On integrable conservation laws

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We study normal forms of scalar integrable dispersive (not necessarily Hamiltonian) conservation laws, via the Dubrovin–Zhang perturbative scheme. Our computations support the conjecture that such normal forms are parametrized by infinitely many arbitrary functions that can be identified with the coefficients of the quasi-linear part of the equation. Moreover, in general, we conjecture that two scalar integrable evolutionary partial differential equations having the same quasi-linear part are Miura equivalent. This conjecture is also consistent with the tensorial behaviour of these coefficients under general Miura transformations.

1. Introduction

The Dubrovin–Zhang perturbative approach is concerned with the classification problem of evolutionary partial differential equations (PDEs) of the form

$$u_t = X(u, u_x, \ldots), \quad i = 1, \ldots, n, \qquad (1.1)$$

where the functions $X(u, u_x, \ldots)$ are polynomials in the jet variables $u_x, u_{xx}, \ldots$. Introducing a rescaling of independent variables of the form $x \to \epsilon x$ and $t \to \epsilon t$, the equation (1.1) takes the form

$$u_t = \sum_{k \geq -1} \epsilon^k F_{k+1}(u, u_x, \ldots, u_{(k+1)}), \qquad (1.2)$$

where the functions F_k are homogeneous differential polynomials of suitable degree, and we adopt the notation $u_{(k)} := \partial_x^k u$. It is also assumed that

$$F_0(u) \equiv 0.$$

Hence, we focus on a class of evolutionary PDEs of the form

$$u_t = \sum_{k \geq 0} \epsilon^k F_{k+1}(u, u_x, \ldots, u_{(k+1)}), \tag{1.3}$$

where the r.h.s is a *formal power series* in ϵ and it does not necessarily truncate. Introducing a gradation such that functions depending on the single variable u have degree zero, and monomials of the form $u_{(k)}$ have degree k, it is straightforward to check that the differential polynomial F_k has degree k. For instance, we have

$$F_1 = V(u)u_x$$

$$F_2 = A(u)u_{xx} + B(u)u_x^2.$$

The Burgers equation

$$u_t = uu_x + \epsilon u_{xx}, \tag{1.4}$$

and the Korteweg–de Vries (KdV) equations

$$u_t = uu_x + \epsilon^2 u_{xxx} \tag{1.5}$$

are two celebrated examples of exactly integrable nonlinear PDEs of the form (1.3). As the r.h.s. of equation (1.3) is allowed to be an infinite power series in ϵ, the class of equations under study also includes non-evolutionary examples such as the Camassa–Holm equation [1]

$$u_t - \epsilon^2 u_{xxt} = -3uu_x + \epsilon^2(uu_{xxx} + 2u_x u_{xx}). \tag{1.6}$$

Indeed, the Camassa–Holm equation (1.6) can be recast in the evolutionary form via the transformation

$$v = u - \epsilon^2 u_{xx} = (1 - \epsilon^2 \partial_x^2)u,$$

whose formal inverse is given by

$$u = (1 - \epsilon^2 \partial_x^2)^{-1} v = 1 + \epsilon^2 v_{xx} + \epsilon^4 v_{xxxx} + \cdots.$$

One of the main problem in the theory of integrable PDEs is to classify equations (or systems of equations) of the form (1.3) up to equivalence under the so-called *Miura transformations*

$$u \to \tilde{u} = M_0(u) + \sum_k \epsilon^k M_k(u, u_x, \ldots),$$

where M_0 is assumed to be invertible and M_k are differential polynomials of degree k.

Hence, the classification problem of integrable equation of the form (1.3) is reformulated in terms of a classification problem of equivalence classes of integrable equations with respect to Miura transformations. The Dubrovin–Zhang perturbative scheme aims at the reconstruction of higher-order *integrable* corrections (both dispersive and dissipative) starting from the quasi-linear PDE of Hopf type (the dispersionless limit)

$$u_t = f(u)u_x.$$

Within this scheme, the various perturbative approaches developed so far mainly differ in the kind of additional structures that are possessed by the dispersionless limit and that are required to be preserved by the perturbation procedure.

Let us consider, for instance, the Hopf equation $u_t = uu_x$. It is clearly integrable as it possesses infinitely many symmetries parametrized by an arbitrary function of one variable $g(u)$. The most general approach to the classification of integrable deformations of the Hopf equation is based on the request that all deformed symmetries $u_\tau = g(u)u_x + \cdots$ commute with the deformed Hopf equation $u_t = uu_x + \cdots$ [2–4].

