F-theory description of this system. As before, we can consider the effects of T-dualizing our suspended brane configuration to a related configuration of D7-branes and O7-planes in type IIB string theory. This leads us to a configuration of seven-branes of SO type wrapping the -2 curves of an A-type singularity. This A-type singularity is, in the IIB description, associated with the IIA NS5-branes used to partition up the interval in the first place.

Turning to the F-theory lift of this description, we need to consider -2 curves intersecting according to the A_{N-1} Dynkin diagram. Each $\mathbb{P}^1$ supports a Kodaira-Tate I_p^* fiber:



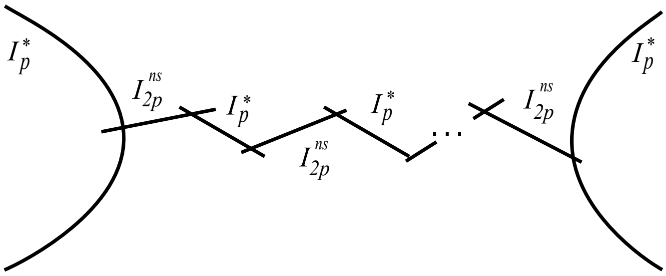
(3.11)

As we already explained, the collision of two such fibers contains conformal matter, given by an $\mathfrak{sp}_p$ gauge theory (coupled to half hypers) wrapping a collapsing exceptional curve:



(3.12)

This type of blowup is the only one that occurs for the configuration of curves we are considering. Proceeding by successive blowups, eventually we obtain the configuration



(3.13)

where each I_{2p}^{ns} (resp. I_p^*) fiber is supported on a compact -1 (resp. -4) curve in the base. Further, each I_{2p}^{ns} is of non-split type, so the associated algebra is $\mathfrak{sp}_p$. For the leftmost and rightmost non-compact curves curves we have an "external" I_p^* fiber, yielding an $\mathrm{SO}(2p+8) \times \mathrm{SO}(2p+8)$ flavor symmetry. The tensor branch of this system can be described by an SO / Sp quiver theory: the alternating $\mathfrak{sp}_p$ and $\mathfrak{so}_{2p+8}$ factors correspond to the alternation of the I_{2p}^{ns} and the I_p^* singular fibers along the chain of intersecting $\mathbb{P}^1$'s of line (3.13). At the intersections of the -4 with the -1 curves we also find localized matter modes in the form of bifundamental half hypers.

Let us stress here a crucial difference with respect to what we have found discussing the M5 probes of an A_{k-1} singularity. For the theory of N M5-branes probing an A_{k-1} singularity, the corresponding F-theory realization involves dressing the -2 curves obtained by resolving the A_{N-1} singularity by I_k fibers, so we find precisely $N-1$ tensor multiplets in the tensor/Coulomb branch of the system. If, instead, we consider the theory of N M5-branes probing a D_{p+4} singularity, we have seen that in the F-theory realization we dress the -2 curves obtained by resolving the A_{N-1} singularity with I_p^* fibers. However, the full resolution does not lead to $N-1$ tensor multiplets, but to $2N-1$. This fact again suggests the existence of a fractional M5 charge: probing the D_{p+4} singularity we find that a full M5-brane is a compound of two objects, and this explains why we find almost twice as many tensor multiplets in both the IIA and IIB descriptions of these 6d systems.

3.4.2 The special case $p = 0$

For generic values of p, we see the M5-branes probing the D_{p+4} singularity lead, in the resolved phase, to a configuration of -4 and -1 curves, where the -4 curves support SO type gauge groups, while the -1 curves support Sp type gauge groups. When $p = 0$, however, each -1 curve supports no gauge group, since $\mathrm{Sp}(0)$ is trivial.

This special case is also closely connected with the "rigid" theories encountered in [12]. Indeed, let us recall that the alternating pattern of compact curves:

$$4, 1, \ldots, 1, 4 \tag{3.14}$$

supports a collection of $\mathfrak{so}_8$ gauge symmetries, each supported on a -4 curve. In the notation of [12] (see also appendix D), this theory comes from the minimal resolution of the endpoint $3A_N3$. To reach the case of M5-branes probing a D-type singularity,

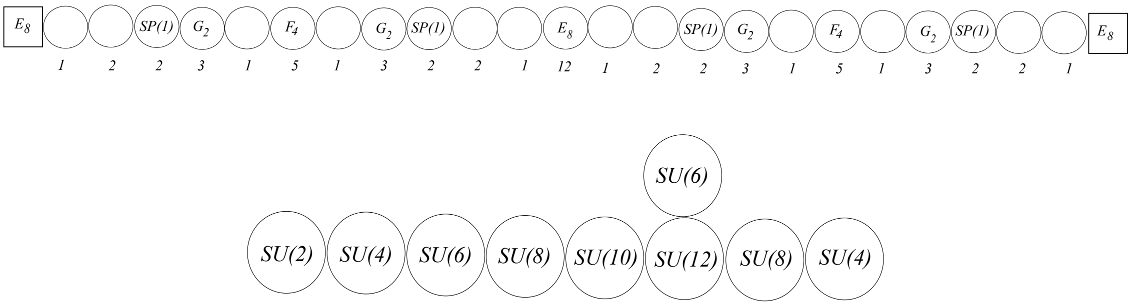


Figure 4. An instance of a novel 5d duality. The 6d theory of two M5-branes probing an E_8 singularity (TOP) compactified on S^1 is dual to the affine $\widehat{E}_8(2)$ 5d quiver system (BOTTOM).

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we decompactify the leftmost and rightmost -4 curves, thus converting them to flavor symmetries.

4 Novel 5d dualities

Though our main focus in this paper is the theory of conformal matter in six dimensions, it is natural to ask what becomes of this system upon further compactification. Here we study the simplest possibility given by compactifying on an S^1. We will argue that this compactified theory is dual to a more conventional 5d quiver gauge theory, as obtained from D-brane probes of an ADE singularity.

To orient ourselves, let us consider the simplest example of this, namely N M5-branes probing an A_k singularity. As we already discussed, this is expected to give a linear quiver gauge theory with $\mathrm{SU}(k+1)^{N-1}$ gauge symmetry, and bifundamental matter. Compactifying on a circle will still give rise to the same quiver theory but now in five dimensions. On the other hand, using the relation between M-theory and type IIA, by viewing the circle as the '11-th' dimension, this is the same as N D4-branes probing the A_k singularity. Using the Douglas-Moore construction [33], we deduce that this should be given by the quiver theory involving $\mathrm{SU}(N)^{k+1}$ arranged along a ring leading to a duality between two different 5d quiver theories. In fact this duality was explained in [38–40] using the horizontal/vertical exchange symmetry [41] of the toric realization of these theories, and used to compute the partition function of the associated strings of the 6d $(1,0)$ SCFT.

For the case of M5-branes probing D-singularities in the same way we get a duality between the quiver chain $SO \times Sp \times SO \ldots$. and the affine D-quiver arising from N D4-branes probing D-singularities. These are already non-trivial dualities. For the E-series we get novel dualities which were not noted before. For example, in the case of 2 M5-branes probing an E_8 singularity the resulting duality is shown in figure 4. Let us note here that the E_8 flavor symmetries are only realized in the strong coupling limit: indeed, the -1 curves at the very left and very right function as "unconventional matter" for the rest of the gauge theory system, which is massive on the tensor branch.

To provide further evidence in support of this duality, we do a basic check: we shall count the number of vector multiplets on the Coulomb branch in the five-dimensional theory obtained by dimensional reduction. We shall then verify that this matches the dimensional reduction of M5-branes, i.e. D4-branes probing our ADE singularity. This will provide a nice check on our proposal that we have correctly identified the degrees of freedom in the 6d theory.[8]

In the five-dimensional theory, we can get vector multiplets in one of two ways. First of all, starting on the tensor branch, we see that each tensor multiplet, upon dimensional reduction, gives a U(1) vector multiplet. Additionally, we see that for each 6d gauge group factor on the tensor branch, we get some additional vector multiplets in 5d. So, we predict the number of vector multiplet to be $r+n_T$ where r is the total rank of the 6d gauge groups and n_T is the number of 6d tensor multiplets. Moreover, the tensor multiplets either arise from the $(N-1)$ "intervals" between the M5-branes and N times the number of tensors for each conformal matter system. Similarly the total rank of the gauge group r arises from two sources: $(N-1)r_G$ from the intervals between M5-branes where r_G is the rank of the G_{ADE}-type singularity, and N times the rank of the gauge group from each conformal matter sector. In a system with N M5-branes, we have partitioned up our interval into $N-1$ compact pieces, so the dimension of the Coulomb branch for the 5d theory is:

$$\dim_{5d\ \mathrm{Coul}}(G,N) = n_{\mathrm{interval}} + n_{\mathrm{cmatter}}, \tag{4.1}$$

The contribution from the intervals is straightforward:

$$n_{\mathrm{interval}} = (N-1)r_G + (N-1). \tag{4.2}$$

The first summand comes from the 6d vector multiplets, and the additional summand of $(N-1)$ comes from the tensor multiplet of each interval. The contribution from the conformal matter sector must be calculated on a case by case basis:

$$A_k : n_{\mathrm{cmatter}} = 0 \tag{4.3}$$
$$D_p : n_{\mathrm{cmatter}} = (p-3)N \tag{4.4}$$
$$E_6 : n_{\mathrm{cmatter}} = (2+3)N \tag{4.5}$$
$$E_7 : n_{\mathrm{cmatter}} = (5+5)N \tag{4.6}$$
$$E_8 : n_{\mathrm{cmatter}} = (10+11)N \tag{4.7}$$

in the E-type cases, we have split up the contribution in the parentheses: the first term is the rank of the gauge group for the conformal matter sector, and the second summand is the total number of tensors. Totalling this up, we get a prediction for the dimension of the Coulomb branch in the lower-dimensional theory:

$$\dim_{5d\ \mathrm{Coul}}(A_k,N) = (k+1)N - (k+1) \tag{4.8}$$
$$\dim_{5d\ \mathrm{Coul}}(D_p,N) = (2p-2)N - (p+1) \tag{4.9}$$
$$\dim_{5d\ \mathrm{Coul}}(E_6,N) = 12N - 7 \tag{4.10}$$
$$\dim_{5d\ \mathrm{Coul}}(E_7,N) = 18N - 8 \tag{4.11}$$
$$\dim_{5d\ \mathrm{Coul}}(E_8,N) = 30N - 9. \tag{4.12}$$

[8] For a similar statement about a different 5d duality see [10, 42].

A succinct formula for all of these cases is:

$$\dim_{5d\ \mathrm{Coul}}(G) = \sum_i (N d_i^{(\widehat{G})} - 1) = N h_G - r_{\widehat{G}}, \tag{4.13}$$

where the sum on i is over all the nodes of the affine Dynkin diagram, with h_G the dual Coxeter number for G, $r_{\widehat{G}}$ the rank of the affine $\widehat{G}$ Dynkin diagram, and $d_i^{(\widehat{G})}$ the affine Dynkin numbers for each node.

On the other hand, we also know that in the IIA description of the D4-brane probe system, there is a quiver gauge theory with connectivity of an affine ADE Dynkin diagram [33]. Each node of the affine Dynkin diagram quiver has gauge group $\prod_i \mathrm{SU}(N d_i^{(\widehat{G})})$. Thus, the dimension of the Coulomb branch for the D4-brane probe theory is nothing other than equation (4.13). This is a rather non-trivial check of the 5d duality and on our proposal as a whole.

Finally, it would be interesting to see whether these dualities can also be extended to the case of flavor symmetries $G \times G'$, where $G \neq G'$. By a similar token, it would be instructive to extend these considerations to lower-dimensional systems, perhaps along the lines of references [34, 43–45].

4.1 Hints of a 6d duality

The close match in five dimensions between D4-brane probes of ADE singularities, and our more "exotic" superconformal matter sectors naturally suggests a lift all the way back to six dimensions.[9] On the one hand, we have a rather conventional 6d quiver gauge theory, of the type studied in [9]. On the other hand, we have a strongly interacting non-Lagrangian conformal fixed point. Indeed, if we compare the superconformal indices on $S^5 \times S^1$ for these two theories, we are guaranteed to get the same answer. This is because the 5d theories on S^5 (after reduction on the S^1) are already dual. While this does not constitute a proof of a 6d duality, it is at least suggestive and would be interesting to understand better.

5 Partial Higgs branches of the $\mathcal{T}(G, N)$ theories

So far, our discussion has focussed on SCFTs which have a geometric avatar in both M- and F-theory. Starting from these "master theories" we can also move down to a number of lower theories by partial Higgsing, that is, by activating operator vevs in the 6d SCFT which break part of the flavor symmetry. Our plan in this section will be to characterize how to pass from our (G, G) theories down to theories with a broken flavor symmetry.

Our interest in this section will be on partial Higgsing of the flavor symmetry. A full Higgsing to the diagonal $G_{\mathrm{diag}} \subset G_L \times G_R$ symmetry group would correspond to moving

[9] For a similar but ultimately different 6d physical system with a completion to a little string theory see [10]. The main distinction with our case is that there the configuration of curves involves $1, 2, 2, \ldots, 2, 2, 1$, while here the configuration of curves is of $2, 2, \ldots, 2, 2$ type. The configuration of curves $1, 2, 2, \ldots, 2, 2, 1$ cannot all be simultaneously contracted at finite distance in moduli space, and so does not correspond to the tensor branch of an SCFT. This is the reason why the two dualities are ultimately very different in nature. However, we find interesting that the main test passed by the dualities proposed in [10] is formally the same as the one in our (4.1).

the M5-branes off the singularity. We are interested instead in keeping all the M5-branes on the singularity, so we exclude this possibility in what follows. This means in particular that the complex geometry of the F-theory compactification will stay put. The resulting class of deformations in F-theory specify "T-brane data" (see e.g. [18, 22]).

The main result from our analysis is that we can characterize the resulting SCFTs as $\mathcal{T}(G, \mu_L, \mu_R, N)$, where N is the number of domain walls, and μ_L and μ_R specify two independent nilpotent elements $\mu_L \in \mathfrak{g}_L$ and $\mu_R \in \mathfrak{g}_R$ (and their orbits), yielding a breaking pattern of the "ambient" flavor symmetry $G_L \times G_R$. There are no anomaly cancellation constraints since the choice of nilpotent element dictates a breaking pattern into the interior of the theory on the tensor branch.

To understand the possible ways to break the flavor symmetry, it is instructive to see the effects of these contributions on the worldvolume theory of the flavor branes. For specificity, we work in terms of the F-theory picture with flavor symmetry G. We seek to understand how background values for fields in the seven-brane show up in the 6d SCFT. What we will show is that these background values induce vevs for operators in the theory. Conversely, a choice of operator vev leads to a specific choice of boundary condition for the flavor branes.

Consider, therefore, the worldvolume theory for our leftmost flavor brane. This is a seven-brane with gauge symmetry G wrapping the curve $\Sigma \simeq \mathbb{C}^*$, that is, a cylinder. To study possible boundary conditions for this system, it is convenient to view this curve as a compact $\mathbb{P}^1$ with two marked points, which we label as p_0 and p_∞. The intersection of the flavor brane with another seven-brane occurs at p_0, while p_∞ is far away from this intersection point.

Allowing for singular behavior for our fields at the marked points, the BPS equations for the seven-brane are governed by the Hitchin system coupled to defects:

$$\overline{\partial}_A \Phi = \sum_p \mu_{\mathbb{C}}^{(p)} \delta_{(p)} \,, \qquad F + [\Phi, \Phi^\dagger] = \sum_p \mu_{\mathbb{R}}^{(p)} \delta_{(p)} \tag{5.1}$$

for an adjoint-valued $(1,0)$ form Φ and the worldvolume connection A, with F its field-strength. Here, we allow for possibly singular behavior at a marked point p. For each point p, $\mu_{\mathbb{C}}^{(p)}$ and $\mu_{\mathbb{R}}^{(p)}$ specify a triplet of moment maps. When matter is localized at an intersection point, the associated sources can be interpreted as vevs for matter fields [22, 29].

Letting z denote a local coordinate on Σ such that $z = 0$ is a marked point, the local behavior of a solution to this system of equations can be determined by solving the holomorphic constraint (i.e. the F-term) modulo complexified gauge transformations. Doing so, we get:

$$\Phi \sim \mu_{\mathbb{C}} \frac{dz}{z} \tag{5.2}$$

where we have presented the solution in a holomorphic gauge so that $A_{(0,1)}$ is trivial (see e.g. [18, 46]). To pass to a unitary gauge in which F- and D-terms modulo unitary gauge transformations are satisfied, we need to conjugate by an appropriate position dependent element $h(z, \overline{z})$ of the complexified group, $\Phi \to h^{-1} \cdot \Phi \cdot h$ and $A_{(0,1)} \to h^{-1} \cdot \overline{\partial} h$. This amounts to replacing $\mu_{\mathbb{C}}$ by a position dependent profile $\mu_{\mathbb{C}}(z, \overline{z})$. This position dependence is related to the fact that the flux profile of the seven-brane yields a funnel solution which opens up near the point p_∞. When we turn to the IIA realization of this system, we will encounter this behavior again as an "ordering constraint" on the partitioning of semi-infinite D6-branes (see also [47]). Solutions to Hitchin system with a simple pole were first considered in [48]. For further discussion, we refer the interested reader to section 3.3 of reference [46]. Higher order poles (i.e. irregular singularities) can also be studied in the same fashion, and correspond to activating vevs for multiple operators [29].

