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Cournot oligopoly with preference interdependence

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How selfish soever man may be supposed, there are evidently some principles in his nature, which interest him in the fortunes of others, and render their happiness necessary to him, though he derives nothing from it, except the pleasure of seeing it.

Adam Smith,
The Theory of Moral Sentiments (1759)

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Contents

Notation

N	number of firms
i, j	indexes of firms, ranging form 1 to N
q_i	strategic output of firm i
$\boldsymbol{q}$	vector of strategic outputs
$\boldsymbol{q}_{-i}$	strategic output level of all firms but the i-th one
Q	aggregate output level of the industry
$\boldsymbol{q}^*$	vector of equilibrium outputs
Q^*	aggregate equilibrium output of the industry
c	constant marginal cost
$p(Q)$	inverse demand function
π_i	profit (material payoff) of firm i
v_i	utility function of firm i
β_{ij}	constant coefficients representing the dependence of utility of firm i from material payoff of firm j
B	coefficient matrix
E_i	set of all firms whose material payoff affects the utility of firm i
$B\boldsymbol{u}$	vector of the overall outgoing degree of social interaction
$\boldsymbol{u}^T B$	vector of the overall ingoing degree of social interaction
$\mathcal{N}$	set of players
$\mathcal{S}_i$	set of strategies of each player i
$\Gamma(\mathcal{N}, \mathcal{S}_i, v_i)$	oligopoly game with interdependent preferences
Γ_0	oligopoly game without interdependent preferences

U	square matrix whose elements are equal to 1
I	identity matrix
$\rho(B)$	spectral radius of matrix B
$\mathrm{BR}_i(\mathbf{q}_{-i})$	reaction or best response function of firm i with respect to the choices of its competitors
Δ_{BR_i}	variation of the best response of firm i after a change in the value of output of an opponent, in the model with interedependent preferences
$\Delta_{BR_i^C}$	variation of the best response of firm i after a change in the value of output of an opponent, in the model without interdependent preferences
$\mathrm{FOE}_{i,j}$	first order effect interdependent effect on the best response of firm i due to the dependence of its utility function on the material payoff of firm j
Δ_i	gross change in the best response of firm i, after a change in the output of firm z, also taking into account the best response of firm j, in the model with interedependent preferences
Δ_i^C	gross change in the best response of firm i, after a change in the output of firm z, also taking into account the best response of firm j, in the model without interdependent preferences
$\mathrm{SOE}_{i,j,z}$	second order interdependent effect on the strategic choice of firm i after a change in z, mediated by the interdependency between firm j and firm z
$\mathrm{SOE}_{i,j,i}$	second order interdependent feedback effect on the strategic choice of firm i after a change in i, mediated by the interdependency between firm i and firm j
$\boldsymbol{\xi}$	vector of Bonacich centrality measures
μ	degree of competitiveness of an oligopoly with interdependent preferences
$\boldsymbol{\sigma}$	vector of market shares
$\boldsymbol{\chi}$	vector of aggregated effect due to any order dependence of social preferences of the industry on firm a given firm
$\tilde{B}$	square matrix corresponding to $(I+B)^{-1}$

$\tilde{\beta}_{ij}$	element of $\tilde{B}$ representing the aggregate any order social interaction of firm i with firm j
$\boldsymbol{\rho}$	vector of intercentrality measures

Chapter 1

Introduction

Suppose that N distinct firms simultaneously and indipendently decide how much to produce of an homogeneous good, i.e. compete on quantities in order to maximize their own material profits. Given these quantity choices, price adjusts to the level that clears the market and is a commonly known decreasing function of total output. Each firm has a cost function, which is common knowledge. The model we briefly described, known as oligopoly market *à la Cournot* [1], opened the way to the study of oligopolies in modern economy. From now on, we may refer to this model as the standard or classical Cournot oligopoly with self-interested firms. Cournot pioneered many other researches on different variants of oligopoly [11], worth mentioning the Bertrand [7] and Stackelberg oligopolies [41]. An oligopoly is an intermediate market form, perhaps the most common, which places itself between to opposing market structures: perfect competition and monopoly. These extremes differ substantially for the number of players involved and their market power, as well as for the equilibrium level of production. In the competitive model, a large number of producers are assumed to act as price takers. Controlling only a small portion of the market they are not able to influence the market price. The equilibrium is characterized for the highest level of production at the lowest price. These assumption may not be good ones when there is only one producer which controls all the supply side and therefore is assumed to be able to make the price. The equilibrium pair is therefore characterized by the lowest production level at the highest price.

[1] Antoine Augustin Cournot was a french mathematician known in economics for the essay *Researches on the Mathematical Principles of the Theory of Wealth* in which he used the application of the formulas and symbols of mathematics in economic analysis. This book was highly criticized and not very successful during Cournot's lifetime but it's nowadays highly influencial in microeconomics.

Competition among firms in an oligopolistic market, such as in a Cournot model, is inherently a setting of strategic interaction. For this reason, the appropriate tool for its analysis is game theory. In Cournot games, the players are the firms, each firm's set of simultaneous actions is the set of its possible outputs (non-negative numbers) and each firm' preferences are represented by its profit, namely players compete with respect to profits [39, 23]. Classical Cournot games are widely used in many economic scenarios to mimic imperfect competition and analyze market power. However, the economic literature also presents theoretical, empirical and experimental contributions that lead to reconsider the standard Cournot model, mainly for two reasons. First, classical game theory, and so Cournot games, generally sets on fundamental hypothesis that agents are self-interested, i.e. aim to maximize their own material payoff and do not take other's welfare state into consideration. According to this assumption, the theory predicts that selection forces favor absolute optimizating (or rational) agents [20] and that a different behavior from selfishness is destined to extintion. Many authors have argued these assumptions are not realistic and poverly descriptive of an oligopolistic market, introducing instead the idea that it is interdependent rather than absolute performance pivotal in the long run survival [2] and that reality seems to suggests that agents immersed into an oligopolistic market are not, or can act as, selfish. Second, the classical game theory analysis predicts an equilibrium outcome which rarely emerges in the experimental literature where agents, instead, agree on other combinations of equilibrium output which range between a wider interval, that goes from the competitive (walrasian) to the collusive (monopolistic) outcome.

Empirical motivation

The explosion of game-theoretic models on oligopoly competition has been followed by a more modest wave of empirical implementation and econometric testing [40]. The empirical literature that focused on the analysis of the phenomenon of the interdependence of preferences provides different specification models that explain a wide variety of real life situations. In some circumstances, very different models can explain the same result, while others more similar models disagree on the conclusions. A simple explanation might be that real life oligopolistic markets are too complex, if we think to the myriad of variables involved, to be modelled with a simple approach as classical Cournot's model does. A tentative to examine methods used to construct and estimate game-theoretic models and a survey of the empirical findings can be found in Slade [40]. Particularly interesting are the cases of partial ownership, partial equity interests[2] [10] as examples of positive interdependence where the fortunes of potential competitors are linked by a positive correlation among material profits. Far more interesting, it is shown that by linking the fortunes of actual or potential competitors or by producing a positive correlation among profits, could result in less output and higher prices than otherwise, even if the ownership shares are relatively small [36]. In this sense, the effects arise because the linking of profits reduces each firm's incentive to compete, and not because of increased opportunities for collusion or changes in the concentration of control. Other scholars confront the case of a Cournot duopoly in the presence or in the absence of cross holdings pointing the attention on the externalities generated by the two firms. They show that when the products of the two firms are substitutes, the market equilibrium exhibits higher quantity produced at lower price for consumers, converting into lower profits for producers but higher consumer surplus. Moreover, the interaction between cross holding and product markets increases economic welfare [16].

