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Capital allocation: standard and beyond

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To my family,

for their support and encouragement.

ABSTRACT

The thesis deals with the problem of capital allocation. After a brief review of the literature and of the standard methods, capital allocation problems with respect to a particular class of risk measures, namely the Haezendonck-Goovaerts (HG) ones [14, 47], are considered. We first generalize the capital allocation rule (CAR) introduced by Xun et al. [67] for Orlicz risk premia [48], using two different approaches, in order to cover HG risk measures. We then provide robust versions of the introduced CARs, both considering the case of ambiguity over the probabilistic model and the one of multiple Young functions, following the scheme of [13].

Further on, we introduce a new approach to face capital allocation problems from the perspective of acceptance sets, by defining the family of sub-acceptance sets. We study the relations between the notions of sub-acceptability and acceptability of a risky position and their impact on the allocation of risk. We define the notion of risk contribution rule and show how in this context it is interpretable as a tool for assessing the contribution of a sub-portfolio to a given portfolio, in terms of acceptability, without necessarily involving a risk measure. Furthermore, we investigate under which conditions on a risk contribution rule a representation of an acceptance set holds in terms of the risk contribution rule itself, thus extending to this setting the interpretation, classical in risk measures theory, of minimal amount required to hedge a risky position.

Finally, we provide a discussion on some possible further extensions of the capital allocation problem. In particular, we discuss the possibility of extending the latter to the framework of intrinsic risk measures [37]. We briefly review the notions and results on intrinsic risk measures, providing a comparison with traditional ones. We later discuss the suitability of the capital allocation problem in this context, as well as that of the properties related to capital allocation rules, considering both the standard setting and the one based on acceptance sets. We derive some results similar to the case of traditional risk measures.

PUBLICATIONS

The dissertation includes a series of joint papers with Francesca Centrone and Emanuela Rosazza Gianin, namely [20–22]. In particular, the contents of [22] constitute Chapter 3, those of [21] are reported in Chapter 4 and the numerical study in [20] constitutes Chapter 6.

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1 INTRODUCTION

Since the first version of the Basel Accord (see [9]), many studies on risk measures and capital requirements have been led both from a theoretical and an empirical point of view. It is well known indeed that the Basel Accord (see [9, 10]) imposes to banks and financial institutions a capital requirement or margin so to be able to face the riskiness due to different sources (market risk, credit risk,...). In the first version of the accord, such a margin had to be measured by means of Value at Risk (VaR for short). The latter is simply the upper α-quantile of the random variable representing the profit and loss or return of a financial position. Even if VaR has been shown to have a lot of drawbacks, it has been used intensively because of its simple interpretation and estimation. Among the different drawbacks, VaR does not encourage diversification of risk in general and it is not able to distinguish different tails but only considers the quantile (for a more detailed study, please see Artzner et al. [4, 5]).

Although VaR is still widely used by practitioners and researchers, Conditional Value at Risk (CVaR for short), also known as Expected Shortfall or Average Value at Risk, is more and more considered; see [1, 5, 39, 60]. It is well known that, compared to Value at Risk, Conditional Value at Risk is a more conservative risk measure, that is, it requires a higher margin, and encourages diversification. In particular, CVaR is a coherent risk measure (see [4, 5]), as it belongs to a peculiar class of risk measures, namely the Haezendonck-Goovaerts (HG) ones. Such risk measures, based on the so-called Orlicz premium introduced by Haezendonck and Goovaerts [48], have been studied in the last decades both from a mathematical point of view and from an actuarial one (see, among others, [12–15, 47, 48]), as they are coherent and generalize CVaR.

It is worth emphasizing that, for VaR or CVaR, a regulator has only to choose a level α of probability. It is financially reasonable, however, to consider also risk measures taking into account preferences and loss aversion of regulators, e. g. in terms of certainty equivalents. A well-known and used risk measure of that kind is the so-called entropic risk measure, defined by means of the certainty equivalent with an exponential utility function (see [8, 39, 41] for more details). We will recall the definitions of these three risk measures afterwards.

Anyway, whatever the risk measure chosen, the main idea and motivation of risk measures is related to capital requirements or margin deposits. Indeed, given a financial position (or, better, its profit and loss or its return) its riskiness is quantified by the minimal cash to be deposited as a guarantee of the position or, in other words, such that the new position is considered as acceptable by the regulator. More precisely, given a position X belonging to some vector space $\mathcal{X}$ and a cash-additive risk measure $\rho\colon \mathcal{X} \to \mathbb{R}$, the riskiness of X by means of ρ is given by

$$\rho(X) = \inf\{m \in \mathbb{R} \mid X + m \in \mathcal{A}\}\,;$$

where $\mathcal{A} \subset \mathcal{X}$ is a set of positions which are acceptable for a specific regulator. Such a set $\mathcal{A}$ is usually termed as *acceptance set* and provided with some minimal properties (see [5, 39]). Roughly speaking, the greater is the riskiness of a position, the higher is the margin to be deposited in order to reach the acceptability of the position (see [5, 39] for details).

Among the many, one of the most relevant problems connected to the use of risk measures in firms and insurances, is the one of *capital allocation*. It consists in, once fixed a suitable risk measure and determined the corresponding risk capital associated to a risky position, finding a division of this aggregate capital among the constituents of the activity, such as business units or various insurance lines. For instance, this problem is particularly meaningful in the context of risk management, or for comparing the return of various business units in order to remunerate managers.

As it can be easily understood, there are many possible ways to allocate the aggregate capital of a company to its sub-units, according to the features one wants to capture and to the properties one wishes to verify. In this respect, a huge literature has grown over the years, and several methods have been proposed (see, for example, [23, 32, 50]), where the different approaches have motivations that can be either axiomatic or financial.

In particular, Kalkbrener [50] defines a *capital allocation rule* as a map whose values depend on the profit and loss or return of both a portfolio and its sub-portfolios, and which is required to satisfy some suitable properties w. r. t. the chosen risk measures, that is, he proposes an axiomatic approach to the problem. More precisely, a capital allocation rule for a monetary risk measure $\rho\colon \mathcal{X} \to \mathbb{R}$ is a map $\Lambda\colon \mathcal{X} \times \mathcal{X} \to \mathbb{R}$ such that $\Lambda(X, X) = \rho(X)$ for every $X \in \mathcal{X}$; where $\Lambda(X, Y)$ is interpreted as the risk contribution of a sub-position X to the risk of the aggregated position Y. For any coherent risk measure, Kalkbrener [50] shows that there exists a capital allocation rule satisfying reasonable properties. When the coherent risk measure ρ is also Gateaux differentiable, Kalkbrener's approach yields the so-called *gradient* or *Euler* allocation, which is largely applied in practice and well-known in the literature (see also Tasche [62, 63]).

Dhaene et al. [32] highlight some of the financial aspects of capital allocation: indeed, some of its core purposes for a firm consist in distributing the cost of capital among the various business units, as well as in being able to make a comparison of their performances through the return of allocated capital. The authors also provide an overview on some of the most used algorithms in practice, namely the *proportional* and the *marginal* ones, which we will review and use as examples throughout the thesis.

Instead, the approach of Centrone and Rosazza Gianin [23], refers both to the axiomatic approach and to the game theoretic stream proposed by Denault [30], where firms are seen as players of a cooperative cost game derived by a risk measure, and the allocation rule is based on the idea of assigning to each player its marginal contribution to the overall risk. Denault's approach is anyway suitable fo coherent and differentiable risk measures, while the capital allocation method proposed in [23] is a generalization of the so-called *Aumann-Shapley* capital allocation rule, suitable also for the class of (quasi-)convex and non-differentiable risk measures.

In view of the above discussion, after providing a review of the different capital allocation methods, we will first focus on capital allocation problems with respect to HG risk measures. In such direction, Xun et al. [67] have recently generalized the contribution to shortfall provided by Overbeck [58] for CVaR, by introducing a capital allocation method that is "tailored" for Orlicz premia and works beyond the special case of CVaR. However, the latter considers only non-negative random variables and depends on the quantile (or VaR) of the aggregated risk Y, which is still somehow connected to the CVaR case. One goal of thesis is therefore to introduce a capital allocation method for HG risk measures which is defined for any pair of random variables (not only positive) and overcomes the special case of CVaR.

Indeed, inspired by [15], we extend the work of Xun et al. [67] by providing capital allocation rules for HG risk measures that are based on generalized quantiles, namely the Orlicz ones (see [15]). We show that such a capital allocation rule satisfies most of the usual properties required for capital allocations and is also reasonable from a financial point of view. We also propose and study an alternative and more general capital allocation rule for HG risk measures that is based on the new concept of linking function, that is, functions which take into account both a position and a sub-position, going beyond the case considered in [67]. A comparison among the approaches here introduced and two popular capital allocation rules, that is the gradient method and the Aumann-Shapley one [23, 50], is also provided. Very recently, a deep analysis on the gradient approach has been done by Gómez and Tang [46] for higher moment risk measures, corresponding to HG risk measures for power Young functions.

Finally, inspired by robust Orlicz premia and robust HG risk measures recently introduced by Bellini et al. [13], we provide some extensions of the proposed methods of capital allocation, in order to cover the ambiguity about the probabilistic model and about the risk perception of the decision-maker. In particular, we first extend the results in [13] by defining and studying robust Orlicz risk premia and HG risk measures for $\alpha \in [0,1)$ (while in [13] the authors assume $\alpha = 0$). We then introduce robust Orlicz quantiles and study their properties, finding out similar results to the non-robust case. By using robust Orlicz quantiles, we then provide robust versions of the presented methods to account for ambiguity over the probability measure to be chosen and for ambiguity over the utility/loss function. We find out that the robust versions work well both for the quantile-based methods and linking based ones, providing results very close to the non-robust case.

So far, we have discussed capital allocation problems associated to risk measures, as it is customary in the literature. Indeed, the definition of capital allocation rule (CAR) provided by Kalkbrener [50] necessarily involved a risk measure as a primary object. However, monetary risk measures are the natural counterpart of acceptance sets (Artzner et al. [5] and Föllmer and Schied [39]) and hence, in the classical sense, any capital allocation rule also takes into account the acceptability of a stand-alone risky position X, allocating no positive capital to acceptable positions.

What is instead missing is the consideration of what happens in terms of acceptability when X is "merged into another position" Y and how this possibly affects the allocation of capital. Indeed, consider a situation where we are provided with a monetary risk measure ρ that qualifies a position X as non-acceptable. If X is anyway considered as a sub-portfolio of another position Y, and we look at the marginal contribution $\rho_Y(X) := \rho(Y) - \rho(Y - X)$, the risk of X can potentially change, and it can become acceptable w. r. t. the monetary risk measure $\rho_Y(\cdot)$, not contributing to the risk of Y.

We wish thus to rephrase the problem of capital allocation in a way that takes into account this eventuality, instead of simply sharing $\rho(Y)$ among its sub-units. In other words, if we consider a portfolio Y, we want to define CARs as maps assigning to each sub-portfolio X of Y a capital that reflects their acceptability as sub-units of Y, and does not necessarily assign a share $\Lambda(X, Y)$ of $\rho(Y)$.

The capital allocation problem is thus disentangled from the use of risk measures, and revisited in terms of a different definition and a newly introduced concept, that is, the one of a *sub-acceptance* family of sets. Under suitable assumptions, we derive capital allocation rules reflecting the above idea starting from acceptance and sub-acceptance sets and, conversely, we show that capital allocation rules having some natural properties give rise to acceptance and sub-acceptance sets in terms of which they can be represented, thus extending to these capital allocation rules the classical interpretation of capital requirement, typical of risk measures. The situation becomes even more interesting when we consider quasi-convex risk measures, where every quasi-convex risk measure is associated to a family of acceptance sets and one speaks of *acceptability at different levels*. In analogy with what happens with monetary risk measures, in this case we will need *families* of sub-acceptance sets.

It turns out that some well known capital allocation rules (such as the Euler and the RORAC ones) are compatible with this approach. Furthermore, the present approach is in line with the general construction of capital requirements in Frittelli and Scandolo [43] and with the systemic risk measures induced by acceptance sets in [17].

As a further discussion, we briefly consider the possibility of generalizing the above results with respect to the chosen ordering among random variables. That is, we drop the standard $\mathbb{P}$-a.s. ordering used throughout the work to consider a general preorder among random variables. We then focus on the first stochastic order and related properties of law invariance.

Indeed, in the standard framework, monetary risk measures which are monotone with respect to the first stochastic order are also law invariant (see [27, 39, 66]). For the latter, only the distribution of the financial position X matters in measuring its risk and this property is reflected on acceptance sets, for the connection with risk measures mentioned above. Since law invariant risk measures and stochastic orders are largely studied (see [39, 42, 52, 66]), we also discuss capital allocation rules in this context, starting from the perspective of acceptance sets.

Very recently, Farkas and Smirnow [37] provide an alternative approach to risk measurement, introducing the so-called *intrinsic risk measures*. The latter differ from standard risk measures as they measure the risk by means of the minimal percentage of the position which should be sold and reinvested in a given "eligible" asset, in order to reach acceptability. Thus, such approach uses only internal resources and does not require any external capital injection to make the position acceptable. As a further extension, we also consider capital allocation problems in this context, both in the standard approach, connected to risk measures, and in the one which starts from acceptance sets.

To sum up, the main contributions of the thesis are the following. Firstly, we provide a review of the literature about capital allocation problems and a comparison among different approaches with a numerical example. We also highlight a peculiar property of capital allocation rules, namely the no-undercut, which turns to be a "core" property in the sense of the game theoretical approach. We then provide capital allocation rules for HG risk measures, overcoming the approach of Xun et al. [67]. Further, we introduce a new approach to capital allocation problems by means of the concepts of sub-acceptance and acceptance sets. Finally, we investigate the possibility of extending the framework to account for general orderings, focusing on the first stochastic one, and also for the recent concept of intrinsic risk measure.

The work is organized as follows. In Chapter 2, we provide the main definitions and results about risk measures and capital allocation, useful as a background for the rest of the work. Chapter 6 provides a further comparison among some well-known capital allocation methods, through a numerical example. In Chapter 3, we provide capital allocation rules for HG risk measures and related results. Chapter 4 is devoted to the new approach of capital allocation, based on acceptance sets and sub-acceptance families. Finally, Chapter 5 discusses the two further extensions mentioned above, that is, to account for general orderings and for intrinsic risk measures. Appendix A collects some mathematical background.

2 RISK MEASURES AND CAPITAL ALLOCATION

In this chapter, we define our setting and recall the main notions and results about risk measures and capital allocation.

We present some well-known risk measures, used in practice and throughout the work as examples. We recall the capital allocation problem, which leads to the formal definition of capital allocation rule. We also recall some capital allocation methods, their definition in term of capital allocation rules and some results from the related literature.

2.1 GENERAL SETTING

In this section, we set the notation and terminology used throughout the thesis, as well as the financial interpretation of the mathematical objects we deal with.

We recall a few concepts of functional analysis, useful to understand the following. Most of them are also presented in Appendix A, but for a proper and detailed treatment of those topics we refer to Aliprantis and Border [2], among many others.

When any assumption of this section is dropped, or any notation is changed, we explicitly mention it in the work.

We consider a one period economy with initial date $t = 0$ and final date (or time horizon) T. We fix a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, where Ω is a fixed set of scenarios, representing the possible states of the world at time T.

A financial position is described by a random variable $X\colon \Omega \to \mathbb{R}$, where $X(\omega)$ represents the profit and loss (P&L) at time T if the scenario $\omega \in \Omega$ is realized. That is, positive values of X are interpreted as gains, while negative ones as losses. However, in Chapter 3, we adopt the actuarial notation about signs; that is, the converse of before: positive values have to be interpreted as losses while negative ones as gains.

Throughout the work, $L^\infty := L^\infty(\Omega, \mathcal{F}, \mathbb{P})$ denotes the space of all $\mathbb{P}$-essentially bounded random variables on $(\Omega, \mathcal{F}, \mathbb{P})$, interpreted as above. Equalities and inequalities between random variables have to be understood to hold $\mathbb{P}$-almost surely. The space L^∞ will be sometimes equipped with the essential supremum norm $\|\cdot\|_\infty$. Under such a norm, L^∞ becomes a Banach space, whose dual can be identified with the space $\mathrm{ba} := \mathrm{ba}(\Omega, \mathcal{F}, \mathbb{P})$, that is, the space of all finitely additive set functions (charges) $\mu\colon \mathcal{F} \to \mathbb{R}$ with finite total variation and absolutely continuous with respect to $\mathbb{P}$ (see also Bhaskara Rao and Bhaskara Rao [16]).

We also use the weak* topology $\sigma(L^\infty, L^1)$, in order to identify the dual space of L^∞ with $L^1 := L^1(\Omega, \mathcal{F}, \mathbb{P})$ and, through the Radon-Nikodym theorem, L^1 with $\mathcal{H}$, the space of all (countably additive) measures absolutely continuous with respect to $\mathbb{P}$. We then define $\mathcal{Q}$ as the space of all probability measures belonging to $\mathcal{H}$.

We only consider financial positions described by random variables belonging to L^∞, even if several results hold also in more general spaces. Thus, risk measures, capital allocation rules and other objects will be defined in such a framework.

2.2 RISK MEASURES

We recall here the main notions and results about risk measures, used throughout the work, and the definition of some well-known risk measures, useful for further examples and widely applied in practice.

We first recall from Föllmer and Schied [39] the standard definitions of monetary, convex and coherent risk measures, and some key results about them.

DEFINITION 2.1. A map $\rho \colon L^\infty \to \mathbb{R}$ is called a *monetary risk measure* if it satisfies the following conditions, for all $X, Y \in L^\infty$:

Monotonicity: if $X \leq Y$ then $\rho(X) \geq \rho(Y)$.

Cash-additivity: $\rho(X + m) = \rho(X) - m$, for all $m \in \mathbb{R}$.

The financial meaning of the required properties is the following. Monotonicity ensures that the risk of a position is reduced when its the P&L is increased. Cash-additivity (sometimes called translation-invariance or cash-invariance) ensures that, when a cash amount is added to a financial position, its risk is reduced by the same amount. In particular, cash-additivity implies $\rho(X + \rho(X)) = 0$ and $\rho(m) = \rho(0) - m$. When the assumption $\rho(0) = 0$ is made, the monetary risk measure ρ is called *normalized*.

DEFINITION 2.2. A monetary risk measure ρ is called a *convex risk measure* if it satisfies the following, for all $X, Y \in L^\infty$ and $\lambda \in [0, 1]$:

Convexity: $\rho(\lambda X + (1 - \lambda)Y) \leq \lambda\rho(X) + (1 - \lambda)\rho(Y)$.

Convexity captures the idea that diversification should not increase the risk. Indeed, if we invest only the fraction λ of the resources on X and the remaining part $1 - \lambda$ on Y, we obtain a position $\lambda X + (1 - \lambda)Y$ which is less risky than the convex combination of the risk of X and Y (see, for more details, Föllmer and Schied [38, 39] and Frittelli and Rosazza Gianin [41]).

DEFINITION 2.3. A convex risk measure ρ is called a *coherent risk measure* if it satisfies the following, for all $X \in L^\infty$:

Positive homogeneity: $\rho(\lambda X) = \lambda \rho(X)$, for all $\lambda \geq 0$.

Positive homogeneity and convexity, together, are equivalent to positive homogeneity and

Subadditivity: $\rho(X + Y) \leq \rho(X) + \rho(Y)$, for all $X, Y \in L^\infty$.

Thus, a coherent risk measure is also referred as a monetary risk measure satisfying positive homogeneity and subadditivity (see, for instance, Artzner et al. [5] and Delbaen [28, 29]).

So far, we have presented risk measures in the cash-additive framework, which has been the standard axiomatic setting for a long time. Recently, different classes of risk measures have been introduced in the literature to overcome some lacks of the cash-additivity framework (see Cerreia-Vioglio et al. [24], El Karoui and Ravanelli [35] and Munari [56] for a proper discussion). We first recall from Farkas et al. [36] and Munari [56] the following.

DEFINITION 2.4. A map $\rho \colon L^\infty \to \overline{\mathbb{R}}$ is called an *S-additive risk measure* if it satisfies monotonicity and the following, for all $X \in L^\infty$:

S-additivity: $\rho(X + mS_T) = \rho(X) - mS_0$, for all $m \in \mathbb{R}$, S eligible;

where an eligible asset can be identified with a couple $S = (S_0, S_T)$ with initial value $S_0 > 0$ and terminal (random) payoff S_T satisfying $\mathbb{P}(S_T \geq 0) = 1$.

By taking $S = (1, \mathbb{1}_\Omega)$ as eligible asset, we see that cash-additivity is a special case of S-additivity. In other words, for an S-additive risk measure, not only cash can be used to hedge the risk of a position but also assets with an almost surely positive payoff. Notice, however, that an S-additive risk measure is not necessarily finite-valued (see Farkas et al. [36] and Munari [56] for further details).

Instead, El Karoui and Ravanelli [35] introduced the following notion, with the aim of weakening cash-additivity.

DEFINITION 2.5. A map $\rho \colon L^\infty \to \mathbb{R}$ is called a *cash-subadditive risk measure* if it satisfies monotonicity and the following, for all $X \in L^\infty$:

Cash-subadditivity: $\rho(X + m) \geq \rho(X) - m$, for all $m \geq 0$.

Cash-subadditive risk measures are suitable to asses the risk in case of stochastic or ambiguous interest rates, as explained by El Karoui and Ravanelli [35], differently from cash-additive risk measures, where the interest rate is usually assumed to be zero or anyway constant (see Artzner et al. [5] and Delbaen [28, 29]).

Among the properties presented so far, both convexity and subadditivity express the idea of diversification of risk (see Artzner et al. [5], Delbaen [28, 29], Föllmer and Schied [38] and Frittelli and Rosazza Gianin [41]). However, as pointed out by Cerreia-Vioglio et al. [24] (see also Drapeau and Kupper [33] or Frittelli and Maggis [40]), once cash-additivity is dropped, the right formulation of diversification is given by the weaker property of quasi-convexity. This gives rise to the following class of risk measures.

DEFINITION 2.6. A map $\rho\colon L^\infty \to \mathbb{R}$ is called a *quasi-convex risk measure* if it satisfies monotonicity and the following, for all $X, Y \in L^\infty$ and $\lambda \in [0,1]$:

Quasi-convexity: $\rho(\lambda X + (1-\lambda)Y) \leq \max\{\rho(X), \rho(Y)\}$.

We finally recall the class of law invariant risk measures, that is, monetary, convex or coherent risk measures satisfying the property contained in the following definition (see, among others, Frittelli and Rosazza Gianin [42], Kusuoka [52] and Weber [66]).

DEFINITION 2.7. A map $\rho\colon L^\infty \to \mathbb{R}$ is called *law invariant* if it satisfies the following, for all $X, Y \in L^\infty$:

Law invariance: if $X \sim Y$ then $\rho(X) = \rho(Y)$.

We recall that the notation $X \sim Y$ means that X and Y have the same distribution under $\mathbb{P}$, i.e. the distribution μ_X of X, defined by $\mu_X(B) := \mathbb{P}(X \in B)$, for every Borel set B, is equal to the distribution μ_Y of Y.

2.2.1 *Risk measures and acceptance sets*

We recall here the notion of acceptance set and the connections between acceptance sets and risk measures (see, for instance, Artzner et al. [5], Farkas et al. [36] and Föllmer and Schied [38, 39]). We put a special emphasis on this part, since acceptance sets play a fundamental role in Chapter 4.

Acceptance sets allow us to split the positions in two categories: the acceptable ones and the unacceptable ones. The first ones do not require extra capital, while the second ones do require it. Notice that one can detect the acceptable positions without necessarily measure their risk before, it is sufficient to consider some features of the positions, as shown in the following examples. Indeed, acceptance sets are often considered as primary objects (see, among others, Farkas et al. [36] and Farkas and Smirnow [37]).

The next definition formalizes the notion of acceptance set (see Farkas et al. [36] or Farkas and Smirnow [37]).

DEFINITION 2.8. A set $\mathcal{A} \subseteq L^\infty$ is called an *acceptance set* if it satisfies the following conditions:

Non-triviality: $\emptyset \neq \mathcal{A} \neq L^\infty$.

Monotonicity: $X \in \mathcal{A}$ and $Y \geq X$ imply $Y \in \mathcal{A}$.

Any position $X \in \mathcal{A}$ is then called acceptable.

It is easy to understand why we require that an acceptance set has to be non-trivial: should we accept a position, say $X = -100$, which leads us (almost-surely) to a loss? Or, conversely, should we require extra capital for a position, say $X = 100$, which leads us (almost-surely) to a profit? Monotonicity, too, is quite easy to understand: if a position is acceptable, so is another one with greater or equal P&L.

As mentioned above, the acceptable positions do not require extra capital to cover future possible losses. They are chosen according to a criterion which involves some features of the positions and which uniquely defines the acceptance set, without necessarily involving any risk measure.

EXAMPLE 2.1. Suppose that a regulator is very averse to risk and does not want that any loss can occur. So, the regulator decides that the only acceptable positions are those with (almost-surely) non-negative P&L. Thus, the acceptance set of the regulator is described by

$$\mathcal{A} = \{X \in L^\infty \mid \mathbb{P}(X \geq 0) = 1\} = L^\infty_+,$$

that is, the positive cone of L^∞. Clearly, $\mathcal{A}$ is an acceptance set according to Definition 2.8: non-triviality and monotonicity are straightforward.

Notice that, once chosen the criterion (being non-negative a.s.), the acceptance set is directly determined (positive cone of L^∞) without measuring the risk of the position or evaluating any other information. Also, only features of the position itself are considered here, we do not involve features of any other position; differently from the case of sub-acceptance set, introduced in Chapter 4.

The acceptance set defined in the previous example is simple and very prudent since it reflects the high risk aversion of the regulator. In the next example, we consider an acceptance set still simple but very tolerant.

EXAMPLE 2.2. Suppose a portfolio manager constructs a high-return portfolio. He or she is not particularly averse to risk and considers acceptable all positions with non-negative expected P&L. In this case, the acceptance set is

$$\mathcal{A} = \{X \in L^\infty \mid \mathbb{E}[X] \geq 0\}.$$

It is clear that $\mathcal{A}$ is an acceptance set according to Definition 2.8: non-triviality and monotonicity follow from the monotonicity of the expected value. As before, we chose a logical criterion, which involves only a feature of the position itself, and the acceptance set is directly determined without any other information about the risk of the position or whatever else.

Although the previous examples are both explanatory, they are not suitable to describe many realistic situations. The acceptance set of Example 2.1 is clearly too prudent while the one of Example 2.2 is clearly too tolerant, furthermore the latter is based only on the expected value. To fill this gap, we provide some more examples which are suitable for different levels of prudence.

Example 2.3. Starting from Example 2.1, we construct an acceptance set which is less prudent and suitable for various situations. Suppose a regulator considers acceptable those positions with a probability of being negative less or equal than a fixed level $\alpha \in (0,1)$. In this case the acceptance set is given by

$$\mathcal{A} = \{X \in L^\infty \mid \mathbb{P}(X \leq 0) \leq \alpha\}$$

which is the acceptance set of the Value at Risk (see, among many others, Artzner et al. [5], Delbaen [28] and Föllmer and Schied [39]). It is straightforward that $\mathcal{A}$ is an acceptance set according to Definition 2.8. Moreover, $\mathcal{A}$ depends on the parameter α, thus it is suitable for several situations. A prudent firm can choose a small α level while a more tolerant one can choose a higher α level.

Example 2.4. Let us move back to Example 2.2, where we have considered acceptable those positions with non-negative expected value. Likewise the previous example, one can instead consider acceptable those positions with expected value greater or equal than a constant $\lambda > 0$. In this case, we obtain the acceptance set

$$\mathcal{A} = \{X \in L^\infty \mid \mathbb{E}[X] \geq \lambda\}$$

which is clearly non-trivial and monotone. It is also suitable for several situations since it depends on the parameter λ.

Moreover, one could also consider the standard deviation σ of the position, as a measure of its risk. In this case, for X such that $\sigma(X) > 0$, we can look at the Sharpe ratio

$$\frac{\mathbb{E}[X]}{\sigma(X)}$$

and consider acceptable those positions with Sharpe ratio greater or equal than $\lambda > 0$. That yields the set

$$\mathcal{A}^* = \{X \in L^\infty \mid \mathbb{E}[X] \geq \lambda\, \sigma(X)\}$$

which looks meaningful but unfortunately it is not an acceptance set because it is not monotone. To see this, consider $X \in L^\infty$ such that $X \sim \mathcal{U}[0, b]$, $b > 0$; that is, X is uniformly distributed in the closed interval $[0, b]$, where b is a positive real number. Then, take $Y = e^X$ so that Y still belongs to L^∞. In this case,

$$\mathbb{E}_b[Y] = \frac{1}{b}(e^b - 1);$$
$$\sigma_b(Y) = \sqrt{\frac{1}{2b}(e^{2b} - 1) - \frac{1}{b^2}(e^b - 1)^2}.$$

Now, notice that $Y \geq 0$ and that $0 \in \mathcal{A}^*$ but for every $\lambda > 0$ we can find $b > 0$ such that

$$\frac{\mathbb{E}_b[Y]}{\sigma_b(Y)} < \lambda$$

because

$$\lim_{b \to +\infty} \frac{\mathbb{E}_b[Y]}{\sigma_b(Y)} = \lim_{b \to +\infty} \frac{\frac{1}{b}(e^b - 1)}{\sqrt{\frac{1}{2b}(e^{2b} - 1) - \frac{1}{b^2}(e^b - 1)^2}} = 0^+.$$

Thus, for every $\lambda > 0$, $\mathcal{A}^*$ is not an acceptance set.

It is well known that there exists the following relation between monetary risk measures and acceptance sets (see Föllmer and Schied [39, Propp. 4.6-4.7]). Indeed, a monetary risk measure ρ induces the *acceptance set of* ρ, that is, the set

$$\mathcal{A}_\rho := \{X \in L^\infty \mid \rho(X) \leq 0\}$$

of positions which are acceptable in the sense that they do not require additional capital. The next proposition shows how the properties of ρ translate into properties of $\mathcal{A}_\rho$.

Proposition 2.1 (Föllmer and Schied [39]). *Let $\rho \colon L^\infty \to \mathbb{R}$. Then the following hold:*

A. *If ρ is a monetary risk measure then $\mathcal{A}_\rho$ is an acceptance set as in Definition 2.8. Moreover, ρ can be recovered from $\mathcal{A}_\rho$, for all $X \in L^\infty$:*

$$\rho(X) = \inf \{m \in \mathbb{R} \mid m + X \in \mathcal{A}_\rho\}.$$

B. *If ρ is a convex risk measure then $\mathcal{A}_\rho$ is also convex.*

C. *If ρ is a coherent risk measure then $\mathcal{A}_\rho$ is also a cone.*

Conversely, one can take an acceptance set $\mathcal{A}$ as primary object, and define the monetary risk measure $\rho_\mathcal{A}$ as the minimal amount m such that $m + X$ becomes acceptable:

$$\rho_\mathcal{A}(X) := \inf \{m \in \mathbb{R} \mid m + X \in \mathcal{A}\}.$$

As before, properties of $\mathcal{A}$ translate into properties of $\rho_\mathcal{A}$ as shown below.

Proposition 2.2 (Föllmer and Schied [39]). *Let $\mathcal{A} \subseteq L^\infty$. Then the following hold:*

a. *If $\mathcal{A}$ is an acceptance set, as in Definition 2.8, then $\rho_\mathcal{A}$ is a monetary risk measure.*

b. *If $\mathcal{A}$ is also convex then ρ is a convex risk measure.*

c. *If $\mathcal{A}$ is also a cone then ρ is a coherent risk measure.*

d. *$\mathcal{A}$ is a subset of $\mathcal{A}_{\rho_\mathcal{A}}$ and if $\mathcal{A}$ is closed then $\mathcal{A} = \mathcal{A}_{\rho_\mathcal{A}}$.*

A similar result holds also for S-additive risk measures (see [56]). Indeed, if ρ is an S-additive risk measure then $\mathcal{A}_\rho$ is non-trivial and monotone. Moreover, ρ can be recovered from $\mathcal{A}_\rho$ as

$$\rho(X) = \inf\left\{ m \in \mathbb{R} \;\middle|\; \frac{m}{S_0} S_T + X \in \mathcal{A}_\rho \right\}.$$

Conversely, given a class $\mathcal{A} \subseteq L^\infty$ of acceptable positions and an eligible asset S, one can define

$$\rho_{\mathcal{A},S}(X) := \inf\left\{ m \in \mathbb{R} \;\middle|\; \frac{m}{S_0} S_T + X \in \mathcal{A} \right\}. \tag{2.1}$$

If $\mathcal{A} \subseteq L^\infty$ is an acceptance set then $\rho_{\mathcal{A},S}$ is an S-additive risk measure.

The correspondence between acceptance sets and risk measures becomes more complex in the quasi-convex case, as illustrated in Chapter 4.

2.2.2 *On the dual representation*

We recall here the so-called dual representation for convex and coherent risk measures. Fore more details, we refer to Artzner et al. [5], Delbaen [28, 29], Föllmer and Schied [38, 39] and Frittelli and Rosazza Gianin [41], among others.

Let ba_1 denote the space of all $\mu \in \mathrm{ba}$ which are positive and normalized, i.e. such that $\mu(A) \geq 0$ for all $A \in \mathcal{F}$ and $\mu(\Omega) = 1$. As in Appendix A, the Choquet integral, with respect to $\mu \in \mathrm{ba}_1$, is denoted by $\mathrm{E}_\mu[\cdot]$. The notation $\mathbb{E}_\mathbb{P}[\cdot]$ is instead devoted to the expectation under $\mathbb{P}$, that is, a Choquet integral with respect to a probability measure (see Appendix A for more details). We still denote by $\mathcal{Q}$ the set of all probability measures absolutely continuous with respect to $\mathbb{P}$.

Proposition 2.3 (Föllmer and Schied [39]). *Any convex risk measure $\rho\colon L^\infty \to \mathbb{R}$ admits the representation*

$$\rho(X) = \max_{\mu \in \mathrm{ba}_1} \{\mathrm{E}_\mu[-X] - \beta(\mu)\}$$

where $\beta(\mu) := \sup_{X \in \mathcal{A}_\rho} \mathrm{E}_\mu[-X]$ *is called the minimal penalty function.*

If ρ *is coherent then*

$$\rho(X) = \max_{\mu \in \mathcal{N}} \mathrm{E}_\mu[-X]$$

where

$$\mathcal{N} := \{\mu \in \mathsf{ba}_1 \mid \beta(\mu) = 0\}.$$

As we can see from the previous proposition, it is always possible to represent a convex (or coherent) risk measure in term of charges, through the Choquet integral. Instead, if we wish a representation in terms of probability measures, through the expectation, we need to require an additional property, as showed in the following proposition.

PROPOSITION 2.4 (Föllmer and Schied [39]). *A convex risk measure* $\rho \colon L^\infty \to \mathbb{R}$ *admits the representation*

$$\rho(X) = \sup_{\mathbb{Q} \in \mathcal{Q}} \{\mathbb{E}_\mathbb{Q}[-X] - \beta(\mathbb{Q})\} \tag{2.2}$$

if any of the following equivalent conditions is satisfied:

A. ρ *is continuous from above:* $X_n \downarrow X \implies \rho(X_n) \uparrow \rho(X)$.

B. ρ *has the Fatou Property: if* (X_n) *is a bounded sequence converging* $\mathbb{P}$*-a. s. to* X *then*

$$\rho(X) \leq \liminf_{n \to +\infty} \rho(X_n).$$

C. ρ *is lower semicontinuous for the weak* topology* $\sigma(L^\infty, L^1)$.

D. *The acceptance set* $\mathcal{A}_\rho$ *of* ρ *is weak* closed in* L^∞.

Moreover, if ρ *is coherent and satisfies any of the above conditions, then*

$$\rho(X) = \sup_{\mathbb{Q} \in \mathcal{M}} \mathbb{E}_\mathbb{Q}[-X]$$

where

$$\mathcal{M} := \{\mathbb{Q} \in \mathcal{Q} \mid \beta(\mathbb{Q}) = 0\}.$$

As we can see from the previous proposition, when a representation in terms of probability measures is used, we cannot guarantee that the supremum is attained. However, for coherent risk measures, it is possible to ensure that the supremum is attained, by requiring a stronger property.

PROPOSITION 2.5 (Föllmer and Schied [39]). *A coherent risk measure* $\rho \colon L^\infty \to \mathbb{R}$ *admits the representation*

$$\rho(X) = \max_{\mathbb{Q} \in \mathcal{M}'} \mathbb{E}_\mathbb{Q}[-X]$$

for some $\mathcal{M}' \subset \mathcal{Q}$*, if any of the following equivalent conditions is satisfied:*

A. *ρ is continuous from below: $X_n \uparrow X \implies \rho(X_n) \downarrow \rho(X)$.*

B. *The set $\mathcal{M}'$ representing ρ is such that the set of densities*

$$\mathcal{D} := \left\{ \frac{d\mathbb{Q}}{d\mathbb{P}} \;\middle|\; \mathbb{Q} \in \mathcal{M}' \right\}$$

is weakly compact in L^1.

See also Delbaen [28, 29] for more details.

2.2.3 *Some well-known risk measures*

We now recall the well-known definitions of VaR, CVaR and entropic risk measure (see, among many others, [5, 39, 49, 60]). Such risk measures are applied in the numerical example of Chapter 6. VaR and CVaR are also used throughout the work as examples.

Definition 2.9. The *Value at Risk (VaR)* of $X \in L^\infty$ at the level $\alpha \in (0,1)$ is defined as

$$\mathrm{VaR}_\alpha(X) := -\inf\{x \in \mathbb{R} \mid \mathbb{P}(X \le x) > \alpha\} = -q^+_\alpha(X),$$

where $q^+_\alpha(X)$ denotes the upper α-quantile of X.

Actually, VaR is defined for any $X \in L^0$. We provided the definition in term of $X \in L^\infty$ only for being coherent with our setting. VaR is cash-additive, monotone and positive homogeneous but it fails to be convex, hence, it does not encourage diversification; see Föllmer and Schied [39] for an example and more details on VaR.

A common risk measure which is close to VaR and overcome the lack of convexity is the following (see Föllmer and Schied [39] and Rockafellar and Uryasev [60]).

Definition 2.10. The *Conditional Value at Risk (CVaR)* of $X \in L^\infty$ at the level $\alpha \in (0,1)$ is defined as

$$\mathrm{CVaR}_\alpha(X) := \inf_{x\in\mathbb{R}} \left\{ \frac{\mathbb{E}\left[(x-X)^+\right]}{\alpha} - x \right\}.$$

Since the infimum in the previous definition is actually attained at any quantile $q_\alpha(X)$ at the level α of $X \in L^\infty$ (see [60]), CVaR can be equivalently formulated as

$$\mathrm{CVaR}_\alpha(X) = \frac{\mathbb{E}\left[(q_\alpha(X)-X)^+\right]}{\alpha} - q_\alpha(X).$$

Moreover, CVaR can be also expressed in terms of the *Average Value of Risk:*

$$\mathrm{CVaR}_\alpha(X) = \frac{1}{\alpha}\int_0^\alpha \mathrm{VaR}_\beta(X)\, d\beta,$$

sometimes called *Expected Shortfall (ES).*

Such a risk measure is a particular case of a larger class of risk measures, namely the *Haezendonck-Goovaerts (HG)* one, treated in detail in Chapter 3.

Differently from VaR, CVaR is a coherent risk measure; hence, in addition to other good properties, it encourages risk diversification.

While VaR can be seen as the maximal loss one can have with a given probability level α, CVaR at level α represents the average of losses exceeding VaR at the same level. So, by definition, the capital requirement evaluated by CVaR_α is always greater than or equal to that evaluated by VaR_α.

We now recall another well-known risk measure, which has the peculiarity of being convex but not positive homogeneous, hence not coherent.

DEFINITION 2.11. The *entropic risk measure* of $X \in L^\infty$ at the level $\alpha \in (0,1)$ is defined as

$$\mathrm{e}_\alpha(X) = \alpha \log \mathbb{E}\left[e^{-X/\alpha}\right]$$

where log denotes the natural logarithm.

The previous definition slightly differs from the common one, since here the parameter α is the reciprocal of the Arrow-Pratt coefficient of absolute risk aversion, while it is customary to let such parameter be exactly the Arrow-Pratt coefficient (see, among others, Föllmer and Schied [39]). That is, in our setting, when α is low the risk aversion is high and vice versa.

Such a risk measure is called entropic because it can be viewed as the maximal expected loss over a set of scenarios penalized by a term given by the relative entropy. That is, the representation in (2.2) holds with minimal penalty function given by the relative entropy of the probability measure appearing in the supremum, with respect to $\mathbb{P}$ (see Föllmer and Schied [39] and Frittelli and Rosazza Gianin [41]).

The reason why this risk measure is quite popular is that it is a convex risk measure fulfilling good properties in a dynamic setting (see, among others, Barrieu and Karoui [8] for details).

2.3 CAPITAL ALLOCATION

We now recall the standard capital allocation problem and the definition of capital allocation rule. We also present some well-known capital allocation methods, used throughout the thesis as examples. Some of them are also applied in the numerical example of Chapter 6.

As before, $X \in L^\infty$ represents the profit and loss of a financial position at a future date T. We henceforth assume that X is an aggregate position and that it is decomposed into sub-units (or business lines) $X_1, \ldots, X_n$ such that $X = \sum_{i=1}^n X_i$. For instance, X can be interpreted as the financial position of a company and the sub-units of X as its branches. Or, similarly, X can be interpreted as a financial portfolio and the sub-units of X as the assets componing the portfolio. We also assume that we are provided with a risk measure ρ, so that $\rho(X)$ represents the riskiness of X.

A *capital allocation problem* consists in finding "suitable" real numbers $k_1, \dots, k_n$ such that $\rho(X) = \sum_{i=1}^n k_i$. Each k_i is then the capital allocated to each sub-unit X_i, that is, the contribution of X_i to the total risk $\rho(X)$ of X.

Capital allocation problems are usually faced in a more general setting (see, for instance, [23, 50]), where the sub-units are not necessarily finite. Indeed, given a generic set of random variables $\mathcal{X}$, we say that $X \in \mathcal{X}$ is a sub-portfolio (or sub-unit) of $Y \in \mathcal{X}$ if there exists $Z \in \mathcal{X}$ such that $Y = X + Z$. Notice that every random variable is a sub-portfolio of any other, whenever $\mathcal{X}$ is a vector space. Since we set $\mathcal{X} = L^\infty$ for most of the work, we simply consider every pair of random variables as a pair of respectively a sub-portfolio and a portfolio.

The standard tool to achieve the purpose of capital allocation problems, that is, to allocate the risk among the sub-units, is commonly defined as follows.

Definition 2.12. Given a risk measure ρ, a *capital allocation rule (CAR)* with respect to ρ is a map $\Lambda_\rho \colon L^\infty \times L^\infty \to \mathbb{R}$ such that for all $X \in L^\infty$

$$\Lambda_\rho(X, X) = \rho(X).$$

We refer the reader to Centrone and Rosazza Gianin [23], Denault [30] and Kalkbrener [50] for more details. The dependence of ρ is usually omitted when it is clear which risk measure is involved.

Notice that, in the previous definition, the risk measure ρ is considered as a primary object. In Chapter 4, we will go beyond this approach.