To characterize these solutions more globally, it is convenient to introduce the complexified connection:

$$\mathcal{A} = A + \Phi + \Phi^\dagger. \tag{5.3}$$

Solutions to Hitchin's system are given by flat complexified connections [49]. In the case at hand where Σ is a cylinder, we see that there is precisely one closed one-cycle to integrate around, so there is one holonomy we get to specify:

$$\mathcal{H} = P\exp\left(-\oint \mathcal{A}\right), \tag{5.4}$$

valued in $G_{\mathbb{C}}$. The conjugacy class for the holonomy $\mathfrak{C}_{\mathcal{H}}$ in $G_{\mathbb{C}}$ is gauge invariant data, and fixes a choice of vacuum. Observe that encircling the point p_∞ and specifying the holonomy there amounts to also specifying the holonomy around p_0 since $\mathcal{H}_\infty \cdot \mathcal{H}_0 = \mathbf{id}$.

Now, in the 6d SCFT, the boundary data associated with $\mu_{\mathbb{C}}$ and $\mu_{\mathbb{R}}$ is interpreted as giving vevs to operators of the theory. Said differently, we see that the background values of the flavor seven-branes generate a non-zero Higgsing in the resulting theory. Conversely, if we ask what the lift of operator vevs in the 6d SCFT translates to in the brane construction, we need to alter the boundary conditions for the Hitchin system fields out at p_∞. Finally, we note that although this fixes the breaking pattern for the flavor symmetry, it is important to note that in unitary gauge, the relative size of the Hitchin system fields at p_∞ and p_0 will in general be different, being controlled by the Hermitian pairing for the Higgs bundle.

Thus, our characterization of vacua boils down to possible choices for boundary conditions at the marked points, which are in turn controlled by $\mu_{\mathbb{C}}$ and $\mu_{\mathbb{R}}$. Now, recall that we are also restricting attention to deformations of the 6d SCFT which cannot be understood as unfolding the singularity associated with the seven-brane. In the decomposition of $\mu_{\mathbb{C}} = \mu_s + \mu_n$ into a semi-simple (i.e. in the Cartan) and nilpotent piece, it is well-known that the semi-simple part shows up as just such an unfolding (see e.g. [29]). For our present purposes, it is therefore enough to focus attention on the case $\mu_s = 0$, so that $\mu_{\mathbb{C}}$ nilpotent. In other words, $\mu_{\mathbb{C}}$ acts as a raising operator, and $\mu_{\mathbb{R}} \propto [\mu_{\mathbb{C}}, \mu_{\mathbb{C}}^\dagger]$ acts as a Cartan generator for an $\mathfrak{su}(2)$ subalgebra of $\mathfrak{g}$, specifying a homomorphism (by abuse of notation) $\mu : \mathfrak{su}(2) \to \mathfrak{g}$.

Summarizing, the basic data associated with a partial Higgs branch of our 6d SCFT is a choice of nilpotent element μ, or even more precisely, its orbit $O_\mu \subset \mathfrak{g}_{\mathbb{C}}$. Conversely, if we specify in $G_{\mathbb{C}}$ a conjugacy class $\mathfrak{C}_{\mathcal{H}}$ for the holonomy, then we have implicitly also fixed a choice of Higgs branch.

Now, since we can independently choose the boundary data for our two flavor branes G_L and G_R, we see that the Higgs branches of our theories are labeled by a pair of homomorphisms (μ_L, μ_R). We shall refer to these theories as $\mathcal{T}(G, \mu_L, \mu_R, N)$. The residual flavor symmetry for $\mathcal{T}(G, \mu_L, \mu_R, N)$ is the commutant of the image $\mu_L(\mathrm{SU}(2)) \times \mu_R(\mathrm{SU}(2)) \subset G_L \times G_R$.[10]

Our characterization of vacua by nilpotent elements of the complexified flavor symmetry is basically a special case of the broader notion of "T-brane data" in an F-theory compactification [18–22]. These are non-abelian intersecting brane configurations in which the adjoint-valued Higgs field of the seven-brane is nilpotent (i.e. upper triangular for $\mathfrak{sl}(n, \mathbb{C})$). As they are nilpotent, such vacua do not appear in holomorphic Casimir invariants, and so are not visible in the complex geometry of the Calabi-Yau threefold. Rather, they can be seen in the limiting behavior of its intermediate Jacobian [22]. What we have just seen is that T-brane data leads to a rich class of 6d SCFTs.[11]

Having illustrated how boundary data of the Hitchin system realization in F-theory feeds into the 6d SCFT, let us now return to the M-theory realization in terms of 6d domain walls in 7d super Yang-Mills theory with gauge group G. Here, our flavor symmetry is supported on a pair of semi-infinite intervals, so it is convenient to label this coordinate as x_6. The two marked points p_∞ and p_0 are now replaced by a choice of boundary conditions for this 7d Yang-Mills sector: one which is far away from all of the compact intervals at $x^6 = x^6_\infty$, and one which touches the various compact intervals. In the worldvolume of the 7d theory, there is a pole for the Nahm equations:

$$\partial_A \Phi^i = \epsilon^{ijk}[\Phi^j, \Phi^k]\,, \tag{5.5}$$

where now Φ^i for $i = x, y, z$ are the triplet of real scalars in 7d SYM, and $\partial_A = \frac{\partial}{\partial x^6} - A_6$ is the worldvolume covariant derivative in the direction x^6 along the semi-infinite interval. Near $x^6 = x^6_\infty$, the fields have asymptotic behavior:

$$\Phi^i \sim \frac{t^i}{x^6 - x^6_\infty}\,, \tag{5.6}$$

where the t^i are Hermitian generators of an $\mathfrak{su}(2)$ subalgebra of the gauge algebra $\mathfrak{g}$. Of course, this is the same data we already encountered in the F-theory description. Further, we see that the pole for the Higgs field in the Hitchin system out at p_∞ is now reflected in a pole for the Φ^i at $x^6 = x^6_\infty$. With this boundary condition fixed, we also see that the profile of the field configuration in 7d SYM will now interpolate from $x^6 = x^6_\infty$ inwards to the first M5-brane, inducing a vev for operators in the 6d SCFT.

5.1 IIA realizations

In the special case where we have a IIA realization, we can be even more explicit. For specificity, we focus on the case $G = \mathrm{SU}(k)$ studied in [13]. In that context, the relevant brane systems are the ones considered in [8, 11, 13]. The relevant brane configurations

[10] This residual symmetry can be extracted from known results in the literature (see e.g. [50]), and was actually already considered in the physics literature in the related context of class S theories in four dimensions; see for example [51, 52].

[11] Such T-brane configurations also provide a way to construct four-dimensional superconformal field theories. These are realized by D3-branes probing a T-brane background. For additional discussion of these worldvolume theories, see [53–57].

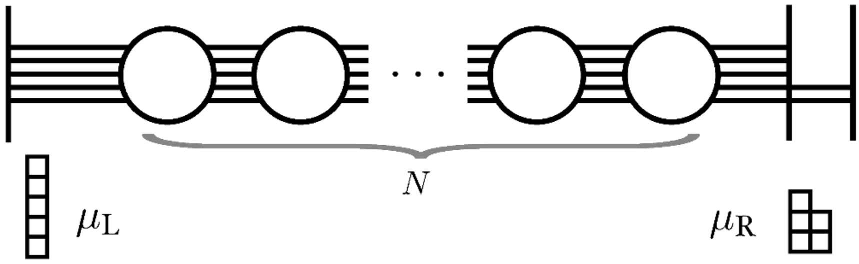

Figure 5. A brane configuration similar to figure 3, where now the D6s end on D8s in two different ways on the two sides, corresponding to two different Young diagrams. This configuration is T-dual to the one in figure 6.

are similar to the ones of our "master theory", except that now we take the D6 to end on several D8s; let $\mu_{\rm L}^a$ ($\mu_{\rm R}^a$) be the number of D6s ending on the a-th D8 on the left (right). When each D6 ends on a separate D8, so that all the $\mu_{\rm L,R}^a = 1$, the D8s impose Dirichlet boundary conditions for the gauge field on the D6s, and Neumann for the scalars.

An example of a more general situation, with not all the $\mu_{\rm L,R}^a = 1$, is depicted for example in figure 5. Actually, however, such a picture is too naive: one now expects that the D6s and the D8s fuse into a single D8/D6 bound state. This is described on the worldvolume of the D6s by a pole for the Nahm equations, just as in equation (5.5).

The presence of the Nahm pole can be interpreted as the fact that the D6s open up into the D8s. To see this, consider first for simplicity the "full" pole where all k D6s end on a single D8, as on the left of figure 5. Notice that for $x^6 \to x^6_\infty$ we have $\Phi_i \Phi_i \to \frac{k(k-1)}{(x^6 - x^6_\infty)^2}\mathbf{id}$; a slice at constant x^6 is well approximated by a fuzzy sphere of radius $\sim \frac{k}{x^6 - x^6_\infty}$. In the more general case where not all the D6s end on the same D8 (as on the right side of figure 5), each D8 represents a Jordan block, whose size is the number μ^i of D6s ending on it; we then have several fuzzy spheres, of radii $\sim \frac{\mu_i}{x^6 - x^6_\infty}$. An exception is the case of a Jordan block of size 1, which is of course simply a zero; in that case, the fuzzy sphere actually has radius 0, and the D6 actually ends on a D8: the two do not fuse together. This block then behaves as in the case we mentioned at the beginning of this subsection: the D8s in this block impose Dirichlet boundary conditions for the gauge field on the D6s, and Neumann for the scalars.

Thus in general we should imagine fuzzy funnels coming out of the NS5 system; for a cartoon (in the case where the NS5s coincide) see figure 5 of reference [13]. Implicit in that cartoon is also the fact that the D8s in pictures such as figure 5 should be ordered so that the $\mu_{\rm L}^a$ and $\mu_{\rm R}^a$ should decrease as one goes outside (this was found in [47] by comparing the moduli spaces of solutions to Nahm equations to the moduli spaces of the corresponding field theories).

A second constraint is that there should be enough NS5s [13]: when one moves the branes around to reach a quiver description, one should not remain with some extra decoupled sectors. This reads

$$N \geq \mu_{\rm L}^1 + \mu_{\rm R}^1 \,, \tag{5.7}$$

where $\mu_{\rm L}^1$ and $\mu_{\rm R}^1$ are the number of D6s ending on the two innermost D8s or, in other words, the tallest columns of the two Young diagrams (this would amount to $N \geq 5+3=8$ in the example of figure 5).

In order to see the connection of the IIA discussion in this subsection to the previous discussion in F-theory it is enough to T-dualize along a direction transverse to the D6s, say x^7, generalizing our discussion in section 3.3.1. The NS5s turn into geometry: each pair of them give rise to a nontrivial two-cycle. The D6s suspended in between the NS5s now become a D7 stack (or in other words D7s with nonabelian gauge group SU(k)) wrapping the two-cycle. This is the tensor branch of the model; the SCFT point is reached by putting the NS5s on top of each other, and in the T-dual picture this amounts to collapsing the corresponding nontrivial two-cycles to zero size. At each of the ends of the diagram, however, things are a little different. Let us first look at the case where each of the D6s ends on a separate D8. In this case, we get an SU(k) flavor symmetry, because no Nahm pole is possible. Further, if we T-dualize, we get a single stack of D7-branes wrapping a non-compact two-cycle.

In more general cases, however, we have seen in IIA that the D6s and the D8s fuse together into a single object, as reflected in non-trivial Nahm pole data. We expect this to have its own counterpart on the IIB side. This can be described on the worldvolume of the former D6 as follows: one of the transverse scalars (say Φ^3) becomes in IIB a gauge field, and we can complexify the two remaining scalars: $\Phi \equiv \Phi_1 + i\Phi_2$. The Nahm equations (5.5) now turn into the (gauge fixed) Hitchin equations (5.1); the presence of a Nahm pole (5.6) turns into the presence of a Hitchin pole. The D7 with a Hitchin pole can now be thought of as the fusion of a system of D7s. The structure of the D8/D6 system with a Nahm pole is therefore encoded in the structure of the block decomposition of the Hitchin field, that, in turn, encode the T-brane data on the F-theory side. The situation is now depicted schematically in figure 6.

In IIA string theory, we can also realize theories with SO and Sp gauge symmetry. Compared with the A-type case, the Type IIA description of the system is almost analogous, the main difference being the presence of the O6$_-$ plane. Again introducing D8s on the left and on the right of the NS5s that impose Dirichlet boundary conditions for the gauge field on the D6s, and Neumann for the scalars we recover the system with $SO_L \times SO_R$ flavor symmetry and the T-dual configuration of I_p^* fibers we discussed in the previous section. Consider now introducing Nahm poles for the D6/D8 systems on the left and on the right. These are specified by a pair of homomorphisms μ_L and μ_R from $\mathfrak{su}(2)$ to $\mathfrak{so}_{2p+8}$. By T-duality these very same homomorphisms characterize the T-brane data on the F-theory side.

5.2 Alternative realizations for some SU(k) cases

Notice that so far we have kept all the NS5s in the region where $F_0 = 0$. In figure 5: there are no D6s ending on any NS5. In general, the net number of D6s ending on a NS5 equals $k_{\rm D6,\,L} - k_{\rm D6,\,L} = n_0 = 2\pi F_0$ (which is an integer by flux quantization). This can be seen either by anomaly cancellation in the 6d field theory, or by a Bianchi identity in the brane picture. Because of this, given any valid D8-D6-NS5 configuration, we can decide to move all the NS5s in the region $F_0 = 0$ as we have done so far. It is also possible to move them all to a

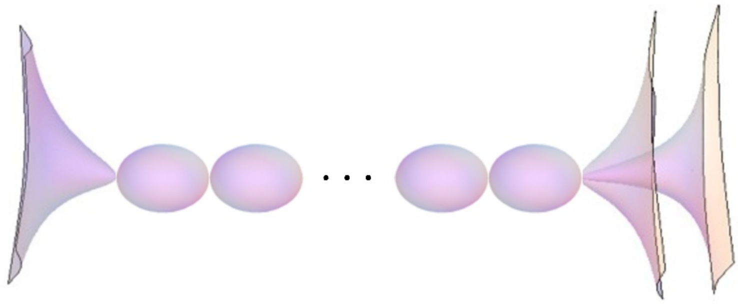

Figure 6. The T-dual in type IIB / F-theory of figure 5. The original description in terms of partitions has been smoothed out to a flux profile over the Hitchin system curve, resulting in a funnel-like solution in the geometry.

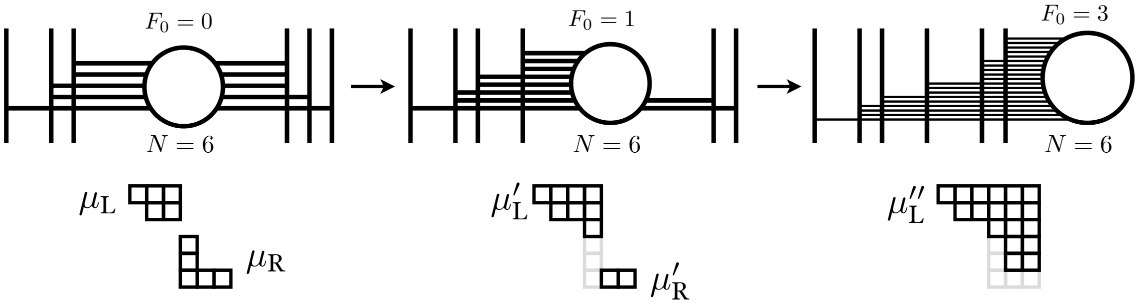

Figure 7. When one moves the NS5s to a region where $F_0 \neq 0$, an intuitive algorithm relates the new Young diagrams $\mu'_{\rm L}$ and $\mu'_{\rm R}$ to the old ones $\mu_{\rm L}$ and $\mu_{\rm R}$. Continuing the process, one can reach a situation where all the NS5s are on one side.

different region, say where $F_0 = 1$. This is not especially natural in general, but it is a useful alternative for some particular theories, namely for those that saturate the bound (5.7).