The classification of integrable conservation laws is based on the simple observation that the Hopf hierarchy consists of conservation laws of the form $u_t = \partial_x(G(u))$ and one may require that the deformation of the integrable hierarchy preserves the form of a conservation law,

i.e. $u_t = \partial_x(G(u) + \cdots)$. The general classification of scalar viscous conservation laws has been recently discussed in reference [5].

A special class of conservation laws is given by Hamiltonian equations. These are equations of the form $u_t = \partial_x(G(u) + \cdots)$ such that the deformed currents $G(u) + \cdots$ can be written as variational derivatives w.r.t. the variable u, i.e.

$$u_t = \partial_x \frac{\delta}{\delta u}\left(\int (h_0(u) + \cdots)\,dx\right).$$

At the dispersionless level, all equations of Hopf hierarchy are Hamiltonian w.r.t. the operator ∂_x that defines a Poisson bracket of hydrodynamic type [6]. This observation suggests to deform the Hamiltonians, requiring that they remain in involution w.r.t. the Poisson bracket. This approach has been first proposed and developed in reference [7].

An alternative classification procedure relies on the observation that Hopf-type equations possess a bi-Hamiltonian structure. This suggests to classify integrable deformations according to the existence of a deformed bi-Hamiltonian structure [3,8–11].

A common feature of these different approaches is that deformations are parametrized by arbitrary functions. Clearly, the numbers of the functional parameters involved crucially depends on the problem at hand. In this paper, following Arsie *et al.* [5], we consider the case of scalar conservation laws extending the analysis to the case of dispersive conservation laws. Besides the undeniable relevance of conservation laws in physical applications, our focus is also motivated by the fact that, within the more general context of systems of PDEs of hydrodynamic type, the class of integrable diagonalizable equations [13] coincide with the class of diagonalizable systems of conservation laws [14].

A key observation of this work is that for scalar evolutionary PDEs the coefficients corresponding to the quasi-linear terms have a tensorial nature. More precisely, given a PDE of the form

$$u_t = X_1(u)u_x + \epsilon(X_2(u)u_{xx} + \cdots) + \epsilon^2(X_3(u)u_{xxx} + \cdots) + \cdots$$

the coefficients $X_1(u)$, $X_2(u)$, etc., of the quasi-linear terms *are invariant* under Miura transformations of the form:

$$u \to v = u + \sum_k \epsilon^k M_k(u_x, u_{xx}, \ldots). \tag{1.7}$$

It is thus natural to expect that these coefficients play a crucial role in the classification problem. We also observe that the above transformations, which can be seen as perturbation of the identity, trivially preserve the dispersionless limit.

Based on the results of this study combined with results already existing in the literature, we formulate conjecture 1.1.

Conjecture 1.1. *Two scalar integrable evolutionary PDEs admitting the same quasi-linear part are Miura equivalent.*

According to conjecture 1.1, the number of free functional parameters appearing in deformations coincide with the number of independent functions in the quasi-linear part of the equation.

2. Tensorial coefficients

Here, we analyse in more detail the transformation properties of quasi-linear terms in evolutionary equations of the form

$$u_t = X(u, u_x, u_{xx}, \ldots) = X_1(u)u_x + \epsilon(X_2(u)u_{xx} + \cdots) + \epsilon^2(X_3(u)u_{xxx} + \cdots) + \mathcal{O}(\epsilon^3), \tag{2.1}$$

under a Miura transformation of the following type

$$u \to v = M_0(u) + \sum_k \epsilon^k M_k(u, u_x, u_{xx}, \ldots). \tag{2.2}$$

Observing that the vector field $X(u, u_x, u_{xx}, \dots)$ in equation (2.1) transforms according to the rule

$$X(u, u_x, \dots) \to \tilde{X}(v, v_x, \dots) = \left(\frac{\partial v}{\partial u} + \frac{\partial v}{\partial u_x} \partial_x + \frac{\partial v}{\partial u_{xx}} \partial_x^2 + \cdots \right) X_{u=u(v, v_x, \dots)} \tag{2.3}$$

where $u = u(v, v_x, \dots)$ is the inverse of Miura transformation (2.2), we show that the coefficients of leading derivatives $X_1(u)$, $X_2(u)$, etc., in (2.1) are not affected by the corrections to the leading part $M_0(u)$ of the Miura transformation (2.2). More precisely, these coefficients transform as tensors w.r.t the leading term of the transformation and in particular are invariant if such a leading term is the identity, i.e.

$$u \to v = u + \sum_k \epsilon^k M_k(u, u_x, u_{xx}, \dots). \tag{2.4}$$

We can prove the following.

