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(Ir-)Regularity of canonical projection operators on some weakly pseudoconvex domains

Alessandro MONGUZZI[1] and Marco M. PELOSO[2]

Abstract. In this paper we discuss some recent results concerning the regularity and irregularity of the Bergman and Szegő projections on some weakly pseudoconvex domains that have the common feature to possess a nontrivial Nebenhülle.

INTRODUCTION

In this note we survey some recent results on the analysis of canonical projection operators, such as the Bergman and Szegő projections, on a family of domains that present some pathological behavior. These domains have the common feature to possess a nontrivial *Nebenhülle*, and they essentially are the worm domain of K. Diederich and J.E. Fornæss, the Hartogs triangle and some of its variants, and some model worm domains introduced by C. Kiselman and studied, among others by D. Barrett, S. Krantz and the authors of this note.

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The worm domain $\mathcal{W}_\mu$ was introduced by K. Diederich and J.E. Fornæss in [25].

$$\mathcal{W}_\mu = \Big\{(z_1, z_2) \in \mathbb{C}^2 \,:\, |z_1 - e^{i\log|z_2|^2}|^2 < 1 - \eta(\log|z_2|^2)\Big\}$$

where η is smooth, even, convex, vanishing on $[-\mu, \mu]$, with $\eta(a) = 1$, and $\eta'(a) > 0$. These properties of η imply that $\mathcal{W}_\mu$ is smooth, bounded and pseudoconvex. Morevover $\mathcal{W}_\mu$ is strictly pseudoconvex at all points $(z_1, z_2) \in \partial\mathcal{W}_\mu$ with $z_1 \neq 0$. The set of points on the boundary $\mathcal{A} = \{(0, z_2) \,:\, \big|\log|z_2|^2\big| \leq \mu\}$ is the *critical annulus*.

In [25] the following important features of $\mathcal{W}_\mu$ were shown:

(I) $\mathcal{W}_\mu$ has non-trivial Nebenhülle (that is, there exists no neighborhood basis of pseudoconvex domains for $\mathcal{W}_\mu$) [Diederich-Fornæss];

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[1]A. Monguzzi, Università di Milano-Bicocca, Dipartimento di Matematica e Applicazioni, Via Roberto Cozzi 55, 20126 Milano, Italy; `alessandro.monguzzi@unimib.it`

[2]M.M. Peloso, Università degli Studi di Milano, Dipartimento di Matematica, Via Cesare Saldini 50, 20133 Milano, Italy. `marco.peloso@unimi.it`

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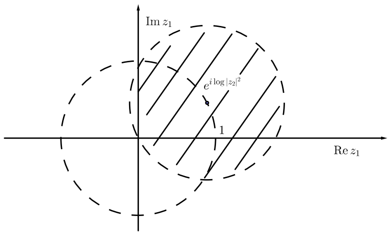

FIGURE 1. Representation of $\mathcal{W}_\mu$ in the $(\operatorname{Re} z_1, \operatorname{Im} z_1)$-plane.

(II) $\mathcal{W}_\mu$ does not admit any plurisubharmonic defining function (that is, a defining function that is plurisubharmonic on the boundary).

Concerning (I), by the Hartogs's extension phenomenon indeed, it follows that if f is holomorphic in a neighborhood of

$$\left\{(0, z_2) \,:\, \left|\log|z_2|^2\right| \leq \pi\right\} \cup \left\{(z_1, z_2) \,:\, \left|\log|z_2|^2\right| = \pi,\ \left|z_1 - e^{i\mu}\right| \leq 1\right\} ,$$

then f is also holomorphic in a neighborhood of the set

$$\left\{(z_1, z_2) \,:\, \left|\log|z_2|^2\right| \leq \pi,\ \left|z_1 - e^{i\mu}\right| \leq 1\right\} .$$

Regarding (II) we recall that, given a domain $\Omega \subseteq \mathbb{C}^n$, a continuous function $\varphi : \Omega \to (-\infty, 0)$ is called a *bounded exhaustion function* if for all $c < 0 \in \mathbb{R}$,

$$\overline{\varphi^{-1}(-\infty, c)} \cap \partial\Omega = \emptyset .$$

In [25] Diederich and Fornæss proved that if $\Omega \subseteq \mathbb{C}^n$ is smooth, bounded and pseudoconvex with defining function ρ, then there exists $\tau \in (0,1]$ such that $-(-\rho)^\tau$ is a bounded stricly plurisubharmonic exhaustion function. Such an exponent $\tau = \tau_\rho$ is called a DF-*exponent* for the defining function ρ. We set

$$\mathrm{DF}(\Omega) = \sup\left\{\tau_\rho \,:\, \rho \text{ defining function of } \Omega\right\} ,$$

and we call this value the *Diederich–Fornæss index* of Ω. In [25], Diedirich and Fornæss proved that

$$\mathrm{DF}(\mathcal{W}_\mu) \leq \frac{\pi}{2\mu} .$$

Since its appearance, research on the properties of the worm domains remained dormant for a number of years. We now consider a still open fundamental problem in analysis and geometry of several complex variables: Given $\mathcal{D}_1, \mathcal{D}_2$ bounded, smooth, pseudoconvex domains and a biholomorphic mapping $\Phi : \mathcal{D}_1 \to \mathcal{D}_2$, does Φ extend smoothly to a diffeormophism of the boundaries? We denote by $\partial\mathcal{D}$ the topological boundary of a given domain $\mathcal{D}$.

Given a domain Ω, let $A^2(\Omega) = L^2(\Omega) \cap \mathrm{Hol}(\Omega)$ be the Bergman space, and $P_\Omega : L^2(\Omega) \to A^2(\Omega)$ be the Bergman projection. A celebrated theorem by S. Bell and E. Ligocka [8], and later improved by S. Bell [7], says that this is the case if one of the two domains satisfies (say $\mathcal{D}_1$) the so-called Condition (R):

$$\text{(R)} \qquad P_{\mathcal{D}_1} : C^\infty(\overline{\mathcal{D}_1}) \to C^\infty(\overline{\mathcal{D}_1}) \qquad \text{is bounded} .$$

In [2] D. Barrett showed that, writing $\mathcal{W}$ in place of $\mathcal{W}_\mu$ for short and denoting the Sobolev space on $\mathcal{W}$ by $W^{2,s}(\mathcal{W})$,

$$P_{\mathcal{W}} : W^{2,s}(\mathcal{W}) \not\to W^{2,s}(\mathcal{W})$$

if $s \geq \pi/2\mu$. Hence, $P_{\mathcal{W}}$ fails to preserve the L^2-Sobolev space $W^{2,s}(\mathcal{W})$, when s is greater or equal to the reciprocal of the windings of the domain $\mathcal{W}$.

Although Barrett's result constituted a major breakthrough, it did not imply that $\mathcal{W}$ failed to satisfy Condition (R). It was M. Christ in [23] to prove that the Neumann operator $\mathcal{N}$ on $\mathcal{W}$ does not preserve $C^\infty(\overline{\mathcal{W}})$; hence $\mathcal{W}$ does not satisfy Condition (R). In fact, by a theorem of Boas–Straube [13], on any given smoothly bounded pseudoconvex domain Ω, $\mathcal{N}$ is globally (exactly) regular if and only if P_Ω is globally (exactly) regular. We say that P_Ω is exactly regular if $P_\Omega : W^{2,s}(\Omega) \to W^{2,s}(\Omega)$ is bounded for all $s > 0$. We say that P_Ω is globally regular if given any $s > 0$, there exists $q = q(s)$ such that $P_\Omega : W^{2,s+q(s)}(\Omega) \to W^{2,s}(\Omega)$ is bounded. In particular, if P_Ω is globally regular, then $P_\Omega : C^\infty(\overline{\Omega}) \to C^\infty(\overline{\Omega})$ is bounded, that is, Ω satisfies condition (R).

The problem of the regularity of the Bergman projection on worm domains has been object of active and intense research and we mention in particular [36, 5, 3, 24, 37, 38]. We also refer the reader to [35] for a detailed account on the subject.