This will actually become clearer when we discuss gravity duals in section 7. Meanwhile, we notice here that there is a nice visual device (applicable whether the bound (5.7) is saturated or not) to see how the T-brane data change when one moves the NS5s to a region where $F_0 \neq 0$; it is depicted in figure 7. Basically, we add a column $(N - \mu_{\rm R}^1)$-box tall to the $\mu_{\rm L}$ Young diagram, and erase the first column to $\mu_{\rm R}$, obtaining two new Young diagrams $\mu'_{\rm L}$ and $\mu'_{\rm R}$. Notice that actually $\mu'_{\rm L}$ is still a Young diagram only thanks to the bound (5.7). Moreover, a new bound $N \geq {\mu'_{\rm L}}^1 + {\mu'_{\rm R}}^1$ is now also satisfied. So in a sense the bound (5.7) is true more generally than in the way we originally formulated it.

In the previous subsection our field theories $\mathcal{T}(\mathrm{SU}(k), \mu_{\rm L}, \mu_{\rm R}, N)$ were presented as produced by Higgsing from the theory $\mathcal{T}(\mathrm{SU}(k), N)$ describing N NS5s on top of k D6s. Now we see that we can actually also think of them as arising by Higgsing from a more general theory $\mathcal{T}(\mathrm{SU}(k_L), \mathrm{SU}(k_R), N)$ describing N NS5s in a region where $F_0 \neq 0$, with $k_{\rm L}$ semi-infinite D6s on their left and $k_{\rm R}$ semi-infinite D6s on their right (where $k_{\rm R} - k_{\rm L} = n_0 N$). As we mentioned, this might actually be more natural for the theories which saturate (5.7), as we will see in section 7. It would be interesting to develop further such alternative characterizations for the SO/Sp type examples which also have IIA realizations.

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6 SCFTs from the Hořava-Witten wall

So far we have discussed $(1,0)$ 6d SCFTs which are realized by M5-branes probing an ADE singularity. Another well known example of $(1,0)$ theories involve M5-branes approaching the Hořava-Witten wall, namely the theory of small E_8 instantons (a.k.a E-string theories) in heterotic string theory [3–5].

The F-theory realization of E-strings has been studied in [24, 58]. In F-theory, the single E-string theory is given by working on the base $\mathcal{O}(-1) \to \mathbb{P}^1$, that is, the base is the local geometry of a single $\mathbb{P}^1$ with self-intersection -1. This arises from blowing up the intersection point $u = v = 0$ in the base of the geometry:

$$y^2 = x^3 + uv^5, \tag{6.1}$$

where the E_8 flavor symmetry is localized at $v = 0$. Multiple M5-branes probing the E_8 wall leads to a similar class of theories. In heterotic string theory, this is the theory of N coincident small E_8 instantons. In F-theory, this is given by a configuration of curves:

$$[E_8]\, \underbrace{1\ 2\ \ldots\ 2}_{N} \tag{6.2}$$

where the -1 curve again enjoys an E_8 flavor symmetry.

The Higgs branch of these theories corresponds (in the heterotic description) to dissolving some of the instantons of this background back into flux. In the F-theory description this is captured by a T-brane configuration for the seven-brane with E_8 symmetry. The moduli space for this system is that of N instantons for E_8 gauge theory on the four-manifold wrapped by the nine-brane.

We can also combine the ingredients of these two classes of $(1,0)$ theories, by considering an ADE singularity intersecting the E_8 wall and bringing in M5-branes to probe this intersection. The aim of this section is to study the resulting $(1,0)$ theory and in particular elucidate its tensor branch and partial Higgs branches.

6.1 Orbifolds

Just as we considered the case of M5-branes probing a singularity, we can also consider the case where the E_8 nine-brane intersects an ADE singularity $\mathbb{C}^2/\Gamma_G$. Such configurations still preserve $(1,0)$ supersymmetry in six dimensions, and are therefore excellent candidates for realizing additional SCFTs. The F-theory realization of these configurations has been studied in [17], and is given by decorating the configuration of line (6.2) by a non-minimal Kodaira-Tate fiber:

$$[E_8]\, \underbrace{\overset{\mathfrak{g}}{1}\ \overset{\mathfrak{g}}{2}\ \ldots\ \overset{\mathfrak{g}}{2}}_{N}\, [G] \tag{6.3}$$

that is, it consists of N compact curves (corresponding to N M5-branes) and one curve which has been decompactified, as denoted by $[G]$. The matter content of these models has been worked out in [10] by requiring anomaly cancellation. We shall refer to this class of theories as $\mathcal{T}(E_8, G_R, N)$, in the obvious notation.

One way to see that this is indeed the correct characterization is to use the standard rules for heterotic / F-theory duality. Starting from heterotic on an ADE singularity, we can instead consider a non-compact elliptic $K3$ surface $T^2 \to K3_{het} \to \mathbb{C}$ with prescribed Kodaira-Tate fiber over a marked point of the base. This elliptic $K3$ is then the gluing region for the stable degeneration limit of a Calabi-Yau threefold in the dual F-theory description:

$$X = X_L \cup_{K3_{het}} X_R, \tag{6.4}$$

so in other words, there is a collision between the $\mathfrak{g}$-type Kodaira-Tate fiber from the right, and the II^* fiber type supported over $[E_8]$.

As presented, (6.3) is not a completely resolved phase of the F-theory geometry. This follows because at the collision of the -1 curve with the $[E_8]$ component, the singularity type passes beyond the allowed order of vanishing for a Kodaira-Tate fiber. To pass to a resolved geometry, we would need to perform further blowups at this point. Alternatively, we can move from the conformal fixed point onto the Higgs branch, passing to a lower theory. Let us now turn to a characterization of each such branch.

6.1.1 Tensor branch

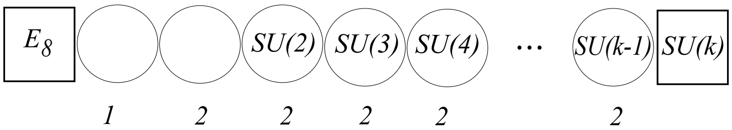
To get started with thinking about the tensor branch, it turns out to be useful to use the M-theory frame. We consider the geometry $\mathbb{R}/\mathbb{Z}_2 \times \mathbb{C}^2/\Gamma_{ADE}$. Note that the locus of the ADE singularity transverse to the 6d spacetime is a half-line $\mathbb{R}/\mathbb{Z}_2$, where the boundary of the half-line is where the singularity intersects the E_8 wall. We can in addition introduce N M5-branes on the singularity half-line. The SCFT is obtained by bringing the M5-branes to the E_8 wall, i.e., the boundary of the half-line. To go to the tensor branch we need to first separate the M5-branes along the half-line. We end up with N segments, each carrying the corresponding ADE gauge symmetry. Between every adjacent interval we get the conformal matter of the ADE type we have already discussed. The only new ingredient is the extra degree of freedom corresponding to the matter localized where the ADE singularity meets the Hořava-Witten wall. This matter will have $E_8 \times G$ global symmetry. This is the main new ingredient we need to understand in the context of this new class of SCFTs. To figure out what this new conformal matter is we will use the F-theory setup.

In the F-theory setup, the theory on the tensor branch is given by performing all possible blowups so that the elliptic fibers are all in Kodaira-Tate form. This has been worked out in [17], though the particular points we emphasize here are somewhat different. The analysis To begin, suppose that we have a $\mathfrak{g} = \mathfrak{su}_k$ type gauge symmetry on each curve, corresponding to an I_k type singularity. Performing a further blowup on the intersection of the -1 curve with the $[E_8]$ locus, we get a new -1 curve, but which now supports an I_{k-1} singularity. Proceeding in this way, we get k additional blowups until we finally reach the configuration:

$$[E_8]\underbrace{1\;\overset{\mathfrak{su}_1}{2}\;\overset{\mathfrak{su}_2}{2}\;\ldots\;\overset{\mathfrak{su}_{k-1}}{2}}_{k}\underbrace{\overset{\mathfrak{su}_k\oplus\mathbf{k}}{2}\;\overset{\mathfrak{su}_k}{2}\;\ldots\;\overset{\mathfrak{su}_k}{2}}_{N}[\mathrm{SU}(k)]. \tag{6.5}$$

with the usual bifundamental matter between the adjacent quivers nodes. Additionally, there is one extra hypermultiplet attached to the leftmost $\mathfrak{su}_k$ factor, i.e. at the "plateau" of $\mathfrak{su}_k$ factors. In F-theory language, this comes from the collision of the zero section with the leftmost $\mathfrak{su}_k$ factor, so that on each such $\mathfrak{su}_k$ there are a total of $2k$ hypermultiplets, as required for 6d gauge anomaly cancelation [10].

Focussing on the region with gauge groups of increasing rank, the new conformal matter system with $E_8\times\mathrm{SU}(k)$ flavor symmetry is the quiver system given by the ramp:



This ramp was also reproduced in IIA language in [11]. There, the end of the ramp corresponds to the presence of an additional D8-brane just before the plateau region to the right. We can view this quiver as arising from the splitting of the intersection point between $\mathrm{SU}(k)$ singularity and the wall to $k+1$ points.

In the case of a D_{p+4}-type orbifold, the initial configuration of curves and singular fibers is:

$$[E_8]\underbrace{\overset{\mathfrak{so}_{2p+8}}{1}\;\overset{\mathfrak{so}_{2p+8}}{2}\;\ldots\;\overset{\mathfrak{so}_{2p+8}}{2}}_{N}[\mathrm{SO}(2p+8)] \tag{6.6}$$

As we already saw in the case of M5-branes probing a D-type singularity, a blowup is required at each collision of I_p^* fibers. Performing the requisite sequence of blowups to reach the resolved phase, we have:

$$[E_8]\underbrace{\overset{\mathfrak{so}_{2p+8}}{1}\;\overset{\mathfrak{so}_{2p+8}}{2}\;\ldots\;\overset{\mathfrak{so}_{2p+8}}{2}}_{N}[\mathrm{SO}(2p+8)]$$

$$\to[E_8]\underbrace{1\;2\;\overset{\mathfrak{sp}_1}{2}\;\overset{\mathfrak{g}_2}{2}\;\overset{\mathfrak{so}_9}{2}\;\ldots\;\overset{\mathfrak{so}_{2p+7}}{2}}_{p+4}\underbrace{\overset{\mathfrak{so}_{2p+8}}{2}\;\overset{\mathfrak{so}_{2p+8}}{2}\;\ldots\;\overset{\mathfrak{so}_{2p+8}}{2}}_{N}[\mathrm{SO}(2p+8)]$$

$$\to[E_8]\underbrace{1\;2\;\overset{\mathfrak{sp}_1}{2}\;\overset{\mathfrak{g}_2}{3}\;\overset{\mathfrak{sp}_0}{1}\;\overset{\mathfrak{so}_9}{4}\;\ldots\;\overset{\mathfrak{so}_{2p+7}}{4}\;\overset{\mathfrak{sp}_p\oplus\frac{1}{2}\mathbf{2p}}{1}}_{2p+5}\underbrace{\overset{\mathfrak{so}_{2p+8}}{4}\;\overset{\mathfrak{sp}_p}{1}\;\overset{\mathfrak{so}_{2p+8}}{4}\;\ldots\;\overset{\mathfrak{so}_{2p+8}}{4}\;\overset{\mathfrak{sp}_p}{1}}_{2N}[\mathrm{SO}(2p+8)]\;;$$

that is, there is again a ramp in the rank of the gauge groups reaching the plateau involving the D-quivers with $D\times D$ conformal matter we have already studied. In this case, there

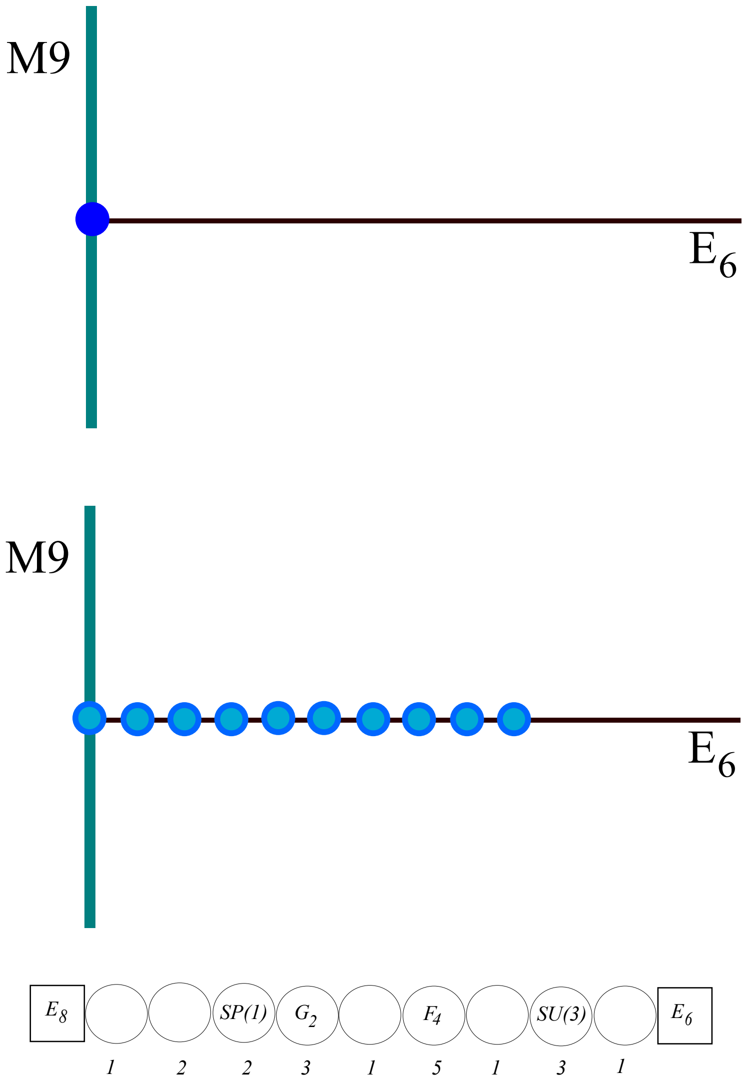


Figure 8. UP: M9-brane at the orbifold E_6 singularity. DOWN: SCFT matter trapped at the intersection in between the M9 and the singularity that gives the (E_8, E_6) ramp. The boundary point has fractionated to 10 points.

is one additional half hypermultiplet attached to the rightmost $\mathfrak{sp}_p$ factor of the ramp. All the rest of the $\mathfrak{sp}$ factors to the left come from the collision of $I_{n-1}^{*,ns}$ and $I_n^{*,ns}$ fibers [17], and so lead to a non-split I_{2n-1} fiber. As explained in [6], this leads to no additional matter multiplets (beyond the half hypers trapped at the SO/Sp intersections), and the gauge symmetry is $\mathfrak{sp}_{n-1}$. Thus, the ramp contains $p+1$ $\mathfrak{sp}_n$ factors which start at the left with $\mathfrak{sp}_0$, and increase one rank at a time until the right of the ramp, with the final $\mathfrak{sp}_p$ factor. Note that in the plateau region, all $\mathfrak{so}_{2p+8}$ factors have $4p$ half hypermultiplets in the fundamental, as required by 6d gauge anomaly cancelation [10]. Again, these ramps can also be reproduced using IIA methods [59]. So the new conformal matter with $E_8 \times \mathrm{SO}(2p+8)$

symmetry is the ramping up quiver with $(2p{+}5)$ tensor multiplets. In particular we can view this as fractionating of the boundary point to $2p+6$ points. In the special case, $p=0$, the ramp truncates to just the $1,2,2,3,1,$ factor, that is, there is no alternating SO/Sp quiver.

Finally, consider the case of M5-branes probing an E-type singularity (see figure 8). In this case, the minimal resolution for N instantons probing the E_6 orbifold is:

$$[E_8]\underbrace{\overset{\mathfrak{e}_6}{1}\;\overset{\mathfrak{e}_6}{2}\;\ldots\;\overset{\mathfrak{e}_6}{2}}_{N}[E_6] \to [E_8]\underbrace{1\;\overset{\mathfrak{sp}_1}{2}\;\overset{\mathfrak{g}_2}{2}\;3\;1\;\overset{\mathfrak{f}_4}{5}\;1\;\overset{\mathfrak{su}_3}{3}\;1}_{9}\underbrace{\overset{\mathfrak{e}_6\oplus\mathbf{27}}{5}\;131\;\overset{\mathfrak{e}_6}{6}\;\ldots\;\overset{\mathfrak{e}_6}{6}\;131}_{4N}[E_6] \tag{6.7}$$

So the new conformal matter system with $E_8\times E_6$ symmetry is the first 9 factors. In particular the intersection point between the E_6 singularity and the wall has fractionated to 10 points. Observe also that in the plateau region, the leftmost E_6 factor also couples to one additional hypermultiplet in the **27** of E_6. This is just an E-type generalization of the phenomenon already encountered for A- and D-type orbifolds in joining the "ramp" to the "plateau" regions.