The literature on industrial and management economics offers also examples of strong intra-group competitions in which one group has the objective to maximize relative profits, that is the difference between its own profits and the material profits obtained by the competing group. This may represent the opposing example of negative preference interdependence between firms. Most research on the relation between executive compensation and company performance has been firmly (if not always explicitly) rooted in agency theory: compensation plans are designed to align the interests of risk-averse self-interested executives with those of shareholders. A major empirical prediction of agency theory concerns the use of relative performance evaluation (RPE) in incentive contracts [24]. For example, it is common for chief-executive officers to be rewarded not simply for their own performance but rather for their performance measured relative to the performances of co-executives [31]. Paying based on relative performance provides essentially the same incentives as paying based on absolute performance, while insulating risk-averse managers from the common shocks. Existing studies of RPE have focused on the implicit relation between CEO cash compensation, company performance, and market and/or industry performance. Gibbons and Murphy document the strongest support for the RPE hypothesis, finding that changes in CEO pay are

[2]Equity interests refer to the ownership of any security, asset or claim for which the return is positively related to the issuing firm's profitability. Thus, the term may include bonds, some kinds of leases, and other financial instruments as well as common and preferred stocks.

positively and significantly related to firm performance, but negatively and significantly related to industry and market performance, ceteris paribus. In addition, they find that CEO performance is more likely to be evaluated relative to aggregate market than relative to industry performances [22]. Other scholars provide evidence on the descriptive validity of this RPE implication of analytical agency models. For example, Antle and Smith find weak support for the use of RPE in the total compensation contracts of 16 of 39 firms in the chemical, aerospace, and electronics industries during 1947 to 1977 [3]. Janakiraman *et al.* find that firms do not find useful (or find it too costly) to filter noise in evaluating the performance of their executives and that the specific form of the agency model used to develop the RPE predictions is not descriptively valid for CEO [28].

Theoretical motivation

Many scholars have found the assumptions of self-interested players whom aim to maximize only their own profits unrealistic and the consequent conclusions unsatisfactory. These scholars first focused on proving the viability of the idea of interdependent preferences. One of the many specifications of interdependent preferences assumes each player is interested into maximize a combination of its own profit and the profit of the opponents [29]. One important point concerns the choice of modelling tools used to represent and well describe the phenomena of the interdependence of preferences. At the best of our knowledge, one of the first to anticipate the concept of interdepent preferences was Edgeworth [3]. In his words, we can imagine that the object which an agent X (whose own utility is π) tends - in a calm, effective moment - to maximize, is not π, but $\pi + \lambda\Pi$ where Π is the profits of its counterpart in the contract and λ is a coefficient of effective sympathy. Cyert and De Groot mutuated this idea in the development of the concept of a coefficient of cooperation and of mutually optimal reaction functions [13]. They provide an extensive generalization of the idea of the effective sympathy from Cournot, with respect to the number of periods considered, showing that for different positive pairs of the coefficients of cooperation and the discount factor an infinite number of equilibria may emerge. Another important contribution to analytically model agents whom are not completely self-interested comes from Rabin who develops a game-theoretic framework, an extension of the Geanakoplos, Pearce and Stacchetti approach [21], that incorporates interdependent preferences into a broad range of economic models [35]. Rabin develops a solution concept, which he called "fairness" equilibrium, that both add new predictions to the economic

[3]Francis Ysidro Edgeworth, in a footnote from 1881, page 53 of *Mathematical Psychics* [15]

models and eliminate some conventional equilibrium predictions. Rabin focuses on a simple model of monopoly pricing, showing that fairness implies a different equilibrium than the one predicted by classic game theory, and his contribution is seminal in exploring the implication of fairness on welfare. Another important and highly cited contribution in modelling interdependence of preferences comes from Levine [30]. His model is similar in spirit to Rabin's, although does differ since he does not focus only on qualitative predictions but examines also the quantitative implications of the theory. This simple model of players that are not completely selfish tries to explain the evidence from the data coming from economic surrogates such as the ultimatum and contribution games among others. Levine's model also represents a novelty with respect to previous literature since considers, in the individual calculus of the utility, both positive and negative weights, which agents place on opponents' monetary payoffs. He calls altruism and spitefulness, respectively. The last, worth mentioning, contribution to modelling interdependent preferences comes from Sethi and Somanathan [38]. They provide an evolutionary theory of reciprocity as an aspect of preference interdependence where it is shown that interdependent preferences, which they call reciprocal preferences, can invade a population of self-interested agents in a class of aggregative games under both assortative and nonassortative matching. Hence, they offer a more flexible specification of interdependency that allows their model to explain a wider variety of experimental results. The theoretical literature born from these contributions took two distinct research directions. The first strand analyzes the maximization of the relative performances of one agent compared to its opponents. In the sense the literature considers the negative effects on the utility of the single player caused by the profits of the competitors. Maximizing the relative performances means that by improving the profits of the opponents the utility of the individual is worsen off [44].

The second strand concerns the existence of positive interests (spillovers) in the individual utility deriving from the performances (profits) of the other agents. Among others, seminal contribution to this literature comes from the work by Cyert and De Groot [12].

Concerning the first strand of models, it generally assumes that agents's objective is to maximize relative rather than absolute performance. By weakening the hypothesis of full rationality and using imitation-based heuristics, these models may lead to the emergence of other equilibria, such as the competitive one, in which the equilibrium outputs are larger than the Cournot-Nash equilibrium outputs under absolute profit maximization. When a firm imitates who is perfoming better, implicitly favors the survival of those that perform better on a relative level while those that perform worse are destined to exit the market. These models show that the likelihood of imitating a more successful action increases in the difference between individual and other's payoff, a results known as Schlag's Rule. Recent contributions have increased economists' interest in imitation. On this topic, seminal contributions come from the works by Vega-Redondo [44] and Schlag [37]. Quite interestingly, these models make different predictions in Cournot games where players evaluate relative performances. The models differ along two different dimensions, the informational structure and the behavioral rule, namely whom agents imitate and how agents imitate. While agents in Vega-Redondo's model observe their immediate competitors, in Schlag's model they observe others who are just like them but play in different groups against different opponents. Additionally, agents in Vega-Redondo's model copy the most successful action of the previous period whenever they can. In contrast, Schlag's agents only imitate in a probabilistic fashion and the probability with which they imitate is proportional to the observed difference in payoffs between own and most successful action [4]. Most notably, in Cournot games, the model from Vega-Redondo predicts the Walrasian outcome while the model from Schlag predicts the Cournot-Nash equilibrium.

According to the second strand of investigation, Cournot models can be seen as a sort of Prisoner's Dilemma [42], where players have an incentive to form a cartel, effectively turning the model into a monopoly. By tacitly colluding using self-imposing strategies, the firms are able to reduce the strategic output which, ceteris paribus, will raise the price and thus increase profits for all firms involved. But standard game theory shows that collusion is not an equilibrium since each player will tend to deviate from the agreed output. Many scholars argue that there is a rational reason why a firm might consider a positive combination of its profits and the material profits of others in the calculus of utility rather than simply trying to maximize its own profits. Each firm realizes that, by placing positive weights on their opponents' material profits, its individual utility will actually be larger than when it tries to maximize only its own profits [12]. Therefore, positive interdependent preferences might lead to the emergence of combinations of collusive equilibria. Worth mentioning, the model from Kopel and Szidarovszky that, even if it is a sort of Tragedy of the commons, considers both the cases of non-cooperative and fully cooperative behavior of the agents. They characterize the relation between the level of cooperation among the agents and their strategic decisions, showing that, only for small levels of cooperation, an increase in cooperation leads to an efficient equilibrium such as collusion.

Experimental motivation

Laboratory experiments on Cournot games have highlighted that the agents strategies, on average, are not necessarily compatible with the Nash equilibrium. Instead, in most cases, the strategic choices fall in the interval between the Nash equilibrium and the competitive equilibrium. These experiments generally differ on the number of the agents involved, the choice of demand and cost function to be used, the information available to the agents and the number of stages of the game. Concerning the information set, generally these models assume incomplete knowledge about some characteristics of the game such as others' strategies and payoffs or the market demand function or other market details. The study of this experimental literature is very important because provides an interpretation of the results of models of Cournot competition which incorporate positive or negative interdependence of preferences and different types of demand functions, such as linear or iso-elastic. As we pointed out already, in most laboratory experiments the choice of the players are almost never compatible with the Nash equilibrium but, instead, fall into the interval between Nash and Walras equilibria. Among the different experiments on Coutnot oligopoly that obtain this result, worth mentioning are the ones from Apesteguia *et al.* [4, 5] and Offerman *et al.* [32]. The formers show, through small experimental variants, that the strategic choices of the agents converge towards competitive equilibria. One difference between these models concerns the type of learning procedure the agents adopt and the number of stages the agents are allowed to play in the game. For instance, Offerman *et al.* examine three types of behavioural dynamics in quantity-setting, i.e. "mimick the successful firm", "follow the exemplary firm" and a rule based on belief learning. The result they obtain is that, theoretically, these three types of rules lead to different equilibria, such as the competitive, the collusive, and the Cournot-Nash outcome, respectively [32]. They run and experiment that, from an informational point of view, is able to represent the three different decisional mechanism and prove the theoretical results.