Main goal of this note is to report on recent progress on the analysis of the (ir-)regularity of the boundary analogue of the Bergman projection, that is, the Szegő projection. Given a smoothly bounded domain $\Omega = \{z : \rho(z) < 0\} \subseteq \mathbb{C}^n$, the Hardy space $H^2(\Omega, d\sigma)$ is defined as

$$H^2(\Omega, d\sigma) = \left\{ f \in \mathrm{Hol}(\Omega) : \sup_{\varepsilon>0} \int_{\partial\Omega_\varepsilon} |f|^2 d\sigma_\varepsilon < \infty \right\},$$

where $\Omega_\varepsilon = \{z : \rho(z) < -\varepsilon\}$ and $d\sigma_\varepsilon$ is the induced surface measure on $\partial\Omega_\varepsilon$.

Then, $H^2(\Omega, d\sigma)$ can be identified with a closed subspace of $L^2(\partial\Omega, d\sigma)$, that we denote by $H^2(\partial\Omega, d\sigma)$, where σ is the induced surface measure on $\partial\Omega$. The Szegő projection is the orthogonal projection

$$S_\Omega : L^2(\partial\Omega, d\sigma) \to H^2(\partial\Omega, d\sigma);$$

see [61].

The note is organized as follows. In Section 1 we recall some further noticeable results concerning the worm domain, some of its generalizations and some ideas involved in the proofs of such results. In Section 2 we discuss the case of Szegő projections on worm domains. Section 4 is devoted to the case of another class of domains, the so-called Hartogs triangles. In Section 5 we present an interesting problem in the theory of 1-dimensional Bergman spaces that arose in the study of orthogonal sets in the Bergman space of the truncated worm domain. We conclude this report with some final remarks and open questions.

1. Generalizations and some open problems on the worm domain

The worm domain $\mathcal{W}$ is still up to today the only known example of a smoothly bounded pseudoconvex domain on which Condition (R) fails. Thus, it is a natural testing ground for the validity of the extendebility to diffeomorphism to the boundary of biholomorphic mappings. The first class of mappings that one is naturally led to consider are the biholomorphic self-maps of $\mathcal{W}$, that is, the *automorphisms* of $\mathcal{W}$, $\mathrm{Aut}(\mathcal{W})$. Clearly, the maps $\Phi(z_1,z_2)=(z_1,e^{i\theta}z_2)$ are in $\mathrm{Aut}(\mathcal{W})$ and extend smoothly to the boundary. The obvious question is: Are there any others? In [22], the author studied the automorphisms group $\mathrm{Aut}(\mathcal{W})$, and claimed that this is the case. Unfortunately, it is generally accepted that there is a gap in the proof and it has not been fixed. Thus, the very interesting question of characterizing the automorphism group $\mathrm{Aut}(\mathcal{W})$ is an open and fundamental question.

Before going any further, we point out that in [5] D. Barrett and S. Şahutoğlu constructed a higher dimensional analogue of the worm domain. Let $n\geq 3$ and for $z\in\mathbb{C}^n$ we write $z=(z_1,z',z_n)\in\mathbb{C}\times\mathbb{C}^{n-2}\times\mathbb{C}$. For $\lambda,\mu>0$ define

$$(1)\quad \mathcal{W}_{\lambda,\mu}=\big\{(z_1,z',z_n)\in\mathbb{C}\times\mathbb{C}^{n-2}\times\mathbb{C} : \big|z_1-e^{i\lambda\log|z_n|^2}\big|^2<1-|z'|^2-\widetilde{\eta}(\log|z_n|^2)\big\},$$

where $\widetilde{\eta}$ is a *particular*, explicit, smooth function which is identically 0 when $e^{-1/2}\leq |z_n|\leq e^{\mu/2}$. The function $\widetilde{\eta}$ is chosen in such a way the domain is smoothly bounded and pseudoconvex, and it is strongly pseudoconvex except at the critical annulus

$$\mathcal{A}=\big\{(z_1,z',z_n)\in\mathbb{C}\times\mathbb{C}^{n-2}\times\mathbb{C}\cap\partial\mathcal{W}_{\lambda,\mu}\ :\ z_1=0,\ z'=0\big\}$$

$$=\big\{(0,0,z_n)\in\mathbb{C}\times\mathbb{C}^{n-2}\times\mathbb{C}\ :\ e^{-\mu/2}\leq|z_n|\leq e^{\mu/2}\big\}\ .$$

Barrett and S. Şahutoğlu proved that the Bergman projection $P_{\mathcal{W}_{\lambda,\mu}}$ fails to preserve the Sobolev spaces $W^{p,s}$, with $p\in[1,\infty)$ and $s\geq 0$, hence including the cases $p\neq 2$, when

$$s\geq\frac{\pi}{2\lambda\mu}+n\Big(\frac12-\frac1p\Big)\ .$$

What is extremely interesting to notice here is that the Bergman projection becomes irregular if either the winding is too "long" (i.e. when μ is large), or is too "fast" (i.e. when λ is large).

For simplicity of presentation, we restricted ourselves to the 2-dimensional case, that is, to the domain $\mathcal{W}=\mathcal{W}_\mu$. However, we point out that the discussion that follows is also valid for the higher dimensional cases of the domains $\mathcal{W}_{\lambda,\mu}$.

Instrumental to Barrett's proof of the irregularity of $P_{\mathcal{W}}$ were two unbounded model worm domains, that we denote by D_μ and D'_μ, where

$$D_\mu=\Big\{(z_1,z_2)\in\mathbb{C}^2\ :\ \mathrm{Re}\big(z_1e^{-i\log|z_2|^2}\big)>0,\ \big|\log|z_2|^2\big|<\mu\Big\}\quad,\quad \mu>\pi\ ,$$

and

$$D'_\mu=\Big\{(z_1,z_2)\in\mathbb{C}^2\ :\ \big|\mathrm{Im}\,z_1-\log|z_2|^2\big|<\frac{\pi}{2},\ \big|\log|z_2|^2\big|<\mu\Big\}\ .$$

Remarks 1.1. The following facts are easy to see:

(i) the domains D'_μ and D_μ are biholomorphically equivalent via the mapping

$$\varphi: D'_\mu\to D_\mu\qquad \varphi(z_1,z_2):=(e^{z_1},z_2)\ ;$$

(ii) for every z_1 fixed the fiber over z_1

$$\Pi(z_1)=\{z_2\in\mathbb{C}\ :\ (z_1,z_2)\in D'_\mu\}$$

is connected, while the same property does not hold for D_μ.

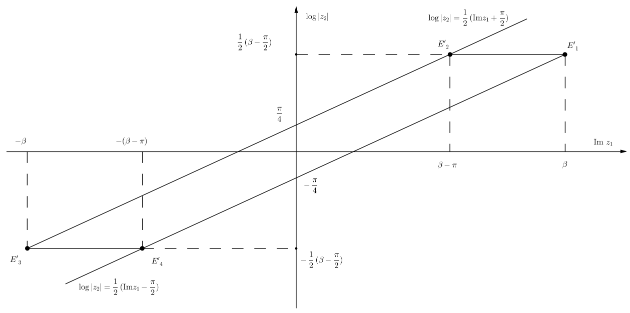

FIGURE 2. Representation of D'_μ in the $(\mathrm{Im}\, z_1, \log|z_2|)$-plane.