In practice, it is customary to require that the whole capital has to be allocated, this property being termed in the literature as

Full allocation: for all $Y_1, \dots, Y_n, Y \in L^\infty$ such that $Y = \sum_{i=1}^n Y_i$,

$$\Lambda(Y, Y) = \sum_{i=1}^n \Lambda(Y_i, Y).$$

Thus, full allocation requires that the sum of the capital allocated to any sub-unit Y_i is equal to the capital allocated to the whole portfolio $Y = \sum_{i=1}^n Y_i$.

Although full allocation is largely required in practice and it is a reasonable property, we will see that in some situations it is not necessary to rely on it. Indeed, we provide in Chapter 4 a detailed discussion on full allocation.

The following properties (stronger than full allocation) are also sometimes used in the literature (see, for example, [50]):

Linear aggregation: for all $Y_1, \dots, Y_n$, $Y \in L^\infty$ and $a_1, \dots, a_n \in \mathbb{R}$ such that $Y = \sum_{i=1}^n a_i Y_i$,

$$\Lambda(Y, Y) = \sum_{i=1}^{n} a_i \Lambda(Y_i, Y).$$

Linearity: for all $X, Y, Z \in L^\infty$ and $a, b \in \mathbb{R}$,

$$\Lambda(aX + bZ, Y) = a\Lambda(X, Y) + b\Lambda(Z, Y).$$

Another desirable feature of a capital allocation rule, is that the capital k_i, allocated to each sub-unit X_i, does not exceed the capital requirement $\rho(X_i)$ of X_i, when considered as a stand-alone unit (*pooling effect*). Such idea is then extended to a general setting and expressed via the property of

No-undercut: $\Lambda(X, Y) \leq \Lambda(X, X)$, for all $X, Y \in L^\infty$.

The latter extends the idea of the pooling effect to any $X \in L^\infty$ viewed as a sub-portfolio of any $Y \in L^\infty$. That is, when Λ satisfies no-undercut, the capital allocated to any X considered as a sub-portfolio of Y does not exceed the capital allocated to X considered as a stand-alone portfolio. In other words, the risk contribution of X does not exceed its risk capital, when Λ is a CAR. Notice that no-undercut is also called no-split (see [64]) or diversification (see [50]). A further discussion on no-undercut will be provided later.

Finally, we recall the following property, which, together with full allocation and no-undercut, is required for a *fair* (also called *coherent*) allocation of risk capital (see Centrone and Rosazza Gianin [23], Denault [30] and Kalkbrener [50]):

Riskless: $\Lambda(a, Y) = -a$, for all $a \in \mathbb{R}$ and $Y \in L^\infty$.

Riskless is easy to understand: it means that the capital allocated to a fixed monetary amount is exactly the opposite of such fixed monetary amount. That is, if $a > 0$ then there is no need to allocate any risk capital to such sub-unit, it actually decreases the aggregated risk by a, so $-a$ should be allocated to the sub-unit. Instead, if $a < 0$ then such sub-unit increases the aggregated risk of $-a$, so the latter amount should be allocated to the sub-unit.

2.3.1 *On the no-undercut property*

It is well-known that the no-undercut property includes some game-theoretical features of stability, since it implies that a sub-portfolio

X has no incentive to be split from the whole portfolio Y, as staying alone would be more costly. Indeed, when the number of sub-portfolios is finite, Denault [30] provided results in such a way; we summarize them as follows.

We fix $X \in L^\infty$, set $N := \{1, \ldots, n\}$ for $n \in \mathbb{N}$ and consider sub-portfolios $X_i \in L^\infty$, $i \in N$, such that $X = \sum_{i \in N} X_i$. We also assume that we are provided with a coherent risk measure ρ and define

$$c(S) := \rho\left(\sum_{i \in S} X_i\right)$$

for all $S \subseteq N$. Then, given a linear CAR Λ with respect to ρ, we define

$$\Gamma := \begin{bmatrix} \Lambda(X_1, X) \\ \Lambda(X_2, X) \\ \vdots \\ \Lambda(X_n, X) \end{bmatrix}.$$

We recall that here the core of a game γ is given by

$$\mathcal{C}_\gamma := \left\{ K \in \mathbb{R}^n \;\middle|\; \sum_{i \in S} K_i \leq \gamma(S) \text{ for all } S \subseteq N \text{ and } \sum_{i \in N} K_i = \gamma(N) \right\}$$

see Denault [30] for more details.

We also recall that, in the setting above and for a linear CAR Λ, the no-undercut can be formulated as

$$\Lambda\left(\sum_{i \in S} X_i, X\right) \leq \rho\left(\sum_{i \in S} X_i\right)$$

for all $S \subseteq N$, see Denault [30]. We have therefore the following result.

Proposition 2.6 (see [30]). *Let Λ be a linear CAR with respect to ρ. Then, Λ satisfies no-undercut if and only if Γ belongs to the core of c.*

There are several ways to extend the discussion to the case where the number of sub-portfolios is not necessarily finite. We propose the following.

We recall that, for a normalized capacity ν, absolutely continuous with respect to $\mathbb{P}$, and $X \in L^\infty$,

$$\rho_\nu(X) := \mathrm{E}_\nu[-X]$$

defines a monetary, positive homogeneous risk measure. Moreover, ρ is also convex, hence coherent, whenever ν is sub-modular (see [39] for more details). See also Appendix A for more details on games and Choquet integrals.

A classical way is defining $\nu := f \circ \mathbb{P}$, where the function $f\colon [0,1] \to [0,1]$ satisfies $f(0) = 0$, $f(1) = 1$. Such a ν is called the *distortion* of the probability measure $\mathbb{P}$ with respect to the *distortion function* f. It is clear that ν is a normalized capacity, moreover if f is concave then ν is sub-modular, see [29].

When ν is also sub-modular, that is, when ρ_ν is coherent, the following representation holds:

$$\rho_\nu(X) = \max_{\mu \in \mathcal{C}_\nu} \mathrm{E}_\mu[-X],$$

see [29]. We henceforth assume that ν is sub-modular and thus that the previous representation holds.

Then, for every fixed $Y \in L^\infty$, we can choose $\mu_Y \in \mathrm{ba}_1$ and define

$$\Lambda(X, Y) := \mathrm{E}_{\mu_Y}[-X]$$

for all $X \in L^\infty$. We recall that here the core of a game γ is defined as

$$\mathcal{C}_\gamma := \{\mu \in \mathrm{ba}(\Omega, \mathcal{F}) \mid \mu(A) \leq \gamma(A), \text{ for all } A \in \mathcal{F},\ \mu(\Omega) = \gamma(\Omega)\}$$

where $\mathrm{ba}(\Omega, \mathcal{F})$ denotes the space of all finitely additive set functions (charges) $\mu\colon \mathcal{F} \to \mathbb{R}$ with finite variation norm (see Appendix A or Marinacci and Montrucchio [54] for more details). The following is therefore straightforward.

Proposition 2.7. *Λ satisfies the no-undercut if and only if $\mu_Y \in \mathcal{C}_\nu$, for all $Y \in L^\infty$. Moreover, Λ is a CAR with respect to ρ_ν if and only if $\mu_Y \in$* $\arg\max_{\mu \in \mathcal{C}_\nu} \mathrm{E}_\mu[-Y]$*, for all $Y \in L^\infty$.*

Proof. Fix $Y \in L^\infty$. If Λ satisfies no-undercut then

$$\mu_Y(A) = \Lambda(-\mathbb{1}_A, Y) \leq \rho_\nu(-\mathbb{1}_A) = \nu(A)$$

for every $A \in \mathcal{F}$. Since $\mu_Y \in \mathrm{ba}_1$ by construction, the thesis follows. Conversely, if $\mu_Y \in \mathcal{C}_\nu$ then

$$\Lambda(X, Y) = \mathrm{E}_{\mu_Y}[-X] \leq \max_{\mu \in \mathcal{C}_\nu} \mathrm{E}_\mu[-X] = \rho_\nu(X)$$

holds for all $X, Y \in L^\infty$; that is, no-undercut holds.

If Λ is a CAR with respect to ρ_ν then

$$\mathrm{E}_{\mu_Y}[-Y] = \Lambda(Y, Y) = \rho_\nu(Y) = \max_{\mu \in \mathcal{C}_\nu} \mathrm{E}_\mu[-Y]$$

thus μ_Y is a maximizer. Conversely, if $\mu_Y \in \arg\max_{\mu \in \mathcal{C}_\nu} \mathrm{E}_\mu[-Y]$ then

$$\Lambda(Y, Y) = \mathrm{E}_{\mu_Y}[-Y] = \max_{\mu \in \mathcal{C}_\nu} \mathrm{E}_\mu[-Y] = \rho_\nu(Y).$$

Since $Y \in L^\infty$ is arbitrary, the thesis follows. □

The previous proposition shows how the no-undercut property is equivalent to μ_Y belonging to the core of ν, thus it extends the result of Denault [30] to the case where the number of sub-portfolios is not necessarily finite.

We point out that an alternative way to prove the previous result is to use the monotonicity of the integral with respect to the game (see Appendix A or Denneberg [31]).

2.3.2 *Some well-known capital allocation methods*

We present here some of the most popular capital allocation principles, used as examples throughout the work. Some of them, namely the *proportional* and the *marginal* one, will be also illustrated in the numerical example of Chapter 6, for VaR, CVaR and for the entropic risk measure.

The first class of capital allocation methods we consider is the class of *proportional* ones (see Dhaene et al. [32]). The method consists in choosing a risk measure ρ and assigning the capital k_i to each sub-portfolio X_i, $i = 1, \dots, n$, via

$$k_i = \frac{\rho(X)}{\sum_{j=1}^{n} \rho(X_j)} \rho(X_i). \tag{2.3}$$

We point out that, by using a proportional allocation method, we get the desired pooling effect whenever the risk measure is such that $\rho(X_i) > 0$ and it is subadditive. The proportional method may be extended to a general framework, in term of capital allocation rules, as follows.

Definition 2.13. Given a monetary normalized risk measure ρ, the *proportional CAR* is given by

$$\Lambda^P_\rho(X, Y) := \frac{\rho(X)}{\rho(X) + \rho(Y - X)} \rho(Y), \quad X, Y \in L^\infty;$$

with the following convention:

$$\text{if } \rho(X) + \rho(Y - X) = 0 \text{ then } \Lambda^P_\rho(X, Y) := 0.$$

It is easy to check that Λ^P_ρ is actually a CAR in the sense of Definition 2.12, since $\Lambda^P_\rho(X, X) = \rho(X)$.

The second class of capital allocation methods we consider starts from the idea of measuring how much a single asset contributes to the total portfolio in terms of risk; that is, it aims at assessing *marginal contributions*. We present the following rule (see Tasche [62]) applied to a chosen risk measure ρ: the capital k_i is attributed to each sub-portfolio X_i, $i = 1, \dots, n$, via

$$k_i = \rho(X) - \rho(X - X_i), \tag{2.4}$$

that is, by the difference of the risk capital of the portfolio with sub-unit i and the risk capital of the portfolio without sub-unit i. Since the sum of marginal risk contributions underestimates the total risk, it is customary to use an adjusted formula, given by

$$k^*_i = \frac{\rho(X)}{\sum_{j=1}^{n} k_j} k_i, \tag{2.5}$$

which ensures to get full allocation. The method presented above is sometimes called the with-or-without allocation (see [62]) and the marginal contributions are sometimes defined in term of directional derivative:

$$MC_i := \lim_{h \to 0} \frac{\rho(X + hX_i) - \rho(X)}{h}.$$

However, we prefer to call marginal method the one given by (2.4), since the last formula above will be termed as *gradient* or *Euler allocation*, as properly explained in the following.

The marginal method (as in (2.4)) may be extended to a general framework, in term of capital allocation rule, as follows.

Definition 2.14. Given a monetary normalized risk measure ρ, the *marginal CAR* is given by

$$\Lambda_\rho^M(X, Y) = \rho(Y) - \rho(Y - X), \quad X, Y \in L^\infty.$$

Likewise the proportional method, it is easy to check that Λ_ρ^M is a CAR in the sense of Definition 2.12, since $\Lambda_\rho^M(X, X) = \rho(X)$.

These are just few of many possible capital allocation methods: we first present them mainly because they are very intuitive, easy to implement through a numerical example, and frequently used in the practice, also for performance measurement purposes. We point out that other very popular methods are inspired to cooperative game theory concepts and principles (Shapley value, the interested reader can see [30]).

In order to investigate how diversification impacts on capital allocation methods, we will also consider the diversification index. For $X = X_1 + \cdots + X_n$ and any risk measure ρ such that $\rho(X_i) > 0$, $i = 1, \ldots, n$, the diversification index is given by

$$\mathrm{DI}_\rho = \frac{\rho(X)}{\sum_{i=1}^n \rho(X_i)}. \tag{2.6}$$

The index shows how much a portfolio is diversified: when DI is close to 0, it means high diversification, when the index is close to 1 it means slight diversification. If the index is above 1 it means that the risk measure is not subadditive.

As a further example, we also introduce the following.

Definition 2.15. Given a risk measure ρ, the total *Return on Risk Adjusted Capital (RORAC)* is defined as

$$R_\rho(Y) := \frac{\mathbb{E}[Y]}{\rho(Y)}, \quad Y \in L^\infty;$$

with the following convention: if $\rho(Y) = 0$ then $R_\rho(Y) := 0$.

In our numerical study (see Chapter 6), we also investigate the contribution of the sub-portfolios to the total RORAC, that is, to the one of the aggregated position. More precisely, the contribution of each sub-portfolio X_i, $i = 1, \dots, n$, to the total RORAC is given by

$$R_i = \frac{\mathbb{E}[X_i]}{k_i}$$

where k_i, $i = 1, \dots, n$, can be obtained by using any capital allocation method. Since the sum of contributions is not equal to the total portfolio RORAC, and thus full allocation is not satisfied, it is customary to use an adjusted formula:

$$R_i^* = \frac{R}{\sum_{j=1}^n R_j} R_i$$

see [62] for details. We formalize the contribution to the total RORAC with the following definition.

DEFINITION 2.16. Given a CAR Λ, with respect to a risk measure ρ, the *RORAC contribution* is given by

$$\Lambda_R(X, Y) := \frac{\mathbb{E}[X]}{\Lambda(X, Y)}, \quad X, Y \in L^\infty;$$

with the following convention: if $\Lambda(X, Y) = 0$ then $\Lambda_R(X, Y) := 0$.

2.3.2.1 *On the gradient allocation*

We now recall the so-called *gradient allocation*, also named as *Euler allocation*, and we briefly summarize the work of Kalkbrener [50]. He showed that for any coherent risk measure there exists a CAR satisfying linearity and no-undercut (diversification in his notation).

More precisely, Kalkbrener [50] firstly shows that any coherent risk measure $\rho \colon \mathcal{X} \to \mathbb{R}$, defined on a linear subspace $\mathcal{X}$ of L^0, can be represented as

$$\rho(X) = \max_{h \in H_\rho} h(-X)$$

where H_ρ is the space of all linear functionals $h \colon \mathcal{X} \to \mathbb{R}$ such that $h(x) \leq \rho(X)$, for all $X \in \mathcal{X}$. Then, for $h_Y \in \arg\max_{h \in H_\rho} h(Y)$, he defines

$$\Lambda_\rho^K(X, Y) := h_Y(-X), \quad X \in L^\infty.$$

It is clear, by construction, that Λ_ρ^K is linear and satisfies no-undercut.

In our setting, since we consider risk measures on L^∞, equipped with the essential supremum norm $\|\cdot\|_\infty$, the functionals belonging to H_ρ are also continuous (see, for instance, Dunford and Schwartz [34, Lemma II.3.4]). Hence, the element of H_ρ are also element of the norm-dual of L^∞, i. e. the space ba. In such a way, we recover the representation of Proposition 2.3. Thus, one can choose a maximizer in the dual representation of Proposition 2.3 for defining a CAR with the same desirable properties as above.

Notice also that H_ρ is the subdifferential of ρ at 0 (see Appendix A or Aliprantis and Border [2]), this justify the terminology used in the following definition, which formalizes the discussion so far.

DEFINITION 2.17. For a coherent risk measure $\rho\colon L^\infty \to \mathbb{R}$, the *subdifferential CAR* is given by

$$\Lambda_\rho^\partial(X,Y) := \mathrm{E}_{\mu_Y^*}[-X], \quad X,Y \in L^\infty;$$

where μ_Y^* is a maximizer in the dual representation of $\rho(Y)$.

Notice that, Λ_ρ^∂ is a family of maps, whenever the maximizer in the dual representation of ρ is not unique.

It then follows that, when ρ is also continuous from below, one can choose a probability measure $\mathbb{Q}_Y^*$ as a maximizer in the dual representation of $\rho(Y)$ and obtain

$$\Lambda_\rho^\partial(X,Y) = \mathbb{E}_{\mathbb{Q}_Y^*}[-X] \quad X,Y \in L^\infty;$$

see also Delbaen [28].

Suppose now that the coherent risk measure ρ is also Gateaux differentiable at X, then its subdifferential reduces to a singleton with the unique element being the gradient of ρ at X (see Appendix A or [2]). This justify the following definition.

DEFINITION 2.18. Given a Gateaux differentiable coherent risk measure $\rho\colon L^\infty \to \mathbb{R}$, the *Euler* or *gradient CAR* is given by

$$\Lambda_\rho^\nabla(X,Y) := \lim_{h\to 0} \frac{\rho(Y+hX)-\rho(Y)}{h} = \mathbb{E}_{\mathbb{Q}_Y^*}[-X], \quad X,Y \in L^\infty;$$

where $\frac{d\mathbb{Q}_Y^*}{d\mathbb{P}}$ is the gradient of ρ at Y.

The gradient allocation is therefore linear (which implies full allocation), satisfies no-undercut and riskless, among others. See Kalkbrener [50] for more details.

We recall anyway that the *Euler method* is sometimes defined as a partial derivative with respect to the weight of an asset in a portfolio. It is called so since the full allocation is given by the validity of Euler's Theorem for coherent and differentiable risk measures (see [62]).

A more general class of capital allocation methods, namely the *Aumann-Shapley* one, has been proposed by Tsanakas [64] and later extended by Centrone and Rosazza Gianin [23] to risk measures that are not necessarily differentiable. Hence, such methods overcome the gradient allocation and represent a generalization.

We define here the method only for coherent risk measures, although it also works with convex and quasi-convex risk measures; see Centrone and Rosazza Gianin [23] for more details.

Definition 2.19. Given a coherent risk measure ρ, which is continuous from below, the *Aumann-Shapley CAR* is defined as

$$\Lambda_{\rho}^{\mathrm{AS}}(X,Y) := \int_0^1 \mathbb{E}_{\mathbb{Q}^*_{\gamma Y}}[-X]\,d\gamma, \quad X,Y \in L^\infty;$$

where $\mathbb{Q}^*_{\gamma Y} \in \arg\max_{\mathbb{Q}\in\mathcal{Q}} \mathbb{E}_{\mathbb{Q}}[\gamma Y]$, $\gamma \in [0,1]$.

As already pointed out by Centrone and Rosazza Gianin [23], $\Lambda^{\rho}_{\mathrm{AS}}$ is a family of maps whenever $\arg\max_{\mathbb{Q}\in\mathcal{Q}} \mathbb{E}_{\mathbb{Q}}[\gamma Y]$ is not a singleton. Instead, when the coherent risk measure ρ is also Gateaux differentiable, $\Lambda_{\rho}^{\mathrm{AS}}$ reduces to the gradient CAR.

3 HAEZENDONCK-GOOVAERTS CAPITAL ALLOCATION RULES

In this chapter, we deal with the problem of capital allocation w. r. t. to Haezendonck-Goovaerts (HG) risk measures [14, 47].

We generalize the capital allocation method introduced by Xun et al. [67] for Orlicz risk premia [48], in order to cover HG risk measures. We first present an approach based on Orlicz quantiles [15] and second a more general one based on the concept of linking function. We study the properties of the proposed CARs for HG risk measures, both in the quantile-based setting and in the linking one. Finally, we provide robust versions of the results, both considering the case of ambiguity over the probabilistic model and the one of multiple Young functions, as in [15].

The chapter is organized as follows: in Section 3.1 we briefly recall some known facts about Orlicz risk premia and HG risk measures; in Section 3.2 we present the capital allocation methods based on Orlicz quantiles; in Section 3.3 we introduce those based on linking functions. Sections 3.4 and 3.5 are instead devoted to the robust versions.

3.1 PRELIMINARIES

Throughout the chapter, we adopt the actuarial notation about signs, that is, positive values have to be interpreted as losses while negative ones as gains. Therefore, in this setting, the Value at Risk of X at level $\alpha \in (0,1)$ is simply the upper α-quantile of X, without changing the sign (compare with Definition 2.9). A similar argument holds for CVaR and for a general coherent risk measure ρ; indeed, here monotonicity means increasing monotonicity, i. e. if $Y \geq X$ implies $\rho(Y) \geq \rho(X)$, while cash-additivity corresponds to $\rho(X+m) = \rho(X) + m$ for all $m \in \mathbb{R}$ and $X \in L^\infty$.

Before recalling the definition and basic results of Orlicz risk premia and Haezendonck-Goovaerts risk measures, we give some preliminaries on Orlicz spaces; see Appendix A for more details or Rao and Ren [59] for a detailed treatment on Young functions and Orlicz spaces.

Let $\Phi\colon [0,+\infty) \to [0,+\infty)$ be a convex and strictly increasing function satisfying $\Phi(0) = 0$ and $\Phi(1) = 1$. Such Φ is called a (normalized) Young function. It follows that Φ is continuous and satisfies $\lim_{x\to+\infty} \Phi(x) = +\infty$.

Given a Young function Φ, the Orlicz space L^Φ is defined as

$$L^\Phi := \left\{ X \in L^0 \;\middle|\; \mathbb{E}\left[\Phi\left(\frac{|X|}{a}\right)\right] < +\infty \text{ for some } a > 0 \right\},$$

where $L^0 := L^0(\Omega, \mathcal{F}, \mathbb{P})$ denotes the space of all random variables on $(\Omega, \mathcal{F}, \mathbb{P})$.

For the Orlicz duality, we also recall the convex conjugate Ψ of Φ, defined as

$$\Psi(y) := \sup_{x \geq 0} \{xy - \Phi(x)\}, \quad y \geq 0.$$

3.1.1 *Orlicz premia and Haezendonck-Goovaerts risk measures*

We now recall the definition and basic results on Orlicz risk premia and Haezendonck-Goovaerts risk measures. See, for details, Bellini and Rosazza Gianin [14], Goovaerts et al. [47] and Haezendonck and Goovaerts [48].

Definition 3.1 (see [48]). Let a Young function Φ be given and let $\alpha \in [0, 1)$ be fixed. The *Orlicz risk premium* of $X \in L^\infty_+$, with $X \neq 0$, is the unique solution $H^\Phi_\alpha(X)$ of

$$\mathbb{E}\left[\Phi\left(\frac{X}{H^\Phi_\alpha(X)}\right)\right] = 1 - \alpha,$$

while, by convention, for $X = 0$, $H^\Phi_\alpha(0) := 0$.

For simplicity of notation, the dependence on Φ is usually omitted in $H^\Phi_\alpha(X)$, i. e. $H_\alpha := H^\Phi_\alpha$, and $H(X) := H_0(X)$.

As the Young function Φ can be seen as a loss function, H_α can be interpreted as a positively homogeneous extension of the certainty equivalent. Indeed, the certainty equivalent (also known as the mean value principle in the actuarial literature) is given by

$$C(X) := \Phi^{-1}(\mathbb{E}[\Phi(X)]), \quad X \in L^\infty,$$

and it is not positively homogeneous, in general. A way to obtain a positively homogeneous premium is to define it as the solution λ of the equation

$$C\left(\frac{X}{\lambda}\right) = 1.$$

Notice that λ coincides with $H(X)$. The generalization to the case $\alpha \neq 0$ comes from the observation that an insurer could sell a fraction $\alpha \in [0, 1]$ of the risk to a reinsurer at a proportional price $\alpha\lambda$. We refer to Bellini and Rosazza Gianin [14], Goovaerts et al. [47] and Haezendonck and Goovaerts [48] for a further discussion.

Notice that a more general definition of Orlicz premia on Orlicz spaces L^Φ has been given by Haezendonck and Goovaerts [48] by means of the Luxemburg norm $\|\cdot\|_\Phi$ (see Rao and Ren [59] for more details on this norm), that is

$$H(X) = \|X\|_\Phi := \inf\left\{k > 0 \;\middle|\; \mathbb{E}\left[\Phi\left(\frac{X}{k}\right)\right] \leq 1\right\}. \tag{3.1}$$

Then, H_α is simply given by (3.1) with $\Phi_\alpha := \frac{\Phi}{1-\alpha}$ instead of Φ.

Haezendonck and Goovaerts [48] proved also that H_α satisfies the following properties:

Monotonicity: if $X \geq Y$ (with $X, Y \in L^\infty_+$), then $H_\alpha(X) \geq H_\alpha(Y)$.

Subadditivity: $H_\alpha(X+Y) \leq H_\alpha(X) + H_\alpha(Y)$ for all $X, Y \in L^\infty_+$.

Positive homogeneity: $H_\alpha(\lambda X) = \lambda H_\alpha(X)$ for all $X \in L^\infty_+$, $\lambda \geq 0$.

Since Orlicz risk premia are defined only for $X \in L^\infty_+$ and fail, in general, to be cash-additive, Haezendonck-Goovaerts risk measures were introduced by Haezendonck and Goovaerts [48] (see also Bellini and Rosazza Gianin [14] and Goovaerts et al. [47]) to extend Orlicz risk premia so to obtain cash-additive risk measures defined on the whole L^∞.

We present here the formulation introduced by Bellini and Rosazza Gianin [14].

Definition 3.2 (see [14]). Let $\alpha \in [0,1)$. The *Haezendonck-Goovaerts risk measure* of $X \in L^\infty$ is defined by

$$\pi_\alpha(X) := \inf_{x\in\mathbb{R}} \pi_\alpha(X, x) \tag{3.2}$$

where

$$\pi_\alpha(X, x) := x + H_\alpha\big((X-x)^+\big).$$

Some properties of π_α were then proved in Bellini and Rosazza Gianin [14] (see also Gao et al. [44]). In particular, $\pi_\alpha(X)$ defines a coherent risk measure in our setting.

Proposition 3.1 (Bellini and Rosazza Gianin [15]). *For any $\alpha \in (0,1)$ and $X \in L^\infty$, the infimum in the definition of the Haezendonck-Goovaerts risk measure $\pi_\alpha(X)$ is attained at some x^*_X. Moreover, π_α admits the following representation:*

$$\pi_\alpha(X) = \max_{\mathbb{Q}\in\mathcal{Q}} \mathbb{E}_\mathbb{Q}[X]$$

where $\mathcal{Q}$ is a subset of $\mathcal{D}^\Psi := \left\{\eta \in L^\Psi_+ \;\middle|\; \mathbb{E}[\eta] = 1\right\}$ and Ψ is the convex conjugate of Φ.

If Φ is also differentiable and $H_\alpha\big((X-\cdot)^+\big)$ is differentiable at x^*_X, then

$$\pi_\alpha(X) = \mathbb{E}_{\mathbb{Q}_X}[X]$$

where

$$\frac{d\mathbb{Q}_X}{d\mathbb{P}} = \frac{\Phi'\left(\frac{(X-x^*_X)^+}{\|(X-x^*_X)^+\|_{\Phi_\alpha}}\right)\mathbb{1}_{\{X>x^*_X\}}}{\mathbb{E}\left[\Phi'\left(\frac{(X-x^*_X)^+}{\|(X-x^*_X)^+\|_{\Phi_\alpha}}\right)\mathbb{1}_{\{X>x^*_X\}}\right]}.$$

3.1.2 *Capital allocation*

As in Chapter 2, a CAR for a monetary risk measure ρ is a map $\Lambda\colon L^\infty \times L^\infty \to \mathbb{R}$ such that $\Lambda(X,X) = \rho(X)$ for all $X \in L^\infty$. However, Brunnermeier and Cheridito [18] pointed out that the equality $\Lambda(X,X) = \rho(X)$ might be not indispensable in some cases, for example when the capital is collected for monitoring purpose. For that reason, the requirement $\Lambda(X,X) = \rho(X)$ can be replaced by $\Lambda(X,X) \leq \rho(X)$ for all $X \in L^\infty$ (in that case, Λ will be called *audacious CAR*) or by $\Lambda(X,X) \geq \rho(X)$ for all $X \in L^\infty$ (*prudential CAR*). See also [23] for a further discussion.

Notice that, throughout the chapter and with an abuse of notation, we still call CAR a map defined on a restricted domain D of $L^\infty \times L^\infty$ (e. g. on $L^\infty_+ \times L^\infty$) and satisfying $\Lambda(X,X) = \rho(X)$ for the corresponding ρ restricted to $X \in D$.

We now recall some standard properties for a CAR which we are going to study throughout the chapter. First of all, we will study the no-undercut property, widely discussed in Chapter 2, and the riskless one (see again Chapter 2). We will also investigate if the presented CARs are monotone increasing in the first variable, that is, we investigate the following property, for all $X, Y, Z \in L^\infty$:

Monotonicity: if $X \geq Z$ then $\Lambda(X,Y) \geq \Lambda(Z,Y)$.

Thus, monotonicity means that the capital allocated to a position with a higher loss has to be greater or equal than the capital allocated to another position with a lower loss.

It is also clear, by definition of capital allocation rule, that a CAR Λ inherits some properties from the underlying risk measure ρ. In particular, if ρ is coherent risk measure then Λ satisfies $\Lambda(X+c, X+c) = \Lambda(X,X) + c$ and $\Lambda(\lambda X, \lambda X) = \lambda \Lambda(X,X)$ for all $c \in \mathbb{R}$, $\lambda \geq 0$, $X \in L^\infty$. It is interesting to study if Λ preserves those properties also for pairs (X,Y), with $X \neq Y$. That is, throughout the chapter we also consider the following properties, for all $X, Y \in L^\infty$:

Cash-additivity: $\Lambda(X+c, Y+c) = \Lambda(X,Y) + c$ for all $c \in \mathbb{R}$.

Positive homogeneity: $\Lambda(\lambda X, \lambda Y) = \lambda \Lambda(X,Y)$ for all $\lambda \geq 0$.

Cash-additivity requires that, whenever we add any cash amount to both the sub-portfolio X and the portfolio Y, the capital allocated to such a pair is exactly that allocated to the pair (X,Y) plus the cash amount. Positive homogeneity requires that the capital allocated to a pair of sub-portfolio and portfolio formed by λ shares of both X and Y is exactly λ times the capital allocated to (X,Y).

Some among the following further properties on Λ can be also required for all $X, Y, Z \in L^\infty$ (see Chapter 4 for details):

1-cash-additivity: $\Lambda(X+c, Y) = \Lambda(X,Y) + c$ for all $c \in \mathbb{R}$.

1-law invariance: if $X \sim Z$ then $\Lambda(X,Y) = \Lambda(Z,Y)$.

1-positive homogeneity: $\Lambda(\lambda X,Y) = \lambda\Lambda(X,Y)$ for all $\lambda \geq 0$.

2-translation-invariance: $\Lambda(X,Y+c) = \Lambda(X,Y)$ for all $c \in \mathbb{R}$.

1-cash-additivity and 1-positive homogeneity have similar interpretations to cash-additivity and positive homogeneity even if, in the present case, they have only impact on the first variable of Λ, i. e. on the sub-portfolio. 1-law invariance requires that the capital allocated to any couple of sub-portfolios with the same distribution is equal. 2-translation-invariance means that the capital allocated to X considered as a sub-portfolio of Y is exactly that one of X viewed as a sub-portfolio of any translation of Y; that is, no matter if we add or remove any cash amount to the portfolio Y, the capital allocated to the sub-portfolios does not change.

As regards the full allocation property (see Chapter 2), we recall that it is always possible to modify a capital allocation rule, as in Dhaene et al. [32], so to guarantee the full allocation property. By defining the risk contribution of each X_i as

$$\widehat{\Lambda}(X_i,X) := \frac{\rho(X)}{\sum_{j=1}^{n} \Lambda(X_j,X)} \Lambda(X_i,X),$$

indeed, $\widehat{\Lambda}$ satisfies the full allocation property.

Another possibility to obtain full allocation is to shift the capital allocation by a suitable exogenous amount (see [18]). That is, one can find suitable c_i, $i = 1,\dots,n$, such that $\sum_{i=1}^{n} \left(\Lambda(Y_i,Y) - c_i\right) = \rho(Y)$ (see [18, 51]).

One possible motivation for the failure of full allocation, already taken into account when introducing prudential and audacious capital allocation rules, is that $\Lambda(X,X) \neq \rho(X)$, although the capital allocation rule is possibly linear in the first variable (see [18]). Another possible reason for a CAR not to satisfy full allocation is instead the lack of linearity in the first variable. In a general framework and for general risk measures, therefore, weaker assumptions than full allocation can turn out to be more suitable for capital allocation problems (see the discussion in Centrone and Rosazza Gianin [23]).

However, we remind to the discussion on full allocation in Chapter 4 for further details.

For the convention about signs explained above, we recall that, for a Gateaux differentiable coherent risk measure ρ, the gradient allocation in this context is given by

$$\Lambda_{\rho}^{\nabla}(X,Y) := \mathbb{E}_{\mathbb{Q}_Y}[X], \tag{3.3}$$

where $\frac{d\mathbb{Q}_Y}{d\mathbb{P}}$ is the gradient of ρ at Y.

Similarly, for a coherent risk measure $\rho(Y) = \sup_{\mathbb{Q}\in\mathcal{Q}} \mathbb{E}_{\mathbb{Q}}[Y]$ whose supremum is attained at some $\mathbb{Q}_Y$, the Aumann-Shapley CAR is given by

$$\Lambda_\rho^{\mathrm{AS}}(X,Y) := \int_0^1 \mathbb{E}_{\mathbb{Q}_{\gamma Y}}[X]\, d\gamma.$$

Quite recently, Xun et al. [67] introduced the following capital allocation rule for Orlicz risk premia. In particular, they defined the risk contribution $H_{Y,\alpha}(X)$ of X as a sub-portfolio of Y as the solution of[1]

$$\mathbb{E}\left[\Phi\left(\frac{X\,\mathbb{1}_{\{Y>\mathrm{VaR}_\alpha(Y)\}}}{H_{Y,\alpha}(X)}\right)\right] = 1-\alpha \tag{3.4}$$

where $\alpha \in [0,1)$. Such definition reduces to the so-called "contribution to shortfall", proposed by Overbeck [58] and given by

$$\mathrm{CS}_\alpha(X,Y) := \mathbb{E}[X \mid Y > \mathrm{VaR}_\alpha(Y)], \tag{3.5}$$

for $\Phi(x) = x$ and for continuous random variables Y. The fact of considering the loss Y only in the event $\{Y > \mathrm{VaR}_\alpha(Y)\}$ comes from the capital allocation approaches based on the insurer's default option. More precisely, such approaches only consider the problem of allocating the loss over a certain threshold ($\mathrm{VaR}_\alpha(Y)$ in this case), motivating it by the fact that the shareholders of a company have limited liability and therefore, in the event of default, they are not obliged to pay when the loss exceeds such fixed threshold. See, for more details, Dhaene et al. [32] and Myers and Read [57].

Therefore, in the present chapter we generalize the CAR proposed by Xun et al. [67] for Orlicz premia in three directions. First, in Section 3.2 we will provide a CAR both for Orlicz risk premia H_α and for HG risk measures π_α not only in terms of VaR_α as in (3.4) but also of Orlicz quantiles (defined in [15]) that seem to be more appropriate for general Φ. Second, in Section 3.4 we extend Xun et al. [67]'s approach and ours so to cover ambiguity over $\mathbb{P}$ or over Φ. Third, a new and more general approach by means of linking functions is also proposed both in the classical and in the robust cases (see Section 3.3 and Section 3.5).

3.2 CAPITAL ALLOCATION VIA ORLICZ QUANTILES

In this section we are going to provide some capital allocation methods both for the Orlicz risk premium and for the HG risk measure by means of the so called Orlicz quantiles introduced by Bellini and Rosazza Gianin [15] (see also Bellini et al. [12]).

1 Differently from the present work, in [67] VaR_α is defined as the lower α-quantile. This different definition, however, is irrelevant for the study.

We recall that for any $\alpha \in (0,1)$ the infimum in Equation (3.2) defining π_α is always attained at some point x^*_α (see Bellini and Rosazza Gianin [14]). Thus, π_α can be written as

$$\pi_\alpha(X) = x^*_\alpha(X) + H_\alpha\big((X - x^*_\alpha(X))^+\big) \tag{3.6}$$

where

$$x^*_\alpha(X) \in \arg\min_{x\in\mathbb{R}} \left\{x + H_\alpha\big((X-x)^+\big)\right\}$$

is called an Orlicz quantile (see Bellini and Rosazza Gianin [15] for details).

Furthermore, in [15] the authors claimed the uniqueness of an Orlicz quantile under the hypothesis of Φ being strictly convex. Unfortunately, we have recently realized (see [22]) that such result does not hold without some additional hypotheses (see also the example below). Therefore, we correct and replace Proposition 3 (c-d) of [15] with Proposition 3.2, whose proof is similar to those of [15, Prop. 3, 11] and of [12, Propp. 1, 5].

Example 3.1. Consider the Orlicz function $\Phi(x) = x^2$. It follows that

$$H_\alpha(X) = \sqrt{\frac{\mathbb{E}[X^2]}{1-\alpha}}, \quad \pi_\alpha(X,x) = x + \sqrt{\frac{\mathbb{E}[((X-x)^+)^2]}{1-\alpha}}.$$

Consider now the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with $\Omega = \{\omega_1, \omega_2\}$, $\mathcal{F} = \mathcal{P}(\Omega)$ and $\mathbb{P}(\{\omega_i\}) = \frac{1}{2}$, $i = 1,2$, and the random variable

$$X = \begin{cases} 4 & \text{on } \omega_1; \\ 8 & \text{on } \omega_2. \end{cases}$$

Hence,

$$\begin{aligned} \pi_\alpha(X,x) &= x + \sqrt{\frac{\mathbb{E}[((X-x)^+)^2]}{1-\alpha}} \\ &= \begin{cases} x + \sqrt{\frac{(4-x)^2+(8-x)^2}{2(1-\alpha)}} & \text{if } x \leq 4; \\ x + \frac{8-x}{\sqrt{2(1-\alpha)}} & \text{if } 4 < x \leq 8; \\ x & \text{if } x > 8. \end{cases} \end{aligned}$$

It follows easily that $\pi_\alpha(X,x)$ is linear in the interval $(4,8]$ (thus, not strictly convex); it is even constant for $\alpha = \frac{1}{2}$. Furthermore, for $\alpha = \frac{1}{2}$ the minimum is not unique but attained at any point of the interval $[4,8]$, where $\pi_\alpha(X,x) \equiv 8$.

It is clear that such example may be extended to more general discrete random variables and also to any $\Phi(x) = x^p$, with $p \in (1, +\infty)$.

In the furthering, we will use the notations

$$x^{*,-}_\alpha(X) := \inf \operatorname*{arg\,min}_{x\in\mathbb{R}} \pi_\alpha(X,x),$$
$$x^{*,+}_\alpha(X) := \sup \operatorname*{arg\,min}_{x\in\mathbb{R}} \pi_\alpha(X,x).$$

PROPOSITION 3.2. *For any $\alpha \in (0,1)$ and $X \in L^\infty$, the set of minimizers is a closed interval, that is*

$$\operatorname*{arg\,min}_{x\in\mathbb{R}} \pi_\alpha(X,x) = \left[x^{*,-}_\alpha(X),\, x^{*,+}_\alpha(X)\right].$$

Moreover, it satisfies the following properties:

Cash-additivity: for all $h \in \mathbb{R}$, $X \in L^\infty$,

$$\left[x^{*,-}_\alpha(X+h),\, x^{*,+}_\alpha(X+h)\right] = \left[x^{*,-}_\alpha(X)+h,\, x^{*,+}_\alpha(X)+h\right].$$

Positive homogeneity: for all $\lambda \geq 0$, $X \in L^\infty$,

$$\left[x^{*,-}_\alpha(\lambda X),\, x^{*,+}_\alpha(\lambda X)\right] = \left[\lambda x^{*,-}_\alpha(X),\, \lambda x^{*,+}_\alpha(X)\right].$$

Riskless: if $X = a \in \mathbb{R}$ then $x^{*,-}_\alpha(X) = x^{*,+}_\alpha(X) = a$.

Boundedness from above: $x^{*,+}_\alpha(X) \leq \operatorname{ess\,sup}(X)$ for all $X \in L^\infty$.

PROOF. Since, for any $\alpha \in (0,1)$ and $X \in L^\infty$, $\pi_\alpha(X,x)$ is finite, convex and $\lim_{x\to\pm\infty} \pi_\alpha(X,x) = +\infty$, from [15, Prop. 3 (a-b)] it follows that the set of minimizers is a closed interval.

Cash-additivity: for any $h \in \mathbb{R}$ and $X \in L^\infty$ it holds that

$$\begin{aligned}\left[x^{*,-}_\alpha(X+h),\, x^{*,+}_\alpha(X+h)\right] &= \operatorname*{arg\,min}_{x\in\mathbb{R}} \pi_\alpha(X+h,x)\\ &= \operatorname*{arg\,min}_{x\in\mathbb{R}} \{\pi_\alpha(X,x-h)+h\}\\ &= \operatorname*{arg\,min}_{x\in\mathbb{R}} \{\pi_\alpha(X,x-h)\}\\ &= \operatorname*{arg\,min}_{x\in\mathbb{R}} \pi_\alpha(X,x)+h\\ &= \left[x^{*,-}_\alpha(X)+h,\, x^{*,+}_\alpha(X)+h\right].\end{aligned}$$

Law invariance: it follows immediately by law invariance of π_α.

Positive homogeneity: take any $\lambda > 0$ and $X \in L^\infty$. Since, by positive homogeneity of H_α,

$$\pi_\alpha(\lambda X,x) = x + H_\alpha\big((\lambda X - x)^+\big) = \lambda \pi_\alpha\Big(X, \frac{x}{\lambda}\Big),$$

it follows that

$$\begin{aligned}\left[x_\alpha^{*,-}(\lambda X),\, x_\alpha^{*,+}(\lambda X)\right] &= \arg\min_{x\in\mathbb{R}} \pi_\alpha(\lambda X, x)\\ &= \arg\min_{x\in\mathbb{R}} \left\{\lambda \pi_\alpha\left(X, \frac{x}{\lambda}\right)\right\}\\ &= \arg\min_{x\in\mathbb{R}} \left\{\pi_\alpha\left(X, \frac{x}{\lambda}\right)\right\}\\ &= \lambda \arg\min_{x\in\mathbb{R}} \pi_\alpha(X, x)\\ &= \left[\lambda x_\alpha^{*,-}(X),\, \lambda x_\alpha^{*,+}(X)\right]\end{aligned}$$

The case $\lambda = 0$ follows by riskless (proved here below).

Riskless: since

$$\pi_\alpha(a) = \inf_{x\in\mathbb{R}} \{x + H_\alpha((a-x)^+)\} = \inf_{x\in\mathbb{R}} \left\{x + \frac{(a-x)^+}{\Phi^{-1}(1-\alpha)}\right\}$$

for any $a \in \mathbb{R}$ and $\Phi^{-1}(1-\alpha) < 1$ for any $\alpha \neq 0$, it follows that the unique minimizer is $x_\alpha^*(a) = a$.

