Poisson quasi-Nijenhuis manifolds and the Toda system

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Abstract

The notion of Poisson quasi-Nijenhuis manifold generalizes that of Poisson-Nijenhuis manifold. The relevance of the latter in the theory of completely integrable systems is well established since the birth of the bi-Hamiltonian approach to integrability. In this note, we discuss the relevance of the notion of Poisson quasi-Nijenhuis manifold in the context of finite-dimensional integrable systems. Generically (as we show by a class of examples with 3 degrees of freedom) the Poisson quasi-Nijenhuis structure is largely too general to ensure Liouville integrability of a system. However, we present a general scheme connecting Poisson quasi-Nijenhuis and Poisson-Nijenhuis manifolds, and we give sufficient conditions such that the spectral invariants of the "quasi-Nijenhuis recursion operator" of a Poisson quasi-Nijenhuis manifold (obtained by deforming a Poisson-Nijenhuis structure) are in involution. Then we prove that the closed (or periodic) n-particle Toda lattice, along with its relation with the open (or non periodic) Toda system, can be framed in such a geometrical structure.

Keywords: Integrable systems; Toda lattices; Poisson quasi-Nijenhuis manifolds; bi-Hamiltonian manifolds.

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1 Introduction

It is well known that Poisson-Nijenhuis (PN) manifolds [12, 10] are an important notion in the theory of integrable systems. Roughly speaking, they are Poisson manifolds $(\mathcal{M}, \pi)$ endowed with a tensor field of type $(1,1)$, say $N : T\mathcal{M} \to T\mathcal{M}$, which is torsionless and compatible (see Section 2) with the Poisson tensor π. They turn out to be bi-Hamiltonian manifolds, with the traces of the powers of N satisfying the Lenard-Magri relations and thus being in involution with respect to the Poisson brackets induced by the Poisson tensors. An example of integrable system that can be studied in the context of PN manifolds is the open (or non periodic) n-particle Toda lattice. (For both the periodic and the non periodic Toda system, see [15] and references therein; see also [3, 13, 14].) The PN structure of the open Toda lattice was presented in [4]. Its Poisson tensor is non degenerate, so that the PN manifold is a symplectic manifold (sometimes it is called an ωN-manifold). This kind of geometrical structure was shown to play an important role in the bi-Hamiltonian interpretation of the separation of variables method (see, e.g., [5, 6]).

Poisson quasi-Nijenhuis (PqN) manifolds are an interesting generalization of PN manifolds. They were introduced in [16], where the requirement about the vanishing of the (Nijenhuis) torsion of N is weakened in a suitable sense, and the relations with quasi-Lie bialgebroid and symplectic Nijenhuis groupoids are investigated. In their Remark 3.13, the authors write: "Poisson Nijenhuis structures arise naturally in the study of integrable systems. It would be interesting to find applications of Poisson quasi-Nijenhuis structures in integrable systems as well." As far as we know, no progress in this direction was made until now.

The purpose of this paper is to discuss the relevance of PqN manifolds in the theory of finite-dimensional integrable systems. To this aim, we first present a class of PqN manifolds clarifying that the involutivity of the traces I_k of the powers of N does not hold in every PqN manifold. Then we consider the case of PqN manifolds that are obtained by deforming a PN manifold $(\mathcal{M}, \pi, N)$ with the help of a closed 2-form Ω, and we identify a set of compatibility conditions between π, N and Ω entailing that the I_k are in involution. (We say in this case that the PqN manifold is *involutive*.) Finally, we interpret the well known integrability of the closed Toda lattice in this framework, showing that its integrals of motion are the traces of the powers of a suitable tensor field $\hat{N}$ of type $(1,1)$, which is a deformation of the recursion operator N of the open Toda system and endows the phase space with the structure of an involutive PqN manifold.

The organization of this paper is the following. In Section 2 we recall the definitions of PN and PqN manifold, and we show how the classical Lenard-Magri recursion relations among the I_k are modified in the PqN case. Subsection 2.1 is devoted to a class of PqN structures on $\mathbb{R}^6$ depending on a potential V and showing that the I_k are in involution only for special choices of V. In Section 3 we present general results clarifying the relations between PN and PqN manifolds, and we identify a class of involutive PqN manifolds. More precisely, we show how a PN structure can be deformed into a PqN structure by means of a closed 2-form, and we give conditions on the deformation such that the PqN manifold turns out to be involutive. These results are applied in Section 4 to the closed Toda system, whose well known integrals of motion are interpreted as involutive deformations of the traces of the powers of the recursion operator of the open Toda system. In the final Appendix we present explicit formulas and computations for the 4-particle closed Toda lattice.

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2 Nijenhuis torsion and Poisson quasi-Nijenhuis manifolds

It is well known that the *Nijenhuis torsion* of a $(1,1)$ tensor field $N : T\mathcal{M} \to T\mathcal{M}$ on a manifold $\mathcal{M}$ is defined as

$$T_N(X,Y) = [NX,NY] - N\left([NX,Y] + [X,NY] - N[X,Y]\right) . \tag{1}$$

It can be written as

$$T_N(X,Y) = (L_{NX}N - NL_XN)Y\,, \tag{2}$$

where, hereafter, L_X denotes the Lie derivative with respect to the vector field X. Hence one arrives at the formula

$$N\,L_XN = L_{NX}N - i_XT_N\,, \tag{3}$$

where i_XT_N is the $(1,1)$ tensor field obviously defined as $(i_XT_N)(Y) = T_N(X,Y)$. We recall that, given a p-form α, with $p \geq 1$, one can construct another p-form $i_N\alpha$ as

$$i_N\alpha(X_1,\ldots,X_p) = \sum_{i=1}^{p} \alpha(X_1,\ldots,NX_i,\ldots,X_p)\,, \tag{4}$$

and that i_N is a derivation of degree zero (if $i_Nf = 0$ for all function f). We also remind [12] that $N : T\mathcal{M} \to T\mathcal{M}$ and a Poisson bivector $\pi : T^*\mathcal{M} \to T\mathcal{M}$ are said to be *compatible* if

$$\begin{aligned} &N\pi = \pi N^*\,, \qquad \text{where } N^* : T^*\mathcal{M} \to T^*\mathcal{M} \text{ is the transpose of } N;\\ &L_{\pi\alpha}(N)X - \pi L_X(N^*\alpha) + \pi L_{NX}\alpha = 0\,, \qquad \text{for all 1-forms } \alpha \text{ and vector fields } X. \end{aligned} \tag{5}$$

Some nice interpretations of these compatibility conditions were given in [9]. We will use one of them in Section 3.

In [16] a *Poisson quasi-Nijenhuis (PqN) manifold* was defined as a quadruple $(\mathcal{M},\pi,N,\phi)$ such that

- the Poisson bivector π and the $(1,1)$ tensor field N are compatible;
- the 3-forms ϕ and $i_N\phi$ are closed;
- $T_N(X,Y) = \pi\,(i_{X\wedge Y}\phi)$ for all vector fields X and Y, where $i_{X\wedge Y}\phi$ is the 1-form defined as $\langle i_{X\wedge Y}\phi, Z\rangle = \phi(X,Y,Z)$.

The bivector field $\pi' = N\pi$ turns out to satisfy the conditions

$$[\pi,\pi'] = 0\,, \qquad [\pi',\pi'] = 2\pi(\phi)\,, \tag{6}$$

where $[\cdot,\cdot]$ is the Schouten bracket (see, e.g., [17]) between bivectors and $\pi(\phi)(\alpha,\beta,\gamma) = \phi(\pi\alpha,\pi\beta,\pi\gamma)$ for any triple of 1-forms (α,β,γ). The following result, also proved in [16], is worth mentioning.