For a given domain Ω in $\mathbb{C}^2$ that is rotationally invariant in the second variable z_2, such as $\mathcal{W}, D_\mu, D'_\mu$, using Fourier expansion in z_2 we can decompose the Bergman space $A^2(\Omega)$ as

$$A^2(\Omega) = \bigoplus_{j \in \mathbf{Z}} \mathcal{H}^j \ , \tag{2}$$

where

$$\mathcal{H}^j = \big\{ F \in A^2 \, : \, F(z_1, e^{i\theta} z_2) = e^{ij\theta} F(z_1, z_2) \big\} \ .$$

If for every z_1 fixed, the fibers $\Pi(z_1) = \{z_2 \, : \, (z_1, z_2) \in \Omega\}$ are connected, then $F \in \mathcal{H}^j$ has the form

$$F(z_1, z_2) = f(z_1) z_2^j \ ,$$

where f is holomorphic. In the case of D'_μ, the fibers $\Pi(z_1)$ are connected and f is holomorphic on the strip $\{|\mathrm{Im}\, z_1| < \mu + \pi/2\}$. Hence, we may write the kernel K' of D'_μ as

$$K'(\zeta, \omega) = \sum_{j=-\infty}^{\infty} K'_j(\zeta_1, \omega_1) \zeta_2^j \overline{\omega}_2^j$$

and using these observations and an explicit computation in 1-dimension, it is possible to compute the Bergman kernel K' of D'_μ quite explicitly. In [2] the kernel K'_{-1} is explicitly computed and it holds

$$K'_{-1}(\zeta, \omega) = \frac{1}{2\pi} \int_{\mathbb{R}} \frac{t^2}{\sinh(2\mu t) \sinh(2\pi t)} \, e^{i(\zeta_1 - \overline{\omega}_1)} \, dt \ .$$

The analysis of the kernels K'_j's for $j \neq -1$ is performed in [36] and it is more difficult since some cancellations that simplify the computations in the case $j = -1$ do not occur for $j \neq -1$. Recalling the transformation rule for the Bergman kernel,

$$P_{D'_\mu}[\varphi'(f \circ \varphi)] = \varphi'[(P_{D_\mu} f) \circ \varphi] \ ,$$

Barrett analyses the kernel K of D_μ and, in particular, the (-1)-component of the kernel, concluding that P_{D_μ} is not exactly regular, that is, P_{D_μ} is not a bounded operator $P_{D_\mu} : W^{s,2}(D_\mu) \to W^{s,2}(D_\mu)$ for s sufficiently large. The same conclusion for the smooth worm $\mathcal{W}$ is then obtained via an exhaustion argument. Setting $\mathcal{W}^\tau = \big\{(z_1,z_2)\in\mathbb{C}^2 \,:\, (z_1/\tau, z_2)\in\mathcal{W}\big\}$, Barrett showed that

$$P_{\mathcal{W}^\tau} f \to P_{D_\mu} f \tag{3}$$

as $\tau\to\infty$. For, if we denote by d_τ the dilation in the first variable by $\tau>0$, $d_\tau(z_1,z_2)=(\tau z_1,z_2)$, then $d_\tau(\mathcal{W})=\mathcal{W}^\tau$ and

$$P_{\mathcal{W}^\tau} = T_\tau^{-1} P_{\mathcal{W}} T_\tau \,. \tag{4}$$

where $T_\tau f(z_1,z_2)=f(\tau z_1,z_2)$. From this relation, it is possible to deduce the boundedness of $P_{\mathcal{W}^\tau}$ from the one of $P_{\mathcal{W}}$. Then, passing to the limit as in (3), we would obtain the boundedness of P_{D_μ}, hence, a contradiction. In order to prove (3), it is necessary the trivial, but important, remark that given any compact set $E\subseteq D_\mu$, there exists $\tau_E>0$ so that for all $\tau\ge\tau_E$, $E\subseteq\mathcal{W}^\tau$.

Thus, the analysis on the domains D'_μ and D_μ not only provided intuition on the case of the smooth, bounded worm domain $\mathcal{W}$, but also it was fundamental in proving the result on the irregularity of the Bergman projection on $\mathcal{W}$ itself.

We now briefly comment on Christ's result [23]. He proved that $\mathcal{W}$ does not satisfy Condition (R), by showing that for all $s>0$ (apart from a discrete set of exceptions) the Neumann operator $\mathcal{N}$ satisfies an *a priori* estimate

$$||\mathcal{N}u||_{W^{2,s}} \le C_{s,j}||u||_{W^{2,s}}$$

valid for every $u\in\mathcal{H}_1^j\cap C^\infty(\overline{\mathcal{W}})$ such that $\mathcal{N}u\in C^\infty(\overline{\mathcal{W}})$. (Here the subscript 1 indicates the fact that u is a $(0,1)$-form.) If $\mathcal{N}: C^\infty(\overline{\mathcal{W}})\to C^\infty(\overline{\mathcal{W}})$ were bounded, such estimates would contradict the irregularity of $P_{\mathcal{W}}$.

We conclude this section by discussing the Diederich–Fornæss index of the worm domain $\mathcal{W}$. In [45, 46] B. Liu proved that $\mathrm{DF}(\mathcal{W})=\pi/2\mu$, see also [34].

2. Hardy spaces on model worm domains

We now consider another canonical kernel and projection of a domain Ω in $\mathbb{C}^n$, the Szegő kernel and projection.

Let

$$\Omega=\big\{z\in\mathbb{C}^n \,:\, \rho(z)<0\big\}\,,$$

where ρ is smooth and $\nabla\rho\neq0$ on $\partial\Omega$. Let $\Omega_\varepsilon=\{\rho(z)<-\varepsilon\}$, and suppose there exists a family of Borel measures $\{\sigma_\varepsilon\}$ on $\overline{\Omega}$ and supported on $\partial\Omega_\varepsilon\subseteq\overline{\Omega}$ such $\sigma_\varepsilon\to\sigma_0=:\sigma$ weakly as $\varepsilon\to0$, that is, for all $f\in C(\overline{\Omega})$, $\int f\,d\sigma_\varepsilon\to\int f\,d\sigma$ as $\varepsilon\to0$. Define the Hardy space $H^2(\Omega,d\sigma)$ as

$$H^2(\Omega,d\sigma)=\Big\{f\in\mathrm{Hol}(\Omega)\,:\,\sup_{\varepsilon>0}\int_{\partial\Omega_\varepsilon}|f(\zeta)|^2d\sigma_\varepsilon(\zeta)<\infty\Big\}\,.$$

Under mild conditions on the family of measures $\{\sigma_\varepsilon\}$, the Hardy $H^2(\Omega,d\sigma)$ is a reproducing kernel Hilbert space and its reproducing kernel is called the Szegő kernel. The classical case is Ω is a smoothly bounded domain and $d\sigma_\varepsilon$ is the induced surface measure on $\partial\Omega_\varepsilon$. In this case, we simply write $H^2(\Omega)$. It is a classical result (see [61]) that under these assumptions, if $f\in H^2(\Omega)$ then f converges *non-tangentially* to a boundary function $\widetilde{f}\in L^2(\partial\Omega)$. In full generality, these latter facts have to be shown to hold true.

Thus, we may define

$$H^2(\partial\Omega, d\sigma) = \Big\{ g \in L^2(\partial\Omega, d\sigma) \,:\, g(\zeta) = \lim_{z\to\zeta} f(z) \text{ non-tangentially,}$$
$$\text{for some } f \in H^2(\Omega, d\sigma) \Big\} .$$

Then, the Szegő projection is the Hilbert space orthogonal projection of $L^2(\partial\Omega, d\sigma)$ onto its (closed) subspace $H^2(\partial\Omega, d\sigma)$, the subspace of boundary values of functions in $H^2(\Omega, d\sigma)$,

$$S_\Omega : L^2(\partial\Omega, d\sigma) \to H^2(\partial\Omega, d\sigma) \quad S_\Omega g(\zeta) = \lim_{z\to\zeta\in\partial\Omega} \int_{b\Omega} g(\zeta')K(z,\zeta')\, d\sigma(\zeta') \; .$$

By definition, the Szegő projection depends on the choice of the measure on the boundary. Another very natural, and thus far little considered, possible choice, is the Fefferman surface measure σ_F, see [4], or any other surface measure $\omega d\sigma$, where ω is a continuous positive function on $\partial\Omega$, [43]. The surface measure σ_F was introduced by C. Fefferman in order to obtain a measure that is *biholomorphic invariant*. To be precise, suppose Ω_1 and Ω_2 are bounded, smooth, pseudoconvex domains that admit a biholomorphic map $\varphi : \Omega_1 \to \Omega_2$ that extends to a smooth C^∞-diffeormophism of the boundary, such as in case one of the two domains satisfies Condition (R). Then, the mapping $\Lambda(f) := \sqrt{\det\varphi'}(f \circ \varphi)$ defines an isometric isomorphism $\Lambda : H^2(\Omega_2, d\sigma_F) \to H^2(\Omega_1, d\sigma_F)$.

