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Research Article

Szegö Kernels and Asymptotic Expansions for Legendre Polynomials

Roberto Paoletti

Dipartimento di Matematica e Applicazioni, Università degli Studi di Milano Bicocca, Via R. Cozzi 55, 20125 Milano, Italy

Correspondence should be addressed to Roberto Paoletti; roberto.paoletti@unimib.it

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We present a geometric approach to the asymptotics of the Legendre polynomials $P_{k,n+1}$, based on the Szegö kernel of the Fermat quadric hypersurface, leading to complete asymptotic expansions holding on expanding subintervals of $[-1,1]$.

1. Introduction

The search for asymptotic expansions and approximations of special functions is a very classical vein of research and is of great relevance in pure mathematics, in numerical analysis, mathematical physics, and the applied sciences (see, for instance, of course with no pretence of completion [1–4]).

The goal of this paper is to develop a geometric approach to the asymptotics of the Legendre polynomial $P_{k,n+1}(t)$ for $k \to +\infty$, with $t = \cos(\vartheta) \in [-1,1]$ and $n \geq 1$ fixed; as is well-known, $P_{k,n+1}(t)$ is the restriction to S^n of the Legendre harmonic, expressed in polar coordinates on the sphere. For thorough discussions and terminology, see, for instance, [1, 3, 5–7]. We obtain an asymptotic expansion holding on expanding subintervals of $[-1,1]$, rather than on fixed subintervals of the form $[-1+\delta, 1-\delta]$ for some given $\delta > 0$, as one typically finds in the literature.

However, the actual point of this work is neither to present essentially new results nor to give an especially economic proof of Legendre asymptotics (the use of Szegö kernel machinery is arguably not more elementary than the traditional approaches). Rather, it is motivated by the following considerations. On the one hand, there is a conceptually very appealing view on spherical harmonics, due to Lebeau and Guillemin, based on the Szegö kernel of the Fermat quadric. On the other hand, in recent years, a considerable amount of work and attention has been devoted to algebro-geometric Szegö kernel asymptotics, which have played a fundamental role in complex geometry. Therefore, it seems *per se* very natural and interesting to illustrate the important conceptual juncture between spherical harmonics and Szegö kernels, by revisiting classical results on Legendre asymptotics in view of these recent developments. In a broader perspective, the application of Szegö kernels to spherical harmonics seems a very promising area; a revisitation of this kind is also instrumental to the development of computational techniques that might be useful in future developments in this direction.

Let us come to a closer description of the content of this paper. There is a tight relation between $P_{k,n+1}(t)$ and the orthogonal projector

$$\mathscr{P}_{k,n} : L^2(S^n) \longrightarrow V_{k,n}, \qquad (1)$$

where $V_{k,n}$ is the space of level-k spherical harmonics on S^n; equivalently, $V_{k,n}$ is the eigenspace of the (positive) Laplace-Beltrami operator on functions on S^n, corresponding to its kth eigenvalue $\lambda_{k,n} = k(k+n-1)$.

Namely, for any choice of an orthonormal basis $(\varrho_{knj})_{j=1}^{N_{k,n}}$ of $V_{k,n}$ the distributional kernel $\mathscr{P}_{k,n}(\cdot,\cdot) \in \mathscr{C}^\infty(S^n \times S^n)$ satisfies

$$\mathscr{P}_{k,n}(\mathbf{q},\mathbf{q}') = \sum_{j=1}^{N_{k,n}} \varrho_{knj}(\mathbf{q}) \cdot \overline{\varrho_{knj}(\mathbf{q}')}, \qquad (2)$$

where $\mathbf{q} \cdot \mathbf{q}' = \mathbf{q}^t \mathbf{q}'$ (we think of $\mathbf{q}$ and $\mathbf{q}'$ as columns vectors), and $N_{k,n}$ is the dimension of $V_{k,n}$. By symmetry considerations, $\mathscr{P}_{k,n}(\mathbf{q},\mathbf{q}')$ only depends on $\mathbf{q}\cdot\mathbf{q}'$. In fact, with the normalization $P_{k,n+1}(1) = 1$,

$$\mathscr{P}_{k,n}\left(\mathbf{q},\mathbf{q}'\right) = \frac{N_{k,n}}{\operatorname{vol}(S^n)} P_{k,n+1}\left(\mathbf{q}\cdot\mathbf{q}'\right). \tag{3}$$

Thus, it is equivalent to give asymptotic expansions for $P_{k,n+1}(\cos(\vartheta))$ and for $\mathscr{P}_{k,n}(\mathbf{q},\mathbf{q}')$ with $\mathbf{q}\cdot\mathbf{q}' = \cos(\vartheta)$.

Since for any $(\mathbf{q},\mathbf{q}') \in S^n \times S^n$ we have

$$\begin{aligned}\mathscr{P}_{k,n}(\mathbf{q},\mathbf{q}) &= \frac{N_{k,n}}{\operatorname{vol}(S^n)},\\ \mathscr{P}_{k,n}\left(\mathbf{q},-\mathbf{q}'\right) &= (-1)^k \mathscr{P}_{k,n}\left(\mathbf{q},\mathbf{q}'\right),\end{aligned} \tag{4}$$

we may assume $\mathbf{q} \neq \pm\mathbf{q}'$. Then there is a unique great circle parametrized by arc length going from $\mathbf{q}$ to $\mathbf{q}'$ in a time $\vartheta \in (0,\pi)$ and $\mathbf{q}^t\mathbf{q}' = \cos(\vartheta)$.

Our geometric approach uses, on the one hand, the specific relation between spherical harmonics on S^n and the Hardy space of the Fermat quadric hypersurface in $\mathbb{P}^n$ [8, 9] and, on the other hand, the off-diagonal scaling asymptotics of the level-k Szegö kernel of polarized projective manifold [10, 11].

The following asymptotic expansions involve a sequence of constants $C_{k,n} > 0$ with a precise geometric meaning [9]. There is a natural algebraic isomorphism between the level-k Szegö kernel of the Fermat quadric $F_n \subset \mathbb{P}^n$ and $V_{k,n}$, given by a push-forward operation; this isomorphism is however unitary only up to an appropriate rescaling, and $C_{k,n}$ is the corresponding scaling factor.

An asymptotic expansion for $C_{k,n}$ is discussed in [9], building on the theory of [8]; an alternative derivation is given in Proposition 2 (with an explicit computation of the leading order term).

In the following, the symbol ~ stands for "has the same asymptotics as."

Theorem 1. *There exist smooth functions A_{nl} and B_{nl} ($l = 1,2,\ldots$) on $[0,\pi]$ such that the following holds. Let us fix $C > 0$ and $\delta \in [0,1/6)$. Then, uniformly in $(\mathbf{q},\mathbf{q}') \in S^n \times S^n$ satisfying $\mathbf{q}^t\mathbf{q}' = \cos(\vartheta)$ with*

$$Ck^{-\delta} < \vartheta < \pi - Ck^{-\delta}, \tag{5}$$

we have for $k \to +\infty$ an asymptotic expansion of the form

$$\begin{aligned}\mathscr{P}_{k,n}\left(\mathbf{q},\mathbf{q}'\right) = \frac{2^{n/2}}{C_{k,n}^2}\left(\frac{1}{\sin(\vartheta)}\right)^{(n-1)/2} &\cdot \left[\cos\left(\alpha_{k,n}(\vartheta)\right)\right.\\ \cdot \mathscr{A}_n(\vartheta,k) + \sin\left(\alpha_{k,n}(\vartheta)\right)&\left.\cdot \mathscr{B}_n(\vartheta,k)\right],\end{aligned} \tag{6}$$

where

$$\begin{aligned}\alpha_{k,n}(\vartheta) &=: k\vartheta + \left(\frac{\vartheta}{2} - \frac{\pi}{4}\right)(n-1),\\ \mathscr{A}_n(\vartheta,k) &\sim 1 + \sum_{l=1}^{+\infty} k^{-l}\frac{A_{nl}(\vartheta)}{\sin(\vartheta)^{6l}},\\ \mathscr{B}_n(\vartheta,k) &\sim \sum_{l=1}^{+\infty} k^{-l}\frac{B_{nl}(\vartheta)}{\sin(\vartheta)^{6l}}.\end{aligned} \tag{7}$$

At the lth step, we have for some constant $C_l > 0$

$$\left|\frac{A_{nl}(\vartheta)}{\sin(\vartheta)^{6l}}\right|, \left|\frac{B_{nl}(\vartheta)}{\sin(\vartheta)^{6l}}\right| \leq C_l k^{-l(1-6\delta)}, \tag{8}$$

and a similar estimate holds for the error term. Hence, the previous is an asymptotic expansion for $\delta \in [0,1/6)$.

As mentioned, the same techniques yield an asymptotic expansion for $C_{n,k}$ (see (6.18) in [9]).

Proposition 2. *For $k \to +\infty$, we have an asymptotic expansion of the form:*

$$\begin{aligned}C_{k,n} \sim \left[\frac{(n-1)!}{2\sqrt{2}} \cdot \operatorname{vol}(S^n)\operatorname{vol}\left(S^{n-1}\right)\right]^{1/2} (\pi k)^{-(n-1)/4}\\ \cdot \left[1 + \sum_{j\geq 1} k^{-j} a_j\right].\end{aligned} \tag{9}$$

If we insert the latter expansion in the one provided by Theorem 1, we obtain the following.

Corollary 3. *With the assumptions and notation of Theorem 1, for $k \to +\infty$, there is an asymptotic expansion*

$$\begin{aligned}\mathscr{P}_{k,n}\left(\mathbf{q},\mathbf{q}'\right) &= \frac{2^{(n+3)/2}}{(n-1)!}\\ &\cdot \frac{1}{\operatorname{vol}(S^n)\operatorname{vol}(S^{n-1})}\left(\frac{\pi k}{\sin(\vartheta)}\right)^{(n-1)/2}\\ &\cdot \left[\cos\left(\alpha_{k,n}(\vartheta)\right)\cdot \mathscr{C}_n(\vartheta,k) + \sin\left(\alpha_{k,n}(\vartheta)\right)\right.\\ &\left.\cdot \mathscr{D}_n(\vartheta,k)\right],\end{aligned} \tag{10}$$

where $\mathscr{C}_n(\vartheta,k)$ and $\mathscr{D}_n(\vartheta,k)$ admit asymptotic expansions similar to those of $\mathscr{A}_n(\vartheta,k)$ and $\mathscr{B}_n(\vartheta,k)$, respectively (of course, with different functions C_{nl} and D_{nl}, $l \geq 1$).

Pairing Corollary 3 with (3), we obtain the following.

Corollary 4. *In the same situation as in Theorem 1, for $k \to +\infty$, there is an asymptotic expansion*

$$\begin{aligned}P_{k,n+1}(\cos(\vartheta)) &= \frac{2^{(n+1)/2}}{\operatorname{vol}(S^{n-1})}\left(\frac{\pi}{\sin(\vartheta)\,k}\right)^{(n-1)/2}\\ &\cdot \left[\cos\left(\alpha_{k,n}(\vartheta)\right)\cdot \mathscr{E}_n(\vartheta,k) + \sin\left(\alpha_{k,n}(\vartheta)\right)\right.\\ &\left.\cdot \mathscr{F}_n(\vartheta,k)\right],\end{aligned} \tag{11}$$

where again $\mathscr{E}_n(\vartheta,k)$ and $\mathscr{F}_n(\vartheta,k)$ admit asymptotic expansions similar to those of $\mathscr{A}_n(\vartheta,k)$ and $\mathscr{B}_n(\vartheta,k)$, respectively.

Let us verify that Corollary 4 fits with the classical asymptotics. For example, when $n = 1$, we obtain

$$P_{k,2}(\cos(\vartheta)) \sim \cos(k\vartheta) + \cdots, \tag{12}$$

so that the leading order term is the kth Chebyshev polynomial. Since it is known that in this case the Legendre polynomial *is* the Chebyshev polynomial ([6], page 11), this is in fact the only term of the expansion.