For the minimal resolution for N instantons probing the E_7 orbifold, we have:

$$[E_8]\underbrace{\overset{\mathfrak{e}_7}{1}\;\overset{\mathfrak{e}_7}{2}\;\ldots\;\overset{\mathfrak{e}_7}{2}}_{N}[E_7] \to [E_8]\underbrace{1\;\overset{\mathfrak{sp}_1}{2}\;\overset{\mathfrak{g}_2}{2}\;3\;1\;\overset{\mathfrak{f}_4}{5}\;1\;\overset{\mathfrak{g}_2}{3}\;\overset{\mathfrak{su}_2}{2}\;1}_{10}\underbrace{\overset{\mathfrak{e}_7\oplus\frac{1}{2}\mathbf{56}}{7}\;12321\;\ldots\;\overset{\mathfrak{e}_7}{8}\;12321}_{6N}[E_7] \tag{6.8}$$

So the new conformal matter system with $E_8\times E_7$ symmetry is the first 10 factors, and the boundary point has fractionated to 11 points. Additionally, there is a half hypermultiplet in the **56** of E_7, precisely on the leftmost node of the plateau, i.e. in the region where we join the ramp to the plateau.

For the minimal resolution for N instantons probing the E_8 orbifold, we have:

$$[E_8]\underbrace{\overset{\mathfrak{e}_8}{1}\;\overset{\mathfrak{e}_8}{2}\;\ldots\;\overset{\mathfrak{e}_8}{2}}_{N}[E_8] \to [E_8]\underbrace{1\;\overset{\mathfrak{sp}_1}{2}\;\overset{\mathfrak{g}_2}{2}\;3\;1\;\overset{\mathfrak{f}_4}{5}\;1\;\overset{\mathfrak{g}_2}{3}\;\overset{\mathfrak{sp}_1}{2}\;2\;1}_{11}\underbrace{\overset{\mathfrak{e}_8\oplus\text{small}}{(11)}\;B_{11}\;\ldots\;\overset{\mathfrak{e}_8}{(12)}\;B_{11}}_{12N}[E_8] \tag{6.9}$$

where B_{11} denotes the configuration of curves for (E_8,E_8) conformal matter. The new conformal matter system at the intersection of the E_8 singularity and the E_8 wall is the system involving the first 11 factors. In other words the boundary point has fractionated to 12 points. Finally, in this case, in the region where we join the ramp to the plateau region, we see that there is the E_8 version of a half hyper, namely a single small instanton on the leftmost node of the plateau region.

6.1.2 Partial Higgs branches

Instead of passing to the resolved phase, we can instead consider moving to a lower theory by passing to partial Higgs branches, as we discussed in the case of M5-branes probing

ADE singularities. In this case, it is simplest to treat all of the theories

$$[E_8]\underbrace{\overset{\mathfrak{g}}{1}\,\overset{\mathfrak{g}}{2}\ldots\overset{\mathfrak{g}}{2}}_{N}[G], \tag{6.10}$$

in a uniform fashion.

Again the partial Higgsing is associated with a choice of breaking pattern for the left and right flavor groups. The right flavor group has exactly the same structure as we have already discussed, and its vacua are characterized by the orbit of a nilpotent element μ_R in the algebra $\mathfrak{g}_{\mathbb{C}}$.

For the left flavor symmetry, that is, the global E_8 factor, the corresponding breaking patterns are characterized somewhat differently. This is because they originate (in heterotic language) from a 10d rather than a 7d Yang-Mills sector. It is simplest to work in terms of the dual heterotic description.[12] There, we are considering heterotic strings on the orbifold $\mathbb{C}^2/\Gamma_G$. E_8 instanton configurations in this case comes with the standard moduli of an instanton, for example the size and position, but also requires specifying boundary data "off at infinity". The boundary of $\mathbb{C}^2/\Gamma_G$ is S^3/Γ_G, and the behavior of the instanton density at infinity is captured by a flat connection at infinity, which translates to a flat E_8 connection on S^3/Γ_G. Such flat bundles are in one to one correspondence with an element $\gamma \in \mathrm{Hom}(\pi_1(S^3/\Gamma_G), E_8) \simeq \mathrm{Hom}(\Gamma_G, E_8)$.

Summarizing, we see that all of these theories are characterized as $\mathcal{T}(E_8, G_R, \gamma_L, \mu_R, N)$.

7 Holographic duals and scaling limits

In this section we study scaling limits for the conformal field theories just constructed in which the number of probe M5-branes becomes large. The most straightforward case to consider is that of N M5-branes in flat space. The near horizon limit for this geometry is well-known, and is given by the 11D supergravity background $AdS_7 \times S^4$ with N units of four-form flux threading the S^4. The probe theory for an ADE singularity is then given by

$$AdS_7 \times S^4/\Gamma_G\,, \tag{7.1}$$

where the ADE subgroup of SU(2) specified by Γ_G has fixed points at the north and south pole of the S^4 [60, 61].

Long ago, the holographic dual of the (1,0) theory of N small instantons was found to be $AdS_7 \times S^4/\mathbb{Z}_2$ (see [62]). In the case of N small instantons probing an ADE singularity,

[12]In the F-theory realization, one must include some additional data beyond just the Hitchin system with E_8 gauge group. The reason is that the Hitchin system on a $\mathbb{P}^1$ with marked points is an appropriate local dual for a smooth $K3_{het}$, since the local geometry of the base in the fibration $T^2 \to K3_{het} \to \mathbb{P}^1$ is $\mathcal{O}(-2) \to \mathbb{P}^1$. Once we allow singularities, however, the reduction to the $\mathbb{P}^1$ will also include extra data associated with the singular fiber of $K3_{het}$. This must be included to fully characterize the moduli space of the flavor seven-brane.

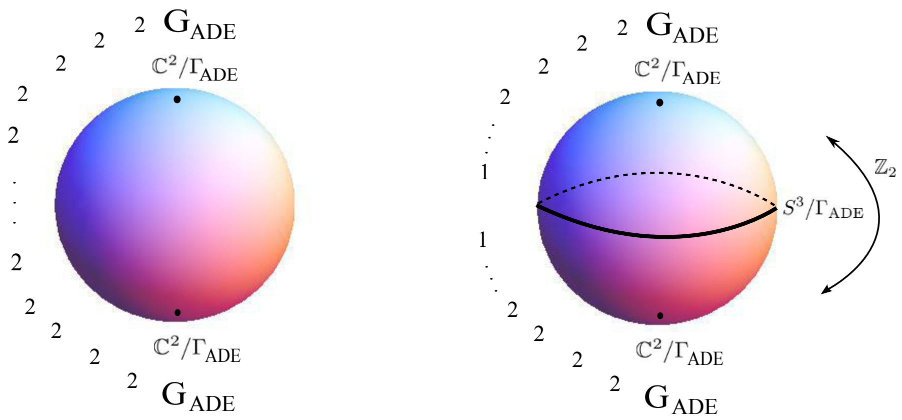


Figure 9. LEFT: a schematic view of the deconstruction of the great arc of S^4/Γ_{ADE}. RIGHT: a schematic view of the deconstruction of the great semi-arc of $S^4/\mathbb{Z}_2 \times \Gamma_{ADE}$.

we instead have the gravity dual

$$AdS_7 \times S^4/\mathbb{Z}_2 \times \Gamma_G \; . \tag{7.2}$$

The $\mathbb{Z}_2$ fixed point locus is the equator of the S^4, while the two fixed points of S^4/Γ_G are now identified. Along the fixed point locus, we also see that there is an E_8 nine-brane wrapped on $AdS_7 \times S^3/\Gamma_G$.

An interesting feature of the F-theory geometry is that the -2 curves in the sequence $2, \ldots, 2$ are literally a deconstruction of a great arc of the associated S^4. More precisely, this is the interval obtained from taking S^4 as an S^3 fibration over a finite interval. Further, in the orbifold S^4/Γ_G, we see that just as the F-theory geometry predicts, there is a flavor symmetry at the north pole, and another at the south pole. These are simply the 7d Super Yang-Mills theories. Starting from such a holographic dual, we can also see that the process of decompactifying leads in the IR to a new dual with a different number of flux units. For example, in the configuration:

$$\underbrace{2, \ldots, 2}_{N}, \underbrace{2, \ldots, 2}_{M}, \tag{7.3}$$

we see that decompactifying curves at the interface of the N and M partitions will decouple the two CFTs. In the gravity dual, this corresponds to introducing a stack of M5-branes located at a specific AdS radius (i.e. the energy scale in the CFT dual), and at a particular point on the great circle of the S^4 (and/or its orbifolds). From this perspective, we can also see that the large N limit of one of our configurations need not yield a semi-classical gravity dual. Indeed, if we attempt to gauge the flavor symmetries, and then push them to strong coupling, we end up collapsing the radius of the AdS space. So in other words, not every large N $(1, 0)$ theory will have a semi-classical gravity dual.

We have also seen that there are a large number of additional SCFTs which can be generated by starting from one of our "master" theories. This is accomplished by moving on

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to the Higgs branch, i.e. by activating vevs for some operators of these theories. As we have already seen, this is captured by a choice of nilpotent element of an algebra, and in the case of the small instanton theories also involves a choice of flat E_8 connection in S^3/Γ_G. It is therefore natural to ask whether these theories with a Higgs branch also have a gravity dual.

First of all, we can see that in most cases, this data will remain hidden from the strict $N=\infty$ description of the gravity dual. This is because the data of the Higgs branches is localized near the north and south poles in the case of the S^4/Γ_G duals, and in the case of the $S^4/\mathbb{Z}_2\times\Gamma_G$ duals is localized near the equator as well. In particular, since the Higgsing only involves a small number of -2 curves (in F-theory language), the actual portion of the great arc sensitive to these effects is of order $1/N$, in units where the great arc between the north and south poles is order one.

To see the effects of the Higgs branches in this limit, we therefore need to take a scaling limit in which

$$\nu=\frac{|\Gamma_G|}{N} \tag{7.4}$$

is held fixed. This can be done for both the A- and D-type orbifold singularities, but not for the E-type singularities. Taking such a limit, we see that we have essentially "stretched out" the contribution from the 7d Super Yang-Mills sector on both the north pole and on the south pole, so we can expect to see the different Higgs branches, i.e. T-brane data directly in the form of the gravity dual solutions.

7.1 The $\mathbb{Z}_k$ case

We can be a little more specific in the case $G=\mathrm{SU}(k)$, $\Gamma_G=\mathbb{Z}_k$. This case was analyzed in [13, 60, 61, 63], and we will review it here. The solutions corresponding to the presence of the T-branes can be described in IIA, by switching on the so-called Romans mass parameter F_0.

First of all, when all T-brane data is switched off, we can reduce $AdS_7\times S^4/\mathbb{Z}_k$ to IIA. This can be done using the characterization of S^4 as an S^3 fibered over an interval I, and observing that the S^3 admits an S^1 fibration $S^1\to S^3\to S^2$, along which the $\mathbb{Z}_k$ acts. Reducing along this S^1, we get a IIA geometry $AdS_7\times Y$, where Y is an S^2 fibration over the interval I. The fibration does not shrink smoothly at the two endpoints of the interval; rather, there are two singularities, which correspond to the presence of two stacks of k D6s and of k $\overline{\text{D6}}$s.[13] Thus Y is not a smooth space; we can think of it as a "football". For more details on this solution, see section 5.1 of reference [63].

Recall that this solution is dual in F-theory to a chain of ordinary D7s, leading to the CFT given in line (3.8). Now let us see what happens if we add T-brane data on the left- and right-most D7s. As we discussed in section 5, these correspond in IIA to brane configurations involving D6s ending on D8s, in a way intuitively summarized by two Young diagrams μ_{L}, μ_{R} which are the same as the T-brane data. It was argued in [13] that the near-horizon limit of those brane configurations is given by the $F_0\neq 0$ solutions in [63]. The internal manifold Y is now no longer a football: its two "spikes" at the North and

[13]This is to be expected, since the S^1 we are reducing along has two zeros on S^4. Note also that the D6s and $\overline{\text{D6}}$s are mutually supersymmetric since they sit at opposite poles of the S^4.

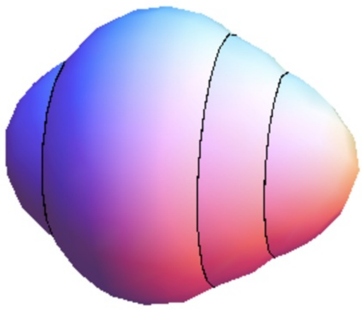

Figure 10. A cartoon of the internal space (topologically an S^3) for one of the solutions in [13, 63]; it represents the near-horizon geometry of the brane configuration in figure 5.

South Pole now expand into several "creases" due to the presence of D8/D6 bound states; we show a cartoon in figure 10. Each of these bound states represents a D8 in a brane picture such as the one in figure 5; the D6 charge of the bound state represents the number of D6s ending on the D8, and a Jordan block in the T-brane Hitchin pole in the IIB picture.

The physical radius r of a D8/D6 with charge μ is determined by a relationship of the type

$$e^{-\phi} r \sim \mu \ . \tag{7.5}$$

This is reminiscent of the Myers effect relationship; indeed we argued in section 5.1 that the D8s and D6s fuse into "fuzzy funnels". Actually equation (7.5) might cause some confusion in the case where $\mu = 1$, since we saw in section 5.1 that this particular case does not correspond to a fuzzy funnel, while in the gravity dual we just described it would appear to have a certain finite radius. However, in the range of applicability of the gravity solution, the radius of a D8/D6 with $\mu = 1$ is always smaller than l_s, thus making it indistinguishable from a D6.

Another case that deserves a special mention is when the bound (5.7) is saturated in the brane picture. In the AdS_7 solution, what happens in this case is that the leftmost and rightmost D8/D6s join, and the region $F_0 = 0$ disappears. It is in this sense that we anticipated in section 5.2 that in these cases it might be more natural to move the NS5s to a region where $F_0 \neq 0$. In that section, we also called $\mathcal{T}(\mathrm{SU}(k_L), \mathrm{SU}(k_R), N)$ the theory engineered by the configuration with unequal numbers of semi-infinite D6s coming out of a stack of NS5s in a region where $F_0 \neq 0$. This case was only briefly mentioned in [13]; the gravity dual in this case looks like an "asymmetric football", with two unequal D6 singularities.

7.2 Adding orientifolds

A similar analysis can also be performed in the presence of orientifolds, for example, the quotient S^4/Γ_D for D-type SU(2) subgroups also admits a scaling limit. Such solutions arise by orientifolding the solutions for the $\mathbb{Z}_k$ case with an orientifold whose space action is the antipodal map on the S^2. We defer a full analysis of the corresponding gravity duals along the lines presented in [13, 63] to future work.

Perhaps more interesting is that some features of an orientifold construction also persist for the theories with an E_8 wall, that is, on the duals of the form $AdS_7 \times S^4/\mathbb{Z}_2 \times \Gamma_G$. We now explain what becomes of the $\mathbb{Z}_2$ invariant wall upon reducing to a IIA configuration. To remain at weak coupling in the reduction, we need to take G to be in the A- or D-series. For illustrative purposes, we focus on the A-series.

In this case, we will now only see half of the football. In eleven dimensions, the "end of the world" boundary supports an E_8 gauge symmetry. When reduced to IIA, this becomes an O8 with a stack of eight D8-branes on top. Only an SO(16) gauge symmetry is visible in the brane construction; the further enhancement to E_8 is due to a strong coupling effect, and is not directly visible in terms of perturbative IIA ingredients. This is very analogous to what is found in the duality between type I and heterotic strings [64]. The presence of the O8-D8 system now changes the theory (3.8) on one of the two sides: it adds a "ramp" of SU groups with linearly decreasing gauge groups, ending with an E-string theory. The result is the theory (6.5). This "ramp" was first found in [17] in F-theory with essentially the computation we presented in section 6.1.1, the matter content was determined in [10]. This computation was then reproduced in [11] directly in IIA.

We can now introduce D8/D6 bound states in this case as well, replacing the D6 singularity of our half-football with several creases corresponding to a single Hitchin pole. This possibility was only briefly mentioned in [13, 63], but presenting such gravity solutions would not be particularly challenging. Starting from the previous case without an O8, one would have to consider configurations where the two partitions $\mu_{\rm L}$, $\mu_{\rm R}$ are equal, and mod out by the reflection that exchanges them.

8 Conclusions

In this paper we have studied aspects of $(1, 0)$ superconformal theories which arise in M-theory when M5-branes probe an ADE singularity, possibly near an E_8 wall. The central theme of this work has been the idea that such theories have tensor branches which involve a generalization of quiver gauge theories in which the matter sector defines a strongly coupled SCFT. This has led us to an identification of specific theories for domain walls in 7d SYM theory, as well as matter systems arising from the collision of an ADE singularity with the E_8 wall of heterotic M-theory. We have also discovered the phenomenon of fractionating of M5-branes on an ADE singularity as well the fractionating of the intersection point of an ADE singularity with the E_8 wall. We have also shown that the partial Higgs branches of these theories are characterized by discrete algebraic data in each flavor symmetry factor which in F-theory is encoded in the choice of a T-brane configuration. Taking scaling limits of these theories, we have also shown when to expect a supergravity dual for these configurations. In the remainder of this section we discuss some further avenues of investigation.