Contrariwise, other experiments show that by increasing the number of interactions (stages of the game), the strategic production outputs gradually decrease towards a collusive equilibrium, without even stop at the Cournot-Nash equilibrium. For example, Friedman's [19] is seminal mainly for two reasons: first, it shows the relevance of long horizons; second, it models a cournotian oligopoly based on an iso-elastic demand function which is a break point into the experimental literature (mainly focused on linear demand functions). When we confront the experimental literature on oligopoly games we start by showing the differences in the construction of the models, then we analyse the outcome of each experiment and finally we report the interpretation from each experimenter.

Apesteguia *et al.*, for instance, come to the conclusion that, by increasing the information set available to agents, learning mechanisms are triggered that allow agents to agree on a Walrasian equilibrium.

On the other side, Friedman claims that increasing the number of stages in the game triggers coordination mechanisms that allow players to understand that by reducing the production output towards the monopolistic level, they can collectively earn more. The model offers a demonstration of a how groups of subjects can learn their way out of dysfunctional heuristics and suggest elements for a new perspective on the emergence of cooperation.

The experiment from Friedman serves as the starting point in motivating our contribution to the literature, so that it might makes sense to dedicate a little more detailed analysis to it. The objective of Friedman *et al.* was originally to propose a model that capture important features of observed adjustment paths that were not covered by the existing experimental literature. In this sense, existing theories accounted for behavior observed in the early and the late phases of a game, but not in the middle phases where output choices tend to drop from the competitive into the collusive region. They first prove an analytical result on the model's convergence to the joint profit maximum (JPM). Their experiment was motivated by the conjecture that, even in a low information environment, the perfectly competitive walrasian outcome (PCW) might be unstable, as players gained experience with the payoff function, they might learn the (myopic) best response and eventually converge to the Cournot-Nash equilibrium. The evidence from their experiment is rather very different. Initially, the agents tend to match the most profitable opponent, pushing the collectivity towards the PCW outcome. After a few dozen periods in the neighborhood of PCW, players start to match the average quantity of other players instead of the most profitable, allowing the collectivity to drift towards more jointly profitable profiles. Most players start to ratchet quantities downward and if prolonged, this behavior eventually leads to the joint profit maximum (JPM). This kind of behaviour is reminiscent of the gradient rule "win-continue, lose-reverse" (WCLR) [4] learning algorithm suggested by Huck *et al.* [25, 26].

[4]The 2003 paper by Huck *et al.* ultimately analyzes a class of dilemma games where agents move on a grid. Each agent determines the direction of the next step on the grid by examining their change in payoff. As long as the payoff increases, a WCLR agent continues to move.

We provide an alternative explanation for the convergence towards different equilibria, other than Nash. In Friedman the players' strategic decision move along the dimension of the volumes of production following the decision mechanism exposed by the WCLR gradient rule. However, the analytical explanation of how agents' decisional mechanism changes, from an imitation-based to a coordination-based heuristic, is not satisfactory.

The novelty is that, the results from Friedman and Apesteguia can be explained with particular interdependent preference structures, without relying on coordination or learning arguments. Depending on how the preference interdependency will be structured, this model can result in different equilibrium choices covering all the spectrum that goes from the walrasian to the monopolistic equilibrium.

Network games

The three research strands mentioned above, experimental, empirical and theoretical, has implicitly introduced a new kind of interaction between the players that is of a different and more sofisticated nature with respect to the classic Cournot model. In Cournot the players interact through the market, in an anonymous and vague way, while the network literature conceptualize the idea of a "social" interaction, indicating all that different bonds that decribe the agents as social economic animals [27].

Following the line of reasoning, a network structure begins to emerge when we introduced laboratory experiments. Tipically, in these experiments, the agents, initially, interact with each other in a highly anonymous way, without having any previous information. Moreover, the experimenters themselves have low or no information about the players regarding, for example, their competitive or collusive predisposition. Without any knowledge about other players, by construction the initial network structure is homogeneous, namely the weights the players place on the other performances all equal to zero. With time passing by, the players begin to learn more about the others and the game itself. They begin to know where the competitive or the collusive equilibrium set and they learn which choices address them in one direction or in the another, being spiteful, on average, lead the aggregate production toward the competitive equilibrium outcome while altruistic choices, on average, lead toward the collusive equilibrium level of production. Hence, the network structure begins to arise with the progress of the game given that the weights of interdependence become more heterogeneous. We can imagine a player i matching a competitive player j and an altruistic player z, evaluating negatively ($\beta_{ij} < 0$) the material payoff of player j while evaluating positively ($\beta_{iz} > 0$) the material payoff of the cooperative player. Therefore, a dynamic adjustment that heterogenizes the β coefficients takes hold and more the social coefficients diverge from zero the more sophisticated the network becomes. It becomes interesting to analyze how a particular structure of social interdependence can influence the aggregate volume of production and how it is shared between the different agents. It is economically interesting both to look at the general dynamic of the system of players as it diverges from a self-interested structure and to look at the single instant picture of a particular interdependent structure to better understand how a specific behaviour on behalf of the players can influence a particular equilibrium to emerge.

The aim of this thesis is to propose and analyze a modelling of Cournot oligopoly in which agents have interdependent preferences, i.e. in which utility of an agent not only depends on its material payoff (profits) but also on the material payoff of the other agents in the market. Furthermore, we want to provide unified framework that provides an alternative to the hypothesis of material self-interest agents and that is able to simultaneously explain a wide variety of experimental results on oligopoly.
To the best of our knowledgewe, we provide an analytical tool which represents a novelty in the literature dedicated to Cournot competition. The family of models we propose can give rise to different equilibria, which can be characterized in terms of different levels of aggregate production falling in the range of outcomes that varies with continuity from the competitive equilibrium to the monopolistic equilibrium. This model may fill, through an organic and unitary specification, some gaps in the literature or better motivate some intuitions and guesses that come from the experimental and empirical results.

The remainder of the thesis is organized as follows. In Chapter 2 we propose the Cournot oligopoly model with interdependence of preferences. The economic setting we consider consists of a market in which a fixed number N of firms produce a homogeneous good and compete in choosing the output quantity. The utility function of each firm depends both on the individual profits and on a linear combination of the material payoffs of some of or all the other firms. The way the preferences of the firms depend on the profits of their competitors defines the network of social interdependence, represented through a (possibly negatively) weighted directed graph. The resulting setting can be described in terms of a static form game. As the role of social interaction becomes more and more relevant, the game diverts from the classic Cournot game (without interdependence of preferences). In Chapter 2 we also introduce simplified scenarios, which are characterized by networks of social interactions with particular structures. The intent of such simplified models is to give a preliminary intuition of the contribution the general setting and the possible outcomes of the game. In particular, we show that even in a very simplified setting the model is able to generate volumes of production coherent with different market forms that fluidly range from the monopolistic to the competitive limit, passing through the classic Cournot oligopoly. Finally, we show that under suitable assumptions on the inverse demand function and on the coefficients of interdependence, the existence and uniqueness of the Nash equilibrium is guaranteed for relevant families of oligopolies, including oligopolies described by a game in which the utility function is concave and the case of isoelastic demand functions.

In Chapter 3 we study the role of preference interdependence on the resulting properties of the Nash equilibrium. Firstly, we focus on the effects in terms of strategic substitutability/complementarity. We examine interdependence effects of order n between two given firms, also taking into account n order feedback effects on each firm. We characterize the Nash equilibrium through the two channels of interaction among firms, namely the market interaction and the social interaction. With respect to this two channels, we show how the social prefrence structure is fundamental to understand the properties of the Nash equilibrium. We show that this can be done in terms of some classical and new measures related to the network structure. We introduce a vector that provides the Bonacich centrality measure of each firm in the network of social interactions and ultimately determines their market share. In addition, we introduce the vector of the aggregated effect due to any order dependence of social preferences of the industry on a given firm that is the level of influence a given firm can exert upon the industry. Finally, we introduce a key element characterizing the oligopoly in terms of its aggregate equilibrium. Such element encompasses the degree of competitiveness that characterizes an oligopoly with interdependent preferences.

