

Microstates of rotating AdS₅ strings

Seyed Morteza Hosseini,^a Kiril Hristov^b and Alberto Zaffaroni^{c,d}

^a*Kavli IPMU (WPI), UTIAS, The University of Tokyo,
Kashiwa, Chiba 277-8583, Japan*

^b*Institute for Nuclear Research and Nuclear Energy, Bulgarian Academy of Sciences,
Tsarigradsko Chaussee 72, 1784 Sofia, Bulgaria*

^c*Dipartimento di Fisica, Università di Milano-Bicocca,
I-20126 Milano, Italy*

^d*INFN, sezione di Milano — Bicocca,
I-20126 Milano, Italy*

E-mail: morteza.hosseini@ipmu.jp, khristov@inrne.bas.bg,
alberto.zaffaroni@mib.infn.it

ABSTRACT: We provide a general formula for the refined topologically twisted index of $\mathcal{N} = 1$ gauge theories living on the world-volume of D3-branes at conical Calabi-Yau singularities in the Cardy limit. The index is defined as the partition function on $T^2 \times S^2_\omega$, with a partial topological twist and a Ω -deformation along S^2 , in the presence of background magnetic fluxes and fugacities for the global symmetries and can be used to study the properties of a class of BPS black strings. To this purpose, we find rotating domain-wall solutions of five-dimensional gauged supergravity interpolating between AdS₅ and a near horizon region consisting of a warped fibration of BTZ over a sphere. We explicitly construct rotating domain-walls that can be embedded in AdS₅ \times S^5 by uplifting a class of four-dimensional rotating black holes. We then provide a microscopic explanation of the entropy of such black holes by using the refined topologically twisted index of $\mathcal{N} = 4$ super Yang-Mills.

KEYWORDS: AdS-CFT Correspondence, Black Holes in String Theory, Supersymmetric Gauge Theory, Conformal Field Theory

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1 Introduction

In this paper we consider theories living on D3-branes sitting at the tip of a Calabi-Yau three-fold cone and the relation of their twisted compactifications on a sphere to AdS black hole and black string physics. The holographic description of these twisted compactifications consists of solutions interpolating between AdS₅ and AdS₃ × S² vacua. These can be interpreted as renormalization group (RG) flows from an ultraviolet (UV) four-dimensional $\mathcal{N} = 1$ conformal field theory (CFT) and an infrared (IR) two-dimensional $\mathcal{N} = (0, 2)$ one. The right-moving central charge of the two-dimensional CFT has been computed in [1–3], and successfully compared with the supergravity result for a variety of models. In [4] the partition function of the boundary theory on $T^2 \times S^2$, the so-called topologically twisted index [5–9], was computed in the Cardy limit $\beta = -2\pi i\tau \rightarrow 0$, where τ is the modular parameter of the torus, and at finite N . It was found that [4]

$$\log Z(\mathbf{n}, \Delta|\beta) = \frac{\pi^2}{6\beta} c_l(\mathbf{n}, \Delta/\pi), \tag{1.1}$$

where $c_l(\mathbf{n}, \Delta/\pi)$ is the left-moving trial central charge of the two-dimensional SCFT (see (3.1)), \mathbf{n} denotes the set of magnetic fluxes of the twisted compactification and Δ denotes the set of chemical potentials for the global symmetries of the theory. If we further compactify the black strings on a circle inside AdS_3 adding a momentum, we obtain a four-dimensional static black hole with a hvLif asymptotic behaviour [10]. As discussed in details in [4, 10–12], a microscopic counting based on (1.1) correctly reproduces the entropy of such black holes, which is just given by the Cardy formula in terms of the *exact* central charge $c_l^{\text{CFT}}(\mathbf{n})$ of the two-dimensional SCFT.

It is the purpose of this paper to extend the previous picture to the rotating case, by computing the relevant quantum field theory partition function and by finding rotating black string solutions that can be embedded in consistent string compactifications.

From the field theory point of view we need to compute the *refined* topologically twisted index [7], which is defined as the partition function of an $\mathcal{N} = 1$ theory on $T^2 \times S_\omega^2$ with a topological A -twist and an Ω -background label by a complex parameter ω on S^2 . It can be also written as the trace

$$Z(\mathbf{n}, y, \zeta|q) = \text{Tr}_{\mathcal{H}_{S_\omega^2 \times S^1}} (-1)^F q^{H_L} \zeta^{2J} \prod_I y_I^{Q_I}, \tag{1.2}$$

where $q = e^{2\pi i\tau}$, $y_I = e^{i\Delta_I}$, and $\zeta = e^{i\omega/2}$ are the fugacities associated to the left-moving Hamiltonian H_L , the flavor charges Q_I and the angular momentum J along S_ω^2 . We will show that, for a generic theory of D3-branes at toric (but not only) conical singularities, the Cardy limit $\beta = -2\pi i\tau \rightarrow 0$ (at finite N) of the topologically twisted index reads

$$\log Z(\mathbf{n}, \Delta, \omega|\beta) = \frac{\pi^2}{6\beta} c_l(\mathbf{n}, \Delta/\pi) - \frac{(2\omega)^2}{27\beta} (3c(\mathbf{n}) - 2a(\mathbf{n})), \tag{1.3}$$

where $c(\mathbf{n})$ and $a(\mathbf{n})$ are the trial central charges (see (3.16)) of the four-dimensional SCFT evaluated as functions of the magnetic fluxes.

From the gravity point of view, we will find a new class of rotating black strings that can be embedded in $\text{AdS}_5 \times S^5$. In five-dimensional language, these are domain walls that interpolate between AdS_5 and a near horizon region consisting of a warped fibration of BTZ over a sphere. To find these solutions, it is convenient to dimensionally reduce them to four dimensions, as also suggested by the Cardy limit we are performing in field theory, and construct the corresponding rotating asymptotically hvLif black holes. Luckily, a large class of dyonic rotating black holes in four-dimensional $\mathcal{N} = 2$ gauged supergravity have been found recently in [13] and we will use these results. Using the 5D/4D relation, we can first find the solutions in the so-called STU model in four-dimensional $\mathcal{N} = 2$ gauged supergravity coupled to vector multiplets, then uplift them to a five-dimensional gauged supergravity, which is a consistent truncation of type IIB on $\text{AdS}_5 \times S^5$, and, finally we could uplift our solutions to type IIB. We will present the general class of solutions for the STU model and, more generally, for symmetric models of gauged supergravity with vector multiplets. Finding solutions in consistent truncations of type IIB compactifications on more general Sasaki-Einstein manifolds is more complicated and we leave it for future work.

We will then show that, for the twisted compactification of $\mathcal{N} = 4$ super Yang-Mills (SYM), the Legendre transform of (1.3) exactly reproduces the entropy of the dyonic rotating black holes that can be embedded in $\text{AdS}_5 \times S^5$. This is another instance of the \mathcal{I} -extremization principle introduced in [14, 15] and that has been successfully used to give a microscopic explanation of the entropy of BPS black holes in diverse dimensions.

In this context, it is also interesting to rewrite the large N limit of (1.3) as

$$\log Z(\mathbf{n}, \Delta, \omega|\beta) = i \sum_I \mathbf{n}_I \frac{\partial \widetilde{\mathcal{W}}(\Delta|\beta)}{\partial \Delta_I} + \frac{i\omega^2}{24} \sum_{I,J,K} \mathbf{n}_I \mathbf{n}_J \mathbf{n}_K \frac{\partial^3 \widetilde{\mathcal{W}}(\Delta|\beta)}{\partial \Delta_I \partial \Delta_J \partial \Delta_K}, \quad (1.4)$$

where

$$\widetilde{\mathcal{W}}(\Delta|\beta) = \frac{16\pi^3 i}{27\beta} a(\Delta/\pi), \quad (1.5)$$

is the on-shell value of the twisted superpotential evaluated on the Bethe vacuum [5, 9, 14] that dominates the index in the large N limit. The first term in (1.4) was already derived in [4]. It appears in the same form in the expression of the three-dimensional topologically twisted index at large N [14, 16, 17]. It has been called *index theorem* and it is the field theory counterpart of the attractor mechanism of gauged supergravity [18–20]. It also appears, in a similar form, in the five-dimensional topologically twisted index at large N [21].

The paper is organized as follows. In section 2 we review the definition of the refined topologically twisted index on $T^2 \times S^2_\omega$. In section 3 we analyse the index in the Cardy limit and derive (1.3). In section 4 we use the 4D/5D connection to explicitly construct black string solutions that can be embedded in $\text{AdS}_5 \times S^5$ and we compute the entropy of the corresponding four-dimensional black holes. In section 5 we compare the field theory and supergravity result finding complete agreement. We conclude in section 6 with discussions and open problems.

2 The refined topologically twisted index

Consider an $\mathcal{N} = 1$ gauge theory with vector and chiral multiplets in a representation $\oplus_I \mathfrak{R}_I$ of the gauge group G , and a non-anomalous $U(1)_R$ symmetry in four dimensions. The topologically twisted index for this class of gauge theories is defined as the (Euclidean) partition function on $T^2 \times \Sigma_{\mathfrak{g}}$, with a partial topological A -twist along the genus \mathfrak{g} Riemann surface $\Sigma_{\mathfrak{g}}$ [7]. It depends on the complex structure of the torus $q = e^{2\pi i \tau}$, fugacities $y_I = e^{i\Delta_I}$ and magnetic fluxes \mathbf{n}_I (that are parameterizing the twist) for the global symmetries of the theory. In the case of $\mathfrak{g} = 0$ one can refine the index by the angular momentum on S^2 and introduce the fugacity $\zeta = e^{i\omega/2}$. The index can be computed using supersymmetric localization and it is given by a matrix integral over the zero mode gauge variables $x = e^{iu}$ parameterizing the Wilson lines on the two directions of the torus

$$u = 2\pi \oint_{\text{A-cycle}} A - 2\pi\tau \oint_{\text{B-cycle}} A, \quad (2.1)$$

which are defined modulo

$$u_i \sim u_i + 2\pi n + 2\pi m\tau, \quad n, m \in \mathbb{Z}. \quad (2.2)$$

The result is summed over a lattice of gauge magnetic fluxes \mathbf{m} (up to gauge transformations) on $\Sigma_{\mathfrak{g}}$ living in the co-root lattice $\Gamma_{\mathfrak{h}}$ of the gauge group.

The *refined* topologically twisted index is thus explicitly given by a contour integral of a meromorphic differential form [7]¹

$$\begin{aligned} Z(\mathbf{m}, x; \mathbf{n}, y, \zeta|q) &= \frac{1}{|\mathfrak{W}|} \sum_{\mathbf{m} \in \Gamma_{\mathfrak{h}}} \int_{\text{JK}} \left(\prod_{\text{Cartan}} \frac{dx}{2\pi i x} \eta(q)^2 \right) (-1)^{\sum_{\alpha>0} \alpha(\mathbf{m})} \prod_{\alpha \in G} \left(\frac{\theta_1(x^\alpha \zeta^{|\alpha(\mathbf{m})|}; q)}{i\eta(q)} \right) \\ &\times \prod_I \prod_{\rho_I \in \mathfrak{R}_I} \prod_{j=-\frac{|B_I|-1}{2}}^{\frac{|B_I|-1}{2}} \left(\frac{i\eta(q)}{\theta_1(x^{\rho_I} y^{\nu_I} \zeta^{2j}; q)} \right)^{\text{sign}(B_I)}, \end{aligned} \quad (2.3)$$

summed over $\mathbf{m} \in \Gamma_{\mathfrak{h}}$, where $B_I = \rho_I(\mathbf{m}) - \nu_I(\mathbf{n}) + 1$ and $|\mathfrak{W}|$ is the order of the Weyl group of G . Here, α denotes the roots of G , and ρ_I, ν_I are the weights of the chiral multiplets under the gauge and flavor symmetry group, respectively. Moreover, $\eta(q)$ is the Dedekind eta function and $\theta_1(x; q)$ is a Jacobi theta function (see appendix A). Given the vanishing of the gauge and the gauge-flavor anomalies, the integrand in (2.3) is a well-defined meromorphic function on the torus.

Finally, the invariance of the superpotential of the theory $W = \sum_a W_a$ under global symmetries imposes the constraint

$$\prod_{I \in W_a} y_I = 1, \quad \text{or} \quad \sum_{I \in W_a} \Delta_I \in 2\pi\mathbb{Z}, \quad (2.4)$$

where the product and the sum are restricted to the fields entering in the monomial W_a . We have a similar constraint on flavor magnetic fluxes

$$\sum_{I \in W_a} \mathbf{n}_I = 2, \quad (2.5)$$

that we call the *twisting condition* and it corresponds to the cancellation of the spin connection by the background R-symmetry gauge field.

3 The Cardy limit

In this section, we analyze the Cardy limit $\tau \rightarrow i0^+$ of the *refined* topologically twisted index of $\mathcal{N} = 1$ theories that are associated with D3-branes at conical toric Calabi-Yau three-fold singularities. When a four-dimensional $\mathcal{N} = 1$ theory is compactified on S^2 with a topological twist, it might flow to a family of $\mathcal{N} = (0, 2)$ SCFTs in the IR that are labeled by a set of magnetic fluxes \mathbf{n}_I parameterizing the twist. The holographic dual

¹Supersymmetric localization selects a particular contour of integration and the final result can be cast in terms of the Jeffrey-Kirwan residue [7].

to such a SCFT is a warped background $\text{AdS}_3 \times_w Y_7$ in type IIB supergravity, where Y_7 is topologically a fibration of Y_5 over S^2 , with Y_5 being the Sasaki-Einstein base of the Calabi-Yau cone.