Theorem 2.1. *Under the Miura transformation (2.2), the coefficients $X_1(u)$, $X_2(u)$, $X_3(u)$, ... of the quasi-linear terms in the right-hand side of (2.1) transform as*

$$X_k(u) \to \tilde{X}_k(v) = X_k(u(v)),$$

where $u(v)$ is the inverse of the dispersionless limit of (2.2).

Proof. First of all, we observe that quasi-linear terms in the differential polynomial

$$\tilde{X}(v, v_x, \dots) = \tilde{X}_1(v)v_x + \epsilon(\tilde{X}_2(v)v_{xx} + \cdots) + \epsilon^2(\tilde{X}_3(v)v_{xxx} + \cdots) + O(\epsilon^3)$$

are completely determined by quasi-linear terms in the differential polynomial

$$\tilde{X}(u, u_x, \dots) = \left(\frac{\partial v}{\partial u} + \frac{\partial v}{\partial u_x} \partial_x + \frac{\partial v}{\partial u_{xx}} \partial_x^2 + \cdots \right) X(u, u_x, \dots)$$

$$= \tilde{X}_1(u)u_x + \epsilon(\tilde{X}_2(u)u_{xx} + \cdots) + \epsilon^2(\tilde{X}_3(u)u_{xxx} + \cdots) + O(\epsilon^3). \tag{2.5}$$

Observing that the inverse of (2.2) is of the form

$$u = N_0(v) + \sum_{k \geq 1} \epsilon^k N_k(v, v_x, \dots), \tag{2.6}$$

it can be easily proved by induction that the term $\tilde{X}_k(v)v_{(k)}$ (i.e. the quasi-linear term in $\tilde{X}(v, v_x, \dots)$ of degree k) is determined by quasi-linear terms in $\tilde{X}(u, u_x, \dots)$ of degree less than or equal to k. Hence, in the following, we focus our analysis on quasi-linear terms of $\tilde{X}(u, u_x, \dots)$ only. Let us now write the Miura transformation (2.2) as

$$u \mapsto v = M_0(u) + \sum_{k \geq 1} \epsilon^k (a_k(u)u_{(k)} + R_k(u, u_x, \dots)), \tag{2.7}$$

where the homogeneous part of the k-th degree $M_k(u, u_x, \dots)$ has been decomposed into the quasi-linear part and the remainder. We check by a direct calculation that quasi-linear terms of $\tilde{X}_m u_{(m)}$ of $\tilde{X}(u, u_x, \dots)$ computed by using the formula (2.5) are not affected by the reminder R_k.

Let us now write the vector field $X(u, u_x, \dots)$ as

$$X = \sum_{l \geq 1} \epsilon^{l-1} X_l(u)u_{(l)} + NQ,$$

where NQ denotes the non-quasi-linear part of $X(u, u_x, \dots)$ and compute the transformed vector field (2.5)

$$\tilde{X}(u, u_x, \dots) = \left(\frac{\partial F_0(u)}{\partial u} + \sum_{k \geq 1} \epsilon^k a_k(u)\partial_x^k + \mathcal{R}_1 \right) \left(\sum_{l \geq 1} \epsilon^{l-1} X_l(u)u_{(l)} + NQ \right),$$

where $\mathcal{R}_1$ accounts for the terms produced by the remainders R_k in (2.7) and terms of the type $\sum_{k \geq 1} \epsilon^k (\partial a_k(u)/\partial u)u_{(k)}$. Observing that the action of $\mathcal{R}_1$ on $X(u, u_x, \dots)$ always produces

non-quasi-linear terms, we have

$$\tilde{X}(u, u_x, \ldots) = \sum_{l \geq 1} \epsilon^{l-1} X_l(u) \left(\frac{\partial F_0(u)}{\partial u} u_{(l)} + \sum_{k \geq 1} \epsilon^k a_k(u) u_{(k+l)} + \text{NQ1} \right) + \text{NQ2}, \tag{2.8}$$

where NQ1 denotes non-quasi-linear terms produced by $\mathcal{R}_1$ and NQ2 stays for remaining non-quasi-linear terms. We can now evaluate explicitly quasi-linear terms in (2.8). Observing that

$$\partial_x^l v = \frac{\partial F_0}{\partial u} u_{(l)} + \sum_k \epsilon^k (a_k(u) u_{(k+l)}) + \mathcal{R},$$

where $\mathcal{R}$ contains products of at least two derivatives of u, we have that the bracket in (2.8)

$$\frac{\partial F_0(u)}{\partial u} u_{(l)} + \sum_{k \geq 1} \epsilon^k a_k(u) u_{(k+l)} + \text{NQ1}$$

is equal to

$$\partial_x^l v + \text{NQ1} - \mathcal{R},$$

and therefore

$$\tilde{X}(v, v_x, \ldots) = \sum_{l \geq 1} \epsilon^{l-1} X_l(u)_{|u=u(v)} \partial_x^l v + \cdots,$$

where the dots stand for non-quasi-linear terms, and $u = u(v)$ is the inverse of the dispersionless part of the Miura transformation. ∎

We have also the following.