Proposition 1 *Let $\mathcal{M}$ be a manifold endowed with a non degenerate Poisson tensor π, a tensor field N of type $(1,1)$, and a closed 3-form ϕ. If $N\pi = \pi N^*$ and conditions (6) are satisfied (with $\pi' = N\pi$), then $(\mathcal{M},\pi,N,\phi)$ is a PqN manifold.*

If $\phi = 0$, then the torsion of N vanishes and $\mathcal{M}$ becomes a *Poisson-Nijenhuis manifold* (see [10] and references therein). The bivector field $\pi' = N\pi$ is in this case a Poisson tensor compatible with π. Moreover, the functions

$$I_k = \frac{1}{k}\,\mathrm{Tr}(N^k)\,, \qquad k = 1,2,\ldots\,, \tag{7}$$

satisfy $dI_{k+1} = N^* dI_k$, entailing the so-called *Lenard-Magri relations*

$$\pi\, dI_{k+1} = \pi'\, dI_k \tag{8}$$

and therefore the involutivity of the I_k (with respect to both Poisson brackets induced by π and π').

For a general PqN manifold $\mathcal{M}$, we will see in the next subsection that such involutivity (with respect to the unique Poisson bracket defined on $\mathcal{M}$, i.e., the one associated with π) does not hold. We will call *involutive* a PqN manifold if the traces (7) of the powers of N are in involution.

To study the involutivity problem, we notice that, for $k \geq 2$ and for a generic vector field X on $\mathcal{M}$,

$$\begin{aligned}
\langle dI_{k+1}, X\rangle &= L_X\left(\frac{1}{k+1}\,\mathrm{Tr}(N^{k+1})\right) = \mathrm{Tr}\left((NL_XN)N^{k-1}\right)\\
&\overset{(3)}{=} \mathrm{Tr}\left(L_{NX}(N)N^{k-1}\right) - \mathrm{Tr}\left((i_XT_N)\,N^{k-1}\right)\\
&= L_{NX}\left(\frac{1}{k}\,\mathrm{Tr}(N^k)\right) - \mathrm{Tr}\left((i_XT_N)\,N^{k-1}\right)\\
&= \langle dI_k, NX\rangle - \mathrm{Tr}\left((i_XT_N)\,N^{k-1}\right)\\
&= \langle N^*dI_k, X\rangle - \mathrm{Tr}\left((i_XT_N)\,N^{k-1}\right)\,.
\end{aligned} \tag{9}$$

So we arrive at the generalized Lenard-Magri relations

$$dI_{k+1} = N^*dI_k - \phi_{k-1}\,, \tag{10}$$

where we used the definition

$$\langle \phi_l, X \rangle = \mathrm{Tr}\left((i_X T_N)\, N^l \right) = \mathrm{Tr}\left(N^l\, (i_X T_N) \right) , \qquad l \geq 0 \, . \tag{11}$$

Notice that this definition, along with (10), was used in [1, 2] for different purposes. Let us compute now the Poisson bracket $\{I_k, I_j\}$ for $k > j \geq 1$:

$$\begin{aligned} \{I_k, I_j\} &= \langle dI_k, \pi\, dI_j \rangle \overset{(10)}{=} \langle N^* dI_{k-1}, \pi\, dI_j \rangle - \langle \phi_{k-2}, \pi\, dI_j \rangle = \langle dI_{k-1}, N\pi\, dI_j \rangle - \langle \phi_{k-2}, \pi\, dI_j \rangle \\ &= \langle dI_{k-1}, \pi\, N^* dI_j \rangle - \langle \phi_{k-2}, \pi\, dI_j \rangle \overset{(10)}{=} \langle dI_{k-1}, \pi\, dI_{j+1} \rangle + \langle dI_{k-1}, \pi\, \phi_{j-1} \rangle - \langle \phi_{k-2}, \pi\, dI_j \rangle \\ &= \{I_{k-1}, I_{j+1}\} - \left(\langle \phi_{j-1}, \pi\, dI_{k-1} \rangle + \langle \phi_{k-2}, \pi\, dI_j \rangle \right) . \end{aligned} \tag{12}$$

Thus, the usual formula

$$\{I_k, I_j\} = \{I_{k-1}, I_{j+1}\} , \tag{13}$$

entailed by the Lenard-Magri relations (8), in the non vanishing torsion case is modified as follows:

$$\{I_k, I_j\} - \{I_{k-1}, I_{j+1}\} = -\langle \phi_{j-1}, \pi\, dI_{k-1} \rangle - \langle \phi_{k-2}, \pi\, dI_j \rangle \, . \tag{14}$$

Actually, one can see that the 1-forms ϕ_l compute the Poisson brackets between the I_j. Indeed, if we consider $k = j + 1$, we obtain from (14)

$$\{I_{j+1}, I_j\} = -\langle \phi_{j-1}, \pi\, dI_j \rangle \, . \tag{15}$$

A necessary condition for the traces of the powers of N to be in involution is thus $\langle \phi_{j-1}, \pi\, dI_j \rangle = 0$ for all $j \geq 1$, which explicitly reads

$$\mathrm{Tr}\left((i_{\pi\, dI_j} T_N)\, N^{j-1} \right) = 0 \, . \tag{16}$$

However, imposing the condition that

$$\langle \phi_k, \pi\, dI_j \rangle = \mathrm{Tr}\left((i_{\pi\, dI_j} T_N)\, N^k \right) = 0 \tag{17}$$

for all k, j (although being clearly sufficient), is too restrictive: indeed, it fails in the simplest non trivial case, namely, the closed Toda system with 4 particles (see the Appendix).

Some further conditions can be written, which explain the above sentence in general. For example, if we take $k = j + 2$ we obtain, still from (14),

$$\{I_{j+2}, I_j\} = \{I_{j+1}, I_{j+1}\} - \langle \phi_{j-1}, \pi\, dI_{j+1} \rangle - \langle \phi_j, \pi\, dI_j \rangle \, . \tag{18}$$

To ensure that $\{I_{j+2}, I_j\}$ be zero, no need that the last two terms in the right-hand side of the above equation be simultaneously vanishing. Indeed, the Toda closed chain with 4 particles is already an example in which these two terms *cancel* each other without vanishing on their own.

2.1 A class of non involutive PqN manifolds

The aim of this subsection is to present a wide class of *non* involutive PqN manifolds. Let us consider, on $\mathcal{M} = \mathbb{R}^6$ with (canonical) variables $(q_1, q_2, q_3, p_1, p_2, p_3)$, the canonical Poisson tensor π and the $(1,1)$ tensor field given by

$$N = \begin{bmatrix} p_1 & 0 & 0 & 0 & 1 & 1 \\ 0 & p_2 & 0 & -1 & 0 & 1 \\ 0 & 0 & p_3 & -1 & -1 & 0 \\ 0 & -V(q_1-q_2) & -V(q_3-q_1) & p_1 & 0 & 0 \\ V(q_1-q_2) & 0 & -V(q_2-q_3) & 0 & p_2 & 0 \\ V(q_3-q_1) & V(q_2-q_3) & 0 & 0 & 0 & p_3 \end{bmatrix}, \tag{19}$$

where V is an arbitrary (differentiable) function of one variable. First of all, we use Proposition 1 to show that π and N define, together with a suitable 3-form ϕ, a PqN structure on $\mathbb{R}^6$. Indeed, if