As we have argued, many 6d SCFTs can be understood in terms of a generalized notion of a quiver, in which the matter sector is itself a strongly coupled SCFT. However, a direct derivation of the matter sector from some putative generalization of brane probes of orbifolds is still to be understood. We have also presented evidence that fractional M5-

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branes can be understood as activating a fractional flux on the singularity, and changing the gauge symmetry. It would be interesting to further understand this.

In the context of F-theory, there are also some immediate generalizations of such theories which involve non-simply laced flavor symmetries. One expects that in the M-theory description, circle reduction with a twist will lead to conformal matter for non-simply laced symmetry factors.

Though our primary focus has been on 6d SCFTs, we have also seen that compactifying some of these theories to lower dimensions leads to novel dual theories for affine quiver gauge theories. It would be interesting to explore this further.

Finally, we have also seen that many of these generalized quiver theories have a holographic dual description. Nevertheless, some aspects, especially the partial Higgs branches for the E-type probe theories are washed away at $N = \infty$. It would be very interesting to see how semi-classical $1/N$ corrections to these duals recover these more detailed structures.

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A Non-Higgsable clusters

In this appendix we briefly review some elements of non-Higgsable clusters in F-theory. For additional discussion, see [12, 27].

The central idea of [27] is to study the minimal singularity type along a given $\mathbb{P}^1$ in the base, which is dictated by the order of vanishing for f and g along the curve. Interestingly, this is fully specified by the self-intersction of such a curve inside the base B. A "non-Higgsable cluster" consists of all such configurations where the singularity type cannot be deformed by a smoothing (i.e. by a Higgsing operation in the field theory). These NHCs have been determined in [27], and consist of a configuration of up to three curves. The NHCs consist of the ADE Dynkin diagrams for -2 curves, as well as some additional cases. For a single curve we can have a self-intersection $-n$ for $2 \leq n \leq 12$, and for two curves, we have a single intersection, with one curve of self-intersection -3 and one with self-intersection -2. For three curves, we have a three node linear graph, with two curves having self-intersection -2, and one curve with self-intersection -3. For each such cluster there is a corresponding gauge group, and possibly some additional matter fields. The six-dimensional theory associated with each type of cluster is:

$$\begin{array}{lcccccccc} \text{Theory:} & \mathfrak{su}_3 & \mathfrak{so}_8 & \mathfrak{f}_4 & \mathfrak{e}_6 & \mathfrak{e}_7 \oplus \frac{1}{2}\mathbf{56} & \mathfrak{e}_7 & \mathfrak{e}_8 \\ \text{Curve:} \;\; 2 & 3 & 4 & 5 & 6 & 7 & 8 & 12 \end{array} \tag{A.1}$$

$$\text{Theory: } \mathfrak{g}_2 \times \mathfrak{su}_2 \oplus \frac{1}{2}(\mathbf{7}+\mathbf{1},\mathbf{2}) \quad / \quad \text{Curves: } 3,2 \tag{A.2}$$

$$\text{Theory: } \mathfrak{g}_2 \times \mathfrak{sp}_1 \oplus \frac{1}{2}(\mathbf{7}+\mathbf{1},\mathbf{2}) \quad / \quad \text{Curves: } 3,2,2 \tag{A.3}$$

$$\text{Theory: } \mathfrak{su}_2 \times \mathfrak{so}_7 \times \mathfrak{su}_2 \oplus \frac{1}{2}(\mathbf{2},\mathbf{8},\mathbf{1}) \oplus \frac{1}{2}(\mathbf{1},\mathbf{8},\mathbf{2}) \quad / \quad \text{Curves: } 2,3,2 \tag{A.4}$$

To form bigger configurations of curves, one then joins these clusters by curves of self-intersection -1. If the -1 curve intersects two curves Σ_L and Σ_R with respective gauge symmetries $\mathfrak{g}_L$ and $\mathfrak{g}_R$, existence of an elliptic fibration with fibers in Kodaira-Tate form requires $\mathfrak{g}_L \times \mathfrak{g}_R \subset \mathfrak{e}_8$. See reference [12] for further discussion of this gluing condition.

B (G_L, G_R) conformal matter

In this appendix we calculate the conformal matter sector associated with a general pairing of flavor symmetries (G_L, G_R), for G_L and G_R a flavor symmetry. We start in an F-theory compactification in which there is a component of the discriminant locus supporting Lie algebras $\mathfrak{g}_L$ and $\mathfrak{g}_R$, respectively. If we cannot reach this configuration from Higgsing of an adjoint-valued field in a higher rank gauge symmetry such that $\mathfrak{g}_L \times \mathfrak{g}_R \subset \mathfrak{g}_{\text{parent}}$, we must blow up the intersection point. Continuing in this fashion, we compute the minimal conformal matter between two such symmetry factors. We focus on the case of ADE flavor symmetries which generate conformal matter, i.e. we exclude the cases of $A \times A$ and $A \times D$ type collisions as they lead to weakly coupled hypermultiplets. Finally, it would be interesting to extend this analysis to non-simply laced algebras.

B.1 $E \times E$ conformal matter

Recall the algorithm for resolving a collision of two E-type loci. We start with two non-compact divisors, and start blowing up the intersection point. The first blowup produces a single -1 curve. Then, if we still do not satisfy the gauging rule outlined in [12], we continue to blow up further. The procedure is completely algorithmic, and we collect the theories on the tensor branch:

(E_8, E_8) :

Gauge Symm:			$\mathfrak{sp}_1$		$\mathfrak{g}_2$		$\mathfrak{f}_4$		$\mathfrak{g}_2$		$\mathfrak{sp}_1$		
Curve:	1	2	2		3	1	5	1	3		2	2	1
Hyper:				$\frac{1}{2}(\mathbf{2},\mathbf{7}+\mathbf{1})$						$\frac{1}{2}(\mathbf{7}+\mathbf{1},\mathbf{2})$			

(B.1)

(E_8, E_7) :

Gauge Symm:			$\mathfrak{sp}_1$		$\mathfrak{g}_2$		$\mathfrak{f}_4$		$\mathfrak{g}_2$		$\mathfrak{su}_2$	
Curve:	1	2	2		3	1	5	1	3		2	1
Hyper:				$\frac{1}{2}(\mathbf{2},\mathbf{7}+\mathbf{1})$						$\frac{1}{2}(\mathbf{7}+\mathbf{1},\mathbf{2})$		

(B.2)

(E_8, E_6) : (B.3)

Gauge Symm:			$\mathfrak{sp}_1$		$\mathfrak{g}_2$		$\mathfrak{f}_4$		$\mathfrak{su}_3$	
Curve:	1	2	2		3	1	5	1	3	1
Hyper				$\frac{1}{2}(\mathbf{2}, \mathbf{7}+\mathbf{1})$						

(E_7, E_7) : (B.4)

Gauge Symm		$\mathfrak{su}_2$		$\mathfrak{so}_7$		$\mathfrak{su}_2$	
Curve	1	2		3		2	1
Hyper			$\frac{1}{2}(\mathbf{2}, \mathbf{8})$		$\frac{1}{2}(\mathbf{8}, \mathbf{2})$		

(E_7, E_6) : (B.5)

Gauge Symm:		$\mathfrak{su}_2$		$\mathfrak{so}_7$		$\mathfrak{su}_2$	
Curve:	1	2		3		2	1
Hyper:			$\frac{1}{2}(\mathbf{2}, \mathbf{8})$		$\frac{1}{2}(\mathbf{8}, \mathbf{2})$		

(E_6, E_6) : (B.6)

Gauge Symm:		$\mathfrak{su}_3$	
Curve:	1	3	1

.

B.2 $E \times A$ conformal matter

Next, consider the case of a collision of an E-type locus with an A-type algebra. This can actually occur in two-different ways in F-theory, so to distinguish them, we shall refer to $E \times A$ matter, and $E \times H$ matter. A-type symmetries arise from a stack of parallel D7-branes in weakly coupled IIB string theory. H-type symmetries arise from a non-perturbative bound state of seven-branes of different (p, q) type. Whereas A_k symmetries exist for arbitrary k, H_k only exists for $1 \leq k \leq 3$.

Consider first the case of an A-type singularity, which is realized by an I_n locus for the discriminant. This has already been worked out in [17], see also [10]. Basically, we consider a collision of two components of the discriminant locus, one supporting an II^* fiber, and the other supporting an I_n singular fiber. Each such collision can be blown up in the base, thereby leading to an exceptional curve with lower singularity type. In the case of the A-series, we have an I_{n-1} fiber after one such blowup. This leads to the minimal resolution on the tensor branch. For a collision with an E_8 seven-brane, this yields:

(E_8, A_{k-1}) : (B.7)

Gauge Symm:		$\mathfrak{su}_1$		$\mathfrak{su}_2$			$\mathfrak{su}_{k-1}$	
Curve:	1	2		2		$\ldots$	2	
Hyper:			$(\overline{\mathbf{1}}, \mathbf{2})$		$(\overline{\mathbf{2}}, \mathbf{3})$			$(\overline{\mathbf{k}-\mathbf{1}}, \mathbf{k})$

where in this case, there is a full hypermultiplet in the bifundamental trapped at the intersection of each A-type gauge group.

Similar considerations hold for collisions with E_7 and E_6. However, in these cases there is no need to fully resolve the collision of an I_n fiber with the singularity, since it can also just lead to ordinary matter. Taking this into account, we get the following collisions:

(E_7, A_{k-1}) : (B.8)

Gauge Symm:		$\mathfrak{su}_1$		$\mathfrak{su}_2$			$\mathfrak{su}_{k-1}$	
Curve:		1		2		$\ldots$	2	
Hyper:	$\frac{1}{2}(\mathbf{56}, \mathbf{1})$		$(\overline{\mathbf{1}}, \mathbf{2})$		$(\overline{\mathbf{2}}, \mathbf{3})$			$(\overline{\mathbf{k}-\mathbf{1}}, \mathbf{k})$

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$$
(E_6, A_{k-1}): \begin{array}{|c|ccccc|}\hline \text{Gauge Symm:} & & \mathfrak{su}_2 & & \mathfrak{su}_{k-1} & \\ \text{Curve:} & & 1 & \dots & 2 & \\ \text{Hyper:} & (\overline{\mathbf{27}}, \mathbf{2}) & & (\overline{\mathbf{2}}, \mathbf{3}) & & (\overline{\mathbf{k-1}}, \mathbf{k}) \\ \hline \end{array} \tag{B.9}
$$

Consider next the case of an H-type singularity, that is, by having an E-type locus collide with a fiber of type II, III or IV. These respectively generate $\mathfrak{su}_1$, $\mathfrak{su}_2$ and $\mathfrak{su}_3$, and so we refer to all of them as H_{n-1} for $\mathfrak{su}_n$. In these special cases, there is initially no minimal singularity type on the exceptional curve after blowing up. Following the minimal resolution algorithm, we have, for $n = 1, 2, 3$ and $k = 1, 2$ the $E \times H$ theories:

$$
(E_8, H_n): \begin{array}{|c|ccccc|}\hline \text{Gauge Symm:} & & & \mathfrak{sp}_1 & \mathfrak{g}_2 & \\ \text{Curve:} & 1 & 2 & 2 & 3 & 1 \\ \text{Hyper:} & & & \multicolumn{2}{c}{\tfrac{1}{2}(\mathbf{2}, \mathbf{7}+\mathbf{1})} & \\ \hline \end{array} \tag{B.10}
$$

$$
(E_7, H_2): \begin{array}{|c|cccc|}\hline \text{Gauge Symm:} & & \mathfrak{su}_2 & \mathfrak{g}_2 & \\ \text{Curve:} & 1 & 2 & 3 & 1 \\ \text{Hyper:} & & \multicolumn{2}{c}{\tfrac{1}{2}(\mathbf{2}, \mathbf{7}+\mathbf{1})} & \\ \hline \end{array} \tag{B.11}
$$

$$
(E_7, H_k): \begin{array}{|c|c|}\hline \text{Gauge Symm:} & \\ \text{Curve:} & 1 \\ \hline \end{array} \tag{B.12}
$$

$$
(E_6, H_n): \begin{array}{|c|c|}\hline \text{Gauge Symm:} & \\ \text{Curve:} & 1 \\ \hline \end{array} \tag{B.13}
$$

Observe that in the case of the $E_7 \times H$ theories, a different number of blowups are required for some of the cases.

B.3 $E \times D$ conformal matter

Consider next the collision of an E-type locus with a D-type locus. In F-theory, a D-type singularity comes from an I_k^* Kodaira-Tate fiber. For E_8, and $k \geq 1$, this yields:

$$
(E_8, D_{k+4}): \begin{array}{|c|ccccc|}\hline \text{Gauge Symm:} & & & \mathfrak{sp}_1 & \mathfrak{g}_2 & \\ \text{Curve:} & 1 & 2 & 2 & 3 & 1 \\ \text{Hyper:} & & & \multicolumn{2}{c}{\tfrac{1}{2}(\mathbf{2}, \mathbf{7}+\mathbf{1})} & \\ \hline \end{array} \times \tag{B.14}
$$

$$
\times \begin{array}{|ccccc|}\hline \mathfrak{so}_9 & & \mathfrak{sp}_1 & & \mathfrak{so}_{11} \\ 4 & & 1 & & \\ & \tfrac{1}{2}(\mathbf{9}, \mathbf{2}) & & \tfrac{1}{2}(\mathbf{2}, \mathbf{11}) & \\ \hline \end{array} \times \dots
$$

$$
\times \begin{array}{|ccccc|}\hline \mathfrak{so}_{2k+7} & & \mathfrak{sp}_k & & \\ & & 1 & & \\ & \tfrac{1}{2}(\mathbf{2k}+\mathbf{7}, \mathbf{2k}) & \tfrac{1}{2}\mathbf{2k} & \tfrac{1}{2}(\mathbf{2k}, \mathbf{2k}+\mathbf{8}) & \\ \hline \end{array} \tag{B.15}
$$

where we have introduced a line break for typographical purposes. The minimal resolutions for the cases E_7 and E_6 correspond to replacing the top line by $1, 2, 3, 1$ and $1, 3, 1$, respectively.[14] The case $k = 0$ follows by omitting the second line. Here, there is a half hypermultiplet trapped at each $\mathfrak{so}/\mathfrak{sp}$ intersection. Furthermore, the $\mathfrak{sp}$ factors increase from $\mathfrak{sp}_1$ up to $\mathfrak{sp}_k$. Finally, for the rightmost $\mathfrak{sp}_k$ factor, there is one additional half hypermultiplet in the fundamental.

B.4 $D \times D$ conformal matter

Finally, we have the collisions of A- or D-type singularities with each other. In all cases other than the collision of two D-type singularities, we get a weakly coupled hypermultiplet. We therefore focus on the D-type collisions. When $k + l$ is even, we have:

$$(D_{k+4}, D_{l+4}): \begin{array}{|c|ccc|} \text{Gauge Symm:} & & \mathfrak{sp}_r & \\ \text{Curve:} & & 1 & \\ \text{Hyper:} & \frac{1}{2}(\mathbf{2l+8}, \mathbf{2r}) & & \frac{1}{2}(\mathbf{2r}, \mathbf{2l+8}) \end{array} \tag{B.16}$$

where $r = (k + l)/2$.

When $k + l$ is odd, the analysis is more subtle, because after blowing up the collision of the I_k^* and I_l^* fibers, we get a non-split I_{k+l-1} fiber. Letting $r = (k + l + 1)/2$, only an $\mathfrak{sp}_{\mathfrak{r}-1}$ can be identified in purely geometric terms [17]. Nevertheless, as explained in [6, 10], 6d anomaly cancelation and consistent Higgsing dictates the structure of the resulting conformal matter sector to be:

$$(D_{k+4}, D_{l+4}): \begin{array}{|c|ccc|} \text{Gauge Symm:} & & \mathfrak{sp}_r & \\ \text{Curve:} & & 1 & \\ \text{Hyper:} & \frac{1}{2}(\mathbf{2l+8}, \mathbf{2r}) & \mathbf{2r} & \frac{1}{2}(\mathbf{2r}, \mathbf{2l+8}) \end{array} \tag{B.17}$$

that is, there is an extra hypermultiplet in the fundamental of $\mathfrak{sp}_r$. This is rather analogous to the fact that in F-theory on Calabi-Yau fourfolds, the structure of a Yukawa point [29] which is transparently realized in gauge theory terms can sometimes be obscure just from the resolution of singular fibers [65].