The right-moving central charge $c_r(\mathbf{n}, r)$ and the gravitational anomaly $k = c_r - c_l$ of the $\mathcal{N} = (0, 2)$ SCFT read [1, 2]

$$c_r(\mathbf{n}, r) = 3 \text{Tr} \gamma_3 R^2(r_I) = -3 \left[\dim G + \sum_I \dim \mathfrak{R}_I(\mathbf{n}_I - 1) (r_I - 1)^2 \right], \quad (3.1)$$

$$k(\mathbf{n}) = \text{Tr} \gamma_3 = - \dim G - \sum_I \dim \mathfrak{R}_I(\mathbf{n}_I - 1),$$

where $R(r_I)$ is the matrix of R-charges for the fermionic fields in the theory and γ_3 is the two-dimensional chirality operator. The massless fermions arise as zero modes of four-dimensional fields. The difference between the number of fermions of opposite chiralities is easily computed using the Riemann-Roch theorem and is equal to $-\dim G$ for the gaugino zero modes and $-\dim \mathfrak{R}_I(\mathbf{n}_I - 1)$ for the zero modes of four-dimensional matter fields in a representation \mathfrak{R}_I [1, 2].² The exact R-charges of the SCFT are obtained by extremizing $c_r(\mathbf{n}, r)$ with respect to a generic assignment of trial R-charges r_I satisfying

$$\sum_{I \in W_a} r_I = 2. \quad (3.2)$$

This principle is known as c -extremization and it has been successfully compared with the prediction of holography for a large class of twisted compactifications [1–3].³ In the large N limit, for theories with a holographic dual, $k = 0$ and $c_l = c_r$ [29].

In the following we will identify the modulus of the torus with the *fictitious*⁴ inverse temperature β and work in the Cardy limit

$$\beta \rightarrow 0 \quad \text{with} \quad \beta \equiv -2\pi i \tau. \quad (3.3)$$

The refined topologically twisted index as $\beta \rightarrow 0$ can be written as

$$Z(\mathbf{m}, u; \mathbf{n}, \Delta, \omega | \beta) \sim \sum_{\mathbf{m} \in \Gamma_{\mathfrak{h}}} \int_{\text{JK}} \left(\prod_{\text{Cartan}} i du e^{-\frac{\pi^2}{3\beta}} \right) \prod_{\alpha \in G} e^{-\frac{1}{\beta} g_2(\alpha(u)) - \frac{\omega^2}{8\beta} \alpha(\mathbf{m})^2} \times \prod_I \prod_{\rho_I \in \mathfrak{R}_I} e^{\frac{1}{\beta} B_I g_2(\rho_I(u) + \nu_I(\Delta)) + \frac{\omega^2}{4\pi^3 \beta} g_3(\pi(B_I + 1))}, \quad (3.4)$$

where we used (A.6), (A.7), and

$$\frac{\text{sign}(B)}{\beta} \sum_{j=-\frac{|B|-1}{2}}^{\frac{|B|-1}{2}} g_2(\rho(u) + \nu_I(\Delta) + j\omega) = \frac{B}{\beta} g_2(\rho(u) + \nu_I(\Delta)) + \frac{\omega^2}{4\pi^3 \beta} g_3(\pi(B + 1)). \quad (3.5)$$

²We use the conventions of [1, 2] that differ by a sign in the definition of γ_3 compared to [4].

³For further developments see [4, 21–28].

⁴The elliptic genus is only counting extremal states and thus the temperature represented by $\text{Im} \tau$ is fictitious.

The polynomial functions $g_s(u)$, $s = 2, 3$, are defined in (A.8). We assumed that $\Re(\rho(u) + \nu_I(\Delta) + j\omega) > 0$. As we will see, this condition is satisfied on the relevant saddle point and in the regime of parameters that lead to a regular black hole, which in particular requires ω to be imaginary.

Let us now focus on the terms proportional to ω^2/β in (3.4), i.e.

$$Z_\omega(\mathbf{m}, u; \mathbf{n}, \Delta|\beta) = \prod_{\alpha \in G} e^{-\frac{\omega^2}{8\beta} \alpha(\mathbf{m})^2} \prod_I \prod_{\rho_I \in \mathfrak{R}_I} e^{\frac{\omega^2}{4\pi^3\beta} g_3(\pi(B_I+1))}. \quad (3.6)$$

Remarkably, the dependence on gauge magnetic fluxes \mathbf{m} drops out of (3.6) due to the following anomaly cancellation conditions:

$$\begin{aligned} \sum_{\alpha \in G} \alpha(\mathbf{m})^2 + \sum_I \sum_{\rho_I \in \mathfrak{R}_I} (\nu_I(\mathbf{n}) - 1) \rho_I(\mathbf{m})^2 &= 0, & \text{U(1)}_R\text{-gauge}^2, \\ \sum_I \sum_{\rho_I \in \mathfrak{R}_I} \rho_I(\mathbf{m})^3 &= 0, & \text{gauge}^3, \\ \sum_I \sum_{\rho_I \in \mathfrak{R}_I} \rho_I(\mathbf{m}) &= 0, & \text{gravitational}^2\text{-gauge}, \\ \sum_I \sum_{\rho_I \in \mathfrak{R}_I} \rho_I(\mathbf{m}) \nu_I(\mathbf{n})^2 &= 0, & \text{U(1)}_R^2\text{-gauge}, \\ \sum_I \sum_{\rho_I \in \mathfrak{R}_I} \rho_I(\mathbf{m}) \nu_I(\mathbf{n}) &= 0, & \text{U(1)}_R\text{-gravitational-gauge}, \end{aligned} \quad (3.7)$$

which are satisfied for all consistent theories of D3-branes at conical singularities.⁵

We therefore find that

$$Z_\omega(u; \mathbf{n}, \Delta|\beta) = \prod_I \prod_{\rho_I \in \mathfrak{R}_I} e^{-\frac{\omega^2}{4\pi^3\beta} g_3(\pi\nu_I(\mathbf{n}))}. \quad (3.8)$$

Hence, the refined twisted index (3.4) can be further simplified to

$$Z(\mathbf{m}, u; \mathbf{n}, \Delta, \omega|\beta) \sim Z_0(\mathbf{m}, u; \mathbf{n}, \Delta|\beta) \prod_I \prod_{\rho_I \in \mathfrak{R}_I} e^{-\frac{\omega^2}{4\pi^3\beta} g_3(\pi\nu_I(\mathbf{n}))}, \quad (3.9)$$

where Z_0 is the *unrefined* twisted index in the Cardy limit [4]

$$Z_0(\mathbf{m}, u; \mathbf{n}, \Delta|\beta) \sim \sum_{\mathbf{m} \in \Gamma_b} \int_{\text{JK}} \left(\prod_{\text{Cartan}} \text{idu} e^{-\frac{\pi^2}{3\beta}} \right) \prod_{\alpha \in G} e^{-\frac{1}{\beta} g_2(\alpha(u))} \prod_I \prod_{\rho_I \in \mathfrak{R}_I} e^{\frac{1}{\beta} B_I g_2(\rho_I(u) + \nu_I(\Delta))}. \quad (3.10)$$

Notice that the correction term is independent of u , Δ and \mathbf{m} , so we can easily write the partition function for generic ω if we know the unrefined twisted index.

The unrefined twisted index in the Cardy limit — and at finite N — has been studied in details in [4]. It can be evaluated by first resumming the geometric series associated with the gauge magnetic fluxes \mathbf{m} . Using the residue theorem, one can reduce the partition function

⁵Notice that, due to (2.5), the fluxes \mathbf{n}_a effectively parameterize an R-symmetry.

to a sum over Bethe vacua, the critical points of the twisted superpotential $\mathcal{W}(u; \Delta|\beta)$ obtained by dimensionally reducing the theory on T^2 [5, 14, 30].⁶ In the Cardy limit, one particular vacuum dominates the partition function. For a theory of D3-branes at conical singularities with $SU(N)$ gauge groups the distribution of eigenvalues was found in [4]. It is the same for all gauge groups and reads

$$u_i - u_j = -\frac{i\beta}{N}(i - j) \quad i, j = 1, \dots, N. \quad (3.11)$$

As shown in [34], this is actually a solution to the Bethe equations for arbitrary β . The unrefined twisted index in the Cardy limit and at finite N can then be compactly written as [4]

$$\log Z_0(\mathbf{n}, \Delta|\beta) = \frac{\pi^2}{6\beta} c_l(\mathbf{n}, \Delta/\pi), \quad \text{as } \beta \rightarrow 0, \quad (3.12)$$

where $c_l(\mathbf{n}, \Delta/\pi)$ is the trial left-moving central charge of the $\mathcal{N} = (0, 2)$ SCFT in the IR.⁷ The result is consistent with the expectation that the topologically twisted index is computing the elliptic genus of the two-dimensional SCFT. (3.12) is indeed nothing else than the supersymmetric version of *Cardy's formula* for the high-temperature behavior of the partition function of a CFT. The result (3.12) was obtained by using (A.6), (A.7) and assumes that $\text{Re}(\Delta_I) \in [0, 2\pi]$ and

$$\sum_{I \in W_a} \Delta_I = 2\pi, \quad (3.13)$$

for each term W_a in the superpotential.⁸ We see that Δ_I/π effectively parameterize a choice of R-charges for the fields of the theory. Notice also that our working condition $\text{Re}(\rho(u) + \nu_I(\Delta) + j\omega) > 0$ is satisfied if ω is imaginary, since the distribution of eigenvalues satisfies (3.11).

Plugging back (3.12) into (3.9) we finally arrive at the following expression for the Cardy limit of the refined twisted index

$$\begin{aligned} \log Z(\mathbf{n}, \Delta, \omega|\beta) &= \frac{\pi^2}{6\beta} c_l(\mathbf{n}, \Delta/\pi) - \frac{\omega^2}{4\pi^3\beta} \sum_I \dim \mathfrak{R}_I g_3(\pi\mathbf{n}_I) \\ &= \frac{\pi^2}{6\beta} c_l(\mathbf{n}, \Delta/\pi) - \frac{\omega^2}{24\beta} (\text{Tr } R^3(\mathbf{n}) - \text{Tr } R(\mathbf{n})) \\ &= \frac{\pi^2}{6\beta} c_l(\mathbf{n}, \Delta/\pi) - \frac{(2\omega)^2}{27\beta} (3c(\mathbf{n}) - 2a(\mathbf{n})), \quad \text{as } \beta \rightarrow 0. \end{aligned} \quad (3.14)$$

In order to write (3.14) we introduced the traces over four-dimensional fermionic fields

$$\begin{aligned} \text{Tr } R^3(r_I) &= \dim G + \sum_I \dim \mathfrak{R}_I (r_I - 1)^3, \\ \text{Tr } R(r_I) &= \dim G + \sum_I \dim \mathfrak{R}_I (r_I - 1), \end{aligned} \quad (3.15)$$

⁶The Bethe vacua approach has been useful in the microscopic counting of states for many black holes and black strings in diverse dimensions [4, 14, 21, 31–33].

⁷It can be obtained from (3.1) as $c_l(\mathbf{n}, \Delta/\pi) = c_r(\mathbf{n}, \Delta/\pi) - k(\mathbf{n})$.

⁸The condition $\text{Re}(\Delta_I) \in [0, 2\pi]$ implies that $\sum_{I \in W_a} \Delta_I$ cannot be zero. Among the possible choices allowed by (2.4), only $\sum_{I \in W_a} \Delta_I = 2\pi$ leads to a physically acceptable solution [4].

where r_I is the R-charge of the I -th chiral multiplet, related by

$$a(r_I) = \frac{9}{32} \text{Tr} R^3(r_I) - \frac{3}{32} \text{Tr} R(r_I), \quad c(r_I) = \frac{9}{32} \text{Tr} R^3(r_I) - \frac{5}{32} \text{Tr} R(r_I), \quad (3.16)$$

to the trial central charges of the four-dimensional $\mathcal{N} = 1$ theory [35]. Notice that, due to the twisting condition (2.5), the fluxes \mathbf{n}_I parameterize an integer choice of R-charges for the chiral fields of the theory.