In Chapter 4 we focus on the weigth matrix B in order to investigate how a change in the structure of interdependent preferences affects the equilibrium. To make this explicit, we introduce another measure related to the social interaction structure, the intercentrality measure ($\boldsymbol{\rho}$) that identifies in a network the player providing the largest contribution to the aggregate outcome. This Chapter

has the twofold objective to understand how a change in the social interaction structure of a single player influences the outcome of the player itself and how a change in the interaction structure of the collectivity affects the collective outcome. In this sense, we focus on the comparative statics of a local change in the preference interdependence structure and the economic effects on the market share of the single firm. In addition, we investigate the comparative statics of a global change in the structure of social preferences and the effect on the industry as a whole in terms of profits.
Chapter 5 bears concluding remarks and highlights possible future developments of the present research strand.
We report all proofs in the Appendix.

Chapter 2

Oligopoly model with interdependent preferences

2.1 Introduction

In this Chapter we present the model with interdependent preferences, namely we consider an oligopolistic market in which N firms produce a homogeneous good and compete in choosing the output quantity given a utility function that depends both on the individual profits and on a linear combination of the material payoffs of some of or all the other firms. The way the preferences of the firms depend on the profits of their competitors defines the interdependence preference structure.
For convenience, we collect all the (positive or negative) weights through which the utility of one firm depends on the material payoffs of the others in a coefficient matrix in order to represent the network of social interdependence through a (possibly negatively) weighted directed graph.
As such coefficients divert from 0, the game diverts from the classic Cournot game with the role of social interaction that becomes more and more relevant as the degrees of social interaction increase.
Moreover, we introduce the overall outgoing or ingoing degree of social interaction, namely two synthetic measures that respectively characterize the overall social interaction of one firm toward the whole industry and the overall social interaction that the industry has toward a given firm.
In order to study the additional effect of considering interdependent preferences and improve the economic interpretation of the analytical results of the next Chapters, we introduce simplified scenarios, which are characterized by networks of social interactions with particular structures. We

first introduce a network with a homogeneous structure of interdependent preferences, namely each firm evaluates its opponents the same way all the opponents do. This way we are able to characterize families of games, in terms of volumes of aggregate production, both in the case of uniform positive interdependence (altrusitic preferences) and the case of uniform negative interdependence (spiteful preferences). Then we introduce a network in which the single firm homogeneously behaves toward each competitor and a network in which the industry homogeneously behaves toward the single firm, namely the weigths the single firm places on opponents' material payoffs are the same and the weights the whole industry places on the single firm's material payoffs are the same, respectively. Finally, we introduce a preference structure in which firms behave on average the same way, namely the summations of social weights of each firm are the same.

The intent of this Chapter is to give a preliminary intuition of the contribution the proposed model can provide in terms of the outcome of the game. In particular, we show that it is able to generate volumes of production coherent with different market forms that fluidly range from the monopolistic to the competitive limit, passing through the classic Cournot oligopoly.

In the last part of this Chapter, we introduce some assumptions on the inverse demand function and on the coefficients of interdependence that guarantee the existence and uniqueness of the Nash equilibrium for relevant families of oligopolies that include relevant examples, such the case of concave oligopolies (i.e. oligopolies described by a game in which the utility function is concave) and the case of isoelastic demand functions.

2.2 The model

We consider an oligopolistic market in which N firms, identified by an index $i \in \{1, 2, \ldots, N\}$, produce a homogeneous good and compete in choosing the output level $q_i \geq 0$. Each firm faces linear cost function with identical constant marginal cost $c > 0$. Prices are determined by the inverse demand function $p : I \to [0, +\infty)$, $Q \mapsto p(Q)$, where I is a suitable domain. We assume that p is continuous on I, twice-differentiable and strictly decreasing on $I \cap [0, b)$ and null on $I \cap [b, +\infty)$, for some $b \in \mathbb{R} \cup \{+\infty\}$. We collect output levels in a vector $\boldsymbol{q} \in [0, +\infty)^N$.

Each firm realizes a profit given by $\pi_i(q_i, Q_{-i}) = q_i(p(Q) - c)$, where $Q_{-i} = \sum_{j=1, j\neq i}^{N} q_j$ is the aggregate quantity produced by all firms but the i-th one and $Q = Q_{-i} + q_i$ is the aggregate output level of the industry. According to [38], in what follows we refer to π_i as the material payoff of firm i and we assume that each firm has interdependent preferences that are described by the utility function

$$v_i = \pi_i(q_i, \boldsymbol{q}_{-i}) + \sum_{j=1, i\neq j}^{N} \beta_{ij}\pi_j(q_i, \boldsymbol{q}_{-i}) = q_i(p(q_i + Q_{-i}) - c) + \sum_{j=1, i\neq j}^{N} \beta_{ij}(q_j(p(q_i + Q_{-i}) - c)) \quad (2.1)$$

where $\boldsymbol{q}_{-i} \in [0, +\infty)^{N-1}$ is the vector collecting the output levels of all firms but the i-th one and β_{ij} are constant coefficients representing the network of dependences among the agents' preferences. Coefficient β_{ij} weights to what extent preferences of firm i depends on the material payoff of firm j.

It is convenient to introduce coefficients $\beta_{ii} = 0$ for $i = 1, \ldots, N$, so weights β_{ij} can be collected in a hollow matrix B

$$B = \begin{bmatrix} 0 & \beta_{12} & \beta_{13} & \cdots & \beta_{1N} \\ \beta_{21} & 0 & \beta_{23} & \cdots & \beta_{2N} \\ \beta_{31} & \beta_{32} & 0 & \cdots & \beta_{3N} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \beta_{N1} & \beta_{N2} & \beta_{N3} & \cdots & 0 \end{bmatrix},$$

which represents a (possibly negatively) weighted directed graph, in which the coefficient related to the edge connecting vertex i to vertex j represents the (positive or negative) weight through which the utility of firm i depends on profit of firm j. We note that setting $\beta_{ii} = 0$ allows dropping condition $i \neq j$ in (2.1). The utility of each firm i is then affected by its own material payoff and by a linear combination of the material payoffs of some of or all the other firms. We denote with

$E_i \subset \{1, 2, \ldots, N\} \setminus \{i\}$ the set of all firms whose material payoff affects the utility of firm i (i.e. $\beta_{ij} \neq 0$ if and only if $j \in E_i$). Set E_i corresponds to the (first degree) neighborhood of node i in the graph described by matrix B. To explicitly show that utility function depends on coefficients β_{ij}, in what follows we write $v_i(q_i, \boldsymbol{q}_{-i}, B)$.

The first evident consequence of preference interdependence is that, depending on the sign of β_{ij}, firm i can achieve the same utility by having smaller own profits if the other firms with which it has interaction have larger (when $\beta_{ij} > 0$) or smaller (when $\beta_{ij} < 0$) profits, since a part of the reduced utility coming from own profits can be compensated by the utility coming from material payoff of other players, as a consequence of the interdependence of preferences. Note this means that the way the utility of firm i changes (i.e. marginal utility $\partial v_i/\partial q_i$) can not be understood just taking into account the way own material payoff changes, but it is affected by a change in the material payoff of any firm in E_i. In particular, it is easy to see that the marginal utility function is

$$\frac{\partial v_i}{\partial q_i} = p(q_i + Q_{-i}) - c + q_i p'(q_i + Q_{-i}) + \sum_{j=1, i \neq j}^{N} \beta_{ij} q_j (p'(q_i + Q_{-i}). \tag{2.2}$$

Since p is a decreasing function, we have that weigths β_{ij} have an opposite effect on the marginal utility $\partial v_i/\partial q_i$ with respect to the effect on the utility v_i. As β_{ij} increases, ceteris paribus, the marginal utility of a firm decreases, while the opposite occurs as β_{ij} decreases.