For $n = 2$, we obtain the formula of Laplace (cfr [12], Section 4.6; [4], (8.01) of Ch. 4; [13], Theorem 8.21.2), but as a full asymptotic expansion holding uniformly on expanding subintervals converging to $[-1, 1]$ at a controlled rate, as above:

$$P_{k,3}(\cos(\vartheta)) \sim \sqrt{\frac{2}{\pi k \sin(\vartheta)}} \cos\left(\left(k + \frac{1}{2}\right)\vartheta - \frac{\pi}{4}\right) + O\left(k^{-3/2+6\delta}\right). \tag{13}$$

For arbitrary n, $P_{k,n+1}$ is a multiple of a Gegenbauer polynomial ([2]; [6], page 16):

$$P_k^{(n/2-1,n/2-1)}(\cos(\vartheta)) = r_{k,n} P_{k,n+1}(\cos(\vartheta)). \tag{14}$$

Given the standardization for $P_k^{(n/2-1,n/2-1)}$ ([14], section 10.8)

$$r_{k,n} = P_k^{(n/2-1,n/2-1)}(1) = \binom{k + \frac{n}{2} - 1}{k} = \frac{(n/2)_k}{k!} = \frac{\Gamma(k+n/2)}{k!\Gamma(n/2)}, \tag{15}$$

where Γ is of course the Gamma function. By (35.31) in [3], for $k \to +\infty$, we have

$$\Gamma\left(k + \frac{n}{2}\right) \sim k^{n/2}\Gamma(k) = k^{n/2}(k-1)!. \tag{16}$$

Therefore,

$$r_{k,n} \sim \frac{k^{n/2}(k-1)!}{k!\Gamma(n/2)} = \frac{k^{n/2-1}}{\Gamma(n/2)}. \tag{17}$$

If we use the well-known formula (see, e.g., (2) of [6])

$$\mathrm{vol}\left(S^{n-1}\right) = \frac{2\pi^{n/2}}{\Gamma(n/2)}, \tag{18}$$

we obtain for $P_k^{(n/2-1,n/2-1)}(\cos(\vartheta))$ as asymptotic expansion with leading order term

$$2^{(n+1)/2} \frac{k^{n/2-1}}{\Gamma\left(\frac{n}{2}\right)} \frac{\Gamma(n/2)}{2\pi^{n/2}} \left(\frac{\pi}{\sin(\vartheta) k}\right)^{(n-1)/2} \cos(\alpha_{k,n}(\vartheta)) = \frac{1}{\sqrt{\pi k}} \frac{1}{\cos(\vartheta/2)^{(n-1)/2} \sin(\vartheta/2)^{(n-1)/2}} \cos(\alpha_{k,n}(\vartheta)), \tag{19}$$

in agreement with (10) on page 198 of [14].

2. Preliminaries

2.1. The Geometric Picture. For the following, see [8, 9].

Let $S_1^n \subset \mathbb{R}^{n+1}$ be the unit sphere, and let us identify the tangent and cotangent bundles of S_1^n by means of the standard Riemannian metric. The unit (co)sphere bundle of S_1^n is given by the incidence correspondence

$$S^*\left(S_1^n\right) \cong S\left(S_1^n\right) = \left\{(\mathbf{q}, \mathbf{p}) \in S_1^n \times S_1^n : \mathbf{q}^t\mathbf{p} = 0\right\}. \tag{20}$$

The Fermat quadric hypersurface in complex projective space is

$$F_n := \left\{[\mathbf{z}] \in \mathbb{P}^n : \mathbf{z}^t\mathbf{z} = 0\right\}; \tag{21}$$

let A be the restriction to F_n of the hyperplane line bundle. Given the standard Hermitian product on $\mathbb{C}^{n+1}$, A is naturally a positive Hermitian line bundle, F_n inherits a Kähler structure ω_{F_n} (the restriction of the Fubini-Study metric), and the spaces of global holomorphic sections of higher powers of A, $H^0(F_n, A^{\otimes k})$ have an induced Hermitian structure.

The affine cone over F_n is $\mathscr{C}_n = \{\mathbf{z}^t\mathbf{z} = 0\} \subset \mathbb{C}^{n+1}$; the intersection $X_1 := \mathscr{C}_n \cap S_1^{2n+1}$ may be viewed as the unit circle bundle in the dual line bundle $A^\vee$. More generally, for any $r > 0$, the intersection

$$X_r := \mathscr{C}_n \cap S_r^{2n+1} \tag{22}$$

with the sphere of radius r is naturally identified with the circle bundle of radius r in $A^\vee$. In particular,

$$X_{\sqrt{2}} = \left\{\mathbf{q} + i\mathbf{p} : \|\mathbf{q}\|^2 = \|\mathbf{p}\|^2 = 1,\ \mathbf{q}^t\mathbf{p} = 0\right\} \tag{23}$$

is diffeomorphic to $S^*(S^n)$ by the map $\beta : (\mathbf{q}, \mathbf{p}) \mapsto \mathbf{q} + i\mathbf{p}$; furthermore, β is equivariant for the natural actions of $O(n+1)$ on $S^*(S^n)$ and $X_{\sqrt{2}}$ defined by, respectively,

$$B \cdot (\mathbf{q}, \mathbf{p}) =: (B\mathbf{q}, B\mathbf{p}), \quad B \cdot (\mathbf{q} + i\mathbf{p}) = B\mathbf{q} + iB\mathbf{p} \quad (B \in O(n+1)). \tag{24}$$

We shall identify $S^*(S^n)$ and $X_{\sqrt{2}}$ and denote the projection by

$$\nu : S^*\left(S^n\right) \cong X_{\sqrt{2}} \longrightarrow S^n, \quad \mathbf{q} + i\mathbf{p} \longmapsto \mathbf{q}. \tag{25}$$

There is also a standard structure action of S^1 on $X_{\sqrt{2}}$, induced by fibrewise scalar multiplication in $A^\vee$, or equivalently in $\mathbb{C}^{n+1}$. The latter action is intertwined by β with the "reverse" geodesic flow on $S^*(S^n) \cong S(S^n)$. The S^1-orbits are the fibers of the circle bundle projection

$$\pi_{\sqrt{2}} : \mathbf{q} + i\mathbf{p} \in X_{\sqrt{2}} \longmapsto [\mathbf{q} + i\mathbf{p}] \in F_n. \tag{26}$$

This holds for any $r > 0$; we shall denote by $\pi_r : X_r \to F_n$ the projection for general $r > 0$.

2.2. The Metric on X_r. Let us dwell on the metric aspect of (22); there are two natural choices of a Riemannian metric on X_r, hence of a Riemannian density, and we need to clarify the relation between the two.

There is an obvious choice of a Riemannian metric g_r' on X_r, induced by the standard Euclidean product on $\mathbb{C}^{n+1}$. With respect to g_r', the S^1 orbits on X_r have length of $2\pi r$. Clearly, g_r' is homogeneous of degree 2 with respect to the dilation $\mu_r : x \in X \mapsto rx \in X_r$, and therefore the corresponding volume form Υ'_{X_r} on X_r is homogeneous of degree $\dim(X) = 2n-1$. That is,

$$\mu_r^* \left(\Upsilon'_{X_r}\right) = r^{2n-1}\Upsilon'_X. \tag{27}$$

An alternative and common choice of a Riemannian structure g_1 on X_1 comes from its structure of a unit circle bundle over F_n. Let $\alpha \in \Omega^1(X_1)$ be the connection 1-form associated with the unique compatible covariant derivative on A, so that $d\alpha = 2\pi_1^*(\omega_{F_n})$. Also, let

$$\begin{aligned} H\left(X_1/F_n\right) &= \ker(\alpha), \\ V\left(X_1/F_n\right) &= \ker\left(d\pi_1\right) \subseteq TX \end{aligned} \tag{28}$$

denote the horizontal and vertical tangent bundles for π_1, respectively. There is a unique Riemannian metric g_1 on X_1 such that π_1 a Riemannian submersion and the S^1-orbits on X_1 have unit length. The corresponding volume form on X_1 is given by

$$\begin{aligned} \Upsilon_{X_1} &= \frac{1}{(n-1)!}\pi_1^*\left(\omega_{F_n}^{\wedge(n-1)}\right) \wedge \frac{1}{2\pi}\alpha \\ &= \frac{1}{2\pi}\pi_1^*\left(\Upsilon_{F_n}\right) \wedge \alpha, \end{aligned} \tag{29}$$

where $\Upsilon_{F_n} = \omega_{F_n}^{\wedge(n-1)}/(n-1)!$ is the symplectic volume form on F_n.

We wish to compare the two Riemannian metrics g_1 and g_1', the corresponding volume forms, Υ'_{X_1} and Υ_{X_1}, and densities, dV_X and $d'V_X$.

Lemma 5. $\Upsilon_{X_1} = (1/2\pi)\Upsilon'_{X_1}$ *and* $dV_{X_1} = (1/2\pi)d'V_{X_1}$.

Proof of Lemma 5. The connection 1-form for the Hopf map $S^{2n+1} \to \mathbb{P}^n$ is

$$\theta = \frac{i}{2}\left(\mathbf{z}^t d\overline{\mathbf{z}}^t - \overline{\mathbf{z}}^t d\mathbf{z}\right); \tag{30}$$

thus, α is the restriction of θ to X_1. Let ω_0 be the standard symplectic structure on $\mathbb{C}^{n+1}$. Since $\theta_{\mathbf{z}}(\mathbf{w}) = \omega_0(\mathbf{z}, \mathbf{w})$, we have $\ker(\theta_{\mathbf{z}}) = \mathbf{z}^{\perp_{\omega_0}}$ (symplectic annihilator). In other words,

$$\ker\left(\theta_{\mathbf{z}}\right) = \left(\operatorname{span}_{\mathbb{R}}(\mathbf{z}) \oplus \mathbf{z}^{\perp_{h_0}}\right) \cap T_{\mathbf{z}}S_1^{2n+1} = \mathbf{z}^{\perp_{h_0}}, \tag{31}$$

where $\mathbf{z}^{\perp_{h_0}}$ is the Hermitian orthocomplement of $\mathbf{z}$ for the standard Hermitian product.

Thus, if $\mathbf{z} \in X_1$, then

$$H_{\mathbf{z}}\left(X_1/F_n\right) = \ker\left(\alpha_{\mathbf{z}}\right) = \mathbf{z}^{\perp_{h_0}} \cap T_{\mathbf{z}}\mathscr{C}_n = \mathbf{z}^{\perp_{h_0}} \cap \overline{\mathbf{z}}^{\perp_{h_0}}. \tag{32}$$

On the other hand, $V_{\mathbf{z}}(X_1/F_n) = \operatorname{span}_{\mathbb{R}}(i\mathbf{z})$. Thus, $V(X_1/F_n)$ and $H(X_1/F_n)$ are orthogonal with respect to both g_1 (by construction) and g_1' (by the previous considerations). Hence, we may compare g_1 and g_1' separately on $H(X_1/F_n)$ and $V(X_1/F_n)$.

On the complex vector bundle $H(X_1/F_n)$, g_1' and g_1 are, respectively, the Euclidean scalar products associated with the restrictions of the $(1,1)$-forms

$$\begin{aligned} \omega_0 &= \frac{i}{2}\partial\overline{\partial}\,\|\mathbf{z}\|^2, \\ \omega_1 &= \frac{i}{2}\partial\overline{\partial}\ln\left(\|\mathbf{z}\|^2\right). \end{aligned} \tag{33}$$

Given that ω_0 and ω_1 agree on TS_1^{2n+2}, $g_1 = g_1'$ on $H(X_1/F_n)$.

On the other hand, both g_1 and g_1' are S^1-invariant, but S^1-orbits on X_1 have length 2π for g_1' and 1 for g_1. Thus, $g_1 = g_1'/2\pi$ on $V(X_1/F_n)$.

The claim follows directly from this. □

2.3. The Szegö Kernel on X_r. The following analysis is based on the equivariant asymptotics of the Szegö kernel of $X_{\sqrt{2}}$ [11]. We refer the reader to [10, 11, 15] for a thorough discussion of Szegö kernels in the algebro-geometric context and to [16, 17] for the basic microlocal theory that underlies the subject (see also the neat discussion of Hardy spaces in [9]). To put things into perspective, however, let us recall that if Y is the boundary of a pseudoconvex domain, its Hardy space $H(Y) \subset L^2(Y)$ is the Hilbert space of square summable boundary values of holomorphic functions. The Szegö projector is then the orthogonal projector $\Pi : L^2(Y) \to H(Y)$; with some abuse of notation, the Szegö kernel $\Pi \in \mathscr{D}'(Y \times Y)$ is the corresponding distributional kernel. A description of Π as a Fourier integral operator was given in [16].

In the special algebro-geometric case, where Y is the dual unit circle bundle of a positive line bundle on a complex projective manifold, $H(Y)$ is the orthogonal direct sum of its isotypical components $H_k(Y)$, $k = 0, 1, 2, \ldots$, under the S^1-action; correspondingly, $\Pi = \bigoplus_k \Pi_k$, where Π_k is the orthogonal projector onto $H_k(Y)$; since $H_k(Y)$ is finite-dimensional, Π_k is a smoothing operator; therefore, its Schwartz kernel is a $\mathscr{C}^\infty$ function on $Y \times Y$. Many local asymptotic properties of Π_k, for $k \to +\infty$, were first discovered in [10, 11, 15] building on the theory of [16]. We shall recall what is needed here shortly.

Let us now come to the specific case in point. For every $r > 0$, X_r is the boundary of a strictly pseudoconvex domain, and as such it carries a CR structure, a Hardy space $H(X_r)$, and a Szegö projector $\Pi_r : L^2(X_r) \to H(X_r)$. We aim to relate the various Π_r's.