Boundedness from above: notice that $Y := X - \operatorname{ess\,sup}(X) \leq 0$, thus

$$0 \geq \pi_\alpha(Y) = x_\alpha^{*,+}(Y) + H_\alpha\left(\left(Y - x_\alpha^{*,+}(Y)\right)^+\right) \geq x_\alpha^{*,+}(Y),$$

by monotonicity of π_α. Therefore, $x_\alpha^{*,+}(X) \leq \operatorname{ess\,sup}(X)$ follows by cash-additivity. □

Notice that Orlicz quantiles fail to be monotone, as they fail to be bounded from below; furthermore, for $\Phi(x) = x$, the upper Orlicz quantile $x_\alpha^{*,+}$ is exactly the Value at Risk at level α, thus it corresponds to a given loss threshold with probability α. Unfortunately, the latter interpretation is lost when we consider a general Young function Φ (see [15] for details).

Going back to capital allocation rules, while the use of VaR_α in the CAR (3.4) proposed by Xun et al. [67] can be justified for $\Phi(x) = x$ by the arguments above on Orlicz quantiles, this is no more the case for general Φ. Roughly speaking, VaR_α can be seen as the "right quantile" for $\Phi(x) = x$ while it is not for a general Φ, where the use of Orlicz quantiles seems to be more appropriate. Motivated by this, we aim at generalizing the CAR of [67] in two directions: first, by replacing VaR_α with an Orlicz quantile in Equation (3.4); second, by defining a CAR for HG risk measures π_α starting from that for Orlicz risk premia H_α so to obtain a CAR defined for any pair $(X, Y) \in L^\infty \times L^\infty$ and not only for X in L^∞_+.

3.2.1 *Capital allocation rules for H_α*

We now generalize the approach of Xun et al. [67] at the level of H_α by means of Orlicz quantiles.

Definition 3.3. Given the Orlicz risk premium H_α, we define the map $\Lambda_H\colon L^\infty_+ \times L^\infty \to \mathbb{R}$ as

$$\Lambda_H(X,Y) := H_\alpha\left(X\mathbb{1}_{\{Y\geq x^*_\alpha(Y)\}}\right) \tag{3.7}$$

where $x^*_\alpha(Y)$ is an Orlicz quantile at level α of Y.

Differently from (3.4), in our definition we consider the loss Y on the event $\{Y \geq x^*_\alpha(Y)\}$ (instead of $\{Y > \mathrm{VaR}_\alpha(Y)\}$). In addition to the use of Orlicz quantiles instead of VaR_α, the choice of considering $\{Y \geq x^*_\alpha(Y)\}$ (instead of a strict inequality) has been done to avoid that, for constant Y, $\Lambda_H(X,Y) = 0$ for all $X \in L^\infty_+$.

Notice that the definition of Λ_H depends on the choice of the Orlicz quantile, thus, roughly speaking, Λ_H can be seen as family of CARs "parameterized" by the Orlicz quantile chosen.

In the following, $x^*_\alpha(Y)$ (used in (3.7)) will be fixed to be the upper Orlicz quantile; that is, we set $x^*_\alpha(Y) := x^{*,+}_\alpha(Y)$. Similar arguments would hold whether the lower Orlicz quantile was fixed.

Notice that $\Lambda_H\colon L^\infty_+ \times L^\infty \to \mathbb{R}^+$. Moreover, for $X \neq 0$ and $\mathbb{P}(Y \geq x^*_\alpha(Y)) > 0$, $\Lambda_H(X,Y)$ is the unique solution of

$$\mathbb{E}\left[\Phi\left(\frac{X\mathbb{1}_{\{Y\geq x^*_\alpha(Y)\}}}{\Lambda_H(X,Y)}\right)\right] = 1-\alpha.$$

Condition $\mathbb{P}(Y \geq x^*_\alpha(Y)) > 0$ is quite commonly satisfied since, by the properties of Orlicz quantiles, $x^*_\alpha(Y) \leq \operatorname{ess\,sup}(Y)$ (see Proposition 3.2). When $\Phi(x) = x$ and Y is a continuous random variable, for instance, $\mathbb{P}(Y \geq \mathrm{VaR}_\alpha(Y)) > 0$ is guaranteed by definition of VaR_α.

As for Orlicz premia, it holds also that

$$\Lambda_H(X,Y) = \inf\left\{k>0 \;\middle|\; \mathbb{E}\left[\Phi\left(\frac{X\mathbb{1}_{\{Y\geq x^*_\alpha(Y)\}}}{k}\right)\right] \leq 1-\alpha\right\}.$$

In the following result we list the main properties satisfied by Λ_H.

Proposition 3.3. *The map Λ_H defined in (3.7) is an audacious CAR for H_α satisfying: no-undercut with respect to H_α (that is, $\Lambda_H(X,Y) \leq H_\alpha(X)$ for all $X \in L^\infty_+, Y \in L^\infty$), monotonicity, 1-law invariance, 1-positive homogeneity and 2-translation-invariance. Moreover, the following holds:*

$$\Lambda_H(a,Y) = a\left(\Phi^{-1}\left(\frac{1-\alpha}{\mathbb{P}(Y\geq x^*_\alpha(Y))}\right)\right)^{-1} \quad \text{for all } a \geq 0,\ Y \in L^\infty.$$

Proof. Audacious CAR, no-undercut and monotonicity follow easily by monotonicity of H_α, while 1-law invariance and 1-positive homogeneity follow from the corresponding properties of H_α.

2-translation-invariance follows because, by cash-additivity of the Orlicz quantile, $\{Y + c \geq x^*_\alpha(Y+c)\} = \{Y \geq x^*_\alpha(Y)\}$ for all $c \in \mathbb{R}$.

As regards the last statement, notice that

$$\Lambda_H(a,Y) = H_\alpha\left(a\mathbb{1}_{\{Y\geq x^*_\alpha(Y)\}}\right) = aH_\alpha\left(\mathbb{1}_{\{Y\geq x^*_\alpha(Y)\}}\right)$$

holds for any $a \geq 0$ and $Y \in L^\infty$ by positive homogeneity of H_α. By taking $A := \{Y \geq x^*_\alpha(Y)\}$, it follows that

$$\begin{aligned}
H_\alpha(\mathbb{1}_A) &= \inf\left\{k > 0 \,\middle|\, \mathbb{E}\left[\Phi\left(\frac{\mathbb{1}_A}{k}\right)\right] \leq 1-\alpha\right\} \\
&= \inf\left\{k > 0 \,\middle|\, \mathbb{E}\left[\Phi\left(\frac{1}{k}\right)\mathbb{1}_A\right] \leq 1-\alpha\right\} \\
&= \inf\left\{k > 0 \,\middle|\, \frac{1}{k} \leq \Phi^{-1}\left(\frac{1-\alpha}{\mathbb{P}(A)}\right)\right\} \\
&= \left(\Phi^{-1}\left(\frac{1-\alpha}{\mathbb{P}(A)}\right)\right)^{-1}
\end{aligned}$$

where the second equality holds because Φ is a Young function while the third one by strict monotonicity of Φ. □

Notice that, for $\Phi(x) = x$ and for Y with $\mathbb{P}(Y = x^*_\alpha(Y)) = 0$, Λ_H reduces to the CAR proposed by Xun et al. [67] since $x^*_\alpha = x^{*,+}_\alpha = \mathrm{VaR}_\alpha$. For $\Phi(x) = x$ and a general Y, however, the two definitions are very similar.

3.2.2 *Different capital allocation rules for π_α*

So far, we have generalized the CAR given by Xun et al. [67] for H_α, by using Orlicz quantiles. In the following, we will propose different CARs for HG risk measures π_α and not only for Orlicz risk premia H_α, again by means of Orlicz quantiles. A comparison among the different CARs here proposed and the classical ones will be also provided.

Starting from the CAR proposed for H_α in (3.7) and from (3.6), we propose the following CAR for π_α.

Definition 3.4. Given π_α, we define the map $\Lambda_\pi \colon L^\infty \times L^\infty \to \mathbb{R}$ as

$$\begin{aligned}
\Lambda_\pi(X,Y) &:= x^*_\alpha(X) + \Lambda_H\big((X - x^*_\alpha(X))^+, Y\big) \\
&= x^*_\alpha(X) + H_\alpha\big((X - x^*_\alpha(X))^+ \mathbb{1}_{\{Y \geq x^*_\alpha(Y)\}}\big)
\end{aligned} \tag{3.8}$$

where $x^*_\alpha(X)$ (resp. $x^*_\alpha(Y)$) is an Orlicz quantile at level α of X (resp. of Y).

Notice that in the previous definition we make use of two different Orlicz quantiles: one for X, the other for Y. The second one plays the role of a threshold for the whole portfolio Y. As for Λ_H, also the definition of Λ_π depends on the choice of the two Orlicz quantiles $x^*_\alpha(X)$ and $x^*_\alpha(Y)$, hence Λ_H can be seen as a family of CARs.

In the following, we will again fix $x^*_\alpha(Z)$ (used in (3.8)) to be the upper Orlicz quantile of Z. Similar arguments would hold whether the lower Orlicz quantiles were fixed.

EXAMPLE 3.2. Consider the case where $\Phi(x) = x$. Then Λ_π reduces to

$$\begin{aligned}\Lambda_\pi(X,Y) &= q_\alpha(X) + \frac{\mathbb{E}\left[(X-q_\alpha(X))^+ \mathbb{1}_{\{Y\geq q_\alpha(Y)\}}\right]}{1-\alpha} \\ &= q_\alpha(X) + \frac{\mathbb{P}(A_{X,Y})}{1-\alpha}\,\mathbb{E}[X-q_\alpha(X) \mid A_{X,Y}]\end{aligned}$$

where $A_{X,Y} := \{X \geq q_\alpha(X), Y \geq q_\alpha(Y)\}$ and $q_\alpha(X)$ (resp. $q_\alpha(Y)$) is the upper α-quantile of X (resp. Y). If X and Y are independent and continuous then $\mathbb{P}(X \geq q_\alpha(X), Y \geq q_\alpha(Y)) = \mathbb{P}(X \geq q_\alpha(X))\mathbb{P}(Y \geq q_\alpha(Y)) = (1-\alpha)^2$, therefore

$$\begin{aligned}\Lambda_\pi(X,Y) &= q_\alpha(X) + (1-\alpha)\mathbb{E}[X-q_\alpha(X) \mid X \geq q_\alpha(X), Y \geq q_\alpha(Y)] \\ &= \alpha\,\mathrm{VaR}_\alpha(X) + (1-\alpha)\mathbb{E}[X \mid X \geq \mathrm{VaR}_\alpha(X), Y \geq \mathrm{VaR}_\alpha(Y)].\end{aligned}$$

In other words, Λ_π is a convex combination of $\mathrm{VaR}_\alpha(X)$ and of a term that is somehow related to the contribution to shortfall (3.5) but taking into account also $\mathrm{VaR}_\alpha(X)$.

PROPOSITION 3.4. *The map Λ_π defined in (3.8) is a CAR for π_α. Furthermore, it satisfies no-undercut, riskless, 1-cash-additivity, 1-law invariance, 1-positive homogeneity, 2-translation-invariance and cash-additivity.*

PROOF. We start to prove that Λ_π is a CAR with respect to π_α. For any $X \in L^\infty$, indeed,

$$\begin{aligned}\Lambda_\pi(X,X) &= x^*_\alpha(X) + H_\alpha\big((X-x^*_\alpha(X))^+ \mathbb{1}_{\{X\geq x^*_\alpha(X)\}}\big) \\ &= x^*_\alpha(X) + H_\alpha\big((X-x^*_\alpha(X))^+\big) \\ &= \pi_\alpha(X).\end{aligned}$$

No-undercut: by monotonicity of H_α, it follows that for all $X, Y \in L^\infty$

$$\begin{aligned}\Lambda_\pi(X,Y) &= x^*_\alpha(X) + H_\alpha\big((X-x^*_\alpha(X))^+ \mathbb{1}_{\{Y\geq x^*_\alpha(Y)\}}\big) \\ &\leq x^*_\alpha(X) + H_\alpha\big((X-x^*_\alpha(X))^+\big) \\ &= \pi_\alpha(X).\end{aligned}$$

Riskless: it follows immediately by riskless of Orlicz quantiles. Indeed, for any $a \in \mathbb{R}$ and $Y \in L^\infty$,

$$\Lambda_\pi(a,Y) = x^*_\alpha(a) + H_\alpha\big((a-x^*_\alpha(a))^+ \mathbb{1}_{\{Y\geq x^*_\alpha(Y)\}}\big) = x^*_\alpha(a) = a.$$

1-cash-additivity: by cash-additivity of Orlicz quantiles, it holds that for any $c \in \mathbb{R}$ and $X, Y \in L^\infty$

$$\begin{aligned}\Lambda_\pi(X+c,Y) &= x^*_\alpha(X+c) + H_\alpha\big((X+c-x^*_\alpha(X+c))^+ \mathbb{1}_{\{Y\geq x^*_\alpha(Y)\}}\big) \\ &= x^*_\alpha(X) + c + H_\alpha\big((X-x^*_\alpha(X))^+ \mathbb{1}_{\{Y\geq x^*_\alpha(Y)\}}\big) \\ &= \Lambda_\pi(X,Y) + c.\end{aligned}$$

1-law invariance: for any $X, Y, Z \in L^\infty$ with $X \sim Z$ it holds that

$$\begin{aligned}\Lambda_\pi(X,Y) &= x^*_\alpha(X) + H_\alpha\big((X - x^*_\alpha(X))^+ \mathbb{1}_{\{Y \geq x^*_\alpha(Y)\}}\big)\\ &= x^*_\alpha(Z) + H_\alpha\big((Z - x^*_\alpha(Z))^+ \mathbb{1}_{\{Y \geq x^*_\alpha(Y)\}}\big)\\ &= \Lambda_\pi(Z,Y)\end{aligned}$$

where the second equality follows from the law invariance of the Orlicz quantile and of H_α.

1-positive homogeneity: by positive homogeneity of Orlicz quantiles and of H_α, it follows that, for any $\lambda \geq 0$ and any $X, Y \in L^\infty$,

$$\begin{aligned}\Lambda_\pi(\lambda X,Y) &= x^*_\alpha(\lambda X) + H_\alpha\big((\lambda X - x^*_\alpha(\lambda X))^+ \mathbb{1}_{\{Y \geq x^*_\alpha(Y)\}}\big)\\ &= \lambda\Big(x^*_\alpha(X) + H_\alpha\big((X - x^*_\alpha(X))^+ \mathbb{1}_{\{Y \geq x^*_\alpha(Y)\}}\big)\Big)\\ &= \lambda\Lambda_\pi(X,Y).\end{aligned}$$

2-translation invariance: for any $c \in \mathbb{R}$ and $X, Y \in L^\infty$ it holds that

$$\begin{aligned}\Lambda_\pi(X,Y+c) &= x^*_\alpha(X) + H_\alpha\big((X - x^*_\alpha(X))^+ \mathbb{1}_{\{Y+c \geq x^*_\alpha(Y+c)\}}\big)\\ &= x^*_\alpha(X) + H_\alpha\big((X - x^*_\alpha(X))^+ \mathbb{1}_{\{Y \geq x^*_\alpha(Y)\}}\big)\\ &= \Lambda_\pi(X,Y)\end{aligned}$$

where the second equality holds because of the cash-additivity of the Orlicz quantile.

Cash-additivity: it follows easily by 2-translation-invariance and 1-cash-additivity. □

Notice that Λ_π fails to be monotone, as a consequence of the same failure of the Orlicz quantiles.

Here below we propose an alternative capital allocation rule for π_α that is in line with the definition of π_α as an infimum and is based, roughly speaking, on a "common quantile" whenever the infimum is attained.

Definition 3.5. Given the Orlicz risk premium H_α, we define the map $\bar{\Lambda}_\pi \colon L^\infty \times L^\infty \to \mathbb{R}$ as

$$\bar{\Lambda}_\pi(X,Y) := \inf_{x \in \mathbb{R}} \left\{ x + H_\alpha\big((X-x)^+ \mathbb{1}_{\{Y \geq x\}}\big) \right\}. \tag{3.9}$$

We first establish some properties satisfied by $\bar{\Lambda}_\pi$, then we will investigate whether the infimum in (3.9) is attained or not.

Proposition 3.5. *$\bar{\Lambda}_\pi$ is a CAR with respect to π_α satisfying: no-undercut, monotonicity, 1-law invariance, cash-additivity and positive homogeneity.*

Proof. Since for any $X \in L^\infty$

$$\begin{aligned}\bar{\Lambda}_\pi(X,X) &= \inf_{x\in\mathbb{R}} \left\{x + H_\alpha\big((X-x)^+ \mathbb{1}_{\{X\geq x\}}\big)\right\} \\ &= \inf_{x\in\mathbb{R}} \left\{x + H_\alpha\big((X-x)^+\big)\right\} = \pi_\alpha(X),\end{aligned}$$

then $\bar{\Lambda}_\pi$ is a CAR with respect to π_α.

No-undercut and monotonicity: they follow by monotonicity of H_α. In particular, by monotonicity of H_α, it follows that, for all $X, Y \in L^\infty$,

$$\begin{aligned}\bar{\Lambda}_\pi(X,Y) &= \inf_{x\in\mathbb{R}} \left\{x + H_\alpha\big((X-x)^+ \mathbb{1}_{\{Y\geq x\}}\big)\right\} \\ &\leq \inf_{x\in\mathbb{R}} \left\{x + H_\alpha\big((X-x)^+\big)\right\} \ = \pi_\alpha(X),\end{aligned}$$

that is no-undercut, while for any $X, Y, Z \in L^\infty$ with $X \leq Z$

$$\begin{aligned}\bar{\Lambda}_\pi(X,Y) &= \inf_{x\in\mathbb{R}} \left\{x + H_\alpha\big((X-x)^+ \mathbb{1}_{\{Y\geq x\}}\big)\right\} \\ &\leq \inf_{x\in\mathbb{R}} \left\{x + H_\alpha\big((Z-x)^+ \mathbb{1}_{\{Y\geq x\}}\big)\right\} \\ &= \bar{\Lambda}_\pi(Z,Y),\end{aligned}$$

that is monotonicity.

1-law invariance: by law invariance of H_α, for any $X, Y, Z \in L^\infty$ with $X \sim Z$ it holds that

$$\begin{aligned}\bar{\Lambda}_\pi(X,Y) &= \inf_{x\in\mathbb{R}} \left\{x + H_\alpha\big((X-x)^+ \mathbb{1}_{\{Y\geq x\}}\big)\right\} \\ &= \inf_{x\in\mathbb{R}} \left\{x + H_\alpha\big((Z-x)^+ \mathbb{1}_{\{Y\geq x\}}\big)\right\} \\ &= \bar{\Lambda}_\pi(Z,Y).\end{aligned}$$

Cash-additivity: for any $c \in \mathbb{R}$ and $X, Y \in L^\infty$ it holds that

$$\begin{aligned}\bar{\Lambda}_\pi(X+c,Y+c) &= \inf_{x\in\mathbb{R}} \left\{x + H_\alpha\big((X+c-x)^+ \mathbb{1}_{\{Y+c\geq x\}}\big)\right\} \\ &= \inf_{y\in\mathbb{R}} \left\{y + c + H_\alpha\big((X-y)^+ \mathbb{1}_{\{Y\geq y\}}\big)\right\} \\ &= \bar{\Lambda}_\pi(X,Y) + c,\end{aligned}$$

hence cash-additivity.

Positive homogeneity: the case of $\lambda = 0$ is immediate. For any $\lambda > 0$ and $X, Y \in L^\infty$, instead,

$$\begin{aligned}\bar{\Lambda}_\pi(\lambda X,\lambda Y) &= \inf_{x\in\mathbb{R}} \left\{x + H_\alpha\big((\lambda X-x)^+ \mathbb{1}_{\{\lambda Y\geq x\}}\big)\right\} \\ &= \inf_{x\in\mathbb{R}} \left\{x + H_\alpha\left(\lambda\left(X - \frac{x}{\lambda}\right)^+ \mathbb{1}_{\{Y\geq \frac{x}{\lambda}\}}\right)\right\} \\ &= \inf_{y\in\mathbb{R}} \left\{\lambda y + \lambda H_\alpha\big((X-y)^+ \mathbb{1}_{\{Y\geq y\}}\big)\right\} \\ &= \lambda\bar{\Lambda}_\pi(X,Y)\end{aligned}$$

where the third equality holds by positive homogeneity of H_α. □

Notice that the infimum is clearly attained whenever $X \leq Y$ because in such case $\bar{\Lambda}_\pi$ coincides with π_α.

As shown in the following example, however, the infimum in Equation (3.9) defining $\bar{\Lambda}_\pi$ may be not attained in general.

To simplify the notation, we set for $\Phi(x) = x$,

$$L_{X,Y}(x) := x + \frac{\mathbb{E}\left[(X-x)^+ \mathbb{1}_{\{Y \geq x\}}\right]}{1-\alpha}.$$

Hence (3.9) becomes

$$\bar{\Lambda}_\pi(X,Y) = \inf_{x \in \mathbb{R}} \left\{ x + \frac{\mathbb{E}\left[(X-x)^+ \mathbb{1}_{\{Y \geq x\}}\right]}{1-\alpha} \right\} = \inf_{x \in \mathbb{R}} L_{X,Y}(x).$$

EXAMPLE 3.3. Take $\Phi(x) = x$ and two random variables X, Y with the following joint distribution:

$$\mathbb{P}(X = k, Y = j) = \frac{1}{9}, \qquad \text{for each } k, j = -1, 0, 1.$$

It is easy to check that

$$L_{X,Y}(x) = \begin{cases} -x\frac{\alpha}{1-\alpha} & \text{if } x \leq -1; \\ x\left(1 - \frac{4}{9(1-\alpha)}\right) + \frac{2}{9(1-\alpha)} & \text{if } -1 < x \leq 0; \\ x\left(1 - \frac{1}{9(1-\alpha)}\right) + \frac{1}{9(1-\alpha)} & \text{if } 0 < x \leq 1; \\ x & \text{if } x > 1. \end{cases}$$

For $\alpha = \frac{1}{9}$, it can be easily seen that $L_{X,Y}$ is not convex in x and that $\inf_{x \in \mathbb{R}} L_{X,Y}(x) = -\frac{1}{4}$ is not attained. A similar result holds also for $\alpha = \frac{5}{9}$, corresponding to $\mathbb{P}(X \geq 0, Y \geq 0) = 1 - \alpha = \frac{4}{9}$, that is "more or less" to the α-quantile of $\min\{X, Y\}$.

For $\alpha = \frac{8}{9}$ or, equivalently, $1 - \alpha = \frac{1}{9} = \mathbb{P}(X \geq 1, Y \geq 1)$, it holds instead that $\inf_{x \in \mathbb{R}} L_{X,Y}(x) = 1$ is attained at any point of the interval $(0, 1]$ but $L_{X,Y}$ is still not convex in x.

We have thus shown that the infimum in Equation (3.9) defining $\bar{\Lambda}_\pi(X,Y)$ may be attained or not. Also, due to non-convexity of $L_{X,Y}$ in x, it is quite hard to obtain a general result for existence of a minimum. We therefore consider the following particular case.

PROPOSITION 3.6. *If X and Y are two continuous random variables in L^∞ and $\Phi(x) = x$, then the infimum in (3.9) is attained at some*

$$x^* \in \left[\operatorname{ess\,inf}(\min\{X, Y\}), \operatorname{ess\,sup}(\max\{X, Y\})\right].$$

PROOF. Assume that X and Y have joint density function $f_{X,Y}$ and that $\Phi(x) = x$. Then

$$L_{X,Y}(x) = x + \frac{1}{1-\alpha} \int_x^{\operatorname{ess\,sup}(X)} \int_x^{\operatorname{ess\,sup}(Y)} (z - x) f_{X,Y}(z, y)\, dz\, dy$$

is continuous in $x \in \mathbb{R}$. Moreover, it is immediate to check that, for $x \leq \operatorname{ess\,inf}(\min\{X,Y\})$,

$$L_{X,Y}(x) = x + \frac{\mathbb{E}[X-x]}{1-\alpha} = -\frac{\alpha}{1-\alpha}x + \frac{\mathbb{E}[X]}{1-\alpha};$$

while $L_{X,Y}(x) = x$ for $x > \operatorname{ess\,sup}(\max\{X,Y\})$. Hence, $L_{X,Y}$ is decreasing in the interval $\big(-\infty, \operatorname{ess\,inf}(\min\{X,Y\})\big)$ and increasing in the interval $\big(\operatorname{ess\,sup}(\max\{X,Y\}); +\infty\big)$. By continuity of $L_{X,Y}$ in x, it follows that there exists (at least) a minimum point belonging to the interval $\big[\operatorname{ess\,inf}(\min\{X,Y\}), \operatorname{ess\,sup}(\max\{X,Y\})\big]$. □

Proposition 3.7. *Assume that X and Y are two independent and continuous random variables in L^∞ and that $\Phi(x) = x$. If x^* is a minimum point in (3.9) satisfying $x^* \in \big(\operatorname{ess\,inf}(\min\{X,Y\}), \operatorname{ess\,sup}(\max\{X,Y\})\big)$ and if the density function f_Y of Y is continuous at x^*, then x^* has to satisfy the following first order condition:*

$$1-\alpha = \mathbb{P}(X \geq x^*)\mathbb{P}(Y \geq x^*) + \mathbb{E}\big[(X-x^*)^+\big] f_Y(x^*). \qquad (3.10)$$

Proof. If X and Y are independent and continuous and $\Phi(x) = x$, then

$$\bar{\Lambda}_\pi(X,Y) = \inf_{x\in\mathbb{R}} \left\{ x + \frac{\mathbb{E}\big[(X-x)^+\big]\mathbb{P}(Y \geq x)}{1-\alpha} \right\}$$

where, by Proposition 3.6, the infimum is attained.

For simplicity of notation, set

$$\begin{aligned} L(x) &:= L_{X,Y}(x), \\ J(x) &:= \mathbb{P}(Y \geq x), \\ K(x) &:= \frac{\mathbb{E}\big[(X-x)^+\big]}{1-\alpha}; \end{aligned}$$

so that $L(x) = x + K(x)J(x)$.

By using similar arguments as in Bellini and Rosazza Gianin [15], the left and right derivatives of L are given by

$$\begin{aligned} L'_-(x) &= 1 + K'_-(x)J(x) + K(x)J'_-(x) \qquad (3.11)\\ &= 1 - \frac{1}{1-\alpha}\mathbb{P}(X \geq x)\mathbb{P}(Y \geq x) - \frac{\mathbb{E}\big[(X-x)^+\big]}{1-\alpha} f_Y(x^-) \\ L'_+(x) &= 1 + K'_+(x)J(x) + K(x)J'_+(x) \qquad (3.12)\\ &= 1 - \frac{1}{1-\alpha}\mathbb{P}(X > x)\mathbb{P}(Y \geq x) - \frac{\mathbb{E}\big[(X-x)^+\big]}{1-\alpha} f_Y(x^+), \end{aligned}$$

where K'_- and K'_+ are given in Bellini and Rosazza Gianin [15] and $f_Y(x^-)$ and $f_Y(x^+)$ denote the left and right limit of f_Y at x.

Since X, Y are continuous, for any $x \in \mathbb{R}$ where f_Y is continuous it holds that

$$\begin{aligned} L'_-(x) &= L'_+(x) \\ &= 1 - \frac{1}{1-\alpha}\mathbb{P}(X \geq x)\mathbb{P}(Y \geq x) - \frac{\mathbb{E}\left[(X-x)^+\right]}{1-\alpha} f_Y(x). \end{aligned}$$

Since L is continuous and differentiable at any point x where f_Y is continuous, a necessary condition for

$$x^* \in \left(\operatorname{ess\,inf}(\min\{X,Y\}), \operatorname{ess\,sup}(\max\{X,Y\})\right)$$

being a minimum point is that $L'(x^*) = 0$, that is satisfying

$$1 - \alpha = \mathbb{P}(X \geq x^*)\mathbb{P}(Y \geq x^*) + \mathbb{E}\left[(X - x^*)^+\right] f_Y(x^*)$$

or, equivalently, by independence of X and Y,

$$1 - \alpha = \mathbb{P}(X \geq x^*, Y \geq x^*) + \mathbb{E}\left[(X - x^*)^+\right] f_Y(x^*). \qquad \square$$

Here below we provide an example of $\bar{\Lambda}_\pi$ when X and Y are independent and continuous random variables.

Example 3.4. Consider two independent random variables X and Y, where X is uniformly distributed on $[-1, 1]$ and Y is uniformly distributed on $[-2, 0]$. Assume that $\Phi(x) = x$ and $\alpha = \frac{1}{2}$. It follows that

$$\bar{\Lambda}_\pi(X, Y) = \min_{x \in \mathbb{R}} \left\{ x + 2\,\mathbb{E}\left[(X - x)^+\right]\mathbb{P}(Y \geq x) \right\},$$

where

$$\begin{aligned} L(x) &= x + 2\mathbb{E}\left[(X-x)^+\right]\mathbb{P}(Y \geq x) \\ &= \begin{cases} -x & \text{if } x \leq -2; \\ x + x^2 & \text{if } -2 < x \leq -1; \\ \frac{3}{4}x + \frac{1}{2}x^2 - \frac{1}{4}x^3 & \text{if } -1 < x \leq 0; \\ x & \text{if } x \geq 0. \end{cases} \end{aligned}$$

By direct computation or by applying (3.11) and (3.12), it follows that $x^* = \frac{2-\sqrt{13}}{3}$ is the unique minimum point of L, satisfying the first order condition (3.10).

We finally propose a last approach to CAR for π_α in terms of Orlicz quantiles. The idea of such an approach is in line with the sub-differential method (see Chapter 2) where $\Lambda_\rho^\partial(X, Y) = \mathbb{E}_{\mathbb{Q}_Y}[X]$ with $\mathbb{Q}_Y$ being an optimal scenario in the dual representation of ρ.[2] In the present case, indeed, we replace in the formulation of $\pi_\alpha(X)$ the Orlicz quantile $x^*_\alpha(Y)$ realizing the infimum in the definition of $\pi_\alpha(Y)$.

2 Remember that the sub-differential approach coincides with the gradient CAR whenever there is a unique optimal scenario, that is, when the risk measure is Gateaux differentiable.

DEFINITION 3.6. Given π_α as in Definition 3.2, we define the map $\widetilde{\Lambda}_\pi \colon L^\infty \times L^\infty \to \mathbb{R}$ as

$$\widetilde{\Lambda}_\pi(X,Y) := x^*_\alpha(Y) + H_\alpha\big((X - x^*_\alpha(Y))^+\big)$$

where $x^*_\alpha(Y)$ is an Orlicz quantile at level α of Y.

In the following, we fix x^*_α to be the upper Orlicz quantile. As previously, similar results would hold when the lower Orlicz quantiles are considered.

Unfortunately, as shown below, $\widetilde{\Lambda}_\pi$ fails to satisfy no-undercut.

PROPOSITION 3.8. *The map $\widetilde{\Lambda}_\pi$ is a CAR with respect to π_α satisfying monotonicity, 1-law invariance and undercut, that is,*

$$\widetilde{\Lambda}_\pi(X,Y) \geq \widetilde{\Lambda}_\pi(X,X) = \pi_\alpha(X) \quad \textit{for all } X,Y \in L^\infty.$$

Furthermore,

$$\widetilde{\Lambda}_\pi(a,Y) = x^*_\alpha(Y) + \frac{(a - x^*_\alpha(Y))^+}{\Phi^{-1}(1-\alpha)} \quad \textit{for all } a \in \mathbb{R},\ Y \in L^\infty. \tag{3.13}$$

PROOF. $\widetilde{\Lambda}_\pi$ is a CAR with respect to π_α since, for any $X \in L^\infty$, it holds that

$$\widetilde{\Lambda}_\pi(X,X) = x^*_\alpha(X) + H_\alpha\big((X - x^*_\alpha(X))^+\big) = \pi_\alpha(X).$$

Monotonicity and 1-law invariance: they follow by monotonicity and law invariance of H_α.

Undercut: for all $X,Y \in L^\infty$

$$\begin{aligned}\pi_\alpha(X) &= x^*_\alpha(X) + H_\alpha\big((X - x^*_\alpha(X))^+\big)\\ &\leq x^*_\alpha(Y) + H_\alpha\big((X - x^*_\alpha(Y))^+\big)\\ &= \widetilde{\Lambda}_\pi(X,Y)\end{aligned}$$

where the inequality holds by

$$x^*_\alpha(X) \in \operatorname*{arg\,min}_{x\in\mathbb{R}} \left\{x + H_\alpha((X-x)^+)\right\},$$

and the latter contains $x^*_\alpha(Y)$.

Finally, (3.13) follows easily by riskless of H_α. □

3.2.3 *Comparison among different approaches and full allocation*

A comparison among the three approaches introduced above and some well known capital allocation rules is provided here below.

PROPOSITION 3.9. *The following relations hold for all $X,Y \in L^\infty$:*

$$\widetilde{\Lambda}_\pi(X,Y) \geq \pi_\alpha(X) \geq \Lambda_\pi(X,Y) \quad \textit{and} \quad \widetilde{\Lambda}_\pi(X,Y) \geq \bar{\Lambda}_\pi(X,Y).$$

PROOF. First of all, for all $X, Y \in L^\infty$,

$$\begin{aligned}\widetilde{\Lambda}_\pi(X,Y) &\geq x^*_\alpha(X) + H_\alpha\big((X - x^*_\alpha(X))^+\big) = \pi_\alpha(X)\\ &\geq x^*_\alpha(X) + H_\alpha\big((X - x^*_\alpha(X))^+ \mathbb{1}_{\{Y \geq x^*_\alpha(Y)\}}\big) = \Lambda_\pi(X,Y)\end{aligned}$$

where the first inequality holds by undercut of $\widetilde{\Lambda}_\pi$ (or by definition of x^*_α) and the second one by no-undercut of Λ_π. Concerning the last inequality, instead, it holds that

$$\begin{aligned}\widetilde{\Lambda}_\pi(X,Y) &= x^*_\alpha(Y) + H_\alpha\big((X - x^*_\alpha(Y))^+\big)\\ &\geq x^*_\alpha(Y) + H_\alpha\big((X - x^*_\alpha(Y))^+ \mathbb{1}_{\{Y \geq x^*_\alpha(Y)\}}\big)\\ &\geq \inf_{x \in \mathbb{R}} \big\{x + H_\alpha\big((X - x))^+ \mathbb{1}_{\{Y \geq x\}}\big)\big\} = \bar{\Lambda}_\pi(X,Y). \qquad \square\end{aligned}$$

As could be expected, $\widetilde{\Lambda}_\pi$ dominates both Λ_π and $\bar{\Lambda}_\pi$: indeed, $\widetilde{\Lambda}_\pi$ depends only on the Orlicz quantile of Y, while Λ_π depends on both the Orlicz quantiles of X and Y, and $\bar{\Lambda}_\pi$ on a common quantile. So, in a certain sense, $\widetilde{\Lambda}_\pi$ does not take into account the possibility that the risks of X and Y can compensate each other and hence assigns to X a higher "cost".

Notice that, by Proposition 3.1 (see also [15, 44]), for Haezendonck-Goovaerts risk measures the gradient capital allocation (3.3) becomes

$$\Lambda^\nabla_\pi(X,Y) = \mathbb{E}_{\mathbb{Q}_Y}[X] = \frac{\mathbb{E}\left[X\,\Phi'\left(\frac{(Y - x^*_\alpha(Y))^+}{\|(Y - x^*_\alpha(Y))^+\|_{\Phi_\alpha}}\right)\mathbb{1}_{\{Y > x^*_\alpha(Y)\}}\right]}{\mathbb{E}\left[\Phi'\left(\frac{(Y - x^*_\alpha(Y))^+}{\|(Y - x^*_\alpha(Y))^+\|_{\Phi_\alpha}}\right)\mathbb{1}_{\{Y > x^*_\alpha(Y)\}}\right]}$$

since

$$\frac{d\mathbb{Q}_Y}{d\mathbb{P}} = \frac{\Phi'\left(\frac{(Y - x^*_\alpha(Y))^+}{\|(Y - x^*_\alpha(Y))^+\|_{\Phi_\alpha}}\right)\mathbb{1}_{\{Y > x^*_\alpha(Y)\}}}{\mathbb{E}\left[\Phi'\left(\frac{(Y - x^*_\alpha(Y))^+}{\|(Y - x^*_\alpha(Y))^+\|_{\Phi_\alpha}}\right)\mathbb{1}_{\{Y > x^*_\alpha(Y)\}}\right]}. \tag{3.14}$$

This allows us to compare Λ_π with the gradient approach and with Aumann-Shapley method. Very recently, Gómez and Tang [46] studied in details the gradient allocation for higher moment risk measures, corresponding to HG risk measures for power Young functions.

PROPOSITION 3.10. *For all $X, Y \in L^\infty$ it holds that*

$$\Lambda_\pi(X,Y) \geq \mathbb{E}\left[\frac{X\,\mathbb{1}_{\{Y \geq x^*_\alpha(Y)\}}}{\Phi^{-1}(1-\alpha)}\right] + x^*_\alpha(X)\left(1 - \frac{\mathbb{P}(Y \geq x^*_\alpha(Y))}{\Phi^{-1}(1-\alpha)}\right).$$

Moreover, if $\Phi(x) = x$ and Y is a continuous random variable, then

$$\Lambda_\pi(X,Y) \geq \Lambda^\nabla_\pi(X,Y) \text{ and } \Lambda_\pi(X,Y) \geq \Lambda^{AS}_\pi(X,Y) \text{ for all } X \in L^\infty.$$

PROOF. For all $X, Y \in L^\infty$ it holds that

$$\begin{aligned}\Lambda_\pi(X,Y) &= x^*_\alpha(X) + H_\alpha\big((X - x^*_\alpha(X))^+ \mathbb{1}_{\{Y \geq x^*_\alpha(Y)\}}\big) \\ &\geq x^*_\alpha(X) + \frac{\mathbb{E}\Big[(X - x^*_\alpha(X))^+ \mathbb{1}_{\{Y \geq x^*_\alpha(Y)\}}\Big]}{\Phi^{-1}(1-\alpha)} \\ &\geq x^*_\alpha(X) + \frac{\mathbb{E}\Big[(X - x^*_\alpha(X)) \mathbb{1}_{\{Y \geq x^*_\alpha(Y)\}}\Big]}{\Phi^{-1}(1-\alpha)} \\ &= \mathbb{E}\left[\frac{X \mathbb{1}_{\{Y \geq x^*_\alpha(Y)\}}}{\Phi^{-1}(1-\alpha)}\right] + x^*_\alpha(X)\left(1 - \frac{\mathbb{P}(Y \geq x^*_\alpha(Y))}{\Phi^{-1}(1-\alpha)}\right) \end{aligned} \tag{3.15}$$

where the first inequality is due to $H_\alpha(Z) \geq \frac{\mathbb{E}[Z]}{\Phi^{-1}(1-\alpha)}$ for $Z \in L^\infty_+$ (see Goovaerts et al. [47] and Haezendonck and Goovaerts [48]).

Since, for $\Phi(x) = x$, (3.14) becomes

$$\frac{d\mathbb{Q}_Y}{d\mathbb{P}} = \frac{\mathbb{1}_{\{Y > q_\alpha(Y)\}}}{\mathbb{P}(Y > q_\alpha(Y))}$$

where q_α is the upper α-quantile, and, for a continuous Y,

$$\frac{d\mathbb{Q}_Y}{d\mathbb{P}} = \frac{\mathbb{1}_{\{Y \geq x^*_\alpha(Y)\}}}{1-\alpha},$$

(3.15) implies $\Lambda_\pi(X,Y) \geq \Lambda^\nabla_\pi(X,Y)$.

Then, for each $\gamma \in (0,1)$, we have

$$\Lambda_\pi(X,Y) \geq \frac{\mathbb{E}\big[X \mathbb{1}_{\{Y > q_\alpha(Y)\}}\big]}{\mathbb{P}(Y > q_\alpha(Y))} = \frac{\mathbb{E}\big[X \mathbb{1}_{\{\gamma Y > q_\alpha(\gamma Y)\}}\big]}{\mathbb{P}(\gamma Y > q_\alpha(\gamma Y))} = \mathbb{E}_{\mathbb{Q}_{\gamma Y}}[X],$$

where the first equality holds by positive homogeneity of the quantile. Thus

$$\Lambda_\pi(X,Y) = \int_0^1 \Lambda_\pi(X,Y)\, d\gamma \geq \int_0^1 \mathbb{E}_{\mathbb{Q}_{\gamma Y}}[X]\, d\gamma = \Lambda^{\mathrm{AS}}_\pi(X,Y). \qquad \square$$

The previous proposition shows that Λ_π exceeds the gradient allocation whenever $\Phi(x) = x$ and Y is a continuous random variable. However, even when Y is not a continuous random variable the same relation may hold, as shown in the following example.

EXAMPLE 3.5. Take $\Phi(x) = x$ and assume the following random variables are independent:

$$X = \begin{cases} -1 & \text{with prob. } \frac{1}{3}; \\ 1 & \text{with prob. } \frac{2}{3}. \end{cases} \qquad Y = \begin{cases} 0 & \text{with prob. } \frac{3}{4}; \\ 2 & \text{with prob. } \frac{1}{4}. \end{cases}$$

Then

$$\Lambda^\nabla_\pi(X,Y) = \frac{\mathbb{E}\big[X \mathbb{1}_{\{Y > q_\alpha(Y)\}}\big]}{\mathbb{P}(Y > q_\alpha(Y))} = \mathbb{E}[X]$$

where $q_\alpha(Y)$ is the upper α-quantile of Y. For any $\alpha > \frac{1}{3}$ we have

$$\begin{aligned}\Lambda_\pi(X,Y) &= q_\alpha(X) + \frac{\mathbb{E}\big[(X - q_\alpha(X))^+\big]\mathbb{P}(Y \geq q_\alpha(Y))}{1-\alpha} \\ &= 1 > \mathbb{E}[X] = \frac{1}{3}.\end{aligned}$$

Finally, we compare $\widetilde{\Lambda}_\pi$ with the gradient and the Aumann-Shapley allocations.

PROPOSITION 3.11. *$\widetilde{\Lambda}_\pi \geq \Lambda_\pi^\nabla$ and $\widetilde{\Lambda}_\pi \geq \Lambda_\pi^{AS}$.*

PROOF. From Proposition 3.9 it follows that for any $X, Y \in L^\infty$

$$\widetilde{\Lambda}_\pi(X,Y) \geq \pi_\alpha(X) = \mathbb{E}_{\mathbb{Q}_X}[X] \geq \mathbb{E}_{\mathbb{Q}_Y}[X] = \Lambda_\pi^\nabla(X,Y)$$

where the second inequality is due to the fact that $\mathbb{Q}_X$ is the maximizer for X. By similar arguments it follows that

$$\begin{aligned}\widetilde{\Lambda}_\pi(X,Y) &= \int_0^1 \widetilde{\Lambda}_\pi(X,Y)\,d\gamma \\ &\geq \int_0^1 \mathbb{E}_{\mathbb{Q}_X}[X]\,d\gamma \\ &\geq \int_0^1 \mathbb{E}_{\mathbb{Q}_{\gamma Y}}[X]\,d\gamma = \Lambda_\pi^{AS}(X,Y)\end{aligned}$$

holds for any $X, Y \in L^\infty$. □

We summarize in Table 3.1 the main properties satisfied by the proposed methods.