Corollary 2.2. *If two evolutionary PDEs are Miura equivalent and have the same dispersionless limit, then their quasi-linear parts coincide.*

Obviously, the converse statement is in general not true. However, we conjecture that it is valid if one restricts to the class of *integrable* equations (see conjecture 1.1).

3. Scalar conservation laws

This section is devoted to the study of integrable scalar conservation laws of the form

$$u_t = \partial_x \left[g(u) + \sum_{k=1}^{\infty} \epsilon^k \omega_k(u, u_x, \ldots) \right], \tag{3.1}$$

where ω_k are differential polynomials of degree k.

For the sake of simplicity, we focus on the case $g(u) = u^2$. The general case can be treated in an analogous way.

In virtue of the conjecture (1.1), equivalence classes (with respect to the action of the Miura group) of integrable scalar equations are labelled by the independent coefficients of the quasi-linear part. Depending on the class of the equations considered, it might happen that only a subset of the coefficients of the quasi-linear part are sufficient to determine all the others. The following analysis provides evidence of the fact that this is the case for the *independent* coefficients of the quasi-linear part in conservation laws of the form (3.1). We will call them *central invariants* by analogy with the central invariants introduced in reference [10].

We follow the approach presented in reference [5] for viscous conservation laws, that is the case $\omega_1 \neq 0$ in (3.1), and we extend it to dispersive conservation laws, where only even powers in the formal parameter ϵ appear in (3.1). The main steps of this approach can be summarized

as follows.

1. Reduce of (3.1) to its normal form

$$u_t = \partial_x \tilde{\omega}_{u^2}^{\mathrm{def}} = \partial_x \left[u^2 + \epsilon a(u) u_x + \sum_{k>1} \epsilon^k \tilde{\omega}_k(u, u_x, \ldots) \right], \qquad (3.2)$$

where

$$\frac{\partial \tilde{\omega}_k}{\partial u_x} = 0, \quad \forall k > 1.$$

This reduction is always possible and it is unique (see [5]).

2. Impose the *integrability* condition, i.e. the requirement that there exists a family of conservation laws

$$u_\tau = \partial_x \omega_f^{\mathrm{def}} = \partial_x \left[f(u) + \sum_{k=1}^\infty \epsilon^k f_k(u, u_x, \ldots) \right] \qquad (3.3)$$

that commute with (3.1). We note that, as shown in [5,15], this is equivalent to requiring that the 1-forms ω_{u^2} and $\omega_{f(u)}$ are in involution w.r.t. the Poisson bracket

$$\{\alpha, \beta\} := \sum_j \partial_x^{j+1} \beta \frac{\partial \alpha}{\partial u_{(j)}} - \partial_x^{j+1} \alpha \frac{\partial \beta}{\partial u_{(j)}} = 0. \qquad (3.4)$$

In general, one imposes the commutativity up to a fixed order in ϵ, and one derives relations that express the terms f_k appearing in (3.3) as functions of the terms $a(u)$ and ω_k appearing in (3.2) and of the leading term $f(u)$ in (3.3). Depending on the structure of the equation (3.2) under consideration, there might be different constraints among $a(u)$ and the coefficients in ω_k, as we will see below. The presence or absence of these constraints will single out the *independent* coefficients of the quasi-linear part.

(a) The viscous case

Let us briefly review the case of a scalar conservation law with viscosity studied in reference [5] and that corresponds to the assumption $a(u) \neq 0$ in (3.2). The procedure outlined above leads to the following.

Theorem 3.1. *Up to $\mathcal{O}(\epsilon^6)$, the quasi-linear part of $\omega_{u^2}^{\mathrm{def}}$*

$$u^2 + \epsilon a(u) u_x + \epsilon^2 b_1(u) u_{xx} + \epsilon^3 c_1(u) u_{xxx} + \epsilon^4 d_1(u) u_{4x} + \epsilon^5 e_1(u) u_{5x} + \mathcal{O}(\epsilon^6) \qquad (3.5)$$

is uniquely determined by $a(u)$. More precisely, we have

$$b_1 = \left(\frac{a^2}{2!} \right)', \quad c_1 = \left(\frac{a^3}{3!} \right)'', \quad d_1 = \left(\frac{a^4}{4!} \right)''', \quad e_1 = \left(\frac{a^5}{5!} \right)'''' \ldots$$

As a consequence, up to order $\mathcal{O}(\epsilon^5)$, $a(u)$ is the only independent coefficient and it is named *viscous central invariant* in reference [5]. Furthermore, the coefficients f_k of the symmetries (3.3) are also completely determined by $a(u)$ and by the leading term of the symmetries $f(u)$.