Let $\mathscr{O}(\mathscr{C}_n \setminus \{\mathbf{0}\})$ be the ring of holomorphic functions on the conic complex manifold $\mathscr{C}_n \setminus \{\mathbf{0}\}$. Let $\mathscr{O}_k(\mathscr{C}_n \setminus \{\mathbf{0}\}) \subset \mathscr{O}(\mathscr{C}_n \setminus \{\mathbf{0}\})$ be the subspace of holomorphic functions of degree of homogeneity k.

For every $r > 0$ and $k = 0, 1, 2, \ldots$ let $H_k(X_r) \subset H(X_r)$ be the finite-dimensional kth isotypical component of $H(X_r)$ with respect to the standard S^1-action. Restriction induces an algebraic isomorphism $\mathscr{O}_k(\mathscr{C}_n \setminus \{\mathbf{0}\}) \to H_k(X_r)$; with a slight abuse of language, we shall denote by the same symbol an element of $H_k(X_r)$ and the corresponding element of $\mathscr{O}_k(\mathscr{C}_n \setminus \{\mathbf{0}\})$.

Suppose that $(s_{kj})_{j=0}^{N_k} \subseteq \mathcal{O}_k(\mathscr{C}_n \setminus \{\mathbf{0}\})$ yields by restriction an orthonormal basis of $H_k(X_1)$:

$$\int_{X_1} s_{kj}(x)\overline{s_{kl}(x)} \mathrm{d}V_{X_1}(x) = \delta_{jl}. \tag{34}$$

Setting $y = rx$ and using (27) together with Lemma 5, we get

$$\begin{aligned}\int_{X_r} s_{kj}(y)\overline{s_{kl}(y)}\frac{1}{2\pi}\mathrm{d}'V_{X_r}(y) &= r^{2n+2k-1}\int_{X_1} s_{kj}(x)\overline{s_{kl}(x)}\frac{1}{2\pi}\mathrm{d}'V_{X_1}(x) \\ &= r^{2n+2k-1}\delta_{jl}.\end{aligned} \tag{35}$$

Therefore, we have the following.

Lemma 6. *If $(s_{kj})_{j=0}^{N_k} \subseteq \mathcal{O}_k(\mathscr{C}_n \setminus \{\mathbf{0}\})$ yields by restriction an orthonormal basis of $H_k(X_1)$ with respect to $\mathrm{d}'V_{X_1}$, then for every $r > 0$*

$$\left(r^{-(k+n-1/2)}s_{kj}\right)_{j=0}^{N_k} \tag{36}$$

yields by restriction an orthonormal basis of $H_k(X_r)$, with respect to $\mathrm{d}'V_{X_r}/2\pi$.

Let now $\Pi_{r,k}$ be the level-k Szegö kernel on X_r, that is, the orthogonal projector

$$\Pi_{r,k} : L^2\left(X_r, \frac{\mathrm{d}'V_{X_r}}{2\pi}\right) \longrightarrow H_k(X_r). \tag{37}$$

By Lemma 6, its Schwartz kernel $\Pi_{r,k} \in \mathscr{C}^\infty(X_r \times X_r)$ is given by

$$\Pi_{r,k}(y, y') = r^{-(2k+2n-1)}\sum_{j=0}^{N_k} s_{kj}(y) \cdot \overline{s_{kj}(y')} \qquad (y, y' \in X_r). \tag{38}$$

When pulled back to X_1, this is (here $x, x' \in X_1$)

$$\begin{aligned}\Pi_{r,k}(rx, rx') &= r^{-(2k+2n-1)}\sum_{j=0}^{N_k} s_{kj}(rx) \cdot \overline{s_{kj}(rx')} \\ &= r^{-(2n-1)}\sum_{j=0}^{N_k} \widehat{s}_{kj}(x) \cdot \overline{\widehat{s}_{kj}(x')} \\ &= r^{1-2n}\Pi_{1,k}(x, x').\end{aligned} \tag{39}$$

In particular,

$$\Pi_{\sqrt{2},k}\left(\sqrt{2}x, \sqrt{2}x'\right) = \frac{\sqrt{2}}{2^n}\Pi_{1,k}(x, x'). \tag{40}$$

We shall make repeated use of the following asymptotic property of $\Pi_{1,k}$, which follows from the microlocal description of Π as an FIO (explicit exponential estimates are discussed in [18]).

Theorem 7. *Let dist_{F_n} be the distance function on F_n associated with the Kähler metric. Given any $C, \epsilon > 0$, uniformly for $x, x' \in X$ satisfying*

$$\mathrm{dist}_{F_n}\left(\pi(x), \pi(x')\right) \geq Ck^{\epsilon-1/2}, \tag{41}$$

we have

$$\Pi_{1,k}(x, x') = O(k^{-\infty}) \tag{42}$$

when $k \to +\infty$.

2.4. Heisenberg Local Coordinates. There are two unit circle bundles in our picture: the Hopf fibration $\pi : S_1^{2n+1} \to \mathbb{P}^n$, and $\pi_1 : X_1 \to F_n$. Clearly, π_1 is the pull-back of π under the inclusion $F_n \hookrightarrow \mathbb{P}^n$. Both S_1^{2n+1} and X_1 are boundaries of strictly pseudoconvex domains and carry a CR structure.

On both S_1^{2n+1} and X_1, we may consider privileged systems of coordinates called *Heisenberg local coordinates* (HLC). In these coordinates, Szegö kernel asymptotics exhibit a "universal" structure [11]; we refer to *ibidem* for a detailed discussion.

Given $\mathbf{z}_0 \in X_1$, a HLC system on X_1 centered at $\mathbf{z}_0$ will be denoted in additive notation:

$$(\theta, \mathbf{v}) \in (-\pi, \pi) \times B_{2n-2}(\mathbf{0}, \delta) \longmapsto \mathbf{z}_0 + (\theta, \mathbf{v}) \in X_1. \tag{43}$$

Here $\theta \in (-\pi, \pi)$ is an "angular" coordinate measuring displacement along the S^1-orbit through $\mathbf{z}_0$ (the fiber through $\mathbf{z}_0$ of $\pi_1 : X_1 \to F_n$); instead, $\mathbf{v} \in B_{2n-2}(\mathbf{0}, \delta) \subseteq \mathbb{R}^{2n-2} \cong \mathbb{C}^{n-1}$ descends to a local coordinate on F_n centered at $m_0 = \pi(x_0)$, inducing a unitary isomorphism $T_{[\mathbf{z}_0]}F_n \cong \mathbb{C}^{n-1}$. We may thus think of $\mathbf{v}$ as a tangent vector in $T_{[\mathbf{z}_0]}F_n$.

Here this additive notation might be misleading, since $X_1 \subset \mathbb{C}^{n+1}$. Therefore, we shall write $\mathbf{z}_0 +_{X_1} (\theta, \mathbf{v})$ for HLC on X_1 centered at $\mathbf{z}_0$. We shall generally abridge notation by writing $\mathbf{z}_0 +_{X_1} \mathbf{v}$ for $\mathbf{z}_0 +_{X_1} (0, \mathbf{v})$.

Similarly, $(\theta, \mathbf{w}) \in (-\pi, \pi) \times B_{2n}(\mathbf{0}, \delta) \mapsto \mathbf{z}_0 +_{S_1^{2n+1}} (\theta, \mathbf{v})$ will denote a system of Heisenberg local coordinates on S_1^{2n+1} centered at $\mathbf{z}_0$. There is in fact a natural choice of HLC on S_1^{2n+1} centered at any $\mathbf{z}_0 \in S_1^{2n+1}$.

Namely, let $(\mathbf{a}_1, \ldots, \mathbf{a}_n)$ be an orthonormal basis of the Hermitian orthocomplement $\mathbf{z}_0^{\perp_h} \subseteq \mathbb{C}^{n+1}$, and for $\mathbf{w} = (w_j) \in \mathbb{C}^n$ let us set

$$\mathbf{z}_0 +_{S_1^{2n+1}} (\theta, \mathbf{w}) := \frac{e^{i\theta}}{\sqrt{1 + \|\mathbf{w}\|^2}}\left(\mathbf{z}_0 + \sum_{j=1}^n w_j \mathbf{a}_j\right). \tag{44}$$

Since there is a canonical unitary identification $\mathbf{z}_0^{\perp_h} \cong T_{[\mathbf{z}_0]}\mathbb{P}^n$, we shall also write this as $\mathbf{z}_0 +_{S_1^{2n+1}} (\theta, \mathbf{v})$ with $(\theta, \mathbf{v}) \in (-\pi, \pi) \times T_{[\mathbf{z}_0]}\mathbb{P}^n$.

If $\mathbf{z}_0 \in X_1$, HLC on X_1 centered at $\mathbf{z}_0$ can be chosen so that they agree to second order with the former HLC on S_1^{2n+1}. More precisely, we may assume that for any $\mathbf{v} \in T_{[\mathbf{z}_0]}F_n \subset T_{[\mathbf{z}_0]}\mathbb{P}^n$ we have

$$\mathbf{z}_0 +_X (\theta, \mathbf{w}) = \mathbf{z}_0 +_{S_1^{2n+1}} (\theta, \mathbf{v} + R_2(\mathbf{v})), \tag{45}$$

where R_2 is a function vanishing to second order at the origin.

Given $\mathbf{v}, \mathbf{w} \in \mathbb{C}^{n+1} \cong \mathbb{R}^{2n+2}$, let us define

$$\psi_2(\mathbf{v}, \mathbf{w}) =: -i\omega_0(\mathbf{v}, \mathbf{w}) - \frac{1}{2}\|\mathbf{v} - \mathbf{w}\|^2; \quad (46)$$

here ω_0 is the standard symplectic structure, and $\|\cdot\|$ is the standard Euclidean norm. We shall make use of the following asymptotic expansion, for which we refer again to [11].

Theorem 8. *Let us fix $C > 0$ and $\epsilon \in (0, 1/6)$. Then for any $\mathbf{z} \in X_1$, and for any choice of HLC on X_1 centered at $\mathbf{z}$, there exist polynomials P_j of degree $\leq 3j$ and parity j on $T_{[\mathbf{z}]}F_n \times T_{[\mathbf{z}]}F_n \cong \mathbb{R}^{2n-2} \times \mathbb{R}^{2n-2}$, such that the following holds. Uniformly in $\mathbf{v}_1, \mathbf{v}_2 \in T_{[\mathbf{z}]}F_n$ with $\|\mathbf{v}_j\| \leq Ck^{\epsilon}$ for $j = 1, 2$, and $\theta_1, \theta_2 \in (-\pi, \pi)$, one has for $k \to +\infty$ the following asymptotic expansion:*

$$\Pi_{1,k}\left(\mathbf{z} + \left(\theta_1, \frac{\mathbf{v}_1}{\sqrt{k}}\right), \mathbf{z} + \left(\theta_2, \frac{\mathbf{v}_2}{\sqrt{k}}\right)\right) \sim \left(\frac{k}{\pi}\right)^{n-1} \cdot e^{ik(\theta_1-\theta_2)+\psi_2(\mathbf{v}_1,\mathbf{v}_2)}\left[1 + \sum_{j=1}^{+\infty} k^{-j/2} P_j(\mathbf{v}_1, \mathbf{v}_2)\right]. \quad (47)$$

In the given range, the above is an asymptotic expansion, since

$$\left|k^{-j/2} P_j(\mathbf{v}_1, \mathbf{v}_2)\right| \leq C_j k^{-(j/2)(1-6\epsilon)}. \quad (48)$$

2.5. $\mathscr{P}_k$ and $\Pi_{\sqrt{2},k}$. As discussed in [9], the push-forward operator $\nu_* : \mathscr{C}^{\infty}(X_{\sqrt{2}}) \to \mathscr{C}^{\infty}(S^n)$ yields by restriction an algebraic isomorphism

$$\mathscr{C}^{\infty}(X_{\sqrt{2}}) \cap H(X_{\sqrt{2}}) \longrightarrow \mathscr{C}^{\infty}(S^n), \quad (49)$$

for every k, and (49) yields by restriction an isomorphism

$$H_k(X_{\sqrt{2}}) \longrightarrow V_k, \quad (50)$$

which is unitary up to a dilation by a constant factor $C_{k,n} > 0$. Thus, we have

$$\|\nu_*(s)\|_{L^2(S^n)} = C_{k,n}\|s\|_{H(X_{\sqrt{2}})} \quad (s \in H_k(X_{\sqrt{2}})). \quad (51)$$

Therefore, if $(\sigma_{kj})_{j=0}^{N_k}$ is an orthonormal basis of $H_k(X_{\sqrt{2}})$, then

$$\left(C_{k,n}^{-1} \cdot \nu_*(\sigma_{kj})\right)_{j=0}^{N_k} \quad (52)$$

is an orthonormal basis of V_k. It follows that $\mathscr{P}_{k,n}$ in (2) is given by

$$\mathscr{P}_{k,n} = \frac{1}{C_{k,n}^2}(\nu \times \nu)_*\left(\Pi_{\sqrt{2},k}\right), \quad (53)$$

where $\nu \times \nu : X_{\sqrt{2}} \times X_{\sqrt{2}} \to S^n \times S^n$ is the product projection.