	Λ_π	$\bar{\Lambda}_\pi$	$\widetilde{\Lambda}_\pi$
No-undercut	yes	yes	no
Comparison $\Lambda_\pi^\nabla / \Lambda_\pi^{AS}$	$\Phi(x) = x$ only	no	no
Monotone	no	yes	yes

Table 3.1: Summary of the properties satisfied by different methods.

It is also easy to see that none of the proposed methods satisfies full allocation, even when the allocation maps are CARs ($\Lambda(X,X) = \rho(X)$), as they are not linear in the first variable. However, one can always modify such CARs in order to get the desired property, as we explained in Section 3.1.

3.3 CAPITAL ALLOCATION VIA LINKING FUNCTIONS

In this section, we introduce an alternative generalization of the work of Xun et al. [67]. Such a generalization shares with the aforementioned work the idea of linking X and Y in the argument of the Orlicz risk premium but does not assume that such link has a specific functional form, while the previous approaches do.

More precisely, looking at the contribution introduced by Xun et al. [67], we can notice that $H_{Y,\alpha}(X)$, solution of (3.4), coincides with the solution $H_\alpha\left(X\mathbb{1}_{\{Y>\mathrm{VaR}_\alpha(Y)\}}\right)$ of

$$\mathbb{E}\left[\Phi\left(\frac{X\mathbb{1}_{\{Y>\mathrm{VaR}_\alpha(Y)\}}}{H_\alpha\left(X\mathbb{1}_{\{Y>\mathrm{VaR}_\alpha(Y)\}}\right)}\right)\right] = 1-\alpha.$$

Thus, the contribution $H_{Y,\alpha}(X)$ introduced by Xun et al. [67] is exactly the Orlicz risk premium H_α of $X\mathbb{1}_{\{Y>\mathrm{VaR}_\alpha(Y)\}}$.

In the present section, we generalize the approach above by considering a general function $f\colon L^\infty_+ \times L^\infty \to L^\infty_+$ "aggregating" the two positions X and Y. The particular case of

$$f(X,Y) := X\mathbb{1}_{\{Y>\mathrm{VaR}_\alpha(Y)\}}$$

corresponds then to the approach of Xun et al. [67]. We introduce therefore the following notion.

Definition 3.7. A map $f\colon L^\infty_+ \times L^\infty \to L^\infty_+$ is said to be *linking* if, for all $X \in L^\infty_+$,

$$f(X,X) = X.$$

Here the interpretation is the following: f has two variables corresponding to the sub-portfolio X and to the portfolio Y, and links them to yield another position $f(X,Y) \in L^\infty_+$ that can be seen as an aggregated position. Then, we can follow the same construction of Xun et al. [67] but considering a general linking function f.

We firstly provide a CAR for the Orlicz risk premium and later its extension to the HG risk measure.

Definition 3.8. Let H_α be the Orlicz risk premium and let $f\colon L^\infty_+ \times L^\infty \to L^\infty_+$ be a linking function. We define H-linking CAR as the map $\Lambda^f_H\colon L^\infty_+ \times L^\infty \to \mathbb{R}^+$ given by

$$\Lambda^f_H(X,Y) := H_\alpha(f(X,Y)), \quad X \in L^\infty_+, Y \in L^\infty.$$

It is clear that Λ^f_H is a capital allocation rule with respect to the Orlicz risk premium if and only if f is linking. Hence, the risk contribution defined by Xun et al. [67] is not a capital allocation rule with respect to the Orlicz risk premium because the function $f(X,Y) = X\mathbb{1}_{\{Y>\mathrm{VaR}_\alpha(Y)\}}$ is not linking.

It is interesting now to study when Λ^f_H satisfies some of the usual properties required for a capital allocation rule.

PROPOSITION 3.12. *Let f be a linking function.*

A. *If $f(X,Y) \leq f(X,X)$ for all $X \in L^\infty_+, Y \in L^\infty$, then Λ^f_H satisfies no-undercut.*

B. *If $f(a,Y) = a$ for all $a \geq 0$ and $Y \in L^\infty$, then Λ^f_H satisfies*

$$\Lambda^f_H(a,Y) = \frac{a}{\Phi^{-1}(1-\alpha)} \quad \text{for all } Y \in L^\infty,\ a \geq 0. \tag{3.16}$$

C. *If f is monotone increasing in the first entry then Λ^f_H is monotone.*

PROOF. A. For all $X \in L^\infty_+, Y \in L^\infty$ it holds that

$$\Lambda^f_H(X,Y) = H_\alpha(f(X,Y)) \leq H_\alpha(f(X,X)) = H_\alpha(X)$$

where the inequality holds by monotonicity of H_α and the last equality by definition of linking function. The no-undercut is then verified.

B. It follows immediately by riskless of H_α.

C. For all $X, Z \in L^\infty_+, Y \in L^\infty$ with $X \leq Z$ it holds that

$$\Lambda^f_H(X,Y) = H_\alpha(f(X,Y)) \leq H_\alpha(f(Z,Y)) = \Lambda^f_H(Z,Y)$$

where the inequality holds by monotonicity both of f and of H_α. □

Notice that (3.16) reduces to the riskless property when $\alpha = 0$.
We now present some examples of explicit formulas for Λ^f_H.

EXAMPLE 3.6. For $\Phi(x) = x$, Λ^f_H becomes

$$\Lambda^f_H(X,Y) = H_\alpha(f(X,Y)) = \frac{\mathbb{E}[f(X,Y)]}{1-\alpha}.$$

By taking the linking function $f(X,Y) = X\mathbb{1}_{\{Y-X \geq q_\alpha(Y-X)\}}$ where q_α is an α-quantile, Λ^f_H reduces to

$$\Lambda^f_H(X,Y) = \frac{\mathbb{E}\left[X\mathbb{1}_{\{Y-X \geq q_\alpha(Y-X)\}}\right]}{1-\alpha}.$$

For continuous random variables X and Y, Λ_H becomes then

$$\Lambda^f_H(X,Y) = \mathbb{E}\left[X \mid Y - X \geq q_\alpha(Y-X)\right],$$

similar to the contribution to shortfall proposed by Overbeck [58].

EXAMPLE 3.7. For $\Phi(x) = x^p$, $p \in (1,+\infty)$, Λ^f_H is given by

$$\Lambda^f_H(X,Y) = \left(\frac{\mathbb{E}[f^p(X,Y)]}{1-\alpha}\right)^{\frac{1}{p}} = \frac{\|f(X,Y)\|_p}{(1-\alpha)^{\frac{1}{p}}}.$$

Then, if we consider again the linking function

$$f(X,Y) = X\mathbb{1}_{\{Y-X\geq x_\alpha^*(Y-X)\}}$$

where x_α^* is an Orlicz quantile, we get

$$\Lambda_H^f(X,Y) = \frac{\left\|X\mathbb{1}_{\{Y-X\geq x_\alpha^*(Y-X)\}}\right\|_p}{(1-\alpha)^{\frac{1}{p}}}.$$

For continuous X and Y, Λ_H^f corresponds then to

$$\Lambda_H^f(X,Y) = \left(\mathbb{E}\left[X^p \mid Y-X\geq x_\alpha^*(Y-X)\right]\right)^{\frac{1}{p}}.$$

We now introduce a CAR for Haezendonck-Goovaerts risk measures π_α based on that for Orlicz premia H_α, by using the same procedure as for the corresponding π_α and H_α.

Definition 3.9. Given the map Λ_H^f of Definition 3.8, we define π-linking CAR as the map $\Lambda_\pi^f\colon L^\infty\times L^\infty\to\mathbb{R}$ given by

$$\begin{aligned}\Lambda_\pi^f(X,Y) &:= \inf_{x\in\mathbb{R}}\left\{x+\Lambda_H^f\left((X-x)^+,(Y-x)^+\right)\right\}\\ &= \inf_{x\in\mathbb{R}}\left\{x+H_\alpha\left(f\left((X-x)^+,(Y-x)^+\right)\right)\right\}.\end{aligned}$$

Proposition 3.13. *Let f be a linking function.*

a. *Λ_π^f is a CAR with respect to π_α.*

b. *If $f(Z,W)\leq f(Z,Z)$ for all $Z\in L_+^\infty, W\in L^\infty$, then Λ_π^f satisfies no-undercut.*

c. *If $f(a,W)=a$ for all $a\geq 0$ and $W\in L^\infty$, then Λ_π^f satisfies riskless.*

d. *If f is monotone increasing in the first entry then Λ_π^f is monotone.*

e. *If $X,Y\in L_+^\infty$ then $\Lambda_\pi^f(X,Y)\leq\Lambda_H^f(X,Y)$.*

Proof. a. By definition of linking function, for all $X\in L^\infty$,

$$\begin{aligned}\Lambda_\pi^f(X,X) &= \inf_{x\in\mathbb{R}}\left\{x+H_\alpha\left(f\left((X-x)^+,(X-x)^+\right)\right)\right\}\\ &= \inf_{x\in\mathbb{R}}\left\{x+H_\alpha\left((X-x)^+\right)\right\}\\ &= \pi_\alpha(X).\end{aligned}$$

b. For all $X,Y\in L^\infty$ it holds that

$$\begin{aligned}\Lambda_\pi^f(X,Y) &= \inf_{x\in\mathbb{R}}\left\{x+H_\alpha\left(f\left((X-x)^+,(Y-x)^+\right)\right)\right\}\\ &\leq \inf_{x\in\mathbb{R}}\left\{x+H_\alpha\left((X-x)^+\right)\right\}\\ &= \pi_\alpha(X)\end{aligned}$$

where the inequality holds by hypothesis and by monotonicity of H_α.

C. By the hypothesis on f and by riskless of π_α (see [15] for details), it follows that

$$\begin{aligned}\Lambda_\pi^f(b,Y) &= \inf_{x\in\mathbb{R}}\left\{x + H_\alpha\big(f\big((b-x)^+,(Y-x)^+\big)\big)\right\}\\ &= \inf_{x\in\mathbb{R}}\left\{x + H_\alpha\big((b-x)^+\big)\right\}\\ &= \pi_\alpha(b) = b\end{aligned}$$

for all $b \in \mathbb{R}$ and $Y \in L^\infty$.

D. For all $X, Y, Z \in L^\infty$ with $X \leq Z$ it holds that

$$\begin{aligned}\Lambda_\pi^f(X,Y) &= \inf_{x\in\mathbb{R}}\left\{x + H_\alpha\big(f\big((X-x)^+,(Y-x)^+\big)\big)\right\}\\ &\leq \inf_{x\in\mathbb{R}}\left\{x + H_\alpha\big(f\big((Z-x)^+,(Y-x)^+\big)\big)\right\}\\ &= \Lambda_\pi^f(Z,Y)\end{aligned}$$

where the inequality holds by monotonicity both of f and of H_α.

E. For all $X, Y \in L^\infty_+$

$$\begin{aligned}\Lambda_\pi^f(X,Y) &= \inf_{x\in\mathbb{R}}\left\{x + H_\alpha\big(f\big((X-x)^+,(Y-x)^+\big)\big)\right\}\\ &\leq H_\alpha\big(f\big(X^+,Y^+\big)\big)\\ &= H_\alpha(f(X,Y))\\ &= \Lambda_H^f(X,Y)\end{aligned}$$

where the inequality holds by taking $x = 0$. □

Notice that, even under suitable hypothesis on f, there is no clear comparison between Λ_π^f and the gradient method or the Aumann-Shapley one.

3.4 ROBUST VERSIONS - QUANTILE-BASED APPROACH

So far, no ambiguity on the choice of the probability measure $\mathbb{P}$ or on the choice of the Young function Φ has been considered. Following the approach of Bellini and Rosazza Gianin [15] who introduced robust Orlicz premia and robust HG risk measures, in this section we provide extensions and robust versions of the previously presented approaches, in order to deal with ambiguity with respect to the probabilistic model $\mathbb{P}$ as well as to the choice of the Young function Φ.

3.4.1 *Ambiguity over* $\mathbb{P}$

Ambiguity over the probabilistic model has been largely considered in decision theory, when facing the problem of maximizing the expected utility. The main feature of ambiguity is that the decision maker may not hold a unique belief about the realization of the future states of the world. Rather, several scenarios of the possible states of the world should be taken into account. This idea is commonly expressed by introducing a set of probability measures, instead of assuming a single one. For a detailed treatment of the argument see, among others, Cerreia-Vioglio et al. [25], Gilboa and Schmeidler [45] and Maccheroni et al. [53].

Throughout this subsection, we assume that $\mathcal{Q}$ is the set of all probability measures absolutely continuous with respect to $\mathbb{P}$ over which there is ambiguity.

Robust versions of Orlicz risk premia were proposed by Bellini and Rosazza Gianin [15] for $\alpha = 0$ following robust versions of expected utility. In particular, the aforementioned authors consider the multiple priors, the variational preferences and the homothetic preferences approaches (see [25, 26, 45, 53] for details on robust versions of expected utility) and show that these three different approaches can be formulated in a unified way.

Since our aim is to generalize the capital allocations for Orlicz premia and HG risk measures introduced in Sections 3.2 and 3.3 by taking into account ambiguity over $\mathbb{P}$, we need to consider robust Orlicz premia and robust HG risk measures for a general $\alpha \in [0, 1)$ so to be able to introduce "robust" Orlicz quantiles. We focus here on the "variational preferences" approach in the general case where $\alpha \in [0, 1)$, referring to [15] for $\alpha = 0$.

We then organize this section as follows. First of all, we generalize to $\alpha \in [0, 1)$ the notions of robust Orlicz premia and robust HG risk measures and the result concerning whether the infimum is attained or not in the definition of robust π_α. At that point, we will be able to introduce and study robust Orlicz quantiles and to define CARs for robust Orlicz premia and robust HG risk measures as in Section 3.2. Notice that the first two points are slight extensions of what done in Bellini and Rosazza Gianin [15] and the proofs can be derived in a similar way.

3.4.1.1 *Robust Orlicz premia and robust HG risk measures for $\alpha \in [0, 1)$*

The following definition extends robust Orlicz risk premia introduced in [15] for $\alpha = 0$.

Definition 3.10. Let a Young function Φ be given and let $\alpha \in [0, 1)$ be fixed. The *robust Orlicz risk premium* of $X \in L^\infty_+$ is defined as

$$H_{c,\alpha}(X) := \inf\left\{ k > 0 \;\middle|\; \sup_{\mathbb{Q}\in\mathcal{Q}} \left\{ \mathbb{E}_{\mathbb{Q}}\left[\Phi\left(\frac{X}{k}\right)\right] - c(\mathbb{Q}) \right\} \leq 1 - \alpha \right\}$$

where $c \colon \mathcal{Q} \to [0, +\infty]$ is convex and lower semicontinuous, satisfying $\inf_{\mathbb{Q}\in\mathcal{Q}} c(\mathbb{Q}) = 0$.

For $\alpha = 0$, we set $H_c := H_{c,0}$. Notice that a further extension is possible by requiring that the infimum of c is only finite but not exactly zero. However, this is beyond the scope of the present work and would produce unnecessary complications in the notation.

When $c\colon \mathcal{Q} \to [0, +\infty]$ is given by

$$c(\mathbb{Q}) = \begin{cases} 0 & \text{if } \mathbb{Q} \in \mathcal{S}; \\ +\infty & \text{if } \mathbb{Q} \notin \mathcal{S}; \end{cases}$$

for $\mathcal{S} \subseteq \mathcal{Q}$, $H_{c,\alpha}$ reduces to the particular case of multiple priors, given by

$$H_{\mathcal{S},\alpha}(X) := \inf\left\{ k > 0 \,\middle|\, \sup_{\mathbb{Q}\in\mathcal{S}} \mathbb{E}_{\mathbb{Q}}\left[\Phi\left(\frac{X}{k}\right)\right] \le 1 - \alpha \right\}.$$

We now slightly generalize the results of [15] concerning the properties of robust Orlicz premia to the case where $\alpha \in [0, 1)$.

PROPOSITION 3.14. *For all $X \in L^\infty_+$ with $X \neq 0$, $H_{c,\alpha}(X)$ is the unique solution of*

$$\sup_{\mathbb{Q}\in\mathcal{Q}} \left\{ \mathbb{E}_{\mathbb{Q}}\left[\Phi\left(\frac{X}{H_{c,\alpha}(X)}\right)\right] - c(\mathbb{Q}) \right\} = 1 - \alpha. \tag{3.17}$$

Moreover, $H_{c,\alpha}$ is monotone, subadditive, positive homogeneous and satisfies $H_{c,\alpha}(b) = \frac{b}{\Phi^{-1}(1-\alpha)}$, for any $b \geq 0$.

PROOF. The proof is similar to [15, Lm 5, Thm 3]. We omit to show that $H_{c,\alpha}(X)$, $X \neq 0$, is the solution of (3.17), since the same argument used in [15], for the case $\alpha = 0$, applies here too. We prove instead the following:

Monotonicity: take $X, Y \in L^\infty_+$, $X, Y \neq 0$, with $X \geq Y$. Then

$$\begin{aligned} &\sup_{\mathbb{Q}\in\mathcal{Q}} \left\{ \mathbb{E}_{\mathbb{Q}}\left[\Phi\left(\frac{Y}{H_{c,\alpha}(Y)}\right)\right] - c(\mathbb{Q}) \right\} \\ &= \sup_{\mathbb{Q}\in\mathcal{Q}} \left\{ \mathbb{E}_{\mathbb{Q}}\left[\Phi\left(\frac{X}{H_{c,\alpha}(X)}\right)\right] - c(\mathbb{Q}) \right\} \\ &\le \sup_{\mathbb{Q}\in\mathcal{Q}} \left\{ \mathbb{E}_{\mathbb{Q}}\left[\Phi\left(\frac{Y}{H_{c,\alpha}(X)}\right)\right] - c(\mathbb{Q}) \right\} \end{aligned}$$

where the equality holds by (3.17) and the inequality by monotonicity of both Φ and the expectation. The thesis therefore follows, since, for any $\mathbb{Q} \in \mathcal{Q}$, $\mathbb{E}_{\mathbb{Q}}\left[\Phi\left(\frac{X}{h}\right)\right]$ is decreasing in $h > 0$.

Subadditivity: take $X, Y \in L^\infty_+$, such that $X, Y \neq 0$, and set

$$\lambda_H := \frac{H_{c,\alpha}(X)}{H_{c,\alpha}(X) + H_{c,\alpha}(Y)} \in (0, 1).$$

Then,

$$\begin{aligned}
&\sup_{\mathbb{Q}\in\mathcal{Q}}\left\{\mathbb{E}_{\mathbb{Q}}\left[\Phi\left(\frac{X+Y}{H_c(X)+H_c(Y)}\right)\right]-c(\mathbb{Q})\right\}\\
&=\sup_{\mathbb{Q}\in\mathcal{Q}}\left\{\mathbb{E}_{\mathbb{Q}}\left[\Phi\left(\lambda_H\frac{X}{H_c(X)}+(1-\lambda_H)\frac{Y}{H_c(Y)}\right)\right]-c(\mathbb{Q})\right\}\\
&\leq\sup_{\mathbb{Q}\in\mathcal{Q}}\left\{\mathbb{E}_{\mathbb{Q}}\left[\lambda_H\Phi\left(\frac{X}{H_c(X)}\right)+(1-\lambda_H)\Phi\left(\frac{Y}{H_c(Y)}\right)\right]-c(\mathbb{Q})\right\}\\
&\leq\lambda_H\sup_{\mathbb{Q}\in\mathcal{Q}}\left\{\mathbb{E}_{\mathbb{Q}}\left[\Phi\left(\frac{X}{H_c(X)}\right)\right]-c(\mathbb{Q})\right\}\\
&\quad+(1-\lambda_H)\sup_{\mathbb{Q}\in\mathcal{Q}}\left\{\mathbb{E}_{\mathbb{Q}}\left[\Phi\left(\frac{Y}{H_c(Y)}\right)\right]-c(\mathbb{Q})\right\}\\
&=1-\alpha=\sup_{\mathbb{Q}\in\mathcal{Q}}\left\{\mathbb{E}_{\mathbb{Q}}\left[\Phi\left(\frac{X+Y}{H_c(X+Y)}\right)\right]-c(\mathbb{Q})\right\},
\end{aligned}$$

where the first inequality holds by convexity of Φ and the last two equalities by (3.17). The thesis then follows, as before.

Positive homogeneity: the case $\lambda=0$ is trivial. Take then $\lambda>0$, for $X\in L^\infty_+$, $X\neq 0$, we have

$$\begin{aligned}
H_{c,\alpha}(\lambda X)&=\inf\left\{k>0\;\middle|\;\sup_{\mathbb{Q}\in\mathcal{Q}}\left\{\mathbb{E}_{\mathbb{Q}}\left[\Phi\left(\frac{\lambda X}{k}\right)\right]-c(\mathbb{Q})\right\}\leq 1-\alpha\right\}\\
&=\inf\left\{\lambda h>0\;\middle|\;\sup_{\mathbb{Q}\in\mathcal{Q}}\left\{\mathbb{E}_{\mathbb{Q}}\left[\Phi\left(\frac{X}{h}\right)\right]-c(\mathbb{Q})\right\}\leq 1-\alpha\right\}\\
&=\lambda H_{c,\alpha}(X)
\end{aligned}$$

where we set $h=\frac{k}{\lambda}$ to obtain the second inequality.

Finally, for $b\geq 0$, we consider

$$\begin{aligned}
H_{c,\alpha}(b)&=\inf\left\{k>0\;\middle|\;\sup_{\mathbb{Q}\in\mathcal{Q}}\left\{\mathbb{E}_{\mathbb{Q}}\left[\Phi\left(\frac{b}{k}\right)\right]-c(\mathbb{Q})\right\}\leq 1-\alpha\right\}\\
&=\inf\left\{k>0\;\middle|\;\Phi\left(\frac{b}{k}\right)\leq 1-\alpha\right\}\\
&=\frac{b}{\Phi^{-1}(1-\alpha)}.
\end{aligned}$$

□

Further on, Bellini and Rosazza Gianin [15] defined the corresponding robust version of the Haezendonck-Goovaerts risk measure by using the same construction of the non-robust case. As before, we consider the more general case of $\alpha\in[0,1)$.

Definition 3.11. Let $H_{c,\alpha}$ be as in Definition 3.10. We define the *robust Haezendonck-Goovaerts risk measure* of $X\in L^\infty$ as

$$\pi_{c,\alpha}(X):=\inf_{x\in\mathbb{R}}\left\{x+H_{c,\alpha}\left((X-x)^+\right)\right\}. \tag{3.18}$$

The case of multiple priors reduces to

$$\pi_{\mathcal{S},\alpha}(X) := \inf_{x\in\mathbb{R}} \left\{ x + H_{\mathcal{S},\alpha}\big((X-x)^+\big) \right\}. \qquad (3.19)$$

For $\alpha = 0$, $\pi_c := \pi_{c,0}$ corresponds to the robust HG risk measure studied by Bellini and Rosazza Gianin [15]. The authors showed that π_c is a coherent risk measure, see [15, Prop. 5]. The following result investigates the case of $\alpha \in [0,1)$.

PROPOSITION 3.15. *For any $\alpha \in [0,1)$, $\pi_{c,\alpha}$ is a coherent risk measure.*

PROOF. The proof follows the scheme of [14, Prop. 12].

Monotonicity: take $X, Y \in L^\infty$, with $X \geq Y$. Then

$$\begin{aligned}\pi_{c,\alpha}(X) &= \inf_{x\in\mathbb{R}} \left\{ x + H_{c,\alpha}\big((X-x)^+\big) \right\} \\ &\geq \inf_{x\in\mathbb{R}} \left\{ x + H_{c,\alpha}\big((Y-x)^+\big) \right\} = \pi_{c,\alpha}(Y),\end{aligned}$$

where the inequality holds by monotonicity of $H_{c,\alpha}$.

Subadditivity: take $X, Y \in L^\infty$, then

$$\begin{aligned}\pi_{c,\alpha}(X+Y) &= \inf_{x\in\mathbb{R}} \left\{ x + H_{c,\alpha}\big((X+Y-x)^+\big) \right\} \\ &= \inf_{x,y\in\mathbb{R}} \left\{ x + y + H_{c,\alpha}\big((X+Y-x-y)^+\big) \right\} \\ &\leq \inf_{x,y\in\mathbb{R}} \left\{ x + y + H_{c,\alpha}\big((X-x)^+ + (Y-y)^+\big) \right\} \\ &\leq \inf_{x,y\in\mathbb{R}} \left\{ x + y + H_{c,\alpha}\big((X-x)^+\big) + H_{c,\alpha}\big((Y-y)^+\big) \right\} \\ &= \inf_{x\in\mathbb{R}} \left\{ x + H_{c,\alpha}\big((X-x)^+\big) \right\} + \inf_{y\in\mathbb{R}} \left\{ y + H_{c,\alpha}\big((Y-y)^+\big) \right\} \\ &= \pi_{c,\alpha}(X) + \pi_{c,\alpha}(Y).\end{aligned}$$

where the inequalities hold by subadditivity of the positive part and of $H_{c,\alpha}$, respectively.

Cash-additivity: take $X \in L^\infty$ and $a \in \mathbb{R}$, then

$$\begin{aligned}\pi_{c,\alpha}(X+a) &= \inf_{x\in\mathbb{R}} \left\{ x + H_{c,\alpha}\big((X+a-x)^+\big) \right\} \\ &= \inf_{y\in\mathbb{R}} \left\{ y + a + H_{c,\alpha}\big((X-y)^+\big) \right\} \\ &= \inf_{y\in\mathbb{R}} \left\{ y + H_{c,\alpha}\big((X-y)^+\big) \right\} + a \\ &= \pi_{c,\alpha}(X) + a,\end{aligned}$$

where we set $y = x - a$ to obtain the second equality.

Positive homogeneity: take $X \in L^\infty$ and $\lambda > 0$, then

$$\begin{aligned}\pi_{c,\alpha}(\lambda X) &= \inf_{x\in\mathbb{R}} \left\{x + H_{c,\alpha}\big((\lambda X - x)^+\big)\right\} \\ &= \inf_{x\in\mathbb{R}} \left\{\lambda x + H_{c,\alpha}\big((\lambda X - \lambda x)^+\big)\right\} \\ &= \inf_{x\in\mathbb{R}} \left\{\lambda x + \lambda H_{c,\alpha}\big((X - x)^+\big)\right\} \\ &= \lambda \pi_{c,\alpha}(X),\end{aligned}$$

where the third equality holds by positive homogeneity of $H_{c,\alpha}$ (and of the positive part). Take now $\lambda = 0$, then

$$\begin{aligned}\pi_{c,\alpha}(0) &= \inf_{x\in\mathbb{R}} \left\{x + H_{c,\alpha}\big((-x)^+\big)\right\} \\ &= \inf_{x\in\mathbb{R}} \left\{x + \frac{(-x)^+}{\Phi^{-1}(1-\alpha)}\right\} \\ &= \inf_{x\in\mathbb{R}} \left\{x\left(1 - \frac{\mathbb{1}_{\{x<0\}}}{\Phi^{-1}(1-\alpha)}\right)\right\} = 0,\end{aligned}$$

where the last equality holds because $\Phi^{-1}(1-\alpha) \le 1$. □

Our aim is now to generalize the approaches based on Orlicz quantiles (and later on linking functions) of Section 3.2 to provide a robust version of CAR. In order to do so, we first need to go back to the definition of Orlicz quantiles and notice that it depends on the particular choice of the probability measure $\mathbb{P}$. Therefore, a robust version of Orlicz quantiles is needed for our purpose. We firstly establish whether the infimum of (3.18) is attained or not. In such a case, then, we will focus on the minimizers.

3.4.1.2 *Existence of the minimum*

To simplify the notation, for $X \in L^\infty$ and $x \in \mathbb{R}$ we set

$$\pi_{c,\alpha}(X, x) := x + H_{c,\alpha}\big((X - x)^+\big). \tag{3.20}$$

so that $\pi_{c,\alpha}(X) = \inf_{x\in\mathbb{R}} \pi_{c,\alpha}(X, x)$. We also set $\pi_c(X, x) := \pi_{c,0}(X, x)$.

We now summarize those properties of $\pi_{c,\alpha}(X, x)$ which will be useful in the following.

PROPOSITION 3.16. *Let $X \in L^\infty$, $\alpha \in [0, 1)$ and $\pi_{c,\alpha}(X, x)$ be given by (3.20).*

A. *$\pi_{c,\alpha}(X, x)$ is convex in $x \in \mathbb{R}$.*

B. *$\pi_{c,\alpha}(X + b, x) = \pi_{c,\alpha}(X, x - b) + b$, for all $x, b \in \mathbb{R}$.*

C. *$\pi_{c,\alpha}(\lambda X, x) = \lambda \pi_{c,\alpha}\big(X, \frac{x}{\lambda}\big)$, for all $\lambda > 0$.*

PROOF. A. Convexity follows by simple computations and properties of $H_{c,\alpha}$. More precisely, for $\lambda \in [0,1]$ and $x, y \in \mathbb{R}$ we have

$$\begin{aligned}
&\lambda \pi_{c,\alpha}(X,x) + (1-\lambda)\pi_{c,\alpha}(X,y) \\
&= \lambda x + \lambda H_{c,\alpha}\big((X-x)^+\big) + (1-\lambda)y + (1-\lambda)H_{c,\alpha}\big((X-y)^+\big) \\
&\geq \lambda x + (1-\lambda)y + H_{c,\alpha}\big(\lambda(X-x)^+ + (1-\lambda)(X-y)^+\big) \\
&\geq \lambda x + (1-\lambda)y + H_{c,\alpha}\big((X-\lambda x-(1-\lambda)y)^+\big) \\
&= \pi_{c,\alpha}(X, \lambda x + (1-\lambda)y),
\end{aligned}$$

where the inequalities hold by convexity of $H_{c,\alpha}$ and of the positive part, respectively.

B. Take $b \in \mathbb{R}$, then

$$\begin{aligned}
\pi_{c,\alpha}(X+b,x) &= x + H_{c,\alpha}\big((X+b-x)^+\big) \\
&= y + H_{c,\alpha}\big((X-y)^+\big) + b \\
&= \pi_{c,\alpha}(X, x-b) + b,
\end{aligned}$$

where we set $y = x - b$ to obtain the second equality.

C. For $\lambda > 0$, we have

$$\begin{aligned}
\pi_{c,\alpha}(\lambda X,x) &= x + H_{c,\alpha}\big((\lambda X-x)^+\big) \\
&= \lambda\big(y + H_{c,\alpha}((X-y)^+)\big) \\
&= \lambda \pi_{c,\alpha}\Big(X, \frac{x}{\lambda}\Big),
\end{aligned}$$

where we set $y = \frac{x}{\lambda}$ to obtain the second equality and we used the positive homogeneity of $H_{c,\alpha}$. □

Similarly to the non robust case, also in the robust case the infimum in (3.18) is always attained for any $\alpha \neq 0$.

PROPOSITION 3.17. *If $\alpha \neq 0$ then the infimum in the definition of $\pi_{c,\alpha}$, given by (3.18), is always attained.*

PROOF. The proof follows the scheme of [14, Prop. 16]. Take $X \in L^\infty$ and $\pi_{c,\alpha}(X,x)$ as in (3.20), $\alpha \in (0,1)$. Since $\pi_{c,\alpha}(X,x)$ is convex in x (see Prop. 3.16) and $\pi_{c,\alpha}(X,x) = x$, for $x \geq \operatorname{ess\,sup}(X)$, it is enough to show that $\pi_{c,\alpha}(X,x)$ is decreasing on some interval, to prove the thesis. Take then $x < \operatorname{ess\,inf}(X)$; we are going to show that there exists a $b_0 \in \mathbb{R}$ such that $\pi_{c,\alpha}(X,x-b) - \pi_{c,\alpha}(X,x) > 0$ for any $b > b_0$.

First, we notice that, for $x < \operatorname{ess\,inf}(X)$ and $b > 0$, we have

$$\pi_{c,\alpha}(X,x-b) - \pi_{c,\alpha}(X,x) = H_{c,\alpha}(X-x+b) - H_{c,\alpha}(X-x) - b.$$

It remains to compare $H_{c,\alpha}(X-x+b)$ and $H_{c,\alpha}(X-x) + b$.

For $b > 0$, set

$$f(b) := \sup_{\mathbb{Q} \in \mathcal{Q}} \left\{ \mathbb{E}_{\mathbb{Q}}\left[\Phi\left(\frac{X-x+b}{H_{c,\alpha}(X-x)+b}\right)\right] - c(\mathbb{Q}) \right\}.$$

On one hand,

$$f(b) \geq \sup_{\mathbb{Q} \in \mathcal{Q}} \left\{ \mathbb{E}_{\mathbb{Q}} \left[\Phi \left(\frac{\operatorname{ess\,inf}(X) - x + b}{H_{c,\alpha}(\operatorname{ess\,sup}(X) - x) + b} \right) \right] - c(\mathbb{Q}) \right\}$$
$$= \Phi \left(\frac{\operatorname{ess\,inf}(X) - x + b}{\frac{\operatorname{ess\,sup}(X) - x}{\Phi^{-1}(1-\alpha)} + b} \right) \xrightarrow[b \to +\infty]{} \Phi(1) = 1,$$

since $\inf_{\mathbb{Q} \in \mathcal{Q}} c(\mathbb{Q}) = 0$, both $H_{c,\alpha}$ and Φ are monotone and Φ is continuous. On the other hand, it follows similarly that

$$f(b) \leq \sup_{\mathbb{Q} \in \mathcal{Q}} \left\{ \mathbb{E}_{\mathbb{Q}} \left[\Phi \left(\frac{\operatorname{ess\,sup}(X) - x + b}{H_{c,\alpha}(\operatorname{ess\,inf}(X) - x) + b} \right) \right] - c(\mathbb{Q}) \right\}$$
$$= \Phi \left(\frac{\operatorname{ess\,sup}(X) - x + b}{\frac{\operatorname{ess\,inf}(X) - x}{\Phi^{-1}(1-\alpha)} + b} \right) \xrightarrow[b \to +\infty]{} \Phi(1) = 1.$$

Therefore,

$$\lim_{b \to +\infty} f(b) = 1 > 1 - \alpha = \sup_{\mathbb{Q} \in \mathcal{Q}} \left\{ \mathbb{E}_{\mathbb{Q}} \left[\Phi \left(\frac{X - x + b}{H_{c,\alpha}(X - x + b)} \right) \right] - c(\mathbb{Q}) \right\}$$

holds because $\alpha \in (0,1)$. Hence, since $\mathbb{E}_{\mathbb{Q}}\left[\Phi\left(\frac{X}{h}\right)\right]$ is decreasing in $h > 0$ for each $\mathbb{Q} \in \mathcal{Q}$, it follows that there exists a $b_0 \in \mathbb{R}$ such that $H_{c,\alpha}(X - x + b) > H_{c,\alpha}(X - x) + b$ for any $b > b_0$. The thesis then follows. □

The result above shows that the infimum of $\pi_{c,\alpha}$ is always attained for $\alpha \neq 0$, similarly to the non-robust case. We now consider the case $\alpha = 0$, starting from the following.

Proposition 3.18. *If $\alpha = 0$ then, for all $X \in L^\infty$, $\pi_c(X, x)$ is increasing in $x \in \mathbb{R}$.*

Proof. Let $X \in L^\infty$ be arbitrarily fixed. For any $x \geq \operatorname{ess\,sup}(X)$ it holds that $\pi_c(X, x) = x$, which is strictly increasing. For any $x < \operatorname{ess\,sup}(X)$ and $b > 0$, it holds that

$$\begin{aligned}
&\pi_c(X, x - b) - \pi_c(X, x) \\
&\leq H_c\big((X - x)^+ + b\big) - H_c\big((X - x)^+\big) - b \\
&\leq H_c\big((X - x)^+\big) + b - H_c\big((X - x)^+\big) - b = 0,
\end{aligned}$$

by $H_c(b) = b$ and by subadditivity of the positive part and of H_c. □

It follows from the proposition above that, for $\alpha = 0$, either the infimum in (3.18) is not attained or it is attained at any point of $(-\infty, x_0]$ for some $x_0 \leq \operatorname{ess\,sup}(X)$.

It is true, instead, that the infimum is always attained for constant random variables, even in the case $\alpha = 0$. Indeed, in the latter case we have, for $a > 0$,

$$\pi_c(a, x) = \begin{cases} a & \text{if } x < a; \\ x & \text{if } x \geq a; \end{cases}$$

and so the infimum is attained at any point of the interval $(-\infty, a]$.

The following result investigates the existence of the minimum for $\alpha = 0$ when $\Phi(x) = x$, in terms of conditions on the penalty function.

PROPOSITION 3.19. *Let $\Phi(x) = x$ and let $X \in L^\infty$ be non-constant. If $\inf_{\mathbb{Q}\in\mathcal{Q}} \frac{c(\mathbb{Q})}{1+c(\mathbb{Q})} > 0$, then the infimum in $\pi_c(X)$ is not attained.*

PROOF. If $\Phi(x) = x$, then $H_c(X) = \sup_{\mathbb{Q}\in\mathcal{Q}} \frac{\mathbb{E}_\mathbb{Q}[X]}{1+c(\mathbb{Q})}$ for all $X \in L^\infty_+$. By (b) of Proposition 3.18, $\pi_c(X, x)$ is increasing in any $x \in \mathbb{R}$.

For each $X \in L^\infty$, it is therefore enough to consider $x \leq \operatorname{ess\,inf}(X)$. Then for any $b > 0$ we have

$$\begin{aligned}
&\pi_c(X, x-b) - \pi_c(X, x) \\
&= H_c(X - x + b) - H_c(X - x) - b \\
&= \sup_{\mathbb{Q}\in\mathcal{Q}} \frac{\mathbb{E}_\mathbb{Q}[X - x + b]}{1 + c(\mathbb{Q})} - \sup_{\mathbb{Q}\in\mathcal{Q}} \frac{\mathbb{E}_\mathbb{Q}[X - x]}{1 + c(\mathbb{Q})} - b \\
&\leq \sup_{\mathbb{Q}\in\mathcal{Q}} \frac{b}{1 + c(\mathbb{Q})} - b \\
&= b \left(\sup_{\mathbb{Q}\in\mathcal{Q}} \frac{1}{1 + c(\mathbb{Q})} - 1 \right) \\
&= -b \inf_{\mathbb{Q}\in\mathcal{Q}} \frac{c(\mathbb{Q})}{1 + c(\mathbb{Q})} < 0
\end{aligned}$$

by hypothesis. So, $\pi_c(X, x)$ is strictly increasing on $(-\infty, \operatorname{ess\,inf}(X))$. The thesis then follows. □

Whenever $\inf_{\mathbb{Q}\in\mathcal{Q}} \frac{c(\mathbb{Q})}{1+c(\mathbb{Q})} = 0$, it can be easily checked that the infimum in $\pi_c(X)$ may be attained or not.

3.4.1.3 *Robust Orlicz quantiles*

Before introducing the notion of robust Orlicz quantiles we present two illustrative and motivating examples.

Example 3.8 (discrete distributions). Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, where $\Omega = \{\omega_1, \omega_2, \omega_3\}$ and $\mathbb{P}(\{\omega_i\}) > 0$, for each $i = 1, 2, 3$. We consider the random variable

$$X = \begin{cases} -4 & \text{on } \omega_1; \\ 4 & \text{on } \omega_2; \\ 8 & \text{on } \omega_3; \end{cases}$$

and the set $\mathcal{S} = \{\mathbb{Q}_1, \mathbb{Q}_2\}$ of probability measures absolutely continuous with respect to $\mathbb{P}$ such that:

$$\mathbb{Q}_1(\omega_1) = \mathbb{Q}_1(\omega_2) = \frac{1}{4};\ \mathbb{Q}_1(\omega_3) = \frac{1}{2}$$
$$\mathbb{Q}_2(\omega_1) = \frac{1}{8};\ \mathbb{Q}_2(\omega_2) = \frac{1}{2};\ \mathbb{Q}_2(\omega_3) = \frac{3}{8}.$$

For $\Phi(x) = x$, it follows that (see Bellini and Rosazza Gianin [15])

$$H_{\mathcal{S},\alpha}(X) = \sup_{\mathbb{Q} \in \mathcal{S}} \frac{\mathbb{E}_{\mathbb{Q}}[X]}{1-\alpha}$$

and

$$\pi_{\mathcal{S},\alpha}(X) = \inf_{x \in \mathbb{R}} \left\{ x + \sup_{\mathbb{Q} \in \mathcal{S}} \frac{\mathbb{E}_{\mathbb{Q}}\left[(X-x)^+\right]}{1-\alpha} \right\}.$$

We then compute

$$x + \frac{\mathbb{E}_{\mathbb{Q}_1}\left[(X-x)^+\right]}{1-\alpha} = \begin{cases} x + \frac{4-x}{1-\alpha} & \text{if } x \le -4; \\ x + \frac{(4-x)\frac{1}{4} + (8-x)\frac{1}{2}}{1-\alpha} & \text{if } -4 < x \le 4; \\ x + \frac{(8-x)\frac{1}{2}}{1-\alpha} & \text{if } 4 < x \le 8; \\ x & \text{if } x > 8; \end{cases}$$

and

$$x + \frac{\mathbb{E}_{\mathbb{Q}_2}\left[(X-x)^+\right]}{1-\alpha} = \begin{cases} x + \frac{\frac{9}{2}-x}{1-\alpha} & \text{if } x \le -4; \\ x + \frac{(4-x)\frac{1}{2} + (8-x)\frac{3}{8}}{1-\alpha} & \text{if } -4 < x \le 4; \\ x + \frac{(8-x)\frac{3}{8}}{1-\alpha} & \text{if } 4 < x \le 8; \\ x & \text{if } x > 8. \end{cases}$$

Thus, it can be easily checked that

$$\arg\max_{\mathbb{Q} \in \mathcal{S}} \mathbb{E}_{\mathbb{Q}}\left[(X-x)^+\right] = \begin{cases} \mathbb{Q}_2 & \text{if } x < 0; \\ \mathbb{Q}_1 & \text{if } 0 < x < 8; \\ \{\mathbb{Q}_1, \mathbb{Q}_2\} & \text{if } x = 0 \text{ or } x \ge 8. \end{cases}$$

For $\alpha = \frac{1}{4}$, it then follows that $\pi_{\mathcal{S},\alpha}(X) = \frac{20}{3}$ and the infimum in (3.19) is attained at any point of the interval $[0,4]$. For $\alpha = \frac{1}{2}$, it follows that $\pi_{\mathcal{S},\alpha}(X) = 8$ and the infimum in (3.19) is attained at any point of the interval $[4,8]$. So that the infimum is not unique in such cases.