Accordingly to (2.1), the preferences of each firm are influenced by two levels of interactions in which firms are involved. If we neglect interdependence among preferences, the utility function is affected by the market interaction among firms through profits (actually, in this case v_i exactly corresponds to the profits): at this level, firms are not individually involved, but each of them influences the final price just depending on the quantity they decide to produce, and not on the basis of the firm's identity. If firm i and firm $j \neq i$ produce the same amount q of good, the influence they exert on the price determination is exactly the same. The network of interdependent preferences introduces an additional level of interaction, in which each firm is possibly involved in a way that is different from that of the other firms. At this level we can say that firms are involved in a network of social interactions, through which each firm has its own neighborhood of firms with which it interacts and to which it is linked, with the neighborhood set possibly ranging from an empty set to the whole industry. Similarly, for each firm i, we have a set of firms whose social preferences depen on the material payoff of firm i. The configurations of outgoing and ingoing links due to social preferences can be, in principle, asymmetric. In an extreme case, preferences of a firm can be affected by the material payoff of all the other firms and, at the same time, its material payoff may not influence the utility function of none of the remaining firms. And, indeed, vice-versa.

Moreover, each firm can behave in a completely heterogeneous way with respect to each firm with which it interacts. Such heterogeneity is described by the size and the sign of each weight β_{ij}, whose absolute value then describes the degree of social interaction of firm i toward firm j. The sign of β_{ij} identifies the kind of social interaction firm i has toward firm j. To this end, we say that firm i is respectively altruistic, selfish and spiteful toward firm j if $\beta_{ij} > 0, \beta_{ij} = 0$ and $\beta_{ij} < 0$, respectively. We stress that such expressions are not intendend to connote a moral or psychological involvements of firms, but they are simply borrowed from the literature about interdependent preferences ([30, 38]). In what follows, when we say that firm i is, for instance, altruistic toward firm j we mean that the preferences of firm i are socially linked to the material payoff of firm j and that the spillover of the material payoff of firm j on the utility of firm i is positive, without entering into details of the reasons for which such spillover is positive.

The distribution of weigths β_{ij}, $j = 1, \ldots, N$ characterizes the social interaction of firm i toward the whole industry, as well as the distribution of weigths β_{ij}, $i = 1, \ldots, N$ characterizes the social interaction that the industry has toward a given firm j. In some cases, it can be useful to summarize these two sets by means of a couple of synthetic measures. To this end, we identify each element of vector[1] $B\boldsymbol{u}$ as the overall outgoing degree of social interaction. Element $(B\boldsymbol{u})_i$ corresponds to the i-th row summation of elements of the weight matrix B, i.e. it aggregates all the weights that firm i places on the material payoff of its competitors. Similarly, we identify each element of vector $\boldsymbol{u}^T B$ as the overall ingoing degree of social interaction. In this case, element $(\boldsymbol{u}^T B)_j$ provides the j-th column summation of elements of weight matrix B, i.e. it aggregates all the weights that all the firms in the industry place on the material payoff of a given firm j. We stress that identical synthetic measures can correspond to completely different weights' distributions, so in most cases they just allow capturing the average, outgoing or ingoing, degree of social interaction.

The following example shows the above mentioned elements.

Example 1. (A general network of social interactions)

[1] With $\boldsymbol{u} \in \mathbb{R}^N$ we denote the vector whose components are $u_i = 1, i = 1, \ldots, N$.

We consider the 7×7 weighted matrix B

$$B = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -0.2 & 0 & 0.5 & 0.2 & 0 & -0.5 & 0 \\ 0.2 & 0.3 & 0 & 0.7 & 0.5 & 0.9 & 0 \\ -0.5 & 0.4 & 0.2 & 0 & -0.3 & 0.7 & 0 \\ -0.1 & -0.15 & -0.19 & 0 & 0 & -0.1 & 0 \\ -0.02 & -0.18 & -0.12 & -0.13 & -0.09 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \tag{2.3}$$

which gives rise to the network shown in Figure 2.1, where the number (in red) on each node corresponds to a firm. A directed/oriented link or edge between two nodes $i \neq j$ depicts the dependence of the utility of firm i on the material payoff of firm $j \neq i$, represented by $\beta_{ij} \neq 0$. The network is oriented in the sense that each arrow indicates the direction along which such dependence realizes. The interdependence between two firms can realize in two ways: it can be unidirectional (in this case the number of links connecting two firms is unique and directed from a node to another one, e.g. as in the case of firms 4 and 5) or not (in this case we have a couple of links, e.g. as in the case of firms 3 and 5). In the network of Figure 2.1, the weight that quantifies the extent of the link between firm i and j is reported above the edge connecting node i to node j, and corresponds to coefficient β_{ij} which represents the magnitude of the "social interest" of player i towards player j.

Each row of matrix B collects the weights that the corresponding firm gives to the opponents' profits. For example, if we focus on the first row (firm 1) we can see that it represents a self-interested firm which opponents profits' weights are all equal to zero, so that the utility of such firm coincides with its own material profits.

A different situation is depicted by the second row where firm $i = 2$ considers in its own utility also the material payoff of the competitors, placing on them both positive and negative weights. For example, firm 2 weights negatively the profits of firm 1 ($\beta_{21} = -0.2$) while evaluates positively the profits of firm 3 ($\beta_{23} = 0.5$). In this case the utility of firm 2 is diminished when firm 1's profits increase, and increased when firm 3's profits increases.

As we already noticed, the graph is oriented. For example, the first node ($N = 1$) has no

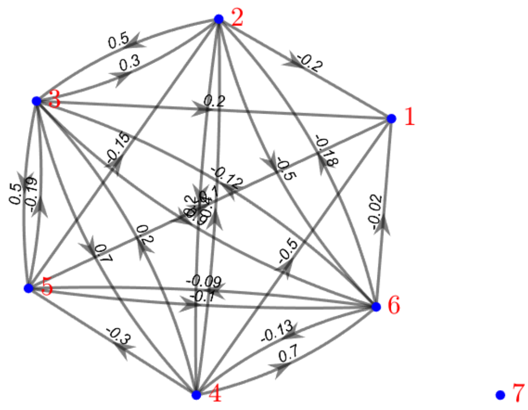


Figure 2.1: Graphical representation of the network described by weight matrix B in 2.3

outgoing but five ingoing links with different signs and weights. This situation perfectly depicts the case, we mentioned above, of a self-interested firm, which nonetheless contributes in some of the opponents' utility.

Moreover, if we compare node 1, node 3 and node 7 we can notice that the number of outgoing and/or ingoing edges is very different and it can be used as a first indication of the level and kind of social interaction of the firm. Firm 7 has no social interactions with the other firms, as its preferences do not take into account the profits of the other firms and its profits are not taken into account in the utility of any of the other firms. The unique channel of interaction of firm 7 with the other firms is the market. Conversely, firm 1 is involved in the network of social interaction. Even if its preferences do not take into account the profits of the other firms, its profits do affect the preferences of all the other firms. Finally, firm 3 is completely involved in the network of social interactions, with both outgoing and ingoing links with the other firms. Its utility derives from a combination of its own profits and a fraction of the profits of every other its opponents. Let consider the column vector $B\mathbf{u}$ resulting from each row summation:

$$B\mathbf{u} = \begin{bmatrix} \sum_{j=1}^{N} \beta_{1j} = 0 \\ \sum_{j=1}^{N} \beta_{2j} = 0 \\ \sum_{j=1}^{N} \beta_{3j} = 2.6 \\ \sum_{j=1}^{N} \beta_{4j} = 0.5 \\ \sum_{j=1}^{N} \beta_{5j} = -0.54 \\ \sum_{j=1}^{N} \beta_{6j} = -0.54 \\ \sum_{j=1}^{N} \beta_{7j} = 0 \end{bmatrix}$$

We can interpret each element as indicating the overall outgoing degree of social interaction of the corresponding firm. A positive value represents a firm as being on average altruistic, a negative value as being on average spiteful and a value equal to zero as being on average self-interested.
A clarification over the terminology used might be useful. For socially altruistic on average, we mean a firm which positively binds its own utility to the material payoff of others; however, a firm may evaluate differently two different firms, in the sense it can be altruistic towards the first one and spiteful towards the other. Let consider firm 1 and firm 2 which can be defined as generally self-interested. Although their social degree coincides the two firms are indeed very different, on average. Looking at the solely outgoing degree on average might be misleading. In this case firm 1 is actually self-interested towards each competitors, while firm 2 is self-interested only on average. The same line of reasoning can be applied to the comparison between firm 3 and 4. They both act altruistically on average, but the former is always acting altruistically with its opponents while the latter only on average. An other interesting insight of the composition of matrix B is given by the comparison between firm 5 and firm 6, which both have the same social degree of spitefulness and both act as spitefully towards each competitors, weighting however in a different ways the profits of each of their competitors. For this reason, it makes sense to identify the general attitude (or

degree) of a firm towards its opponents.