More explicitly, for $\mathbf{q} \in S^n$ let $S(\mathbf{q}^{\perp}) \cong S^{n-1}$ be the unit sphere centered at the origin in the orthocomplement $\mathbf{q}^{\perp}$, and let $dV_{S(\mathbf{q}^{\perp})}$ be the Riemannian density on $S(\mathbf{q}^{\perp})$; then

$$\mathscr{P}_{k,n}(\mathbf{q}_0, \mathbf{q}_1) = \frac{1}{C_{k,n}^2}\int_{S(\mathbf{q}_0^{\perp})}\int_{S(\mathbf{q}_1^{\perp})} \Pi_{\sqrt{2},k}\left(\mathbf{q}_0 + i\mathbf{p}, \mathbf{q}_1 + i\mathbf{p}'\right) dV_{S(\mathbf{q}_0^{\perp})}(\mathbf{p})\, dV_{S(\mathbf{q}_1^{\perp})}(\mathbf{p}'). \quad (54)$$

2.6. $\Pi_{r,k}$ and Conjugation. Conjugation $\sigma : \mathbf{z} \mapsto \overline{\mathbf{z}}$ in $\mathbb{C}^{n+1}$ leaves invariant the affine cone $\mathscr{C}_n$ and every X_r. Furthermore, it yields a Riemannian isometry of X_r into itself. For $f \in \mathscr{O}(\mathscr{C}_n \setminus \{\mathbf{0}\})$, let us set

$$f^{\sigma}(\mathbf{z}) =: \overline{f(\overline{\mathbf{z}})}. \quad (55)$$

If $f \in \mathscr{O}_k(\mathscr{C}_n \setminus \{\mathbf{0}\})$, then $f^{\sigma} \in \mathscr{O}_k(\mathscr{C}_n \setminus \{\mathbf{0}\})$.

Hence, if $(s_{kj})_j \subseteq \mathscr{O}_k(\mathscr{C}_n \setminus \{\mathbf{0}\})$ yields by restriction an orthonormal basis of $H_k(X_r)$, then so does $(s_{kj}^{\sigma})_j$. Thus, for any $\mathbf{z}_0, \mathbf{z}_1 \in X_r$, we have

$$\begin{aligned}\Pi_{r,k}(\overline{\mathbf{z}}_0, \overline{\mathbf{z}}_1) &= \sum_j s_{kj}(\overline{\mathbf{z}}_0) \cdot \overline{s_{kj}(\overline{\mathbf{z}}_1)} \\ &= \sum_j \overline{s_{kj}^{\sigma}(\mathbf{z}_0)} \cdot s_{kj}^{\sigma}(\mathbf{z}_1) = \Pi_{rk}(\mathbf{z}_1, \mathbf{z}_0) \\ &= \overline{\Pi_{rk}(\mathbf{z}_0, \mathbf{z}_1)}.\end{aligned} \quad (56)$$

3. Proof of Theorem 1

Proof of Theorem 1. Given $\mathbf{q}_0, \mathbf{q}_1 \in S^{n-1}$ with $\mathbf{q}_1 \neq \pm\mathbf{q}_0$, let γ_+ be the unique unit speed geodesic on S^n such that $\gamma_+(0) = \mathbf{q}_0$ and $\gamma_+(\vartheta) = \mathbf{q}'$ for some $\vartheta \in (0, \pi)$. Then

$$\begin{aligned}\mathbf{p}_0 &:= \dot{\gamma}_+(0) \in S^{n-1}(\mathbf{q}_0^{\perp}), \\ \mathbf{p}_1 &:= \dot{\gamma}_+(\vartheta) \in S^{n-1}(\mathbf{q}_1^{\perp}).\end{aligned} \quad (57)$$

The reverse geodesic $\gamma_-(\vartheta) =: \gamma(-\vartheta)$ satisfies $\gamma_-(0) = \mathbf{q}$, $\dot{\gamma}_-(0) = -\mathbf{p}_0$ and $\gamma_-(\vartheta') = \mathbf{q}'$ for a unique $\vartheta' = -\vartheta \in (-\pi, 0)$.

Although they project down to the same locus in S^n, γ_+ and γ_- correspond to distinct fibers of the circle bundle projection $\pi : X(\sqrt{2}) \to F_n$. Let us express the (co)tangent lift $\widetilde{\gamma}_{\pm}$ of the geodesics $\gamma_{\pm}$ in complex coordinates, and set $\mathbf{p}_1 = \dot{\gamma}_+(\vartheta)$. Then

$$\widetilde{\gamma}_{\pm}(\theta) = \gamma_{\pm}(\theta) + i\dot{\gamma}_{\pm}(\theta) = e^{-i\theta}(\mathbf{q}_0 \pm i\mathbf{p}_0) = \mathbf{q}_1 \pm i\mathbf{p}_1. \quad (58)$$

In view of (26), we have

$$\mathbf{q}_0 \pm i\mathbf{p}_0, \mathbf{q}_1 \pm i\mathbf{p}_1 \in \pi_{\sqrt{2}}^{-1}([\mathbf{q}_0 \pm i\mathbf{p}_0]). \quad (59)$$

On the other hand, $[\mathbf{q}_0 + i\mathbf{p}_0] \neq [\mathbf{q}_0 - i\mathbf{p}_0] \in F_n$, since $\mathbf{q}_0 + i\mathbf{p}_0$ and $\mathbf{q}_0 - i\mathbf{p}_0$ are linearly independent in $\mathbb{C}^{n+1}$.

Thus, we have the following.

Lemma 9. *Suppose $\mathbf{q}_0, \mathbf{q}_1 \in S^n$ and $\mathbf{q}_1 \neq \pm\mathbf{q}_0$. Then the only points $[\mathbf{z}] \in F_n$ such that*

$$\begin{aligned}\nu^{-1}(\mathbf{q}_0) \cap \pi_{\sqrt{2}}^{-1}([\mathbf{z}]) &\neq \emptyset, \\ \nu^{-1}(\mathbf{q}_1) \cap \pi_{\sqrt{2}}^{-1}([\mathbf{z}]) &\neq \emptyset\end{aligned} \quad (60)$$

are

$$\begin{aligned}[\mathbf{z}_+] &= [\mathbf{q}_0 + i\mathbf{p}_0], \\ [\mathbf{z}_-] &= [\mathbf{q}_0 - i\mathbf{p}_0].\end{aligned} \quad (61)$$

By Theorem 7, for fixed $\mathbf{p}$ and $\mathbf{p}'$ and $k \to +\infty$, we have

$$\Pi_{\sqrt{2},k}\left(\mathbf{q}_0 + i\mathbf{p}, \mathbf{q}_1 + i\mathbf{p}'\right) = O\left(k^{-\infty}\right), \tag{62}$$

unless $\mathbf{p} = \pm\mathbf{p}_0$ and $\mathbf{p}' = \pm\mathbf{p}_1$. Therefore, for a fixed $\vartheta \in (0,\pi)$ integration in (54) may be localized in a small neighborhood of $(\pm\mathbf{p}_0, \pm\mathbf{p}_1)$, perhaps at the cost of disregarding a negligible contribution to the asymptotics.

Since however we are allowing ϑ to approach 0 or π at a controlled rate, we need to give a more precise quantitative estimate of how small the previous neighborhood may be chosen when $k \to +\infty$.

To this end, let us introduce some further notation. Given linearly independent $\mathbf{a}, \mathbf{b} \in S^n$, let us set

$$\begin{aligned} R(\mathbf{a}, \mathbf{b}) &=: \operatorname{span}_{\mathbb{R}}(\mathbf{a}, \mathbf{b}) \subseteq \mathbb{R}^{n+1}, \\ R(\mathbf{a}, \mathbf{b})_{\mathbb{C}} &:= R(\mathbf{a}, \mathbf{b}) \otimes \mathbb{C} = \operatorname{span}_{\mathbb{C}}(\mathbf{a}, \mathbf{b}) \subseteq \mathbb{C}^{n+1}. \end{aligned} \tag{63}$$

Furthermore, for $\|\mathbf{v}\| \le 1$, we shall set

$$S_{\pm}(\mathbf{v}) =: -1 \pm \sqrt{1 - \|\mathbf{v}\|^2}. \tag{64}$$

A straightforward computation yields the following.

Lemma 10. *Assume that $\mathbf{q}_0 + i\mathbf{p}_0 \in X_{\sqrt{2}}$ and $\mathbf{q}_1 + i\mathbf{p}_1 = e^{-i\vartheta}(\mathbf{q}_0 + i\mathbf{p}_0)$ with $\vartheta \in (0,\pi)$. Then any $\mathbf{p} \in S^{n-1}(\mathbf{q}_0^{\perp})$ with $\mathbf{p}_0^t\mathbf{p} \ge 0$, respectively, $\mathbf{p}_0^t\mathbf{p} \le 0$, may be written uniquely in the form*

$$\mathbf{p} = \left(1 + S_+(\mathbf{v})\right)\mathbf{p}_0 + \mathbf{v}, \tag{65}$$

respectively

$$\mathbf{p} = \left(1 + S_-(\mathbf{v})\right)\mathbf{p}_0 + \mathbf{v}, \tag{66}$$

where $\mathbf{v} \in \mathbf{q}_0^{\perp} \cap \mathbf{p}_0^{\perp} = R(\mathbf{q}_0, \mathbf{q}_1)^{\perp}$ (the Euclidean orthocomplement) has norm ≤ 1 and

$$S_{\pm}(\mathbf{v}) =: -1 \pm \sqrt{1 - \|\mathbf{v}\|^2}. \tag{67}$$

Proposition 11. *Let us fix $C > 0$, $\delta \in (0, 1/6)$ and $\epsilon > \delta$. Then there exist constants $D, \epsilon_1 > 0$ such that the following holds. Suppose that*

(1) $Ck^{-\delta} < \vartheta < \pi - Ck^{-\delta}$;

(2) $\mathbf{q}_j + i\mathbf{p}_j \in X_{\sqrt{2}}$ *for* $j = 0, 1$;

(3) $\mathbf{q}_1 + i\mathbf{p}_1 = e^{-i\vartheta}(\mathbf{q}_0 + i\mathbf{p}_0)$;

(4) $\mathbf{v}_j \in \mathbf{q}_0^{\perp} \cap \mathbf{q}_1^{\perp}$ *for* $j = 0, 1$;

(5) $1 \ge \max\{\|\mathbf{v}_0\|, \|\mathbf{v}_1\|\} \ge Ck^{\epsilon - 1/2}$;

(6) $\mathbf{p}_j' = (1 + S_j(\mathbf{v}_j))\mathbf{p}_j + \mathbf{v}_j \in S^{n-1}(\mathbf{q}_j^{\perp})$ *for* $j = 0, 1$, *where* S_j *can be either one of* $S_{\pm}$ *(Lemma 10).*

Then

$$\operatorname{dist}_{F_n}\left(\left[\mathbf{q}_0 + i\mathbf{p}_0'\right], \left[\mathbf{q}_1 + i\mathbf{p}_1'\right]\right) \ge Dk^{\epsilon_1 - 1/2} \tag{68}$$

for every $k \gg 0$.

In view of Theorem 7, Proposition 11 implies the following.

Corollary 12. *Uniformly in the range of Proposition 11, we have*

$$\Pi_{\sqrt{2},k}\left(\mathbf{q}_0 + i\mathbf{p}_0', \mathbf{q}_1 + i\mathbf{p}_1'\right) = O\left(k^{-\infty}\right). \tag{69}$$

Proof of Proposition 11. Let us set for $\gamma \in [-\pi, \pi]$

$$\Phi\left(\gamma, \mathbf{p}_0', \mathbf{p}_1'\right) =: e^{-i\gamma}\left(\mathbf{q}_0 + i\mathbf{p}_0'\right) - \left(\mathbf{q}_1 + i\mathbf{p}_1'\right). \tag{70}$$

Let $\operatorname{dist}_{F_n}$ be the restriction to F_n of the distance function on $\mathbb{P}^n$. Then

$$\begin{aligned} &\operatorname{dist}_{F_n}\left(\left[\mathbf{q}_0 + i\mathbf{p}_0'\right], \left[\mathbf{q}_1 + i\mathbf{p}_1'\right]\right) \\ &\quad = \frac{1}{\sqrt{2}} \min\left\{\left\|\Phi\left(\gamma, \mathbf{p}_0', \mathbf{p}_1'\right)\right\| : \gamma \in [0, 2\pi]\right\}. \end{aligned} \tag{71}$$

The factor in front is needed because while the Hopf map $S_1^{2n+1} \to \mathbb{P}^n$ is a Riemannian submersion, the projection $S_{\sqrt{2}}^{2n+1} \to \mathbb{P}^n$ is so only after a constant rescaling of the metric.