When we consider non-smooth domains, such as D_μ and D'_μ, and more noticeably the polydisk, it is perhaps more natural, and certainly interesting to study Hardy spaces defined by integration over the so-called *distinguished boundary*. Given a domain $\Omega \subseteq \mathbb{C}^n$, we call the distinguished boundary, and we denote it by $d_b(\Omega)$,the set

$$d_b(\Omega) = \Big\{ \zeta \in \partial\Omega \,:\, \sup_{z\in\partial\Omega} |f(z)| \le \sup_{\zeta\in d_b(\Omega)} |f(\zeta)| \text{ for all } f \in H^\infty(\Omega) \Big\} \tag{5}$$

where $H^\infty(\Omega)$ is the spaces of holomorphic functions on Ω that are bounded. Then, we consider the induced measure on $d_b(\Omega)$ and denote it by $d\beta$.

We now describe the main results that we have obtained on the regularity of Szegő projections on model worm domains. We first consider the case of D'_μ and the induced surface measure $d\beta$. The following result is in [52].

Theorem 2.1. *The Szegő projection $\mathcal{S}$, initially defined on the dense subspace $W^{s,p}(\partial D'_\mu) \cap L^2(\partial D'_\mu, d\sigma)$, extends to a bounded operator*

$$\mathcal{S} : W^{s,p}(\partial D'_\mu) \to W^{s,p}(\partial D'_\mu) \; ,$$

for $1 < p < \infty$ and $s \ge 0$.

The proof of such result relies on explicit computations on the boundary of D'_μ, which can be written as union of four pieces that have intersection of null measure. The Szegő projection can be correspondingly written as sum of 16 different integral operators. For each of these operators we apply a decomposition similar to (2) and obtain an explicit expression and thus write them as composition of a bounded Fourier multiplier and an operator of Hilbert-type.

It is worth to remark that the boundedness of the corresponding Szegő projection on D_μ is still unexplored and it would be significant to study such (ir)-regularity.

We now turn to the case of the Szegő projection on the distinguished boundaries ([50]). More precisely, denote by $d_b(D'_\mu)$ and $d_b(D_\mu)$ the distinguished boundaries of the domains D'_μ and D_μ, resp., and by $\mathscr{S}'$ and $\mathscr{S}$ the corresponding Szegő projections, resp. We point out that in this setting, the operators $\mathscr{S}'$ and $\mathscr{S}$ are given by singular integrals over $d_b(D'_\mu)$ and $d_b(D_\mu)$, resp.

The case of D'_μ was considered in [49], where the main result is the following

Theorem 2.2. *The Szegő projection $\mathscr{S}'$, initially defined on the dense subspace $W^{s,p}(d_b(D'_\mu), d\beta) \cap L^2(d_b(D'_\mu), d\beta)$, extends to a bounded operator*

$$\mathscr{S}' : W^{s,p}(d_b(D'_\mu), d\beta) \to W^{s,p}(d_b(D'_\mu), d\beta) ,$$

for $1 < p < \infty$ and $s \geq 0$.

The proof of this result follows from explicit computations of the Szegő projection of suitably defined Hardy spaces on the distinguished boundary of D'_μ. Such a boundary is the union of four different connected components which are mutually disjoint and the Szegő projection turns out to be a linear combination of bounded Fourier multiplier operators. A detailed analysis of the Szegő kernel associated to $\mathscr{S}'$ is performed in [48].

With a similar proof the analogous result for the Szegő projection $\mathscr{S}$ on the distinguished boundary of D_μ is studied and we now describe it with greater details. For $(t, s) \in (0, \pi/2) \times [0, \mu)$ consider the domain

$$D_{t,s} = \left\{(z_1, z_2) \in \mathbb{C}^2 \,:\, \left|\arg\, z_1 - \log|z_2|^2\right| < t,\ \left|\log|z_2|^2\right| < s\right\} .$$

Then, the domains $\{D_{t,s}\}_{t,s}$ constitute a family of approximating domains for D_μ. The distinguished boundary of these domains is given by

$$d_b(D_{t,s}) = \left\{(z_1, z_2) \in \mathbb{C}^2 \,:\, \left|\arg\, z_1 - \log|z_2|^2\right| = t,\ \left|\log|z_2|^2\right| = s\right\} .$$

Consequently, for $1 \leq p < \infty$, we define the Hardy space $H^p(d_b(D_\beta), d\beta)$ defined by

$$H^p(d_b(D_\beta), d\beta) = \bigg\{ f \in \mathrm{Hol}(D_\beta) \,:\, \|f\|^p_{H^p(d_b(D_\beta),d\beta)} = \sup_{(t,s)\in(0,\frac{\pi}{2})\times[0,\mu)} \|f\|^p_{L^p(d_b(D_{t,s}),d\beta)} < \infty \bigg\} ,$$

where, denoting by $d\beta_{t,s}$ the induced measure on $d_b(D_{t,s})$,

$$\begin{aligned}\|f\|^p_{L^p(d_b(D_{t,s}),d\beta)} &= \int_{d_b(D_{t,s})} |f|^p \, d\beta_{t,s} \\ &= \int_0^\infty \int_0^{2\pi} |f\big(re^{i(s+t)}, e^{s/2}e^{i\theta}\big)|^p \, e^{s/2} d\theta dr \\ &+ \int_0^\infty \int_0^{2\pi} |f\big(re^{i(s-t)}, e^{s/2}e^{i\theta}\big)|^p \, e^{s/2} d\theta dr \\ &+ \int_0^\infty \int_0^{2\pi} |f\big(re^{-i(s+t)}, e^{-s/2}e^{i\theta}\big)|^p \, e^{-s/2} d\theta dr \\ &+ \int_0^\infty \int_0^{2\pi} |f\big(re^{-i(s-t)}, e^{-s/2}e^{i\theta}\big)|^p \, e^{-s/2} d\theta dr .\end{aligned}$$

The main results in [53] are the following. The first result provides the sharp interval of values of p for which the Szegő projection $\mathscr{S}$ on the distinguished boundary of D_μ is bounded. We recall that we set $\nu = \pi/2\mu$, so that $\nu = \nu_\mu$ tends to 0 as μ becomes large.

Theorem 2.3. *The Szegő projection $\mathscr{S}$, initially defined on the dense subspace $L^p(d_b(D_\mu),d\beta)\cap L^2(d_b(D_\mu),d\beta))$, extends to a bounded operator*

$$\mathscr{S}: L^p(d_b(D_\mu),d\beta)) \to L^p(d_b(D_\mu),d\beta))$$

if and only if $2/(1+\nu) < p < 2/(1-\nu)$.

The next result concerns with the sharp boundedness of $\mathscr{S}$ on the L^2-Sobolev spaces on $d_b(D_\mu)$.

Theorem 2.4. *The Szegő projection $\mathscr{S}$ defines a bounded operator*

$$\mathscr{S}: W^{s,2}(d_b(D_\mu),d\beta)) \to W^{s,2}(d_b(D_\mu),d\beta)$$

if and only if $0 \le s < \nu/2$.

In the case of Sobolev norms with $p \neq 2$ we do not have a complete characterization of the mapping properties of $\mathscr{S}$, but we have a partial result.