In the large N limit, $a = c$ and $c_l = c_r$. We can thus further simplify (3.14) and write

$$\log Z(\mathbf{n}, \Delta, \omega|\beta) = \frac{\pi^2}{6\beta} \left(c_r(\mathbf{n}, \Delta/\pi) - \frac{8\omega^2}{9\pi^2} a(\mathbf{n}) \right), \quad \text{as } \beta \rightarrow 0, N \gg 1. \quad (3.17)$$

For theories associated with D3-branes at toric Calabi-Yau singularities, we can always choose a convenient parameterization of the general R-charges of the chiral fields satisfying (2.5) in terms of a minimal set of d quantities r_a with

$$\sum_{a=1}^d r_a = 2, \quad (3.18)$$

where d is the number of global symmetries of the theory [36, 37]. In this convenient parameterization, the central charge is a homogeneous function of degree three [38]

$$a(r) \equiv \frac{9N^2}{64} \sum_{a,b,c=1}^d c_{abc} r_a r_b r_c, \quad (3.19)$$

where c_{abc} are proportional to the 't Hooft anomaly coefficients. For a toric quiver $c_{abc} = |\det(v_a, v_b, v_c)|$, where $v_a \in \mathbb{Z}^3$ are the integer vectors defining the toric diagram.⁹ Moreover, as shown in [4, 28], in the large N limit the two-dimensional central charge can be written as

$$c_r(\mathbf{n}, r) = -\frac{32}{9} \sum_{a=1}^d \mathbf{n}_a \frac{\partial a(r_a)}{\partial r_a} = -\frac{3N^2}{2} \sum_{a,b,c=1}^d c_{abc} \mathbf{n}_a r_b r_c. \quad (3.20)$$

It is convenient, also for comparison with other microstate counting for black holes and black strings, to express all quantities in terms of the *on-shell* value of the effective twisted superpotential of the theory, $\widetilde{\mathcal{W}}(\Delta|\beta)$, which is by definition $\mathcal{W}(u; \Delta|\beta)$ evaluated on the Bethe vacuum solution (3.11). It is related to the a central charge of the four-dimensional $\mathcal{N} = 1$ theory as [4]

$$\widetilde{\mathcal{W}}(\Delta|\beta) = \frac{16\pi^3 i}{27\beta} a(\Delta/\pi) = \frac{iN^2}{12\beta} \sum_{a,b,c} c_{abc} \Delta_a \Delta_b \Delta_c, \quad \text{for } N \gg 1. \quad (3.21)$$

Combining (3.17) and (3.20) we can finally write the compact expression

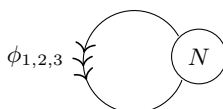
$$\log Z(\mathbf{n}, \Delta, \omega|\beta) = i \sum_{a=1}^d \mathbf{n}_a \frac{\partial \widetilde{\mathcal{W}}(\Delta|\beta)}{\partial \Delta_a} + \frac{i\omega^2}{24} \sum_{a,b,c=1}^d \mathbf{n}_a \mathbf{n}_b \mathbf{n}_c \frac{\partial^3 \widetilde{\mathcal{W}}(\Delta|\beta)}{\partial \Delta_a \partial \Delta_b \partial \Delta_c}. \quad (3.22)$$

The first term in (3.22) was already derived in [4, section 5.3].

We now present two simple examples, $\mathcal{N} = 4$ super Yang-Mills and the Klebanov-Witten theory [39].

⁹We redefined $c_{abc}^{\text{here}} \equiv \frac{2}{N^2} c_{abc}^{\text{there}}$ in comparison with [28, eq. (2.7)].

$\mathcal{N} = 4$ super Yang-Mills. Our first example is the $\mathcal{N} = 4$ super Yang-Mills (SYM) theory whose $\mathcal{N} = 1$ quiver is depicted below.



$$(3.23)$$

Here the circular node is a $SU(N)$ gauge group and the loop around it denotes the adjoint chiral multiplets ϕ_a , $a = 1, 2, 3$, interacting through the cubic superpotential

$$W = \text{Tr}(\phi_3[\phi_1, \phi_2]) . \quad (3.24)$$

We denote the chemical potentials associated to the chiral fields ϕ_a by Δ_a , and assign the flux \mathbf{n}_a to each chiral field. Then, the invariance of (3.24) under the global symmetries of the theory imposes the constraint

$$\sum_{a=1}^3 \Delta_a = 2\pi . \quad (3.25)$$

A similar constraint exists on the magnetic fluxes \mathbf{n}_a , see (2.5). The trial central charges, at finite N , are easily computed from (3.16) and read¹⁰

$$a(r) = c(r) = \frac{27(N^2 - 1)}{32} r_1 r_2 r_3 , \quad (3.26)$$

with $\sum_{a=1}^3 r_a = 2$. The only nonzero anomaly coefficients are $c_{123} = 1$ and permutations thereof. For $\mathcal{N} = 4$ SYM compactified on S^2 the gravitational anomaly of the two-dimensional $\mathcal{N} = (0, 2)$ theory is zero, $k = 0$, and thus $c_l = c_r$. The trial right-moving central charge c_r is given by (3.1),

$$c_r(\mathbf{n}, r) = -3(N^2 - 1)(r_1 r_2 \mathbf{n}_3 + r_2 r_3 \mathbf{n}_1 + r_1 r_3 \mathbf{n}_2) . \quad (3.27)$$

Extremizing (3.27) with respect to r_a , $a = 1, 2$, we obtain

$$\bar{r}_a^{\mathcal{N}=4} = \frac{4\mathbf{n}_a(\mathbf{n}_a - 1)}{\Theta_{\mathcal{N}=4}} , \quad a = 1, 2 , \quad (3.28)$$

where we defined

$$\Theta_{\mathcal{N}=4} \equiv \mathbf{n}_1^2 + \mathbf{n}_2^2 + \mathbf{n}_3^2 - 2(\mathbf{n}_1 \mathbf{n}_2 + \mathbf{n}_1 \mathbf{n}_3 + \mathbf{n}_2 \mathbf{n}_3) . \quad (3.29)$$

Plugging (3.28) back into (3.27) we find the exact central charge of the 2D CFT as

$$c_r^{\text{CFT}}(\mathbf{n}) = 12(N^2 - 1) \frac{\mathbf{n}_1 \mathbf{n}_2 \mathbf{n}_3}{\Theta_{\mathcal{N}=4}} . \quad (3.30)$$

Finally, the refined twisted index in the Cardy limit is given by (3.14),

$$\log Z_{\mathcal{N}=4}(\mathbf{n}, \Delta | \beta) = -\frac{(N^2 - 1)}{8\beta} (4\Delta_1 \Delta_2 \mathbf{n}_3 + 4\Delta_2 \Delta_3 \mathbf{n}_1 + 4\Delta_1 \Delta_3 \mathbf{n}_2 + \omega^2 \mathbf{n}_1 \mathbf{n}_2 \mathbf{n}_3) . \quad (3.31)$$

¹⁰For $\mathcal{N} = 4$ SYM we have $\text{Tr} R(r_a) = 0$.

The Klebanov-Witten theory. Our second example is the twisted compactification of the Klebanov-Witten theory [39]. It is an $\mathcal{N} = 1$ gauge theory with gauge group $SU(N) \times SU(N)$ and two pairs of bi-fundamental chiral fields $\{A_i, B_j\}$, $i, j = 1, 2$, in the representations $(\mathbf{N}, \overline{\mathbf{N}})$ and $(\overline{\mathbf{N}}, \mathbf{N})$, respectively. The matter content is described by the quiver diagram


(3.32)

and there is a quartic superpotential

$$W = \text{Tr}(A_1 B_1 A_2 B_2 - A_1 B_2 A_2 B_1). \quad (3.33)$$

We assign chemical potentials Δ_a and fluxes \mathbf{n}_a , $a = 1, \dots, 4$, to the fields $\{A_1, A_2, B_1, B_2\}$ in this order. The invariance of the superpotential (3.33) under the global symmetries of the theory imposes the constraint

$$\sum_{a=1}^4 \Delta_a = 2\pi. \quad (3.34)$$

The twisting condition (2.5) also reads

$$\sum_{a=1}^4 \mathbf{n}_a = 2. \quad (3.35)$$

The trial central charges of the four-dimensional theory (at finite N) are given by (3.16),

$$a(r) = \frac{27N^2}{32} \sum_{a<b<c} r_a r_b r_c - \frac{3}{8}, \quad c(r) = \frac{27N^2}{32} \sum_{a<b<c} r_a r_b r_c - \frac{1}{4}, \quad (3.36)$$

where again r_a parameterize an R-symmetry of the theory and $\sum_{a=1}^4 r_a = 2$. For later convenience we introduce the function

$$\Theta_{\text{KW}}(\mathbf{n}) = \frac{\varsigma(\mathbf{n})}{\vartheta(\mathbf{n})}, \quad (3.37)$$

where we defined

$$\varsigma(\mathbf{n}) \equiv \frac{1}{3} \left(4 \sum_{a=1}^4 \mathbf{n}_a^3 - \sum_{a,b,c=1}^4 \mathbf{n}_a \mathbf{n}_b \mathbf{n}_c \right) \sum_{a<b<c} \mathbf{n}_a \mathbf{n}_b \mathbf{n}_c, \quad \vartheta(\mathbf{n}) \equiv \sum_{\substack{a<b \\ (\neq c)}} \mathbf{n}_a \mathbf{n}_b \mathbf{n}_c^2. \quad (3.38)$$

The trial central charges of the two-dimensional $\mathcal{N} = (0, 2)$ theory in the IR read (3.1),

$$c_r(\mathbf{n}, r) = -3N^2 \sum_{\substack{a<b \\ (\neq c)}} r_a r_b \mathbf{n}_c + 6, \quad k = 2, \quad (3.39)$$

$$c_l(\mathbf{n}, r) = c_r(\mathbf{n}, r) - k.$$

Extremizing c_r in (3.39) with respect to r_a , $a = 1, 2, 3$, under the constraint $\sum_{a=1}^4 r_a = 2$, yields the critical points

$$\bar{r}_a^{\text{KW}} = \frac{12 \mathbf{n}_a^2 \left(\mathbf{n}_a^2 - \frac{1}{2} \sum_{a=1}^4 \mathbf{n}_a^2 \right) - \sum_{a < b < c < d} \mathbf{n}_a \mathbf{n}_b \mathbf{n}_c \mathbf{n}_d}{\mathbf{n}_a \left(4 \sum_{a=1}^4 \mathbf{n}_a^3 - \sum_{a,b,c=1}^4 \mathbf{n}_a \mathbf{n}_b \mathbf{n}_c \right)}, \quad a = 1, 2, 3. \quad (3.40)$$

Substituting this back into (3.39) we obtain

$$c_r^{\text{CFT}}(\mathbf{n}) = \frac{12N^2}{\Theta_{\text{KW}}} \sum_{a < b < c} \mathbf{n}_a \mathbf{n}_b \mathbf{n}_c + 6. \quad (3.41)$$

Finally, the refined twisted index in the Cardy limit is given by (3.14),

$$\log Z_{\text{KW}}(\mathbf{n}, \Delta | \beta) = -\frac{N^2}{2\beta} \sum_{\substack{a < b \\ (\neq c)}} \Delta_a \Delta_b \mathbf{n}_c + \frac{2\pi^2}{3\beta} - \frac{\omega^2 N^2}{8\beta} \sum_{a < b < c} \mathbf{n}_a \mathbf{n}_b \mathbf{n}_c. \quad (3.42)$$

4 Rotating black strings in AdS₅

In this section we consider rotating black string solutions in five-dimensional gauged supergravity.¹¹ Some of these solutions can be embedded in AdS₅ × S⁵ and represent the holographic dual of the field theory setting discussed in the previous section for $\mathcal{N} = 4$ SYM. It is convenient, both from supergravity and field theory perspective, to perform a dimensional reduction to a four-dimensional black hole which carries momentum along the compactification circle. As we will see, it is the entropy of such black holes that can be directly compared with the field theory microscopic counting performed in section 3.

As shown in [10], asymptotically AdS₅ black string solutions of five-dimensional gauged supergravity can be reduced to four-dimensional solutions in a gauged supergravity *without* a maximally symmetric vacuum solution. In four dimensions the resulting black hole solutions can be thought of as having runaway asymptotics of the hvLif type with particular exponents as described in more detail in [13, section 3.1]. A similar construction was successfully used to relate the Gutowski-Reall black holes in AdS₅ and their generalizations to four-dimensional black holes [41].

Here we want to generalize the original construction presented in [10] by allowing for rotation. Since the 5D/4D relation actually provides a one-to-one map, we take the approach of first writing down the solutions in four dimensions, and then uplifting them to five dimensions. The four-dimensional black holes preserve supersymmetry with a twist, and the general solutions of this type that include rotation were recently found in [13] for arbitrary symmetric models of gauged supergravity with vector multiplets. Here we will discuss a particular subclass of symmetric models and will explicitly write down the

¹¹To be more precise, we consider black strings with spherical topology and rotation along the sphere. For rotating *hyperbolic* black strings with noncompact horizon, see [40].

relevant solutions of [13]. We will discuss in particular the STU model that can be embedded in $\text{AdS}_5 \times S^5$. Unfortunately, more general compactifications $\text{AdS}_5 \times \text{SE}_5$ based on Sasaki-Einstein manifolds are not included in this construction since the known consistent truncations contain hypermultiplets, see [42–44]. Additionally, these truncations do not yet include all abelian isometries of the internal spaces to allow for an interesting comparison with field theory.