C ADE subgroups of SU(2)

In this appendix we summarize some of the relevant properties of discrete subgroups of SU(2), and the corresponding orbifold singularities which they generate. For additional discussion, see [66]. We first start by considering the set of generators of the exceptional binary polyhedral groups as subgroups of SU(2). For each generator, the subscript indicates the order of the element. Also, we let $\xi_{(k)} = \exp(2\pi i/k)$ denote a primitive kth root of unity.

- $\mathbb{A}_k$, the cyclic group of order $k + 1$, with generator:

$$\omega_{(k+1)} \equiv \begin{pmatrix} \xi_{(k+1)} & 0 \\ 0 & \xi_{(k+1)}^{-1} \end{pmatrix}, \tag{C.1}$$

[14] We thank T. Rudelius for alerting us to a typo in a previous version of this statement.

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- $\mathbb{D}_p$, for $p \geq 4$, the binary dihedral group of order $4p-8$, with generators:

$$\omega_{(2p)} \equiv \begin{pmatrix} \xi_{(2p)} & 0 \\ 0 & \xi_{(2p)}^{-1} \end{pmatrix} \quad \text{and} \quad \tau_{(4)} \equiv \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \tag{C.2}$$

- $\mathbb{T}$, the binary tetrahedral group has order 24, with generators:

$$\omega_{(4)} = \begin{pmatrix} \xi_{(4)} & 0 \\ 0 & \xi_{(4)}^{-1} \end{pmatrix} \quad \text{and} \quad \kappa_{(6)} = \frac{1}{\sqrt{2}} \begin{pmatrix} \xi_{(8)}^7 & \xi_{(8)}^7 \\ \xi_{(8)}^5 & \xi_{(8)} \end{pmatrix}, \tag{C.3}$$

- $\mathbb{O}$, the binary octahedral group has order 48, with the same generators as $\mathbb{T}$, as well as an additional generator:

$$\omega_{(4)} = \begin{pmatrix} \xi_{(4)} & 0 \\ 0 & \xi_{(4)}^{-1} \end{pmatrix}, \quad \kappa_{(6)} = \frac{1}{\sqrt{2}} \begin{pmatrix} \xi_{(8)}^7 & \xi_{(8)}^7 \\ \xi_{(8)}^5 & \xi_{(8)} \end{pmatrix}, \quad \omega_{(8)} = \begin{pmatrix} \xi_{(8)} & \\ 0 & \xi_{(8)}^{-1} \end{pmatrix} \tag{C.4}$$

- $\mathbb{I}$, the binary icosahedral group has order 120, with generators:

$$\omega_{(10)} = - \begin{pmatrix} \xi_{(5)}^3 & 0 \\ 0 & \xi_{(5)}^2 \end{pmatrix} \quad \text{and} \quad \kappa_{(4)} = \frac{1}{\xi_{(5)}^2 - \xi_{(5)}^3} \begin{pmatrix} \xi_{(5)} + \xi_{(5)}^4 & 1 \\ 1 & -\xi_{(5)} - \xi_{(5)}^4 \end{pmatrix}. \tag{C.5}$$

For each of these discrete subgroups of SU(2), we get a corresponding orbifold singularity $\mathbb{C}^2/\Gamma$. We summarize the corresponding hypersurface, discrete subgroup, and order of the subgroup in the following list (see e.g. [66]):

	singularity	Γ	$\lvert\Gamma\rvert$
A_k	$y^2 = x^2 + z^{k+1}$	$\mathbb{Z}_{k+1}$	$k+1$
D_p	$y^2 = x^2 z + z^{p-1}$	$\mathbb{D}_{p-2}$	$4p-8$
E_6	$y^2 = x^3 + z^4$	$\mathbb{T}$	24
E_7	$y^2 = x^3 + xz^3$	$\mathbb{O}$	48
E_8	$y^2 = x^3 + z^5$	$\mathbb{I}$	120

(C.6)

D 6d (1, 0) minimal models of type $\alpha A_N \beta$

In this appendix we study the emergence of conformal matter sectors as building blocks of the 6d (1, 0) models classified in [12]. We consider in detail the case of the A_N models. The models of D_N type and exceptional outliers can be treated in a similar way.

Let us start by introducing some notations. Let us denote C^* the mirror of the configuration C, e.g. for $C = 2, 2, 3, 1, 5$, $C^* = 5, 1, 3, 2, 2$. Notice that a given configuration is palindromic iff $C^* = C$. Moreover, le us write C^N for a given configuration of curves that is repeated N times, e.g. $(1, 4)^3 = 1, 4, 1, 4, 1, 4$. In the following tables we describe the minimal resolutions of the minimal models of type $\alpha\, A_N\, \beta^*$. For all pairs (α, β) one can form out of

$$\mathcal{I} \equiv \{7,\, 3\,3,\, 2\,4,\, 2\,2\,3,\, 2\,2\,2\,3,\, 2\,2\,2\,2\,3\} \quad \text{and} \quad \mathcal{J} \equiv \{6,\, 5,\, 4,\, 3,\, 2,\, 2\,3,\, 2\,2,\, 2\,2\,2,\, 2\,2\,2\,2\}$$

The models of type $\alpha\, A_N\, \beta^*$ behave literally as *generalized linear quivers*. Each model has a central core or *plateaux* built of generalized exceptional bifundamentals and two external tails on the left and on the right that complete these systems without (non-abelian) flavor symmetries in a superconformal fashion. Recall from [12] that these models are non-Higgsable, therefore the only allowed bifundamentals are those without a Higgs branch. As we discussed in the main body of the text these are the exceptional bifundamentals

$$\begin{aligned} B(E_8) &\equiv 1,2,2,3,1,5,1,3,2,2,1 \\ B(E_7) &\equiv 1,2,3,2,1 \\ B(E_6) &\equiv 1,3,1 \end{aligned} \tag{D.1}$$

and the bifundamental with $\mathrm{SO}(8) \times \mathrm{SO}(8)$ flavor symmetry we have discussed in Section 3.4.2. The infinite series that we find can be organized in four types, according to the type of bifundamental that occurs in the core of the generalized linear quiver. For pairs in $\mathcal{I} \times \mathcal{I}$ and $\mathcal{I} \times \mathcal{J}$ only bifundamentals $B(E_8)$ and $B(E_7)$ shows up; the other two types arises only when one considers pairs in $\mathcal{J} \times \mathcal{J}$. The descendents 2, 22, 222, and 2222 can be grouped in the same A_N family of type αA_N, however, as we will see below, typically, $\alpha 2$, $\alpha 22$, and $\alpha 222$, behaves differently from $\alpha 2222$, and only in this last case the systems develops a specific type. We refer to these models as "isolated" below.

In the tables below the we list the types, blow ups, number of curves in the minimal blow up of the base, N_T, algebras for all A_N minimal models. In addition, in the column marked by a W, we give also the "wannabe" flavor symmetry obtained by making the leftmost and rightmost cycles non-compact.

Typically the models of type E_8 have many $\mathfrak{sp}_1$ factors in their Lie algebras. To avoid lenghty and redundant tables, in this case we have marked with a $\star$ the models in which there are genuine $\mathfrak{su}_2$ factors (e.g. nHc's of type $3,2$ or $2,3$) and used the isomorphism of the two rank one Lie algebras. This however happens rarely, as one can see explicitly from the blow ups, typically when α or β equals $2,2,2,3$.

Moreover, we have noticed that all models of type E_7 occur when α or β equals 5 or $2,2,3$. Similarly, models of type *III* or *IV* arise only if α or β are 3, 4, or $2,3$. We have marked with a ♠ models with $\frac{1}{2}$ hypers in the **56** of E_7.

Series	Type	Blow up	N_T	Algebra	W	
$7\,A_N\,7$	E_8	$12, (B(E_8), 12)^{N+1}$	$12(N+1)+1$	$(\mathfrak{e}_8)^{N+2} \oplus (\mathfrak{sp}_1 \oplus \mathfrak{g}_2)^{2N+2} \oplus (\mathfrak{f}_4)^{N+1}$	$\mathfrak{e}_8 \oplus \mathfrak{e}_8$	
$2\,4\,A_N\,7$	E_8	$3, 2, 2, 1, 12, (B(E_8), 12)^{N+1}$	$12(N+1)+5$	$(\mathfrak{e}_8)^{N+2} \oplus (\mathfrak{sp}_1 \oplus \mathfrak{g}_2)^{2N+3} \oplus (\mathfrak{f}_4)^{N+1}$	$\mathfrak{g}_2 \oplus \mathfrak{e}_8$	
$3\,3\,A_N\,7$	E_8	$5, 1, 3, 2, 2, 1, 12, (B(E_8), 12)^{N+1}$	$12(N+1)+7$	$(\mathfrak{e}_8)^{N+2} \oplus (\mathfrak{sp}_1 \oplus \mathfrak{g}_2)^{2N+3} \oplus (\mathfrak{f}_4)^{N+2}$	$\mathfrak{f}_4 \oplus \mathfrak{e}_8$	
$2\,2\,3\,A_N\,7$	E_8	$3, 1, 5, 1, 3, 2, 2, 1, 12, (B(E_8), 12)^{N+1}$	$12(N+1)+9$	$\mathfrak{su}_3 \oplus (\mathfrak{e}_8)^{N+2} \oplus (\mathfrak{sp}_1 \oplus \mathfrak{g}_2)^{2N+3} \oplus (\mathfrak{f}_4)^{N+2}$	$\mathfrak{su}_3 \oplus \mathfrak{e}_8$	
$2\,2\,2\,3\,A_N\,7$	E_8	$2, 3, 1, 5, 1, 3, 2, 2, 1, 12, (B(E_8), 12)^{N+1}$	$12(N+1)+10$	$(\mathfrak{e}_8)^{N+2} \oplus (\mathfrak{sp}_1 \oplus \mathfrak{g}_2)^{2N+4} \oplus (\mathfrak{f}_4)^{N+2}$	$\mathfrak{su}_2 \oplus \mathfrak{e}_8$	$\star$
$2\,2\,2\,2\,3\,A_N\,7$	E_8	$2, 2, 3, 1, 5, 1, 3, 2, 2, 1, 12, (B(E_8), 12)^{N+1}$	$12(N+1)+11$	$(\mathfrak{e}_8)^{N+2} \oplus (\mathfrak{sp}_1 \oplus \mathfrak{g}_2)^{2N+4} \oplus (\mathfrak{f}_4)^{N+2}$	$\mathfrak{e}_8$	
$2\,4\,A_N\,4\,2$	E_8	$3, 2, 2, 1, 12, (B(E_8), 12)^{N+1}, 1, 2, 2, 3$	$12(N+1)+9$	$(\mathfrak{e}_8)^{N+2} \oplus (\mathfrak{sp}_1 \oplus \mathfrak{g}_2)^{2N+4} \oplus (\mathfrak{f}_4)^{N+1}$	$\mathfrak{g}_2 \oplus \mathfrak{g}_2$	
$3\,3\,A_N\,4\,2$	E_8	$5, 1, 3, 2, 2, 1, 12, (B(E_8), 12)^{N+1}, 1, 2, 2, 3$	$12(N+1)+11$	$(\mathfrak{e}_8)^{N+2} \oplus (\mathfrak{sp}_1 \oplus \mathfrak{g}_2)^{2N+4} \oplus (\mathfrak{f}_4)^{N+2}$	$\mathfrak{f}_4 \oplus \mathfrak{g}_2$	
$2\,2\,3\,A_N\,4\,2$	E_8	$3, 1, 5, 1, 3, 2, 2, 1, 12, (B(E_8), 12)^{N+1}, 1, 2, 2, 3$	$12(N+2)+1$	$\mathfrak{su}_3 \oplus (\mathfrak{e}_8)^{N+2} \oplus (\mathfrak{sp}_1 \oplus \mathfrak{g}_2)^{2N+4} \oplus (\mathfrak{f}_4)^{N+2}$	$\mathfrak{su}_3 \oplus \mathfrak{g}_2$	
$2\,2\,2\,3\,A_N\,4\,2$	E_8	$2, 3, 1, 5, 1, 3, 2, 2, 1, 12, (B(E_8), 12)^{N+1}, 1, 2, 2, 3$	$12(N+2)+2$	$(\mathfrak{e}_8)^{N+2} \oplus (\mathfrak{sp}_1 \oplus \mathfrak{g}_2)^{2N+5} \oplus (\mathfrak{f}_4)^{N+2}$	$\mathfrak{su}_2 \oplus \mathfrak{g}_2$	$\star$
$2\,2\,2\,2\,3\,A_N\,4\,2$	E_8	$2, 2, 3, 1, 5, 1, 3, 2, 2, 1, 12, (B(E_8), 12)^{N+1}, 1, 2, 2, 3$	$12(N+2)+3$	$(\mathfrak{e}_8)^{N+2} \oplus (\mathfrak{sp}_1 \oplus \mathfrak{g}_2)^{2N+5} \oplus (\mathfrak{f}_4)^{N+2}$	$\mathfrak{g}_2$	

Series	Type	Blow up	N_T	Algebra	W	
$3\,3\,A_N\,3\,3$	E_8	$5,1,3,2,2,1,12,(B(E_8),12)^{N+1}, 1,2,2,3,1,5$	$12(N+2)+1$	$(\mathfrak{e}_8)^{N+2} \oplus (\mathfrak{sp}_1 \oplus \mathfrak{g}_2)^{2N+4} \oplus (\mathfrak{f}_4)^{N+3}$	$\mathfrak{f}_4 \oplus \mathfrak{f}_4$	
$2\,2\,3\,A_N\,3\,3$	E_8	$3,1,5,1,3,2,2,1,12,(B(E_8),12)^{N+1}, 1,2,2,3,1,5$	$12(N+2)+3$	$\mathfrak{su}_3 \oplus (\mathfrak{e}_8)^{N+2} \oplus (\mathfrak{sp}_1 \oplus \mathfrak{g}_2)^{2N+4} \oplus (\mathfrak{f}_4)^{N+3}$	$\mathfrak{su}_3 \oplus \mathfrak{f}_4$	
$2\,2\,2\,3\,A_N\,3\,3$	E_8	$2,3,1,5,1,3,2,2,1,12,(B(E_8),12)^{N+1}, 1,2,2,3,1,5$	$12(N+2)+4$	$(\mathfrak{e}_8)^{N+2} \oplus (\mathfrak{sp}_1 \oplus \mathfrak{g}_2)^{2N+5} \oplus (\mathfrak{f}_4)^{N+3}$	$\mathfrak{su}_2 \oplus \mathfrak{f}_4$	$\star$
$2\,2\,2\,2\,3\,A_N\,3\,3$	E_8	$2,2,3,1,5,1,3,2,2,1,12,(B(E_8),12)^{N+1}, 1,2,2,3,1,5$	$12(N+2)+5$	$(\mathfrak{e}_8)^{N+2} \oplus (\mathfrak{sp}_1 \oplus \mathfrak{g}_2)^{2N+5} \oplus (\mathfrak{f}_4)^{N+3}$	$\mathfrak{f}_4$	
$2\,2\,3\,A_N\,3\,2\,2$	E_7	$2,3,2,1,8,(B(E_7),8)^{N+1},1,2,3,2$	$6(N+1)+9$	$(\mathfrak{e}_7)^{N+2} \oplus (\mathfrak{so}_7)^{N+3} \oplus (\mathfrak{su}_2)^{2N+6}$	$\mathfrak{su}_2 \oplus \mathfrak{su}_2$	
$2\,2\,2\,3\,A_N\,3\,2\,2$	E_8	$2,3,1,5,1,3,2,2,1,12,(B(E_8),12)^{N+1}, 1,2,2,3,1,5,1,3$	$12(N+2)+6$	$\mathfrak{su}_3 \oplus (\mathfrak{e}_8)^{N+2} \oplus (\mathfrak{sp}_1 \oplus \mathfrak{g}_2)^{2N+5} \oplus (\mathfrak{f}_4)^{N+3}$	$\mathfrak{su}_2 \oplus \mathfrak{su}_3$	$\star$
$2\,2\,2\,2\,3\,A_N\,3\,2\,2$	E_8	$2,2,3,1,5,1,3,2,2,1,12,(B(E_8),12)^{N+1}, 1,2,2,3,1,5,1,3$	$12(N+2)+7$	$\mathfrak{su}_3 \oplus (\mathfrak{e}_8)^{N+2} \oplus (\mathfrak{sp}_1 \oplus \mathfrak{g}_2)^{2N+5} \oplus (\mathfrak{f}_4)^{N+3}$	$\mathfrak{su}_3$	
$2\,2\,2\,3\,A_N\,3\,2\,2\,2$	E_8	$2,3,1,5,1,3,2,2,1,12,(B(E_8),12)^{N+1}, 1,2,2,3,1,5,1,3,2$	$12(N+2)+7$	$(\mathfrak{e}_8)^{N+2} \oplus (\mathfrak{sp}_1 \oplus \mathfrak{g}_2)^{2N+6} \oplus (\mathfrak{f}_4)^{N+3}$	$\mathfrak{su}_2 \oplus \mathfrak{su}_2$	$\star$
$2\,2\,2\,2\,3\,A_N\,3\,2\,2\,2$	E_8	$2,2,3,1,5,1,3,2,2,1,12,(B(E_8),12)^{N+1}, 1,2,2,3,1,5,1,3,2$	$12(N+2)+8$	$(\mathfrak{e}_8)^{N+2} \oplus (\mathfrak{sp}_1 \oplus \mathfrak{g}_2)^{2N+6} \oplus (\mathfrak{f}_4)^{N+3}$	$\mathfrak{su}_2$	$\star$
$2\,2\,2\,2\,3\,A_N\,3\,2\,2\,2\,2$	E_8	$2,2,3,1,5,1,3,2,2,1,12,(B(E_8),12)^{N+1}, 1,2,2,3,1,5,1,3,2,2$	$12(N+2)+9$	$(\mathfrak{e}_8)^{N+2} \oplus (\mathfrak{sp}_1 \oplus \mathfrak{g}_2)^{2N+6} \oplus (\mathfrak{f}_4)^{N+3}$		

Table 1. Minimal resolutions: $(\mathcal{I},\mathcal{I})$ infinite families i.e. rigid theories.