We are reduced to proving that in the given range there exist constants $D, \epsilon_1 > 0$ such that for every $k \gg 0$ and $\gamma \in [0, 2\pi]$

$$\left\|\Phi\left(\gamma, \mathbf{p}_0', \mathbf{p}_1'\right)\right\| \ge Dk^{\epsilon_1 - 1/2}. \tag{72}$$

We have

$$\begin{aligned} \Phi\left(\gamma, \mathbf{p}_0', \mathbf{p}_1'\right) &= e^{-i\gamma}\left(\mathbf{q}_0 + i\mathbf{p}_0 + iS_0(\mathbf{v}_0)\mathbf{p}_0 + i\mathbf{v}_0\right) \\ &\quad - \left(e^{-i\vartheta}\left(\mathbf{q}_0 + i\mathbf{p}_0\right) + iS_1(\mathbf{v}_1)\mathbf{p}_1 + i\mathbf{v}_1\right) \\ &= \left(A\mathbf{q}_0 + B\mathbf{p}_0\right) + i\left[e^{-i\gamma}\mathbf{v}_0 - \mathbf{v}_1\right], \end{aligned} \tag{73}$$

where

$$\begin{aligned} A &:= \left(e^{-i\gamma} - e^{-i\vartheta}\right) + iS_1(\mathbf{v}_1)\sin(\vartheta) = \cos(\gamma) - \cos(\vartheta) \\ &\quad + i\left[-\sin(\gamma) + \sin(\vartheta)\left(1 + S_1(\mathbf{v}_1)\right)\right], \end{aligned} \tag{74}$$

$$\begin{aligned} B &:= i\left(e^{-i\gamma} - e^{-i\vartheta}\right) + i\left(e^{-i\gamma}S_0(\mathbf{v}_0) - S_1(\mathbf{v}_1)\cos(\vartheta)\right) \\ &= \sin(\gamma)\left(1 + S_0(\mathbf{v}_0)\right) - \sin(\vartheta) + i\left(\cos(\gamma)S_0(\mathbf{v}_0)\right. \\ &\quad \left. - S_1(\mathbf{v}_1)\cos(\vartheta) + \cos(\gamma) - \cos(\vartheta)\right). \end{aligned} \tag{75}$$

Regarding the two summands on the last line of (73), we have

$$\begin{aligned} &A\mathbf{q}_0 + B\mathbf{p}_0 \in R\left(\mathbf{q}_0, \mathbf{q}_1\right)_{\mathbb{C}}, \\ &\qquad i\left[e^{-i\gamma}\mathbf{v}_0 - \mathbf{v}_1\right] \in R\left(\mathbf{q}_0, \mathbf{q}_1\right)_{\mathbb{C}}^{\perp_h}, \end{aligned} \tag{76}$$

where $\perp_h$ denotes the Hermitian orthocomplement. Hence,

$$\begin{aligned} \left\|\Phi\left(\gamma, \mathbf{p}_0', \mathbf{p}_1'\right)\right\|^2 &\ge \left\|e^{-i\gamma}\mathbf{v}_0 - \mathbf{v}_1\right\|^2 \\ &\ge \left(1 - \left|\cos(\gamma)\right|\right)\left[\left\|\mathbf{v}_0\right\|^2 + \left\|\mathbf{v}_1\right\|^2\right]. \end{aligned} \tag{77}$$

Since $1 - |\cos(\gamma)|$ vanishes exactly to second order at $\gamma = 0, \pi, 2\pi$, there exists $D > 0$ such that for $\gamma \in [0, 2\pi]$, we have

$$1 - \left|\cos(\gamma)\right| \geq D^2 \min\left\{\gamma^2, (\gamma-\pi)^2, (\gamma-2\pi)^2\right\}. \tag{78}$$

Given this and (77), we conclude that, under the present hypothesis,

$$\begin{aligned}\left\|\Phi\left(\gamma, \mathbf{p}_0', \mathbf{p}_1'\right)\right\| &\geq D \min\left\{\gamma, \left|\gamma-\pi\right|, 2\pi-\gamma\right\} \max\left\{\|\mathbf{v}\|, \left\|\mathbf{v}'\right\|\right\} \\ &\geq CD \min\left\{\gamma, \left|\gamma-\pi\right|, 2\pi-\gamma\right\} k^{\epsilon-1/2}.\end{aligned} \tag{79}$$

Let us now pick δ' with $\epsilon > \delta' > \delta$ and assume

$$\min\left\{\gamma, \left|\gamma-\pi\right|, 2\pi-\gamma\right\} \geq k^{-\delta'}. \tag{80}$$

Then

$$\left\|\Phi\left(\gamma, \mathbf{p}, \mathbf{p}'\right)\right\| \geq CDk^{(\epsilon-\delta')-1/2}. \tag{81}$$

This establishes (72) with $\epsilon_1 = \epsilon - \delta'$, in the case where (80) holds. Thus, we are reduced to assuming

$$\min\left\{\gamma, \left|\gamma-\pi\right|, 2\pi-\gamma\right\} \leq k^{-\delta'}. \tag{82}$$

Then we also have $|\sin(\gamma)| \leq k^{-\delta'}$. Let us then look at the first summand on the last line of (73). We have an Hermitian orthogonal direct sum

$$\begin{aligned}R\left(\mathbf{q}_0, \mathbf{q}_1\right)_{\mathbb{C}} &= R\left(\mathbf{q}_0, \mathbf{p}_0\right)_{\mathbb{C}} \\ &= \operatorname{span}_{\mathbb{C}}\left(\mathbf{q}_0\right) \oplus \operatorname{span}_{\mathbb{C}}\left(\mathbf{p}_0\right).\end{aligned} \tag{83}$$

On the other hand, since $\sin(\vartheta)$ vanishes exactly to first order at $\vartheta = 0$ and $\vartheta = \pi$, there exists $E > 0$ such that for $\vartheta \in (0, \pi)$ under the assumptions of the lemma we have

$$\sin(\vartheta) \geq E \min\{\vartheta, \pi-\vartheta\} \geq ECk^{-\delta}. \tag{84}$$

Hence, in view of (75), we have for some $D_1 > 0$ and $k \gg 0$

$$\begin{aligned}\left\|\Phi\left(\gamma, \mathbf{p}, \mathbf{p}'\right)\right\| &\geq \left|A\mathbf{q}_0 + B\mathbf{p}_0\right| \geq |B| \geq |\Re(B)| \\ &= \left|\sin(\gamma)\left(1 + S_0\left(\mathbf{v}_0\right)\right) - \sin(\vartheta)\right| \\ &\geq |\sin(\vartheta)| - k^{-\delta'} \geq ECk^{-\delta} - k^{-\delta'} \\ &\geq \frac{1}{2}ECk^{-\delta} \geq \frac{1}{2}ECk^{-1/6},\end{aligned} \tag{85}$$

since $\delta' > \delta$ and $\delta < 1/6$. This establishes (72) with $\epsilon_1 = 1/3$ when (82) holds.

The proof of Proposition 11 is complete. □

Equations (65) and (66) parametrize neighborhoods of $\mathbf{p}_0$ and $-\mathbf{p}_0$, respectively. Therefore, Proposition 11 implies that in (54) only a negligible contribution to the asymptotics is lost, if integration in $\mathbf{p}$ and $\mathbf{p}'$ yields by restriction the shrinking neighborhoods of $\pm\mathbf{p}_0$ and $\pm\mathbf{p}_1$, of radii $O(k^{\epsilon-1/2})$.

This may be rephrased as follows. Let $\varrho \in \mathscr{C}_0^\infty(\mathbb{R}^{n+1})$ be even, supported in a small neighborhood of the origin, and identically equal to one in a smaller neighborhood of the origin. Then the asymptotics of (54) are unchanged, if the integrand is multiplied by

$$\begin{aligned}&\left[\varrho\left(k^{1/2-\epsilon}\left(\mathbf{p}-\mathbf{p}_0\right)\right) + \varrho\left(k^{1/2-\epsilon}\left(\mathbf{p}+\mathbf{p}_0\right)\right)\right] \\ &\quad \cdot \left[\varrho\left(k^{1/2-\epsilon}\left(\mathbf{p}'-\mathbf{p}_1\right)\right) + \varrho\left(k^{1/2-\epsilon}\left(\mathbf{p}'+\mathbf{p}_1\right)\right)\right].\end{aligned} \tag{86}$$

In this way, the integrand splits into four summands. In fact, only two of these are nonnegligible for $k \to +\infty$. Namely, consider the summand containing the factor

$$\varrho\left(k^{1/2-\epsilon}\left(\mathbf{p}-\mathbf{p}_0\right)\right)\varrho\left(k^{1/2-\epsilon}\left(\mathbf{p}'+\mathbf{p}_1\right)\right). \tag{87}$$

On its support, $\mathbf{p}$ lies in a shrinking neighborhood of $\mathbf{p}_0$ and $\mathbf{p}'$ in a shrinking neighborhood of $-\mathbf{p}_1$. Therefore, on the same support $\mathbf{q}_0 + i\mathbf{p}$ lies in a shrinking neighborhood of $\mathbf{q}_0 + i\mathbf{p}_0$, and $\mathbf{q}_1 - i\mathbf{p}'$ lies in a shrinking neighborhood of $\mathbf{q}_1 - i\mathbf{p}_1$. Since

$$\begin{aligned}&\frac{1}{\sqrt{2}}\left(\mathbf{q}_0 + i\mathbf{p}_0\right) \wedge \frac{1}{\sqrt{2}}\left(\mathbf{q}_1 - i\mathbf{p}_1\right) \\ &\quad = \frac{1}{2}\left(\mathbf{q}_0 + i\mathbf{p}_0\right) \wedge e^{i\vartheta}\left(\mathbf{q}_0 - i\mathbf{p}_0\right) = -i\mathbf{q}_0 \wedge \mathbf{p}_0\end{aligned} \tag{88}$$

has unit norm, on the support of (87) $[\mathbf{q}_0 + i\mathbf{p}]$ and $[\mathbf{q}_1 + i\mathbf{p}']$ remain at a distance $\geq 2/3$, say, in projective space. This implies that as $k \to +\infty$

$$\Pi_{\sqrt{2},k}\left(\mathbf{q}_0 + i\mathbf{p}, \mathbf{q}_1 + i\mathbf{p}'\right) = O\left(k^{-\infty}\right) \tag{89}$$

uniformly in $(\mathbf{p}, \mathbf{p}')$ in the support of (87). A similar argument applies to the summand containing the factor

$$\varrho\left(k^{1/2-\epsilon}\left(\mathbf{p}+\mathbf{p}_0\right)\right)\varrho\left(k^{1/2-\epsilon}\left(\mathbf{p}'-\mathbf{p}_1\right)\right). \tag{90}$$

Thus, we may rewrite (54) as follows:

$$\mathscr{P}_{k,n}\left(\mathbf{q}_0, \mathbf{q}_1\right) \sim \mathscr{P}_{k,n}\left(\mathbf{q}_0, \mathbf{q}_1\right)_+ + \mathscr{P}_{k,n}\left(\mathbf{q}_0, \mathbf{q}_1\right)_-, \tag{91}$$

where

$$\begin{aligned}\mathscr{P}_{k,n}\left(\mathbf{q}_0, \mathbf{q}_1\right)_\pm &:= \frac{1}{C_{k,n}^2} \int_{S(\mathbf{q}_0^\perp)} \int_{S(\mathbf{q}_1{}^\perp)} \varrho\left(k^{1/2-\epsilon}\left(\mathbf{p} \mp \mathbf{p}_0\right)\right) \\ &\quad \cdot \varrho\left(k^{1/2-\epsilon}\left(\mathbf{p}' \mp \mathbf{p}_1\right)\right) \cdot \Pi_{\sqrt{2},k}\left(\mathbf{q}_0 + i\mathbf{p}, \mathbf{q}_1\right. \\ &\quad \left. + i\mathbf{p}'\right) dV_{S(\mathbf{q}_0^\perp)}(\mathbf{p})\, dV_{S(\mathbf{q}_1{}^\perp)}\left(\mathbf{p}'\right).\end{aligned} \tag{92}$$

As a further reduction, we need only deal with one of $\mathscr{P}_{k,n}(\mathbf{q}_0, \mathbf{q}_1)_\pm$.

Lemma 13. $\mathscr{P}_{k,n}(\mathbf{q}_0, \mathbf{q}_1)_\pm = \overline{\mathscr{P}_k(\mathbf{q}_0, \mathbf{q}_1)_\mp}$.