$$\pi' = N\pi = \begin{bmatrix} 0 & -1 & -1 & p_1 & 0 & 0 \\ 1 & 0 & -1 & 0 & p_2 & 0 \\ 1 & 1 & 0 & 0 & 0 & p_3 \\ -p_1 & 0 & 0 & 0 & -V(q_1-q_2) & -V(q_3-q_1) \\ 0 & -p_2 & 0 & V(q_1-q_2) & 0 & -V(q_2-q_3) \\ 0 & 0 & -p_3 & V(q_3-q_1) & V(q_2-q_3) & 0 \end{bmatrix}, \tag{20}$$

then one can easily show that $[\pi, \pi'] = 0$, so that the first of (6) holds. After computing $[\pi', \pi']$, we have that the 3-form ϕ such that $[\pi', \pi'] = 2\pi(\phi)$ turns out to be

$$\begin{aligned} \phi = & \big(V'(q_1-q_2) - V(q_1-q_2)\big)\, d(p_1+p_2) \wedge dq_2 \wedge dq_1 \\ & + \big(V'(q_2-q_3) - V(q_2-q_3)\big)\, d(p_2+p_3) \wedge dq_3 \wedge dq_2 \\ & - \big(V'(q_3-q_1) + V(q_3-q_1)\big)\, d(p_1+p_3) \wedge dq_3 \wedge dq_1 \\ & - 2V'(q_3-q_1) dp_2 \wedge dq_3 \wedge dq_1 \,, \end{aligned} \tag{21}$$

which is clearly closed. Hence we can conclude by Proposition 1 that $(\mathbb{R}^6, \pi, N, \phi)$ is a PqN manifold for every choice of the function V.

Consider now the functions $H_k = \frac{1}{2} I_k = \frac{1}{2k} \operatorname{Tr}(N^k)$. We have that $H_1 = p_1 + p_2 + p_3$ and

$$H_2 = \frac{1}{2}({p_1}^2 + {p_2}^2 + {p_3}^2) + V(q_1-q_2) + V(q_2-q_3) + V(q_3-q_1)\,, \tag{22}$$

which can be obviously thought of as the Hamiltonian of three interacting particles of equal mass. It is easily seen that $\{H_1, H_2\} = \{H_1, H_3\} = 0$, while the Poisson bracket

$$\begin{aligned} \{H_2, H_3\} = & V(q_1-q_2)\big(V'(q_2-q_3) - V'(q_3-q_1)\big) + V(q_2-q_3)\big(V'(q_3-q_1) - V'(q_1-q_2)\big) \\ & + V(q_3-q_1)\big(V'(q_1-q_2) - V'(q_2-q_3)\big) \end{aligned} \tag{23}$$

does not vanish for any function V (for example, one can easily check that it is different from zero if $V(x) = 1/x$). However, involutivity holds in the cases $V(x) = \mathrm{e}^x$ (to be discussed in the next sections) and $V(x) = 1/x^2$ (corresponding to the Calogero model).

In conclusion, given a PqN manifold, further conditions on (π, N, ϕ) are needed to guarantee that the functions I_k are in involution. We will present a set of such conditions in the following section.

3 Relations between PN and PqN manifolds, and an involution theorem

In this section we present general results concerning the connection between PN and PqN manifolds. In particular, we explain how to deform a PN structure into a PqN structure, and we give conditions on the deformation entailing that the PqN manifold is involutive.

First of all, we recall that, given a tensor field $N : T\mathcal{M} \to T\mathcal{M}$, the usual Cartan differential can be modified as follows,

$$\begin{aligned}(d_N\alpha)(X_0, \ldots, X_q) = &\sum_{j=0}^{q}(-1)^j L_{NX_j}\left(\alpha(X_0, \ldots, \hat{X}_j, \ldots, X_q)\right) \\ &+ \sum_{i<j}(-1)^{i+j}\alpha([X_i, X_j]_N, X_0, \ldots, \hat{X}_i, \ldots, \hat{X}_j, \ldots, X_q)\,,\end{aligned} \tag{24}$$

where α is a q-form, the X_i are vector fields, and $[X, Y]_N = [NX, Y] + [X, NY] - N[X, Y]$. Note that $d_N f = N^* df$ for all $f \in C^\infty(\mathcal{M})$. Moreover,

$$d_N = i_N \circ d - d \circ i_N\,, \tag{25}$$

where i_N is given by (4), and consequently $d \circ d_N + d_N \circ d = 0$. Finally, $d_N^2 = 0$ if and only if the torsion of N vanishes.

We also remind that one can define a Lie bracket between the 1-forms on a Poisson manifold $(\mathcal{M}, \pi)$ as

$$[\alpha, \beta]_\pi = L_{\pi\alpha}\beta - L_{\pi\beta}\alpha - d\langle\beta, \pi\alpha\rangle\,, \tag{26}$$

and that this Lie bracket can be uniquely extended to all forms on $\mathcal{M}$ in such a way that

(K1) $[\eta, \eta']_\pi = -(-1)^{(q-1)(q'-1)}[\eta', \eta]_\pi$ if η is a q-form and η' is a q'-form;

(K2) $[\alpha, f]_\pi = i_{\pi\, df}\,\alpha = \langle\alpha, \pi\, df\rangle$ for all $f \in C^\infty(M)$ and for all 1-forms α;

(K3) if η is a q-form, then $[\eta, \cdot]_\pi$ is a derivation of degree $q-1$ of the wedge product, that is,

$$[\eta, \eta' \wedge \eta'']_\pi = [\eta, \eta']_\pi \wedge \eta'' + (-1)^{(q-1)q'}\eta' \wedge [\eta, \eta'']_\pi \tag{27}$$

if η' is a q'-form and η'' is any differential form.

This extension is a *graded* Lie bracket, in the sense that (besides (K1)) the graded Jacobi identity holds:

$$(-1)^{(q_1-1)(q_3-1)}[\eta_1,[\eta_2,\eta_3]_\pi]_\pi + (-1)^{(q_2-1)(q_1-1)}[\eta_2,[\eta_3,\eta_1]_\pi]_\pi + (-1)^{(q_3-1)(q_2-1)}[\eta_3,[\eta_1,\eta_2]_\pi]_\pi = 0 \tag{28}$$

if q_i is the degree of η_i. It is sometimes called the Koszul bracket — see, e.g., [7] and references therein.

It was proved in [9] that the compatibility conditions (5) between a Poisson tensor π and a tensor field $N : T\mathcal{M} \to T\mathcal{M}$ hold if and only if d_N is a derivation of $[\cdot,\cdot]_\pi$, that is,

$$d_N[\eta,\eta']_\pi = [d_N\eta,\eta']_\pi + (-1)^{(q-1)}[\eta, d_N\eta']_\pi \tag{29}$$

if η is a q-form and η' is any differential form. In particular, taking $N = Id$, one has that the Cartan differential d is always a derivation of $[\cdot,\cdot]_\pi$. Moreover, if ϕ is any 3-form,

$$d_N^2 = [\phi,\cdot]_\pi \quad \text{if and only if} \quad \begin{cases} T_N(X,Y) = \pi\,(i_{X\wedge Y}\phi) & \text{for all vector fields } X, Y \\ i_{(\pi\alpha)\wedge(\pi\beta)\wedge(\pi\gamma)}(d\phi) = 0 & \text{for all 1-forms } \alpha,\beta,\gamma, \end{cases} \tag{30}$$

see [16]. We are now ready to state

Theorem 2 *Suppose that $(\mathcal{M},\pi,\phi,N)$ is a PqN manifold and that there exists a closed 2-form Ω such that*

$$d_N\Omega + \frac{1}{2}[\Omega,\Omega]_\pi = -\phi\,. \tag{31}$$

Let $\hat{N} = N - \pi\,\Omega^\flat$, where $\Omega^\flat : T\mathcal{M} \to T^*\mathcal{M}$ is defined as usual by $\Omega^\flat(X) = i_X\Omega$. Then $(\mathcal{M},\pi,\hat{N})$ is a PN manifold.