Let consider the row vector $\mathbf{u}^T B$ resulting from each column summation:

$$\mathbf{u}^T B = \begin{bmatrix} \sum_{i=1}^{N} \beta_{i1} & \sum_{i=1}^{N} \beta_{i2} & \sum_{i=1}^{N} \beta_{i3} & \sum_{i=1}^{N} \beta_{i4} & \sum_{i=1}^{N} \beta_{i5} & \sum_{i=1}^{N} \beta_{i6} & \sum_{i=1}^{N} \beta_{i7} \end{bmatrix}$$

$$= \begin{bmatrix} -0.62 & 0.37 & 0.39 & 0.77 & 0.11 & 1 & 0 \end{bmatrix}$$

Each value in the row vector $\mathbf{u}^T B$ indicates the ingoing social degree of interaction, i.e. how each firm in the network is taken into account on average in the opponents' utilities. For example, the first element indicates that firm 1 is on average considered negatively in the opponents' utility. On the opposite side, the second elements indicates that firm 2 positively contributes, on average, to its opponents' utilities. Finally, the last element identifies a firm that, on average, does not contributes to its opponents' utilities.

Each firm tries to maximize its own utility by choosing the quantity to produce. Such setting can be described by a game $\Gamma = (\mathcal{N}, S_i, v_i(q_i, \boldsymbol{q}_{-i}, B))$, where $\mathcal{N} = \{1, 2, \ldots, N\}$ is the set of players, $S_i \subset [0, +\infty)$ is the set of strategies of each player i and function v_i defined in (2.1) is the utility function for the i-th firm, for $i \in \mathcal{N}$. A particular case among the games belonging to the class of all games defined in the previous way is the classical Cournot game, namely game $\Gamma_0 = (\mathcal{N}, S_i, v_i(q_i, \boldsymbol{q}_{-i}, O)) = (\mathcal{N}, S_i, \pi_i(q_i, \boldsymbol{q}_{-i}))$ obtained setting B equal to the null matrix O. In Γ_0 firms chooses the quantity to produce in order to maximize material payoff, i.e. profits.

As already discussed, the first straightforward effect of considering interdependent preferences is that we can identify an additional channel of interaction among firms, along with the usual market interaction. Such latter channel is the unique one that is present in the classical Cournot game Γ_0 and establishes a "global", market related, form of interaction among all firms, mediated by the common inverse price function through the aggregate output level. Interdependent preferences establish another, possibly local or even one-to-one form of interaction, described by the distribution of coefficients β_{ij}. As such coefficients divert from 0, game Γ diverts from Γ_0, with the role of social interaction that becomes more and more relevant as $|\beta_{ij}|$ increase.

2.3 Particular structures of social interaction

As it will become evident in what follows, the general framework described by game Γ allows for the description of a wide range of situations. In order to simplify and improve the economic interpretation of the analytical results that will be provided in the remainder of the thesis, it is convenient to introduce some simpified scenarios, which are characterized by networks of social interactions with particular structures.

The first and simplest structure we consider consists of a "homogeneous" weight distribution. In this setting, the preferences of all firms are affected by the profits of any one of their competitors by the same extent β. The matrix describing the netwotk of social interactions is then $B = \beta(U - I)$ where U is the $N \times N$ matrix whose elements are equal to 1, and I is the $N \times N$ identity matrix. For example, if $N = 5$ we have the following 5×5 matrix

$$B = \begin{bmatrix} 0 & \beta & \beta & \beta & \beta \\ \beta & 0 & \beta & \beta & \beta \\ \beta & \beta & 0 & \beta & \beta \\ \beta & \beta & \beta & 0 & \beta \\ \beta & \beta & \beta & \beta & 0 \end{bmatrix} \tag{2.4}$$

to which corresponds the network described in Figure 2.2.

The present scenario is very unsophisticated and it is even unappropriate to speak about "structure" of social interaction, because of the very regular distribution of weights. However, it deserves some investigation as it allows casting a first glance on the possible equilibrium configurations described by game Γ.

Proposition 1. *Let us consider an oligopoly for which the network of social interaction is described by a matrix B in which $\beta_{ij} = \beta$ for any $i \neq j$, $1 \leq i, j \leq N$. Let p be an inverse demand function for which game $\Gamma = (\mathcal{N}, S_i, v_i(q_i, \boldsymbol{q}_{-i}, B))$ has an internal equilibrium $\boldsymbol{q}^*(\beta)$ for any β in $(-1/(N-1), 1)$.*

We have that as $\beta \to 1^-$ the aggregate equilibrium output level $Q^*(\beta)$ converges to the equilibrium output level $Q^*(\beta)_M$ of a monopoly in which p is the inverse demand function.

We have that as $\beta \to \left(-\frac{1}{N-1}\right)^+$ the equilibrium aggregate output level $Q^*(\beta)$ converges to the

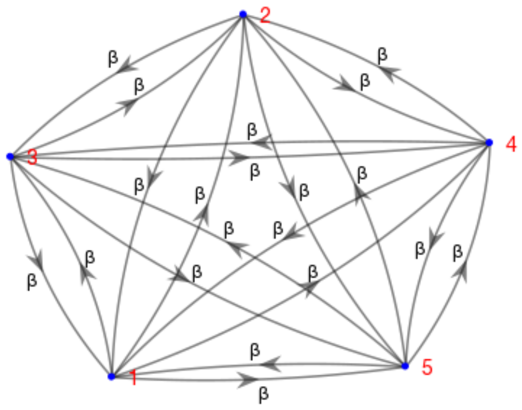


Figure 2.2: Graphical representation of the network described by weight matrix B in 2.4

aggregate equilibrium output level $Q^*(\beta)_C$ *of a competitive market in which* p *is the inverse demand function.*

Moreover, on increasing β *in* $(-1/(N-1), 1)$ *we have that the aggregate equilibrium output level* $Q^*(\beta)$ *is a continuous function that monotonically varies from* $Q^*(\beta)_M$ *to* $Q^*(\beta)_C$*.*

An oligopoly is usually described as a market structure that is dominated by a few firms and is characterized by an intermediate degree of competition, lying between monopoly (just one firm, minimum competition degree) and perfect competition (many firms, maximum competition degree). The family of games Γ considered in Proposition 1 describe oligopolies that provide a continuum of outcomes (identified by the industry output levels) that are characterized by a level of competition that ranges between such extrema. As weights β_{ij} approach 1, the setting with interdependent preferences tends to describe the setting in which a social planner coordinates the agents' production in order to maximize the aggregate industry profits and just looking at the aggregate output level at the equilibrium corresponds to that of a monopoly market. Similarly, as weights β_{ij} approach $-1/(N-1)$, the aggregate output level at the equilibrium corresponds to that of a competitive market.

In game Γ, the transition between the monopolistic and competitive markets (aggregate) equilibria does not (only) occurs on increasing of the number of suppliers populating the market, but it takes place, for any given number of firms, as the distribution of weights describing interaction among firms decrease from the uniform distribution $\beta_{ij} = 1$ to the uniform distribution $\beta_{ij} = -\frac{1}{N-1}$. We stress that even such a very simplified setting is able to represent all the possible situations, in terms of aggregate equilibrium outcomes, ranging from the monopolistic limit to the competitive limit scenarios.

The previous proposition also provides two intrinsic bounds on weights β_{ij}, leading to the following assumption

Assumption 1. $-\frac{1}{N-1} < \beta_{ij} < 1$,

so that, with coefficients in such range, we can compare the aggregate Nash equilibrium of any game Γ to the (aggregate) output levels of a monopoly and of a competitive market. In terms of aggregate output levels (and hence of degree of competitiveness), games Γ allow for a continuous transition between two extreme market situations. Accordingly, we can address such two extreme situations as the "monopolistic limit" and the "competitive limit" of sequences of games Γ.