4.1 The 5D/4D relation

Consider a five-dimensional $\mathcal{N} = 2$ gauged supergravity theory coupled to n_V vector multiplets. The bosonic fields are the metric, n_V abelian gauge fields $A_{(5)}^i$ and real scalar fields L^i ($i = 1, \dots, n_V$) parameterizing the manifold $\frac{1}{6}c_{ijk}L^iL^jL^k = 1$. Here c_{ijk} is a fully symmetric tensor appearing in the Chen-Simons terms and it corresponds to the 't Hooft anomaly coefficients of the dual $\mathcal{N} = 1$ four-dimensional CFT [38, 45]. The rules for reducing the bosonic fields along the circle x^5 are the following [46–49]:

$$\begin{aligned} ds_{(5)}^2 &= e^{2\varphi} ds_{(4)}^2 + e^{-4\varphi} \left(dx^5 - A_{(4)}^0\right)^2, & dx^5 &= dy, \\ A_{(5)}^i &= A_{(4)}^i + \mathbb{R}e z^i \left(dx^5 - A_{(4)}^0\right), & & (4.1) \\ L^i &= e^{2\varphi} \mathbb{I}m z^i, & e^{-6\varphi} &= \frac{1}{6}c_{ijk} \mathbb{I}m z^i \mathbb{I}m z^j \mathbb{I}m z^k. \end{aligned}$$

Here $ds_{(4)}^2$ denotes the four-dimensional line element, $A_{(4)}^I$ ($I = 0, i$) are the four-dimensional abelian gauge fields and z^i are the complex scalar fields in four dimensions. The four-dimensional theory has n_V abelian vector multiplets, parameterizing a special Kähler manifold \mathcal{M} with metric $g_{i\bar{j}}$, in addition to the gravity multiplet (thus a total of $n_V + 1$ gauge fields and n_V complex scalars). The scalar manifold is defined by the prepotential $\mathcal{F}(X^I)$, which is a homogeneous holomorphic function of sections X^I ,

$$\mathcal{F}(X^I) = \frac{1}{6} \frac{c_{ijk} X^i X^j X^k}{X^0}. \quad (4.2)$$

In $\mathcal{N} = 2$ gauged supergravity in four dimensions the $U(1)_R$ symmetry, rotating the gravitini, is gauged by a linear combination of the abelian gauge fields. The coefficients are called Fayet-Iliopoulos (FI) parameters g_I and n_V of them can be directly read off from the five-dimensional theory: g_i .¹² The last coefficient, g_0 , measuring how the Kaluza-Klein gauge potential $A_{(4)}^0$ enters the R-symmetry, we choose to be vanishing since a standard Kaluza-Klein reduction suffices our purposes here (as opposed to a more general Scherk-Schwarz reduction, see [41]).

4.2 The 4D rotating solutions: asymptotics and near horizon

The definition of the four-dimensional models, as well as the subsequent solution, are most commonly written in a duality covariant language in terms of symplectic vectors. We also

¹²In consistent models one can always apply an electric-magnetic duality transformation so that the corresponding gauging becomes purely electric, i.e., $g^I = 0$.

extensively use the formalism where quantities are written in terms of the quartic invariant I_4 as explained in [13]. In particular, the model we are interested in is specified by the prepotential (4.2), where the constant tensor c_{ijk} further satisfies the identity

$$\frac{4}{3} \delta_{i(l} c_{mpq)} = c_{ijk} c_{j'(lm} c_{pq)k'} \delta^{jj'} \delta^{kk'}, \quad (4.3)$$

from the requirement that the scalar manifold is a symmetric space. This guarantees the existence of a rank 4 symplectic tensor, which upon contraction with a generic charge vector $\Gamma = \{p^I; q_I\}$ defines the quartic form

$$I_4(\Gamma) = -(p^0 q_0 + p^i q_i)^2 + \frac{2}{3} q_0 c_{ijk} p^i p^j p^k - \frac{2}{3} p^0 c^{ijk} q_i q_j q_k + c_{ijk} p^j p^k c^{ilm} q_l q_m, \quad (4.4)$$

invariant under symplectic transformations. Subsequently we will use the quartic invariant form coming from the contraction of the different symplectic objects in order to fully characterize the solution. In the above formula, indices are raised and lowered with the Kähler metric on the scalar manifold that depends on the prepotential (4.2), see [50] for the complete set of conventions we follow. The condition (4.3) ensures that the inverse tensor c^{ijk} also has constant entries and therefore the quartic form is not scalar dependent.¹³

The FI parameters can be encoded in a gauging vector

$$G = \{g^I = 0; g_0 = 0, g_i\}, \quad (4.5)$$

and we can evaluate the quartic invariant and its first derivative on the vector G ,

$$I_4(G) = 0, \quad (I'_4)_0(G) = -\frac{\partial I_4(G)}{\partial g^0} = \frac{2}{3} c^{ijk} g_i g_j g_k, \quad (4.6)$$

where the first derivative I'_4 also transforms covariantly as a vector. The vanishing of $I_4(G)$ means that no AdS₄ vacuum can exist in this supergravity model, and instead the nonzero $I'_4(G)$ signifies the hvLif asymptotics already anticipated to uplift to AdS₅ when using the rules (4.1).

Now, let us focus our attention to the near horizon geometry of the four-dimensional black hole we are interested in. We want to generalize the solution discussed in [10] and first found in [19] (and in [51] for the case with extra electric charges) to include angular momentum \mathcal{J} . Therefore, we are interested in a slightly restricted set of electromagnetic charges,

$$\Gamma = \{p^0 = 0, p^i; q_0, q_i\}, \quad (4.7)$$

which, together with \mathcal{J} will eventually specify the complete supersymmetric solution. In terms of the five-dimensional rotating string, the p^i and q_i are the magnetic and electric charges, while q_0 is the momentum added along the compactification direction.¹⁴ The

¹³It is not difficult to see that the anomaly coefficients of a theory of D3-branes at a toric singularity satisfy (4.3) only in the case of $\mathcal{N} = 4$ SYM. This is another reason why the class of solutions discussed in this section cannot be immediately used for studying black strings in general Sasaki-Einstein compactifications.

¹⁴Comparing with the dual field theory setting of section 3, p^i are associated with the fluxes n_a , the electric charges q_i are conjugated to the chemical potentials Δ_a and q_0 is conjugated to β . The precise dictionary is spelled out in section 5.

general near horizon geometry with a twist and rotation was written down in [13, section 4]. Here we just repeat the main ingredients and solve the attractor equation that determines explicitly all quantities. Note that the chosen set of charges in general will lead to a non-trivial NUT charge in the solution, so below we will restrict one of the electric charges q_i in a specific way in order to ensure regularity and write down the full solution in a compact form.

The metric can be written in the form

$$ds_4^2 = -e^{2u} (r dt + \omega_0)^2 + e^{-2u} \left(\frac{dr^2}{r^2} + v^2 \left(\frac{d\theta^2}{\Delta(\theta)} + \Delta(\theta) \sin^2(\theta) d\phi^2 \right) \right), \quad (4.8)$$

where

$$e^{-2u} = \sqrt{I_4(\mathcal{I}_0)}, \quad v \mathcal{I}_0 = \mathcal{H}_0 + j G \cos(\theta), \quad v = \langle G, \mathcal{H}_0 \rangle = g_i \beta^i. \quad (4.9)$$

In the last equality we already used the definition of symplectic inner product and the parameterization of the vector \mathcal{H}_0 ,

$$\mathcal{H}_0 = \{0, \beta^i; \beta_0, \beta_i\}, \quad (4.10)$$

parallel to the charge vector Γ . All remaining metric functions, as well as the scalars, are uniquely fixed in terms of the vector \mathcal{H}_0 . For the choice of gauging and electromagnetic charges, we find from [13]

$$\Delta(\theta) = 1, \quad \omega_0 = -\frac{j}{v} \sin^2(\theta) d\phi, \quad (4.11)$$

where we already imposed regularity of the metric, i.e. vanishing NUT charge, which we check below, and used

$$I_4(\Gamma, G, G, G) = I_4(\mathcal{H}_0, G, G, G) = 0. \quad (4.12)$$

The symplectic sections at the horizon, in a suitable gauge, are fixed by

$$\{X^I; F_I\} = -\frac{1}{2\sqrt{I_4(\mathcal{I}_0)}} I_4'(\mathcal{I}_0) + i \mathcal{I}_0, \quad (4.13)$$

and we can recover the physical scalars $z^i = X^i/X^0$. We need to impose one standard constraint among the magnetic charges stemming from the twisting condition,

$$\langle G, \Gamma \rangle = g_i p^i = -1. \quad (4.14)$$

Finally, the main attractor equation to be solved for \mathcal{H}_0 in terms of the charges reads

$$\Gamma = \frac{1}{4} I_4'(\mathcal{H}_0, \mathcal{H}_0, G) + \frac{1}{2} j^2 I_4'(G), \quad (4.15)$$

together with the equation determining the angular momentum

$$\mathcal{J} = -\frac{j}{2} \left(\langle I_4'(G), I_4'(\mathcal{H}_0) \rangle - \frac{1}{2} I_4(\mathcal{H}_0, \mathcal{H}_0, G, G) \langle G, \mathcal{H}_0 \rangle \right). \quad (4.16)$$

In principle, (4.15) and (4.16) can be solved for \mathcal{H}_0 and j in terms of the conserved charges Γ, \mathcal{J} . This determines the complete solution. However, the system of equations is not always easy to simplify in full generality. In order to do so here, we impose the following restriction on the charges,

$$I_4(\Gamma, \Gamma, \Gamma, G) = 0, \tag{4.17}$$

which also ensures that we have a vanishing NUT charge. The above relation will generally fix one of the electric charges q_i in terms of other conserved charges. Using (4.17), together with $I_4(G) = 0$ and a multitude of special identities of the quartic invariant, spelled out in [52, appendix A.3], we can find a general solution to (4.15) and then plug it in (4.16) to finally arrive at the solution

$$\begin{aligned} \mathcal{H}_0 &= -\frac{1}{2\sqrt{\Pi}} \left(\frac{1}{2} I_4'(\Gamma, \Gamma, G) + \frac{(I_4(\Gamma) + \mathcal{J}^2)}{\Theta} I_4'(G) \right), \\ j &= -\frac{\mathcal{J}}{\sqrt{\Pi}}, \quad v = \langle G, \mathcal{H}_0 \rangle = \frac{\Theta}{\sqrt{\Pi}}, \end{aligned} \tag{4.18}$$

where we defined for shorthand the combinations

$$\Theta \equiv -\frac{1}{4} I_4(\Gamma, \Gamma, G, G), \quad \Pi \equiv \frac{1}{2} \langle I_4'(\Gamma), I_4'(G) \rangle - \frac{1}{4} I_4(\Gamma, \Gamma, G, G). \tag{4.19}$$

Note that now we can evaluate any physical quantity of the solution in terms of the conserved charges. The near horizon solution can be uniquely extended to fix the full black hole flow to the asymptotic region using the formulae given in [13]. Here we are particularly interested in the entropy, which in terms of the parameters \mathcal{H}_0 and j has the simple expression

$$S = \frac{A}{4G_N} = \frac{\pi}{G_N} \sqrt{I_4(\mathcal{H}_0) - j^2}, \tag{4.20}$$

and in terms of the conserved charges, using more I_4 identities from [52],

$$S = \frac{\pi}{G_N} \sqrt{\frac{-I_4(\Gamma) - \mathcal{J}^2}{\Theta}}. \tag{4.21}$$

We can also define the chemical potential conjugate to the angular momentum as in [13],

$$w \equiv \frac{j}{v \sqrt{I_4(\mathcal{H}_0) - j^2}} = -\frac{\mathcal{J}}{\sqrt{\Theta(-I_4(\Gamma) - \mathcal{J}^2)}}. \tag{4.22}$$

We finish the discussion of the general near horizon solution with the formula determining the electric gauge fields,

$$A_{(4)}^I = -\frac{\Theta^5 I_4'(\mathcal{I}_0)^I}{\Pi^2 \Xi(\theta)} (r dt + \omega_0) - p^I \cos(\theta) d\phi, \tag{4.23}$$

where we have used the solution for \mathcal{H}_0 and j to simplify the formula, and for brevity we defined the quantity

$$\Xi(\theta) \equiv (-I_4(\Gamma) - \mathcal{J}^2) + \frac{\Theta}{\Pi} \mathcal{J}^2 \sin^2(\theta). \tag{4.24}$$

4.3 STU model

In the special case when $n_V = 3$ with $c_{123} = 1$ (and cyclic permutations), and gaugings $g_i = 1$, $i = 1, 2, 3$, we have the so-called STU model. Upon uplift to five dimensions, it is also embeddable in maximal gauged supergravity and follows from reduction of type IIB supergravity on S^5 .

In this model the rotating solutions we described above are a generalization of the black string solutions of [2]. In this case we find that

$$\begin{aligned} I_4^{\text{STU}}(\Gamma) &= 4q_0 p^1 p^2 p^3 - \sum_{i=1}^3 (p^i q_i)^2 + 2 \sum_{i < j} q_i p^i q_j p^j, \\ \Theta^{\text{STU}} &= (p^1)^2 + (p^2)^2 + (p^3)^2 - 2(p^1 p^2 + p^1 p^3 + p^2 p^3), \\ \Pi^{\text{STU}} &= (-p^1 + p^2 + p^3)(p^1 - p^2 + p^3)(p^1 + p^2 - p^3), \end{aligned} \tag{4.25}$$

under the constraint

$$p^1 + p^2 + p^3 = -1, \tag{4.26}$$

in exact accordance with [2] for the case of spherical topology, $\mathfrak{g} = 0$. Notice that

$$\Theta^{\text{STU}} - \Pi^{\text{STU}} = 8p^1 p^2 p^3. \tag{4.27}$$

4.3.1 Magnetic case

Let us first for simplicity consider the magnetic case, $q_i = 0$. The constraint (4.17) is then satisfied automatically. The Bekenstein-Hawking entropy in this case reads

$$S^{\text{STU}} = \frac{\pi}{G_N} \sqrt{\frac{-4q_0 p^1 p^2 p^3 - \mathcal{J}^2}{\Theta^{\text{STU}}}}, \tag{4.28}$$

and the region of positivity (the charge parameter space where regular solutions exist) was analyzed and shown to be non-empty in [2]. The inclusion of the angular momentum naturally decreases or increases this region depending on the sign of Θ^{STU} . Note however that due to the constraint (4.26) the solution does not allow a limit of vanishing charges.