Proof of Lemma 13. Let us apply the change of integration variable $\mathbf{p} \mapsto -\mathbf{p}$ and $\mathbf{p}' \mapsto -\mathbf{p}'$ and apply (56). Since ϱ is even, we get

$$\begin{aligned}\mathscr{P}_{k,n}(\mathbf{q}_0,\mathbf{q}_1)_- &= \frac{1}{C_{k,n}^2}\int_{S(\mathbf{q}_0^\perp)}\int_{S(\mathbf{q}_1{}^\perp)} \varrho\left(k^{1/2-\epsilon}(\mathbf{p}-\mathbf{p}_0)\right) \\ &\cdot \varrho\left(k^{1/2-\epsilon}(\mathbf{p}'-\mathbf{p}_1)\right) \\ &\cdot \overline{\Pi_{\sqrt{2},k}(\mathbf{q}_0+i\mathbf{p},\mathbf{q}_1+i\mathbf{p}')}dV_{S(\mathbf{q}_0^\perp)}(\mathbf{p})\,dV_{S(\mathbf{q}_1{}^\perp)}(\mathbf{p}') \\ &= \overline{\mathscr{P}_k(\mathbf{q}_0,\mathbf{q}_1)_+}.\end{aligned} \tag{93}$$

□

Lemma 13 and (91) imply

$$\mathscr{P}_{k,n}(\mathbf{q}_0,\mathbf{q}_1) \sim 2\Re\left(\mathscr{P}_k(\mathbf{q}_0,\mathbf{q}_1)_+\right). \tag{94}$$

In the definition of $\mathscr{P}_k(\mathbf{q}_0,\mathbf{q}_1)_+$, integration is over a shrinking neighborhood of $(\mathbf{p}_0,\mathbf{p}_1) \in S(\mathbf{q}_0^\perp)\times S(\mathbf{q}_1{}^\perp)$. We can thus make use of the parametrization (65) and write in (92):

$$\begin{aligned}\mathbf{p} &= \mathbf{p}_0 + A(\mathbf{v}_0), \\ \mathbf{p}' &= \mathbf{p}_1 + A(\mathbf{v}_1),\end{aligned} \tag{95}$$

where we have set

$$A(\mathbf{v}_j) =: \mathbf{v}_j + S_+(\mathbf{v}_j)\,\mathbf{p}_j. \tag{96}$$

It is also harmless to replace $\mathbf{p}-\mathbf{p}_j$ by $\mathbf{v}_j$ in the rescaled cut-offs in (92). Let us also set $\mathbf{z}_j = \mathbf{q}_j + i\mathbf{p}_j$ and recall that $\mathbf{z}_1 = e^{-i\vartheta}\mathbf{z}_0$. We then obtain

$$\begin{aligned}&\mathscr{P}_{k,n}(\mathbf{q}_0,\mathbf{q}_1)_+ \\ &\quad= \frac{1}{C_{k,n}^2}\int_{\mathbf{q}_0^\perp\cap\mathbf{q}_1^\perp}\int_{\mathbf{q}_0^\perp\cap\mathbf{q}_1^\perp}\varrho\left(k^{1/2-\epsilon}\mathbf{v}_0\right)\varrho\left(k^{1/2-\epsilon}\mathbf{v}_1\right) \\ &\quad\cdot \Pi_{\sqrt{2},k}\left(\mathbf{z}_0 + iA(\mathbf{v}_0),\mathbf{z}_1 + iA(\mathbf{v}_1)\right) \\ &\quad\cdot \mathscr{V}(\mathbf{v}_0,\mathbf{v}_1)\,d\mathbf{v}_0 d\mathbf{v}_1,\end{aligned} \tag{97}$$

where $\mathscr{V}(\mathbf{0},\mathbf{0}) = 1$.

Let us pass to rescaled integration variables $\mathbf{v}_j \mapsto \mathbf{v}_j/\sqrt{k}$ in (97). Then

$$\begin{aligned}&\mathscr{P}_{k,n}(\mathbf{q}_0,\mathbf{q}_1)_+ \\ &\quad= \frac{k^{1-n}}{C_{k,n}^2}\int_{\mathbf{q}_0^\perp\cap\mathbf{q}_1^\perp}\int_{\mathbf{q}_0^\perp\cap\mathbf{q}_1^\perp}\varrho\left(k^{-\epsilon}\mathbf{v}_0\right)\varrho\left(k^{-\epsilon}\mathbf{v}_1\right) \\ &\quad\cdot \Pi_{\sqrt{2},k}\left(\mathbf{z}_0 + \frac{i}{\sqrt{k}}A_{0k}(\mathbf{v}_0),\mathbf{z}_1 + \frac{i}{\sqrt{k}}A_{1k}(\mathbf{v}_1)\right) \\ &\quad\cdot \mathscr{V}\left(\frac{\mathbf{v}_0}{\sqrt{k}},\frac{\mathbf{v}_1}{\sqrt{k}}\right)d\mathbf{v}_0 d\mathbf{v}_1,\end{aligned} \tag{98}$$

with

$$A_{jk}(\mathbf{v}) \coloneqq \mathbf{v} + \sqrt{k}\cdot S_+\left(\frac{\mathbf{v}}{\sqrt{k}}\right)\mathbf{p}_j. \tag{99}$$

Let us consider the Szegö term in the integrand. In view of (40), this is

$$\begin{aligned}&\Pi_{\sqrt{2},k}\left(\mathbf{z}_0 + \frac{i}{\sqrt{k}}A_{0k}(\mathbf{v}_0),\mathbf{z}_1 + \frac{i}{\sqrt{k}}A_{1k}(\mathbf{v}_1)\right) = \frac{\sqrt{2}}{2^n} \\ &\quad\cdot e^{ik\vartheta}\Pi_{1,k}\left(\frac{\mathbf{z}_0}{\sqrt{2}} + \frac{1}{\sqrt{k}}\frac{iA_{0k}(\mathbf{v}_0)}{\sqrt{2}},\frac{\mathbf{z}_0}{\sqrt{2}}\right. \\ &\quad\left. + \frac{1}{\sqrt{k}}\frac{ie^{i\vartheta}A_{1k}(\mathbf{v}_1)}{\sqrt{2}}\right).\end{aligned} \tag{100}$$

Now the sums in the previous expression are just algebraic sums in $\mathbb{C}^{n+1}$; in order to apply the scaling asymptotics of Theorem 8, we need to first express the argument of (100) in terms of local Heisenberg coordinates on X_1 centered at $\mathbf{z}_0/\sqrt{2}$.

Lemma 14. *Suppose $\mathbf{z} = \mathbf{q} + i\mathbf{p} \in X_1$ and choose a system of HLC on X_1 centered at $\mathbf{z}$. Then for $\delta\mathbf{p} \sim \mathbf{0} \in \mathbb{R}^{n+1}$ and $e^{i\vartheta} \in S^1$ such that $\mathbf{z} + ie^{i\vartheta}\delta\mathbf{p} \in X_1$ we have*

$$\mathbf{z} + ie^{i\vartheta}\delta\mathbf{p} = \mathbf{z} +_{X_1}\left(0, ie^{i\vartheta}\delta\mathbf{p} + R_2(\theta;\delta\mathbf{p})\right), \tag{101}$$

for a suitable smooth function $R_2(\theta;\cdot)$ vanishing to second order at the origin (in $\mathbf{v}$).

Proof of Lemma 14. In view of (45), it suffices to prove the statement on S_1^{2n+1}, working with the HLC (44). Since $\mathbf{z}, \mathbf{z} + ie^{i\vartheta}\delta\mathbf{p} \in \mathscr{C}_n$, we have

$$\begin{aligned}0 &= \mathbf{z}^t\mathbf{z} + 2ie^{i\vartheta}\mathbf{z}^t\delta\mathbf{p} - e^{2i\vartheta}\delta\mathbf{p}^t\delta\mathbf{p} \\ &= 2ie^{i\vartheta}\mathbf{z}^t\delta\mathbf{p} - e^{2i\vartheta}\|\delta\mathbf{p}\|^2,\end{aligned} \tag{102}$$

so that $i\mathbf{z}^t\delta\mathbf{p} = e^{i\vartheta}\|\delta\mathbf{p}\|^2/2$.

Let us look for $\beta > 0$ and $\mathbf{h} \in \mathbf{z}^{\perp_h}$ (Hermitian orthocomplement) such that

$$\mathbf{z} + ie^{i\vartheta}\delta\mathbf{p} = \beta(\mathbf{z} + \mathbf{h}). \tag{103}$$

If this is possible at all, then necessarily $\beta = 1/\|\mathbf{z} + \mathbf{h}\|$, as $\|\mathbf{z} + ie^{i\vartheta}\delta\mathbf{p}\| = 1$. Then

$$\mathbf{z} + ie^{i\vartheta}\delta\mathbf{p} = \mathbf{z} +_{S_1^{2n+1}}(0,\mathbf{h}). \tag{104}$$

Assuming that (103) may be solved, then, taking the Hermitian product with $\mathbf{z}$ on both sides of (103) and using (102), we get

$$\begin{aligned}\beta &= \mathbf{z}^t\left(\overline{\mathbf{z}} - ie^{-i\vartheta}\delta\mathbf{p}\right) = 1 - ie^{-i\vartheta}\mathbf{z}^t\delta\mathbf{p} = 1 - \frac{1}{2}\|\delta\mathbf{p}\|^2 \\ &> 0.\end{aligned} \tag{105}$$

With this value of β, let us set

$$\mathbf{h} \coloneqq \frac{1}{\beta}\left(\mathbf{z} + ie^{i\theta}\delta\mathbf{p}\right) - \mathbf{z}, \tag{106}$$

so that (103) is certainly satisfied. We need to verify that $\mathbf{h} \in \mathbf{z}^{\perp_h}$. Indeed we have

$$\mathbf{h}^t\overline{\mathbf{z}} = \frac{1}{\beta}\left(1 + ie^{i\theta}\delta\mathbf{p}^t\overline{\mathbf{z}}\right) - 1 = \frac{1}{\beta}\left(1 - \frac{1}{2}\|\delta\mathbf{p}\|^2\right) - 1 = 0. \tag{107}$$

Since $\mathbf{h} = ie^{i\theta}\delta\mathbf{p} + R_2(\delta\mathbf{p})$, the proof of the lemma is complete. □

Notice that $\mathbf{h}$ is given for $\delta\mathbf{p} \sim \mathbf{0}$ by an asymptotic expansion in homogeneous polynomials of increasing degree in $\delta\mathbf{p}$ of the form

$$\mathbf{h} \sim ie^{i\theta}\delta\mathbf{p} + \frac{1}{2}\|\delta\mathbf{p}\|^2\mathbf{z} + \frac{i}{2}e^{i\theta}\|\delta\mathbf{p}\|^2\delta\mathbf{p} + \frac{1}{4}\|\delta\mathbf{p}\|^4\mathbf{z} + \cdots. \tag{108}$$

This holds on S_1^{2n+1}, but a similar expansion obviously holds on X_1, possibly with modified terms in higher degree.

Let us apply Lemma 14 with $\mathbf{z} = \mathbf{z}_0/\sqrt{2}$ and $\delta\mathbf{p}_j = e^{i\theta}(A_{jk}(\mathbf{v}_j)/\sqrt{2})/\sqrt{k}$ (we will set $\theta = 0$ for $j = 0$ and $\theta = \vartheta$ for $j = 1$). To this end, let us note that in view of (99) for $k \to +\infty$ there is an asymptotic expansion of the form

$$A_{jk}(\mathbf{v}) \sim \sum_{j\geq 0}\frac{1}{k^{l/2}}P_{j,l+1}(\mathbf{v}), \tag{109}$$

where $P_{j,l}$ is a homogeneous (vector valued) polynomial function of degree l and $P_{j1}(\mathbf{v}) = \mathbf{v}$. Hence,

$$\delta\mathbf{p}_j = \frac{e^{i\theta}}{\sqrt{2k}}A_{jk}(\mathbf{v}_j) \sim \frac{e^{i\theta}}{\sqrt{2}}\sum_{j\geq 0}\frac{1}{k^{(l+1)/2}}P_{j,l+1}(\mathbf{v}_j) = \frac{e^{i\theta}}{\sqrt{k}}\frac{\mathbf{v}_j}{\sqrt{2}} + \cdots. \tag{110}$$

Making use of (110) in (108) we obtain

$$\mathbf{h}_{kj} \sim \sum_{l\geq 1}\frac{1}{k^{l/2}}Q_{jl}(\theta;\mathbf{v}) = \frac{1}{\sqrt{k}}\left(ie^{i\theta}\frac{\mathbf{v}_j}{\sqrt{2}} + \sum_{l\geq 1}\frac{1}{k^{l/2}}Q_{j,l+1}(\theta;\mathbf{v}_j)\right), \tag{111}$$

where $Q_{jl}(\theta;\cdot)$ is a homogeneous polynomial function of degree l and we have emphasized the dependence on k.