For $\Phi(x) = x^2$, it follows that (see Bellini and Rosazza Gianin [15])

$$H_{\mathcal{S},\alpha}(X) = \sup_{\mathbb{Q}\in\mathcal{S}} \sqrt{\frac{\mathbb{E}_{\mathbb{Q}}[X^2]}{1-\alpha}}$$

and

$$\pi_{\mathcal{S},\alpha}(X) = \inf_{x\in\mathbb{R}} \left\{ x + \sup_{\mathbb{Q}\in\mathcal{S}} \sqrt{\frac{\mathbb{E}_{\mathbb{Q}}[((X-x)^+)^2]}{1-\alpha}} \right\}.$$

We then compute

$$x + \sqrt{\frac{\mathbb{E}_{\mathbb{Q}_1}[((X-x)^+)^2]}{1-\alpha}} = \begin{cases} x + \sqrt{\frac{x^2-8x+40}{1-\alpha}} & \text{if } x \leq -4; \\ x + \sqrt{\frac{\frac{3}{4}x^2-10x+36}{1-\alpha}} & \text{if } -4 < x \leq 4; \\ x + \frac{8-x}{\sqrt{2(1-\alpha)}} & \text{if } 4 < x \leq 8; \\ x & \text{if } x > 8; \end{cases}$$

and

$$x + \sqrt{\frac{\mathbb{E}_{\mathbb{Q}_2}[((X-x)^+)^2]}{1-\alpha}} = \begin{cases} x + \sqrt{\frac{x^2-9x+34}{1-\alpha}}; & \text{if } x \leq -4; \\ x + \sqrt{\frac{\frac{7}{8}x^2-10x+32}{1-\alpha}} & \text{if } -4 < x \leq 4; \\ x + \sqrt{\frac{3}{8(1-\alpha)}}(8-x) & \text{if } 4 < x \leq 8; \\ x & \text{if } x > 8. \end{cases}$$

Thus, it holds that

$$\arg\max_{\mathbb{Q}\in\mathcal{S}} \mathbb{E}_{\mathbb{Q}}\left[\left((X-x)^+\right)^2\right] = \begin{cases} \mathbb{Q}_2 & \text{if } x < -6; \\ \mathbb{Q}_1 & \text{if } -6 < x < 8; \\ \{\mathbb{Q}_1, \mathbb{Q}_2\} & \text{if } x = -6 \text{ or } x \geq 8. \end{cases}$$

For $\alpha = \frac{1}{2}$, we have $\pi_{\mathcal{S},\alpha}(X) = 8$ and the infimum in (3.19) is attained at any point of the interval $[4,8]$. For $\alpha = \frac{5}{8}$, we have $\pi_{\mathcal{S},\alpha}(X) = 8$ and the infimum in (3.19) is attained at $x^* = 8$. Notice that, for $\alpha = \frac{1}{2}$, the minimum point is not unique, even if Φ is strictly convex.

EXAMPLE 3.9 (continuous distributions). Given a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and a random variable $X \in L^\infty$, consider the set $\mathcal{S} = \{\mathbb{Q}_1, \mathbb{Q}_2, \mathbb{Q}_3\}$ of probability measures absolutely continuous with respect to $\mathbb{P}$ such that:

- Under $\mathbb{Q}_1$, X has density function $f_X(x) = \frac{1}{2}\mathbb{1}_{[-1,1]}$, that is, X has Uniform distribution on $[-1,1]$.
- Under $\mathbb{Q}_2$, X has density function $f_X(x) = (1-|x|)\,\mathbb{1}_{[-1,1]}$, that is, X has symmetric Triangular distribution on $[-1,1]$.
- Under $\mathbb{Q}_3$, X has density function $f_X(x) = \frac{3}{4}(1-x^2)\,\mathbb{1}_{[-1,1]}$.

Take now $\Phi(x) = x$. As before, we have

$$H_{\mathcal{S},\alpha}(X) = \sup_{\mathbb{Q}\in\mathcal{S}} \frac{\mathbb{E}_{\mathbb{Q}}[X]}{1-\alpha} \quad \text{and} \quad \pi_{\mathcal{S},\alpha}(X,x) = x + \sup_{\mathbb{Q}\in\mathcal{S}} \frac{\mathbb{E}_{\mathbb{Q}}\left[(X-x)^+\right]}{1-\alpha}.$$

Therefore, we compute

$$\mathbb{E}_{\mathbb{Q}_1}\left[(X-x)^+\right] = \begin{cases} -x & \text{if } x \leq -1; \\ \frac{(1-x)^2}{4} & \text{if } -1 < x \leq 1; \\ 0 & \text{if } x > 1; \end{cases}$$

$$\mathbb{E}_{\mathbb{Q}_2}\left[(X-x)^+\right] = \begin{cases} -x & \text{if } x \leq -1; \\ \frac{-x^2|x|+3x^2-3x+1}{6} & \text{if } -1 < x \leq 1; \\ 0 & \text{if } x > 1; \end{cases}$$

$$\mathbb{E}_{\mathbb{Q}_3}\left[(X-x)^+\right] = \begin{cases} -x & \text{if } x \leq -1; \\ \frac{-x^4+6x^2-8x+3}{16} & \text{if } -1 < x \leq 1; \\ 0 & \text{if } x > 1; \end{cases}$$

so that the sup is attained at $\mathbb{Q}_1$ and

$$\pi_{\mathcal{S},\alpha}(X,x) = x + \frac{\mathbb{E}_{\mathbb{Q}_1}\left[(X-x)^+\right]}{1-\alpha} = \begin{cases} -\frac{\alpha}{1-\alpha}x & \text{if } x \leq -1; \\ x + \frac{(1-x)^2}{4(1-\alpha)} & \text{if } -1 < x \leq 1; \\ x & \text{if } x > 1. \end{cases}$$

For $\alpha = \frac{1}{2}$, it follows that $\pi_{\mathcal{S},\alpha}(X) = \frac{1}{2}$ and the infimum in (3.19) is attained at $x^* = 0$. For $\alpha = \frac{1}{4}$, it follows that $\pi_{\mathcal{S},\alpha}(X) = \frac{1}{4}$ and the infimum in (3.19) is attained at $x^* = -\frac{1}{2}$.

Take now $\Phi(x) = x^2$. It follows that

$$H_{\mathcal{S},\alpha}(X) = \sup_{\mathbb{Q}\in\mathcal{S}} \sqrt{\frac{\mathbb{E}_{\mathbb{Q}}[X^2]}{1-\alpha}},$$

$$\pi_{\mathcal{S},\alpha}(X,x) = x + \sup_{\mathbb{Q}\in\mathcal{S}} \sqrt{\frac{\mathbb{E}_{\mathbb{Q}}\left[((X-x)^+)^2\right]}{1-\alpha}}.$$

Therefore,

$$\mathbb{E}_{\mathbb{Q}_1}\left[((X-x)^+)^2\right] = \begin{cases} \frac{1}{3}+x^2 & \text{if } x \leq -1; \\ \frac{-x^3+3x^2-3x+1}{6} & \text{if } -1 < x \leq 1; \\ 0 & \text{if } x > 1; \end{cases}$$

$$\mathbb{E}_{\mathbb{Q}_2}\left[((X-x)^+)^2\right] = \begin{cases} \frac{1}{6}+x^2 & \text{if } x \leq -1; \\ \frac{x^3|x|-4x^3+6x^2-4x+1}{12} & \text{if } -1 < x \leq 1; \\ 0 & \text{if } x > 1; \end{cases}$$

$$\mathbb{E}_{\mathbb{Q}_3}\left[((X-x)^+)^2\right] = \begin{cases} \frac{1}{5}+x^2 & \text{if } x \leq -1; \\ \frac{x^5-10x^3+20x^2-15x+4}{40} & \text{if } -1 < x \leq 1; \\ 0 & \text{if } x > 1; \end{cases}$$

so that the sup is attained at $\mathbb{Q}_1$ and

$$\pi_{\mathcal{S},\alpha}(X,x) = x + \sqrt{\frac{\mathbb{E}_{\mathbb{Q}_1}\left[((X-x)^+)^2\right]}{1-\alpha}}$$

$$= \begin{cases} x+\sqrt{\frac{3x^2+1}{3(1-\alpha)}} & \text{if } x \leq -1; \\ x+\sqrt{\frac{-x^3+3x^2-3x+1}{6(1-\alpha)}} & \text{if } -1 < x \leq 1; \\ x & \text{if } x > 1. \end{cases}$$

For $\alpha = \frac{1}{2}$, it follows that $\pi_{\mathcal{S},\alpha}(X) = \frac{5}{9}$ and the infimum in (3.19) is attained at $x^* = -\frac{1}{3}$. For $\alpha = \frac{1}{4}$, it follows that $\pi_{\mathcal{S},\alpha}(X) = \frac{1}{3}$ and the infimum in (3.19) is attained at $x^* = -1$. While, for instance, if $\alpha = \frac{1}{9}$, it follows that $\pi_{\mathcal{S},\alpha}(X) = \frac{1}{2\sqrt{6}}$ and the infimum in (3.19) is attained at $x^* = -\frac{2}{3}\sqrt{6}$. Therefore, in such a case we have $x^* < -1 = \operatorname{ess\,inf}(X)$. The latter also shows that the minimizers of (3.20) are not monotone: by simply taking $Y = -1$, we obtain that $y^* = -1$ while $x^* < -1$, even though $X \geq Y$.

The examples above motivate us to extend the definition of Orlicz quantiles (depending on the probability $\mathbb{P}$ given a priori) to the present setting dealing with ambiguity on the choice of $\mathbb{P}$. Differently from the non-robust case, where for $\Phi(x) = x$ the minimizers x^*_α reduce to classical quantiles with respect to $\mathbb{P}$, in the present setting (corresponding to ambiguity and to multiple priors - penalized by c or not) the minimizers take into account all the multiple priors, hence they can be interpreted as "robust quantiles". Referring to the previous example, indeed, while $q_{\mathbb{Q}_1,\alpha} = [-4,4]$ and $q_{\mathbb{Q}_2,\alpha} = \{4\}$ for $\alpha = \frac{1}{4}$, $q_{\mathbb{Q}_1,\alpha} = [4,8]$ and $q_{\mathbb{Q}_2,\alpha} = \{4\}$ for $\alpha = \frac{1}{2}$ (where $q_{\mathbb{Q},\alpha}$ denotes the set of α-quantiles with respect to $\mathbb{Q}$), in the robust case $q_{\mathcal{S},\alpha} = [0,4]$ for

$\alpha = \frac{1}{4}$ and $q_{\mathcal{S},\alpha} = [4, 8]$ for $\alpha = \frac{1}{2}$ (where $q_{\mathcal{S},\alpha}$ denotes the set of minimizers in the multiple prior case, later called robust α-quantiles).

In Proposition 3.17, we established that, for any $\alpha \neq 0$, the infimum in $\pi_{c,\alpha}$ is always attained. Motivated also by the previous examples, it looks then natural to follow the same scheme of the non-robust case and call any

$$x^*_{c,\alpha}(X) \in \arg\min_{x\in\mathbb{R}} \pi_{c,\alpha}(X, x)$$

a *robust Orlicz quantile* at level α of X.

We now study in detail the properties of robust Orlicz quantiles, using the notations

$$x^{*,-}_{c,\alpha}(X) := \inf \arg\min_{x\in\mathbb{R}} \pi_{c,\alpha}(X, x),$$
$$x^{*,+}_{c,\alpha}(X) := \sup \arg\min_{x\in\mathbb{R}} \pi_{c,\alpha}(X, x).$$

Proposition 3.20. *For any $\alpha \in (0, 1)$ and $X \in L^\infty$, the set of robust Orlicz quantiles is a closed interval satisfying cash-additivity, positive homogeneity and riskless. Moreover, robust Orlicz quantiles are bounded from above, i. e. $x^{*,+}_{c,\alpha}(X) \leq \operatorname{ess\,sup}(X)$ for all $X \in L^\infty$.*

Proof. The proof follows from Propositions 3.16 and 3.17, similarly to the non-robust case. □

Since robust Orlicz quantiles satisfy most of the properties of the non-robust ones, we extend now the definitions of CARs given in Section 3.2 to the robust case.

Definition 3.12. Given the robust Orlicz risk premium $H_{c,\alpha}$ and the robust HG risk measure $\pi_{c,\alpha}$, we define $\Lambda^c_H \colon L^\infty_+ \times L^\infty \to \mathbb{R}^+$ as

$$\Lambda^c_H(X, Y) := H_{c,\alpha}\big(X \mathbb{1}_{\{Y \geq x^*_{c,\alpha}(Y)\}}\big)$$

and the map $\Lambda^c_\pi \colon L^\infty \times L^\infty \to \mathbb{R}$ as

$$\begin{aligned}\Lambda^c_\pi(X, Y) &:= x^*_{c,\alpha}(X) + \Lambda^c_H\big((X - x^*_{c,\alpha}(X))^+, Y\big)\\ &= x^*_{c,\alpha}(X) + H_{c,\alpha}\big((X - x^*_{c,\alpha}(X))^+ \mathbb{1}_{\{Y \geq x^*_{c,\alpha}(Y)\}}\big)\end{aligned}$$

where $x^*_{c,\alpha}(X)$ is a robust Orlicz quantile at level α of X.

As previously, in the following we fix $x^*_{c,\alpha}(X)$ to be the upper robust Orlicz quantile at level α of X.

Proposition 3.21. *The map Λ^c_H is an audacious CAR for $H_{c,\alpha}$ satisfying no-undercut, monotonicity, 1-positive homogeneity and 2-translation-invariance, while Λ^c_π is a CAR for $\pi_{c,\alpha}$ satisfying no-undercut, riskless, 1-cash-additivity, 1-positive homogeneity, 2-translation-invariance and cash-additivity.*

PROOF. The proof can be driven from the non-robust case. □

Similarly to the non-robust case, it is possible to extend also $\bar{\Lambda}_\pi$ of Definition 3.5 and $\widetilde{\Lambda}_\pi$ of Definition 3.6 to the robust case.

3.4.2 *The case of multiple* Φ

In the previous section, we have considered robust versions of Orlicz premia that overcome the ambiguity over the true probabilistic model $\mathbb{P}$. Here, instead, we consider the situation whereby the decision-maker is uncertain about the Young function to be used, while we assume there is only one probabilistic model $\mathbb{P}$. As before, we follow the scheme of Bellini and Rosazza Gianin [15] and take a worst-case approach for the multiplicity of possible Young functions.

We begin by clarifying which set of Young functions is suitable for the purpose.

DEFINITION 3.13. A non-empty set $\mathcal{P}$ of Young functions, equipped with the pointwise order (i. e. $\Psi \geq \Phi :\Longleftrightarrow \Psi(x) \geq \Phi(x),\ \forall x > 0$), is said to be proper if $(\sup \mathcal{P})(x) = \sup_{\Phi \in \mathcal{P}} \Phi(x) < +\infty,\ \forall x > 0$.

Notice that $\sup \mathcal{P}$ is still a Young function, whenever $\mathcal{P}$ is proper. Before going further, we provide an example of such a set $\mathcal{P}$.

EXAMPLE 3.10. Consider the set

$$\mathcal{P} = \left\{ \Phi_p(x) = x^p,\ x \geq 0 \,\middle|\, 1 \leq p \leq k \right\}$$

for some fixed $k > 1$. It is clear that each element of $\mathcal{P}$ is a Young function, moreover

$$(\sup \mathcal{P})(x) = \begin{cases} x & \text{if } 0 \leq x \leq 1; \\ x^k & \text{if } x > 1; \end{cases}$$

is still a Young function even if it does not belong to $\mathcal{P}$. Suppose now we slightly modify the previous set and define

$$\mathcal{P}' = \left\{ \Phi_p(x) = x^p,\ x \geq 0 \,\middle|\, 1 \leq p < +\infty \right\}.$$

Each element of $\mathcal{P}'$ is still a Young function but now we have

$$(\sup \mathcal{P}')(x) = \begin{cases} x & \text{if } 0 \leq x \leq 1; \\ +\infty & \text{if } x > 1; \end{cases}$$

which is no longer finite for any $x > 1$.

We generalize here below to $\alpha \in [0, 1)$ the definition introduced by Bellini and Rosazza Gianin [15] for $\alpha = 0$.

Definition 3.14. Let $\mathcal{P}$ be a proper set of Young functions and let $\alpha \in [0,1)$ be fixed. The *Φ-robust Orlicz risk premium* of $X \in L^\infty_+$ is defined as

$$H_{\mathcal{P},\alpha}(X) := \inf\left\{ k > 0 \,\middle|\, \sup_{\Phi \in \mathcal{P}} \mathbb{E}\left[\Phi\left(\frac{X}{k}\right)\right] \le 1-\alpha \right\}. \qquad (3.21)$$

It then follows that $H_{\mathcal{P}}(0) = 0$. For $\alpha = 0$, $H_{\mathcal{P}} := H_{\mathcal{P},0}$ reduces to that of Bellini and Rosazza Gianin [15].

We now consider the properties of $H_{\mathcal{P},\alpha}$, in the case $\alpha \in (0,1)$. We will use the notation $H^\Phi_\alpha(X) = H_\alpha(X)$, introduced in the beginning, for any Young function Φ.

The leading result, fundamental to prove any following one, is contained in the next proposition.

Proposition 3.22. *Let $\mathcal{P}$ be a proper set of Young functions. Then, for all $X \in L^\infty_+$, it holds that*

$$H_{\mathcal{P},\alpha}(X) = H^{\sup \mathcal{P}}_\alpha(X).$$

Proof. Take $X \in L^\infty_+$. Bellini and Rosazza Gianin [15, Prop. 25 (b)] shows that $H_{\mathcal{P}}(X) = \sup_{\Phi\in\mathcal{P}} H^\Phi(X)$ holds; since the same argument remains valid for $\alpha \neq 0$, we assume such result here too. Therefore, we only need to prove that $\sup_{\Phi\in\mathcal{P}} H^\Phi_\alpha(X) = H^{\sup\mathcal{P}}_\alpha(X)$. To this end, since $\mathcal{P}$ is proper, it is enough to prove that H^Φ_α is monotone increasing in Φ. Take any $\Psi, \Phi \in \mathcal{P}$, with $\Psi \ge \Phi$, then

$$\mathbb{E}\left[\Phi\left(\frac{X}{H^\Psi_\alpha(X)}\right)\right] \le \mathbb{E}\left[\Psi\left(\frac{X}{H^\Psi_\alpha(X)}\right)\right] = 1-\alpha = \mathbb{E}\left[\Phi\left(\frac{X}{H^\Phi_\alpha(X)}\right)\right]$$

hence $H^\Psi_\alpha(X) \ge H^\Phi_\alpha(X)$, since $\mathbb{E}\left[\Phi\left(\frac{X}{h}\right)\right]$ is decreasing in $h > 0$. □

The following result is then straightforward.

Proposition 3.23. *Let $\mathcal{P}$ be a proper set of Young functions. Then $H_{\mathcal{P},\alpha}$ is monotone, subadditive and positive homogeneous. Moreover, if $\sup\mathcal{P}$ is strictly convex then $H_{\mathcal{P},\alpha}$ also satisfy the following, for $X, Y \in L^\infty_+$:*

Strict monotonicity: if $X \ge Y$, $\mathbb{P}(X > Y) > 0$ then $H_{\mathcal{P},\alpha}(X) > H_{\mathcal{P},\alpha}(Y)$.

Strict subadditivity: if $X \neq Y$, $X \neq 0 \neq Y$ and at least one of them is not constant then $H_{\mathcal{P},\alpha}(X+Y) < H_{\mathcal{P},\alpha}(X) + H_{\mathcal{P},\alpha}(Y)$.

Proof. By Proposition 3.22, we have $H_{\mathcal{P},\alpha} = H^{\sup\mathcal{P}}_\alpha$, where $\sup\mathcal{P}$ is a Young function, since $\mathcal{P}$ is proper. All the results then follow by [14, Prop. 2], since they hold for any Young function. □

We now provide an example of a proper set of Young functions $\mathcal{P}$ such that $\sup\mathcal{P}$ is strictly convex.

Example 3.11. Consider the set

$$\mathcal{P} = \left\{ \Phi_\beta(x) = \frac{e^{\beta x} - 1}{e^{\beta} - 1}, \, x \geq 0 \;\middle|\; 0 < \alpha < \beta \leq \gamma \right\}$$

for some fixed $\gamma > 0$. Each member of $\mathcal{P}$ is a strictly convex Young function, moreover

$$(\sup \mathcal{P})(x) = \begin{cases} \dfrac{e^{\alpha x} - 1}{e^{\alpha} - 1} & \text{if } 0 \leq x \leq 1; \\ \dfrac{e^{\gamma x} - 1}{e^{\gamma} - 1} & \text{if } x > 1; \end{cases}$$

is still a strictly convex Young function.

Similarly to the case of ambiguity over $\mathbb{P}$, we provide a Φ-robust version of the Haezendonck-Goovaerts risk measure as an extension of that introduced in [15] for $\alpha = 0$.

Definition 3.15. Let $H_{\mathcal{P},\alpha}$ be defined by (3.21) and $\alpha \in [0,1)$. The *Φ-robust Haezendonck-Goovaerts risk measure* of $X \in L^\infty$ is defined as

$$\pi_{\mathcal{P},\alpha}(X) := \inf_{x \in \mathbb{R}} \left\{ x + H_{\mathcal{P},\alpha}\big((X - x)^+\big) \right\}.$$

Thanks to Proposition 3.22, the following properties of $\pi_{\mathcal{P},\alpha}$ are straightforward.

Proposition 3.24. *For each $\alpha \in [0,1)$, $\pi_{\mathcal{P},\alpha}$ is a law invariant coherent risk measure.*

Proof. By Proposition 3.22, we have $H_{\mathcal{P},\alpha} = H_\alpha^{\sup \mathcal{P}}$, where $\sup \mathcal{P}$ is a Young function, since $\mathcal{P}$ is proper. The result then follows by [14, Prop. 12], since it holds for any Young function. □

As in the case of ambiguity about the probabilistic model, to pursue our purpose we need to establish if the infimum of $\pi_{\mathcal{P},\alpha}$ is attained or not. However, in this case, it is clear that infimum is always attained for $\alpha \neq 0$, since $\pi_{\mathcal{P},\alpha}$ is simply π_α with $\sup \mathcal{P}$ as Young function.

We define, indeed, a *Φ-robust Orlicz quantile* at level α of X as any

$$x^*_{\mathcal{P},\alpha}(X) \in \arg\min_{x \in \mathbb{R}} \pi_{\mathcal{P},\alpha}(X, x).$$

It is clear that Φ-robust Orlicz quantiles satisfy the same properties of non-robust ones, as in Proposition 3.2. Therefore, we can define Φ-robust CARs with the same properties of the non-robust ones.

Definition 3.16. Given the Φ-robust Orlicz risk premium $H_{\mathcal{P},\alpha}$ and the Φ-robust HG risk measure $\pi_{\mathcal{P},\alpha}$, we define $\Lambda_H^{\mathcal{P}} \colon L^\infty_+ \times L^\infty \to \mathbb{R}^+$ as

$$\Lambda_H^{\mathcal{P}}(X, Y) := H_{\mathcal{P},\alpha}\big(X \mathbb{1}_{\{Y \geq x^*_{\mathcal{P},\alpha}(Y)\}}\big).$$

and $\Lambda_\pi^{\mathcal{P}}\colon L^\infty \times L^\infty \to \mathbb{R}$ as

$$\begin{aligned}\Lambda_\pi^{\mathcal{P}}(X,Y) &:= x^*_{\mathcal{P},\alpha}(X) + \Lambda_H^{\mathcal{P}}\big((X - x^*_{\mathcal{P},\alpha}(X))^+, Y\big)\\ &= x^*_{\mathcal{P},\alpha}(X) + H_{\mathcal{P},\alpha}\big((X - x^*_{\mathcal{P},\alpha}(X))^+ \mathbb{1}_{\{Y \geq x^*_{\mathcal{P},\alpha}(Y)\}}\big)\end{aligned}$$

where $x^*_{\mathcal{P},\alpha}(X)$ is a Φ-robust Orlicz quantile at level α of X.

As previously, we fix $x^*_{\mathcal{P},\alpha}(X)$ to be the upper Φ-robust Orlicz quantile at level α of X.

The following is then straightforward.

Proposition 3.25. *$\Lambda_H^{\mathcal{P}}$ is an audacious CAR for $H_{\mathcal{P},\alpha}$ which satisfies no-undercut, monotonicity, 1-law invariance, 1-positive homogeneity and 2-translation-invariance; while $\Lambda_\pi^{\mathcal{P}}$ is a CAR for $\pi_{\mathcal{P},\alpha}$ satisfying no-undercut, riskless, 1-cash-additivity, 1-law invariance, 1-positive homogeneity, 2-translation-invariance and cash-additivity.*

Proof. Thanks to Proposition 3.22, the proof is similar to the non-robust case and thus omitted here. □

As previously, it is possible to extend also $\bar{\Lambda}_\pi$ of Definition 3.5 and $\widetilde{\Lambda}_\pi$ of Definition 3.6 to the Φ-robust case.

3.5 ROBUST VERSIONS - LINKING-BASED APPROACH

We provide here robust versions of the linking based approaches, both considering the ambiguity over the probabilistic model and the uncertainty over the Young function to be chosen.

Definition 3.17. Let $H_{c,\alpha}$ be the robust Orlicz risk premium, given by Definition 3.10, and let $f\colon L_+^\infty \times L^\infty \to L_+^\infty$ be a linking function. We define the robust H-linking CAR as the map $\Lambda_H^{c,f} : L_+^\infty \times L^\infty \to \mathbb{R}_+$ given by

$$\Lambda_H^{c,f}(X,Y) := H_{c,\alpha}(f(X,Y)), \quad X \in L_+^\infty, Y \in L^\infty.$$

It is clear that $\Lambda_H^{c,f}$ is a capital allocation rule with respect to the robust Orlicz risk premium, since f is linking. Moreover, for $f(X,Y) \neq 0$, $\Lambda_H^{c,f}(X,Y)$ is the unique solution of

$$\sup_{\mathbb{Q}\in\mathcal{Q}} \left\{ \mathbb{E}_{\mathbb{Q}}\left[\Phi\left(\frac{f(X,Y)}{\Lambda_H^{c,f}(X,Y)}\right)\right] - c(\mathbb{Q}) \right\} = 1.$$

We now study when $\Lambda_H^{c,f}$ satisfies some of the usual properties required for a capital allocation rule. The result turns out to be very similar to the non-robust case.

PROPOSITION 3.26. *Let f be a linking function.*

A. *If $f(X,Y) \leq f(X,X) = X$ for any $X \in L^\infty_+$ and $Y \in L^\infty$, then $\Lambda^{c,f}_H$ satisfies no-undercut.*

B. *If $f(a,Y) = a$ for any $a \geq 0$ and $Y \in L^\infty$, then $\Lambda^{c,f}_H$ satisfies riskless.*

C. *If f is monotone increasing in the first entry then $\Lambda^{c,f}_H$ is monotone.*

PROOF. The proof is the same as in the non-robust case and omitted here. □

We now provide a robust version of the π-linking CAR and study its properties. The results are again very similar to the non-robust case, due to the properties of $H_{c,\alpha}$.

DEFINITION 3.18. Given the map $\Lambda^{c,f}_H$ of Definition 3.17, we define the robust π-linking CAR as the map $\Lambda^{c,f}_\pi : L^\infty \times L^\infty \to \mathbb{R}$ given by

$$\begin{aligned}\Lambda^{c,f}_\pi(X,Y) &:= \inf_{x\in\mathbb{R}} \left\{x + \Lambda^{c,f}_H\left((X-x)^+,(Y-x)^+\right)\right\} \\ &= \inf_{x\in\mathbb{R}} \left\{x + H_{c,\alpha}\left(f\left((X-x)^+,(Y-x)^+\right)\right)\right\}.\end{aligned}$$

PROPOSITION 3.27. *Let f be a linking function.*

A. *$\Lambda^{c,f}_\pi$ is a CAR with respect to π_c.*

B. *If $f(Z,W) \leq f(Z,Z) = Z$ for any $Z \in L^\infty_+$ and $W \in L^\infty$, then $\Lambda^{c,f}_\pi$ satisfies no-undercut.*

C. *If $f(a,W) = a$ for any $a \geq 0$ and $W \in L^\infty$, then $\Lambda^{c,f}_\pi$ satisfies riskless.*

D. *If f is monotone increasing in the first entry then $\Lambda^{c,f}_\pi$ is monotone.*

E. *If $X,Y \in L^\infty_+$ then $\Lambda^{c,f}_\pi(X,Y) \leq \Lambda^{c,f}_H(X,Y)$.*

PROOF. The proof is the same as in the non-robust case and omitted here. □

We now provide a Φ-robust version of the H-linking CAR, given by Definition 3.8, and study its properties.

DEFINITION 3.19. Let $H_{\mathcal{P},\alpha}$ be given by (3.21), and let $f\colon L^\infty_+ \times L^\infty \to L^\infty_+$ be a linking function. We define the Φ-robust H-linking CAR as the map $\Lambda^{\mathcal{P},f}_H : L^\infty_+ \times L^\infty \to \mathbb{R}_+$ given by

$$\Lambda^{\mathcal{P},f}_H(X,Y) := H_{\mathcal{P}}(f(X,Y)), \quad X \in L^\infty_+, Y \in L^\infty.$$

It is clear that $\Lambda_H^{\mathcal{P},f}$ is a capital allocation rule with respect to the robust Orlicz risk premium, since f is linking. Moreover, for $f(X,Y) \neq 0$, $\Lambda_H^{\mathcal{P},f}(X,Y)$ is the unique solution of

$$\sup_{\Phi \in \mathcal{P}} \mathbb{E}\left[\Phi\left(\frac{f(X,Y)}{\Lambda_H^{\mathcal{P},f}(X,Y)}\right)\right] = 1.$$

We now study when $\Lambda_H^{\mathcal{P},f}$ satisfies some of the usual properties required for a capital allocation rule. As before, we obtain results very similar to the non-robust case.

PROPOSITION 3.28. *Let f be a linking function.*

A. *If $f(X,Y) \leq f(X,X) = X$ for any $X \in L_+^\infty$ and $Y \in L^\infty$, then $\Lambda_H^{\mathcal{P},f}$ satisfies no-undercut.*

B. *If $f(a,Y) = a$ for any $a \geq 0$ and $Y \in L^\infty$, then $\Lambda_H^{\mathcal{P},f}$ satisfies riskless.*

C. *If f is monotone increasing in the first entry then $\Lambda_H^{\mathcal{P},f}$ is monotone.*

PROOF. The proof is the same as in the non-robust case and omitted here. □

In the following, we provide a Φ-robust version of the π-linking CAR and study its properties. Once again, the results are very similar to the non-robust case, thanks to the properties of $H_{\mathcal{P},\alpha}$.

DEFINITION 3.20. Let $\Lambda_H^{\mathcal{P},f}$ be as in Definition 3.19. We define the Φ-robust π-linking CAR as the map $\Lambda_\pi^{\mathcal{P},f} : L^\infty \times L^\infty \to \mathbb{R}$ given by

$$\begin{aligned}\Lambda_\pi^{\mathcal{P},f}(X,Y) &:= \inf_{x\in\mathbb{R}} \left\{x + \Lambda_H^{\mathcal{P},f}\left((X-x)^+, (Y-x)^+\right)\right\} \\ &= \inf_{x\in\mathbb{R}} \left\{x + H_{\mathcal{P},\alpha}\left(f\left((X-x)^+, (Y-x)^+\right)\right)\right\}.\end{aligned}$$

PROPOSITION 3.29. *Let f be a linking function.*

A. *$\Lambda_\pi^{\mathcal{P},f}$ is a CAR with respect to $\pi_{\mathcal{P}}$.*

B. *If $f(Z,W) \leq f(Z,Z) = Z$ for any $Z \in L_+^\infty$ and $W \in L^\infty$, then $\Lambda_\pi^{\mathcal{P},f}$ satisfies no-undercut.*

C. *If $f(a,W) = a$ for any $a \geq 0$ and $W \in L^\infty$, then $\Lambda_\pi^{\mathcal{P},f}$ satisfies riskless.*

D. *If f is monotone increasing in the first entry then $\Lambda_\pi^{\mathcal{P},f}$ is monotone.*

E. *If $X, Y \in L_+^\infty$ then $\Lambda_\pi^{\mathcal{P},f}(X,Y) \leq \Lambda_H^{\mathcal{P},f}(X,Y)$.*

PROOF. The proof is the same as in the non-robust case and omitted here. □

4 CAPITAL ALLOCATION RULES AND ACCEPTANCE SETS

In this chapter, we discuss a new approach to face capital allocation problems from the perspective of acceptance sets, by introducing the notion of sub-acceptance family.

We first provide an example motivating the introduction of the notion of sub-acceptability, and later a formal definition, studying the relations with acceptability. We then define the notion of risk contribution rule, a tool which is similar to capital allocation rules but does not involve any risk measure. We show that, in this context, risk contribution rules are suitable for assessing the contribution of a sub-portfolio to a given portfolio, in term of acceptability. Moreover, we study under which conditions on a risk contribution rule a representation of an acceptance set holds in terms of the risk contribution rule itself, extending the interpretation of minimal amount required to hedge a risky position, classical in risk measures theory. Finally, we discuss some generalizations of the previous results to account for S-additive and quasi-convex risk measures.

The chapter is organized as follows. In Section 4.1 we introduce the notion of sub-acceptance sets, while in Section 4.2 we define the notion of risk contribution rule and prove the main results. Finally, Section 4.3 contains extensions to the S-additive and quasi-convex cases.

4.1 ACCEPTANCE AND SUB-ACCEPTANCE SETS

In the classical approach to capital allocation, given a position Y and a sub-portfolio X, $\Lambda(X, Y)$ reflects $\rho(Y)$. However, the capital allocation problem can be seen from another standpoint, as the following example shows.

Example 4.1. Suppose we are provided with a normalized monetary risk measure ρ to quantify the riskiness of financial positions, together with its acceptance set $\mathcal{A}_\rho$. Given a portfolio $Y \in \mathcal{A}_\rho$ we can look for those positions which do not increment the risk of Y, that is belonging to the set

$$\mathcal{A}_{Y,\rho} = \{X \in L^\infty \mid \rho(Y) - \rho(Y - X) \leq 0\}.$$

Roughly speaking, $\mathcal{A}_{Y,\rho}$ is formed by positions such that the risk of the portfolio containing the position is at most equal to the risk of the portfolio without the position.

Note that $\rho_Y(\cdot) := \rho(Y) - \rho(Y - \cdot)$ is still a normalized monetary risk measure and evaluates the riskiness of X as a sub-portfolio of Y. In this case, $\mathcal{A}_{Y,\rho}$ can be viewed as the set of all acceptable positions with respect to ρ_Y, i. e. the acceptance set of ρ_Y.

Notice that it is possible to find a position Z which is not acceptable according to $\mathcal{A}_\rho$ but belongs to $\mathcal{A}_{Y,\rho}$. A simple example is the following. For the probability space $(\Omega = [-1, 1], \mathcal{F} = \mathcal{B}(\Omega), \mathbb{P} = \frac{\lambda}{2})$, where λ is the Lebesgue measure on $[-1, 1]$, consider the random variables $Y = \frac{1}{2}$ and $Z = \mathbb{1}_{[0,1]} - \mathbb{1}_{[-1,0)}$. Then, for $\rho(\cdot) := \operatorname{ess\,sup}(-\cdot)$ we have that $\rho(Z) = 1$; hence $Z \notin \mathcal{A}_\rho$ but

$$\rho(Y) - \rho(Y - Z) = -\frac{1}{2} - \frac{1}{2} = -1 < 0,$$

so $Z \in \mathcal{A}_{Y,\rho}$.

The previous example shows that there may exist some positions which do not contribute to the risk of the portfolio, even if they require extra capital when considered as stand-alone portfolios. Hence, in that case, ρ may be not enough to establish whether a position is acceptable or not as a sub-portfolio of Y, but only to measure the riskiness of positions by itself. It would be more suitable, instead, to measure the risk of sub-portfolios by using ρ_Y and not to allocate any part of the risk capital to those sub-portfolios belonging to $\mathcal{A}_Y$. The relevance of this fact and the lack of literature about it lead us to formalize the idea with the following definition.

Definition 4.1. Let $\mathcal{A}$ be an acceptance set, in the sense of Definition 2.8. A family of sets $(\mathcal{A}_Y)_{Y \in L^\infty}$ is called a *sub-acceptance family* of $\mathcal{A}$ if the following properties hold:

A. $\mathcal{A}_Y$ is an acceptance set for every $Y \in L^\infty$.

B. $\mathcal{A} = \{Y \in L^\infty \mid Y \in \mathcal{A}_Y\}$.

Any $\mathcal{A}_Y$ is called a *sub-acceptance set* of Y and any position $X \in \mathcal{A}_Y$ is called *sub-acceptable* with respect to Y.

Condition A of the previous definition means that the positions belonging to $\mathcal{A}_Y$ are acceptable with respect to a fixed position Y, that is when they are considered as sub-portfolios of Y. This implies also that the sub-acceptance criterion, i. e. the one which leads us to detect $\mathcal{A}_Y$, involves features of both the position itself and of Y. Condition B requires that Y is acceptable if and only if it belongs to $\mathcal{A}_Y$, that is, if and only if it is sub-acceptable with respect to itself.

In the following, we provide two examples of sub-acceptance family, pointing out that the criterion defining the above family depends also on the fixed acceptance set $\mathcal{A}$.

EXAMPLE 4.2. Consider the acceptance set of Example 2.2, that is

$$\mathcal{A} = \{Y \in L^\infty \mid \mathbb{E}[Y] \geq 0\}.$$

Starting from this simple acceptance set, we can fix a portfolio $Y \in L^\infty$ and consider a position $X \in L^\infty$ to be sub-acceptable whenever the expected P&L of $X + Y$ is still non-negative. This yields the sub-acceptance set

$$\mathcal{A}_Y = \{X \in L^\infty \mid \mathbb{E}[X + Y] \geq 0\}.$$

Notice that a position belongs to $\mathcal{A}_Y$ when the expected P&L of X is less or equal than the P&L of $-Y$. So, when Y is acceptable, we consider as sub-acceptable those positions with expectation greater or equal than the negative number $\mathbb{E}[-Y]$. Hence, in such case, some sub-acceptable positions have a negative expectation. We point out that, according to the chosen criterion, those positions are still sub-acceptable with respect to Y, while they would not be acceptable if they were considered as stand-alone portfolios.

The collection of sets given by $\mathcal{A}_Y$ is actually a sub-acceptance family in the sense of Definition 4.1. Non-triviality and monotonicity follows straightforwardly from the properties of the expectation and $\mathcal{A}$ can be recognized from the family via

$$\begin{aligned}\mathcal{A} &= \{Y \in L^\infty \mid Y \in \mathcal{A}_Y\} \\ &= \{Y \in L^\infty \mid \mathbb{E}[2Y] \geq 0\} \\ &= \{Y \in L^\infty \mid \mathbb{E}[Y] \geq 0\}.\end{aligned}$$

Suppose now we slightly modify $\mathcal{A}$ and consider the acceptance set of Example 2.4:

$$\mathcal{A}' = \{Y \in L^\infty \mid \mathbb{E}[Y] \geq \lambda\} \text{ for some } \lambda > 0.$$

In this case, $(\mathcal{A}_Y)_{Y \in L^\infty}$ is no more a sub-acceptance family of $\mathcal{A}'$, since Condition B of Definition 4.1 fails. Indeed,

$$\{Y \in L^\infty \mid Y \in \mathcal{A}_Y\} = \{Y \in L^\infty \mid \mathbb{E}[Y] \geq 0\} \neq \mathcal{A}'.$$

EXAMPLE 4.3. We now consider the acceptance set of Example 2.3:

$$\mathcal{A} = \{X \in L^\infty \mid \mathbb{P}(X \leq 0) \leq \alpha\}$$

for a fixed level $\alpha \in (0, 1)$. We will use the same construction of the previous example to define a sub-acceptance family. Hence, once fixed $Y \in L^\infty$, we consider sub-acceptable those positions $X \in L^\infty$ for which the probability that the sum $X + Y$ becomes negative is less or equal than the fixed level α. In this case, we get the sub-acceptance set

$$\mathcal{A}_Y = \{X \in L^\infty \mid \mathbb{P}(X + Y \leq 0) \leq \alpha\}$$

which gives rise to a sub-acceptance family, as in Definition 4.1. Non-triviality and monotonicity are straightforward and $\mathcal{A}$ is actually recoverable from the family. Notice that, as before, $\mathcal{A}_Y$ is detected by a criterion which involves both the sub-portfolio X and the position Y.

4.2 RISK CONTRIBUTION RULES

Acceptance and sub-acceptance sets are tools to detect whether a position needs to be covered by extra capital or not, both when considered as a stand-alone portfolio and when considered as a sub-portfolio of another position. We now provide a tool suitable for assessing the contribution of a sub-portfolio to a given portfolio in terms of acceptability. As shown in the previous section, we need to go beyond the standard approach by linking directly capital allocation rules (or risk contribution rules) and (sub-)acceptance sets. To this aim, we define a map Λ, where $\Lambda(X,Y)$ is interpreted as the risk contribution (or the capital allocated) of X as a sub-portfolio of Y.

DEFINITION 4.2. A function $\Lambda\colon L^\infty \times L^\infty \to \mathbb{R}$ is called a *risk contribution rule* if it satisfies the following properties, for all $X, Y \in L^\infty$:

1-cash-additivity: $\Lambda(X+c,Y) = \Lambda(X,Y) - c$, for all $c \in \mathbb{R}$.

Normalization: $\Lambda(0,Y) = 0$.

Notice that risk contributions are essentially similar to capital allocation rules but they do not satisfy in general full allocation and, moreover, may not depend on a risk measure.

1-cash-additivity means that if we add a cash amount c to the sub-portfolio X, its risk contribution decreases exactly of c. Notice that some known capital allocation rules in the literature satisfy 1-cash additivity, as for example those based on directional derivatives and extensions (see Centrone and Rosazza Gianin [23], Denault [30] and Kalkbrener [50]). Normalization property is quite clear: there is no reason to allocate any capital to a position which yields an almost surely null profit and loss.

Let us now consider the following example, based on the two well-known marginal and proportional methods (see Chapter 2).

EXAMPLE 4.4. Let ρ be a normalized monetary risk measure. We recall that the marginal method is given by

$$\Lambda_\rho^M(X,Y) = \rho(Y) - \rho(Y-X), \quad X, Y \in L^\infty;$$

see Definition 2.14. While the proportional method is given by

$$\Lambda_\rho^P(X,Y) = \frac{\rho(X)}{\rho(X)+\rho(Y-X)}\rho(Y), \quad X, Y \in L^\infty;$$

provided that $\rho(X)+\rho(Y-X) \neq 0$; see Definition 2.13.

It is easy to check that Λ_ρ^M is a CAR (in the sense of Definition 2.12) satisfying normalization and 1-cash-additivity, thanks to the cash-additivity of ρ. Thus, Λ_ρ^M is also a risk contribution rule according to Definition 4.2. While Λ_ρ^P is normalized but not 1-cash-additive and hence it is not a risk contribution rule, despite it is a capital allocation rule, because $\Lambda_\rho^P(X,X) = \rho(X)$.

The following additional property will be sometimes required:

Cash-additivity: $\Lambda(Y+c, Y+c) = \Lambda(Y,Y) - c$, for all $c \in \mathbb{R}, Y \in L^\infty$.

Note that the cash-additivity property is automatically satisfied in the standard case when a monetary risk measure is involved, while this does not necessarily hold for 1-cash-additivity.