In particular, if (M,π,N) is a PN manifold, Ω a closed 2-form such that

$$d_N\Omega + \frac{1}{2}[\Omega,\Omega]_\pi = 0\,, \tag{32}$$

and $\hat{N} = N - \pi\,\Omega^\flat$, then $(M,\pi,\hat{N})$ is still a PN manifold.

Proof. First of all we show that $d_{\pi\,\Omega^\flat} = -[\Omega,\cdot]_\pi$. This follows from the fact that both are derivations (with respect to the wedge product) anti-commuting with d, and they coincide on functions. Indeed, for all $f \in C^\infty(\mathcal{M})$,

$$d_{\pi\,\Omega^\flat} f = (\pi\,\Omega^\flat)^* df = (\Omega^\flat\pi)df = i_{\pi\,df}\,\Omega = -[\Omega, f]_\pi,$$

where the last equality holds for every 2-form Ω and can be easily checked to be a consequence of (K2) and (K3).

Hence $d_{\hat{N}} = d_N - d_{\pi\,\Omega^\flat} = d_N + [\Omega,\cdot]_\pi$ is a derivation of $[\cdot,\cdot]_\pi$ (since π and N are compatible and $[\cdot,\cdot]_\pi$ satisfies (28)), so that π and $\hat{N}$ are compatible too.

Finally, equivalence (30) and formula (31) imply that $d_{\hat{N}}^2 = 0$, meaning that the torsion of $\hat{N}$ vanishes. We conclude that $(\mathcal{M},\pi,\hat{N})$ is a PN manifold. □

In the terminology of [8], Theorem 2 describes how to deform a quasi-Lie bialgebroid into a Lie bialgebroid by means of the so called *twist*. A kind of converse of Theorem 2 is given by

Theorem 3 *Let (M, π, N) be a PN manifold and let Ω be a closed 2-form such that*

$$[d_N\Omega, \Omega]_\pi = 0 \,. \tag{33}$$

If

$$\phi = d_N\Omega + \frac{1}{2}[\Omega, \Omega]_\pi \tag{34}$$

and $\hat{N} = N - \pi\,\Omega^\flat$, then $(M, \pi, \hat{N}, \phi)$ is a PqN manifold.

Proof. First we note that condition (33) guarantees that the 3-form ϕ defined by (34) satisfies $d_N\phi = 0$ and $d\phi = 0$. Thanks to (25), it follows that $i_N\phi$ is closed. Since $d_{\hat{N}} = d_N - d_{\pi\,\Omega^\flat} = d_N + [\Omega, \cdot]_\pi$, the compatibility between π and $\hat{N}$ can be shown as in the proof of Theorem 2. Finally, using (34) and $d_N^2 = 0$, we can prove that $d_{\hat{N}}^2 = [\phi, \cdot]_\pi$. To conclude, it suffices to use equivalence (30). □

Remark 4 To clarify the relation between the torsions of $\hat{N}$, N, and $\pi\,\Omega^\flat$, we recall that

$$\langle df, T_N(X,Y)\rangle = \left(d_N^2 f\right)(X,Y) \tag{35}$$

for any function f, tensor field N of type $(1,1)$, and vector fields X, Y. Since $d_{\hat{N}} = d_N - d_{\pi\,\Omega^\flat} = d_N + [\Omega, \cdot]_\pi$, we have that

$$\begin{aligned}\langle df, T_{\hat{N}}(X,Y)\rangle &= \left(d_{\hat{N}}^2 f\right)(X,Y) \\ &= \left(d_N^2 f\right)(X,Y) + [\Omega, d_N f]_\pi(X,Y) + \left(d_N[\Omega, f]_\pi\right)(X,Y) + [\Omega, [\Omega, f]_\pi]_\pi(X,Y) \\ &= \langle df, T_N(X,Y)\rangle + [d_N\Omega, f]_\pi(X,Y) + [\Omega, [\Omega, f]_\pi]_\pi(X,Y)\,,\end{aligned} \tag{36}$$

where in the last equality we have used (29). The first term in the last row of (36), quadratic in N, vanishes in the hypotheses of Theorem 3. The second term is linear in N, while the third one is $\langle df, T_{\pi\,\Omega^\flat}(X,Y)\rangle$ and can be written, using the properties (K1) and (28) of the Koszul bracket, as

$$\langle df, T_{\pi\,\Omega^\flat}(X,Y)\rangle = \frac{1}{2}\left[[\Omega, \Omega]_\pi, f\right]_\pi(X,Y)\,. \tag{37}$$

Thanks to (34), the last two terms in the last row of (36) give $[\phi, f]_\pi(X,Y)$, so that we obtain

$$\langle df, T_{\hat{N}}(X,Y)\rangle = [\phi, f]_\pi(X,Y) = \langle df, \pi(i_{X\wedge Y}\phi)\rangle\,, \tag{38}$$

that is, the third requirement in the definition of PqN manifolds.

We finally notice that in Theorem 6 we will assume that $[\Omega, \Omega]_\pi = 0$, so that the torsion of $\pi\,\Omega^\flat$ will vanish in that case.

Remark 5 To the best of our knowledge equation (32) was first introduced and studied by Liu, Weinstein and Xu in their work on the theory of Manin triples for Lie algebroids, see Section 6 of [11]. These authors, starting from a Poisson manifold $(\mathcal{M}, \pi)$ and the corresponding *standard*

Courant algebroid structure on $T^*\mathcal{M}\oplus T\mathcal{M}$, showed that for $N = Id$ every solution of (32) defines a Dirac subbundle $\Gamma_\Omega \subset T^*\mathcal{M} \oplus T\mathcal{M}$ transversal to $T^*\mathcal{M}$. Moreover, they proved that every solution of

$$d\Omega = 0 \quad \text{and} \quad [\Omega, \Omega]_\pi = 0 \tag{39}$$

defines a new Poisson structure π' on $\mathcal{M}$ compatible with π and induced by a torsionless operator, defining in this way a Poisson-Nijenhuis structure on $\mathcal{M}$. It is worth to mention that the second equation in (39) was studied in depth by Vaisman in [18], where its solutions were named complementary 2-forms of the (underlying) Poisson structure.

We are now ready to state the main result of this paper. Indeed, in the following theorem we identify a suitable set of compatibility conditions between π, N and Ω implying the involutivity of the traces of the powers of the deformed tensor field $\hat{N}$.