As the oligopoly is a market structure lying between the monopoly and the competitive market, the family of admissible matrices B through which oligopolistic game Γ with interdependent preferences is defined has to lie between two extremal matrices

$$\underline{B} = \begin{bmatrix} 0 & -\dfrac{1}{N-1} & -\dfrac{1}{N-1} & \cdots & -\dfrac{1}{N-1} \\ -\dfrac{1}{N-1} & 0 & -\dfrac{1}{N-1} & \cdots & -\dfrac{1}{N-1} \\ -\dfrac{1}{N-1} & -\dfrac{1}{N-1} & 0 & \cdots & -\dfrac{1}{N-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -\dfrac{1}{N-1} & -\dfrac{1}{N-1} & -\dfrac{1}{N-1} & \cdots & 0 \end{bmatrix}, \quad \bar{B} = \begin{bmatrix} 0 & 1 & 1 & \cdots & 1 \\ 1 & 0 & 1 & \cdots & 1 \\ 1 & 1 & 0 & \cdots & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & 1 & \cdots & 0 \end{bmatrix}.$$

In the next Sections, we will come back on such aspects. We highlight that, in the case of a duopoly, Assumption 1 provides $-1 < \beta_{ij} < 1$ and we find the same symmetric bound on weights that are used in [38] and in the literature strand about oligopolies ([10]). In general situations, the bound provided by Assumption 1 is asymmetric, with potentially larger positive than negative weights in absolute value.

The second simplified newtork we consider is obtained setting $\beta_{ij} = \beta_i$ for $\beta_{ij} = \beta_i \in (-1/(N-1), 1)$ for $i, j = 1, \ldots, N$ and $i \neq j$. For example, in the case of $N = 5$ firms the corresponding

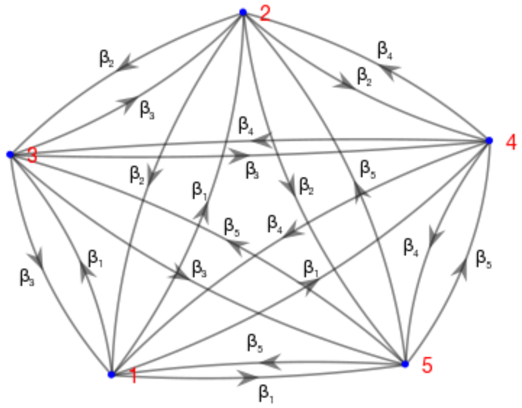


Figure 2.3: Graphical representation of the network described by weight matrix B in 2.5

matrix is

$$B = \begin{bmatrix} 0 & \beta_1 & \beta_1 & \beta_1 & \beta_1 \\ \beta_2 & 0 & \beta_2 & \beta_2 & \beta_2 \\ \beta_3 & \beta_3 & 0 & \beta_3 & \beta_3 \\ \beta_4 & \beta_4 & \beta_4 & 0 & \beta_4 \\ \beta_5 & \beta_5 & \beta_5 & \beta_5 & 0 \end{bmatrix} \tag{2.5}$$

whose graph is reported in Figure 2.3

In such setting we have that each firm has a homogeneous behavior with respect to all the other firms in the industry, but the behavior of each firm can be different with respect to that of the other firms. It is possible to say that firms are heterogeneous but each of them homogeneously takes into account all its competitors. Firm i has social preferences that are either uniformly altruistic, selfish or spiteful toward any other firm j, with a constant degree of social interaction. However, preferences of firm i can be different from those of firm j.

The third simplified newtork we consider is close to the second one, and it is described in terms of its transposed matrix. This structure is obtained setting $\beta_{ij} = \beta_i$ for $\beta_{ij} = \beta_j \in (-1/(N-1), 1)$

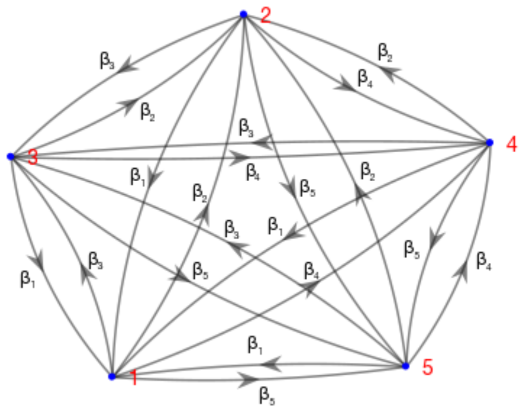


Figure 2.4: Graphical representation of the network described by weight matrix B in 2.6

for $i, j = 1, \ldots, N$ and $i \neq j$. For example, in the case of $N = 5$ firms the corresponding matrix is

$$B = \begin{bmatrix} 0 & \beta_2 & \beta_3 & \beta_4 & \beta_5 \\ \beta_1 & 0 & \beta_3 & \beta_4 & \beta_5 \\ \beta_1 & \beta_2 & 0 & \beta_4 & \beta_5 \\ \beta_1 & \beta_2 & \beta_3 & 0 & \beta_5 \\ \beta_1 & \beta_2 & \beta_3 & \beta_4 & 0 \end{bmatrix} \tag{2.6}$$

whose graph is reported in Figure 2.4

In this setting, each firm has a heterogeneos behavior with respect to all the other firms in the industry, but the behavior of the industry toward a given firm is homogeneous. We can say that firms are heterogeneous but each of them homogeneously is taken into account by all its competitors. All firms j have social preferences that are independent of firm i and they are either uniformly altruistic, selfish or spiteful toward the firm i, with a constant degree of social interaction. However, preferences toward firm i can be different from those toward firm j.

In the last simplified newtork we consider we have that the overall outgoing degree of social interaction is the same for all firms, i.e. vector $B\boldsymbol{u}$ has identical elements. In this case no restriction is imposed on each weight $\beta_{ij} \in (-1/(N-1), 1)$, the summation of each row just have to provide

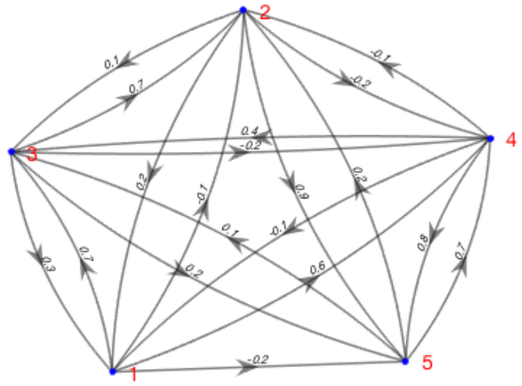


Figure 2.5: Graphical representation of the network described by weight matrix B in 2.7

the same value. All firms have, on average, the same overall degree of altruism or spitefullnes (or they can even be selfish), but the way the social preferences of firm i are influenced by those of firm j can be different on varying i and j. As an example, we can consider for $N = 5$ matrix

$$B = \begin{bmatrix} 0 & -0.1 & 0.7 & 0.6 & -0.2 \\ 0.2 & 0 & 0.1 & -0.2 & 0.9 \\ 0.3 & 0.7 & 0 & -0.2 & 0.2 \\ -0.1 & -0.1 & 0.4 & 0 & 0.8 \\ 0 & 0.2 & 0.1 & 0.7 & 0 \end{bmatrix} \tag{2.7}$$

whose graph is reported in Figure 2.5

Note that we have

$$B \cdot \boldsymbol{u} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \tag{2.8}$$

2.4 Existence and uniqueness of Nash equilibria with interdependent preferences

In this Section we show games described by $\Gamma = (\mathcal{N}, S_i, v_i(q_i, \boldsymbol{q}_{-i}, B))$ provide a well-posed framework to work with, in terms of the existence and/or uniqueness of Nash equilibria. To this end, we want to generalize the conditions under which existence and/or uniqueness of Nash equilibrium in classical game Γ_0 is guaranteed for particular Cournotian oligopoly models without interdependent preferences. To this end, following e.g.[33, 45, 8] we consider two settings, respectively consisting of "concave" oligopolies (i.e. for which assumptions on inverse demand function and network of interdependent preferences guarantee the concavity of the utility function) and oligopolies with isoelastic demand function, as an economically relevant crucial example of a setting that provides a game in which the best response functions are not monotonic.

We start considering the family of oligopolies for which the payoff function is concave. In such setting, to guarantee the uniqueness of the equilibrium, it is necessary to introduce a bound on the maximum possible strategy chosen by the agents. On the contrary, without such assumption, it is possible to see that multiple equilibria can occur even without interdependent preferences (see e.g. [33]). To this end, we introduce the capacity limit $L_i > 0$ for each firm $i \in \mathcal{N}$, which represents the maximum output level that each firm is able to supply. For more details about such aspects we refer to [33, 8].