If we explicitly evaluate the sections on the horizon from (4.13) as functions of the spherical coordinate θ , we obtain

$$\begin{aligned} X^0(\theta) &= \frac{2p^1 p^2 p^3}{\sqrt{\Theta^{\text{STU}} \Xi^{\text{STU}}(\theta)}}, \\ X^i(\theta) &= \frac{p^i \mathcal{J} \cos(\theta)}{\sqrt{\Theta^{\text{STU}} \Xi^{\text{STU}}(\theta)}} + i \frac{p^i (1 + 2p^i)}{\Theta^{\text{STU}}}, \quad i = 1, 2, 3, \end{aligned} \tag{4.29}$$

where in the absence of electric charges q_i we find

$$\Xi^{\text{STU}}(\theta) \equiv (-4q_0 p^1 p^2 p^3 - \mathcal{J}^2) + \frac{\Theta^{\text{STU}}}{\Pi^{\text{STU}}} \mathcal{J}^2 \sin^2(\theta). \tag{4.30}$$

Moreover, the chemical potential conjugate to the angular momentum is given by

$$w^{\text{STU}} = -\frac{\mathcal{J}}{\sqrt{\Theta^{\text{STU}} (-4q_0 p^1 p^2 p^3 - \mathcal{J}^2)}}. \tag{4.31}$$

The values of the sections X^I at the south pole $\theta = 0$ can be written as

$$\begin{aligned} X_{\text{SP}}^0 &\equiv X^0(\theta)\Big|_{\theta=0} = -\frac{2w^{\text{STU}}p^1p^2p^3}{\mathcal{J}}, \\ X_{\text{SP}}^i &\equiv X^i(\theta)\Big|_{\theta=0} = -w^{\text{STU}}p^i + i\frac{p^i(1+2p^i)}{\Theta^{\text{STU}}}, \quad i = 1, 2, 3. \end{aligned} \quad (4.32)$$

Their values at the north pole $\theta = \pi$ can be obtained by sending $w^{\text{STU}} \rightarrow -w^{\text{STU}}$ and $\mathcal{J} \rightarrow -\mathcal{J}$. These become important later when we compare the entropy (4.28) with the refined twisted index of $\mathcal{N} = 4$ SYM.

Finally, we can explicitly write down the gauge fields that complete the specification of the supersymmetric background,

$$\begin{aligned} A_{(4)}^0 &= \frac{2(\Theta^{\text{STU}})^2 p^1 p^2 p^3}{\Pi^{\text{STU}} \Xi^{\text{STU}}(\theta)} (r dt + \omega_0^{\text{STU}}), \\ A_{(4)}^i &= \frac{(\Theta^{\text{STU}})^2 \mathcal{J} p^i \cos(\theta)}{\Pi^{\text{STU}} \Xi^{\text{STU}}(\theta)} (r dt + \omega_0^{\text{STU}}) - p^i \cos(\theta) d\phi, \end{aligned} \quad (4.33)$$

with the one-form

$$\omega_0^{\text{STU}} = \frac{\mathcal{J}}{\Theta^{\text{STU}}} \sin^2(\theta) d\phi. \quad (4.34)$$

4.3.2 Dyonic case

Adding non-vanishing electric charges q_i is straightforward given our general results above. The constraint (4.17) can be solved in this case by fixing one of the electric charges, e.g.

$$q_3 = -\frac{q_1 p^1 (1 + 2p^1) + q_2 p^2 (1 + 2p^2)}{p^3 (1 + 2p^3)}. \quad (4.35)$$

The entropy is given by

$$S_{\text{dyonic}}^{\text{STU}} = \frac{\pi}{G_{\text{N}}} \sqrt{\frac{-I_4^{\text{STU}}(\Gamma) - \mathcal{J}^2}{\Theta^{\text{STU}}}}, \quad (4.36)$$

in terms of the quantities defined in (4.25). The chemical potential conjugate to the angular momentum is given by

$$w_{\text{dyonic}}^{\text{STU}} = -\frac{\mathcal{J}}{\sqrt{\Theta^{\text{STU}}(-I_4^{\text{STU}}(\Gamma) - \mathcal{J}^2)}}, \quad (4.37)$$

while the sections, using explicitly the relation (4.35), read

$$\begin{aligned} X_{\text{dyonic}}^0(\theta) &= \frac{2p^1 p^2 p^3}{\sqrt{\Theta^{\text{STU}} \Xi_{\text{dyonic}}^{\text{STU}}(\theta)}}, \\ X_{\text{dyonic}}^i(\theta) &= \frac{p^i \mathcal{J} \cos(\theta)}{\sqrt{\Theta^{\text{STU}} \Xi_{\text{dyonic}}^{\text{STU}}(\theta)}} + i\frac{p^i(1+2p^i)}{\Theta^{\text{STU}}} \\ &\quad + \frac{2p^1 p^2 p^3}{(1+2p^i)\sqrt{\Theta^{\text{STU}} \Xi_{\text{dyonic}}^{\text{STU}}(\theta)}} \left(q_i - \sum_{i=1}^3 q_i \right), \quad i = 1, 2, 3. \end{aligned} \quad (4.38)$$

4.4 The 5D uplift

It is also straightforward to uplift to five dimensions the solutions described above, using the general rules (4.1). In particular we can directly look at the STU model. We focus only on some noteworthy features of the resulting near horizon solutions for rotating black strings, which have the standard locally AdS₅ asymptotics with a spatial boundary $S^1 \times S^2$, and upon a further uplift on S^5 become ten-dimensional solutions of type IIB supergravity.

The five-dimensional near horizon solution can be summarized as follows. The scalars L^i are constant (θ independent) and coincide with the scalars in the static case as presented in [2],

$$L^i = -\frac{p^i(1+2p^i)}{(p^1 p^2 p^3 \Pi^{\text{STU}})^{1/3}}. \tag{4.39}$$

The gauge fields in the purely magnetic case are given by

$$A_{(5)}^i = -p^i \cos(\theta) \left(d\phi - \frac{\mathcal{J}}{2p^1 p^2 p^3} dy \right), \tag{4.40}$$

where y is the fifth coordinate that is topologically a non-contractible circle. In the dyonic case the gauge fields are supplemented with additional constant Wilson lines $\alpha^i dy$ with the constraint $\sum_i \alpha^i = 0$ as discussed in [51], where you can find the map between the charges q_i and the parameters α^i . Notice that the electric charges from a five-dimensional perspective do not alter the scalars, meaning that the attractor mechanism is qualitatively different than its four-dimensional analogue and fixes not only the scalars, but also some components of the gauge fields.

The metric, as already anticipated in the introduction, is a fibration of BTZ and S^2 where in the limit $\mathcal{J} \rightarrow 0$ the fibration becomes trivial. The BTZ part is of course locally isometric to AdS₃, but the global difference is important in order to keep the relation to four dimensions well-defined, and, therefore, to ensure that our construction here is correct. Note that the length scales of the BTZ and the S^2 do not depend on the angular momentum \mathcal{J} , which only leads to the nontrivial fibration. Thus, the length scale associated with the two-dimensional right-moving central charge (3.30) remains unchanged,

$$R_{\text{BTZ}} R_{S^2}^2 = \frac{8p^1 p^2 p^3}{\Theta^{\text{STU}}}. \tag{4.41}$$

Let us finally note that the static magnetic near horizon limit of the black strings of [2] was already found in [53] from a classification of supersymmetric geometries. The generalizations here with electric charges and rotation were not considered in the classification as it is crucial to allow for a globally BTZ spacetime.

5 \mathcal{I} -extremization principle and microstates counting

We now consider the degeneracy of supersymmetric states of the $\mathcal{N} = 1$ gauge theories on $S_\omega^2 \times T^2$ discussed in section 3. The chemical potentials Δ_a are conjugate to the flavor charges, ω to the angular momentum on S_ω^2 and β to the momentum along one of the cycles of the torus T^2 . In a theory specified by the magnetic fluxes \mathfrak{n}_a , the number of

supersymmetric states $d_{\text{micro}}(\mathbf{n}_a, e_a, J|e_0)$ with electric charges e_a , $a = 1, \dots, d$, angular momentum J and momentum e_0 is then given by the Fourier/Laplace transform of the refined index (3.14) with respect to the independent chemical potentials Δ_a , $a = 1, \dots, d$, ω , and β ,

$$d_{\text{micro}} = -\frac{i}{(2\pi)^{d+1}} \oint \prod_{a=1}^d d\Delta_a \int_{i\mathbb{R}} d\beta \int d\omega \delta\left(2\pi - \sum_a \Delta_a\right) Z(\mathbf{n}, \Delta, \omega|\beta) e^{-i\sum_a \Delta_a e_a - i\omega J + \beta e_0}. \quad (5.1)$$

We are considering theories of D3-branes at conical singularities and using the R-symmetry parameterization introduced in section 3. The delta function in (5.1) imposes the analogous of constraint (3.18). Due to this constraint, d_{micro} only depends on the $d-1$ combinations $e_a - e_d$, $a = 1, \dots, d-1$, corresponding to the flavor symmetries of the theory. In the Cardy limit ($\beta \rightarrow 0$), d_{micro} can be evaluated by a saddle point approximation

$$\log d_{\text{micro}}(\mathbf{n}_a, e_a, J|e_0) \equiv \log Z(\mathbf{n}, \Delta, \omega|\beta) - i \sum_a \Delta_a e_a - i\omega J + \beta e_0 \Big|_{\bar{\Delta}_a, \bar{\omega}, \bar{\beta}}, \quad (5.2)$$

where $\bar{\Delta}_a$, $\bar{\omega}$, and $\bar{\beta}$ are the critical points of the functional

$$\mathcal{I}(\mathbf{n}, \Delta, \omega|\beta) \equiv \log Z(\mathbf{n}, \Delta, \omega|\beta) - i \sum_a \Delta_a e_a - i\omega J + \beta e_0, \quad (5.3)$$

under the constraint $\sum_{a=1}^d \Delta_a = 2\pi$, with respect to Δ_a , ω , and β ,

$$\begin{aligned} \partial_{\Delta_a} \mathcal{I}(\mathbf{n}, \Delta, \omega|\beta) \Big|_{\bar{\Delta}_a, \bar{\omega}, \bar{\beta}} &= 0, & a = 1, \dots, d-1, \\ \partial_{\omega} \mathcal{I}(\mathbf{n}, \Delta, \omega|\beta) \Big|_{\bar{\Delta}_a, \bar{\omega}, \bar{\beta}} &= \partial_{\beta} \mathcal{I}(\mathbf{n}, \Delta, \omega|\beta) \Big|_{\bar{\Delta}_a, \bar{\omega}, \bar{\beta}} = 0. \end{aligned} \quad (5.4)$$

This procedure is the so-called \mathcal{I} -*extremization* principle that has been used to give a microscopic explanation of the entropy (density) of BPS black holes (strings) [4, 11, 14, 15, 21, 54] in diverse dimensions. It contains two basic pieces of information:

1. extremizing the index unambiguously determines the exact R-symmetry of the SCFT in the IR;
2. the value of the index at its critical points is the (possibly regularized) number of ground states.

We now apply this counting to the STU black holes discussed in section 4.3.

5.1 The case with $e_a = 0$

Consider for simplicity $e_a = 0$, $a = 1, \dots, d$. Now let us plug back the value for the refined topologically twisted index in the Cardy limit (and at finite N) (3.14) into (5.3), obtaining

$$\mathcal{I}(\mathbf{n}, \Delta, \omega|\beta) = \frac{\pi^2}{6\beta} c_l(\mathbf{n}, \Delta/\pi) - \frac{(2\omega)^2}{27\beta} (3c(\mathbf{n}) - 2a(\mathbf{n})) - i\omega J + \beta e_0, \quad (5.5)$$

Extremizing it with respect to Δ_a sets the trial left-moving central charge $c_l(\mathbf{n}, \Delta/\pi)$ to its *exact* value $c_l^{\text{CFT}}(\mathbf{n})$ in the IR — in accordance with the c -extremization principle [1, 2].¹⁵ Then, the extremization of $\mathcal{I}(\mathbf{n}, \Delta, \omega|\beta)$ with respect to β and ω yields.

$$\begin{aligned}\bar{\beta} &= \pi \sqrt{\frac{c_l^{\text{CFT}}(\mathbf{n})}{6} \left(e_0 - \frac{27J^2}{16(3c(\mathbf{n}) - 2a(\mathbf{n}))} \right)^{-1}}, \\ \bar{\omega} &= -\frac{27i}{8(3c(\mathbf{n}) - 2a(\mathbf{n}))} \bar{\beta} J.\end{aligned}\tag{5.6}$$

Notice that the extremal value of ω is purely imaginary thus justifying the assumptions made in the derivation of (3.14) in section 3. Substituting (5.6) back into (5.3) we finally obtain

$$\log d_{\text{micro}}(\mathbf{n}, J|e_0) = 2\pi \sqrt{\frac{c_l^{\text{CFT}}(\mathbf{n})}{6} \left(e_0 - \frac{27J^2}{16(3c(\mathbf{n}) - 2a(\mathbf{n}))} \right)}.\tag{5.7}$$

In the case of $J = 0$, this is the familiar Cardy formula for a two-dimensional CFT [55].