Thus, we obtain for $j = 0$ (with $\theta = 0$) that

$$\frac{\mathbf{z}_0}{\sqrt{2}} + \frac{1}{\sqrt{k}}\frac{iA_{0k}(\mathbf{v}_0)}{\sqrt{2}} = \frac{\mathbf{z}_0}{\sqrt{2}} +_{X_1}(0, \mathbf{h}_{k0}), \tag{112}$$

where

$$\mathbf{h}_{k0} \sim \frac{1}{\sqrt{k}}\left(i\frac{\mathbf{v}_0}{\sqrt{2}} + \sum_{l\geq 1}\frac{1}{k^{l/2}}Q_{0,l+1}(0;\mathbf{v}_0)\right) = \frac{1}{\sqrt{k}}\mathbf{a}_{k0}, \tag{113}$$

with $\mathbf{a}_{k0}$ defined by the latter equality. Similarly, for $j = 1$ (with $\theta = \vartheta$), we have

$$\frac{\mathbf{z}_0}{\sqrt{2}} + \frac{1}{\sqrt{k}}\frac{ie^{i\vartheta}A_{1k}(\mathbf{v}_1)}{\sqrt{2}} = \frac{\mathbf{z}_0}{\sqrt{2}} +_{X_1}(0, \mathbf{h}_{k1}), \tag{114}$$

where

$$\mathbf{h}_{k1} \sim \frac{1}{\sqrt{k}}\left(ie^{i\vartheta}\frac{\mathbf{v}_1}{\sqrt{2}} + \sum_{l\geq 1}\frac{1}{k^{l/2}}Q_{1,l+1}(\vartheta;\mathbf{v}_1)\right) = \frac{1}{\sqrt{k}}\mathbf{a}_{k1}. \tag{115}$$

Let us return to (100). In view of Theorem 8, we get

$$\begin{aligned}\Pi_{1,k}&\left(\frac{\mathbf{z}_0}{\sqrt{2}} + \frac{1}{\sqrt{k}}\frac{iA_{0k}(\mathbf{v}_0)}{\sqrt{2}}, \frac{\mathbf{z}_0}{\sqrt{2}} + \frac{1}{\sqrt{k}}\frac{ie^{i\vartheta}A_{1k}(\mathbf{v}_1)}{\sqrt{2}}\right)\\ &= \Pi_{1,k}\left(\frac{\mathbf{z}_0}{\sqrt{2}} +_{X_1}\frac{1}{\sqrt{k}}\mathbf{a}_{k0}, \frac{\mathbf{z}_0}{\sqrt{2}} +_{X_1}\frac{1}{\sqrt{k}}\mathbf{a}_{k1}\right)\\ &\sim \left(\frac{k}{\pi}\right)^{n-1}e^{\psi_2(\mathbf{a}_{k0},\mathbf{a}_{k1})}\cdot\left[1 + \sum_{b=1}^{+\infty}k^{-b/2}P_b(\mathbf{a}_{k0},\mathbf{a}_{k1})\right].\end{aligned} \tag{116}$$

We have

$$\psi_2(\mathbf{a}_{k0},\mathbf{a}_{k1}) \sim \frac{1}{2}\psi_2(\mathbf{v}_0, e^{i\vartheta}\mathbf{v}_1) + \sum_{l\geq 1}\frac{1}{k^{l/2}}\widetilde{Q}_{l+2}(\vartheta;\mathbf{v}_0,\mathbf{v}_1), \tag{117}$$

where $\widetilde{Q}_l(\vartheta;\cdot,\cdot)$ is a homogeneous $\mathbb{C}$-valued polynomial of degree l. For any $r \geq 1$ and $l_1,\ldots,l_r \geq 1$, we have

$$\prod_{j=1}^{r}\frac{1}{k^{l_j/2}}\widetilde{Q}_{l_j+2}(\vartheta;\mathbf{v}_0,\mathbf{v}_1) = \frac{1}{k^{\sum_{j=1}^r l_j/2}}\widehat{Q}_{\sum_{j=1}^r l_j+2r}(\vartheta;\mathbf{v}_0,\mathbf{v}_1), \tag{118}$$

where $\widehat{Q}_{\sum_{j=1}^r l_j+2r}(\vartheta;\cdot,\cdot)$ is homogeneous of degree $\sum_{j=1}^r l_j + 2r$. Since $l_j \geq 1$ for every j, we have $\sum_{j=1}^r l_j + 2r \leq 3\sum_{j=1}^r l_j$.

One can see from this that

$$e^{\psi_2(\mathbf{a}_{k0},\mathbf{a}_{k1})} \sim e^{(1/2)\psi_2(\mathbf{v}_0,e^{i\vartheta}\mathbf{v}_1)}\left[1 + \sum_{l\geq 1}\frac{1}{k^{l/2}}B_l(\vartheta;\mathbf{v}_0,\mathbf{v}_1)\right], \tag{119}$$

where $B_l(\vartheta;\cdot,\cdot)$ is a polynomial of degree $\leq 3l$, having the same parity as l.

Similarly, recalling that P_b has the same parity as b and degree $\leq 3b$, each summand $k^{-b/2}P_b(\mathbf{a}_{k0}, \mathbf{a}_{k1})$ in (116) gives rise to an asymptotic expansion in terms of the form

$$\frac{1}{k^{b/2}}\prod_{a=1}^{r}\frac{1}{k^{l_a/2}}R_{l_a+1}(\vartheta;\mathbf{v}_0,\mathbf{v}_1) = \frac{1}{k^{(b+\sum_{a=1}^{r}l_a)/2}}\widetilde{R}_{\sum_{a=1}^{r}l_a+r}(\vartheta;\mathbf{v}_0,\mathbf{v}_1), \quad (120)$$

where $R_l(\vartheta;\cdot,\cdot)$ and $\widetilde{R}_l(\vartheta;\cdot,\cdot)$ are homogeneous polynomials of the given degree, $r \leq 3b$, and $b-r$ is even. Then $3(b+\sum_{a=1}^{r}l_a) \geq \sum_{a=1}^{r}l_a + r$, and $(b+\sum_{a=1}^{r}l_a) - (\sum_{a=1}^{r}l_a + r) = b-r$ is also even. Hence, each summand $k^{-b/2}P_b(\mathbf{a}_{k0},\mathbf{a}_{k1})$ $(b \geq 1)$ yields an asymptotic expansion of the form

$$\sum_{l\geq 1}k^{-l/2}T_{bl}(\vartheta;\mathbf{v}_0,\mathbf{v}_1), \quad (121)$$

where again each T_{bl} has the same parity as l and degree $\leq 3l$.

Putting this all together, we obtain an asymptotic expansion for the integrand in (98).

Lemma 15. *For $l \geq 0$, there exist polynomials $Z_l(\vartheta;\cdot,\cdot)$ of degree $\leq 3l$ and parity $(-1)^l$, with $Z_0(\vartheta;\cdot,\cdot)=1$, such that*

$$\Pi_{\sqrt{2},k}\left(\mathbf{z}_0+\frac{i}{\sqrt{k}}A_{0k}(\mathbf{v}_0),\mathbf{z}_1+\frac{i}{\sqrt{k}}A_{1k}(\mathbf{v}_1)\right)\cdot\mathscr{V}\left(\frac{\mathbf{v}_0}{\sqrt{k}},\frac{\mathbf{v}_1}{\sqrt{k}}\right)\sim\frac{\sqrt{2}}{2^n}e^{ik\vartheta}\left(\frac{k}{\pi}\right)^{n-1}\cdot e^{(1/2)\psi_2(\mathbf{v}_0,e^{i\vartheta}\mathbf{v}_1)}\sum_{l\geq 0}\frac{1}{k^{l/2}}Z_l(\vartheta;\mathbf{v}_0,\mathbf{v}_1). \quad (122)$$

Proof of Lemma 15. The previous arguments yield an asymptotic expansion of the given form for the first factor. We need only multiply the latter expansion by the Taylor expansion of the second factor. □

Since integration in (98) takes place over a poly-ball or radius $O(k^{\epsilon})$ in $(\mathbf{q}_0^{\perp}\cap\mathbf{q}_1^{\perp})^2$, the expansion may be integrated term by term. In addition, given that the exponent and the cut-offs are even functions of $(\mathbf{v}_0,\mathbf{v}_1)$, only terms of even parity yield a nonzero integral. Hence, we may discard the half-integer powers and obtain

$$\mathscr{P}_k(\mathbf{q}_0,\mathbf{q}_1)_+ \sim \frac{k^{1-n}}{C_{k,n}^2}\frac{\sqrt{2}}{2^n}e^{ik\vartheta}\left(\frac{k}{\pi}\right)^{n-1}\sum_{l\geq 0}k^{-l}\widehat{P}_l(\vartheta)_+, \quad (123)$$

where

$$\widehat{P}_l(\vartheta)_+ := \int_{\mathbf{q}_0^{\perp}\cap\mathbf{q}_1^{\perp}}\int_{\mathbf{q}_0^{\perp}\cap\mathbf{q}_1^{\perp}}\varrho(k^{-\epsilon}\mathbf{v}_0)\,\varrho(k^{-\epsilon}\mathbf{v}_1)\cdot e^{(1/2)\psi_2(\mathbf{v}_0,e^{i\vartheta}\mathbf{v}_1)}Z_{2l}(\vartheta;\mathbf{v}_0,\mathbf{v}_1)\,d\mathbf{v}_0 d\mathbf{v}_1. \quad (124)$$

We can slightly simplify the previous asymptotic expansion as follows. First, as emphasized the dependence on $(\mathbf{q}_0,\mathbf{q}_1)$ is of course only through the angle ϑ. In particular, in (124) nothing is lost by assuming that $\mathbf{q}_0$ and $\mathbf{q}_1$ span the 2-plane $\{\mathbf{0}\}\times\mathbb{R}^2\subseteq\mathbb{R}^{n+1}$, and therefore that $\mathbf{q}_0^{\perp}\cap\mathbf{q}_1^{\perp}=\mathbb{R}^{n-1}\times\{\mathbf{0}\}$.

Furthermore, given (46), we have

$$\psi_2(\mathbf{v}_0,e^{i\vartheta}\mathbf{v}_1) = -i\sin(\vartheta)\,\mathbf{v}_0{}^t\mathbf{v}_1 - \frac{1}{2}\left\|\mathbf{v}_0-\cos(\vartheta)\mathbf{v}_1\right\|^2 - \frac{1}{2}\sin(\vartheta)^2\left\|\mathbf{v}_1\right\|^2. \quad (125)$$

With the change of variables

$$\begin{pmatrix}\mathbf{v}_0\\ \mathbf{v}_1\end{pmatrix} = \sqrt{2}\begin{pmatrix}\mathbf{b}_0+\cot(\vartheta)\mathbf{b}_1\\ \left(\dfrac{1}{\sin(\vartheta)}\right)\mathbf{b}_1\end{pmatrix} \quad (126)$$

we obtain

$$\psi_2(\mathbf{v}_0,e^{i\vartheta}\mathbf{v}_1) = -\frac{1}{2}\left\|\mathbf{b}_0\right\|^2 - i\mathbf{b}_0^t\mathbf{b}_1 - \frac{1}{2}(1+2i\cot(\vartheta))\left\|\mathbf{b}_1\right\|^2. \quad (127)$$

Since $Z_{2l}(\vartheta,\cdot,\cdot)$ is even and has degree $\leq 6l$, we can write

$$Z_{2l}\left(\vartheta;\mathbf{b}_0+\cot(\vartheta)\mathbf{b}_1,\frac{\mathbf{b}_1}{\sin(\vartheta)}\right) = \frac{1}{\sin(\vartheta)^{6l}}T_l(\vartheta;\mathbf{b}_0,\mathbf{b}_1), \quad (128)$$

where $T_l(\vartheta;\cdot,\cdot)$ is an even polynomial of degree $\leq 6l$, with smooth bounded coefficients for $\vartheta\in[0,\pi]$. Thus,

$$\widehat{P}_l(\vartheta)_+ = \left(\frac{2}{\sin(\vartheta)}\right)^{n-1}\frac{1}{\sin(\vartheta)^{6l}}\cdot\int_{\mathbb{R}^{n-1}}\int_{\mathbb{R}^{n-1}}e^{-(1/2)\|\mathbf{b}_0\|^2-i\mathbf{b}_0^t\mathbf{b}_1-(1/2)(1+2i\cot(\vartheta))\|\mathbf{b}_1\|^2}\cdot\varrho\left(k^{-\epsilon}\sqrt{2}(\mathbf{b}_0+\cot(\vartheta)\mathbf{b}_1)\right)\cdot\varrho\left(k^{-\epsilon}\sqrt{2}\sin(\vartheta)^{-1}\mathbf{b}_1\right)T_l(\vartheta;\mathbf{b}_0,\mathbf{b}_1)\cdot d\mathbf{b}_0 d\mathbf{b}_1. \quad (129)$$

There is a constant $C>0$ such that the support of

$$1-\varrho\left(k^{-\epsilon}\sqrt{2}(\mathbf{b}_0+\cot(\vartheta)\mathbf{b}_1)\right)\cdot\varrho\left(k^{-\epsilon}\sqrt{2}\sin(\vartheta)^{-1}\mathbf{b}_1\right) \quad (130)$$

is contained in the locus, where $\|\mathbf{b}_0,\mathbf{b}_1\|\geq Ck^{\epsilon}\sin(\vartheta)$. Under the assumptions of the theorem, this implies, perhaps for a different constant $C>0$, that $\|\mathbf{b}_0,\mathbf{b}_1\|\geq Ck^{\epsilon-\delta}$. On the other hand, the exponent in (129) satisfies

$$\left|-\frac{1}{2}\left\|\mathbf{b}_0\right\|^2 - i\mathbf{b}_0^t\mathbf{b}_1 - \frac{1}{2}(1+2i\cot(\vartheta))\left\|\mathbf{b}_1\right\|^2\right| \leq -\frac{1}{2}\left(\left\|\mathbf{b}_0\right\|^2+\left\|\mathbf{b}_1\right\|^2\right). \quad (131)$$

Given that $\epsilon>\delta$ (statement of Proposition 11), we conclude that only a negligible contribution to the asymptotics is lost, if the cut-off function is omitted and integration is now extended to all of $\mathbb{R}^{n-1} \times \mathbb{R}^{n-1}$.