This result suggests the following.

Conjecture 3.2. *The quasi-linear part of a viscous conservation law (3.2) is uniquely determined by $a(u)$.*

Accordingly, in the case of scalar viscous conservation laws, the main conjecture can be formulated as follows.

Conjecture 3.3. *Two integrable viscous conservation laws (3.2) admitting the same viscous central invariant $a(u)$ are Miura equivalent.*

Therefore, if this conjecture is true, for scalar viscous conservation laws (3.2), there exists only one independent coefficient in the quasi-linear part of the equation.

(b) Dispersive case

The case of dispersive conservation laws arises as a branching in the classification procedure that corresponds to the choice $a(u) = 0$. The difference with the viscous case (see theorem 3.1) is remarkable as the classification suggests the existence of infinitely many free functional parameters. Let us assume, for the sake of simplicity, that the coefficients in front of all odd powers of ϵ vanish (we will justify this assumption later). In this case, the current in (3.2) reads as

$$\omega_{u^2}^{\text{def}} = u^2 + \epsilon^2 b_1(u) u_{xx} + \epsilon^4 [c_1(u) u_{4x} + c_2(u) u_{xx}^2]$$

$$+ \epsilon^6 [d_1(u) u_{6x} + d_2(u) u_{xx} u_{4x} + d_3(u) u_{xxx}^2 + d_4(u) u_{xx}^3]$$

$$+ \epsilon^8 [e_1(u) u_{8x} + e_2(u) u_{xx} u_{6x} + e_3(u) u_{5x} u_{xxx} + e_4(u) u_{4x}^2$$

$$+ e_5(u) u_{4x} u_{xx}^2 + e_6(u) u_{xxx}^2 u_{xx} + e_7(u) u_{xx}^4] + \cdots, \tag{3.6}$$

and the current in (3.3)

$$\omega_f^{\text{def}} = f(u) + \epsilon^2 [B_1(u) u_{xx} + B_2(u) u_x^2]$$

$$+ \epsilon^4 [C_1(u) u_{4x} + C_2(u) u_x u_{xxx} + C_3(u) u_{xx}^2 + C_4(u) u_x^2 u_{xx} + C_5(u) u_x^4]$$

$$+ \epsilon^6 [D_1(u) u_{6x} + D_2(u) u_x u_{5x} + D_3(u) u_{xx} u_{4x} + D_4(u) u_x^2 u_{4x} + D_5(u) u_{xxx}^2$$

$$+ D_6(u) u_x u_{xx} u_{xxx} + D_7(u) u_x^3 u_{xxx} + D_8(u) u_{xx}^3 + D_9(u) u_{xx}^2 u_x^2 + D_{10}(u) u_{xx} u_x^4$$

$$+ D_{11}(u) u_x^6] + \cdots. \tag{3.7}$$

The integrability condition, that is the involutivity conditions on the associated 1-forms

$$\{\omega_{u^2}^{\text{def}}, \omega_{f(u)}^{\text{def}}\} = 0, \quad \forall f(u)$$

up to the order $\mathcal{O}(\epsilon^{12})$, gives the following set of constraints:

At order ϵ^0, no conditions are enforced.

At order ϵ^2, B_1 and B_2 are expressed in terms of b_1 and f.

At order ϵ^4, C_1, C_2, C_3, C_4 and C_5 are expressed in terms of b_1, c_1, c_2 and f.

At order ϵ^6, the terms D_i, $i = 1, \ldots, 11$ are given as functions of $b_1, c_1, c_2, d_1, d_2, d_3, d_4$ and f.

At order ϵ^8, the terms E_i, $i = 1, \ldots, 22$ are expressed as functions of the small letters (coefficients of $\omega_{u^2}^{\text{def}}$) and f. Moreover, there appear constraints that express c_2 in terms of b_1, c_1 and d_1

$$c_2 = \frac{1}{144} \frac{1}{b_1^2} \left[117 \left(\frac{\partial^2 b_1}{\partial u^2} \right) b_1^3 - 84 b_1^2 \left(\frac{\partial b_1}{\partial u} \right)^2 + 670 \left(\frac{\partial b_1}{\partial u} \right) c_1 - 330 b_1^2 \left(\frac{\partial c_1}{\partial u} \right) + 560 b_1 d_1 - 800 c_1^2 \right] \tag{3.8}$$

and d_3 and d_4 in terms of b_1, c_1, d_1, d_2 and e_2, e_3, e_4.