In order to compare (5.7) with the entropy of the STU black hole (4.28), we need to go to large N for which $c_l = c_r$ and $c = a$. Thus,

$$\log d_{\text{micro}}(\mathbf{n}, J|e_0) = 2\pi \sqrt{\frac{c_r^{\text{CFT}}(\mathbf{n})}{6} \left(e_0 - \frac{27J^2}{16a(\mathbf{n})} \right)}, \quad N \gg 1.\tag{5.8}$$

Plugging back the values for $c_r^{\text{CFT}}(\mathbf{n})$ and $a(\mathbf{n})$ of $\mathcal{N} = 4$ SYM, see (3.30) and (3.26), into (5.8), we find that

$$\log d_{\text{micro}}^{\mathcal{N}=4}(\mathbf{n}, J|e_0) = 2\sqrt{2}\pi \sqrt{\frac{N^2 e_0 \mathbf{n}_1 \mathbf{n}_2 \mathbf{n}_3 - 2J^2}{\Theta_{\mathcal{N}=4}}}, \quad N \gg 1,\tag{5.9}$$

where $\Theta_{\mathcal{N}=4}$ is given in (3.29). Moreover, the values of the critical points (5.6) read

$$\begin{aligned}\bar{\beta}^{\mathcal{N}=4} &= \sqrt{2}\pi \frac{N^2 \mathbf{n}_1 \mathbf{n}_2 \mathbf{n}_3}{\sqrt{\Theta_{\mathcal{N}=4}(N^2 e_0 \mathbf{n}_1 \mathbf{n}_2 \mathbf{n}_3 - 2J^2)}}, \\ \bar{\omega}^{\mathcal{N}=4} &= -\frac{4i}{N^2 \mathbf{n}_1 \mathbf{n}_2 \mathbf{n}_3} \bar{\beta}^{\mathcal{N}=4} J.\end{aligned}\tag{5.10}$$

Using the $\text{AdS}_5/\text{CFT}_4$ relation between gravitational and SCFT parameters at large N ,

$$\frac{\pi}{2G_{\text{N}}^{(5)}} = N^2,\tag{5.11}$$

where $G_{\text{N}}^{(5)} = 2\pi G_{\text{N}}$, and upon identifying

$$\begin{aligned}\mathbf{n}_a &\equiv -2p^a, & a &= 1, 2, 3, \\ e_0 &\equiv \frac{1}{G_{\text{N}}} q_0, & J &\equiv -\frac{1}{2G_{\text{N}}} \mathcal{J},\end{aligned}\tag{5.12}$$

¹⁵The gravitational anomaly k is independent of Δ_a , and extremizations of c_l or c_r are equivalent.

we find that the entropy of the STU black hole (4.28) matches *precisely* our microscopic expression for the number of ground states (5.9):

$$S^{\text{STU}} = \log d_{\text{micro}}^{\mathcal{N}=4}(\mathbf{n}, J|e_0). \quad (5.13)$$

We further observe that

$$\bar{\omega}^{\mathcal{N}=4} = -2\pi i w^{\text{STU}}, \quad (5.14)$$

and the values of the scalars at the south pole (4.32) (or north pole) are mapped to the critical points (5.10). Explicitly, we have

$$\begin{aligned} X_{\text{SP, NP}}^0 &= -\frac{1}{\pi} \bar{\beta}^{\mathcal{N}=4}, \\ X_{\text{SP, NP}}^a &= \frac{i}{2\pi} \left(\bar{\Delta}_a^{\mathcal{N}=4} \pm \frac{\bar{\omega}^{\mathcal{N}=4}}{2} \mathbf{n}_a \right), \quad a = 1, 2, 3, \end{aligned} \quad (5.15)$$

where $\bar{\Delta}_a$, $a = 1, 2$, are given in (3.28), $\bar{\Delta}_3 = 2\pi - \bar{\Delta}_1 - \bar{\Delta}_2$ and \pm refers to SP and NP, respectively.¹⁶

5.2 Dyonic $\mathcal{N} = 4$ super Yang-Mills

In this section we evaluate (5.1) for $\mathcal{N} = 4$ SYM keeping e_a , $a = 1, 2, 3$, nonzero. Explicitly, we write

$$\begin{aligned} d_{\text{micro}} &= -\frac{i}{(2\pi)^4} \oint \prod_{a=1}^3 d\Delta_a \\ &\times \int_{i\mathbb{R}} d\beta \int d\omega \delta\left(2\pi - \sum_a \Delta_a\right) e^{\frac{\pi^2}{6\beta} c_r(\mathbf{n}, \Delta/\pi) - \frac{(2\omega)^2}{27\beta} a(\mathbf{n}) - i \sum_{a=1}^3 \Delta_a e_a - i\omega J + \beta e_0}, \end{aligned} \quad (5.16)$$

where $c_l(\mathbf{n}, \Delta/\pi)$ and $a(\mathbf{n})$ are given in (3.27) and (3.26), respectively. Again, in a saddle point approximation we shall extremize

$$\mathcal{I}(\mathbf{n}, \Delta, \omega|\beta) = \frac{\pi^2}{6\beta} c_r(\mathbf{n}, \Delta/\pi) - \frac{(2\omega)^2}{27\beta} a(\mathbf{n}) - i \sum_{a=1}^3 \Delta_a e_a - i\omega J + \beta e_0, \quad (5.17)$$

under the constraint $\sum_{a=1}^3 \Delta_a = 2\pi$, with respect to Δ_1 , Δ_2 , ω , and β . The critical points of the \mathcal{I} -functional (5.17) read

$$\begin{aligned} \frac{\bar{\Delta}_1}{2\pi} &= \frac{\mathbf{n}_1(\mathbf{n}_1 - \mathbf{n}_2 - \mathbf{n}_3)}{\Theta_{\mathcal{N}=4}} - i \frac{\sqrt{2} \mathbf{n}_1 \mathbf{n}_2 \mathbf{n}_3 (2e_1 \mathbf{n}_1 - e_2(\mathbf{n}_1 + \mathbf{n}_2 - \mathbf{n}_3) - e_3(\mathbf{n}_1 - \mathbf{n}_2 + \mathbf{n}_3))}{\Theta_{\mathcal{N}=4}^{3/2} \sqrt{\mathcal{Q} \mathbf{n}_1 \mathbf{n}_2 \mathbf{n}_3 - 2J^2}}, \\ \frac{\bar{\Delta}_2}{2\pi} &= \frac{\mathbf{n}_2(-\mathbf{n}_1 + \mathbf{n}_2 - \mathbf{n}_3)}{\Theta_{\mathcal{N}=4}} - i \frac{\sqrt{2} \mathbf{n}_1 \mathbf{n}_2 \mathbf{n}_3 (2e_2 \mathbf{n}_2 - e_1(\mathbf{n}_1 + \mathbf{n}_2 - \mathbf{n}_3) - e_3(-\mathbf{n}_1 + \mathbf{n}_2 + \mathbf{n}_3))}{\Theta_{\mathcal{N}=4}^{3/2} \sqrt{\mathcal{Q} \mathbf{n}_1 \mathbf{n}_2 \mathbf{n}_3 - 2J^2}}, \\ \bar{\beta} &= \frac{\pi \sqrt{2} N^2 \mathbf{n}_1 \mathbf{n}_2 \mathbf{n}_3}{\sqrt{\Theta_{\mathcal{N}=4}(\mathcal{Q} \mathbf{n}_1 \mathbf{n}_2 \mathbf{n}_3 - 2J^2)}}, \quad \bar{\omega} = -\frac{4\sqrt{2} i \pi}{\sqrt{\Theta_{\mathcal{N}=4}(\mathcal{Q} \mathbf{n}_1 \mathbf{n}_2 \mathbf{n}_3 - 2J^2)}} J, \end{aligned} \quad (5.18)$$

¹⁶Recall that $r_a = \Delta_a/\pi$ parameterize an R-symmetry of the theory.

where we defined

$$\mathcal{Q} \equiv e_0 N^2 - \frac{2((e_1 - e_2)((e_1 - e_3)\mathbf{n}_1 - (e_2 - e_3)\mathbf{n}_2) + (e_1 - e_3)(e_2 - e_3)\mathbf{n}_3)}{\Theta_{\mathcal{N}=4}}. \quad (5.19)$$

Notice again that the extremal value of ω is purely imaginary in agreement with the assumptions made in section 3. Plugging the critical points (5.18) back into the \mathcal{I} -functional (5.17) we find

$$\begin{aligned} \mathcal{I} \Big|_{\text{crit.}}(\mathbf{n}_a, e_a, J|e_0) &= 2\sqrt{2}\pi \sqrt{\frac{\mathcal{Q}\mathbf{n}_1\mathbf{n}_2\mathbf{n}_3 - 2J^2}{\Theta_{\mathcal{N}=4}}} \\ &+ 2\pi i \frac{e_1\mathbf{n}_1(-\mathbf{n}_1 + \mathbf{n}_2 + \mathbf{n}_3) + e_2\mathbf{n}_2(\mathbf{n}_1 - \mathbf{n}_2 + \mathbf{n}_3) + e_3\mathbf{n}_3(\mathbf{n}_1 + \mathbf{n}_2 - \mathbf{n}_3)}{\Theta_{\mathcal{N}=4}}. \end{aligned} \quad (5.20)$$

Demanding the reality of the \mathcal{I} -functional at its critical point fixes one of the charges, say e_3 , in terms of the others. We can then write

$$e_3 = \frac{e_1\mathbf{n}_1(\mathbf{n}_1 - \mathbf{n}_2 - \mathbf{n}_3) + e_2\mathbf{n}_2(-\mathbf{n}_1 + \mathbf{n}_2 - \mathbf{n}_3)}{(\mathbf{n}_1 + \mathbf{n}_2 - \mathbf{n}_3)\mathbf{n}_3}, \quad (5.21)$$

and, therefore,

$$\log d_{\text{micro}}(\mathbf{n}_a, e_a, J|e_0) = 2\sqrt{2}\pi \sqrt{\frac{\mathcal{Q}\mathbf{n}_1\mathbf{n}_2\mathbf{n}_3 - 2J^2}{\Theta_{\mathcal{N}=4}}}. \quad (5.22)$$

Quite remarkably, the constraint (5.21) is *precisely* the one that we obtained on the supergravity side, see (4.35), where we identify

$$e_a = \frac{1}{2G_{\text{N}}} q_a, \quad a = 1, 2, 3. \quad (5.23)$$

It would be very interesting to give a first principle derivation of this constraint. Finally, using (5.11), (5.12), and (5.23), we find that the Bekenstein-Hawking entropy of four-dimensional rotating *dyonic* black holes (4.36) matches (5.22):

$$S_{\text{dyonic}}^{\text{STU}} = \log d_{\text{micro}}(\mathbf{n}_a, e_a, J|e_0). \quad (5.24)$$

Notice that the expression for the entropy matches only after imposing the constraint (5.21).¹⁷ We observe that the relations (5.14) and (5.15) hold also in this case.

It is interesting to rewrite the combination (5.19) in terms of purely field theoretic data, in particular 't Hooft flavor anomalies. The chemical potentials Δ_a are conjugated to a basis R_a , $a = 1, 2, 3$, of R-symmetries that assign charge 2 to the a -th chiral multiplet of $\mathcal{N} = 4$ SYM and zero to the others. We can choose two independent flavor symmetries $Q_1 = (R_1 - R_3)/2$ and $Q_2 = (R_2 - R_3)/2$. The 't Hooft anomaly matrix is easily computed from the multiplicity of fermionic zero-modes¹⁸ and reads

$$A_{AB} = \text{Tr} \gamma_3 Q_A Q_B = N^2 \begin{pmatrix} 2 - \mathbf{n}_1 - \mathbf{n}_3 & 1 - \mathbf{n}_3 \\ 1 - \mathbf{n}_3 & 2 - \mathbf{n}_2 - \mathbf{n}_3 \end{pmatrix}. \quad (5.25)$$

¹⁷In particular, $\mathcal{Q}\mathbf{n}_1\mathbf{n}_2\mathbf{n}_3 = -2N^4 I_4^{\text{STU}}(\Gamma)/\pi^2$ holds only if the charges satisfy (5.21).

¹⁸Recall that the difference between the number of fermionic zero-modes of opposite chiralities is equal to $-\dim \mathfrak{R}_I(\mathbf{n}_I - 1)$ for four-dimensional matter fields in a representation \mathfrak{R}_I .