4.2.1 *From acceptance sets to risk contribution rules*

We now investigate the connections between risk contribution rules and acceptance sets. To this aim, take an acceptance set $\mathcal{A}$, a sub-acceptance family $(\mathcal{A}_Y)_{Y\in L^\infty}$ and define, for all $X, Y \in L^\infty$,

$$\Lambda_{\mathcal{A}}(X,Y) := \inf\{m \in \mathbb{R} \mid m + X \in \mathcal{A}_Y\}. \tag{4.1}$$

The subscript $\mathcal{A}$ will be omitted when no misunderstandings can arise. Here, $\Lambda(X,Y)$ can be interpreted as the capital allocated to X (or the risk contribution of X), considered as a sub-portfolio of Y, in terms of the minimum amount of capital which should be added to X to make it sub-acceptable. Notice that, in general, $\Lambda(Y,Y)$ does not define the minimum amount of capital which should be added to Y to make it acceptable but only the minimum amount of capital m to make $m+Y$ sub-acceptable with respect to $\mathcal{A}_Y$. However, under additional conditions on the sub-acceptance family, the previous property is fulfilled. We thus introduce the following notion.

DEFINITION 4.3. A sub-acceptance family $(\mathcal{A}_Y)_{Y\in L^\infty}$ is said to be *translation invariant* if it satisfies the following property:

Translation invariance: $\mathcal{A}_Y = \mathcal{A}_{Y+m}$ for all $m \in \mathbb{R}$, $Y \in L^\infty$.

Translation invariance can be interpreted as follows: no matter if we add or remove a fixed amount of capital m to the portfolio Y, the sub-acceptable positions keep being so. This property can be too restrictive, even if it works for some capital allocation methods, as we are going to show in the following examples.

EXAMPLE 4.5. Consider the set of Example 4.1:

$$\mathcal{A}_Y = \{X \in L^\infty \mid \rho(Y) - \rho(Y-X) \le 0\}, \quad Y \in L^\infty;$$

for a given (normalized) monetary risk measure ρ. By cash-additivity of ρ, it follows that $(\mathcal{A}_Y)_{Y\in L^\infty}$ is translation invariant. Indeed, for any $m \in \mathbb{R}$ and $Y \in L^\infty$ it holds that

$$\begin{aligned}\mathcal{A}_{Y+m} &= \{X \in L^\infty \mid \rho(Y+m) - \rho(Y+m-X) \le 0\} \\ &= \{X \in L^\infty \mid \rho(Y) - \rho(Y-X) \le 0\} = \mathcal{A}_Y.\end{aligned}$$

Example 4.6. Consider instead the set discussed in Example 4.2:

$$\mathcal{A}_Y = \{X \in L^\infty \mid \mathbb{E}[X+Y] \geq 0\}, \quad Y \in L^\infty.$$

It is easy to check that $(\mathcal{A}_Y)_{Y\in L^\infty}$ is not translation invariant. However, the following inclusions hold for any $Y \in L^\infty$:

$$\mathcal{A}_{Y+m} \subseteq \mathcal{A}_Y \quad \text{if } m < 0,$$
$$\mathcal{A}_{Y+m} \supseteq \mathcal{A}_Y \quad \text{if } m > 0.$$

To continue our study, we need to define the following property which an acceptance set $\mathcal{A}$ can fulfill or not:

No certain losses: $\inf\{m \in \mathbb{R} \mid m \in \mathcal{A}\} = 0.$

No certain losses means that the smallest constant random variable which is acceptable is 0, i. e. no positions with a (certain) negative profit and loss can be acceptable. We will show in the following that no certain losses is strictly related to the normalization property of a risk contribution rule.

We are now ready to state a result generalizing the one true for risk measures; see Chapter 2 or, for more details, Föllmer and Schied [39, Prop. 4.7].

Proposition 4.1. *If $\mathcal{A}$ is an acceptance set, $(\mathcal{A}_Y)_{Y\in L^\infty}$ is a sub-acceptance family and they both satisfy no certain losses property, then Λ defined in (4.1) is a risk contribution rule.*

Moreover, if the sub-acceptance family is also translation invariant then

$$\Lambda(Y,Y) = \inf\{m \in \mathbb{R} \mid m + Y \in \mathcal{A}\}, \quad \textit{for all } Y \in L^\infty.$$

Proof. Finiteness of $\Lambda(X,Y)$: by the essential boundedness of X and the monotonicity of $\mathcal{A}_Y$ it holds that

$$\{m \in \mathbb{R} \mid m + X \in \mathcal{A}_Y\} \supseteq \{m \in \mathbb{R} \mid m + \operatorname{ess\,inf} X \in \mathcal{A}_Y\} \neq \emptyset.$$

No certain losses implies that $\Lambda(X,Y) < +\infty$. Moreover, by similar arguments,

$$\Lambda(X,Y) \geq \inf\{m \in \mathbb{R} \mid m + \operatorname{ess\,sup} X \in \mathcal{A}_Y\} = -\operatorname{ess\,sup} X > -\infty$$

by essential boundedness of X, monotonicity of $\mathcal{A}_Y$ and no certain losses of $\mathcal{A}_Y$.

1-cash-additivity: for any $X, Y \in L^\infty$ and $c \in \mathbb{R}$ it holds that

$$\begin{aligned}\Lambda(X+c,Y) &= \inf\{m \in \mathbb{R} \mid m + X + c \in \mathcal{A}_Y\} \\ &= \inf\{k \in \mathbb{R} \mid k + X \in \mathcal{A}_Y\} - c \\ &= \Lambda(X,Y) - c\end{aligned}$$

by taking $k = m + c$.

Normalization: no certain losses of $\mathcal{A}_Y$ implies that

$$\Lambda(0,Y) = \inf\{m \in \mathbb{R} \mid m \in \mathcal{A}_Y\} = 0 \quad \text{for all } Y \in L^\infty.$$

Monotonicity: fix $Y \in L^\infty$ and consider $Z \geq X$ (with $Z, X \in L^\infty$). By monotonicity of $\mathcal{A}_Y$,

$$\{m \in \mathbb{R} \mid m + Z \in \mathcal{A}_Y\} \supseteq \{m \in \mathbb{R} \mid m + X \in \mathcal{A}_Y\},$$

hence $\Lambda(Z,Y) \leq \Lambda(X,Y)$.

It remains to prove the last statement. For any $Y \in L^\infty$ it holds that

$$\begin{aligned}\Lambda(Y,Y) &= \inf\{m \in \mathbb{R} \mid m + Y \in \mathcal{A}_Y\} \\ &= \inf\{m \in \mathbb{R} \mid m + Y \in \mathcal{A}_{Y+m}\} \\ &= \inf\{m \in \mathbb{R} \mid m + Y \in \mathcal{A}\},\end{aligned}$$

where the second equality holds by translation invariance and the last one by definition of sub-acceptance family. □

Remark 4.1. Notice that, when the sub-acceptance family is translation invariant, $\Lambda(Y,Y)$ defines exactly the minimum amount of capital which should be added to Y to make it acceptable, even if the acceptance set $\mathcal{A}$ is not involved in the definition of Λ.

We will give now some examples of risk contribution rules associated to the acceptance and sub-acceptance sets presented in the previous section.

Example 4.7. Consider the acceptance set and the sub-acceptance family given by

$$\begin{aligned}\mathcal{A} &= \{X \in L^\infty \mid \rho(X) \leq 0\} \\ \mathcal{A}_Y &= \{X \in L^\infty \mid \rho(Y) - \rho(Y - X) \leq 0\}\end{aligned}$$

for ρ being a monetary risk measure and $Y \in L^\infty$. By cash-additivity of ρ, $\Lambda_{\mathcal{A}}$ defined in (4.1) becomes

$$\begin{aligned}\Lambda_{\mathcal{A}}(X,Y) &= \inf\{m \in \mathbb{R} \mid m + X \in \mathcal{A}_Y\} \\ &= \inf\{m \in \mathbb{R} \mid \rho(Y) - \rho(Y - (X + m)) \leq 0\} \\ &= \inf\{m \in \mathbb{R} \mid \rho(Y) - \rho(Y - X) \leq m\} \\ &= \rho(Y) - \rho(Y - X),\end{aligned}$$

hence corresponding to the marginal method; see Chapter 2 and the references therein.

Moreover, $\Lambda_{\mathcal{A}}$ is a risk contribution rule. Indeed, 1-cash-additivity is immediate and normalization follows by no certain losses property of $\mathcal{A}_Y$. Furthermore, it is also a CAR: $\Lambda_{\mathcal{A}}(Y,Y) = \rho(Y)$ follows by translation invariance of $\mathcal{A}_Y$.

Example 4.8. Consider the acceptance set and the sub-acceptance family given by

$$\mathcal{A} = \{Y \in L^\infty \mid \mathbb{P}(Y \le 0) \le \alpha\}$$
$$\mathcal{A}_Y = \{X \in L^\infty \mid \mathbb{P}(X + Y \le 0) \le \alpha\}$$

for some $\alpha \in (0, 1)$ and for any $Y \in L^\infty$. Then

$$\begin{aligned}\Lambda_{\mathcal{A}}(X, Y) &= \inf\{m \in \mathbb{R} \mid m + X \in \mathcal{A}_Y\} \\ &= \inf\{m \in \mathbb{R} \mid \mathbb{P}(X + m + Y \le 0) \le \alpha\} \\ &= -\sup\{k \in \mathbb{R} \mid \mathbb{P}(X + Y \le k) \le \alpha\} \\ &= -q^+_\alpha(X + Y) = \mathrm{VaR}_\alpha(X + Y),\end{aligned}$$

where $q^+_\alpha(Z) := \inf\{m \in \mathbb{R} \mid \mathbb{P}(Z \le m) > \alpha\}$ is the upper α-quantile and $\mathrm{VaR}_\alpha(Z)$ is the Value at Risk at level α of Z (see Definition 2.9). Hence, $\Lambda_{\mathcal{A}}$ is 1-cash-additive but not normalized, so it is not a risk contribution rule. Moreover,

$$\Lambda_{\mathcal{A}}(Y, Y) = -2q^+_\alpha(Y) \ne \inf\{m \in \mathbb{R} \mid m + Y \in \mathcal{A}\} = -q^+_\alpha(Y).$$

Indeed, the sub-acceptance family is not translation invariant.

We now define some properties on acceptance sets corresponding to those already introduced on risk contribution rules.

First of all, it may be reasonable to require that any acceptable positions is also sub-acceptable for every portfolio, that is, to require the following property:

$\mathcal{A}$-no-undercut: $\mathcal{A} \subseteq \mathcal{A}_Y$ for all $Y \in L^\infty$.

As shown in the following result, $\mathcal{A}$-no-undercut corresponds to no-undercut of the associated Λ.

Proposition 4.2. *Let $\mathcal{A}$ be an acceptance set and let $(\mathcal{A}_Y)_{Y \in L^\infty}$ be a translation invariant sub-acceptance family. If the family $(\mathcal{A}_Y)_{Y \in L^\infty}$ satisfies $\mathcal{A}$-no-undercut, then $\Lambda_{\mathcal{A}}$ defined in (4.1) satisfies no-undercut.*

Proof. Given arbitrary $X, Y \in L^\infty$, $\mathcal{A}$-no-undercut implies

$$\{m \in \mathbb{R} \mid m + X \in \mathcal{A}\} \subseteq \{m \in \mathbb{R} \mid m + X \in \mathcal{A}_Y\}.$$

Hence,

$$\Lambda_{\mathcal{A}}(X, Y) \le \inf\{m \in \mathbb{R} \mid m + X \in \mathcal{A}\} = \Lambda_{\mathcal{A}}(X, X)$$

where the last equality holds by translation invariance of the sub-acceptance family. □

We conclude this section by discussing the compatibility of our approach with the Euler's allocation and the RORAC (see Chapter 2 or [6, 18, 50, 62] for more details).

EXAMPLE 4.9. Let $\mathcal{Q}$ be a set of probability measures that are absolutely continuous with respect to $\mathbb{P}$ and let $\rho(Y) = \sup_{\mathbb{Q}\in\mathcal{Q}} \mathbb{E}_{\mathbb{Q}}[-Y]$ be a coherent risk measure.

Consider the acceptance set and the sub-acceptance family given by

$$\mathcal{A} = \{X \in L^\infty \mid \mathbb{E}_{\mathbb{Q}_X}[-X] \leq 0\}$$
$$\mathcal{A}_Y = \{X \in L^\infty \mid \mathbb{E}_{\mathbb{Q}_Y}[-X] \leq 0\}$$

for a given $Y \in L^\infty$ and $\mathbb{Q}_Y \in \arg\max_{\mathbb{Q}\in\mathcal{Q}} \mathbb{E}_{\mathbb{Q}}[-Y]$. In other words, we assume that in the representation of ρ the supremum is always attained.

It follows then that $\Lambda_{\mathcal{A}}$ coincides with the Euler capital allocation. Indeed,

$$\Lambda_{\mathcal{A}}(X,Y) = \inf\{m \in \mathbb{R} \mid m + X \in \mathcal{A}_Y\} = \mathbb{E}_{\mathbb{Q}_Y}[-X],$$

that is the Euler (or gradient) allocation of Definition 2.18 (see [6, 18, 23, 28, 50, 62] for more details).

EXAMPLE 4.10. Consider an acceptance set $\mathcal{A}$, a sub-acceptance family $(\mathcal{A}_Y)_{Y\in L^\infty}$ and the associated $\Lambda_{\mathcal{A}}$ defined in (4.1). It is then possible to define the RORAC risk contribution rule induced by $\Lambda_{\mathcal{A}}$ as

$$\Lambda_{R,\mathcal{A}}(X,Y) := \frac{\mathbb{E}[X]}{\Lambda_{\mathcal{A}}(X,Y)}$$

similarly to Definition 2.16, where a CAR Λ with respect to a risk measure ρ is given. However, $\Lambda_{R,\mathcal{A}}$ fails to satisfy 1-cash-additivity, hence it is not a risk contribution in general. Normalization holds whenever we assume that $\frac{0}{0} = 0$.

Furthermore, even if $\Lambda_{\mathcal{A}}$ satisfies no-undercut, the same is no more true for the associated RORAC risk contribution rule. Indeed, whenever $\mathbb{E}[X] \geq 0$,

$$\Lambda_{R,\mathcal{A}}(X,Y) = \frac{\mathbb{E}[X]}{\Lambda_{\mathcal{A}}(X,Y)} \geq \frac{\mathbb{E}[X]}{\Lambda_{\mathcal{A}}(X,X)} = \Lambda_{R,\mathcal{A}}(X,X).$$

4.2.2 *From risk contribution rules to acceptance sets*

So far, we have defined a risk contribution rule starting from an acceptance set and a sub-acceptance family and studied some properties of that risk contribution rule corresponding to those required for the sets. We now investigate the converse.

Let us start with the case when Λ is a risk contribution rule that is also induced by a monetary risk measure ρ such that $\Lambda(X,X) = \rho(X)$ for all $X \in L^\infty$.

Consider the acceptance set $\mathcal{A}_\rho$ of ρ, that is, the set of all $X \in L^\infty$ such that $\Lambda(X, X) \leq 0$. Then define, for all $Y \in L^\infty$,

$$\mathcal{A}_Y := \{X \in L^\infty \mid \Lambda(X, Y) \leq 0\} .$$

Every position X in $\mathcal{A}_Y$ is sub-acceptable in the sense that it does not need any capital injection when seen as a sub-portfolio of Y. Notice that $\mathcal{A}_\rho = \{Y \in L^\infty \mid Y \in \mathcal{A}_Y\}$.

The following representation result is then straightforward.

Proposition 4.3. *If Λ is a risk contribution rule induced by a monetary risk measure ρ, then*

$$\begin{aligned}\Lambda(X, Y) &= \inf\{m \in \mathbb{R} \mid m + X \in \mathcal{A}_Y\} , \\ \Lambda(Y, Y) &= \inf\{m \in \mathbb{R} \mid m + Y \in \mathcal{A}\}\end{aligned}$$

for any $X, Y \in L^\infty$.

If, moreover, Λ is monotone, then $\mathcal{A}_Y$ is an acceptance set for any $Y \in L^\infty$, and $\Lambda(\cdot, Y) = \rho_Y(\cdot)$ is a monetary risk measure satisfying $\rho_Y(Y) = \rho(Y)$.

Proof. If Λ is a risk contribution rule then, by 1-cash-additivity,

$$\begin{aligned}\inf\{m \in \mathbb{R} \mid m + X \in \mathcal{A}_Y\} &= \inf\{m \in \mathbb{R} \mid \Lambda(m + X, Y) \leq 0\} \\ &= \Lambda(X, Y)\end{aligned}$$

holds for any $X, Y \in L^\infty$.

Moreover, $\Lambda(Y, Y) = \rho(Y) = \inf\{m \in \mathbb{R} \mid m + Y \in \mathcal{A}\}$, where the former equality holds since Λ is induced by ρ, while the latter follows from the relation between monetary risk measures and acceptance sets. If Λ is monotone, the monotonicity of each $\mathcal{A}_Y$ follows straightforwardly. □

The modified monetary risk measure ρ_Y reflects the "true" risk of X as a sub-portfolio of Y. This risk contribution rule is not linear in general, but this is justified by the fact that we are not trying to share the risk $\rho(Y)$ among the various sub-units of Y but to assign to each sub-unit exactly its risk contribution as a sub-unit of Y.

We now investigate if the previous representation result still holds true for a general Λ not necessarily induced by a monetary risk measure. Unfortunately, this is not the case without imposing some additional properties on the risk contribution rule. The main problems are related to the lack of cash-additivity and of monotonicity, which are instead automatically fulfilled in the standard framework, whenever a monotone risk measure is involved.

There are several ways to fill those gaps: in the following, we will discuss and investigate the different properties to be required to obtain results similar to Proposition 4.3.

Given a risk contribution rule Λ, we define the following sets:

$$\mathcal{A}_\Lambda := \{Y \in L^\infty \mid \Lambda(Y,Y) \le 0\} \tag{4.2}$$

$$\mathcal{A}_{Y,\Lambda} := \{X \in L^\infty \mid \Lambda(X,Y) \le 0\}, \quad Y \in L^\infty, \tag{4.3}$$

where the subscript Λ will be omitted when it is clear which risk contribution rule is involved.

Proposition 4.4. *If Λ is a risk contribution rule satisfying monotonicity, cash-additivity and no-undercut, then $\mathcal{A}$ defined in (4.2) is an acceptance set and $(\mathcal{A}_Y)_{Y\in L^\infty}$ given by (4.3) is a sub-acceptance family with respect to $\mathcal{A}$. Moreover, Λ can be written as:*

$$\Lambda(X,Y) = \begin{cases} \inf\{m \in \mathbb{R} \mid m + X \in \mathcal{A}_Y\} & \text{if } X \neq Y \\ \inf\{m \in \mathbb{R} \mid m + Y \in \mathcal{A}\} & \text{if } X = Y. \end{cases} \tag{4.4}$$

Proof. Non triviality of $\mathcal{A}$: first of all, $\mathcal{A} \neq \emptyset$ since $0 \in \mathcal{A}$ by normalization. In order to check $\mathcal{A} \neq L^\infty$, let us consider $c < \Lambda(Y,Y)$. Then, by cash-additivity of Λ, $\Lambda(Y+c, Y+c) = \Lambda(Y,Y) - c > 0$ so that $Y + c \notin \mathcal{A}$. A similar argument clearly holds for each $\mathcal{A}_Y$.

Monotonicity of each $\mathcal{A}_Y$: consider $X \in \mathcal{A}_Y$ and $Z \geq X$ with $Z, X \in L^\infty$. Then, by monotonicity of Λ and (4.3), it follows that

$$\Lambda(Z,Y) \le \Lambda(X,Y) \le 0,$$

hence $Z \in \mathcal{A}_Y$.

Monotonicity of $\mathcal{A}$: take $X \in \mathcal{A}$ and $Y \geq X$. Then

$$\Lambda(Y,Y) \le \Lambda(X,Y) \le \Lambda(X,X) \le 0 \tag{4.5}$$

where the first inequality holds by monotonicity of Λ, the second one by no-undercut and the last one because $X \in \mathcal{A}$. Therefore, $Y \in \mathcal{A}$ and $\mathcal{A}$ is an acceptance set. Since

$$\mathcal{A} = \{Y \in L^\infty \mid Y \in \mathcal{A}_Y\} = \{Y \in L^\infty \mid \Lambda(Y,Y) \le 0\},$$

then $(\mathcal{A}_Y)_{Y\in L^\infty}$ is a sub-acceptance family with respect to $\mathcal{A}$.

It remains to show that Λ can be represented as in (4.4). Consider, firstly, the case where $X \neq Y$. Then

$$\begin{aligned} \Lambda(X,Y) &= \inf\{m \in \mathbb{R} \mid \Lambda(X,Y) \le m\} \\ &= \inf\{m \in \mathbb{R} \mid \Lambda(X+m,Y) \le 0\} \\ &= \inf\{m \in \mathbb{R} \mid X + m \in \mathcal{A}_Y\} \end{aligned}$$

where the second equality holds by 1-cash-additivity of Λ and the last one by definition of $\mathcal{A}_Y$. Finally, by cash-additivity of Λ and by definition of $\mathcal{A}$, for any $Y \in L^\infty$ it holds that

$$\begin{aligned} \Lambda(Y,Y) &= \inf\{m \in \mathbb{R} \mid \Lambda(Y,Y) \le m\} \\ &= \inf\{m \in \mathbb{R} \mid \Lambda(Y+m,Y+m) \le 0\} \\ &= \inf\{m \in \mathbb{R} \mid Y + m \in \mathcal{A}\}. \end{aligned}$$

□

It is worth mentioning that other properties on a risk contribution rule could guarantee the same thesis of the previous result. Monotonicity is clearly needed to prove that each $\mathcal{A}_Y$ is monotone and there are no significant alternatives. Notice that no-undercut is required to fill the lack of the following property:

Full monotonicity: if $Y \geq X$ $(Y, X \in L^\infty)$, then $\Lambda(Y, Y) \leq \Lambda(X, X)$.

The previous property is automatically satisfied in the standard capital allocation framework when Λ is induced by a monotone risk measure. However, full monotonicity follows from monotonicity and no-undercut, as we can see from inequalities in (4.5).

Notice, moreover, that Λ satisfying no-undercut does not imply the same property on acceptance and sub-acceptance sets.

4.2.3 *Full allocation*

This section is devoted to a discussion of a crucial property for capital allocations, namely full allocation (see Chapter 2 or [50] for more details).

Different motivations can be given that would lead to including or not full-allocation in capital allocation models: in the following we will give some reasons for not requiring this property, but also will explore some consequences of assuming it in our framework.

First, for capital allocations induced by risk measures requiring or not full allocation would depend also on how general the considered risk measure is. As shown by Kalkbrener [50, Thm 4.2], indeed, for Λ induced by a risk measure ρ full allocation, combined with no-undercut and $\Lambda(Y, Y) = \rho(Y)$, implies subadditivity of ρ, while linearity and no-undercut imply its positive homogeneity. In other words, we cannot expect that a capital allocation related to general risk measures satisfies both full allocation and no-undercut. Although the combination of the two properties is especially significant and used in practice (see, for example, [19, 63]), in the literature the problem of capital allocation has been faced also beyond the context of coherent risk measures. Indeed, for example, Tsanakas [64] extends the Aumann-Shapley capital allocation method to the context of convex risk measures.

Moreover, as remarked by Brunnermeier and Cheridito [18], the need of imposing full allocation or not to a capital allocation would depend also on the purpose of such a capital allocation. When such a capital allocation is mainly used to monitor the position, full allocation may not be required. Indeed, the authors introduce a convex measure ρ of systemic risk and study capital allocation w. r. t. some popular methods, such as the marginal ones, not satisfying in general full allocation. Some other well-known capital allocations, for instance the RORAC, do not satisfy full allocation. Furthermore, it is always possible to modify a capital allocation Λ so to guarantee full allocation, as we showed in Chapter 3, but at the cost of losing some other properties.

We also point out that, in order to make capital allocation, it is not always requested to start with the specification of a given risk measures. In fact, as pointed out in [6], Remark 2.4, the total capital may be adjusted according to different possible scenarios.

Since our aim is to focus on capital allocations (or, better, risk contribution rules) that are not necessarily related to risk measures but, even in such a case, the risk measure is quite general and not necessarily coherent, we have not assumed full allocation in general. To better clarify our point about dispensing of full allocation, consider for instance the case of a convex risk measure ρ. In general, there may exist a portfolio $Y = Y_1 + Y_2$ where $\rho(Y) > \rho(Y_1) + \rho(Y_2)$. As pointed out in [23], if Λ_ρ is a capital allocation satisfying no-undercut and $\Lambda(Y,Y) = \rho(Y)$, then $\Lambda(Y,Y) = \rho(Y) > \rho(Y_1) + \rho(Y_2) \geq \Lambda(Y_1,Y) + \Lambda(Y_2,Y)$; that is, full allocation fails. Since the no-undercut property guarantees that there is no incentive to split a sub-portfolio from the whole portfolio, it would be therefore more reasonable to replace full allocation with the following property, introduced by Centrone and Rosazza Gianin [23]:

Sub-allocation: for all $Y_1, \ldots, Y_n$, $Y \in L^\infty$ such that $Y = \sum_{i=1}^n Y_i$,

$$\Lambda(Y,Y) \geq \sum_{i=1}^n \Lambda(Y_i,Y).$$

While for full allocation the capital requirement $\Lambda(Y,Y)$ is fully divided into the different business lines (or sub-units) $Y_1, \ldots, Y_n$, for sub-allocation there is some undivided cost that is not shared between the sub-units because it is due to some fixed costs to be faced by the whole portfolio (such as taxes, common costs, ...) that is not proportional to the riskiness of any business line. Furthermore, the undivided $\Lambda(Y,Y) - \sum_{i=1}^n \Lambda(Y_i,Y) > 0$ can be also interpreted as an extra security requirement.

It is clear that sub-allocation is weaker than full allocation. As shown in Centrone and Rosazza Gianin [23, Prop. 4], there exists a CAR induced by a monetary convex risk measure and satisfying no-undercut and sub-allocation.

Another property which is compatible with no-undercut for Λ that are induced by monetary convex risk measures is the following:

1-weak convexity: for all $Y_1, \ldots, Y_n$, $Y \in L^\infty$ and $\alpha_i \in [0,1]$ such that $\sum_{i=1}^n \alpha_i = 1$ and $Y = \sum_{i=1}^n \alpha_i Y_i$,

$$\Lambda(Y,Y) \leq \sum_{i=1}^n \alpha_i \Lambda(Y_i,Y).$$

The name of the previous property is justified by the fact that it is a sort of convexity in the first variable, holding for weighted Y_i's summing to Y.

The following results investigate the impact (on the sub-acceptance family) of imposing full allocation or the other weaker conditions on a risk contribution rule Λ.

PROPOSITION 4.5 (Full allocation). A. *If $\Lambda\colon L^\infty \times L^\infty \to \mathbb{R}$ is a map which satisfies full allocation, then each $\mathcal{A}_{Y,\Lambda}$ satisfies the following condition for all $Y_1, \dots, Y_n$, $Y \in L^\infty$:*

$$Y_i \in \mathcal{A}_{Y,\Lambda} \text{ for each } i = 1, \dots, n \text{ with } \sum_{i=1}^n Y_i = Y \implies \sum_{i=1}^n Y_i \in \mathcal{A}_{Y,\Lambda}.$$

B. *If $(\mathcal{A}_Y)_{Y\in L^\infty}$ is a family of acceptance sets satisfying the following condition, for all $Y_1, \dots, Y_n$, $Y \in L^\infty$ with $\sum_{i=1}^n Y_i = Y$:*

$$m_i + Y_i \in \mathcal{A}_Y \text{ for each } i = 1, \dots, n \iff \sum_{i=1}^n (m_i + Y_i) \in \mathcal{A}_Y,$$

then $\Lambda_{\mathcal{A}}$ satisfies full allocation.

PROOF. A. Let $Y_1, \dots, Y_n$, $Y \in L^\infty$ be such that $\sum_{i=1}^n Y_i = Y$ and $Y_i \in \mathcal{A}_{Y,\Lambda}$ for each $i = 1, \dots, n$. By full allocation of Λ, it follows that

$$\Lambda(Y, Y) = \Lambda\left(\sum_{i=1}^n Y_i, Y\right) = \sum_{i=1}^n \Lambda(Y_i, Y) \leq 0,$$

hence also $Y = \sum_{i=1}^n Y_i \in \mathcal{A}_{Y,\Lambda}$.

B. It can be proved similarly as in Frittelli and Scandolo [43, Prop. 3.3]. It follows, indeed, that

$$\begin{aligned}
\sum_{i=1}^n \Lambda_{\mathcal{A}}(Y_i, Y) &= \sum_{i=1}^n \inf \{m_i \in \mathbb{R} \mid m_i + Y_i \in \mathcal{A}_Y\} \\
&= \inf \left\{ \sum_{i=1}^n m_i \in \mathbb{R} \,\middle|\, m_i + Y_i \in \mathcal{A}_Y \text{ for all } i = 1, \dots, n \right\} \\
&= \inf \left\{ \sum_{i=1}^n m_i \in \mathbb{R} \,\middle|\, \sum_{i=1}^n (m_i + Y_i) \in \mathcal{A}_Y \right\} \\
&= \Lambda_{\mathcal{A}}(Y, Y),
\end{aligned}$$

where the third equality is due to the hypothesis on $\mathcal{A}_Y$. □

Notice that the condition in A implies that, under full allocation, if a position Y is not acceptable, then every decomposition of Y into sub-positions will contain at least a sub-position which is not sub-acceptable. Moreover, the condition in B in the previous result implies that, if $(\mathcal{A}_Y)_{Y\in L^\infty}$ is a sub-acceptance family, a position Y is acceptable if and only if every sub-position of Y is also sub-acceptable. So, in a certain sense, the concept of sub-acceptability can highlight that full allocation is quite a strong requirement.

PROPOSITION 4.6 (Sub-allocation). A. *If $\Lambda\colon L^\infty \times L^\infty \to \mathbb{R}$ is a map that satisfies sub-allocation, then each $\mathcal{A}_{Y,\Lambda}$ satisfies the following condition for all $Y_1, \dots, Y_n$, $Y \in L^\infty$:*

$$Y_i \notin \mathcal{A}_{Y,\Lambda} \text{ for each } i = 1, \dots, n \text{ with } \sum_{i=1}^{n} Y_i = Y \implies \sum_{i=1}^{n} Y_i \notin \mathcal{A}_{Y,\Lambda}.$$

B. *If each $\mathcal{A}_Y$ is monotone and each $\mathcal{A}_Y^c \cap L^\infty$ is subadditive, i.e. for all $X_1, \dots, X_n, Y \in L^\infty$ it holds that*

$$X_i \notin \mathcal{A}_Y \text{ for each } i = 1, \dots, n \implies \sum_{i=1}^{n} X_i \notin \mathcal{A}_Y,$$

then $\Lambda_{\mathcal{A}}$ satisfies sub-allocation.

Note that the necessary condition on $\mathcal{A}_Y^c \cap L^\infty$ in item A is weaker than subadditivity since it is satisfied only for $Y_1, \dots, Y_n$, $Y \in L^\infty$ such that $\sum_{i=1}^{n} Y_i = Y$.

PROOF. A. Let $Y_1, \dots, Y_n$, $Y \in L^\infty$ be such that $\sum_{i=1}^{n} Y_i = Y$ and $Y_i \notin \mathcal{A}_{Y,\Lambda}$ for each $i = 1, \dots, n$. By sub-allocation of Λ it follows that

$$\Lambda(Y, Y) = \Lambda\left(\sum_{i=1}^{n} Y_i, Y\right) \geq \sum_{i=1}^{n} \Lambda(Y_i, Y) > 0,$$

hence also $Y = \sum_{i=1}^{n} Y_i \notin \mathcal{A}_{Y,\Lambda}$.

B. It follows that

$$\begin{aligned}
\sum_{i=1}^{n} \Lambda_{\mathcal{A}}(Y_i, Y) &= \sum_{i=1}^{n} \inf\{m_i \in \mathbb{R} \mid m_i + Y_i \in \mathcal{A}_Y\} \\
&= \sum_{i=1}^{n} \sup\{m_i \in \mathbb{R} \mid m_i + Y_i \notin \mathcal{A}_Y\} \\
&= \sup\left\{\sum_{i=1}^{n} m_i \in \mathbb{R} \;\middle|\; m_i + Y_i \notin \mathcal{A}_Y \text{ for all } i = 1, \dots, n\right\} \\
&\leq \sup\left\{\sum_{i=1}^{n} m_i \in \mathbb{R} \;\middle|\; \sum_{i=1}^{n} (m_i + Y_i) \notin \mathcal{A}_Y\right\} \\
&= \Lambda_{\mathcal{A}}(Y, Y),
\end{aligned}$$

where the inequality is due to subadditivity of $\mathcal{A}_Y^c$ while the second equality to monotonicity of each $\mathcal{A}_Y$. □

Proposition 4.7 (1-weak convexity). a. *If $\Lambda\colon L^\infty \times L^\infty \to \mathbb{R}$ is a map that satisfies 1-weak convexity, then each $\mathcal{A}_{Y,\Lambda}$ satisfies the following condition for all $Y_1, \dots, Y_n$, $Y \in L^\infty$:*

$$Y_i \in \mathcal{A}_{Y,\Lambda} \text{ for all } i = 1, \dots, n \text{ with } \sum_{i=1}^n \alpha_i Y_i = Y \implies \sum_{i=1}^n \alpha_i Y_i \in \mathcal{A}_{Y,\Lambda}.$$

b. *If each $\mathcal{A}_Y$ is convex, then $\Lambda_{\mathcal{A}}$ satisfies 1-weak convexity.*

Proof. a. Let $Y_1, \dots, Y_n$, $Y \in L^\infty$ and $\alpha_1, \dots, \alpha_n \in [0,1]$ be such that $\sum_{i=1}^n \alpha_i = 1$, $\sum_{i=1}^n \alpha_i Y_i = Y$ and $Y_i \in \mathcal{A}_{Y,\Lambda}$ for each $i = 1, \dots, n$. By 1-weak convexity of Λ it follows that

$$\Lambda(Y,Y) = \Lambda\left(\sum_{i=1}^n \alpha_i Y_i, Y\right) \leq \sum_{i=1}^n \alpha_i \Lambda(Y_i, Y) \leq 0,$$

hence also $Y = \sum_{i=1}^n \alpha_i Y_i \in \mathcal{A}_{Y,\Lambda}$.

b. It can be proved as in Frittelli and Scandolo [43, Prop. 3.3]. By convexity of each $\mathcal{A}_Y$, indeed,

$$\begin{aligned}
&\sum_{i=1}^n \alpha_i \Lambda_{\mathcal{A}}(Y_i, Y) \\
&= \sum_{i=1}^n \alpha_i \inf\{m_i \in \mathbb{R} \mid m_i + Y_i \in \mathcal{A}_Y\} \\
&= \inf\left\{\sum_{i=1}^n \alpha_i m_i \in \mathbb{R} \;\middle|\; m_i + Y_i \in \mathcal{A}_Y \text{ for each } i = 1, \dots, n\right\} \\
&\geq \inf\left\{\sum_{i=1}^n \alpha_i m_i \in \mathbb{R} \;\middle|\; \sum_{i=1}^n \alpha_i (m_i + Y_i) \in \mathcal{A}_Y\right\} \\
&= \inf\{m \in \mathbb{R} \mid m + Y \in \mathcal{A}_Y\} = \Lambda_{\mathcal{A}}(Y,Y).
\end{aligned}$$

□

In conclusion, while full allocation seems to be a too strong requirement for general risk contribution rules, sub-allocation or 1-weak convexity seem to be more appropriate.

4.3 Some extensions

So far, we focused on the cash-additive case, that is related to a 1-cash-additive CAR or to a translation invariant sub-acceptance family. In the following, we generalize the approach above to the case where translation invariance of the acceptance family either holds with respect to a reference asset (not necessarily a risk-free asset) or is dropped. More precisely, we will focus both on the S-additive case and on the quasi-convex case.

4.3.1 *S-additivity*

As pointed out by Farkas et al. [36] and Munari [56], the idea of the milestone work of Artzner et al. [5] is to measure the risk of a position by describing how close or far from acceptability the position is, given a "reference instrument" that does not necessarily correspond to a cash account. In our framework, capital allocation rules assess the capital to be allocated to a sub-portfolio by means of the distance to a sub-acceptance set, which is, in some cases, related to the risk of the sub-portfolio. Therefore, in general, it is too restrictive to impose the cash-additivity assumption to capital allocation rules. Following the approach of Farkas et al. [36] and Munari [56] who introduced the so-called S-additive risk measures, we would like to admit the possibility to make a portfolio acceptable or sub-acceptable by adding not necessarily cash but also shares of a "suitable" asset.

Fix now a time horizon T and an asset S given by $S = (S_0, S_T)$, where $S_0 \in \mathbb{R}$ is the initial value and $S_T \in L^\infty$ is the value of S at time T. We assume the existence of a financial market where assets are traded. We recall the following definition to clarify which are the "suitable" assets we wish to add to sub-portfolios in order to reach acceptability.

Definition 4.4. (see Farkas and Smirnow [37]) Given a time horizon $T \geq 0$ and an acceptance set $\mathcal{A}$, an asset $S = (S_0, S_T)$ is called *eligible* if $S_T \in \mathcal{A}$ and its initial value S_0 is strictly positive.

In the following, S will denote, with an abuse of notation, both the asset and its terminal value S_T, while $\mathcal{E}$ will denote the set of all eligible assets. The previous definition slightly differs from the one of Farkas et al. [36] and Munari [56], where they require the same condition on S_0 but a different one on S_T, i. e. $\mathbb{P}(S_T \geq 0) = 1$.

Our aim is now to investigate whether the results of the previous section can be generalized to the present case where we introduce the following definition of S-risk contribution rules.

Definition 4.5. A function $\Lambda\colon L^\infty \times L^\infty \to \mathbb{R}$ is called an *S-risk contribution rule* if it satisfies the following properties, for all $X, Y \in L^\infty$:

1-S-additivity: $\Lambda(X + mS, Y) = \Lambda(X, Y) - mS_0$, for all $m \in \mathbb{R}, S \in \mathcal{E}$.

Normalization: $\Lambda(0, Y) = 0$.

Compared to risk contribution rules of Definition 4.2, in S-risk contribution rules the assumption of 1-cash-additivity has been replaced by 1-S-additivity. Therefore, for S-risk contribution rules, not only cash amounts $m \in \mathbb{R}$ are suitable to reduce the capital allocated to a sub-portfolio but also eligible assets $S \in \mathcal{E}$.

Similarly to the previous section, given an acceptance set $\mathcal{A}$ and a sub-acceptance family $(\mathcal{A}_Y)_{Y\in L^\infty}$ we define, for all $X, Y \in L^\infty$,

$$\Lambda_{\mathcal{A}}(X,Y) := \inf\left\{ m \in \mathbb{R} \;\middle|\; \frac{m}{S_0}S + X \in \mathcal{A}_Y \right\}, \tag{4.6}$$

where the subscript $\mathcal{A}$ will be omitted when no misunderstandings can arise. Before going further, we need to introduce the following properties for a sub-acceptance family $(\mathcal{A}_Y)_{Y\in L^\infty}$:

S-translation invariance: $\mathcal{A}_Y = \mathcal{A}_{Y+mS}$ for all $m \in \mathbb{R}$ and $S \in \mathcal{E}$.

S-no certain losses: $\inf\left\{ m \in \mathbb{R} \;\middle|\; \frac{m}{S_0}S \in \mathcal{A} \right\} = 0$ for all $S \in \mathcal{E}$.

S-translation invariance property means that those positions which are sub-acceptable with respect to a given portfolio Y are also sub-acceptable with respect to any sum $Y + mS$ where S is eligible and m is any cash amount. In other words, no matter if we add or remove any quantity (even negative) of eligible asset S to the portfolio Y, the sub-acceptable positions keep being so. S-no certain losses property, instead, requires that the smallest share of eligible asset which is acceptable is 0, i. e. no short positions on S can be acceptable.

Proposition 4.8. *If $\mathcal{A}$ is an acceptance set, $(\mathcal{A}_Y)_{Y\in L^\infty}$ is a sub-acceptance family and they both satisfy no certain losses property, then $\Lambda_{\mathcal{A}}$ defined in (4.6) is a monotone S-risk contribution rule.*

Moreover, if the sub-acceptance family is also S-translation invariant then

$$\Lambda_{\mathcal{A}}(Y,Y) = \inf\left\{ m \in \mathbb{R} \;\middle|\; \frac{m}{S_0}S + Y \in \mathcal{A} \right\} \quad \textit{for all } Y \in L^\infty.$$

Proof. Finiteness of $\Lambda_{\mathcal{A}}(X,Y)$: since $\mathcal{A}_Y$ is an acceptance set (hence it is monotone and $\mathcal{A}_Y \neq \emptyset, L^\infty$) and $X \in L^\infty$,

$$\left\{ m \in \mathbb{R} \;\middle|\; \frac{m}{S_0}S + X \in \mathcal{A}_Y \right\} \neq \emptyset, \mathbb{R}.$$

Hence $\Lambda_{\mathcal{A}}(X,Y) \in \mathbb{R}$.

1-S-additivity: for any $X, Y \in L^\infty$, S eligible and $k \in \mathbb{R}$ we consider

$$\begin{aligned}
\Lambda_{\mathcal{A}}(X + kS; Y) &= \inf\left\{ m \in \mathbb{R} \;\middle|\; \frac{m}{S_0}S + X + kS \in \mathcal{A}_Y \right\} \\
&= \inf\left\{ (c-k)S_0 \in \mathbb{R} \mid cS + X \in \mathcal{A}_Y \right\} \\
&= \inf\left\{ cS_0 \in \mathbb{R} \mid cS + X \in \mathcal{A}_Y \right\} - kS_0 \\
&= \inf\left\{ \beta \in \mathbb{R} \;\middle|\; \frac{\beta}{S_0}S + X \in \mathcal{A}_Y \right\} - kS_0 \\
&= \Lambda_{\mathcal{A}}(X,Y) - kS_0.
\end{aligned}$$

Normalization: S-no certain losses implies, for every Y, that

$$\Lambda_{\mathcal{A}}(0,Y) = \inf\left\{m \in \mathbb{R} \;\middle|\; \frac{m}{S_0}S \in \mathcal{A}_Y\right\} = 0.$$

Monotonicity: fix any $X, Y \in L^\infty$ and consider $Z \geq X$. By monotonicity of $\mathcal{A}_Y$,

$$\left\{m \in \mathbb{R} \;\middle|\; \frac{m}{S_0}S + Z \in \mathcal{A}_Y\right\} \supseteq \left\{m \in \mathbb{R} \;\middle|\; \frac{m}{S_0}S + X \in \mathcal{A}_Y\right\}$$

hence $\Lambda_{\mathcal{A}}(Z,Y) \leq \Lambda_{\mathcal{A}}(X,Y)$.

Finally, by S-translation invariance of $\mathcal{A}_Y$ and by definition of sub-acceptance family it follows that for any $Y \in L^\infty$

$$\begin{aligned}\Lambda_{\mathcal{A}}(Y,Y) &= \inf\left\{m \in \mathbb{R} \;\middle|\; \frac{m}{S_0}S + Y \in \mathcal{A}_Y\right\} \\ &= \inf\left\{m \in \mathbb{R} \;\middle|\; \frac{m}{S_0}S + Y \in \mathcal{A}_{Y+\frac{m}{S_0}S}\right\} \\ &= \inf\left\{m \in \mathbb{R} \;\middle|\; \frac{m}{S_0}S + Y \in \mathcal{A}\right\}.\end{aligned}$$ □

Assume now that an S-risk contribution rule Λ is given. We can wonder which properties are fulfilled by the acceptance sets induced by Λ. To this aim, we introduce the following property:

S-additivity: $\Lambda(Y + mS, Y + mS) = \Lambda(Y,Y) - mS_0$, for all $m \in \mathbb{R}$, $S \in \mathcal{E}, Y \in L^\infty$;

generalizing cash-additivity of Λ.