Theorem 2.5. *Let $s > 0$ and $p \in (1,\infty)$. If the operator $\mathscr{S}$, initially defined on the dense subspace $W^{s,p}(d_b(D_\mu),d\beta)) \cap L^2(d_b(D_\mu),d\beta)$, extends to a bounded operator $\mathscr{S}: W^{s,p}(d_b(D_\mu),d\beta) \to W^{s,p}(d_b(D_\mu),d\beta)$, then*

$$-\frac{\nu_\beta}{2} \le s + \frac{1}{2} - \frac{1}{p} \le \frac{\nu_\beta}{2} \,.$$

Assuming $p \ge 2$ we obtain the stronger condition

$$0 \le s + \frac{1}{2} - \frac{1}{p} < \frac{\nu_\beta}{2} \,.$$

The main fact used in the proofs is that we can write the Szegő projection $\mathscr{S}$ as a sum of Mellin–Fourier multiplier operators which we now briefly describe. In order to do so we introduce some notation. We set $\mathbb{X}$ to denote either $\mathbb{R}$ or $\mathbb{T}$, and, accordingly, $\widehat{\mathbb{X}} = \mathbb{R}$, or $\mathbb{Z}$, respectively, where we denote by $\widehat{\ }$ the Fourier transform or Fourier series on $\mathbb{R}$ and $\mathbb{T}$, resp. Instead, we denote by $\mathscr{F}$ the Fourier transform on $\mathbb{R}\times\mathbb{X}$, given by

$$\mathscr{F}f(\xi_1,\xi_2) = \int_{\mathbb{R}\times\mathbb{X}} f(x_1,x_2)e^{-i(x_1\xi_+x_2\xi_2)}\,dx_1dx_2$$

when f is absolutely integrable. We consider the Fourier multiplier operator given by

$$T_m(f) = \mathscr{F}^{-1}\big(m\mathscr{F}f\big)$$

when m is a bounded measurable function on $\mathbb{R}\times\widehat{\mathbb{X}}$. We say that a bounded function m on $\mathbb{R}\times\mathbb{X}$ is a *bounded Fourier multiplier* on $L^p(\mathbb{R}\times\mathbb{X})$ if $T_m : L^p(\mathbb{R}\times\mathbb{X}) \to L^p(\mathbb{R}\times\mathbb{X})$ is bounded.

Given a function $\varphi \in C^\infty_c\big((0,\infty)\times\mathbb{X}\big)$ we define the operator

$$\mathcal{C}_p\varphi(x,y) = e^{(1/p)(x)}\varphi(e^x,y) \,.$$

It is clear that $\mathcal{C}_p$ extends to an isometry of $L^p\big((0,\infty)\times\mathbb{X}\big)$ onto $L^p\big((0,\infty)\times\mathbb{X}\big)$.

For $a,b\in\mathbb{R}$, with $0<a<b<1$, we denote by $S_{a,b}$ the vertical strip in the complex plane

$$S_{a,b} = \big\{z\in\mathbb{C} : a < \operatorname{Re} z < b\big\} \,.$$

Given a bounded measurable function m defined on $S_{a,b}\times\mathbb{X}$, when $a<1/p<b$ we write

$$m_p(\xi_1,\xi_2) = m(\frac{1}{p} - i\xi_1,\xi_2) \,.$$

Finally, we define an operator acting on functions defined on $(0,+\infty)\times\mathbb{X}$ as

$$\mathscr{T}_{m,p}=\mathcal{C}_p^{-1}T_{m_p}\mathcal{C}_p\ . \tag{6}$$

We call such an operator a Mellin–Fourier multiplier operator, the reason for which will soon be clear. A similar class of operators was studied by Rooney [59]. Incidentally, we believe that this class of operators is of its own interest.

Theorem 2.6. *With the above notation, let $m:S_{a,b}\times\mathbb{X}\to\mathbb{C}$ be continuous and such that*

(i) $m(\cdot,\xi_2)\in\mathrm{Hol}(S_{a,b})$ *and bounded in every closed substrip of $S_{a,b}$, for every $\xi_2\in\mathbb{X}$ fixed;*
(ii) *for every q such that $a<1/q<b$, m_q is a bounded Fourier multiplier on $L^q(\mathbb{R}\times\mathbb{X})$.*

Then, for $a<1/p<b$, $\mathscr{T}_{m,p}=\mathscr{T}_m$ is independent of p and

$$\mathscr{T}_m:L^p((0,+\infty)\times\mathbb{X})\to L^p((0,+\infty)\times\mathbb{X})$$

is bounded.

In the course of the proof we show that if m satisfies the hypotheses of the theorem, then

$$\mathcal{C}_p^{-1}T_{m_p}\mathcal{C}_p(\varphi)=\mathscr{F}_2^{-1}M_1^{-1}\big(m(M_1\mathscr{F}_2\varphi)\big)\ , \tag{7}$$

where M_1 denotes the Mellin transform in the first variable, that is,

$$M_1\varphi(z,\zeta_2)=\int_0^{+\infty}t^{z-1}\varphi(t,\zeta_2)\,dt\ ,$$

and $\mathcal{F}_2$ denotes the Fourier transform in the second variable. Equality (7) clearly giustify the fact that the operator $\mathscr{T}_m$ a Mellin–Fourier multiplier operator: it is a Mellin transformation in the first variable, a Fourier transformation in the second variable, followed by multiplication by m and then the inverses of the Mellin and Fourier transforms. We also point that, if $m,\tilde{m}$ satisfy the assumptions in the theorem, then $\mathscr{T}_m\mathscr{T}_{\tilde{m}}=\mathscr{T}_{m\tilde{m}}$, and thus it is reasonable to call these operators *multipliers*.

Once we explictly write the Szegő projection $\mathscr{S}$ as a linear combination of Mellin–Fourier multiplier operators, we are able to study its regularity by also exploiting the regularity of $\mathscr{S}'$. We recall that, unlike in the case of the Bergman projection, in the Szegő setting, in general, there is no transformation rule for the Szegő projection under biholomorphic mappings. Nonetheless, we are able to prove a transformation rule for the projections $\mathscr{S}$ and $\mathscr{S}'$. Recall that D'_μ and D_μ are biholomorphically equivalent via the map

$$\begin{aligned}\varphi^{-1}:\quad & D_\beta\to D'_\beta\\ & (z_1,z_2)\mapsto(\mathrm{Log}(z_1e^{-i\log|z_2|^2})+i\log|z_2|^2,z_2)\ ,\end{aligned} \tag{8}$$

where Log denotes the principal branch of the complex logarithm. Setting

$$\psi_p(z_1,z_2):=e^{-(i/p)\log|z_2|^2}(z_1e^{-i\log|z_2|^2})^{-1/p}$$

we obtain that

$$\mathscr{S}'(\Lambda^{-1}f)=\Lambda^{-1}(\mathscr{S})\ ,$$

where $\Lambda f:=\psi_p(f\circ\varphi^{-1})$.

3. Other results on the regularity of Szegő projections.

Mapping properties of the Szegő projection on other function spaces have been studied for various classes of smooth bounded domains and the are several positive results. The Szegő projection S_Ω turns out to be bounded on the Lebesgue–Sobolev spaces $W^{s,p}(\partial\Omega)$ for $1 < p < \infty$ and $s \geq 0$ in the case of strictly pseudoconvex domains [58], domains of finite type in $\mathbb{C}^2$ [55] and convex domains of finite type in $\mathbb{C}^n$ [47]. The exact regularity of S_Ω, that is, the boundedness $S_\Omega : W^{s,2}(\partial\Omega) \to W^{s,2}(\partial\Omega)$ for every $s \geq 0$, holds when Ω is a Reinhardt domain [9, 62], a domain with partially transverse symmetries [11], a pseudoconvex domain satisfying Catlin's property $(\mathcal{P})$ [10], a complete Hartogs domain in $\mathbb{C}^2$ [12], or a domain with a plurisubharmonic defining function on the boundary [14]. We also mention that, if Ω is bounded, C^2 and strongly pseudoconvex in $\mathbb{C}^n$, the Szegő projection P_Ω again extends to bounded operator on $L^p(\partial\Omega)$ for $1 < p < \infty$, [42, 43].

There are also examples of domains Ω on which the Szegő projection P_Ω is less regular. L. Lanzani and E. M. Stein described the (ir-)regularity of P_Ω on Lebesgue spaces in the case of planar simply connected domains, [41, Thm. 2.1]. In particular they showed that if Ω has Lipschitz boundary, then $P_\Omega : L^p(\partial\Omega) \to L^p(\partial\Omega)$ if and only if $p'_\Omega < p < p_\Omega$, where p_Ω depends only on the Lipschitz constant of $\partial\Omega$. More recently, S. Munasinghe and Y.E. Zeytuncu provided an example of a piecewise smooth, bounded pseudoconvex domain in $\mathbb{C}^2$ on which the Szegő projection P_Ω is unbounded on $L^p(\partial\Omega)$ for every $p \neq 2$ [54]. The same result on tube domains over irreducible self-dual cones of rank greater than 1 has been known for a number of years, [6].