Theorem 6 *Let $(\mathcal{M}, \pi, N)$ be a PN manifold, Ω a closed 2-form on $\mathcal{M}$ such that $[\Omega, \Omega]_\pi = 0$, $\hat{N} = N - \pi\,\Omega^\flat$, and $I_k = \frac{1}{k}\operatorname{Tr}(\hat{N}^k)$. Suppose that*

1. *$d_N\Omega = dI_1 \wedge \Omega$;*
2. *$i_{Y_k}\Omega = 0$, where $Y_k = (\hat{N})^{k-1}X_1 - X_k$ and $X_k = \pi\, dI_k$;*
3. *$\{I_1, I_k\} = 0$ for all $k \geq 2$.*

Then

- *i) $(\mathcal{M}, \pi, \hat{N}, d_N\Omega)$ is a PqN manifold;*
- *ii) $\{I_j, I_k\} = 0$ for all $j, k \geq 1$.*

Proof. Assertion i) follows from Theorem 3 and the fact that $[\Omega, \Omega]_\pi = 0$ implies $[d_N\Omega, \Omega]_\pi = 0$. To prove assertion ii), we start noticing that

$$T_{\hat{N}}(X, Y) = \pi(i_{X\wedge Y} d_N\Omega)\,. \tag{40}$$

This follows from the fact that $(\mathcal{M}, \pi, \hat{N}, d_N\Omega)$ is a PqN manifold and from the third requirement in the definition of PqN manifolds — see also (38), where $\phi = d_N\Omega$. Hence we have that

$$\begin{aligned} T_{\hat{N}}(X, Y) &= \pi(i_{X\wedge Y} d_N\Omega) = \pi(i_Y i_X(dI_1 \wedge \Omega)) = \pi(i_Y(\langle dI_1, X\rangle\Omega - dI_1 \wedge i_X\Omega)) \\ &= \pi(\langle dI_1, X\rangle i_Y\Omega - \langle dI_1, Y\rangle i_X\Omega + i_Y i_X\Omega dI_1) \\ &= \langle dI_1, X\rangle(\pi\,\Omega^\flat)(Y) - \langle dI_1, Y\rangle(\pi\,\Omega^\flat)(X) + \Omega(X, Y)X_1 \end{aligned} \tag{41}$$

for all vector fields X, Y, so that

$$i_X T_{\hat{N}} = \langle dI_1, X\rangle\,\pi\,\Omega^\flat - (\pi\,\Omega^\flat)(X) \otimes dI_1 + X_1 \otimes i_X\Omega\,. \tag{42}$$

Now we use assumption 3, that is, $\langle dI_1, X_j\rangle = 0$, to obtain

$$i_{X_j} T_{\hat{N}} = -(\pi\,\Omega^\flat)(X_j) \otimes dI_1 + X_1 \otimes i_{X_j}\Omega \tag{43}$$

and therefore, by the definition (11) of the 1-forms ϕ_k,

$$\begin{aligned}\langle \phi_k, X_j \rangle &= \mathrm{Tr}\left(\hat{N}^k(i_{X_j} T_{\hat{N}})\right) = \mathrm{Tr}\left(\hat{N}^k(-(\pi\,\Omega^\flat)(X_j) \otimes dI_1 + X_1 \otimes i_{X_j}\Omega)\right) \\ &= -\,\mathrm{Tr}\left((\hat{N}^k \pi\,\Omega^\flat)(X_j) \otimes dI_1\right) + \mathrm{Tr}\left((\hat{N}^k X_1) \otimes i_{X_j}\Omega\right)\,.\end{aligned} \tag{44}$$

Both summands coincide with $\Omega(X_j, \hat{N}^k X_1)$. This is easily seen for the second summand, since $\mathrm{Tr}(X \otimes \alpha) = \langle \alpha, X\rangle$ for all vector fields X and 1-forms α. As far as the first one is concerned,

$$\begin{aligned}\mathrm{Tr}\left((\hat{N}^k \pi\,\Omega^\flat)(X_j) \otimes dI_1\right) &= \langle dI_1, (\hat{N}^k \pi\,\Omega^\flat)(X_j)\rangle = \langle dI_1, (\pi(\hat{N}^*)^k \Omega^\flat)(X_j)\rangle \\ &= -\langle ((\hat{N}^*)^k \Omega^\flat)(X_j), X_1\rangle = -\langle \Omega^\flat(X_j), \hat{N}^k X_1\rangle = -\Omega(X_j, \hat{N}^k X_1)\,.\end{aligned} \tag{45}$$

Therefore we have obtained the formula

$$\langle \phi_k, X_j\rangle = 2\Omega(X_j, \hat{N}^k X_1)\,. \tag{46}$$

To prove that the traces I_k of the powers of $\hat{N}$ are in involution it suffices to show that the additional term, appearing in (14), to the usual Lenard-Magri recursion relations for the Poisson brackets between the I_k vanishes. Actually, this additional term is

$$\langle \phi_{j-1}, \pi\, dI_{k-1}\rangle + \langle \phi_{k-2}, \pi\, dI_j\rangle \tag{47}$$

and it reads, thanks to (46),

$$2\Omega(X_{k-1}, N^{j-1} X_1) + 2\Omega(X_j, N^{k-2} X_1)\,. \tag{48}$$

Now, thanks to assumption 2, we can substitute $N^{i-1}X_1$ with X_i in the previous expression, showing that it vanishes. Hence we obtain that the Lenard-Magri recursion relations (13) hold also in this case, leading to the involutivity of the I_k. □

4 The closed Toda lattice case

In this section we show that the results obtained in the previous one can be applied to the Toda lattice. More precisely, we show how to deform the well known PN structure of the open Toda lattice to obtain an involutive PqN structure for the closed one.

First of all, we recall from [4] that $\mathbb{R}^{2n}$ can endowed with the PN structure given by the canonical Poisson tensor π (in the canonical coordinates q_i, p_i) and the (torsion free) tensor field

$$\begin{aligned}N = &\sum_{i=1}^{n} p_i\,(\partial_{q_i} \otimes dq_i + \partial_{p_i} \otimes dp_i) + \sum_{i<j}\left(\partial_{q_i} \otimes dp_j - \partial_{q_j} \otimes dp_i\right) \\ &+ \sum_{i=1}^{n-1} \mathrm{e}^{q_i - q_{i+1}}\left(\partial_{p_{i+1}} \otimes dq_i - \partial_{p_i} \otimes dq_{i+1}\right)\,,\end{aligned} \tag{49}$$

and that the traces of the powers of N are the integrals of motion of the *open* Toda chain. For example,

$$\frac{1}{2}\,\mathrm{Tr}(N) = \sum_{i=1}^{n} p_i\,, \qquad \frac{1}{4}\,\mathrm{Tr}(N^2) = \frac{1}{2}\sum_{i=1}^{n} p_i^2 + \sum_{i=1}^{n-1} \mathrm{e}^{q_i - q_{i+1}} \tag{50}$$

are respectively the total momentum and the energy.

Next we show that a suitable 2-form Ω can be defined in such a way to apply Theorem 6 to deform the PN manifold above into an involutive PqN manifold connected with the closed Toda chain.