We can then write (5.19) as

$$\frac{\mathcal{Q}}{N^2} = e_0 + \frac{1}{2} \sum_{A,B=1}^2 \tilde{e}_A A_{AB}^{-1} \tilde{e}_B, \quad (5.26)$$

where $\tilde{e}_A = e_A - e_3$ are the flavor charges. We see that the entropy depends only on a particular combination of e_0 and electric flavor charges. As we will see, this result extends to more general quivers. The combination (5.26) is familiar from the Rademacher expansion for asymptotically flat black holes [56, 57], where it is a consequence of the transformation properties of the elliptic genus of the two-dimensional CFT. Our topologically twisted index is expected to compute the elliptic genus of the IR CFT and relate to the quantum entropy of black holes not just in the Cardy limit. The result, at finite N , is actually a meromorphic function of Δ_a , and has a complicated structure in terms of modular forms. We expect that, for large N , this structure simplifies and gives results similar to those for asymptotically flat black holes. In the latter case the meromorphicity of the elliptic genus leads to interesting behavior, related to wall-crossing phenomena (see for example [58]). It would be interesting to see if the same happens for the black holes discussed here once quantum corrections are taken into account.

5.3 Microstate counting for a generic quiver

Although there is no gravitational dual at the moment, for completeness we count the degeneracy of states for a generic theory of D3-branes at toric conical singularities.

We must extremize the analog of (5.17),

$$\mathcal{I}(\mathbf{n}, \Delta, \omega | \beta) = \frac{\pi^2}{6\beta} c_r(\mathbf{n}, \Delta/\pi) - \frac{(2\omega)^2}{27\beta} a(\mathbf{n}) - i \sum_{a=1}^d \Delta_a e_a - i\omega J + \beta e_0, \quad (5.27)$$

with respect to ω , β and d chemical potential Δ_a subject to the constraint $\sum_{a=1}^d \Delta_a = 2\pi$. It is convenient to introduce a basis of flavor symmetries $Q_A = (R_A - R_d)/2$, $A = 1, \dots, d-1$, where R_a is the basis of R-symmetries conjugated to the chemical potentials Δ_a . Choosing a reference R-symmetry r_a^0 , we can parameterize

$$\begin{aligned} \frac{\Delta_a}{\pi} &= r_a^0 + \delta_a, & a &= 1, \dots, d, \\ \sum_{a=1}^d r_a^0 &= 2, & \sum_{a=1}^d \delta_a &= 0, \end{aligned} \quad (5.28)$$

so that the two-dimensional R-symmetry matrix reads

$$R(\Delta_a/\pi) = R_0 + \sum_{A=1}^{d-1} \delta_A Q_A, \quad R_0 \equiv R(r_a^0), \quad (5.29)$$

and the two-dimensional trial central charge c_r becomes

$$3 \operatorname{Tr} \gamma_3 R(\Delta_a/\pi)^2 = 3 \operatorname{Tr} \gamma_3 R_0^2 + 3 \sum_{A,B=1}^{d-1} \delta_A A_{AB} \delta_B + 6 \sum_{A=1}^{d-1} k_A \delta_A, \quad (5.30)$$

where

$$A_{AB} = \text{Tr } \gamma_3 Q_A Q_B, \quad k_A = \text{Tr } \gamma_3 R_0 Q_A. \quad (5.31)$$

Notice that A_{AB} is the 't Hooft anomaly matrix and is a function of the fluxes \mathbf{n}_a . Plugging it back into (5.27), and extremizing with respect to β , ω and the $d - 1$ independent parameters δ_A we find

$$\begin{aligned} \bar{\beta} &= \pi \sqrt{\frac{\text{Tr } \gamma_3 R_0^2 - k_A A_{AB}^{-1} k_B}{2(e_0 + \frac{1}{2} \tilde{e}_A A_{AB}^{-1} \tilde{e}_B - \frac{27}{16a(\mathbf{n})} J^2)}} = \pi \sqrt{\frac{c_r^{\text{CFT}}(\mathbf{n})}{6(e_0 + \frac{1}{2} \tilde{e}_A A_{AB}^{-1} \tilde{e}_B - \frac{27}{16a(\mathbf{n})} J^2)}}, \\ \bar{\omega} &= -\frac{27i\bar{\beta}}{8a(\mathbf{n})} J, \quad \bar{\delta}_B = iA_{AB}^{-1} \left(\frac{\bar{\beta}}{\pi} \tilde{e}_A + ik_A \right), \quad A, B = 1, \dots, d-1, \end{aligned} \quad (5.32)$$

where $\tilde{e}_A = e_A - e_d$ and, here and below, summation over repeated indices is understood. In the first line we reconstructed the exact central charge $c_r^{\text{CFT}}(\mathbf{n})$ by observing that the result must reduce to (5.6) for $e_a = 0$. We also find the critical value of the functional (5.27)

$$\mathcal{I} \Big|_{\text{crit.}}(\mathbf{n}_a, e_a, J|e_0) = 2\pi \sqrt{\frac{c_r^{\text{CFT}}(\mathbf{n})}{6} \left(e_0 + \frac{1}{2} \tilde{e}_A A_{AB}^{-1} \tilde{e}_B - \frac{27}{16a(\mathbf{n})} J^2 \right)} + \pi i (\tilde{e}_A A_{AB}^{-1} k_B - r_a^0 e_a). \quad (5.33)$$

By identifying the degeneracy of states with the real part of the \mathcal{I} -functional, we finally find

$$\text{Re } \log d_{\text{micro}}(\mathbf{n}_a, e_a, J|e_0) = 2\pi \sqrt{\frac{c_r^{\text{CFT}}(\mathbf{n})}{6} \left(e_0 + \frac{1}{2} \tilde{e}_A A_{AB}^{-1}(\mathbf{n}) \tilde{e}_B - \frac{27}{16a(\mathbf{n})} J^2 \right)}. \quad (5.34)$$

We see that the field theory entropy can be completely written in terms of anomalies and central charges of the four- and two-dimensional theories. It also depends on a very particular combinations of electric charges and J . It would be interesting to see if, in the dual black hole, the imaginary part of (5.33) vanishes as a consequence of the gravitational BPS constraint on electric charges (4.17), as it happens for $\mathcal{N} = 4$ SYM.

6 Discussion and outlook

In this paper we have studied a class of rotating black strings that can be embedded in $\text{AdS}_5 \times S^5$. In five-dimensional language, these are domain walls that interpolate between AdS_5 and a near horizon region consisting of a warped fibration of BTZ over a sphere, thus generalizing the solutions of [2] by adding rotation. Upon compactification along the BTZ circle, we find a rotating black hole with hvLif asymptotics. We have successfully matched the entropy of such black holes with a field theory computation based on the refined topologically twisted index of $\mathcal{N} = 4$ SYM on $T^2 \times S_\omega^2$ in the Cardy limit. We have also computed the index in the Cardy limit at finite N for a generic $\mathcal{N} = 1$ SCFT living on the world-volume of D3-branes at toric (and not only) conical singularities.

Many interesting questions remain open. It would be very interesting to generalize the gravity solutions that we have found in this paper to a generic compactification based on Sasaki-Einstein manifolds and compare the entropy of the resulting four-dimensional black holes with the field theory prediction of section 5.3. There are various obstacles to do that.

First of all, consistent truncations to a five-dimensional gauged supergravity containing only vector multiplets exist just for $\text{AdS}_5 \times S^5$. The available truncations on $\text{AdS}_5 \times \text{SE}_5$ [42–44], where SE_5 is a five-dimensional Sasaki-Einstein manifold, involve hypermultiplets as well and are more difficult to treat. Such truncations typically contain massive vector multiplets and the hypermultiplets themselves have a vacuum expectation value that Higgses other vectors. Therefore, the number of vector multiplets in the truncation needs not to be equal to the number of global symmetries of the dual SCFT, since the latter correspond to massless vectors in the bulk. It would be tempting to speculate that, as far as the near horizon region is concerned, we should be able to use an effective supergravity containing only massless vector multiplets associated with the global symmetries. This happens, for example, in the matching of the entropy of some magnetically charged AdS_4 black holes [31, 32, 59]. Since a five-dimensional gauged supergravity with only vector multiplets is completely specified by the tensor c_{ijk} , associated with the 't Hooft anomalies of the dual CFT, the entropy will be also a function of c_{ijk} , J and the magnetic and electric charges. Indeed we found that the field theory prediction for the entropy given in section 5.3 can be written just in terms of anomalies. However, we cannot immediately compare with our supergravity results. The solutions found in this paper assume the validity of (4.3), which, unfortunately, is not satisfied by the anomaly coefficients of a generic $\mathcal{N} = 1$ SCFT of D3-branes at singularities. From this perspective, it would be very interesting to remove the technical assumption (4.3).

It would be also very interesting to see what part of our analysis can be extended to the refined topologically twisted index on $S^1 \times S_\omega^2$, which is much more difficult to analyse from the field theory point of view. The refined topologically twisted index in three dimensions is supposed to count the microstates of the magnetically charged and rotating asymptotically AdS_4 black holes found in [13]. We hope to report soon on the subject.

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A Asymptotic behavior of elliptic functions near $q = 1$

The Dedekind eta function is defined by

$$\eta(q) = \eta(\tau) = q^{\frac{1}{24}} \prod_{n=1}^{\infty} (1 - q^n), \quad \text{Im } \tau > 0, \quad (\text{A.1})$$

and it has the following modular properties

$$\eta(\tau + 1) = e^{\frac{i\pi}{12}} \eta(\tau), \quad \eta\left(-\frac{1}{\tau}\right) = \sqrt{-i\tau} \eta(\tau). \quad (\text{A.2})$$

Here, $q = e^{2\pi i\tau}$. The Jacobi theta function is also defined by

$$\begin{aligned} \theta_1(x; q) = \theta_1(u; \tau) &= -iq^{\frac{1}{8}} x^{\frac{1}{2}} \prod_{k=1}^{\infty} (1 - q^k) (1 - xq^k) (1 - x^{-1}q^{k-1}) \\ &= -i \sum_{n \in \mathbb{Z}} (-1)^n e^{iu(n+\frac{1}{2})} e^{\pi i\tau(n+\frac{1}{2})^2}, \end{aligned} \quad (\text{A.3})$$

where $x = e^{iu}$ and q is as before. (A.3) has simple zeros in u at $u = 2\pi\mathbb{Z} + 2\pi\tau\mathbb{Z}$ and no poles. Its modular properties read

$$\theta_1(u; \tau + 1) = e^{\frac{i\pi}{4}} \theta_1(u; \tau), \quad \theta_1\left(\frac{u}{\tau}; -\frac{1}{\tau}\right) = -i\sqrt{-i\tau} e^{\frac{iu^2}{4\pi\tau}} \theta_1(u; \tau). \quad (\text{A.4})$$

Furthermore, (A.3) satisfies the following useful formula

$$\theta_1(q^m x; q) = (-1)^{-m} x^{-m} q^{-\frac{m^2}{2}} \theta_1(x; q), \quad m \in \mathbb{Z}. \quad (\text{A.5})$$

The asymptotic behavior of (A.1) and (A.3) near $q = 1$ ($\tau = i0$) can be easily derived by using their modular properties (A.2) and (A.4), respectively. This was already done in [4, appendix A] and here we only quote the final result, *i.e.*

$$\begin{aligned} \log \eta(\beta) &= -\frac{1}{2} \log\left(\frac{\beta}{2\pi}\right) - \frac{\pi^2}{6\beta} + \mathcal{O}\left(e^{-1/\beta}\right), \\ \log \theta_1(u; \beta) &= -\frac{\pi^2}{2\beta} - \frac{u^2}{2\beta} - \frac{1}{2} \log\left(\frac{\beta}{2\pi}\right) + \frac{\pi}{\beta} u \text{sign}[\text{Re}(u)] + \log(\text{sign}[\text{Re}(u)]) + \mathcal{O}\left(e^{-1/\beta}\right). \end{aligned} \quad (\text{A.6})$$

Here we identified the complex structure of the torus τ with the *fictitious* inverse temperature β as $\tau \equiv i\beta/(2\pi)$. Therefore, for $\text{Re}(u) > 0$,

$$\log\left(\frac{i\eta(q)}{\theta_1(x; q)}\right) \sim \frac{1}{\beta} g_2(u) + \frac{i\pi}{2}, \quad \text{as } \beta \rightarrow 0, \quad (\text{A.7})$$

where we defined the polynomial functions

$$g_2(u) = \frac{u^2}{2} - \pi u + \frac{\pi^2}{3}, \quad g_3(u) = \frac{u^3}{6} - \frac{\pi}{2} u^2 + \frac{\pi^2}{3} u. \quad (\text{A.8})$$

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References

- [1] F. Benini and N. Bobev, *Exact two-dimensional superconformal R-symmetry and c-extremization*, *Phys. Rev. Lett.* **110** (2013) 061601 [[arXiv:1211.4030](https://arxiv.org/abs/1211.4030)] [[INSPIRE](https://inspirehep.net/literature/1211403)].
- [2] F. Benini and N. Bobev, *Two-dimensional SCFTs from wrapped branes and c-extremization*, *JHEP* **06** (2013) 005 [[arXiv:1302.4451](https://arxiv.org/abs/1302.4451)] [[INSPIRE](https://inspirehep.net/literature/1302445)].