In a recent paper [44] Lanzani and Stein announced a result concerning the L^p continuity of the Szegő projection attached to the smooth worm domain $\mathcal{W}_\mu$ with respect to the induced surface measure $\mathrm{d}\sigma$ on $\partial\mathcal{W}_\mu$. In particular, they announced that for any $p \neq 2$ there is a $\mu = \mu(p)$ such that the Szegő projection is not bounded $P_{\mathcal{W}_\mu} : L^p(\partial\mathcal{W}_\mu) \to L^p(\partial\mathcal{W}_\mu)$.

It is reasonable to think that the culprit of the (ir-)regularity of both the Bergman and Szegő projection on the worm domain $\mathcal{W}_\mu$ is the presence of the critical annulus $\mathcal{A} = \{(0, z_2) : \left|\log|z_2|^2\right| \leq \mu\}$ in the boundary $\partial\mathcal{W}_\mu$; see, for instance, [15]. For this reason, it would be interesting to study the Szegő projection of $\mathcal{W}_\mu$ with respect to the Fefferman measure $d\sigma_F$. In fact, as we now see, the Fefferman measure $d\sigma_F$ is given by a smooth density ω times $d\sigma$ and the density ω vanishes identically on the critical annulus $\mathcal{A}$. In detail, given $\Omega = \{z \in \mathbb{C}^n : \rho(z) < 0\}$, the Fefferman surface area measure ([31, pg. 259], [4]) on $\partial\Omega$ is defined by

$$d\sigma_F = c_n \sqrt[n+1]{M(\rho)}\, \frac{d\sigma}{\|\nabla\rho\|} \tag{9}$$

where $M(\rho)$ is the Fefferman Monge–Ampére operator

$$M(\rho) = -\det \begin{pmatrix} \rho & \rho_{\overline{k}} \\ \rho_j & \rho_{j\overline{k}} \end{pmatrix}_{1\leq j,k\leq n}.$$

It can be proved (see also [4, Section 2]) that the definition of $d\sigma_F$ does not depend on the defining function ρ and that there exists a sesqui-holomorphic kernel $S(z, \zeta)$

such that, for every $f \in H^2(\Omega)$,

$$f(z) = \int_{\partial\Omega} f(\zeta) S(z,\zeta)\, d\sigma_F(\zeta) .$$

Hence, Hardy spaces and Szegő projections with respect to the Fefferman measure can be defined and investigated. In the case of the worm domain $\mathcal{W}_\mu$ the defining function is

$$\rho(z_1,z_2) = |z_1|^2 - 2\mathrm{Re}(z_1 e^{-i\log|z_2|^2}) + \eta(\log|z_2|^2) ,$$

therefore, setting $R(z_1,z_2) = \mathrm{Re}(iz_1 e^{-i\log|z_2|^2})$,

$$M(\rho)(z_1,z_2) = \begin{pmatrix} 0 & z_1 - e^{i\log|z_2|^2} & \frac{1}{\overline{z_2}}\big(2R(z_1,z_2)+\eta'(\log|z_2|^2)\big) \\ \overline{z_1} - e^{-i\log|z_2|^2} & 1 & \frac{i}{\overline{z_2}} e^{-i\log|z_2|^2} \\ \frac{1}{z_2}\big(2R(z_1,z_2)+\eta'(\log|z_2|^2)\big) & -\frac{i}{z_2} e^{i\log|z_2|^2} & \frac{1}{|z_2|^2}\big(2R(z_1,z_2)+\eta''(\log|z_2|^2)\big) \end{pmatrix} .$$

When we restrict the matrix $M(\rho)$ to the critical annulus $\mathcal{A}$ we get

$$\det M(\rho)(0,z_2) = \det \begin{pmatrix} 0 & -e^{i\log|z_2|^2} & 0 \\ -e^{-i\log|z_2|^2} & 1 & \frac{i}{\overline{z_2}}\, e^{-i\log|z_2|^2} \\ 0 & -\frac{i}{z_2}\, e^{i\log|z_2|^2} & 0 \end{pmatrix} = 0 .$$

Since the boundary of the domain D_μ (similarly, of D'_μ) is Levi flat, that is, the Levi form of its defining function is identically zero at every point of bD_μ, the density of the Fefferman measure on bD_μ is identically zero. This can be easily verified by explicitly computing $M(\rho)$ for $\rho(z_1,z_2) = \mathrm{Re}(z_1 e^{-i\log|z_2|^2})$. Thus, the Szegő projection on $\mathcal{W}_\mu$ with respect to the Fefferman area measure cannot be investigated exploiting the model domains, but it must be directly approached. This certainly is an interesting direction for future research.

4. Hartogs triangles

Another class of domains on which it is interesting to test and study the regularity of the Bergman and Szegő projection is the one of generalized Hartogs triangles. Given a real parameter $\gamma > 0$, the generalized Hartogs triangle $\mathbb{H}_\gamma$ is defined as

$$\mathbb{H}_\gamma = \big\{(z_1,z_2) \in \mathbb{C}^2 : |z_1|^\gamma < |z_2| < 1\big\} .$$

This family of domains were recently introduced in [27] and the value $\gamma = 1$ corresponds to the classical Hartogs triangle $\mathbb{H}$ ([60]). The Hartogs triangle is a simple, but not trivial, model domain on which it is worth to test several conjectures. It turns out that $\mathbb{H}$ is a source of counterexamples in complex analysis. For instance, as the worm domain $\mathcal{W}$, the Hartogs triangle has non-trivial Nebenhülle. However, unlike $\mathcal{W}$, the domain $\mathbb{H}$ is not smooth; on the contrary, it is highly singular at the point $z_1 = z_2 = 0$. This pathological geometry affects the L^p behavior of the

Bergman projection $P_{\mathbb{H}}$ and it turns out that $P_{\mathbb{H}}$ extends to a bounded operator $P_{\mathbb{H}} : L^p(\mathbb{H}) \to L^p(\mathbb{H})$ if and only if $p \in (4/3, 4)$ ([16]). This result has been extended to the case of generalized Hartogs triangle in a series of paper by L. D. Edholm and J. D. McNeal and it holds that $P_{\mathbb{H}_\gamma}$ extends to a bounded operator $L^p(\mathbb{H}_\gamma) \to L^p(\mathbb{H}_\gamma)$ for a restricted range of $p \in (1, \infty)$ whenever $\gamma \in \mathbb{Q}$, but $P_{\mathbb{H}_\gamma}$ is unbounded on $L^p(\mathbb{H}_\gamma)$ for any $p \neq 2$ whenever γ is irrational ([27, 28, 29]). The Sobolev (ir-)regulariy of $P_{\mathbb{H}_\gamma}$ has been investigated as well and we refer the reader to the very recent paper [30].

In addition to the aforementioned papers, we mention also the recent papers [33, 32], where weighted L^p and endpoint estimates for the classical Hartogs triangle are obtained via dyadic harmonic analysis techniques, and [21, 17, 18, 19, 20], where some other generalizations of the Hartogs triangle and the associated weighted Bergman projections are investigated.

The definition of a Hardy space H^2 on $\mathbb{H}$, hence the definition of a Szegő projection on $\mathbb{H}$, is not canonical due to the geometry of the domain. We now recall the definition of a candidate Hardy space on the classical Hartogs triangle which is introduced by the first author in a recent paper [51].