Theorem 7 *Let us consider the above defined PN manifold* $(\mathbb{R}^{2n}, \pi, N)$ *and the closed 2-form* $\Omega = \mathrm{e}^{q_n - q_1} dq_n \wedge dq_1$ *on* $\mathbb{R}^{2n}$*. Then*

i) $[\Omega, \Omega]_\pi = 0$*;*

ii) $d_N \Omega = dI_1 \wedge \Omega$*, where* $I_k = \frac{1}{k} \operatorname{Tr} \hat{N}^k$ *and* $\hat{N} = N - \pi\, \Omega^\flat$*;*

iii) $i_{Y_k} \Omega = 0$*, where* $Y_k = (\hat{N})^{k-1} X_1 - X_k$ *and* $X_k = \pi\, dI_k$*;*

iv) $\{I_1, I_k\} = 0$ *for all* $k \geq 2$*.*

Proof.

i) can be easily proved by writing $\Omega = d\,(\mathrm{e}^{q_n - q_1} dq_1)$ and taking into account that the Cartan differential d is a derivation of $[\cdot,\cdot]_\pi$.

ii) follows from $d \circ d_N + d_N \circ d = 0$, $d_N f = N^* df$ and

$$N^* dq_1 = p_1 dq_1 + \sum_{i=2}^{n} dp_i\,, \qquad N^* dq_n = p_n dq_n - \sum_{i=1}^{n-1} dp_i\,. \tag{51}$$

iii) Applying π to both members of (10), written for $\hat{N}$, one easily finds that $\hat{N} X_l - X_{l+1} = \pi\, \phi_{l-1}$. Then we have

$$Y_k = \sum_{l=1}^{k-1} \left(\hat{N}^{k-l} X_l - \hat{N}^{k-l-1} X_{l+1} \right) = \sum_{l=1}^{k-1} \hat{N}^{k-l-1} \left(\hat{N} X_l - X_{l+1} \right) = \sum_{l=1}^{k-1} \hat{N}^{k-l-1} \pi\, \phi_{l-1}\,, \tag{52}$$

so that

$$Y_k = \pi \left(\sum_{l=1}^{k-1} (\hat{N}^*)^{k-l-1} \phi_{l-1} \right) = \pi \left(\sum_{l=0}^{k-2} (\hat{N}^*)^{k-l-2} \phi_l \right). \tag{53}$$

Therefore, the condition $i_{Y_k} \Omega = 0$, that is, $\langle dq_n, Y_k \rangle = \langle dq_1, Y_k \rangle = 0$, becomes

$$\sum_{l=0}^{k-2} \langle \phi_l, \hat{N}^{k-l-2} \partial_{p_n} \rangle = \sum_{l=0}^{k-2} \langle \phi_l, \hat{N}^{k-l-2} \partial_{p_1} \rangle = 0\,. \tag{54}$$

Recall now the definition

$$\langle \phi_l, X \rangle = \operatorname{Tr} \left(\hat{N}^l (i_X T_{\hat{N}}) \right) \tag{55}$$

of the 1-forms ϕ_l and formula (42), that is,

$$i_X T_{\hat{N}} = \langle dI_1, X \rangle\, \pi\, \Omega^\flat - (\pi\, \Omega^\flat)(X) \otimes dI_1 + X_1 \otimes i_X \Omega\,. \tag{56}$$

Then, for all $k \geq 2$ and $l = 0, \dots, k-2$, we have that

$$
\begin{aligned}
\langle \phi_l, \hat{N}^{k-l-2}\partial_{p_n}\rangle &= \mathrm{Tr}\left(\hat{N}^l (i_{\hat{N}^{k-l-2}\partial_{p_n}} T_{\hat{N}})\right) \\
&= \mathrm{Tr}\left[\hat{N}^l\left(\langle dI_1, \hat{N}^{k-l-2}\partial_{p_n}\rangle\, \pi\,\Omega^\flat - (\pi\,\Omega^\flat \hat{N}^{k-l-2})(\partial_{p_n}) \otimes dI_1 + X_1 \otimes i_{\hat{N}^{k-l-2}\partial_{p_n}}\Omega\right)\right] \\
&= \langle dI_1, \hat{N}^{k-l-2}\partial_{p_n}\rangle\, \mathrm{Tr}(\hat{N}^l \pi\,\Omega^\flat) - \langle dI_1, (\hat{N}^l \pi\,\Omega^\flat \hat{N}^{k-l-2})(\partial_{p_n})\rangle + \Omega(\hat{N}^{k-l-2}\partial_{p_n}, \hat{N}^l X_1) \\
&= \langle dI_1, \hat{N}^{k-l-2}\partial_{p_n}\rangle\, \mathrm{Tr}(\hat{N}^l \pi\,\Omega^\flat) + 2\Omega(\hat{N}^{k-l-2}\partial_{p_n}, \hat{N}^l X_1)\,.
\end{aligned}
\tag{57}
$$

Let us compute the three terms appearing in (57):

(1) $\langle dI_1, \hat{N}^{k-l-2}\partial_{p_n}\rangle = -\langle dI_1, \hat{N}^{k-l-2}(\pi dq_n)\rangle = \langle dq_n, \hat{N}^{k-l-2}X_1\rangle$.

(2) $\mathrm{Tr}(\hat{N}^l\pi\,\Omega^\flat) = \langle dq_1, (\hat{N}^l\pi\,\Omega^\flat)(\partial_{q_1})\rangle + \langle dq_n, (\hat{N}^l\pi\,\Omega^\flat)(\partial_{q_n})\rangle = -e^{q_n-q_1}\big(\langle dq_n, \hat{N}^l\partial_{p_1}\rangle - \langle dq_1, \hat{N}^l\partial_{p_n}\rangle\big) = 2e^{q_n-q_1}\langle dq_1, \hat{N}^l\partial_{p_n}\rangle$.

(3) $\Omega(\hat{N}^{k-l-2}\partial_{p_n}, \hat{N}^l X_1) = e^{q_n-q_1}\left[\langle dq_n, \hat{N}^{k-l-2}\partial_{p_n}\rangle\langle dq_1, \hat{N}^l X_1\rangle - \langle dq_1, \hat{N}^{k-l-2}\partial_{p_n}\rangle\langle dq_n, \hat{N}^l X_1\rangle\right]$.

Then we proved that

$$
\begin{aligned}
\langle \phi_l, \hat{N}^{k-l-2}\partial_{p_n}\rangle = 2e^{q_n-q_1}\Big[&\langle dq_n, \hat{N}^{k-l-2}\partial_{p_n}\rangle\langle dq_1, \hat{N}^l X_1\rangle - \langle dq_1, \hat{N}^{k-l-2}\partial_{p_n}\rangle\langle dq_n, \hat{N}^l X_1\rangle \\
&+\langle dq_1, \hat{N}^l\partial_{p_n}\rangle\langle dq_n, \hat{N}^{k-l-2}X_1\rangle\Big]\,.
\end{aligned}
\tag{58}
$$

It follows that, for all $k \geq 2$,

$$
\begin{aligned}
\langle dq_n, Y_k\rangle &= \langle dq_n, \pi\sum_{l=0}^{k-2}(\hat{N}^*)^{k-l-2}\phi_l\rangle = \sum_{l=0}^{k-2}\langle \phi_l, \hat{N}^{k-l-2}\partial_{p_n}\rangle \\
&= 2e^{q_n-q_1}\sum_{l=0}^{k-2}\langle dq_n, \hat{N}^{k-l-2}\partial_{p_n}\rangle\langle dq_1, \hat{N}^l X_1\rangle\,,
\end{aligned}
\tag{59}
$$

proving that if $\langle dq_n, \hat{N}^j\partial_{p_n}\rangle = 0$ for all $j \geq 1$, then $\langle dq_n, Y_k\rangle = 0$ for all $k \geq 1$. A similar computation shows that $\langle dq_1, Y_k\rangle = 0$ is implied by $\langle dq_1, \hat{N}^j\partial_{p_1}\rangle = 0$. Hence we are left with proving that the entries $(1, n+1)$ and $(n, 2n)$ of $\hat{N}^k$ vanish for all $k \geq 1$. But this follows from the fact that the $n \times n$ block in the upper right corner of $\hat{N}^k$ is skewsymmetric, since $\hat{N}^k\pi = \pi\,(\hat{N}^*)^k$.

iv) For all $k \geq 2$, we have that $\{I_1, I_k\} = -\langle dI_k, X_1\rangle = 0$, since $X_1 = 2\sum_{i=1}^n \partial_{q_i}$ and $\hat{N}$ (and hence its traces) depends only on the differences $q_i - q_{i+1}$. □