Proposition 4.9. *If Λ is an S-risk contribution rule satisfying monotonicity, S-additivity and no-undercut, then the corresponding $\mathcal{A}$ and $(\mathcal{A}_Y)_{Y\in L^\infty}$ are, respectively, an acceptance set and a sub-acceptance family with respect to $\mathcal{A}$. Moreover, Λ is given by*

$$\Lambda(X,Y) = \begin{cases} \inf\left\{m \in \mathbb{R} \;\middle|\; \frac{m}{S_0}S + X \in \mathcal{A}_Y\right\} & \text{if } X \neq Y; \\ \inf\left\{m \in \mathbb{R} \;\middle|\; \frac{m}{S_0}S + Y \in \mathcal{A}\right\} & \text{if } X = Y. \end{cases} \tag{4.7}$$

Proof. $\mathcal{A} \neq \emptyset, L^\infty$: $0 \in \mathcal{A}$ by normalization of Λ. Given an arbitrary $S \in \mathcal{E}$ there exists $m \in \mathbb{R}$ such that $mS_0 < \Lambda(Y,Y)$. S-additivity implies then

$$\Lambda(Y + mS, Y + mS) = \Lambda(Y,Y) - mS_0 > 0,$$

hence $Y + mS \notin \mathcal{A}$. Non triviality of any $\mathcal{A}_Y$ can be checked similarly.

Monotonicity of each $\mathcal{A}_Y$: consider $X \in \mathcal{A}_Y$ and $Z \geq X$. Then, by monotonicity of Λ,

$$\Lambda(Z, Y) \leq \Lambda(X, Y) \leq 0.$$

Hence $Z \in \mathcal{A}_Y$.

Monotonicity of $\mathcal{A}$: take $X \in \mathcal{A}$ and $Y \geq X$ then, by monotonicity and no-undercut of Λ,

$$\Lambda(Y, Y) \leq \Lambda(X, Y) \leq \Lambda(X, X) \leq 0,$$

where the last inequality is due to $X \in \mathcal{A}$. Therefore $Y \in \mathcal{A}$ and $\mathcal{A}$ is an acceptance set. Since

$$\mathcal{A} = \{Y \in L^\infty \mid Y \in \mathcal{A}_Y\} = \{Y \in L^\infty \mid \Lambda(Y, Y) \leq 0\},$$

$(\mathcal{A}_Y)_{Y \in L^\infty}$ is a sub-acceptance family with respect to $\mathcal{A}$.

It remains to show that Λ can be represented as in (4.7). For any $X, Y \in L^\infty$ with $X \neq Y$ it holds that

$$\begin{aligned}\Lambda(X, Y) &= \inf\{m \in \mathbb{R} \mid \Lambda(X, Y) \leq m\} \\ &= \inf\left\{m \in \mathbb{R} \;\middle|\; \Lambda\left(X + \frac{m}{S_0}S, Y\right) \leq 0\right\} \\ &= \inf\left\{m \in \mathbb{R} \;\middle|\; X + \frac{m}{S_0}S \in \mathcal{A}_Y\right\},\end{aligned}$$

where the second equality holds by 1-S-additivity and the last one by definition of $\mathcal{A}_Y$. Finally, by S-additivity, it follows that, for any $Y \in L^\infty$,

$$\begin{aligned}\Lambda(Y, Y) &= \inf\{m \in \mathbb{R} \mid \Lambda(Y, Y) \leq m\} \\ &= \inf\left\{m \in \mathbb{R} \;\middle|\; \Lambda\left(Y + \frac{m}{S_0}S, Y + \frac{m}{S_0}S\right) \leq 0\right\} \\ &= \inf\left\{m \in \mathbb{R} \;\middle|\; Y + \frac{m}{S_0}S \in \mathcal{A}\right\}.\end{aligned}$$

□

Notice that when the only eligible asset is the risk-free asset with $S_T = S_0$, the previous results reduce to Proposition 4.1 and to Proposition 4.4, respectively.

4.3.2 *Quasi-convex case*

We now consider the case of families of sub-acceptance sets in quite a general framework. This is in line with the approach of quasi-convex risk measures where no cash-additivity is assumed on the risk measure and, consequently, neither on the family of acceptance sets. See Cerreia-Vioglio et al. [24], Drapeau and Kupper [33] and Frittelli and Maggis [40] for a detailed treatment on quasi-convex risk measures.

As pointed out in Drapeau and Kupper [33], in the case of quasi-convex risk measures the one-to-one correspondence between risk measures and acceptance sets (see Chapter 2) is no more true but has to be formulated in terms of acceptance sets at different levels. Differently from the cash-additive case where only the set at level 0 is relevant since all the other sets can be obtained from it by translation invariance, in the quasi-convex case the whole family of acceptance sets at different levels is needed.

Let $(\mathcal{A}_{Y,m})_{Y \in L^\infty, m \in \mathbb{R}}$ be a family of sub-acceptance sets at different levels $m \in \mathbb{R}$. That is:

- For any fixed $Y \in L^\infty$, $(\mathcal{A}_{Y,m})_{m \in \mathbb{R}}$ is a family of acceptance sets at the level m; i.e. every $\mathcal{A}_{Y,m}$ is an acceptance set parametrized by $m \in \mathbb{R}$.
- For any fixed $m \in \mathbb{R}$, $(\mathcal{A}_{Y,m})_{Y \in L^\infty}$ is a monotone sub-acceptance family with respect to an acceptance set $\mathcal{A}^m$. More precisely, $\mathcal{A}_{Y,m}$ is monotone increasing in $Y \in L^\infty$.

Roughly speaking, the level $m \in \mathbb{R}$ can be seen as a degree of acceptability. Given such a family, we define, for all $X, Y \in L^\infty$,

$$\Lambda_{\mathcal{A}}(X,Y) := \inf \{ m \in \mathbb{R} \mid X \in \mathcal{A}_{Y,m} \}, \tag{4.8}$$

and consider the following result.

PROPOSITION 4.10. *Let $(\mathcal{A}_{Y,m})_{Y \in L^\infty, m \in \mathbb{R}}$ be a family of sub-acceptance sets at the level $m \in \mathbb{R}$. Then $\Lambda_{\mathcal{A}}$ defined in (4.8) satisfies the following properties:*

i. *Decreasing monotonicity in $X \in L^\infty$.*

ii. *Decreasing monotonicity in $Y \in L^\infty$.*

iii. *Normalization, whenever $0 \in \mathcal{A}_{Y,0}$ and $0 \notin \mathcal{A}_{Y,m}$ for all $m < 0$.*

Furthermore, the following hold:

a. *If each $\mathcal{A}_{Y,m}$ is convex, then $\Lambda_{\mathcal{A}}$ is quasi-convex in the first variable.*

b. *If $\mathcal{A}_{Y,m} \subseteq \mathcal{A}_{Y,m+c} + c$ for all $m \in \mathbb{R}$, $c \geq 0$ and $Y \in L^\infty$, then $\Lambda_{\mathcal{A}}$ is cash-subadditive in the first variable.*

c. *If $\mathcal{A}_{Y,m} = \mathcal{A}_{Y,m+c} + c$ for all $m, c \in \mathbb{R}$ and $Y \in L^\infty$, then $\Lambda_{\mathcal{A}}$ is 1-cash-additive.*

PROOF. i. Take any $Z \geq X$. By monotonicity of $\mathcal{A}_{Y,m}$, it follows that

$$\begin{aligned}\Lambda_{\mathcal{A}}(X,Y) &= \inf \{ m \in \mathbb{R} \mid X \in \mathcal{A}_{Y,m} \} \\ &\geq \inf \{ m \in \mathbb{R} \mid Z \in \mathcal{A}_{Y,m} \} = \Lambda_{\mathcal{A}}(Z,Y).\end{aligned}$$

ii. It follows similarly, by monotonicity of $\mathcal{A}_{Y,m}$ in Y.

III. It is immediate.

The proofs of A and B are similar to those in Drapeau and Kupper [33]. We include them for reader's convenience.

A. Let $\alpha \in [0,1]$ and $X, Y, Z \in L^\infty$ be arbitrarily fixed. Assume now that $X, Z \in \mathcal{A}_{Y,\bar{m}}$ for some $\bar{m} \in \mathbb{R}$. It follows that both $\Lambda_\mathcal{A}(X,Y) \leq \bar{m}$ and $\Lambda_\mathcal{A}(Z,Y) \leq \bar{m}$. Then, by convexity of $\mathcal{A}_{Y,\bar{m}}$, also $\alpha X + (1-\alpha)Z \in \mathcal{A}_{Y,\bar{m}}$. Consequently, $\Lambda_\mathcal{A}(\alpha X + (1-\alpha)Z, Y) \leq \bar{m}$. By a well-known result on quasi-convex functionals, it follows that $\Lambda_\mathcal{A}(\cdot, Y)$ is quasi-convex for any $Y \in L^\infty$.

B. For any $m \in \mathbb{R}$, $c \geq 0$ and $X, Y \in L^\infty$ it holds that

$$\begin{aligned}
\Lambda_\mathcal{A}(X+c, Y) &= \inf\{m \in \mathbb{R} \mid X + c \in \mathcal{A}_{Y,m}\} \\
&\geq \inf\{m \in \mathbb{R} \mid X + c \in (\mathcal{A}_{Y,m+c} + c)\} \\
&= \inf\{m \in \mathbb{R} \mid X \in \mathcal{A}_{Y,m+c}\} \\
&= \inf\{m \in \mathbb{R} \mid X \in \mathcal{A}_{Y,m}\} - c \\
&= \Lambda_\mathcal{A}(X,Y) - c,
\end{aligned}$$

where the inequality above is due to the assumption $\mathcal{A}_{Y,m} \subseteq \mathcal{A}_{Y,m+c} + c$ for $c \leq 0$.

C. It can be proved similarly to item B. □

Notice that, thanks to the previous result, it holds that $\Lambda_\mathcal{A}(X + c, Y) \leq \Lambda_\mathcal{A}(X,Y)$ for any $c \geq 0$ and $X, Y \in L^\infty$ (by monotonicity in the first variable). Moreover, normalization and cash-subadditivity (whenever satisfied) imply that $\Lambda_\mathcal{A}(c, Y) \geq -c$ for any $c \geq 0$ and Y, while $\Lambda_\mathcal{A}(c,Y) \leq -c$ for any $c < 0$ and Y.

So far, we have defined a risk contribution rule starting from a family of sub-acceptance sets at different levels. We are now going to investigate the converse. Consider a risk contribution rule Λ, not necessarily satisfying cash-additivity, and define

$$\begin{aligned}
\mathcal{A}_{Y,m} &:= \{X \in L^\infty \mid \Lambda(X,Y) \leq m\} \\
\mathcal{A}^m &:= \{Y \in L^\infty \mid \Lambda(Y,Y) \leq m\}
\end{aligned} \tag{4.9}$$

for all $m \in \mathbb{R}$ and $Y \in L^\infty$.

PROPOSITION 4.11. *If Λ is a monotone risk contribution rule, then the corresponding $\mathcal{A}_{Y,m}$ and $\mathcal{A}^m$ defined as in (4.9) satisfy the following properties:*

I. *For each fixed $m \in \mathbb{R}$: $(\mathcal{A}_{Y,m})_{Y \in L^\infty}$, is a sub-acceptance family of*

$$\mathcal{A}^m = \{Y \in L^\infty \mid Y \in \mathcal{A}_{Y,m}\}.$$

II. *For each fixed $Y \in L^\infty$:*

A. *$\mathcal{A}_{Y,m}$ is monotone for every $m \in \mathbb{R}$.*

B. *$\mathcal{A}_{Y,m}$ is monotone in $m \in \mathbb{R}$ w. r. t. set inclusion.*

C. *$\mathcal{A}_{Y,m}$ is convex whenever $\Lambda(X, Y)$ is quasi-convex in $X \in L^\infty$.*

PROOF. I. We start to prove the properties once $m \in \mathbb{R}$ is fixed arbitrarily. We have only to check the first statement since the second is immediate. $\mathcal{A}_{Y,m} \neq \emptyset$ follows immediately by the assumptions on Λ implying that $-m \in \mathcal{A}_{Y,m}$ for any $m \in \mathbb{R}$ (since, by 1-cash-additivity, $\Lambda(m, Y) = \Lambda(0, Y) - m = -m$). $\mathcal{A}_{Y,m} \neq L^\infty$: again by the assumptions on Λ it follows that $-\bar{m} \notin \mathcal{A}_{Y,m}$ for any $\bar{m} > m$, hence the thesis.

II. Let now $Y \in L^\infty$ be fixed and let $m \in \mathbb{R}$ be arbitrary.

A. Assume that $X \in \mathcal{A}_{Y,m}$ and $Z \geq X$. By monotonicity of Λ in the first variable, it follows that $\Lambda(Z, Y) \leq \Lambda(X, Y) \leq m$. Hence, $Z \in \mathcal{A}_{Y,m}$.

B. It follows immediately by the definition of $\mathcal{A}_{Y,m}$ in (4.9).

C. It follows immediately as in B. □

5 FURTHER EXTENSIONS

In this chapter, we discuss some possible further extensions of the standard capital allocation setting, similarly to the previous chapter. However, here the topics only represent ideas for further works.

We first briefly discuss a generalization with respect to the ordering among random variables. That is, we drop the $\mathbb{P}$-a.s. order and assume a general preorder; we then focus on the first stochastic one (see Föllmer and Schied [39]). We find connections between the properties of acceptance sets and those of capital allocation rules very close to the results of the previous chapter.

Later, we discuss the capital allocation problem in the setting of intrinsic risk measures (see Farkas and Smirnow [37]). We begin with a discussion on intrinsic measures and then we state the capital allocation problem in this context. We further try to understand if extensions of the standard results and methods apply here. Finally, we consider the connections between capital allocation rules in the intrinsic context and acceptance sets, following the scheme of the previous chapter.

The chapter is organized as follows. In Section 5.1 we discuss the generalization with respect to the ordering; while in Section 5.2 we provide the discussion on intrinsic risk measures and the extension to capital allocation rules.

5.1 CAPITAL ALLOCATION AND GENERAL PREORDERS

So far, we have only considered capital allocation problems assuming the $\mathbb{P}$-a.s. ordering between random variables. In this section, we briefly discuss the possibility of extending the framework to a general preorder and later we focus on the first stochastic order (see Föllmer and Schied [39]). In particular, we discuss the extension of the results of Chapter 4. We refer to Appendix A for the standard notions about orderings, or, for more details, to Aliprantis and Border [2].

It is easy to see that the results presented in Chapter 4 still hold when we consider a general preorder, with the only requirement of being a vector preorder; that is, compatible with the vector structure of L^∞. Indeed, monotonicity, such as the one required for the acceptance sets and risk measures, may be considered with respect to a general preorder instead of the $\mathbb{P}$-a.s. order, with essentially the same meaning.

Therefore, given a preorder $\succeq$ on L^∞, a subset $\mathcal{A} \subseteq L^\infty$ is called an acceptance set with respect to $\succeq$ if it is non-trivial and $\succeq$-monotone:

$$X \in \mathcal{A}, Y \succeq X \implies Y \in \mathcal{A}.$$

The same applies to sub-acceptance families. A monetary risk measure with respect to $\succeq$ is a cash-additive map $\rho \colon L^\infty \to \mathbb{R}$ which is $\succeq$-monotone (decreasing):

$$X, Y \in L^\infty, Y \succeq X \implies \rho(Y) \leq \rho(X).$$

The same applies to monotone capital allocation or risk contribution rules.

Notice that acceptance sets and risk measures with respect to general preorders have been already considered in Arduca et al. [3], Farkas et al. [36] and Munari [56].

It is then clear that results such as those of Proposition 4.1 or Proposition 4.3 still hold with the minimal requirement of $\succeq$ being a vector preorder. The latter is needed to allow us summing up constant random variables to reach acceptability, that is, to ensure that there exist $m \in \mathbb{R}$ such that $X + m \in \mathcal{A}$; where $X \in L^\infty$ and $\mathcal{A}$ is an acceptance set.

5.1.1 *Stochastic orders*

We now focus on a particular ordering, namely the first stochastic one, and study their impact on our results.

We recall that $X \in L^\infty$ dominates $Y \in L^\infty$ in the first stochastic order, written $X \succeq_1 Y$, if and only if for all increasing functions $u \colon \mathbb{R} \to \mathbb{R}$,

$$\mathbb{E}[u(X)] \geq \mathbb{E}[u(X)]$$

or, equivalently, if and only if for all $x \in \mathbb{R}$,

$$F_X(x) \leq F_Y(x)$$

where $F_X(x) := \mathbb{P}(X \leq x)$ denotes the (cumulative) distribution function of X. See Appendix A or Bäuerle and Müller [11], Dana [27], Föllmer and Schied [39] and Müller et al. [55] for more details.

Under the first stochastic order, monetary risk measures turns out to be law invariant, as in Definition 2.7 (see Bäuerle and Müller [11], Dana [27], Wang et al. [65] and Weber [66]). While acceptance sets satisfy the following property:

$\mathcal{A}$-law invariance: if $X \sim Y, Y \in \mathcal{A}$ then $X \in \mathcal{A}$.

The notation $X \sim Y$ means that X and Y are equal in distribution, as we recalled in Chapter 2.

We are interested in studying capital allocation problems under the first stochastic order. That is, when only the distribution of the positions matters in allocating the capital or assessing the risk of the sub-portfolios. To this end, we introduce the following property for a risk contribution rule Λ:

1-law invariance: if $X \sim Z$ then $\Lambda(X,Y) = \Lambda(Z,Y)$ for all $Y \in L^\infty$.

The latter means that the risk contribution of X as a sub-portfolio of Y is equal to the risk contribution of any sub-portfolio Z of Y which has the same distribution of X; that is, for 1-law invariant risk contribution rule, only the distribution of the sub-portfolios matters in assessing their risk contribution.

Before going further, we need to introduce another property, which reflects the situation whereby Λ is linked with a law invariant risk measure:

Law invariance: if $X, Y \in L^\infty$, $X \sim Y$ then $\Lambda(X,X) = \Lambda(Y,Y)$.

We can now consider the following result.

PROPOSITION 5.1. *Let $\mathcal{A}$ be an acceptance set and $(\mathcal{A}_Y)_{Y \in L^\infty}$ be a sub-acceptance family, both with respect to the first stochastic order $\succeq_1$. Then $\Lambda_{\mathcal{A}}$ as in (4.1) is 1-law invariant. Moreover, if the sub-acceptance family is translation invariant then $\Lambda_{\mathcal{A}}$ is law invariant.*

PROOF. Fix $Y \in L^\infty$. If $\mathcal{A}_Y$ is an acceptance set with respect to $\succeq_1$ then it is also $\mathcal{A}$-law invariant. Hence, every $m \in \mathbb{R}$ such that $X + m \in \mathcal{A}_Y$ is also such that $Z + m \in \mathcal{A}_Y$, whenever $Z \sim X$. This implies

$$\{m \in \mathbb{R} \mid m + X \in \mathcal{A}_Y\} \subseteq \{m \in \mathbb{R} \mid m + Z \in \mathcal{A}_Y\}.$$

But also the converse is true: reversing the role of X and Z we get

$$\{m \in \mathbb{R} \mid m + X \in \mathcal{A}_Y\} \supseteq \{m \in \mathbb{R} \mid m + Z \in \mathcal{A}_Y\}.$$

Thus the two sets are equal and

$$\inf\{m \in \mathbb{R} \mid m + X \in \mathcal{A}_Y\} = \inf\{m \in \mathbb{R} \mid m + Z \in \mathcal{A}_Y\}$$
$$\Lambda_{\mathcal{A}}(X,Y) = \Lambda_{\mathcal{A}}(Z,Y).$$

By replacing $\mathcal{A}_Y$ with $\mathcal{A}$, we obtain that $\Lambda_{\mathcal{A}}$ is also law invariant when the sub-acceptance family is translation invariant (see Proposition 4.1). □

Now, we consider the impact of a (1-)law invariant risk contribution rule on the acceptance set and the sub-acceptance family induced by such a risk contribution rule.

PROPOSITION 5.2. *If Λ is a 1-law invariant risk contribution rule then each member of the sub-acceptance family given by (4.3) is $\mathcal{A}$-law invariant. Moreover, if Λ is also law invariant then the acceptance set given by (4.2) is $\mathcal{A}$-law invariant.*

PROOF. Fix $Y \in L^\infty$, take $X \in \mathcal{A}_Y$ and $Z \sim X$. It follows that

$$\Lambda(Z, Y) = \Lambda(X, Y) \leq 0$$

where the equality holds by 1-law invariance and the inequality by definition of $\mathcal{A}_Y$; thus $Z \in \mathcal{A}_Y$. Now take $X \in \mathcal{A}$ and $Y \sim X$, then

$$\Lambda(Y, Y) = \Lambda(X, X) \leq 0$$

where the equality holds by law invariance and the inequality by definition of $\mathcal{A}$; thus $Y \in \mathcal{A}$. □

So far, we have only considered the first stochastic order and shown that it is strictly related to the law invariance property, even in the context of capital allocation. However, it is possible to consider higher stochastic orders, such as the second one, that are not strictly related to the law invariance property and does not depend only on the chosen probability measure.

Moreover, we point out that 1-law invariance holds only for those Λ defined by means of a linear functional, that is, only when it is possible to separate the impact of X and Y. For instance, this happens for $\Lambda_{\mathcal{A}}$ derived as in (4.1) from a sub-acceptance family defined by means of the expected value, as in Example 4.2. Otherwise, Λ is not expected to be 1-law invariant with respect to the reference probability measure but only with respect to the conditional probability given Y; consider indeed the setting of Example 4.3. Thus, a possible extension is to consider a sort of conditional 1-law invariance, that is, with respect the conditional probability given Y instead of the reference probability measure.

5.2 INTRINSIC CAPITAL ALLOCATION RULES

We discuss here the problem of capital allocation with respect to the recent concept of intrinsic risk measure, proposed by Farkas and Smirnow [37].

In particular, we first discuss the meaning of the capital allocation problem when an intrinsic risk measure is involved, providing also some considerations on the intrinsic approach for risk measurement. We later define the notion of intrinsic capital allocation rule and discuss which properties are suitable (from a financial point of view) for this object. We finally discuss the intrinsic capital allocation problem from the point of view of acceptance sets, without involving any (intrinsic) risk measure, following the scheme of Chapter 4.

5.2.1 *Intrinsic risk measures*

In the traditional risk measure theory, given an acceptance set $\mathcal{A}$, the induced risk measure $\rho_{\mathcal{A}}$ is interpreted as a tool to quantify the minimal amount of money which should be added to a position to reach acceptability. In the generalized framework of Farkas et al. [36] and Munari [56], not only cash is considered to reach acceptability but also eligible assets. Anyway, both the approaches require an external source of capital which should be injected in the position.

The main idea of the intrinsic approach is instead to sell a part of the position and reinvest the gain in a given eligible asset, in order to reach acceptability. That is, in the intrinsic approach, only internal resources are used. Before going further, we provide some considerations of the two approaches. The traditional one, as highlighted by Farkas and Smirnow [37], has two main lacks:

- the procedure to acquire external capital is not addressed;
- the possibility of failing to acquire capital is not considered.

For instance, it is possible that the company does not have any liquidity to purchase the eligible asset (or to directly hedge the position) and it is not able to acquire such capital quickly. In that case, the idea of selling a part of the position to get some liquidity for buying the eligible asset, that is the intrinsic approach, looks interesting. However, we notice two main lacks of the intrinsic approach as well:

- the case of indivisible positions is not addressed;
- the possibility of failing to sell (a part of) the position is not considered.

In particular, when X represents the net asset value (assets minus liabilities) of a firm, it is not possible in general to sell a part of X, because it is not possible to sell out liabilities. Therefore, the procedure works only when X represents only the asset value of the firm (without considering the liabilities). Notice that Baes et al. [7] proposed a similar approach that works in the last case: looking for the proportion of each asset to sell in order to create optimal portfolios of eligible assets to reach acceptability.

We therefore claim that intrinsic risk measures cannot substitute the traditional ones, but they might be used as a complement. More precisely, for each risky position to be measured, one could both consider the intrinsic approach and the traditional one, then apply the one which is more suitable for the specific case.

We now recall from Farkas and Smirnow [37] the definition of intrinsic risk measure, as well as some key results.

Here, we consider financial positions as couples $X = (X_0, X_T)$ belonging to the product space $\mathbb{R}_{++} \times L^\infty$, where X_0 represents the value of the position at the initial time 0, and X_T its the payoff, at time $T \geq 0$. We will use the two following orderings on $\mathbb{R}_{++} \times L^\infty$:

Element-wise: $X \geq_{\text{el}} Y :\iff X_0 \geq Y_0$ and $X_T \geq Y_T$.

Return-wise: $X \geq_{\text{re}} Y :\iff \frac{X_T}{X_0} \geq \frac{Y_T}{Y_0}$.

Definition 5.1 (Farkas and Smirnow [37]). Given an acceptance set $\mathcal{A}$ containing 0 and an eligible asset S, an *intrinsic risk measure* is a map $R_{\mathcal{A},S}\colon \mathbb{R}_{++} \times L^\infty \to [0,1]$ defined by

$$R_{\mathcal{A},S}(X) := \inf \left\{ \lambda \in [0,1] \,\middle|\, (1-\lambda)X_T + \lambda \frac{X_0}{S_0} S_T \in \mathcal{A} \right\}.$$

We will simply write R rather than $R_{\mathcal{A},S}$ when the reference to the acceptance set and the eligible asset is clear. Actually, Farkas and Smirnow [37] do not require $0 \in \mathcal{A}$ in the definition of intrinsic risk measure, but they point out that for well-definedness of R, $\mathcal{A}$ must be either a cone or contain 0. Therefore, we include the latter in the previous definition.

In the previous definition, $R_{\mathcal{A},S}(X)$ is interpreted as the smallest $\lambda \in [0,1]$ such that selling the fraction λ of X and investing the monetary amount λX_0 in the eligible asset S yields an acceptable position (see Farkas and Smirnow [37] for more details).

We now recall some results on intrinsic risk measures.

Proposition 5.3 (Farkas and Smirnow [37]). *Let $\mathcal{A}$ be an acceptance set containing 0, S be an eligible asset and $X, Y \in L^\infty \times \mathbb{R}_{++}$. Then, the following hold:*

a. *R is $\geq_{\mathrm{el}}$-monotone decreasing: if $X \geq_{\mathrm{el}} Y$ then $R(X) \leq R(Y)$.*

b. *If $\mathcal{A}$ is conic then R is $\geq_{\mathrm{re}}$-monotone decreasing.*

c. *If $\mathcal{A}$ is convex then R is quasi-convex.*

Proposition 5.4 (Farkas and Smirnow [37]). *Let $\mathcal{A}$ be an acceptance set. If $\mathcal{A}$ is closed and conic then $R_{\mathcal{A},S}$ admits the following representation:*

$$R_{\mathcal{A},S}(X) = \frac{\left(\rho_{\mathcal{A},S}(X_T)\right)^+}{X_0 + \rho_{\mathcal{A},S}(X_T)}$$

where $\rho_{\mathcal{A},S}$ is given by (2.1). Moreover, under the same assumptions, $R_{\mathcal{A},S}$ is scale invariant:

$$R_{\mathcal{A},S}(\alpha X) = R_{\mathcal{A},S}(X), \text{ for each } \alpha > 0 \text{ and } X \in \mathbb{R}_{++} \times L^\infty.$$

Proposition 5.5 (Farkas and Smirnow [37]). *Let $\mathcal{A}$ be an acceptance set. If $\mathcal{A}$ is $\sigma(L^\infty, L^1)$-closed, convex and it contains 0 then, for any eligible asset S, $R_{\mathcal{A},S}$ admits the following representation:*

$$R_{\mathcal{A},S}(X) = \sup_{\mathbb{Q}\in\mathcal{Q}} \frac{\left(\mathbb{E}_{\mathbb{Q}}[-X_T] - \beta(\mathbb{Q})\right)^+}{\frac{X_0}{S_0}\mathbb{E}_{\mathbb{Q}}[S_T] - \mathbb{E}_{\mathbb{Q}}[X_T]}$$

where $\beta(\mathbb{Q}) := \sup_{X_T \in \mathcal{A}} \mathbb{E}_{\mathbb{Q}}[-X_T]$, for all $\mathbb{Q} \in \mathcal{Q}$.

We add to the previous results the following one, not contained in [37].

PROPOSITION 5.6. *Let $\mathcal{A}$ be an acceptance set. If $\mathcal{A}$ is closed, convex, conic and it contains* 0 *then* R *is subadditive.*

PROOF. The result follows by quasi-convexity and scale invariant of R. More precisely, for $X, Y \in \mathbb{R}_{++} \times L^\infty$ the following holds:

$$R(X+Y) = R\left(\frac{1}{2}(X+Y)\right) \leq \max\{R(X), R(Y)\} \leq R(X) + R(Y),$$

where the first equality is due to scale invariant, the first inequality by quasi-convexity and the last one because $R(\cdot)$ is positive. □

5.2.2 *Intrinsic capital allocation*

We discuss here the problem of capital allocation in the intrinsic framework. More formally, we assume that an intrinsic risk measure R, as in Definition 5.1, is given and we think of assigning a share of $R(Y)$ to the sub-units of Y. We remember that $R(Y) \in [0,1]$, so it can be interpreted as the percentage of the aggregated position Y that should be sold and reinvested in a given eligible asset to reach acceptability. With this idea in mind, it makes sense to look for which (part of the) sub-units should be sold in order to ensure that the "new" aggregated position reaches the acceptability.

We therefore restate the capital allocation problem as follows. We assume that $X = (X_0, X_T) \in \mathbb{R}_{++} \times L^\infty$ is an aggregate position and that it is decomposed into sub-units

$$X^1 = (X_0^1, X_T^1), \dots, X^n = (X_0^n, X_T^n)$$

all belonging to $\mathbb{R}_{++} \times L^\infty$; that is, $X = \sum_{i=1}^n X^i$. We also assume that an acceptable $\mathcal{A}$ and eligible asset S are given, so that $R := R_{\mathcal{A},S}$ is well defined. In such context, a capital allocation problem consists in finding "suitable" real numbers $l_1, \dots, l_n$, belonging to $[0,1]$, such that

$$R(X)X_0 = \sum_{i=1}^n l_i X_0^i.$$

Here, $R(X)X_0$ is the monetary amount obtained by selling the fraction $R(X)$ of the aggregated position X. The latter, if invested in the eligible asset, ensures the acceptability of the new aggregated position. Such monetary amount $R(X)X_0$ should be the sum of the monetary amounts obtained by selling the fractions l_i of the sub-positions $X^i, i = 1, \dots, n$. Each l_i is then the percentage of the sub-unit X^i which should be sold and reinvested in the eligible asset to ensure that the new aggregated position reaches acceptability.

We now provide the following definition, which gives us the tool to face capital allocation problems in the intrinsic framework. We then discuss some properties reasonable for such a tool, in order to clarify the term "suitable" mentioned above.

Definition 5.2. Let $\mathcal{A}$ be an acceptance set containing 0, S be an eligible asset and R be an intrinsic risk measure as in Definition 5.1. An *intrinsic capital allocation rule (ICAR)* with respect to R, is a map $L_R\colon (\mathbb{R}_{++} \times L^\infty) \times (\mathbb{R}_{++} \times L^\infty) \to [0,1]$ such that, for all $X \in \mathbb{R}_{++} \times L^\infty$,

$$L_R(X, X) = R(X).$$

We will simply write L when the reference to the intrinsic risk measure is clear.

The previous definition is very close to the one of capital allocation rule (Definition 2.12). This is because the requirement $L_R(X,X) = R(X)$ makes sense in this context as well. Consider indeed the capital allocation problem presented above: if X is the only sub-unit of itself, then the problem reduces to finding a suitable $l \in [0,1]$ such that $R(X)X_0 = lX_0$; thus such l must be equal to $R(X)$.

Here, $L(X,Y)$ represents the fraction of the position X, viewed as a sub-portfolio of Y, which should be sold and reinvested in the eligible asset to ensure that the new aggregated position reaches acceptability. In some sense, $L(X,Y)$ reflects the "intrinsic risk" of Y, equal, by definition, to $L(Y,Y)$.

We now discuss the main properties to be required for a ICAR, starting from those commonly required for CARs.

Full allocation. This property, as stated in Chapter 2, does not apply in this context, since here we work with general risk measures which furthermore yield numbers in $[0,1]$. However, it can be restated as follows: for all $Y^1, \dots, Y^n$, $Y \in \mathbb{R}_{++} \times L^\infty$ such that $Y = \sum_{i=1}^n Y^i$,

$$L(Y,Y)Y_0 = \sum_{i=1}^{n} L(Y^i, Y)Y_0^i.$$

That is, we require that the sum of the monetary amounts obtained by selling the fractions $L(Y^i, Y)$ of the sub-positions $Y^i, i = 1, \dots, n$ is equal to the monetary amount obtained by selling the fraction $L(Y,Y)$ of the aggregated position Y.

No-undercut. It looks meaningful to require the no-undercut here too, since we still do not want the sub-units to split from the company. Moreover, it is meaningful to require that the percentage of a position, viewed as sub-unit of an aggregated position, to be sold to reach acceptability should be less than that of the sub-unit viewed as a stand-alone portfolio. Thus, no-undercut is still meant as

$$L(X,Y) \leq L(X,X)$$

for all $X, Y \in \mathbb{R}_{++} \times L^\infty$.

Riskless. It does not apply here since we are working with percentages. Moreover, it may be needed to sell only a part of a constant position $a \in \mathbb{R}_{++} \times \mathbb{R}$ to guarantee the acceptability of the aggregated one, so there are no significant alternatives to this property.

We now consider a simple example of ICAR, derived from the marginal method of capital allocation.

EXAMPLE 5.1. We recall that the marginal CAR is given by Definition 2.14. We rephrase the method in the intrinsic framework and we discuss its meaning.

Assume that $\mathcal{A}$ is an acceptance set containing 0, S is an eligible asset and R is an intrinsic risk measure. Assume also that, for all $X, Y \in \mathbb{R}_{++} \times L^\infty$,

$$X - Y := (|X_0 - Y_0|, X_T - Y_T) \in \mathbb{R}_{+} \times L^\infty$$

and, for all $Z_T \in L^\infty$,

$$R((0, Z_T)) := \begin{cases} 0 & \text{if } Z_T \in \mathcal{A}; \\ 1 & \text{if } Z_T \notin \mathcal{A}; \end{cases}$$

so that $R(X - Y)$ makes sense for all $X, Y \in \mathbb{R}_{++} \times L^\infty$.

We then define the marginal ICAR as

$$L_R^M(X, Y) := \big(R(Y) - R(Y - X)\big)^+, \quad X, Y \in \mathbb{R}_{++} \times L^\infty.$$

It is clear that the method is an intrinsic capital allocation rule with respect to R, because the assumptions above imply $L_R^M(X, X) = R(X)$ for all $X \in \mathbb{R}_{++} \times L^\infty$. Moreover, whenever $\mathcal{A}$ is also convex and conic, that is, whenever R is subadditive, the marginal method satisfy no-undercut.

As regards full allocation, we can use a trick similar to the standard case. Indeed,

$$\widehat{L}_R^M(X, Y) := \frac{R(Y) Y_0}{L_R^M(X, Y) X_0 + L_R^M(Y - X, Y)(Y - X)_0} L_R^M(X, Y)$$

satisfies full allocation, as stated above.

5.2.3 *ICAR and acceptance sets*

We now rephrase the intrinsic capital allocation setting, provided above, from the perspective of acceptance sets and sub-acceptance families, as in Chapter 4.

We start from acceptance sets, define a map induced from those and study how the properties on sets impact on the map. We then investigate the converse.

Given a family $(\mathcal{A}_{Y_T})_{Y_T \in L^\infty}$ of sets and a position $S \in \mathbb{R}_{++} \times L^\infty$, we define, for all $X, Y \in \mathbb{R}_{++} \times L^\infty$,

$$L_{\mathcal{A},S}(X, Y) := \inf \left\{ \lambda \in [0, 1] \;\middle|\; (1 - \lambda) X_T + \lambda \frac{X_0}{S_0} S_T \in \mathcal{A}_{Y_T} \right\}. \quad (5.1)$$

As usual, we will simply write L when no misunderstandings can arise.

The following definition introduce a property for sub-acceptance families which is similar to translation invariance but suitable for this context.

Definition 5.3. A sub-acceptance family $(\mathcal{A}_{Y_T})_{Y_T \in L^\infty}$ is said to be *intrinsic invariant* if it satisfies the following property:

Intrinsic invariance: $\mathcal{A}_{Y_T} = \mathcal{A}_{\alpha Y_T + \beta S_T}$ for all $\alpha, \beta \geq 0$ and $S \in \mathcal{E}$.

Thus, for intrinsic invariance sub-acceptance families the positions which are sub-acceptable with respect to Y_T are also so with respect to any combination of Y_T and the payoff of the eligible asset S_T, with weights $\alpha, \beta \geq 0$.

When $\mathcal{A}$ is an acceptance set and $(\mathcal{A}_{Y_T})_{Y_T \in L^\infty}$ is a sub-acceptance family, we say that $S \in \mathbb{R}_{++} \times L^\infty$ is eligible for the family if $S_T \in \mathcal{A}_{Y_T}$ for all $Y_T \in L^\infty$. In particular, the latter implies $S_T \in \mathcal{A}_{S_T}$, thus, by definition of sub-acceptance family, $S_T \in \mathcal{A}$ and so S is also eligible.

We can now state the following result.

Proposition 5.7. *If $\mathcal{A}$ is an acceptance set, $(\mathcal{A}_{Y_T})_{Y_T \in L^\infty}$ is a sub-acceptance family such that $0 \in \mathcal{A}_{Y_T}$ for all $Y_T \in L^\infty$ and $S \in \mathbb{R}_{++} \times L^\infty$ is eligible for the family, then L as in (5.1) satisfies the following:*

A. $L(X, Y) \in [0, 1]$ *for all* $X, Y \in \mathbb{R}_{++} \times L^\infty$.

B. *There exists* $X \in \mathbb{R}_{++} \times L^\infty$ *such that* $L(X, Y) > 0$ *for all* $Y \in \mathbb{R}_{++} \times L^\infty$.

C. $L\big((X_0, 0), Y\big) = 0$ *for all* $Y \in \mathbb{R}_{++} \times L^\infty$ *and* $X_0 \in \mathbb{R}_{++}$.

D. *Decreasing* $\geq_{\mathrm{el}}$*-monotonicity.*

Moreover, if the sub-acceptance family is also intrinsic invariant then

$$L_{\mathcal{A},S}(Y, Y) = \inf \left\{ \lambda \in [0, 1] \;\middle|\; (1 - \lambda) Y_T + \lambda \frac{Y_0}{S_0} S_T \in \mathcal{A} \right\}.$$

Proof. A. Fix $Y \in \mathbb{R}_{++} \times L^\infty$. Notice that $L_{\mathcal{A}}(X, Y) = R_{\mathcal{A}_{Y_T}, S}(X)$ for all $X \in \mathbb{R}_{++} \times L^\infty$ and S eligible. The thesis then easily follows.

B. It follows by non-triviality of each $\mathcal{A}_{Y_T}$. In particular, $\mathcal{A}_{Y_T} \neq L^\infty$ implies $L(Z, Y) > 0$ for some $Z \in \mathbb{R}_{++} \times L^\infty$.

C. It follows easily because each $\mathcal{A}_{Y_T}$ contains 0.

D. Fix $X, Y, Z \in \mathbb{R}_{++} \times L^\infty$ and assume $Z \geq_{\mathrm{el}} X$. Then

$$(1 - \lambda) Z_T + \lambda \frac{Z_0}{S_0} S_T \geq (1 - \lambda) X_T + \lambda \frac{X_0}{S_0} S_T$$

holds for any $\lambda \in [0,1]$ and S eligible. By monotonicity of $\mathcal{A}_{Y_T}$,

$$\left\{\lambda \in [0,1] \,\middle|\, (1-\lambda)X_T + \lambda\frac{X_0}{S_0}S_T \in \mathcal{A}_{Y_T}\right\}$$
$$\subseteq \left\{\lambda \in [0,1] \,\middle|\, (1-\lambda)Z_T + \lambda\frac{Z_0}{S_0}S_T \in \mathcal{A}_{Y_T}\right\}$$

hence $L(Z,Y) \leq L(X,Y)$.

As regards the last statement, for any $Y \in \mathbb{R}_{++} \times L^\infty$ it holds that

$$\begin{aligned} L(Y,Y) &= \inf\left\{\lambda \in [0,1] \,\middle|\, (1-\lambda)Y_T + \lambda\frac{Y_0}{S_0}S_T \in \mathcal{A}_{Y_T}\right\} \\ &= \inf\left\{\lambda \in [0,1] \,\middle|\, (1-\lambda)Y_T + \lambda\frac{Y_0}{S_0}S_T \in \mathcal{A}_{(1-\lambda)Y_T+\lambda\frac{Y_0}{S_0}S_T}\right\} \\ &= \inf\left\{\lambda \in [0,1] \,\middle|\, (1-\lambda)Y_T + \lambda\frac{Y_0}{S_0}S_T \in \mathcal{A}\right\}, \end{aligned}$$

where the second equality holds by intrinsic invariance and the last one by definition of sub-acceptance family. □

We now provide a result which links the $\mathcal{A}$-no-undercut of a sub-acceptance family and the no-undercut of the map L induced by such family, as in Chapter 4.

PROPOSITION 5.8. *Let $\mathcal{A}$ be an acceptance set and let $(\mathcal{A}_{Y_T})_{Y_T \in L^\infty}$ be an intrinsic invariant sub-acceptance family. If the family $(\mathcal{A}_{Y_T})_{Y_T \in L^\infty}$ satisfies $\mathcal{A}$-no-undercut, then $L_\mathcal{A}$ satisfies no-undercut.*

PROOF. Fix $X, Y \in \mathbb{R}_{++} \times L^\infty$, $\mathcal{A}$-no-undercut implies

$$\left\{\lambda \in [0,1] \,\middle|\, (1-\lambda)X_T + \lambda\frac{X_0}{S_0}S_T \in \mathcal{A}\right\}$$
$$\subseteq \left\{\lambda \in [0,1] \,\middle|\, (1-\lambda)X_T + \lambda\frac{X_0}{S_0}S_T \in \mathcal{A}_{Y_T}\right\}.$$

Hence,

$$\begin{aligned} L_\mathcal{A}(X,Y) &= \inf\left\{\lambda \in [0,1] \,\middle|\, (1-\lambda)X_T + \lambda\frac{X_0}{S_0}S_T \in \mathcal{A}_{Y_T}\right\} \\ &\leq \inf\left\{\lambda \in [0,1] \,\middle|\, (1-\lambda)X_T + \lambda\frac{X_0}{S_0}S_T \in \mathcal{A}\right\} \\ &= L_\mathcal{A}(X,X), \end{aligned}$$

where the last equality holds thanks to intrinsic invariance of the sub-acceptance family. □

Conversely, given a map $L\colon (\mathbb{R}_{++} \times L^\infty) \times (\mathbb{R}_{++} \times L^\infty) \to [0,1]$, we define the following sets, for $Y_T \in L^\infty$:

$$\begin{aligned} \mathcal{A}_L &:= \{Y_T \mid Y = (Y_0, Y_T) \in \mathbb{R}_{++} \times L^\infty,\ L(Y,Y) = 0\} \\ \mathcal{A}_{Y_T,L} &:= \{X_T \mid X = (X_0, X_T) \in \mathbb{R}_{++} \times L^\infty,\ L(X,Y) = 0\} \end{aligned} \tag{5.2}$$

where the subscript L will be omitted when it is clear which map is involved. Notice that $\mathcal{A}_L$ and $\mathcal{A}_{Y_T,L}$ have basically the same meaning of $\mathcal{A}_\Lambda$ and $\mathcal{A}_{Y,\Lambda}$ (see (4.2) and (4.3)). In this context, the requirement ≤ 0 does not make any sense, so it is replaced by $= 0$ which looks meaningful, thinking at the interpretation of intrinsic risk measures or intrinsic capital allocation rules.