Let $\nu > -1$ be a real parameter, let $\mathbb{D}$ be the unit disc in the complex plane and let us consider the classical weighted Bergman spaces $A^2_\nu(\mathbb{D})$ defined as the space of holomorphic functions in $\mathbb{D}$ endowed with the norm

$$\|f\|^2_{A^2_\nu(\mathbb{D})} = (\nu + 1) \int_{\mathbb{D}} |f(z)|^2 (1 - |z^2|)^\nu \, dz \, .$$

It is a well-known fact that

$$\|f\|^2_{H^2(\mathbb{D})} = \lim_{\nu \to -1^+} \|f\|^2_{A^2_\nu(\mathbb{D})}$$

where $H^2(\mathbb{D})$ is the Hardy space in $\mathbb{D}$, that is, the space of holomorphic functions in $\mathbb{D}$ endowed with the norm

$$\|f\|^2_{H^2(\mathbb{D})} := \sup_{0<r<1} \frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^2 \, d\theta \, .$$

Notice that if $K_{\mathbb{D}}(z, w) = (1 - \overline{z}w)^{-2}$ denotes the reproducing kernel of $A^2(\mathbb{H})$, the unweighted Bergman space, then

$$K_{\mathbb{D}}^{-\nu/2}(z, z) = (1 - |z|^2)^\nu \approx \delta^\nu(z) \, ,$$

where $\delta(z)$ is the distance of $z \in \mathbb{D}$ from the topological boundary $\partial\mathbb{D}$. Therefore, we analogously define the weighted Bergman space $A^2_\nu(\mathbb{H})$ as the space of holomorphic functions on $\mathbb{H}$ such that

$$\|f\|^2_{A^2_\nu(\mathbb{H})} := C_\nu \int_{\mathbb{H}} |f(z)|^2 K^{-\nu/2}(z, z) \, dz \, , \tag{10}$$

where C_ν is a positive constant to be chosen, dz denotes the Lebesgue measure in $\mathbb{C}^2$ and $K(z, w)$ is the reproducing kernel of the unweighted Bergman space $A^2(\mathbb{H})$ and it is given by

$$K(z, w) = K\big((z_1, z_2), (w_1, w_2)\big) = \frac{1}{2z_2\overline{w}_2 \left(1 - \dfrac{z_1\overline{w}_1}{z_2\overline{w}_2}\right)(1 - z_2\overline{w}_2)^2} \, .$$

The following proposition is proved in [26, Theorems 3.1.4 and 3.1.5] and describes the diagonal behavior of the kernel K.

Proposition 4.1 ([26]). *The following facts hold true.*

(i) *Let $\delta(z)$ be the distance of z to $\partial\mathbb{H}$, the topological boundary of $\mathbb{H}$. Then,*

$$K(z,z) \approx \delta(z)^{-2}$$

as z tends to the origin.

(ii) *Let p be any point in the distinguished boundary $d_b(\mathbb{H})$. For any number $\beta \in (2,4]$ there exists a path $\gamma : [1/2,1] \to \overline{\mathbb{H}}$ such that $\gamma(1) = p$ and for all $u \in [1/2,1)$,*

$$K(\gamma(u),\gamma(u)) \approx \delta(\gamma(u))^{-\beta} \ .$$

In [51] the L^p regularity of the weighted Bergman projection P_ν is completely characterized.

Theorem 4.2 ([51], Theorem 1). *Let $\nu > -1$ and let P_ν be the weighted Bergman projection densely defined on $L^2_\nu(\mathbb{H}) \cap L^p_\nu(\mathbb{H})$ for $p \in (1,+\infty)$. Then, we have the following:*

(i) *if $\nu > 0$ and $\nu \neq 2n, n \in \mathbb{N}$, the weighted Bergman projection P_ν extends to a bounded operator $P_\nu : L^p_\nu(\mathbb{H}) \to L^p_\nu(\mathbb{H})$ if and only if*

$$p \in \big(2 - \frac{\nu - 2[\nu/2]}{2+\nu-[\nu/2]}\ ,\ 2 + \frac{\nu-2[\nu/2]}{2+[\nu/2]}\big)\ ;$$

(ii) *if $\nu = 2n, n \in \mathbb{N}_0$, the weighted Bergman projection P_ν extends to a bounded operator $P_\nu : L^p_\nu(\mathbb{H}) \to L^p_\nu(\mathbb{H})$ if and only if*

$$p \in \Big(2 - \frac{2}{3+n}\ ,\ 2 + \frac{2}{1+n}\Big)\ ;$$

(iii) *if $-1 < \nu < 0$, the weighted Bergman projection P_ν extends to a bounded operator $P_\nu : L^p_\nu(\mathbb{H}) \to L^p_\nu(\mathbb{H})$ if and only if*

$$p \in \Big(2 - \frac{2+\nu}{3+\nu}\ ,\ 4+\nu\Big)\ .$$

The proof of this result follows from an explicit computation of the weighted kernel K_ν and an application of classical the Schur's lemma to the operator with positive kernel $|K_\nu|$.

The Hardy space $H^2(\mathbb{H})$ is then defined as the limit space corresponding to the value $\nu = -1$ of the parameter. In particular, $H^2(\mathbb{H})$ is defined in a way such that

$$K_{H^2(\mathbb{H})}(z,w) = \lim_{\nu\to -1^+} K_\nu(z,w)$$

and

$$\|f\|^2_{H^2} = \lim_{\nu\to-1^+} \|f\|^2_{A^2_\nu}\ .$$

It turns out that

$$H^2(\mathbb{H}) := \bigg\{ f \in \mathrm{Hol}(\mathbb{H})\ :\ \sup_{(s,t)\in(0,1)\times(0,1)} \frac{1}{4\pi^2} \int_{d_b(\mathbb{H}_{st})} |f|^2\, d\sigma_{st} < +\infty \bigg\}\ ,$$

where $d\sigma_{st}$ denotes the induced surface measure on $d_b(\mathbb{H}_{st})$, the distinguished boundary of the domain

$$\mathbb{H}_{st} = \big\{ (z_1,z_2) \in \mathbb{C}^2\ :\ |z_1|/s < |z_2| < t \big\} \subsetneq \mathbb{H}$$

for $(s,t) \in (0,1)\times(0,1)$. In particular, $d_b(\mathbb{H}_{st}) = \big\{(z_1,z_2)\in\mathbb{C}^2 : |z_1|/s = |z_2| = t\big\}$. We endow $H^2(\mathbb{H})$ with the norm

$$\|f\|^2_{H^2} := \sup_{(s,t)\in(0,1)\times(0,1)} \frac{1}{4\pi^2}\int_{d_b(\mathbb{H}_{st})} |f|^2\, d\sigma_{st}$$
$$= \sup_{(s,t)\in(0,1)\times(0,1)} \frac{1}{4\pi^2}\int_0^{2\pi}\int_0^{2\pi} |f(ste^{i\theta}, te^{i\gamma})|^2 st^2\, d\theta d\gamma\ .$$

The Hardy space $H^2(\mathbb{H})$ can be identified with a closed subspace of $L^2(d_b(\mathbb{H}))$, which we denote by $H^2(d_b(\mathbb{H}))$, hence a Szegő projection $S\colon L^2(d_b(\mathbb{H}))\to H^2(d_b(\mathbb{H}))$ is well-defined and can be investigated. In particular the following holds.

Theorem 4.3 ([51], Theorem 2)**.** *The Szegő projection S densely defined on $L^2(d_b(\mathbb{H}))\cap L^p(d_b(\mathbb{H}))$ extends to a bounded operator $S : L^p(d_b(\mathbb{H}))\to L^p(d_b(\mathbb{H}))$ for any $p\in(1,+\infty)$.*

In comparison with Theorem 4.2, the L^p regularity of the Szegő projection is surprising and unexpected. The reason of this result may be found in the fact that the Hardy space considered, even if it is naturally defined, turns out to be modeled only on the distinguished boundary $d_b(\mathbb{H})$ of $\mathbb{H}$ and not on the whole topological boundary $\partial\mathbb{H}$. Therefore, we loose track of the origin $(0,0)$, the most pathological point of $\partial\mathbb{H}$.

A further investigation of Hardy spaces on $\mathbb{H}$ and the extension of the results in [51] to the case of generalized Hartogs triangle certainly is an interesting direction for future research.