It is easy to check that the deformed tensor field $\hat{N} = N - \pi\,\Omega^\flat$ is given by

$$
\begin{aligned}
\hat{N} = &\sum_{i=1}^n p_i\,(\partial_{q_i}\otimes dq_i + \partial_{p_i}\otimes dp_i) + \sum_{i<j}\left(\partial_{q_i}\otimes dp_j - \partial_{q_j}\otimes dp_i\right) \\
&+\sum_{i=1}^{n-1}\mathrm{e}^{q_i-q_{i+1}}\left(\partial_{p_{i+1}}\otimes dq_i - \partial_{p_i}\otimes dq_{i+1}\right) - \mathrm{e}^{q_n-q_1}\left(\partial_{p_1}\otimes dq_n - \partial_{p_n}\otimes dq_1\right)\,,
\end{aligned}
\tag{60}
$$

while $\phi = d_N\Omega = dI_1 \wedge \Omega = \mathrm{e}^{q_n-q_1}\,(dI_1 \wedge dq_1 \wedge dq_n) = dI_1 \wedge d\mathrm{e}^{q_n} \wedge d\mathrm{e}^{-q_1} = d\,(I_1\, d\mathrm{e}^{q_n} \wedge d\mathrm{e}^{-q_1})$. The functions $I_k = \frac{1}{k}\,\mathrm{Tr}(\hat{N}^k)$ are the integrals of motion of the *closed* Toda chain. For example,

$$
\frac{1}{2}\,\mathrm{Tr}(\hat{N}) = \frac{1}{2}\,\mathrm{Tr}(N) = \sum_{i=1}^n p_i\,, \qquad \frac{1}{4}\,\mathrm{Tr}(\hat{N}^2) = \sum_{i=1}^n\left(\frac{1}{2}p_i^2 + \mathrm{e}^{q_i-q_{i+1}}\right)\,,
\tag{61}
$$

where $q_{n+1} = q_1$. As we have already seen in Section 2, many features of the usual picture of PN manifolds are lost in the case, since the functions I_k do not fulfill the Lenard-Magri relations. For example, $\hat{N}^* dI_1 \neq dI_2$, so that $\hat{N} X_1 \neq X_2$. However, the involutivity of the I_k can be seen as a consequence of Theorem 6.

Remark 8 We can use Theorem 2 to come back to the PN structure of the open Toda chain starting from the PqN structure $(\pi, \hat{N}, \phi)$ of the closed Toda chain. It suffices to consider the 2-form

$$\hat{\Omega} = -\Omega = -e^{q_n - q_1} dq_n \wedge dq_1 = -d\left(e^{q_n - q_1} dq_1\right) , \tag{62}$$

since $[\hat{\Omega}, \hat{\Omega}]_\pi = [\Omega, \Omega]_\pi = 0$ and $d_N \hat{\Omega} = -d_N \Omega = -\phi$, so that (31) is satisfied.

Appendix: The 4-particle closed Toda case

In this appendix we give more explicit formulas concerning the closed Toda lattice and we justify some assertions done in Section 2, before the beginning of Subsection 2.1.

In the canonical variables $(q_1, q_2, q_3, q_4, p_1, p_2, p_3, p_4)$, we have that

$$N = \begin{bmatrix} p_1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & p_2 & 0 & 0 & -1 & 0 & 1 & 1 \\ 0 & 0 & p_3 & 0 & -1 & -1 & 0 & 1 \\ 0 & 0 & 0 & p_4 & -1 & -1 & -1 & 0 \\ 0 & -\mathrm{e}^{q_1 - q_2} & 0 & 0 & p_1 & 0 & 0 & 0 \\ \mathrm{e}^{q_1 - q_2} & 0 & -\mathrm{e}^{q_2 - q_3} & 0 & 0 & p_2 & 0 & 0 \\ 0 & \mathrm{e}^{q_2 - q_3} & 0 & -\mathrm{e}^{q_3 - q_4} & 0 & 0 & p_3 & 0 \\ 0 & 0 & \mathrm{e}^{q_3 - q_4} & 0 & 0 & 0 & 0 & p_4 \end{bmatrix} \tag{63}$$

and $\Omega = \mathrm{e}^{q_4 - q_1} dq_4 \wedge dq_1$, so that $\pi\, \Omega^\flat = \mathrm{e}^{q_4 - q_1} \left(\partial_{p_4} \otimes dq_1 - \partial_{p_1} \otimes dq_4\right)$ is a rank-2 tensor. It can be checked that its torsion vanishes, while that of

$$\hat{N} = N - \pi\, \Omega^\flat = \begin{bmatrix} p_1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & p_2 & 0 & 0 & -1 & 0 & 1 & 1 \\ 0 & 0 & p_3 & 0 & -1 & -1 & 0 & 1 \\ 0 & 0 & 0 & p_4 & -1 & -1 & -1 & 0 \\ 0 & -\mathrm{e}^{q_1 - q_2} & 0 & -\mathrm{e}^{q_4 - q_1} & p_1 & 0 & 0 & 0 \\ \mathrm{e}^{q_1 - q_2} & 0 & -\mathrm{e}^{q_2 - q_3} & 0 & 0 & p_2 & 0 & 0 \\ 0 & \mathrm{e}^{q_2 - q_3} & 0 & -\mathrm{e}^{q_3 - q_4} & 0 & 0 & p_3 & 0 \\ \mathrm{e}^{q_4 - q_1} & 0 & \mathrm{e}^{q_3 - q_4} & 0 & 0 & 0 & 0 & p_4 \end{bmatrix} \tag{64}$$

turns out to be

$$T_{\hat{N}} = \mathrm{e}^{q_4 - q_1} \left(\partial_{p_1} \otimes dq_4 \wedge dI_1 - \partial_{p_4} \otimes dq_1 \wedge dI_1 - X_1 \otimes dq_1 \wedge dq_4\right) , \tag{65}$$

where $X_1 = \pi\, dI_1$. This is consistent with formula (41). Moreover, one can check that $T_N(X,Y) = \pi\,(i_{X\wedge Y}\phi)$ is satisfied with

$$\phi = dI_1 \wedge \Omega = \mathrm{e}^{q_4-q_1}\,(dI_1 \wedge dq_1 \wedge dq_4) = dI_1 \wedge d\mathrm{e}^{q_4} \wedge d\mathrm{e}^{-q_1} = d\left(I_1\, d\mathrm{e}^{q_4} \wedge d\mathrm{e}^{-q_1}\right)\,. \tag{66}$$

If we put $H_k = \frac{1}{2} I_k = \frac{1}{2k}\,\mathrm{Tr}(\hat{N}^k)$, with $k = 1,2,3,4$, then we obtain the constants of the motion of the 4-particle closed Toda chain. Here, by "constants of the motion of the 4-particle closed Toda chain" we mean those obtained by taking traces of the powers of the well known Lax matrix (see, e.g., [15])

$$L = \begin{bmatrix} p_1 & \mathrm{e}^{\frac{1}{2}(q_1-q_2)} & 0 & \mathrm{e}^{\frac{1}{2}(q_4-q_1)} \\ \mathrm{e}^{\frac{1}{2}(q_1-q_2)} & p_2 & \mathrm{e}^{\frac{1}{2}(q_2-q_3)} & 0 \\ 0 & \mathrm{e}^{\frac{1}{2}(q_2-q_3)} & p_3 & \mathrm{e}^{\frac{1}{2}(q_3-q_4)} \\ \mathrm{e}^{\frac{1}{2}(q_4-q_1)} & 0 & \mathrm{e}^{\frac{1}{2}(q_3-q_4)} & p_4 \end{bmatrix}\,. \tag{67}$$