At order ϵ^{10}, all the terms F_i, $i = 1, \ldots, 42$ are expressed as functions of the small letters (coefficients of $\omega_{u^2}^{\text{def}}$) and f. Moreover, there appears a constraint that gives d_2 in terms of b_1, c_1, d_1 and e_1. Furthermore, the coefficients e_2, e_4, e_5, e_7, f_4 and f_6 also are determined in terms of the other coefficients of $\omega_{u^2}^{\text{def}}$.

At order ϵ^{12}, all the coefficients G_i, $i = 1, \ldots, 77$ are expressed as functions of the small letters (coefficients of $\omega_{u^2}^{\text{def}}$) and f. Moreover, there appear constraints that give e_3, e_6, part of the $f_i's$ and part of $g_i's$ in terms of the remaining coefficients of $\omega_{u^2}^{\text{def}}$, namely those coefficients that are still free.

Summarizing, taking into account computations up to order 12, we have the following situation:

— at the order 2 b_1 is free,

— at the order 4 c_1 is free, c_2 is a function of b_1, c_1, d_1,

— at the order 6 d_1 is free and d_2, d_3, d_4 are functions of b_1, c_1, d_1, e_1 and e_2, e_3, e_4, but e_2, e_3, e_4, owing to constraints appearing at higher orders, can be expressed in terms of b_1, c_1, d_1, e_1 and f_1.

The above results are summarized in the following.

Theorem 3.4. *Up to $\mathcal{O}(\epsilon^{12})$, all the coefficients of the quasi-linear part of (3.6) are independent. Moreover, up to $\mathcal{O}(\epsilon^6)$, all the small letters are uniquely determined in terms of b_1, c_1, d_1, e_1 and f_1.*

For the sake of simplicity, we have imposed from the very beginning that in the case of $a(u) = 0$ only even powers of ϵ are present. However, one can check that this assumption is not restrictive and it follows directly from computations, at least up to the sixth order.

The above results lead to the following.

Conjecture 3.5. *The quasi-linear part of an integrable dispersive conservation law contains only even powers of ϵ and all the coefficients of the quasi-linear part are independent.*

(c) Are all dispersive conservation laws Hamiltonian?

In reference [7], it was proved that Hamiltonian conservation laws can be reduced to the form

$$u_t = \partial_x \left[\frac{u^2}{2} + \frac{\epsilon^2}{24}(2cu_{xx} + c'u_x^2) + \epsilon^4(2pu_{(4)} + 4p'u_xu_{xxx} + 3p'u_{xx}^2 + 2p''u_x^2u_{xx}) + \mathcal{O}(\epsilon^6) \right], \quad (3.9)$$

where $p(u)$ and $c(u)$ are arbitrary functions. Equation (3.9) is brought to the normal form (3.2), up to $\mathcal{O}(\epsilon^6)$,

$$u_t = \partial_x \left[\frac{u^2}{2} + \frac{\epsilon^2}{12}c(u)u_{xx} + \epsilon^4 \left(2p(u)u_{(4)} + \frac{4c'(u)^2 + 3cc''}{1152}u_{xx}^2 \right) + \mathcal{O}(\epsilon^6) \right] \quad (3.10)$$

by means of the Miura transformation

$$\tilde{u} = u + \epsilon\partial_x(\epsilon\alpha(u)u_x + \epsilon^2(\beta_0(u)u_{xxx} + \beta_1(u)u_xu_{xx} + \beta_2(u)u_x^3) + \mathcal{O}(\epsilon^4)),$$

where the coefficients $\alpha(u), \beta_0(u), \beta_1(u)$ and $\beta_2(u)$ are given by

$$\alpha(u) = -\frac{c'(u)}{24}$$

$$\beta_0(u) = -p'(u) + \frac{c'(u)^2 - c(u)c''(u)}{384}$$

$$\beta_1(u) = -\frac{p''(u)}{2} + \frac{5c'(u)c''(u) - 6c(u)c'''(u)}{1152}$$

$$\beta_2(u) = \frac{3c''(u)^2 + 2c'(u)c'''(u) - 4c(u)c^{(IV)}(u)}{3456}.$$

Hence, the coefficients of the normal form (3.6) can be uniquely expressed in terms of the invariants $c(u)$ and $p(u)$. In particular, we have

$$b_1(u) = \frac{c(u)}{12}$$

$$c_1(u) = 2p(u)$$

$$c_2(u) = \frac{4c'(u)^2 + 3c(u)c''(u)}{1152}.$$

Then, the relation (3.8) results in a constraint on the coefficient d_1 that is consequently no longer free, being uniquely determined in terms of the two functional parameters $c(u)$ and $p(u)$, i.e.