- [3] F. Benini, N. Bobev and P.M. Cricigno, *Two-dimensional SCFTs from D3-branes*, *JHEP* **07** (2016) 020 [[arXiv:1511.09462](#)] [[INSPIRE](#)].
- [4] S.M. Hosseini, A. Nedelin and A. Zaffaroni, *The Cardy limit of the topologically twisted index and black strings in AdS₅*, *JHEP* **04** (2017) 014 [[arXiv:1611.09374](#)] [[INSPIRE](#)].
- [5] N.A. Nekrasov and S.L. Shatashvili, *Bethe/Gauge correspondence on curved spaces*, *JHEP* **01** (2015) 100 [[arXiv:1405.6046](#)] [[INSPIRE](#)].
- [6] C. Closset and I. Shamir, *The $\mathcal{N} = 1$ Chiral Multiplet on $T^2 \times S^2$ and Supersymmetric Localization*, *JHEP* **03** (2014) 040 [[arXiv:1311.2430](#)] [[INSPIRE](#)].
- [7] F. Benini and A. Zaffaroni, *A topologically twisted index for three-dimensional supersymmetric theories*, *JHEP* **07** (2015) 127 [[arXiv:1504.03698](#)] [[INSPIRE](#)].
- [8] M. Honda and Y. Yoshida, *Supersymmetric index on $T^2 \times S^2$ and elliptic genus*, [arXiv:1504.04355](#) [[INSPIRE](#)].
- [9] C. Closset, H. Kim and B. Willett, *$\mathcal{N} = 1$ supersymmetric indices and the four-dimensional A-model*, *JHEP* **08** (2017) 090 [[arXiv:1707.05774](#)] [[INSPIRE](#)].
- [10] K. Hristov, *Dimensional reduction of BPS attractors in AdS gauged supergravities*, *JHEP* **12** (2014) 066 [[arXiv:1409.8504](#)] [[INSPIRE](#)].
- [11] S.M. Hosseini, *Black hole microstates and supersymmetric localization*, Ph.D. Thesis, Milan Bicocca University, (2018), [arXiv:1803.01863](#) [[INSPIRE](#)].
- [12] A. Zaffaroni, *Lectures on AdS Black Holes, Holography and Localization*, [arXiv:1902.07176](#) [[INSPIRE](#)].
- [13] K. Hristov, S. Katmadas and C. Toldo, *Rotating attractors and BPS black holes in AdS₄*, *JHEP* **01** (2019) 199 [[arXiv:1811.00292](#)] [[INSPIRE](#)].
- [14] F. Benini, K. Hristov and A. Zaffaroni, *Black hole microstates in AdS₄ from supersymmetric localization*, *JHEP* **05** (2016) 054 [[arXiv:1511.04085](#)] [[INSPIRE](#)].
- [15] F. Benini, K. Hristov and A. Zaffaroni, *Exact microstate counting for dyonic black holes in AdS₄*, *Phys. Lett. B* **771** (2017) 462 [[arXiv:1608.07294](#)] [[INSPIRE](#)].
- [16] S.M. Hosseini and A. Zaffaroni, *Large N matrix models for 3d $\mathcal{N} = 2$ theories: twisted index, free energy and black holes*, *JHEP* **08** (2016) 064 [[arXiv:1604.03122](#)] [[INSPIRE](#)].
- [17] S.M. Hosseini and N. Mekareeya, *Large N topologically twisted index: necklace quivers, dualities and Sasaki-Einstein spaces*, *JHEP* **08** (2016) 089 [[arXiv:1604.03397](#)] [[INSPIRE](#)].
- [18] S. Ferrara, R. Kallosh and A. Strominger, *$N = 2$ extremal black holes*, *Phys. Rev. D* **52** (1995) R5412 [[hep-th/9508072](#)] [[INSPIRE](#)].
- [19] S.L. Cacciatori and D. Klemm, *Supersymmetric AdS₄ black holes and attractors*, *JHEP* **01** (2010) 085 [[arXiv:0911.4926](#)] [[INSPIRE](#)].
- [20] G. Dall'Agata and A. Gnechchi, *Flow equations and attractors for black holes in $N = 2$ U(1) gauged supergravity*, *JHEP* **03** (2011) 037 [[arXiv:1012.3756](#)] [[INSPIRE](#)].
- [21] S.M. Hosseini, I. Yaakov and A. Zaffaroni, *Topologically twisted indices in five dimensions and holography*, *JHEP* **11** (2018) 119 [[arXiv:1808.06626](#)] [[INSPIRE](#)].
- [22] P. Karndumri and E. O Colgain, *Supergravity dual of c-extremization*, *Phys. Rev. D* **87** (2013) 101902 [[arXiv:1302.6532](#)] [[INSPIRE](#)].

- [23] A. Amariti and C. Toldo, *Betti multiplets, flows across dimensions and c-extremization*, *JHEP* **07** (2017) 040 [[arXiv:1610.08858](#)] [[INSPIRE](#)].
- [24] A. Amariti, L. Cassia and S. Penati, *Surveying 4d SCFTs twisted on Riemann surfaces*, *JHEP* **06** (2017) 056 [[arXiv:1703.08201](#)] [[INSPIRE](#)].
- [25] A. Amariti, L. Cassia and S. Penati, *c-extremization from toric geometry*, *Nucl. Phys. B* **929** (2018) 137 [[arXiv:1706.07752](#)] [[INSPIRE](#)].
- [26] C. Couzens, J.P. Gauntlett, D. Martelli and J. Sparks, *A geometric dual of c-extremization*, *JHEP* **01** (2019) 212 [[arXiv:1810.11026](#)] [[INSPIRE](#)].
- [27] J.P. Gauntlett, D. Martelli and J. Sparks, *Toric geometry and the dual of c-extremization*, *JHEP* **01** (2019) 204 [[arXiv:1812.05597](#)] [[INSPIRE](#)].
- [28] S.M. Hosseini and A. Zaffaroni, *Proving the equivalence of c-extremization and its gravitational dual for all toric quivers*, *JHEP* **03** (2019) 108 [[arXiv:1901.05977](#)] [[INSPIRE](#)].
- [29] M. Henningson and K. Skenderis, *The holographic Weyl anomaly*, *JHEP* **07** (1998) 023 [[hep-th/9806087](#)] [[INSPIRE](#)].
- [30] C. Closset, H. Kim and B. Willett, *Supersymmetric partition functions and the three-dimensional A-twist*, *JHEP* **03** (2017) 074 [[arXiv:1701.03171](#)] [[INSPIRE](#)].
- [31] S.M. Hosseini, K. Hristov and A. Passias, *Holographic microstate counting for AdS₄ black holes in massive IIA supergravity*, *JHEP* **10** (2017) 190 [[arXiv:1707.06884](#)] [[INSPIRE](#)].
- [32] F. Benini, H. Khachatryan and P. Milan, *Black hole entropy in massive Type IIA*, *Class. Quant. Grav.* **35** (2018) 035004 [[arXiv:1707.06886](#)] [[INSPIRE](#)].
- [33] F. Benini and P. Milan, *Black holes in 4d $\mathcal{N} = 4$ Super-Yang-Mills*, [arXiv:1812.09613](#) [[INSPIRE](#)].
- [34] J. Hong and J.T. Liu, *The topologically twisted index of $\mathcal{N} = 4$ super-Yang-Mills on $T^2 \times S^2$ and the elliptic genus*, *JHEP* **07** (2018) 018 [[arXiv:1804.04592](#)] [[INSPIRE](#)].
- [35] D. Anselmi, D.Z. Freedman, M.T. Grisaru and A.A. Johansen, *Nonperturbative formulas for central functions of supersymmetric gauge theories*, *Nucl. Phys. B* **526** (1998) 543 [[hep-th/9708042](#)] [[INSPIRE](#)].
- [36] A. Butti and A. Zaffaroni, *R-charges from toric diagrams and the equivalence of a-maximization and Z-minimization*, *JHEP* **11** (2005) 019 [[hep-th/0506232](#)] [[INSPIRE](#)].
- [37] S. Benvenuti and M. Kruczenski, *From Sasaki-Einstein spaces to quivers via BPS geodesics: $L^{p,q|r}$* , *JHEP* **04** (2006) 033 [[hep-th/0505206](#)] [[INSPIRE](#)].
- [38] S. Benvenuti, L.A. Pando Zayas and Y. Tachikawa, *Triangle anomalies from Einstein manifolds*, *Adv. Theor. Math. Phys.* **10** (2006) 395 [[hep-th/0601054](#)] [[INSPIRE](#)].
- [39] I.R. Klebanov and E. Witten, *Superconformal field theory on three-branes at a Calabi-Yau singularity*, *Nucl. Phys. B* **536** (1998) 199 [[hep-th/9807080](#)] [[INSPIRE](#)].
- [40] M. Azzola, D. Klemm and M. Rabbiosi, *AdS₅ black strings in the STU model of FI-gauged $N = 2$ supergravity*, *JHEP* **10** (2018) 080 [[arXiv:1803.03570](#)] [[INSPIRE](#)].
- [41] S.M. Hosseini, K. Hristov and A. Zaffaroni, *An extremization principle for the entropy of rotating BPS black holes in AdS₅*, *JHEP* **07** (2017) 106 [[arXiv:1705.05383](#)] [[INSPIRE](#)].
- [42] J.P. Gauntlett and O. Varela, *Consistent Kaluza-Klein reductions for general supersymmetric AdS solutions*, *Phys. Rev. D* **76** (2007) 126007 [[arXiv:0707.2315](#)] [[INSPIRE](#)].

- [43] D. Cassani and A.F. Faedo, *A supersymmetric consistent truncation for conifold solutions*, *Nucl. Phys. B* **843** (2011) 455 [[arXiv:1008.0883](#)] [[INSPIRE](#)].
- [44] I. Bena, G. Giecold, M. Graña, N. Halmagyi and F. Orsi, *Supersymmetric Consistent Truncations of IIB on $T^{1,1}$* , *JHEP* **04** (2011) 021 [[arXiv:1008.0983](#)] [[INSPIRE](#)].
- [45] S. Ferrara and A. Zaffaroni, *$N = 1$, $N = 2$ 4-D superconformal field theories and supergravity in AdS_5* , *Phys. Lett. B* **431** (1998) 49 [[hep-th/9803060](#)] [[INSPIRE](#)].
- [46] L. Andrianopoli, S. Ferrara and M.A. Lledó, *Scherk-Schwarz reduction of $D = 5$ special and quaternionic geometry*, *Class. Quant. Grav.* **21** (2004) 4677 [[hep-th/0405164](#)] [[INSPIRE](#)].
- [47] D. Gaiotto, A. Strominger and X. Yin, *New connections between 4-D and 5-D black holes*, *JHEP* **02** (2006) 024 [[hep-th/0503217](#)] [[INSPIRE](#)].
- [48] K. Behrndt, G. Lopes Cardoso and S. Mahapatra, *Exploring the relation between 4-D and 5-D BPS solutions*, *Nucl. Phys. B* **732** (2006) 200 [[hep-th/0506251](#)] [[INSPIRE](#)].
- [49] G. Lopes Cardoso, J.M. Oberreuter and J. Perz, *Entropy function for rotating extremal black holes in very special geometry*, *JHEP* **05** (2007) 025 [[hep-th/0701176](#)] [[INSPIRE](#)].
- [50] L. Andrianopoli et al., *$N = 2$ supergravity and $N = 2$ superYang-Mills theory on general scalar manifolds: Symplectic covariance, gaugings and the momentum map*, *J. Geom. Phys.* **23** (1997) 111 [[hep-th/9605032](#)] [[INSPIRE](#)].
- [51] K. Hristov and S. Katmadas, *Wilson lines for AdS_5 black strings*, *JHEP* **02** (2015) 009 [[arXiv:1411.2432](#)] [[INSPIRE](#)].
- [52] N. Halmagyi, *Static BPS black holes in AdS_4 with general dyonic charges*, *JHEP* **03** (2015) 032 [[arXiv:1408.2831](#)] [[INSPIRE](#)].
- [53] H.K. Kunduri and J. Lucietti, *Near-horizon geometries of supersymmetric AdS_5 black holes*, *JHEP* **12** (2007) 015 [[arXiv:0708.3695](#)] [[INSPIRE](#)].
- [54] A. Cabo-Bizet, V.I. Giraldo-Rivera and L.A. Pando Zayas, *Microstate counting of AdS_4 hyperbolic black hole entropy via the topologically twisted index*, *JHEP* **08** (2017) 023 [[arXiv:1701.07893](#)] [[INSPIRE](#)].
- [55] J.L. Cardy, *Operator Content of Two-Dimensional Conformally Invariant Theories*, *Nucl. Phys. B* **270** (1986) 186 [[INSPIRE](#)].
- [56] R. Dijkgraaf, J.M. Maldacena, G.W. Moore and E.P. Verlinde, *A black hole Farey tail*, [hep-th/0005003](#) [[INSPIRE](#)].
- [57] J. Manschot and G.W. Moore, *A Modern Farey Tail*, *Commun. Num. Theor. Phys.* **4** (2010) 103 [[arXiv:0712.0573](#)] [[INSPIRE](#)].
- [58] A. Dabholkar, S. Murthy and D. Zagier, *Quantum Black Holes, Wall Crossing and Mock Modular Forms*, [arXiv:1208.4074](#) [[INSPIRE](#)].
- [59] N. Bobev, V.S. Min and K. Pilch, *Mass-deformed ABJM and black holes in AdS_4* , *JHEP* **03** (2018) 050 [[arXiv:1801.03135](#)] [[INSPIRE](#)].