PROPOSITION 5.9. *Let* $L\colon (\mathbb{R}_{++} \times L^\infty) \times (\mathbb{R}_{++} \times L^\infty) \to [0,1]$ *be such that the following are satisfied:*

A. *There exists* $X \in \mathbb{R}_{++} \times L^\infty$ *such that* $L(X,Y) > 0$ *for all* $Y \in \mathbb{R}_{++} \times L^\infty$.

B. $L\big((X_0,0),Y\big) = 0$ *for all* $Y \in \mathbb{R}_{++} \times L^\infty$ *and* $X_0 \in \mathbb{R}_{++}$.

C. *Decreasing* $\geq_{\mathrm{el}}$*-monotonicity.*

D. *Decreasing* $\geq_{\mathrm{el}}$*-full monotonicity:*

$$X \geq_{\mathrm{el}} Y \implies L(X,X) \leq L(Y,Y).$$

Then, $\mathcal{A}$ *and* $(\mathcal{A}_{Y_T})_{Y_T \in L^\infty}$ *given by* (5.2) *are, respectively, an acceptance set containing* 0 *and a sub-acceptance family such that* $0 \in \mathcal{A}_{Y_T}$ *for all* $Y_T \in L^\infty$.

PROOF. $\mathcal{A}_{Y_T} \neq \emptyset, L^\infty$: it follows easily by A. and B. Moreover, B. implies that $0 \in \mathcal{A}_{Y_T}$.

Monotonicity of each $\mathcal{A}_{Y_T}$: it follows by decreasing $\geq_{\mathrm{el}}$-monotonicity of L. In particular, take $X, Z \in \mathbb{R}_{++} \times L^\infty$ such that $Z_T \in \mathcal{A}_{Y_T}$ and $X \geq_{\mathrm{el}} Z$. Then

$$L(X,Y) \leq L(Z,Y) = 0$$

holds because $Z_T \in \mathcal{A}_{Y_T}$ and L is decreasing $\geq_{\mathrm{el}}$-monotone. Thus, $L(X,Y) \in [0,1]$ implies $L(X,Y) = 0$, so $X_T \in \mathcal{A}_{Y_T}$.

$\mathcal{A} \neq \emptyset, L^\infty$: A. implies $L(X,X) > 0$ for some $X \in \mathbb{R}_{++} \times L^\infty$ and B. implies $L\big((X_0,0),(X_0,0)\big) = 0$ so that $0 \in \mathcal{A}$ and $\mathcal{A} \neq L^\infty$.

Monotonicity of $\mathcal{A}$: it follows by decreasing $\geq_{\mathrm{el}}$-full monotonicity of L, similarly to the monotonicity of each $\mathcal{A}_{Y_T}$. □

As we can see from the previous proposition, we do not obtain a representation of L in term of the acceptance sets, as in the standard case. Moreover, we did not involve any eligible asset in the previous proposition.

As pointed out in the beginning, these are just a few notes on the topic, containing some ideas which could be developed in a further work.

6 A NUMERICAL AND COMPARATIVE STUDY

In this chapter, we make a short survey on the problem of capital allocation through the use of risk measures and we apply some of the most popular capital allocation methods to a portfolio of risky positions. In particular, we consider the proportional and the marginal methods, presented in Chapter 2, together with the Value at Risk, the Conditional Value at Risk and the entropic risk measure. Indeed, this chapter can be considered as an appendix of Chapter 2, with the aim of illustrating some of the standard methods of capital allocation, in order to better understand their connections with risk measures. For this reason, we do not provide here numerical examples of the new capital allocation rules introduced in Chapter 3 and Chapter 4, as they could be subject of a dedicated work.

Since such risk measures are law invariant, that is, the capital requirement of a risky position only depends on its distribution, the same holds for the proportional and marginal method. The situation is different if we use such methods with the covariance as a risk measure, that is, if we set $\rho(X_i) := \mathrm{Cov}(X_i, X)$ for any sub-unit X_i of the fixed position X. In this case, the dependence among the P&Ls of the various sub-units matters. Therefore, we also include this method in the survey.

6.1 DATA COLLECTION AND ANALYSIS

We apply the capital allocation methods mentioned above to a portfolio of five stocks of the FTSE-MIB index, chosen in different sectors: Atlantia (ATL), Brembo (BRE), Eni (ENI), Intesa San Paolo (ISP) and Telecom Italia (TIT). We collected from Bloomberg five years of daily closing prices of the stocks listed above, in the period December 2013-2018, obtaining a sample of 1269 observations for each asset.

We model the daily P&L instead of daily prices, i.e. each stock is represented by the random variable

$$X_i := S_t^i - S_{t-1}^i$$

where S_t^i is the price at day t of the i-th stock, $i = 1, \dots, 5$. The portfolio X is simply given by $X := \sum_{i=1}^5 X_i$; that is, we buy one unit of each stock. Figure 6.1 shows the dynamic of the portfolio prices and of the portfolio P&Ls; some descriptive statistics of the P&Ls of the stocks and of the portfolio are reported in Table 6.1.

Looking at Figure 6.1, we notice some high peaks followed by a drop, this shows high volatility of data; to be more precise, we analyze Table 6.1. The means are close to zero, in particular for Intesa

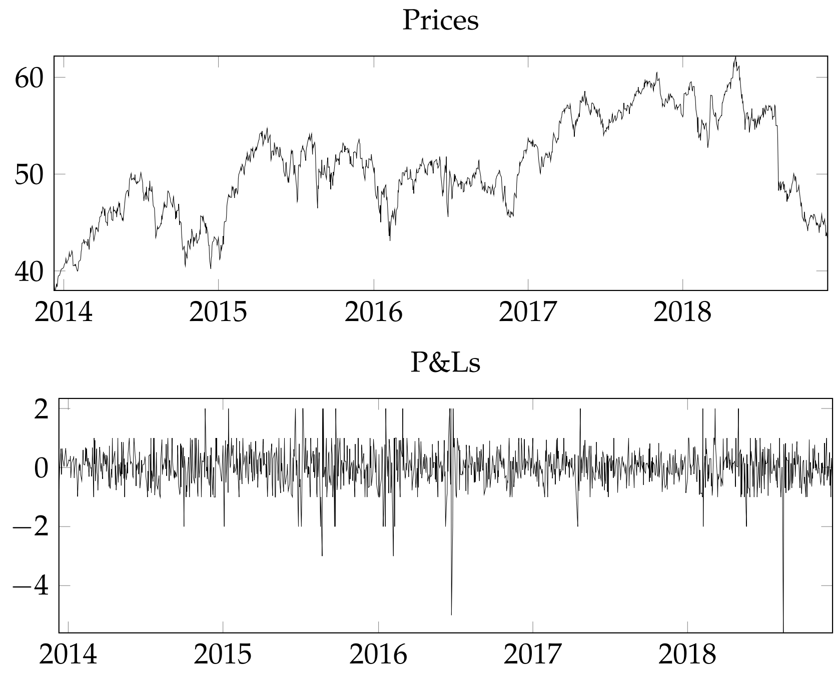

Figure 6.1: Daily portfolio prices and P&Ls.

	ATL	BRE	ENI	ISP	TIT	Port
Mean	0.0011	0.0045	-0.0023	0.0002	-0.0001	0.0034
StDev	0.3638	0.1618	0.2291	0.0547	0.0188	0.6538
Min	-5.2400	-0.7440	-1.3400	-0.5180	-0.1375	-5.5933
Max	1.3000	0.8840	0.8500	0.3100	0.1010	2.3360
Skew	-2.3574	0.2630	-0.2875	-0.5681	-0.0890	-0.8073
Kurt	36.3471	6.0586	5.3169	11.1909	6.9823	9.8177

Table 6.1: Daily P&L descriptive statistics.

and Telecom. This is reasonable since we consider one-day P&Ls. Standard deviations and ranges confirm high volatility of the portfolio P&Ls, since the first three stocks have a high standard deviation. Skewness is positive for Brembo, while it is negative for the others and far from zero for Atlantia. Kurtosis is very high, in particular for Atlantia: this can be also seen from the minimum P&L which Atlantia performed in the considered period. Skewness and Kurtosis highlight how the data are far from being normally distributed, taking into account that Normal distribution has zero Skewness and Kurtosis equal 3. Rather, they seem to come from heavy-tailed distributions. In such situations, therefore, it may happen that VaR does not encourage diversification of risk.

6.2 RISK CAPITAL COMPUTATION

We apply the considered risk measures to each stock and to the whole portfolio, using the historical simulation method (see for instance Jorion [49]); that is, we replace the theoretical distribution of the P&L with the observed time series and we compute risk measures using these data. To illustrate the procedure, we show how we compute the historical VaR, i. e. how we sample the empirical quantile. We first take each time series and sort the data concerning daily P&Ls from the smallest to the largest, we then assign to each price a weight of 1/1268, where 1268 is the number of observed daily P&Ls. We compute the empirical cumulative distribution function by computing cumulative weights: starting from the smallest P&L, we sum the weight of the previous P&L to the weight of the current one, until the last observed P&L. Then we set $\alpha = 0.01$ and look for the smallest value which has a cumulative weight greater than 0.01; changing the sign of this value, we obtain the VaR at the level 0.01. We compute in a similar way the other risk measures, letting $\alpha = 0.01$; this means, for the entropic risk measure, a high risk aversion and so a more conservative risk measure. We also compute the diversification index, given by Equation (2.6), for each risk measure. The results we obtained are shown in Table 6.2.

	ATL	BRE	ENI	ISP	TIT	Port	DI
VaR_α	0.8951	0.4216	0.5600	0.1453	0.0480	1.5721	0.7595
CVaR_α	1.4950	0.5155	0.7815	0.2045	0.0660	2.4820	0.8104
e_α	5.1685	0.6725	1.2685	0.4465	0.0662	5.5218	0.7244
Var	0.1356	0.0262	0.0525	0.0030	0.0004	0.4275	1.9644

Table 6.2: Daily risk measures of stocks and portfolio ($\alpha = 0.01$).

Looking at Table 6.2, we notice that the entropic risk measure is the most conservative one; this is due to the small α value we set, as we explained before. A diversification effect is obtained for the first three risk measures, despite VaR and the entropic risk measure are, in general, not subadditive. We check this simply by looking at the diversification index: the first three risk measures have a DI less than 1, hence they are subadditive in this example. In particular, the entropic risk measure obtained the highest diversification effect. The diversification effect is not achieved from the variance, which is super-additive in this example, in fact it has a diversification index greater than 1. However, being convex, the variance should not penalize diversification (see the discussion on convex risk measures provided in Chapter 2). The reason is that we defined (as it is customary) the diversification index in terms of sums and not convex combinations. This shows how different diversification concepts can sometimes conflict with each other.

6.3 RISK CAPITAL ALLOCATION

In the previous section, we have computed the risk capital of each stock and of the portfolio. The latter will now be allocated to each stock, while the first one will be useful to evaluate the pooling effect (see Chapter 2). We compute the capital allocations using the proportional and the marginal methods (see again Chapter 2). We also include the RORAC contributions as a further comparison.

6.3.1 *Proportional methods*

We first apply the proportional methods. The results are shown in Table 6.3.

	ATL	BRE	ENI	ISP	TIT
VaR_α	0.6798 (43.2%)	0.3202 (20.4%)	0.4253 (27%)	0.1103 (7%)	0.0365 (2.4%)
CVaR_α	1.2116 (48.8%)	0.4178 (16.8%)	0.6334 (25.5%)	0.1658 (6.7%)	0.0535 (2.2%)
e_α	3.7442 (67.8%)	0.4872 (8.8%)	0.9190 (16.6%)	0.3235 (5.9%)	0.0480 (0.9%)
Cov	0.2123 (49.7%)	0.0666 (15.6%)	0.1154 (27%)	0.0259 (6%)	0.0072 (1.7%)

Table 6.3: Daily proportional capital allocations of stocks ($\alpha = 0.01$).

Looking at Table 6.3, we notice that, for the first three capital allocation methods, the risk capital allocated to each stock considered as an element of the portfolio does not exceed the risk capital allocated to the stock considered as a stand-alone portfolio. To check this, we simply compare the results of Table 6.3 to the results of Table 6.2; since each value of the first three rows of Table 6.3 is less than the respective value of Table 6.2, the pooling effect mentioned above is obtained. This follows straightforwardly from the diversification effect we obtained in Table 6.2: as on our data the considered risk measures turn out to behave subadditively and as $\rho(X_i) \geq 0$ for all i, Equation (2.3) shows that risk capitals allocated via proportional allocation methods benefit from the pooling effect. In particular, the proportional method based on the entropic risk measure has benefited from the highest pooling effect. The reason is clear, since the entropic risk measure has the highest diversification index and the proportional methods allocate the capital via $K_i = \mathrm{DI}_\rho \cdot \rho(X_i)$, the allocated capital by using the entropic risk measure is, for each unit of risk capital $\rho(X_i)$, less than the capital allocated via proportional methods based on different risk measures. Since variance is super-additive in this example, the pooling effect is not obtained from this risk measure and the risk capital allocated to each stock considered as an element of the portfolio exceeds the risk capital allocated to the stock considered as a stand-alone portfolio. The full allocation property is satisfied for each risk measure: summing by row the values in Table 6.3 we obtain exactly the last column of Table 6.2; that is, the sum of risk capitals allocated to each stock is equal to the risk capital allocated to the portfolio using the respective risk measure. Furthermore, the results of Table 6.3 show also that all the capital allocation rules here considered agree in putting more weight on Atlantia than on others, reflecting the large risk capital assigned to this single stock. Moreover, also the ranking of capital allocation weights across the different sub-units is more or less the same for all the different rules that have been considered.

6.3.2 *Marginal methods, RORAC contributions and comparison*

So far, we have considered proportional capital allocations using VaR, CVaR, the entropic risk measure and covariance. We investigate now what happens with marginal or RORAC methods and we compare the results with those of proportional methods. A priori we could expect that the marginal method would distribute differently the capital to be allocated by putting more weight on the riskier assets.

Here below (see tables 6.4, 6.5, 6.6 and 6.7) we present the results obtained by computing the risk capital allocated to each stock, via marginal methods and the contribution of stocks to the total portfolio RORAC. Each table reports the risk capital allocated to each stock using both proportional methods and marginal ones and the contribution of each stock to the total portfolio RORAC, for any single risk measure. For what concerns the contribution to the total RORAC, we compute the contributions of stocks by using just the proportional allocation methods.

	ATL	BRE	ENI	ISP	TIT
Proportional	0.6798 (43.2%)	0.3202 (20.4%)	0.4253 (27%)	0.1103 (7%)	0.0365 (2.4%)
Marginal	0.7755 (49.3%)	0.2850 (18.1%)	0.3633 (23.1%)	0.0928 (5.9%)	0.0555 (3.6%)
RORAC	0.0004 (17.1%)	0.0033 (154%)	-0.0013 (-58.7%)	0.0004 (20.5%)	-0.0007 (-32.9%)

Table 6.4: VaR daily contributions of stocks.

	ATL	BRE	ENI	ISP	TIT
Proportional	1.2116 (48.8%)	0.4178 (16.8%)	0.6334 (25.5%)	0.1658 (6.7%)	0.0535 (2.2%)
Marginal	1.3998 (56.4%)	0.2518 (10.1%)	0.6140 (24.7%)	0.1651 (6.7%)	0.0514 (2.1%)
RORAC	0.0002 (12.1%)	0.0020 (148.6%)	-0.0007 (-49.7%)	0.0002 (17.2%)	-0.0004 (-28.2%)

Table 6.5: CVaR daily contributions of stocks.

Looking at tables 6.4, 6.5, 6.6 and 6.7 we notice not too significant differences between the proportional methods and the marginal one: among different risk measures, both methods agree in putting more weight on Atlantia than on others and the ranking of capital allocation weights across the different stocks is the same for both methods. Nevertheless, apart from the case of covariance that however is not really a risk measure, it is worth to emphasize that our "intuition" concerning marginal contributions was correct. Compared to proportional capital allocations, indeed, marginal contributions put more weight (in terms of capital allocation) on Atlantia, that is, on the riskiest asset in the portfolio. Among different risk measures, the ranking of the contributions to the total RORAC is still the same: Brembo gives the best contribution, which is even more than the total RORAC, and Eni gives the worst contribution, which is negative; i. e. it is not worth having such an asset in the portfolio, since it reduces the total RORAC.

	ATL	BRE	ENI	ISP	TIT
Proportional	3.7442 (67.8%)	0.4872 (8.8%)	0.9190 (16.6%)	0.3235 (5.9%)	0.0480 (0.9%)
Marginal	4.9476 (89.6%)	0.1788 (3.2%)	0.3283 (6%)	0.0658 (1.1%)	0.0013 (0.1%)
RORAC	0.0000 (5.3%)	0.0010 (171.2%)	-0.0003 (-46%)	0.0001 (11.8%)	-0.0003 (-42.3%)

Table 6.6: Entropic daily contributions of stocks.

	ATL	BRE	ENI	ISP	TIT
Proportional	0.2123 (49.7%)	0.0666 (15.6%)	0.1154 (27%)	0.0259 (6%)	0.0072 (1.7%)
Marginal	0.1938 (45.3%)	0.0718 (16.8%)	0.1195 (28%)	0.0328 (7.7%)	0.0094 (2.2%)
RORAC	0.0009 (11%)	0.0117 (148.2%)	-0.0034 (-43.4%)	0.0014 (17.5%)	-0.0026 (-33.3%)

Table 6.7: Covariance daily contributions of stocks.

Risk capitals allocated via marginal methods benefit from the pooling effect for the first three risk measures, except the capital allocated to Telecom using VaR: this amount is larger than VaR of Telecom considered as a stand-alone portfolio. As well as for proportional methods, the marginal method based on the entropic risk measure has benefited from the highest pooling effect. Despite this result it is not evident from marginal methods' formula, the data confirm: comparing the values of Table 6.2 with those of tables 6.4, 6.5 and 6.6, we can notice that the marginal method based on the entropic risk measure has the highest difference between the risk capital of the titles and the capital allocated to them by using this method. The pooling effect is not achieved by the covariance marginal allocation method, as well as for the proportional one, as we noted above. The full allocation property for marginal allocation methods is, of course, satisfied for each risk measure since we use the adjusted formulation of Equation (2.5). By the same argument, the sum of RORAC contributions is equal to the total portfolio RORAC, for each risk measure.

To sum up, considering the numerical example above, we cannot conclude that a given method allocates always more or less capital than another. However, for the risk measures examined the results obtained by proportional and marginal methods are substantially very different from those of the RORAC method. Even if proportional and marginal contribution methods seem to provide similar results, marginal one better reacts and takes into account riskier assets.

A

MATHEMATICAL BACKGROUND

The appendix is devoted to collect some mathematical background about notations and results used throughout the work. However, the topics presented here are just summarized and not discussed in detail; we mainly refer to Aliprantis and Border [2], for a proper discussion. Many results can be also found in Dunford and Schwartz [34] or in Rudin [61]. Further references will be given for specific topics.

The focus is on those topics which are mostly used throughout the work. However, we skip the basics of set theory and those of vector spaces, even if largely used throughout the work, since they are assumed to be well known. The reader can still refer to [2, 34, 61] for such topics.

A.1 ORDERINGS

Let $\mathcal{X}$ be any non-empty set. A binary relation $\succeq$ on $\mathcal{X}$ is a preorder if it satisfies the following conditions, for all $X, Y, Z \in \mathcal{X}$:

Reflexivity: $X \succeq X$.

Transitivity: if $X \succeq Y$ and $Y \succeq Z$ then $X \succeq Z$.

A preorder $\succeq$ induces an asymmetric relation $\succ$ via

$$X \succ Y :\Longleftrightarrow Y \not\succeq X$$

and an equivalence relation $\sim$ via

$$X \sim Y :\Longleftrightarrow X \succeq Y \text{ and } Y \succeq X.$$

The notations $X \preceq Y :\Longleftrightarrow X \not\succ Y$ and $X \prec Y :\Longleftrightarrow X \not\succeq Y$ are also commonly used. A preorder $\succeq$ on $\mathcal{X}$ is called:

Order if it is antisymmetric: $X \succeq Y$ and $Y \succeq X$ imply $X = Y$ (i. e. X and Y coincide, $X, Y \in \mathcal{X}$).

Total if it is complete: for all $X, Y \in \mathcal{X}$, either $X \succeq Y$ or $Y \succeq X$ or both must hold.

Let $\mathcal{X}$ be a vector space over $\mathbb{R}$. A preorder $\succeq$ on $\mathcal{X}$ is said to be a vector preorder if $X \succeq Y$ implies $X + Z \succeq Y + Z$ for any $Z \in \mathcal{X}$ and $\lambda X \succeq \lambda Y$ for any $\lambda \geq 0$. Similarly, when $\succeq$ is an order on $\mathcal{X}$, we say that it is a vector order if the above condition is satisfied.

Let $\succeq$ be a vector order on $\mathcal{X}$. The positive cone induced by $\succeq$ is given by

$$\mathcal{X}_+ := \{X \in \mathcal{X} \mid X \succeq 0\}.$$

Notice that $\mathcal{X}_+$ is a convex cone. We will use the notation $\overline{\mathcal{X}} := \mathcal{X} \cup \{\pm\infty\}$, where $+\infty \succ x \succ -\infty$, for every $x \in \mathcal{X}$.

An upper bound of $A \subseteq \mathcal{X}$ is an element $x \in \overline{\mathcal{X}}$ satisfying $x \succeq y$ for all $y \in A$. Similarly, a lower bound of A is an element $x \in \overline{\mathcal{X}}$ satisfying $x \preceq y$ for all $y \in A$. The supremum of A, denoted by $\sup A$, is its least upper bound and the infimum of A, denoted by $\inf A$, is its greatest lower bound. Notice that the supremum and the infimum are unique because $\succeq$ is an order. Moreover, note that $\sup A = -\inf -A$, for any $A \subseteq \mathcal{X}$.

A.2 TOPOLOGY

Let $\mathcal{X}$ be a topological vector space over $\mathbb{R}$ with topology τ. That is, a vector space $\mathcal{X}$ and a topology τ such that the two vector operations (sum and multiplication by a scalar) are continuous with respect to τ.

Whenever τ' is another topology on $\mathcal{X}$ and $\tau' \subseteq \tau$, we say that τ' is coarser than τ, or, equivalently, that τ is finer than τ'.

We denote by $\mathcal{X}'$ the topological dual of $\mathcal{X}$; that is, the set of all linear functionals $\ell \colon \mathcal{X} \to \mathbb{R}$ which are continuous with respect to τ. Notice that $\mathcal{X}'$ is itself vector space over $\mathbb{R}$, when endowed with the pointwise operations.

A dual pair is a pair $(\mathcal{X}, \mathcal{Y})$ of vector spaces, together with a bilinear functional $\langle \cdot, \cdot \rangle \colon \mathcal{X} \times \mathcal{Y} \to \mathbb{R}$, called the duality of the dual pair $(\mathcal{X}, \mathcal{Y})$, that separates the points of $\mathcal{X}$ and $\mathcal{Y}$. That is: if $\langle x, y \rangle = 0$ for all $y \in \mathcal{Y}$, then $x = 0$; if $\langle x, y \rangle = 0$ for all $x \in \mathcal{X}$, then $y = 0$.

The topology denoted by $\sigma(\mathcal{X}, \mathcal{X}')$ is called the weak topology and it is the coarsest topology on $\mathcal{X}$ that makes every functional $\ell \in \mathcal{X}'$ continuous. Similarly, the weak* topology $\sigma(\mathcal{X}', \mathcal{X})$ is the coarsest topology on $\mathcal{X}'$ that makes the maps $T_x \colon \mathcal{X}' \to \mathbb{R}$, defined by $T_x(\ell) := \ell(x)$, $x \in \mathcal{X}$, continuous.

Whenever $\mathcal{X}'$ is isomorphic to a set $\mathcal{Y}$, it is customary to write $\sigma(\mathcal{X}, \mathcal{Y})$ or $\sigma(\mathcal{Y}, \mathcal{X})$, instead of $\sigma(\mathcal{X}, \mathcal{X}')$ and $\sigma(\mathcal{X}', \mathcal{X})$, respectively.

A.3 FUNCTIONALS

Let $\mathcal{X}$ be a topological vector space over $\mathbb{R}$, equipped with a vector preorder $\succeq$. Consider a functional $f \colon \mathcal{X} \to \mathbb{R}$. We say that f is $\succeq$-monotone increasing (resp. decreasing) if for all $x, y \in \mathcal{X}$

$$x \succeq y \implies f(x) \geq f(y) \text{ (resp. } f(x) \leq f(y)\text{)}.$$

We simply say that f is monotone (increasing or decreasing) whenever it is clear that we refer to $\succeq$.

We say that f is convex if for all $x, y \in \mathcal{X}$ and $\lambda \in [0, 1]$

$$f\big(\lambda x + (1-\lambda) y\big) \leq \lambda f(x) + (1-\lambda) f(y).$$

A functional f is concave whenever $-f$ is convex. We say that f is positive homogeneous if for all $x \in \mathcal{X}$ and $\lambda \geq 0$

$$f(\lambda x) = \lambda f(x).$$

We say that f is subadditive if for all $x, y \in \mathcal{X}$

$$f(x+y) \leq f(x) + f(y).$$

Notice that convexity and positive homogeneity are equivalent to subadditivity and positive homogeneity. We use the notation

$$\{f \leq c\} := \{x \in \mathcal{X} \mid f(x) \leq c\}\,;$$

and similarly $\{f \geq c\}$, $\{f < c\}$, $\{f > c\}$, $\{f = c\}$ are meant as above.

We say that f is lower semicontinuous if $\{f \leq c\}$ is closed for each $c \in \mathbb{R}$. Equivalently, f is lower semicontinuous if for each net $x_\alpha \to x$ in $\mathcal{X}$ we have

$$f(x) \leq \liminf_\alpha f(x_\alpha).$$

Given a convex functional $f\colon \mathcal{X} \to \mathbb{R}$, we say that $\ell \in \mathcal{X}$ is a subgradient of f at $x \in \mathcal{X}$ if it satisfies

$$f(y) - f(x) \geq \ell(y - x)$$

for all $y \in \mathcal{X}$. The set of all subgradients at x is called the subdifferential of f at x and denoted by $\partial f(x)$. If $\partial f(x)$ is non-empty, we say that f is subdifferentiable at x.

The directional derivative of f at $x \in \mathcal{X}$, in the direction $y \in \mathcal{X}$, is given by

$$Df(x;y) := \lim_{t \to 0} \frac{f(x+ty) - f(x)}{t}.$$

The function f is said to be Gateaux differentiable at $x \in \mathcal{X}$, with Gateaux derivative $\ell \in \mathcal{X}'$, if $Df(x;y) = \ell(y)$ for all $y \in \mathcal{X}$. That is, if the directional derivative is linear in the direction. If f is Gateaux differentiable at $x \in \mathcal{X}$, then its subdifferential at x is a singleton, with the unique element being the Gateaux derivative, usually called the gradient of f at x and denoted by $\nabla f(x)$.

A.4 GAMES AND CHOQUET INTEGRALS

For this topic, we also refer to Bhaskara Rao and Bhaskara Rao [16], Denneberg [31] and Marinacci and Montrucchio [54].

Let $(\Omega, \mathcal{F})$ be a measurable space. A game is a set function $\nu\colon \mathcal{F} \to \mathbb{R}$ satisfying $\nu(\emptyset) = 0$. A game ν is called:

Positive if $\nu(A) \geq 0$ for all $A \in \mathcal{F}$.

Capacity if it is monotone: $\nu(A) \leq \nu(B)$ whenever $A \subseteq B$, with $A, B \in \mathcal{F}$.

Charge if it is finitely additive: $\nu(A \cup B) = \nu(A) + \nu(B)$ for all pairwise disjoint sets $A, B \in \mathcal{F}$.

Measure if it is countably additive: $\nu\left(\bigcup_{i=1}^{+\infty} A_i\right) = \sum_{i=1}^{+\infty} \nu(A_i)$ for all pairwise disjoint sequence of sets $(A_i)_{i \in \mathbb{N}} \subseteq \mathcal{F}$.[1]

Normalized if $\nu(\Omega) = 1$.

Probability measure if it is a normalized positive measure.

Submodular if $\nu(A \cup B) + \nu(A \cap B) \leq \nu(A) + \nu(B)$ for all $A, B \in \mathcal{F}$.

Let $\mathbb{P}$ be a probability measure on $(\Omega, \mathcal{F})$ and define $\nu := f \circ \mathbb{P}$, where the function $f\colon [0,1] \to [0,1]$ satisfies $f(0) = 0$, $f(1) = 1$. Such a ν is called the distortion of the probability measure $\mathbb{P}$ with respect to the distortion function f. It is clear that ν is a normalized capacity, moreover if f is concave then ν is sub-modular, see [39].

For the measurable function $X\colon \Omega \to \mathbb{R}$, the game $\nu\colon \mathcal{F} \to \mathbb{R}$ and any $t \in \mathbb{R}$, we recall the notation

$$\nu(X \leq t) := \nu\big(\{\omega \in \Omega \mid X(\omega) \leq t\}\big).$$

We assume that the notation above still holds whenever we replace $\leq$ with $\geq, <, >, =$ or whenever we replace t with a set $B \in \mathcal{B}(\mathbb{R})$ and $\leq$ with $\in$.

Given a game ν, its variation norm is given by

$$\|\nu\| := \sup \sum_{i=1}^{n} |\nu(A_i) - \nu(A_{i-1})|$$

where the supremum is taken over all finite chains $\emptyset = A_0 \subseteq A_1 \subseteq \cdots \subseteq A_n = \Omega$. We denote by $\mathrm{ba}(\Omega, \mathcal{F})$ the space of all charges with finite variation norm.

Given two games, ν and μ, we say that ν is absolutely continuous with respect to μ, written $\nu \ll \mu$, whenever $\mu(A) = 0$ implies $\nu(A) = 0$, for all $A \in \mathcal{F}$.

The core of a game ν is defined as

$$\mathcal{C}_\nu := \{\mu \in \mathrm{ba}(\Omega, \mathcal{F}) \mid \mu(A) \leq \nu(A), \text{ for all } A \in \mathcal{F}, \, \mu(\Omega) = \nu(\Omega)\}\,.$$

Let $B(\mathcal{F})$ be the set of all bounded $\mathcal{F}$-measurable functions. That is, bounded functions $X\colon \Omega \to \mathbb{R}$ such that $X^{-1}(B) \in \mathcal{F}$ for all $B \in \mathcal{B}(\mathbb{R})$, letting $\mathcal{B}(\mathbb{R})$ denote the Borel σ-algebra on $\mathbb{R}$. The Choquet integral of $X \in B(\mathcal{F})$ with respect to a capacity ν is defined as

$$\int X \, d\nu := \int_{-\infty}^{0} \big(\nu(X \geq t) - \nu(\Omega)\big) \, dt + \int_{0}^{+\infty} \nu(X \geq t) \, dt.$$

The Choquet integral, with respect to ν, is also denoted by $\mathrm{E}_\nu[\cdot]$, or simply $\mathrm{E}[\cdot]$, when it is clear which game is involved. The notation

1 Notice that some authors ([2] for instance) define a measure as a positive countably additive game.

$\mathbb{E}_{\mathbb{P}}[\cdot]$, or simply $\mathbb{E}[\cdot]$, is devoted to the expectation under $\mathbb{P}$, that is, a Choquet integral with respect to a probability measure.

The Choquet integral is monotone, positive homogeneous and also translation invariant. If ν is sub-modular, then the Choquet integral is also subadditive. Moreover, it is finitely additive when ν is a charge and coincides with the standard Lebesgue integral when ν is a positive measure. Furthermore, the Choquet integral is monotone with respect to the game, that is, if ν and μ are two games such that $\nu(A) \leq \mu(A)$ for all $A \in \mathcal{F}$, then $\mathrm{E}_\nu[X] \leq \mathrm{E}_\mu[X]$ for all $X \in B(\mathcal{F})$.

A.5 SPACES OF FUNCTIONS

Given a measurable space $(\Omega, \mathcal{F})$, we denote by $\mathcal{L}^0$ the (vector) space of all real-valued $\mathcal{F}$-measurable functions on Ω. Let $\mathcal{X} \subseteq \mathcal{L}^0$ and $\mathbb{P}$ be a probability measure. The triplet $(\Omega, \mathcal{F}, \mathbb{P})$ is called a probability space and the functions $X \in \mathcal{X}$ are also called random variables.

The pointwise ordering on $\mathcal{X}$ is defined by

$$X \geq_{\mathrm{P}} Y :\Longleftrightarrow X(\omega) \geq Y(\omega) \text{ for all } \omega \in \Omega.$$

The pointwise relation is an order but it is not total. The $\mathbb{P}$-almost surely ($\mathbb{P}$-a.s.) ordering on $\mathcal{X}$ is defined by

$$X \geq_{\mathbb{P}} Y :\Longleftrightarrow \mathbb{P}(X \geq Y) = 1.$$

The $\mathbb{P}$-a.s. ordering is a preorder on $\mathcal{X}$ but it is neither an order nor total. Indeed, $X \geq_{\mathbb{P}} Y$ and $Y \geq_{\mathbb{P}} X$ imply just $X =_{\mathbb{P}} Y$, that is $\mathbb{P}(X = Y) = 1$, but X and Y may not coincide. Moreover, for some $X, Y \in \mathcal{X}$ it happens that neither $X \geq_{\mathbb{P}} Y$ nor $Y \geq_{\mathbb{P}} X$ hold.

The quotient space of $\mathcal{L}^0$ with respect to the $\mathbb{P}$-a.s. equivalence relation is denoted by L^0. As usual, we do not distinguish between equivalence classes in L^0 and their representative elements in $\mathcal{L}^0$.

Given a random variable $X \in \mathcal{X}$, its essential supremum is defined as

$$\operatorname{ess\,sup}(X) := \inf \{c \in \mathbb{R} \mid \mathbb{P}(X \leq c) = 1\};$$

similarly, its essential infimum is defined as

$$\operatorname{ess\,inf}(X) := \sup \{c \in \mathbb{R} \mid \mathbb{P}(X \geq c) = 1\}.$$

We denote by L^∞ the subspace of L^0 consisting of all functions that are $\mathbb{P}$-a.s. bounded; that is, $\operatorname{ess\,sup}|X| < +\infty$ for all $X \in L^\infty$. The space L^∞ becomes a Banach lattice under the essential supremum norm:

$$\|X\|_\infty := \operatorname{ess\,sup}|X|.$$

The dual of L^∞ can be identified with $\mathrm{ba} := \mathrm{ba}(\Omega, \mathcal{F}, \mathbb{P})$, the set of all finitely additive set functions $\mu\colon \mathcal{F} \to \mathbb{R}$ (charges) that have finite total variation and that are absolutely continuous with respect to $\mathbb{P}$. For any $X \in L^\infty$ and $\mu \in \mathrm{ba}$, the duality is given by

$$\langle X, \mu \rangle = \mathrm{E}_\mu[X].$$

However, under the weak* topology $\sigma(L^\infty, L^1)$, the dual of L^∞ can be identified with L^1, the space of all random variables belonging to L^0 whose absolute value is integrable (see below). Through the Radon-Nikodym theorem, it is then possible to identify L^1 with the space of all (countably additive) measures absolutely continuous with respect to $\mathbb{P}$.

We denote by L^p, $p \in (0, +\infty)$, the subspace of L^0 consisting of all random variables satisfying $\mathbb{E}[|X|^p] < +\infty$. We do not consider the case $p \in (0,1)$. For $p \in [1, +\infty)$, the space L^p is a Banach lattice under the norm

$$\|X\|_p := \left(\mathbb{E}[|X|^p]\right)^{\frac{1}{p}}.$$

The dual of L^p can be identified with L^q for $q = \frac{p}{1-p}$, using the convention $\frac{1}{0} := +\infty$. In particular, the dual of L^1 can be identified with L^∞. For any $X \in L^p$ and $Y \in L^q$, the duality is given by

$$\langle X, Y \rangle = \mathbb{E}[XY].$$

The indicator function of the set $A \in \mathcal{F}$ is the measurable function $\mathbb{1}_A \colon \Omega \to \{0, 1\}$ defined by

$$\mathbb{1}_A(\omega) := \begin{cases} 1 & \text{if } \omega \in A; \\ 0 & \text{otherwise.} \end{cases}$$

Notice that, for $\lambda \in \mathbb{R}$, $\lambda \mathbb{1}_\Omega$ is the function constantly equal to λ. In such a case, we simply write λ instead of $\lambda \mathbb{1}_\Omega$. The indicator function is monotone with respect to set inclusion, subadditive with respect to set union and additive when the sets are disjoint.

The positive part is the operator $\cdot^+ \colon \mathcal{X} \to \mathcal{X}_+$ defined by

$$X^+ := \max\{X, 0\} = \begin{cases} X & \text{if } X > 0; \\ 0 & \text{otherwise.} \end{cases}$$

The positive part can be also written as $X^+ = X \mathbb{1}_{\{X>0\}}$. The positive part is monotone, positive homogeneous and subadditive, with respect to the pointwise order.

Given $\alpha \in (0,1)$, an α-quantile of $X \in \mathcal{X}$ is any number $q \in \mathbb{R}$ satisfying

$$\mathbb{P}(X \leq q) \geq \alpha \geq \mathbb{P}(X < q).$$

The set of α-quantiles is a closed interval where

$$q_\alpha^- := \sup\{x \in \mathbb{R} \mid \mathbb{P}(X < x) < \alpha\}$$

is called the lower α-quantile of X and

$$q_\alpha^+ := \inf\{x \in \mathbb{R} \mid \mathbb{P}(X \leq x) > \alpha\}$$

is called the upper α-quantile of X.

A.6 STOCHASTIC ORDERS

For this topic, we refer to Dana [27], Föllmer and Schied [39] and Müller et al. [55]. Notice that, in [39] the second stochastic order is called "uniform preferences".

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and $X, Y \in L^1$, that is, the space of $\mathbb{P}$-integrable random variable. The first stochastic order is defined as

$$X \succeq_1 Y :\Longleftrightarrow \mathbb{E}[u(X)] \geq \mathbb{E}[u(X)]$$

for all monotone increasing functions $u\colon \mathbb{R} \to \mathbb{R}$. We recall that

$$X \succeq_1 Y \iff F_X(x) \leq F_Y(x) \text{ for all } x \in \mathbb{R};$$

where $F_X(x) := \mathbb{P}(X \leq x)$ denotes the distribution function of X. Notice that X and Y are equal in distribution if and only if $X \succeq_1 Y$ and $Y \succeq_1 X$. The equality in distribution is denoted by $\sim$. Therefore, a functional $f\colon L^1 \to \mathbb{R}$ which is $\succeq_1$-monotone is also distribution invariant, that is, $f(X) = f(Y)$ whenever $X \sim Y$.

The second stochastic order is defined as

$$X \succeq_2 Y :\Longleftrightarrow \mathbb{E}[u(X)] \geq \mathbb{E}[u(X)]$$

for all monotone increasing and concave functions $u\colon \mathbb{R} \to \mathbb{R}$. We recall that

$$X \succeq_2 Y \iff \int_{-\infty}^{t} F_X(x)\,dx \leq \int_{-\infty}^{t} F_Y(x)\,dx \text{ for all } t \in \mathbb{R}.$$

Here the interpretation is the following: $X \succeq_1 Y$ means that any agent which is a profit maximizer (increasing utility function) prefers X to Y, while $X \succeq_2 Y$ means that any agent which is a profit maximizer and risk averse (increasing and concave utility function) prefers X to Y.

It is clear that the following relations hold among the two stochastic orders and the $\mathbb{P}$-a.s. relation, for all $X, Y \in L^1$.

$$X \geq_{\mathbb{P}} Y \implies X \succeq_1 Y \implies X \succeq_2 Y.$$

Moreover, it also clear that whenever $f\colon L^1 \to \mathbb{R}$ is $\succeq_2$-monotone increasing (resp. decreasing) then it is $\geq_{\mathbb{P}}$-monotone increasing (resp. decreasing).

A.7 ORLICZ SPACES

Here, we also refer to Rao and Ren [59], for more details.

Let $\Phi\colon [0, +\infty) \to [0, +\infty)$ be a convex and strictly increasing function satisfying $\Phi(0) = 0$ and $\Phi(1) = 1$. Such Φ is called a (normalized) Young function. It follows that Φ is continuous and satisfies $\lim_{x \to +\infty} \Phi(x) = +\infty$.

Given a Young function Φ, the Orlicz space L^Φ is defined as

$$L^\Phi := \left\{ X \in L^0 \;\middle|\; \mathbb{E}\left[\Phi\left(\frac{|X|}{a}\right)\right] < +\infty \text{ for some } a > 0 \right\}.$$

The space L^Φ is a Banach lattice under the Luxembourg norm:

$$\|X\|_\Phi := \inf \left\{ \lambda > 0 \;\middle|\; \mathbb{E}\left[\Phi\left(\frac{|X|}{\lambda}\right)\right] \leq 1 \right\}.$$

The Orlicz heart M^Φ as

$$M^\Phi := \left\{ X \in L^0 \;\middle|\; \mathbb{E}\left[\Phi\left(\frac{|X|}{a}\right)\right] < +\infty \text{ for all } a > 0 \right\}.$$

We recall that $L^\Phi = M^\Phi$ whenever satisfies the so-called Δ_2 condition, that is, whenever there exists $x_0 > 0$ and $\alpha > 0$ such that

$$x > x_0 \implies \Phi(2x) \leq \alpha \Phi(x).$$

For the Orlicz duality, it is also useful to consider the convex conjugate Ψ of Φ, defined for $y \geq 0$ as

$$\Psi(y) := \sup_{x \geq 0} \{xy - \Phi(x)\}.$$

The Orlicz norm of $X \in L^\Phi$ is defined as

$$\|X\|_\Phi^* := \sup_{\|Y\|_\Psi \leq 1} \mathbb{E}[|XY|]$$

for Ψ being the convex conjugate of Φ. The dual space of $(M^\Phi, \|\cdot\|_\Phi)$ can be identified with $(L^\Psi, \|\cdot\|_\Psi^*)$.

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DECLARATION

I declare that the thesis has been composed by myself and that the work has not be submitted for any other degree or professional qualification. I confirm that the work submitted is my own, except those parts of jointly-authored publications which I included. My contribution and those of the other authors to this work have to be meant as equally divided. I confirm that appropriate credit has been given in this thesis where reference has been made to the work of others.

Milan, December 28, 2020

Gabriele Canna