$$d_1(u) = \frac{1}{7}\frac{p(u)}{c(u)}\left[480p(u) - \frac{67}{4}c'(u)\right] + \frac{1}{112}c(u)\left[11p'(u) + \frac{13}{720}c'(u)^2 - \frac{7}{960}c(u)c''(u)\right].$$

Comparing this result with the one presented in theorem 3.4, it follows that integrable hierarchies of dispersive conservation laws, in general, are not Hamiltonian with respect to the Poisson operator ∂_x or with respect to a Poisson operator related to ∂_x by a Miura transformation.

4. Examples

The classification procedure discussed above turns out to be consistent with alternative definitions of integrability (e.g. S-integrability, existence of a bi-Hamiltonian structure, Backlund transformations) and reproduces a number of relevant examples known in the literature. Given the general integrable conservation law in the form

$$u_t = \partial_x[u^2 + \epsilon^2 b_1(u)u_{xx} + \epsilon^4(c_1(u)u_{4x} + \ldots) + \epsilon^6(d_1(u)u_{6x} + \ldots) + \epsilon^8(e_1(u)u_{8x} + \ldots) + \ldots], \quad (4.1)$$

for particular choices of the free functional parameters, we easily recover a few examples well known in the literature. The list below is not meant to be complete.

(a) KdV equation

In the particular case of constant central invariants, we could reproduce two important examples. The choice

$$b_1 = \text{const.} \quad c_1(u) = d_1(u) = \cdots = 0$$

gives the celebrated KdV equation.

(b) Nonlinear intermediate long-wave equation (Hodge–KdV equation)

The following example of constant central invariants

$$b_1 = \frac{|B_2|}{2} \quad c_1 = \frac{|B_4|}{4} \quad d_1 = \frac{|B_6|}{6} \ldots,$$

where $B_2 = \frac{1}{6}, B_4 = -\frac{1}{30}, \ldots$ are Bernoulli numbers, corresponds to the nonlinear intermediate long-wave equation [16]. This equation appeared also in the study of Gromow–Witten invariants in topological field theory [17,18]. In this framework, it is known as Hodge–KdV equation.

(c) Camassa–Holm and Degasperis–Procesi equations

In the case of linear central invariants, we have the Camassa–Holm equation

$$u_t - \epsilon^2 u_{xxt} = -3uu_x + \epsilon^2(uu_{xxx} + 2u_x u_{xx})$$

and Degasperis–Procesi equation [19]

$$u_t - \epsilon^2 u_{xxt} = -4uu_x + \epsilon^2(uu_{xxx} + 3u_x u_{xx}).$$

Note that these two equations do not appear in the evolutionary form, but can be brought to the evolutionary form via formal inversion of the operator $1 - \partial_x^2$. They can also be reduced to the same dispersionless limit via a rescaling of the dependent variable u. In both cases, the central invariants have the form

$$b_1(u) = c_1(u) = d_1(u) = \cdots = cu,$$

but the value of constant c is -2 in the first case and -3 in the second case, and thus as a consequence of conjecture 1.1 they are not expected to be Miura equivalent.

We conclude this section by presenting a list of examples of integrable equations sharing one and the same quasi-linear part and that are known to be Miura equivalent.

(d) Korteweg–de Vries versus mKdV equation

Let us consider the KdV equation

$$u_t = 2uu_x + \epsilon^2 u_{xxx}, \tag{4.2}$$

and the modified KdV equation in the form

$$v_t = -3v^2 v_x + \epsilon^2 v_{xxx}. \tag{4.3}$$

We observe that, introducing the transformation of the dependent variable $v = \pm\sqrt{-2w/3}$, the equation (4.3) takes the form

$$w_t = 2ww_x + \epsilon^2 \left(w_{xxx} - \frac{3}{2w} w_x w_{xx} + \frac{3}{4w^2} w_x^3 \right), \tag{4.4}$$

whose quasi-linear part coincides with KdV equation (4.2). Hence, the conjecture 1.1 is consistent with the very well-known fact that there exists the Miura transformation mapping the equation (4.2) into (4.3) [20], that is explicitly given by

$$u = -\tfrac{3}{2}(v^2 + \epsilon\sqrt{2}v_x).$$

(e) Korteweg–de Vries versus Gardner equation

The Gardner equation,

$$v_t = \partial_x(v^2 - \alpha v^3 + \epsilon^2 v_{xx}), \tag{4.5}$$

is completely integrable, and it is known to be Miura equivalent to the KdV equation (4.2) via the following invertible transformation [21]