5. Orthogonal sets and the Müntz-Szász problem for the Bergman space

It would be ideal to be able to obtain the asymptotic expansion of the Bergman and Szegő kernels on the worm domain $\mathcal{W}$. A fundamental step in this direction would be to obtain the explicit expression of Bergman and Szegő kernel on the *truncated* worm domain

$$\mathcal{W}' = \Big\{(z_1,z_2)\in\mathbb{C}^2 : |z_1 - e^{i\log|z_2|^2}|^2 < 1,\ |\log|z_2|^2| < \mu\Big\}\ . \tag{11}$$

Obviously, this domain coincides with $\mathcal{W}$ when we select $\eta = \chi_{|t|>\mu}$, and $\mathcal{W}'$ is bounded, non-smooth, and its boundary contains the same critical annulus $\mathcal{A}$ as $\mathcal{W}$. In analogy with the case of the unit bidisk $\mathbb{D}^2$, we are led to look for an orthonormal basis of mononials. In the case of $\mathcal{W}'$, as well as of $\mathcal{W}$, the following functions resemble the monomials $z_1^j z_2^k$, $j,k\in\mathbb{N}$, where $\mathbb{N}$ denotes the set of non-negative integers. We set

$$E_\eta(z) = e^{\eta L(z)}\ ,$$

where

$$L(z) = \log\big(z_1 e^{-i\log|z_2|^2}\big) + i\log|z_2|^2\ ,$$

and log denotes the principal branch of the logarithm, so that

$$E_\eta(z_1,z_2) = \big(z_1 e^{-i\log|z_2|^2}\big)^\eta e^{i\eta\log|z_2|^2}\ .$$

Now we define constants $\gamma_{\alpha\beta} = h(\alpha-\overline{\beta})$, where

$$h(z) = \frac{\sinh[\mu(j+1+iz)]}{j+1+iz}\ .$$

The following is Proposition 3.1 in [38].

Proposition 5.1. *Let $\mu > 0$. For $\alpha \in \mathbb{C}$ and $j \in \mathbb{Z}$ let $F_{\alpha,j}(z_1, z_2) = E_\alpha(z) z_2^j$. Then $F_{\alpha,j} \in A^2(\mathcal{W}'_\mu)$ if and only if $\operatorname{Re}\alpha > -1$. Moreover, if $\operatorname{Re}\alpha, \operatorname{Re}\beta > -1$ then*

$$\langle F_{\alpha,j}, F_{\beta,j} \rangle_{A^2(\mathcal{W}'_\mu)} = (2\pi)^2 \gamma_{\alpha\beta} \frac{\Gamma(\alpha + \overline{\beta} + 2)}{\Gamma(\alpha + 2)\Gamma(\overline{\beta} + 2)} .$$

In particular, $\langle F_{\alpha,j}, F_{\beta,j} \rangle_{A^2(\mathcal{W}'_\mu)} = 0$ if and only if

$$\alpha - \overline{\beta} = 2k\nu + i(j+1) \qquad \text{with } k \in \mathbb{Z} \setminus \{0\} . \tag{12}$$

Thus, if $c > -1$ and $\ell \in \mathbb{N}$, and we set

$$H_{\ell,j}(z_1, z_2) = E_{c_0 + \nu\ell + i(j+1)/2}(z) z_2^j , \tag{13}$$

the next corollary follows.

Corollary 5.2. *Each of the two sets*

$$\big\{H_{2k,j} , j \in \mathbb{Z}, k \in \mathbb{N}\big\}, \quad \text{and} \quad \big\{H_{2k+1,j} , j \in \mathbb{Z}, k \in \mathbb{N}\big\} , \tag{14}$$

is an orthogonal system in $A^2(\mathcal{W}'_\mu)$.

Thus, we are led to consider the following problem. We set $\Delta = \{\zeta : |\zeta - 1| < 1\}$ and consider a set of functions $\{\zeta^{\lambda_k}\}$, $k = 1, 2, \cdots$. We call the *Müntz–Szász problem for the Bergman space* the question of determining necessary and sufficient condition for such a set to be a *complete set* in $A^2(\Delta)$, that is, its linear span to be dense in $A^2(\Delta)$. The following is Theorem 3.1 in [38], that gives a sufficient condition for the solution of the Müntz–Szász problem for the Bergman space.

Theorem 5.3. *For $k \in \mathbb{N}$, $0 < a < 1$, $c_0 > -1$ and $b \in \mathbb{R}$, let $\lambda_k = ak + c_0 + ib$. Then $\{\zeta^{\lambda_k}\}$ is a complete set in $A^2(\Delta)$.*

As a consequence we obtain the following density result in $A^2(\mathcal{W}'_\mu)$, which is Theorem 3.1 in [38].

Theorem 5.4. *Let $\mu > \pi/2$. Let $H_{\ell,j}(z_1, z_2)$ be as in (13). Then $\{H_{\ell,j}\}_{j\in\mathbb{Z}, \ell\in\mathbb{N}}$, is a complete set in $A^2(\mathcal{W}'_\mu)$.*

Notice that the set $\{H_{\ell,j} : j \in \mathbb{Z}, \ell \in \mathbb{N}\}$ is the union of the two sets in (14). However, such that set is not an orthogonal set, and we cannot compute the Bergman kernel from such complete set.

We now divert a bit from our main course to discuss the question of solving the Müntz–Szász problem. This was done in [56, 57], however without finding a complete solution. In [57] it is proved that the Müntz–Szász problem for the Bergman space is equivalent to characterizing the sets of uniqueness of the Hilbert space of holomorphic functions $\mathcal{M}^2_\omega(\mathcal{R})$ which is the space of holomorphic functions on the right half-plane $\mathcal{R}$ such that:

(H) $f \in H^2(S_b)$ for every $0 < b < \infty$;
(B) $f \in L^2(\overline{\mathcal{R}}, d\omega)$;

where $H^2(S_b)$ denotes the standard Hardy space on the vertical strip $\{z \in \mathbb{C} : 0 < \operatorname{Im} z < b\}$, and ω is the measure on $\overline{\mathcal{R}}$

$$\omega = \sum_{n=0}^{+\infty} \frac{2^n}{n!} \, \delta_{\frac{n}{2}}(x) \otimes dy .$$

Observe thag ω is a translation invariant measure in $\overline{\mathcal{R}}$. A quite interesting fact is that such space $\mathcal{M}^2_\omega(\mathcal{R})$ is closely related to a space of holomorphic functions descovered by T. Kriete and D. Trutt, [39, 40]. The following are the main results in [57] on this problem.

Theorem 5.5. *Let* $\{z_j\} \subseteq \mathcal{R}$*,* $1 \le |z_j| \to +\infty$*. The following properties hold.*

(i) *If* $\{z_j\}$ *has exponent of convergence* 1 *and upper density* $d^+ < 1/2$*, then* $\{z_j\}$ *is a zero-set for* $\mathcal{M}^2_\omega(\mathcal{R}) \cap \mathrm{Hol}(\overline{\mathcal{R}})$*.*

(ii) *If* $\{z_j\}$ *is a zero-set for* $\mathcal{M}^2_\omega(\mathcal{R}) \cap \mathrm{Hol}(\overline{\mathcal{R}})$*, then*

$$\limsup_{R\to+\infty} \frac{1}{\log R} \sum_{|z_j|\le R} \mathrm{Re}\left(\frac{1}{z_j}\right) \le \frac{2}{\pi}\,. \tag{15}$$

We observe that part (ii) in the above theorem follows from a generalization of the classical Carleman's formula in the right half-plane.

Theorem 5.6. *A sequence* $\{z_j\}$ *of points in* $\mathcal{R}$ *such that* $\mathrm{Re}\, z_j \ge \varepsilon_0$*, for some* $\varepsilon_0 > 0$ *and that violates condition* (15)*, is a set of uniqueness for* $\mathcal{M}^2_\omega(\mathcal{R})$*.*

As a consequence, if $\{z_j\}$ *is a sequence as above, the set of powers* $\{\zeta^{z_j-1}\}$ *is a complete set in* $A^2(\Delta)$*.*

Final Remarks

It is worth mentioning that in [37] the authors considered the unbounded worm domain $\mathcal{W}_\infty$. In [37] it is proved that the Bergman space $A^2(\mathcal{W}_\infty)$ is non-trivial and the Bergman projection is unbounded on $W^{s,p}(\mathcal{W}_\infty)$ for all $p \ne 2$ and $s > 0$.

Moreover, in [1] yet another interesting point of view of the pathological behavior of the worm domain is considered, in connection with the theory of spacetime singularities associated to the Fefferman metric.

Many questions remain unswered and thus analysis on worm domains is, and we believe it will remain, a very active area of research. It touches function theory and geometry of domains in several complex variables, holomorphic function spaces, distribution of zeros of entire functions, regolarity of integral operators, hypoellipticity of partial differential operators, to name the most significant.

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