Theory	Blow up	N_T	Algebra	W
7 2	10,1,2,2,3	5	$\mathfrak{e}_8 \oplus \mathfrak{sp}_1 \oplus \mathfrak{g}_2$	$\mathfrak{e}_8 \oplus \mathfrak{g}_2$
7 2 2	11,1,2,2,3,1,5,1,3	9	$\mathfrak{e}_8 \oplus \mathfrak{sp}_1 \oplus \mathfrak{g}_2 \oplus \mathfrak{f}_4 \oplus \mathfrak{su}_3$	$\mathfrak{e}_8 \oplus \mathfrak{su}_3$
7 2 2 2	11,1,2,2,3,1,5,1,3,2	10	$\mathfrak{e}_8 \oplus \mathfrak{sp}_1 \oplus \mathfrak{g}_2 \oplus \mathfrak{f}_4 \oplus \mathfrak{g}_2 \oplus \mathfrak{su}_2$	$\mathfrak{e}_8 \oplus \mathfrak{su}_2$
2 4 2	3,1,6,1,3	5	$\mathfrak{e}_6 \oplus (\mathfrak{su}_3)^2$	$\mathfrak{su}_3 \oplus \mathfrak{su}_3$
2 4 2 2	3,2,1,8,1,2,3,2	8	$\mathfrak{e}_7 \oplus (\mathfrak{su}_2)^3 \oplus \mathfrak{g}_2 \oplus \mathfrak{so}_7$	$\mathfrak{g}_2 \oplus \mathfrak{su}_2$
2 4 2 2 2	3,2,2,1,11,1,2,2,3,1,5,1,3,2	14	$\mathfrak{e}_8 \oplus (\mathfrak{sp}_1 \oplus \mathfrak{g}_2)^2 \oplus \mathfrak{f}_4 \oplus \mathfrak{g}_2 \oplus \mathfrak{su}_2$	$\mathfrak{g}_2 \oplus \mathfrak{su}_2$
3 3 2	5,1, 3,1,6,1,3	7	$\mathfrak{e}_6 \oplus (\mathfrak{su}_3)^2 \oplus \mathfrak{f}_4$	$\mathfrak{su}_3 \oplus \mathfrak{su}_3$
3 3 2 2	5,1, 3,2,1,8,1,2,3,2	10	$\mathfrak{e}_7 \oplus (\mathfrak{su}_2)^3 \oplus \mathfrak{g}_2 \oplus \mathfrak{so}_7 \oplus \mathfrak{f}_4$	$\mathfrak{f}_4 \oplus \mathfrak{su}_2$
3 3 2 2 2	5,1, 3,2,2,1,11,1,2,2,3,1,5,1,3,2	16	$\mathfrak{e}_8 \oplus (\mathfrak{sp}_1 \oplus \mathfrak{g}_2)^2 \oplus (\mathfrak{f}_4)^2 \oplus \mathfrak{g}_2 \oplus \mathfrak{su}_2$	$\mathfrak{f}_4 \oplus \mathfrak{su}_2$
2 2 3 2	2,3,1, 5,1,3	6	$\mathfrak{su}_2 \oplus \mathfrak{g}_2 \oplus \mathfrak{f}_4 \oplus \mathfrak{su}_3$	$\mathfrak{su}_2 \oplus \mathfrak{su}_3$
2 2 3 2 2	2,3,1, 5,1,3,2	7	$(\mathfrak{su}_2 \oplus \mathfrak{g}_2)^2 \oplus \mathfrak{f}_4$	$\mathfrak{su}_2 \oplus \mathfrak{su}_2$
2 2 3 2 2 2	2,3,1, 5,1,3,2,2	8	$\mathfrak{su}_2 \oplus \mathfrak{sp}_1 \oplus (\mathfrak{g}_2)^2 \oplus \mathfrak{f}_4$	$\mathfrak{su}_2$
2 2 2 3 2	2,2,3,1, 5,1,3	7	$\mathfrak{sp}_1 \oplus \mathfrak{g}_2 \oplus \mathfrak{f}_4 \oplus \mathfrak{su}_3$	$\mathfrak{su}_3$
2 2 2 3 2 2 2	2,2,3,1, 5,1,3,2,2	9	$(\mathfrak{sp}_1 \oplus \mathfrak{g}_2)^2 \oplus \mathfrak{f}_4$	

Table 2. Minimal resolutions: $(\mathcal{I}, \mathcal{J})$ isolated theories.

Series	Type	Blow up	N_T	Algebra	W	
$7\,A_N\,6$	E_8	$12,(B(E_8),12)^N,B(E_8),11$	$12(N+1)+1$	$(\mathfrak{e}_8)^{N+2}\oplus(\mathfrak{sp}_1\oplus\mathfrak{g}_2)^{2N+2}\oplus(\mathfrak{f}_4)^{N+1}$	$\mathfrak{e}_8\oplus\mathfrak{e}_8$	
$2\,4\,A_N\,6$	E_8	$3,2,2,1,12,(B(E_8),12)^N,B(E_8),11$	$12(N+1)+5$	$(\mathfrak{e}_8)^{N+2}\oplus(\mathfrak{sp}_1\oplus\mathfrak{g}_2)^{2N+3}\oplus(\mathfrak{f}_4)^{N+1}$	$\mathfrak{e}_8\oplus\mathfrak{g}_2$	
$3\,3\,A_N\,6$	E_8	$5,1,3,2,2,1,12,(B(E_8),12)^N,B(E_8),11$	$12(N+1)+7$	$(\mathfrak{e}_8)^{N+2}\oplus(\mathfrak{sp}_1\oplus\mathfrak{g}_2)^{2N+3}\oplus(\mathfrak{f}_4)^{N+2}$	$\mathfrak{e}_8\oplus\mathfrak{f}_4$	
$2\,2\,3\,A_N\,6$	E_8	$3,1,5,1,3,2,2,1,12,(B(E_8),12)^N,B(E_8),11$	$12(N+1)+9$	$\mathfrak{su}_3\oplus(\mathfrak{e}_8)^{N+2}\oplus(\mathfrak{sp}_1\oplus\mathfrak{g}_2)^{2N+3}\oplus(\mathfrak{f}_4)^{N+2}$	$\mathfrak{e}_8\oplus\mathfrak{su}_3$	
$2\,2\,2\,3\,A_N\,6$	E_8	$2,3,1,5,1,3,2,2,1,12,(B(E_8),12)^N,B(E_8),11$	$12(N+1)+10$	$(\mathfrak{e}_8)^{N+2}\oplus(\mathfrak{sp}_1\oplus\mathfrak{g}_2)^{2N+4}\oplus(\mathfrak{f}_4)^{N+2}$	$\mathfrak{e}_8\oplus\mathfrak{su}_2$	$\star$
$2\,2\,2\,2\,3\,A_N\,6$	E_8	$2,2,3,1,5,1,3,2,2,1,12,(B(E_8),12)^N,B(E_8),11$	$12(N+1)+11$	$(\mathfrak{e}_8)^{N+2}\oplus(\mathfrak{sp}_1\oplus\mathfrak{g}_2)^{2N+4}\oplus(\mathfrak{f}_4)^{N+2}$	$\mathfrak{e}_8$	
$7\,A_N\,5$	E_8	$12,(B(E_8),12)^N,B(E_8),10$	$12(N+1)+1$	$(\mathfrak{e}_8)^{N+2}\oplus(\mathfrak{sp}_1\oplus\mathfrak{g}_2)^{2N+2}\oplus(\mathfrak{f}_4)^{N+1}$	$\mathfrak{e}_8\oplus\mathfrak{e}_8$	
$2\,4\,A_N\,5$	E_8	$3,2,2,1,12,(B(E_8),12)^N,B(E_8),10$	$12(N+1)+5$	$(\mathfrak{e}_8)^{N+2}\oplus(\mathfrak{sp}_1\oplus\mathfrak{g}_2)^{2N+3}\oplus(\mathfrak{f}_4)^{N+1}$	$\mathfrak{e}_8\oplus\mathfrak{g}_2$	
$3\,3\,A_N\,5$	E_8	$5,1,3,2,2,1,12,(B(E_8),12)^N,B(E_8),10$	$12(N+1)+7$	$(\mathfrak{e}_8)^{N+2}\oplus(\mathfrak{sp}_1\oplus\mathfrak{g}_2)^{2N+3}\oplus(\mathfrak{f}_4)^{N+2}$	$\mathfrak{e}_8\oplus\mathfrak{f}_4$	
$2\,2\,3\,A_N\,5$	E_7	$2,3,2,1,8,(B(E_7),8)^{N+1}$	$6(N+1)+5$	$(\mathfrak{e}_7)^{N+2}\oplus(so_7)^{N+2}\oplus(\mathfrak{su}_2)^{2N+4}$	$\mathfrak{e}_7\oplus\mathfrak{su}_2$	
$2\,2\,2\,3\,A_N\,5$	E_8	$2,3,1,5,1,3,2,2,1,12,(B(E_8),12)^N,B(E_8),10$	$12(N+1)+10$	$(\mathfrak{e}_8)^{N+2}\oplus(\mathfrak{sp}_1\oplus\mathfrak{g}_2)^{2N+4}\oplus(\mathfrak{f}_4)^{N+2}$	$\mathfrak{e}_8\oplus\mathfrak{su}_2$	$\star$
$2\,2\,2\,2\,3\,A_N\,5$	E_8	$2,2,3,1,5,1,3,2,2,1,12,(B(E_8),12)^N,B(E_8),10$	$12(N+1)+11$	$(\mathfrak{e}_8)^{N+2}\oplus(\mathfrak{sp}_1\oplus\mathfrak{g}_2)^{2N+4}\oplus(\mathfrak{f}_4)^{N+2}$	$\mathfrak{e}_8$	

Series	Type	Blow up	N_T	Algebra	W	
$7\,A_N\,4$	E_8	$12,(B(E_8),12)^N,1,2,2,3,1,5,1,3,2,1,8$	$12(N+1)$	$\mathfrak{e}_7 \oplus (\mathfrak{e}_8)^{N+1} \oplus (\mathfrak{sp}_1 \oplus \mathfrak{g}_2)^{2N+2} \oplus (\mathfrak{f}_4)^{N+1}$	$\mathfrak{e}_8 \oplus \mathfrak{e}_7$	⋆
$2\,4\,A_N\,4$	E_8	$3,2,2,1,12,(B(E_8),12)^N,$ $1,2,2,3,1,5,1,3,2,1,8$	$12(N+1)+4$	$\mathfrak{e}_7 \oplus (\mathfrak{e}_8)^{N+1} \oplus (\mathfrak{sp}_1 \oplus \mathfrak{g}_2)^{2N+3} \oplus (\mathfrak{f}_4)^{N+1}$	$\mathfrak{e}_7 \oplus \mathfrak{g}_2$	⋆
$3\,3\,A_N\,4$	E_8	$5,1,3,2,2,1,12,(B(E_8),12)^N,$ $1,2,2,3,1,5,1,3,2,1,8$	$12(N+1)+6$	$\mathfrak{e}_7 \oplus (\mathfrak{e}_8)^{N+1} \oplus (\mathfrak{sp}_1 \oplus \mathfrak{g}_2)^{2N+3} \oplus (\mathfrak{f}_4)^{N+2}$	$\mathfrak{e}_7 \oplus \mathfrak{f}_4$	⋆
$2\,2\,3\,A_N\,4$	E_7	$2,3,2,1,(8,B(E_7))^{N+1},7$	$6(N+1)+5$	$(\mathfrak{e}_7)^{N+2} \oplus (\mathfrak{so}_7)^{N+2} \oplus (\mathfrak{su}_2)^{2N+4}$	$\mathfrak{e}_7 \oplus \mathfrak{su}_2$	♠
$2\,2\,2\,3\,A_N\,4$	E_8	$2,3,1,5,1,3,2,2,1,12,(B(E_8),12)^N,$ $1,2,2,3,1,5,1,3,2,1,8$	$12(N+1)+9$	$\mathfrak{e}_7 \oplus (\mathfrak{e}_8)^{N+1} \oplus (\mathfrak{sp}_1 \oplus \mathfrak{g}_2)^{2N+4} \oplus (\mathfrak{f}_4)^{N+2}$	$\mathfrak{e}_7 \oplus \mathfrak{su}_2$	⋆⋆
$2\,2\,2\,2\,3\,A_N\,4$	E_8	$2,2,3,1,5,1,3,2,2,1,12,(B(E_8),12)^N,$ $1,2,2,3,1,5,1,3,2,1,8$	$12(N+1)+10$	$\mathfrak{e}_7 \oplus (\mathfrak{e}_8)^{N+1} \oplus (\mathfrak{sp}_1 \oplus \mathfrak{g}_2)^{2N+4} \oplus (\mathfrak{f}_4)^{N+2}$	$\mathfrak{e}_7$	⋆
$7\,A_N\,2\,2\,2\,2$	E_8	$(12,B(E_8))^N,11,1,2,2,3,1,5,1,3,2,2$	$12N+11$	$(\mathfrak{e}_8)^{N+1} \oplus (\mathfrak{sp}_1 \oplus \mathfrak{g}_2)^{2N+2} \oplus (\mathfrak{f}_4)^{N+1}$	$\mathfrak{e}_8$	
$2\,4A_N\,2\,2\,2\,2$	E_8	$3,2,2,1,(12,B(E_8))^N,11,1,2,2,3,1,5,1,3,2,2$	$12(N+1)+3$	$(\mathfrak{e}_8)^{N+1} \oplus (\mathfrak{sp}_1 \oplus \mathfrak{g}_2)^{2N+3} \oplus (\mathfrak{f}_4)^{N+1}$	$\mathfrak{g}_2$	
$3\,3\,A_N\,2\,2\,2\,2$	E_8	$5,1,3,2,2,(12,B(E_8))^N,$ $11,1,2,2,3,1,5,1,3,2,2$	$12(N+1)+4$	$(\mathfrak{e}_8)^{N+1} \oplus (\mathfrak{sp}_1 \oplus \mathfrak{g}_2)^{2N+3} \oplus (\mathfrak{f}_4)^{N+2}$	$\mathfrak{f}_4$	
$2\,2\,3\,A_N\,2\,2\,2\,2$	E_7	$2,3,2,1,(8,B(E_7))^N,8,1,2,3,1,5,1,3,2,2$	$6N+14$	$(\mathfrak{e}_7)^{N+1} \oplus (\mathfrak{so}_7)^{N+1} \oplus (\mathfrak{su}_2)^{2N+3}$ $\oplus(\mathfrak{g}_2)^2 \oplus \mathfrak{sp}_1 \oplus \mathfrak{f}_4$	$\mathfrak{su}_2$	
$2\,2\,2\,3\,A_N\,2\,2\,2\,2$	E_8	$2,3,1,5,1,3,2,2,(12,B(E_8))^N,$ $11,1,2,2,3,1,5,1,3,2,2$	$12(N+1)+7$	$(\mathfrak{e}_8)^{N+1} \oplus (\mathfrak{sp}_1 \oplus \mathfrak{g}_2)^{2N+4} \oplus (\mathfrak{f}_4)^{N+2}$	$\mathfrak{su}_2$	⋆
$2\,2\,2\,2\,3\,A_N\,2\,2\,2\,2$	E_8	$2,2,3,1,5,1,3,2,2,(12,B(E_8))^N,$ $11,1,2,2,3,1,5,1,3,2,2$	$12(N+1)+8$	$(\mathfrak{e}_8)^{N+1} \oplus (\mathfrak{sp}_1 \oplus \mathfrak{g}_2)^{2N+4} \oplus (\mathfrak{f}_4)^{N+2}$		