$$u = v - \tfrac{3}{2}\alpha v^2 - \tfrac{3}{2}\sqrt{2\alpha}\epsilon v_x.$$

We note that this transformation clearly does not preserve the dispersionless limit. In order to verify the consistency of the conjecture 1.1 with the above classical result, it is first necessary to reduce the dispersionless part of Gardner's equation (4.5) to the Hopf equation. This is done by introducing the variable w such that $v = (1 \pm \sqrt{1 - 6\alpha w})/(3\alpha)$, and the equation (4.5) takes the form

$$w_t = 2ww_x + \epsilon^2 \left(w_{xxx} + \frac{9\alpha}{1 - 6\alpha w} w_x w_{xx} + \frac{27\alpha^2}{(1 - 6\alpha w)^2} w_x^3 \right).$$

We immediately see that the above equation and the KdV equation (4.2) share one and the same quasi-linear part.

(f) Sawada–Kotera versus Kaup–Kuperschmidt equation

A direct comparison of Sawada–Kotera(SK)

$$u_t = \partial_x[\tfrac{5}{3}u^3 + \epsilon^2(5uu_{xx}) + \epsilon^4(u_{xxxx})]$$

and Kaup–Kuperschmidt (KK) equations

$$u_t = \partial_x[\tfrac{5}{3}u^3 + \epsilon^2(5uu_{xx} + \tfrac{15}{4}u_x^2) + \epsilon^4(u_{xxxx})]$$

clearly show that these two equation possess the same quasi-linear part and as a consequence of conjecture (1.1) they are expected to be Miura equivalent. Indeed, such a Miura transformation exists and was discovered by Fordy & Gibbons [22].

5. Concluding remarks

In this paper, we focused our attention on the study of equivalence classes of integrable dispersive conservation laws with respect to Miura transformations. The analysis of transformation properties of quasi-linear terms in the deformation and the results from the perturbative approach

Table 1. Summary of known and conjectured results in the scalar case.

type of deformations	no. invariants	references
general viscous deformations	2 (conjecture)	[3]
general dispersive deformations	∞ (conjecture)	this paper and [2]
viscous conservation laws	1 (conjecture)	[5]
dispersive conservation laws	∞ (conjecture)	this paper
Hamiltonian conservation laws	∞ (conjecture)	[6,23]
bi-Hamiltonian structures	1 (proved in [12])	[8,9,11,12,24]

suggest that equations sharing one and the same quasi-linear part are also Miura equivalent. This seems to be a general principle.

Table 1 aims at providing a summary of currently known classification results of scalar-integrable PDEs. We have also included the case of bi-Hamiltonian structures although in this case the invariant parameter is not directly related to the quasi-linear part of the corresponding equations.

Apart from the bi-Hamiltonian case where results at any order are already available, other results have been proved so far only up to a certain order in the deformation parameter ϵ. The number of independent functions parametrizing the quasi-linear part depends on the type of deformations. In the case of dispersive conservation laws apparently, the coefficients of the quasi-linear part can be arbitrarily chosen. A similar freedom has been observed for Hamiltonian conservation laws [23]. Note that Miura transformations involved in this case are canonical (this is not a restrictive choice as a consequence of an important theorem proved in [25]), and the comparison with dispersive conservation laws is not immediate (as explained in §3c).

Viscous deformations turn out to be much more rigid being parametrized by a single function of one variable for viscous conservation laws [5] and by two functions for general viscous equations [3].

We point out that although there is a certain flexibility in the choice of the functional parameters that characterize the deformation, it is more convenient to choose invariant parameters as they allow a direct comparison between different equations not necessarily brought to their normal form.

Apparently, the case of general dispersive equations is more general than the case of dispersive conservation laws. However, according to the main conjecture 1.1, integrable dispersive equations should be parametrized by the coefficients of the quasi-linear part. Because no additional free parameters are available, this would lead to the conclusion that general dispersive-integrable equations are necessarily conservative.

We conclude by mentioning that the case of integrable systems of conservation laws of the form

$$u_t^i = \partial_x[f^i(\mathbf{u}) + \epsilon(A_j^i(\mathbf{u})u_x^j) + \epsilon^2(B_j^i(\mathbf{u})u_{xx}^j + C_{jk}^i(\mathbf{u})u_x^j u_x^k) + \mathcal{O}(\epsilon^3)], \quad i = 1, \dots, n,$$

is completely open. For instance, even the generalization of the notion of normal form is not straightforward. Moreover, it is not difficult to check that the coefficients of the quasi-linear part do not transform as tensors under a general Miura transformation. We plan to tackle this case in a future publication.

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