We also have that

$$\hat{\pi}' = \hat{N}\pi = \begin{bmatrix} 0 & -1 & -1 & -1 & p_1 & 0 & 0 & 0 \\ 1 & 0 & -1 & -1 & 0 & p_2 & 0 & 0 \\ 1 & 1 & 0 & -1 & 0 & 0 & p_3 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 & p_4 \\ -p_1 & 0 & 0 & 0 & 0 & -\mathrm{e}^{q_1-q_2} & 0 & -\mathrm{e}^{q_4-q_1} \\ 0 & -p_2 & 0 & 0 & \mathrm{e}^{q_1-q_2} & 0 & -\mathrm{e}^{q_2-q_3} & 0 \\ 0 & 0 & -p_3 & 0 & 0 & \mathrm{e}^{q_2-q_3} & 0 & -\mathrm{e}^{q_3-q_4} \\ 0 & 0 & 0 & -p_4 & \mathrm{e}^{q_4-q_1} & 0 & \mathrm{e}^{q_3-q_4} & 0 \end{bmatrix}, \tag{68}$$

while the corresponding *Poisson* tensor for the open Toda lattice is

$$\pi' = N\pi = \begin{bmatrix} 0 & -1 & -1 & -1 & p_1 & 0 & 0 & 0 \\ 1 & 0 & -1 & -1 & 0 & p_2 & 0 & 0 \\ 1 & 1 & 0 & -1 & 0 & 0 & p_3 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 & p_4 \\ -p_1 & 0 & 0 & 0 & 0 & -\mathrm{e}^{q_1-q_2} & 0 & 0 \\ 0 & -p_2 & 0 & 0 & \mathrm{e}^{q_1-q_2} & 0 & -\mathrm{e}^{q_2-q_3} & 0 \\ 0 & 0 & -p_3 & 0 & 0 & \mathrm{e}^{q_2-q_3} & 0 & -\mathrm{e}^{q_3-q_4} \\ 0 & 0 & 0 & -p_4 & 0 & 0 & \mathrm{e}^{q_3-q_4} & 0 \end{bmatrix}. \tag{69}$$

It holds

$$\hat{\pi}' = \pi' + \mathrm{e}^{q_4-q_1}\partial_{p_4} \wedge \partial_{p_1}\,, \tag{70}$$

and the Schouten bracket of $\hat{\pi}'$ with itself is

$$[\hat{\pi}', \hat{\pi}'] = 2\mathrm{e}^{q_4-q_1}\,(X_1 \wedge \partial_{p_4} \wedge \partial_{p_1})\,. \tag{71}$$

Then we can verify that the second of (6) is satisfied if ϕ is given by (66).

Finally, we explicitly show that the functions I_2, I_3, I_4 are in involution, as stated in Theorem 6. Taking (15) and (46) into account, we obtain

$$\{I_2, I_3\} = \langle \phi_1, X_2 \rangle = 2\Omega(X_2, \hat{N} X_1)\,, \qquad \{I_3, I_4\} = \langle \phi_2, X_3 \rangle = 2\Omega(X_3, \hat{N}^2 X_1)\,. \tag{72}$$

Since Ω vanishes on the vector fields $Y_k = \hat{N}^{k-1} X_1 - X_k$, it holds

$$\{I_2, I_3\} = 2\Omega(X_2, X_2) = 0\,, \qquad \{I_3, I_4\} = 2\Omega(X_3, X_3) = 0\,. \tag{73}$$

As far as $\{I_2, I_4\}$ is concerned, thanks to (18) and (46) it can be written as

$$\{I_2, I_4\} = \langle \phi_1, X_3 \rangle + \langle \phi_2, X_2 \rangle = 2\Omega(X_3, \hat{N} X_1) + 2\Omega(X_2, \hat{N}^2 X_1). \tag{74}$$

Hence

$$\{I_2, I_4\} = 2\Omega(X_3, X_2) + 2\Omega(X_2, X_3)\,, \tag{75}$$

which clearly vanishes. Notice however that, e.g.,

$$\langle \phi_1, X_3 \rangle = 2\Omega(X_3, \hat{N} X_1) = 2\Omega(X_3, X_2)$$

is *not* vanishing by itself, as anticipated in Section 2.

References

[1] Bogoyavlenskij, O.I., *Theory of Tensor Invariants of Integrable Hamltonian Systems. I. Incompatible Poisson Structures*, Commun. Math. Phys. **180** (1996), 529–586.

[2] Bogoyavlenskij, O.I., *Necessary Conditions for Existence of Non-Degenerate Hamiltonian Structures*, Commun. Math. Phys. **182** (1996), 253–290.

[3] Damianou, P., *On the bi-Hamiltonian structure of Bogoyavlensky-Toda lattices*, Nonlinearity **17** (2004), 397–413.

[4] Das, A., Okubo, S., *A systematic study of the Toda lattice*, Ann. Physics **190** (1989), 215–232.

[5] Falqui, G., Magri, F., Pedroni, M., *Bihamiltonian geometry and separation of variables for Toda lattices*, J. Nonlinear Math. Phys. **8** (2001), suppl., 118–127.

[6] Falqui, G., Pedroni, M., *Separation of variables for bi-Hamiltonian systems*, Math. Phys. Anal. Geom. **6** (2003), 139–179.

[7] Fiorenza, D., Manetti, M., *Formality of Koszul brackets and deformations of holomorphic Poisson manifolds*, Homology Homotopy Appl. **14** (2012), 63–75.

[8] Iglesias-Ponte, D., Laurent-Gengoux, C., Xu, P., *Universal lifting theorem and quasi-Poisson groupoids*, J. Eur. Math. Soc. (JEMS) **14** (2012), 681–731.

[9] Kosmann-Schwarzbach, Y., *The Lie Bialgebroid of a Poisson-Nijenhuis Manifold*, Lett. Math. Phys. **38** (1996), 421–428.

[10] Kosmann-Schwarzbach, Y., Magri, F., *Poisson-Nijenhuis structures*, Ann. Inst. Henri Poincaré **53** (1990), 35–81.

[11] Liu, Z-J., Weinstein, A., Xu, P., *Manin Triples for Lie Bialgebroids*, J. Differential Geom. **45** (1997), 547–574.

[12] Magri, F., Morosi, C., Ragnisco, O., *Reduction techniques for infinite-dimensional Hamiltonian systems: some ideas and applications*, Comm. Math. Phys. **99** (1985), 115–140.

[13] Morosi, C., Pizzocchero, L., *R-Matrix Theory, Formal Casimirs and the Periodic Toda Lattice*, J. Math. Phys. **37** (1996), 4484–4513.

[14] Okubo, S., *Integrability condition and finite-periodic Toda lattice*, J. Math. Phys. **31** (1990), 1919–1928.

[15] Perelomov, A.M., *Integrable systems of classical mechanics and Lie algebras. Vol. I*, Birkhäuser Verlag, Basel, 1990.

[16] Stiénon, M., Xu, P., *Poisson Quasi-Nijenhuis Manifolds*, Commun. Math. Phys. **270** (2007), 709–725.

[17] Vaisman, I., *The Geometry of Poisson Manifolds*, Birkäuser Verlag, Basel, 1994.

[18] Vaisman, I., *Complementary* 2*-forms of Poisson structures*, Compositio Math. **101** (1996), 55–75.