Series	Type	Blow up	N_T	Algebra	W	
$7\,A_N\,3$	E_8	$(12, B(E_8))^N, 11, 1, 2, 2, 3, 1, 5$	$12N + 7$	$(\mathfrak{e}_8)^{N+1} \oplus (\mathfrak{sp}_1 \oplus \mathfrak{g}_2)^{2N+1} \oplus (\mathfrak{f}_4)^{N+1}$	$\mathfrak{e}_8 \oplus \mathfrak{f}_4$	
$2\,4\,A_N\,3$	E_8	$3, 2, 2, 1, (12, B(E_8))^N, 11, 1, 2, 2, 3, 1, 5$	$12N + 11$	$(\mathfrak{e}_8)^{N+1} \oplus (\mathfrak{sp}_1 \oplus \mathfrak{g}_2)^{2N+2} \oplus (\mathfrak{f}_4)^{N+1}$	$\mathfrak{g}_2 \oplus \mathfrak{f}_4$	
$3\,3\,A_N\,3$	E_8	$5, 1, 3, 2, 2, 1, (12, B(E_8))^N, 11, 1, 2, 2, 3, 1, 5$	$12(N+1) + 1$	$(\mathfrak{e}_8)^{N+1} \oplus (\mathfrak{sp}_1 \oplus \mathfrak{g}_2)^{2N+2} \oplus (\mathfrak{f}_4)^{N+2}$	$\mathfrak{f}_4 \oplus \mathfrak{f}_4$	
$2\,2\,3\,A_N 3$	E_7	$2, 3, 2, 1, 8, (B(E_7), 8)^N, 1, 2, 3, 1, 5$	$6N + 10$	$(\mathfrak{e}_7)^{N+1} \oplus (\mathfrak{so}_7)^{N+1} \oplus (\mathfrak{su}_2)^{2N+3} \oplus (\mathfrak{g}_2) \oplus \mathfrak{f}_4$	$\mathfrak{su}_2 \oplus \mathfrak{f}_4$	
$2\,2\,2\,3\,A_N\,3$	E_8	$2, 3, 1, 5, 1, 3, 2, 2, 1, (12, B(E_8))^N, 11, 1, 2, 2, 3, 1, 5$	$12(N+1) + 4$	$(\mathfrak{e}_8)^{N+1} \oplus (\mathfrak{sp}_1 \oplus \mathfrak{g}_2)^{2N+3} \oplus (\mathfrak{f}_4)^{N+2}$	$\mathfrak{su}_2 \oplus \mathfrak{f}_4$	$\star$
$2\,2\,2\,2\,3\,A_N\,3$	E_8	$2, 2, 3, 1, 5, 1, 3, 2, 2, 1, (12, B(E_8))^N, 11, 1, 2, 2, 3, 1, 5$	$12(N+1) + 5$	$(\mathfrak{e}_8)^{N+1} \oplus (\mathfrak{sp}_1 \oplus \mathfrak{g}_2)^{2N+3} \oplus (\mathfrak{f}_4)^{N+2}$	$\mathfrak{f}_4$	
$7\,A_N\,3\,2$	E_8	$(12, B(E_8))^N, 11, 1, 2, 2, 3$	$12N + 5$	$(\mathfrak{e}_8)^{N+1} \oplus (\mathfrak{sp}_1 \oplus \mathfrak{g}_2)^{2N+1} \oplus (\mathfrak{f}_4)^{N}$	$\mathfrak{e}_8 \oplus \mathfrak{g}_2$	
$2\,4\,A_N\,3\,2$	E_8	$3, 2, 2, 1(12, B(E_8))^N, 11, 1, 2, 2, 3$	$12N + 9$	$(\mathfrak{e}_8)^{N+1} \oplus (\mathfrak{sp}_1 \oplus \mathfrak{g}_2)^{2N+2} \oplus (\mathfrak{f}_4)^{N}$	$\mathfrak{g}_2 \oplus \mathfrak{g}_2$	
$3\,3\,A_N\,3\,2$	E_8	$5, 1, 3, 2, 2, 1(12, B(E_8))^N, 11, 1, 2, 2, 3$	$12N + 11$	$(\mathfrak{e}_8)^{N+1} \oplus (\mathfrak{sp}_1 \oplus \mathfrak{g}_2)^{2N+2} \oplus (\mathfrak{f}_4)^{N+1}$	$\mathfrak{f}_4 \oplus \mathfrak{g}_2$	
$2\,2\,3\,A_N 3\,2$	E_7	$2, 3, 2, 1, 8, (B(E_7), 8)^{N+1}, 1, 2, 3$	$6(N+1) + 8$	$(\mathfrak{e}_7)^{N+2} \oplus (\mathfrak{so}_7)^{N+2} \oplus (\mathfrak{su}_2)^{2N+5} \oplus (\mathfrak{g}_2)$	$\mathfrak{su}_2 \oplus \mathfrak{g}_2$	
$2\,2\,2\,3\,A_N\,3\,2$	E_8	$2, 3, 1, 5, 1, 3, 2, 2, 1(12, B(E_8))^N,$ $11, 1, 2, 2, 3$	$12(N+1) + 2$	$(\mathfrak{e}_8)^{N+1} \oplus (\mathfrak{sp}_1 \oplus \mathfrak{g}_2)^{2N+3} \oplus (\mathfrak{f}_4)^{N+1}$	$\mathfrak{su}_2 \oplus \mathfrak{g}_2$	$\star$
$2\,2\,2\,2\,3\,A_N\,3\,2$	E_8	$2, 2, 3, 1, 5, 1, 3, 2, 2, 1(12, B(E_8))^N,$ $11, 1, 2, 2, 3$	$12(N+1) + 3$	$(\mathfrak{e}_8)^{N+1} \oplus (\mathfrak{sp}_1 \oplus \mathfrak{g}_2)^{2N+3} \oplus (\mathfrak{f}_4)^{N+1}$	$\mathfrak{g}_2$	$\star$

Table 3. Minimal resolutions: $(\mathcal{I}, \mathcal{J})$ infinite families.

Theory	Blow up	N_T	Algebra	W
6 3	10,1,2,2,3,1,5	7	$\mathfrak{e}_8 \oplus \mathfrak{sp}_1 \oplus \mathfrak{g}_2 \oplus \mathfrak{f}_4$	$\mathfrak{e}_8 \oplus \mathfrak{f}_4$
6 2	8,1,2,3	4	$\mathfrak{e}_7 \oplus \mathfrak{su}_2 \oplus \mathfrak{g}_2$	$\mathfrak{e}_7 \oplus \mathfrak{g}_2$
6 2 2	8,1,2,3,2	5	$\mathfrak{e}_7 \oplus (\mathfrak{su}_2)^2 \oplus \mathfrak{so}_7$	$\mathfrak{e}_7 \oplus \mathfrak{su}_2$
6 2 2 2	10, 1, 2, 2, 3, 1, 5, 1, 3, 2	10	$\mathfrak{e}_8 \oplus \mathfrak{su}_2 \oplus (\mathfrak{g}_2)^2 \oplus \mathfrak{sp}_1 \oplus \mathfrak{f}_4$	$\mathfrak{e}_8 \oplus \mathfrak{su}_2$
6 2 2 2 2	10, 1, 2, 2, 3, 1, 5, 1, 3, 2, 2	11	$\mathfrak{e}_8 \oplus (\mathfrak{sp}_1 \oplus \mathfrak{g}_2)^2 \oplus \mathfrak{f}_4$	$\mathfrak{e}_8$
5 2	6,1,3	3	$\mathfrak{e}_6 \oplus \mathfrak{su}_3$	$\mathfrak{e}_6 \oplus \mathfrak{su}_3$
5 2 2	7,1,2,3,2	5	$\mathfrak{e}_7 \oplus (\mathfrak{su}_2)^2 \oplus \mathfrak{so}_7$	$\mathfrak{e}_6 \oplus \mathfrak{su}_2$ ♠
5 2 2 2	8,1,2,3,1,5,1,3,2	9	$\mathfrak{e}_7 \oplus (\mathfrak{su}_2 \oplus \mathfrak{g}_2)^2 \oplus \mathfrak{f}_4$	$\mathfrak{e}_7 \oplus \mathfrak{su}_2$
4 2	5,1,3	3	$\mathfrak{f}_4 \oplus \mathfrak{su}_3$	$\mathfrak{f}_4 \oplus \mathfrak{su}_3$
4 2 2	5,1,3,2	4	$\mathfrak{f}_4 \oplus \mathfrak{su}_2 \oplus \mathfrak{g}_2$	$\mathfrak{f}_4 \oplus \mathfrak{su}_2$
4 2 2 2	5,1,3,2,2	5	$\mathfrak{f}_4 \oplus \mathfrak{sp}_1 \oplus \mathfrak{g}_2$	$\mathfrak{f}_4$

Table 4. Minimal resolutions: $(\mathcal{J}, \mathcal{J})$ isolated theories.

Series	Type	Blow up	N_T	Algebra	W	
$6\,A_N\,6$	E_8	$11,(B(E_8),12)^N,B(E_8),11$	$12(N+1)+1$	$(\mathfrak{e}_8)^{N+2}\oplus(\mathfrak{sp}_1\oplus\mathfrak{g}_2)^{2N+2}\oplus(\mathfrak{f}_4)^{N+1}$	$\mathfrak{e}_8\oplus\mathfrak{e}_8$	
$6\,A_N\,5$	E_8	$11,(B(E_8),12)^N,B(E_8),10$	$12(N+1)+1$	$(\mathfrak{e}_8)^{N+2}\oplus(\mathfrak{sp}_1\oplus\mathfrak{g}_2)^{2N+2}\oplus(\mathfrak{f}_4)^{N+1}$	$\mathfrak{e}_8\oplus\mathfrak{e}_8$	
$6\,A_N\,4$	E_8	$11,(B(E_8),12)^N,1,2,2,3,1,5,1,3,2,1,8$	$12(N+1)$	$\mathfrak{e}_7\oplus(\mathfrak{e}_8)^{N+1}\oplus(\mathfrak{sp}_1\oplus\mathfrak{g}_2)^{2N+2}\oplus(\mathfrak{f}_4)^{N+1}$	$\mathfrak{e}_8\oplus\mathfrak{e}_7$	$\star$
$6\,A_N\,3\,2$	E_8	$11,(B(E_8),12)^N,B(E_8),11,1,2,2,3$	$12(N+1)+5$	$(\mathfrak{e}_8)^{N+2}\oplus(\mathfrak{sp}_1\oplus\mathfrak{g}_2)^{2N+3}\oplus(\mathfrak{f}_4)^{N+1}$	$\mathfrak{e}_8\oplus\mathfrak{g}_2$	
$6\,A_N\,2\,3$	E_8	$11,(B(E_8),12)^N,B(E_8),11,1,2,2,3,1,5$	$12(N+1)+7$	$(\mathfrak{e}_8)^{N+2}\oplus(\mathfrak{sp}_1\oplus\mathfrak{g}_2)^{2N+3}\oplus(\mathfrak{f}_4)^{N+2}$	$\mathfrak{e}_8\oplus\mathfrak{f}_4$	
$6\,A_N\,2\,2\,2\,2\,2$	E_8	$11,(B(E_8),12)^N,B(E_8),11,1,2,2,3,1,5,1,3,2,2$	$12(N+1)+11$	$(\mathfrak{e}_8)^{N+2}\oplus(\mathfrak{sp}_1\oplus\mathfrak{g}_2)^{2N+4}\oplus(\mathfrak{f}_4)^{N+2}$	$\mathfrak{e}_8$	
$5\,A_N\,5$	E_7	$(8,B(E_7))^{N+1},8$	$6(N+1)+1$	$(\mathfrak{e}_7)^{N+2}\oplus(\mathfrak{su}_2)^{2N+2}\oplus(\mathfrak{so}_7)^{N+1}$	$\mathfrak{e}_7\oplus\mathfrak{e}_7$	
$5\,A_N\,4$	E_7	$(8,B(E_7))^{N+1},7$	$6(N+1)+1$	$(\mathfrak{e}_7)^{N+2}\oplus(\mathfrak{su}_2)^{2N+2}\oplus(\mathfrak{so}_7)^{N+1}$	$\mathfrak{e}_7\oplus\mathfrak{e}_7$	♠
$5\,A_N\,3$	E_7	$(8,B(E_7))^N,8,1,2,3,1,5$	$6N+6$	$(\mathfrak{e}_7)^{N+1}\oplus(\mathfrak{su}_2)^{2N+1}\oplus(\mathfrak{so}_7)^N\oplus\mathfrak{g}_2\oplus\mathfrak{f}_4$	$\mathfrak{e}_7\oplus\mathfrak{f}_4$	
$5\,A_N\,2\,2\,2\,2$	E_7	$(8,B(E_7))^N,8,1,2,3,1,5,1,3,2,2$	$6(N+2)+4$	$(\mathfrak{e}_7)^{N+1}\oplus(\mathfrak{su}_2)^{2N+1}\oplus(\mathfrak{so}_7)^N\oplus(\mathfrak{g}_2)^2\oplus\mathfrak{f}_4\oplus\mathfrak{sp}_1$	$\mathfrak{e}_7$	
$5\,A_N\,3\,2$	E_7	$(8,B(E_7))^{N+1},8,1,2,3$	$6(N+1)+4$	$(\mathfrak{e}_7)^{N+2}\oplus(\mathfrak{su}_2)^{2N+3}\oplus(\mathfrak{so}_7)^{N+1}\oplus\mathfrak{g}_2$	$\mathfrak{e}_7\oplus\mathfrak{g}_2$	

Series	Type	Blow up	N_T	Algebra	W
$4\,A_N\,4$	E_6	$(6, B(E_6))^{N+1}, 6$	$4(N+1)+1$	$(\mathfrak{e}_6)^{N+2} \oplus (\mathfrak{su}_3)^{N+1}$	$\mathfrak{e}_6 \oplus \mathfrak{e}_6$
$4\,A_N\,3$	E_6	$(6, B(E_6))^{N+1}, 5$	$4(N+1)+1$	$(\mathfrak{e}_6 \oplus \mathfrak{su}_3)^{N+1} \oplus \mathfrak{f}_4$	$\mathfrak{e}_6 \oplus \mathfrak{f}_4$
$4\,A_N\,3\,2$	E_6	$(6, B(E_6))^{N+1}, 6, 1, 3$	$4(N+1)+3$	$(\mathfrak{e}_6 \oplus \mathfrak{su}_3)^{N+2}$	$\mathfrak{e}_6 \oplus \mathfrak{su}_3$
$4\,A_N\,2\,2\,2\,2$	E_6	$(6, B(E_6))^{N+1}, 5, 1, 3, 2, 2$	$4(N+2)+1$	$(\mathfrak{e}_6 \oplus \mathfrak{su}_3)^{N} \oplus \mathfrak{f}_4 \oplus \mathfrak{sp}_1 \oplus \mathfrak{g}_2$	$\mathfrak{e}_6$
$3\,A_N\,3$	D_4	$4, (1, 4)^{N+1}$	$2N+5$	$(\mathfrak{so}_8)^{N+2}$	$\mathfrak{so}_8 \oplus \mathfrak{so}_8$
$3\,A_N\,3\,2$	E_6	$5, (B(E_6), 6)^{N+1}, 1, 3$	$4(N+1)+3$	$\mathfrak{f}_4 \oplus (\mathfrak{su}_3)^{N+2} \oplus (\mathfrak{e}_6)^{N+1}$	$\mathfrak{f}_4 \oplus \mathfrak{su}_3$
$3\,A_N\,2\,2\,2$	D_4	$(4, 1)^{N+1}, 3, 2, 2$	$2(N+2)+1$	$(\mathfrak{so}_8)^{N+1} \oplus \mathfrak{sp}_1 \oplus \mathfrak{g}_2$	$\mathfrak{so}_8$
$2\,3\,A_N\,3\,2$	E_6	$3, 1, 6, (B(E_6), 6)^{N+1}, 1, 3$	$4(N+1)+5$	$(\mathfrak{e}_6)^{N+2} \oplus (\mathfrak{su}_3)^{N+3}$	$\mathfrak{su}_3 \oplus \mathfrak{su}_3$
$2\,3\,A_N\,2\,2\,2\,2$	E_6	$3, 1, (6, B(E_6))^{N+1}, 5, 1, 3, 2, 2$	$4(N+1)+7$	$(\mathfrak{e}_6)^{N+1} \oplus (\mathfrak{su}_3)^{N+2} \oplus \mathfrak{su}_2 \oplus \mathfrak{g}_2 \oplus \mathfrak{f}_4$	$\mathfrak{su}_3$

Table 5. Minimal resolutions: $(\mathcal{J}, \mathcal{J})$ infinite families.

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