We can thus rewrite (123) as follows:

$$\mathscr{P}_k(\mathbf{q}_0, \mathbf{q}_1)_+ \sim \frac{1}{C_{k,n}^2} \frac{\sqrt{2}}{2} e^{ik\vartheta} \left(\frac{1}{\pi \sin(\vartheta)}\right)^{n-1} \cdot \sum_{l \geq 0} k^{-l} \frac{1}{\sin(\vartheta)^{6l}} \widetilde{P}_l(\vartheta)_+, \tag{132}$$

where

$$\widetilde{P}_l(\vartheta)_+ = \int_{\mathbb{R}^{n-1}} \int_{\mathbb{R}^{n-1}} e^{-(1/2)\|\mathbf{b}_0\|^2 - i\mathbf{b}_0^t \mathbf{b}_1 - (1/2)(1+2i\cot(\vartheta))\|\mathbf{b}_1\|^2} T_l(\vartheta; \mathbf{b}_0, \mathbf{b}_1)\, d\mathbf{b}_0 d\mathbf{b}_1. \tag{133}$$

Let us set $B_\vartheta = (1 + i\cot(\vartheta)) I_{n-1}$. The leading order coefficient is

$$\widetilde{P}_0(\vartheta)_+ = \int_{\mathbb{R}^{n-1}} e^{-(1/2)(1+2i\cot(\vartheta))\|\mathbf{b}_1\|^2} \cdot \left[\int_{\mathbb{R}^{n-1}} e^{-i\mathbf{b}_0^t \mathbf{b}_1 - (1/2)\|\mathbf{b}_0\|^2} d\mathbf{b}_0\right] d\mathbf{b}_1 = (2\pi)^{(n-1)/2} \cdot \int_{\mathbb{R}^{n-1}} e^{-(1/2)(2+2i\cot(\vartheta))\|\mathbf{b}_1\|^2} d\mathbf{b}_1 = 2^{(n-1)/2} \pi^{n-1} \cdot \frac{1}{\sqrt{\det(B_\vartheta)}} = \left(\sqrt{2}\pi\right)^{n-1} \cdot \sin(\vartheta)^{(n-1)/2} e^{i(\vartheta/2 - \pi/4)(n-1)}. \tag{134}$$

Given (134), (132), and (94), $\mathscr{P}_k(\mathbf{q}_0, \mathbf{q}_1)$ has an asymptotic expansion for $k \to +\infty$ with leading order term

$$\frac{2^{n/2}}{C_{k,n}^2} \frac{1}{\sin(\vartheta)^{(n-1)/2}} \cos\left(k\vartheta + \left(\frac{\vartheta}{2} - \frac{\pi}{2}\right)(n-1)\right). \tag{135}$$

For any l, we can write

$$\widetilde{P}_l(\vartheta)_+ = \int_{\mathbb{R}^{n-1}} e^{-(1/2)(1+2i\cot(\vartheta))\|\mathbf{b}_1\|^2} \left[\int_{\mathbb{R}^{n-1}} e^{-i\mathbf{b}_0^t \mathbf{b}_1 - (1/2)\|\mathbf{b}_0\|^2} T_l(\vartheta; \mathbf{b}_0, \mathbf{b}_1)\, d\mathbf{b}_0\right] d\mathbf{b}_1 = \int_{\mathbb{R}^{n-1}} e^{-(1/2)(2+2i\cot(\vartheta))\|\mathbf{b}_1\|^2} \mathscr{T}_l(\vartheta; \mathbf{b}_1)\, d\mathbf{b}_1, \tag{136}$$

where $\mathscr{T}_l(\vartheta; \cdot)$ is an even polynomial of degree $\leq 6l$.

Let us introduce the Fourier transform

$$\mathscr{F}(\mathbf{c}) = \int_{\mathbb{R}^{n-1}} e^{-(1/2)(2+2i\cot(\vartheta))\|\mathbf{b}_1\|^2 - i\mathbf{b}_1^t \mathbf{c}} d\mathbf{b}_1 = (2\pi)^{(n-1)/2} \sin(\vartheta)^{(n-1)/2} \cdot e^{i(\vartheta/2 - \pi/4)(n-1)} e^{-(1/2)(2+2i\cot(\vartheta))^{-1}\|\mathbf{c}\|^2}. \tag{137}$$

Then (136) is the result of applying an even differential polynomial $P_l(D_{\mathbf{c}})$ of degree $\leq 6l$ to $\mathscr{F}(\mathbf{c})$ and then evaluating the result at $\mathbf{c} = \mathbf{0}$.

Given this and (135), we conclude that

$$\mathscr{P}_k(\mathbf{q}_0, \mathbf{q}_1) = \frac{2^{n/2}}{C_{k,n}^2} \left(\frac{1}{\sin(\vartheta)}\right)^{(n-1)/2} \cdot \left[\cos\left(k\vartheta + \left(\frac{\vartheta}{2} - \frac{\pi}{4}\right)(n-1)\right) \cdot A(\vartheta) + \sin\left(k\vartheta + \left(\frac{\vartheta}{2} - \frac{\pi}{4}\right)(n-1)\right) \cdot B(\vartheta)\right], \tag{138}$$

where

$$A(\vartheta) \sim 1 + \sum_{l=1}^{+\infty} k^{-l} \frac{A_l(\vartheta)}{\sin(\vartheta)^{6l}}, \quad B(\vartheta) \sim \sum_{l=1}^{+\infty} k^{-l} \frac{B_l(\vartheta)}{\sin(\vartheta)^{6l}}, \tag{139}$$

with A_l and B_l smooth functions of ϑ on $[0, 2\pi]$.

Proof of Theorem 1 is complete.

4. Proof of Proposition 2

Proof of Proposition 2. The diagonal restriction $\mathscr{P}_{k,n}(\mathbf{q}, \mathbf{q})$ may be computed in two different ways. On the one hand, since $\mathscr{P}_{k,n}(\mathbf{q}, \mathbf{q})$ is constant, we have

$$\mathscr{P}_{k,n}(\mathbf{q}, \mathbf{q}) = \frac{N_{k,n}}{\mathrm{vol}(S^n)} = \frac{2}{\mathrm{vol}(S^n)} \frac{k^{n-1}}{(n-1)!} + O\left(k^{n-2}\right). \tag{140}$$

On the other hand, (54) with $\mathbf{q}_0 = \mathbf{q}_1 = \mathbf{q}$ yields

$$\mathscr{P}_{k,n}(\mathbf{q}, \mathbf{q}) = \frac{1}{C_{k,n}^2} \int_{S(\mathbf{q}^\perp)} \int_{S(\mathbf{q}^\perp)} \Pi_{\sqrt{2},k}\left(\mathbf{q} + i\mathbf{p}, \mathbf{q} + i\mathbf{p}'\right) dV_{S(\mathbf{q}^\perp)}(\mathbf{p})\, dV_{S(\mathbf{q}^\perp)}\left(\mathbf{p}'\right) = \frac{1}{C_{k,n}^2} \int_{S(\mathbf{q}^\perp)} F_k(\mathbf{q}, \mathbf{p})\, dV_{S(\mathbf{q}^\perp)}(\mathbf{p}), \tag{141}$$

where

$$F_k(\mathbf{q}, \mathbf{p}) =: \int_{S(\mathbf{q}^\perp)} \Pi_{\sqrt{2},k}\left(\mathbf{q} + i\mathbf{p}, \mathbf{q} + i\mathbf{p}'\right) dV_{S(\mathbf{q}^\perp)}\left(\mathbf{p}'\right). \tag{142}$$

Again, integration in $dV_{S(\mathbf{q}^\perp)}(\mathbf{p}')$ localizes in a shrinking neighborhood of $\mathbf{p}$. Hence, we may let

$$\mathbf{p}' = \mathbf{p} + A(\mathbf{v}), \quad A(\mathbf{v}) = \mathbf{v} + S_+(\mathbf{v})\,\mathbf{p}, \tag{143}$$

where $\mathbf{v} \in \mathbf{q}^{\perp} \cap \mathbf{p}^{\perp}$, and introduce the cut-off $\varrho(k^{1/2-\epsilon}\mathbf{v})$. Passing to rescaled coordinates, and setting $\mathbf{z} = \mathbf{q} + i\mathbf{p}$, we get

$$F_k(\mathbf{q},\mathbf{p}) = \frac{1}{k^{(n-1)/2}} \int_{\mathbf{q}^{\perp}\cap\mathbf{p}^{\perp}} \varrho\left(k^{-\epsilon}\mathbf{v}\right) \cdot \Pi_{\sqrt{2},k}\left(\mathbf{z}, \mathbf{z} + \frac{i}{\sqrt{k}} A_k(\mathbf{v})\right) \mathscr{V}\left(\frac{\mathbf{v}}{\sqrt{k}}\right) d\mathbf{v}, \tag{144}$$

where

$$A_k(\mathbf{v}) = \mathbf{v} + \sqrt{k} S_+\left(\frac{\mathbf{v}}{\sqrt{k}}\right)\mathbf{p}, \qquad \mathscr{V}(\mathbf{0}) = 1. \tag{145}$$

By Lemma 15 (with $\mathbf{z} = \mathbf{z}_0 = \mathbf{z}_1$, $\mathbf{v}_0 = \mathbf{0}$, $\mathbf{v}_1 = \mathbf{v}$, $\vartheta = 0$), we have

$$\Pi_{\sqrt{2},k}\left(\mathbf{z}, \mathbf{z} + \frac{i}{\sqrt{k}} A_k(\mathbf{v})\right) \cdot \mathscr{V}\left(\frac{\mathbf{v}}{\sqrt{k}}\right) \sim \frac{\sqrt{2}}{2^n}\left(\frac{k}{\pi}\right)^{n-1} e^{-(1/4)\|\mathbf{v}\|^2} \sum_{l\geq 0} \frac{1}{k^{l/2}} Z_l(\mathbf{v}), \tag{146}$$

for certain polynomials Z_l of degree $\leq 3l$ and parity $(-1)^l$, with $Z_0(\cdot) = 1$.

As before, the expansion may be integrated term by term and, by parity, only the summands with l even yield a nonzero contribution. In addition, only a negligible contribution is lost if the cut-off is omitted and integration is extended to all of $\mathbf{q}^{\perp} \cap \mathbf{p}^{\perp} \cong \mathbb{R}^{n-1}$. Therefore,

$$F_k(\mathbf{q},\mathbf{p}) \sim \frac{\sqrt{2}}{2^n} \frac{k^{(n-1)/2}}{\pi^{n-1}} \sum_{l\geq 0} k^{-l} \int_{\mathbb{R}^{n-1}} e^{-(1/4)\|\mathbf{v}\|^2} Z_{2l}(\mathbf{v})\, d\mathbf{v} = \frac{1}{\sqrt{2}}\left(\frac{k}{\pi}\right)^{(n-1)/2} + \cdots. \tag{147}$$

Inserting this in (141), we obtain an asymptotic expansion

$$\mathscr{P}_{k,n}(\mathbf{q},\mathbf{q}) \sim \frac{\operatorname{vol}\left(S^{n-1}\right)}{C_{k,n}^2} \frac{1}{\sqrt{2}}\left(\frac{k}{\pi}\right)^{(n-1)/2} + \cdots. \tag{148}$$

Comparing (140) and (148), we obtain an asymptotic expansion in descending powers of k, of the form

$$C_{k,n} \sim \left[\frac{\operatorname{vol}\left(S^n\right)\operatorname{vol}\left(S^{n-1}\right)}{2\sqrt{2}} \cdot (n-1)!\right]^{1/2} (\pi k)^{-(n-1)/4} + \cdots. \tag{149}$$

□

Conflicts of Interest

The author declares that there are no conflicts of interest regarding the publication of this paper.

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