To provide a suitably rich family of oligopolies that both include relevant examples and at the same time for which existence and uniqueness of the Nash equilibrium is guaranteed, we introduce some assumptions on the inverse demand function p and on coefficients β_{ij}. In what follows, the set of the oligopolies that fulfill the following Assumptions will be identified with $\mathcal{O}$.

Assumption 2. *For any $q_i \in [0, L_i], i \in \mathcal{N}$ and for $Q \in \left[0, \sum_{k=1}^{N} L_i\right]$ we have $p'(Q) < 0$ and*

$$p''(Q)q_i + p'(Q) + \sum_{j=1}^{N} \beta_{ij} p''(Q) q_j \leq 0, \quad i \in \mathcal{N} \tag{2.9}$$

The previous condition is the generalization to the case of interdependent preferences of decreasing marginal revenue condition $p''(Q)q_i + p'(Q) < 0$, which is given for concave oligopolies without preferences' interdependence (see [45]).

We stress that, recalling Assumption 1, Assumption 2 uniformly holds with respect to β_{ij} under suitable sufficient conditions, which depend on the concavity of function p. In particular, if $p(Q)$

is convex, condition (2.9) is guaranteed by

$$p''(Q)z + p'(Q) < 0 \text{ for } z \in \left[0, \sum_{k=1}^{N} L_i\right] \tag{2.10}$$

while if $p(Q)$ is concave condition (2.9) is guaranteed by

$$-p''(Q)\frac{Q}{N-1} + p'(Q) < 0 \text{ for } Q \in \left[0, \sum_{k=1}^{N} L_i\right]. \tag{2.11}$$

Note that for a convex p, condition (2.10) actually reduces to classical decreasing marginal revenue condition (it just have to hold on the extended interval $\left[0, \sum_{k=1}^{N} L_i\right]$ instead of $[0, L_i]$). Conversely, for a concave p, in game Γ_0 the decreasing marginal revenue condition would hold without any additional condition, which instead has to be imposed in Γ to obtain condition (2.9). However, it's worth noting that sufficient condition (2.11) is increasingly less restrictive as N increases.

Concerning the admissible distributions of weights, Assumption 1 just provides a first restriction on the economically relevant values of β_{ij}. However, the resulting set of weights is still too wide to guarantee existence and/or uniqueness of the Nash equilibrium of Γ. If we applied to game Γ the assumption that in the literature of games on networks allows obtaining existence and uniqueness of the Nash equilibrium (see [6]), we would impose $\rho(B) < 1$. However, the family of oligopolies described by games obtained adopting such condition would be too restricted. For instance, it would be not possible to consider a sequence of games Γ approaching the monopolistic limit, as in such case we must necessarily have $\rho(B) > 1$ in a neighborhood of the limit. Moreover, we stress that the above mentioned condition is applied in the literature to a situation in which $\beta_{ij} \leq 0$.

As it is evident in the proofs of the following propositions and accordingly to the literature, the problem of studying the existence and uniqueness of the Nash equilibrium of Γ can be rephrased into a linear complementarity problem (from now one, LCP) (see e.g. [33, 43]). For an LCP, well-posendess is guaranteed if the matrix associated to the corresponding problem is a P-matrix, i.e. a matrix in which all the principal minors are strictly positive (for a survey about P-matrices we refer to [43]). As we can see from the proofs of the following propositions, the matrix associated to the linear complementarity problem arising from the optimization problem related to the Nash equilibrium of game Γ is matrix $I + B$. This leads to the assumption

Assumption 3. *Matrix $I + B$ is a P-matrix.*

The previous Assumption can be seen as a generalization of assumption $\rho(B) < 1$ (as such condition, when $-\frac{1}{N-1} < \beta_{ij} \leq 0$, guarantees that $I + B$ is a P-matrix) and has basically the same economic interpretation: local complementaries has to be small enough to avoid the emergence of a non-finite equilibrium solution. The first relevant consequence of Assumption 3 is that $I + B$ is an invertible matrix, which will play a key role on the characterization of the internal Nash equilibrium in terms of the inverse of $I + B$. In addition, it bears several interesting properties that allows studying (and characterizing) the family of oligopolies in $\mathcal{O}$. For example, starting from a game in $\mathcal{O}$ it is possible to vary coefficients with continuity to obtain Γ_0, which indeed belongs to $\mathcal{O}$. This guarantees that any oligopoly in $\mathcal{O}$ can be studied by considering a continuous family of oligopolies in $\mathcal{O}$ with progressively larger coefficients, always starting from the purely selfish scenario. In particular, if an oligopoly of N firms belongs to $\mathcal{O}$, all the oligopolies obtained rescaling the coefficients of some (possibly one or even all) firms by any coefficient $\beta \in [0, 1]$ will describe oligopolies belonging to $\mathcal{O}$. Moreover, if an oligopoly of N firms belongs to $\mathcal{O}$, also the oligopolies obtained removing one firm has to belong to $\mathcal{O}$.

Now we consider the existence and uniqueness of Nash equilibrium in the case of concave oligopoly.

Proposition 2. *Under Assumptions 1-3, game Γ has at least a Nash equilibrium $\boldsymbol{q}^*$ with $q_i^* \in [0, L_i]$. If $q_i^* < L_i$ for each $i \in \mathcal{N}$, then the equilibrium is unique. Moreover, in the particular case of linear demand function, game Γ always has a unique Nash equilibrium.*

The previous Assumptions guarantee a setting for which the Nash equilibrium exists, and if it belongs to $[0, L_i)^N$, it is also unique (this is the case in which the capacity limit of no firms coincides with its equilibrium output level).

Assumption 3 is a suitable setting also for "non-concave" oligopolies, as for example in the relevant case of isoelastic demand function.

Proposition 3. *Under Assumptions 3, if $p(Q) = 1/Q$, game Γ has a unique Nash equilibrium $\boldsymbol{q}^*$.*

We stress that the equilibrium provided by the previous proposition can be also a boundary equilibrium, and this just depends on the network of social interactions among firms.

The case of isoelastic demand function, derived from Cobb–Douglas preferences, has been first considered by Dana and Montrucchio [14], who discusses duopoly games where firms maximize their discounted sum of profits and uses Markov-perfect equilibrium strategies, and later by Puu [34] and Ahmed and Agiza [1], who study the dynamics and the stability of Nash equilibria of two competing firms in a market and of n competitors in a Cournot game, respectively. In the case of a duopoly, Puu finds expression for the Nash-Cournot equilibrium

$$\boldsymbol{q}^* = \left[\frac{b}{(a+b)^2}, \frac{a}{(a+b)^2}\right] \tag{2.12}$$

where a, b are individual constant unit costs of the two firms.

Ahmed and Agiza generalize the previous result to the case of an n-firms oligopoly, obtaining

$$\boldsymbol{q}^* = \left[\frac{(N-1)(C-(N-1)c_1)}{C^2}, \frac{(N-1)(C-(N-1)c_2)}{C^2}, \cdots, \frac{(N-1)(C-(N-1)c_n)}{C^2}\right] \tag{2.13}$$

where $C = \sum_{i=1}^{N} c_i$ and c_i is the constant unit cost of the firm i.

Although these models consider the case of potentially heterogeneous firms in the cost function, we also report the equilibrium values for the case of homogeneous cost c between firms. In the case of Puu's model we have

$$\boldsymbol{q}^* = \left[\frac{1}{4c}, \frac{1}{4c}\right] \tag{2.14}$$

while in the case of Ahmed and Agiza' model we have

$$\boldsymbol{q}^* = \left[\frac{N-1}{N^2 c}, \frac{N-1}{N^2 c}, \cdots, \frac{N-1}{N^2 c}\right] \tag{2.15}$$

We stress the fact that both models only depend on the demand and cost function without considering a structure of interdependence of preferences.

However, these models can serve as a comparison when we will characterize the equilibrium with isoelastic demand function and interdependence of preferences.

2.5 Conclusions

In this Chapter we introduced an oligopolistic market in which N firms produce a homogeneous good and compete in choosing the output quantity given the individual interdependent preferences structure described by a utility function that depends both on the individual profits and on a linear combination of the profits of some of or all the other firms. The introduction of a interdependent preferences structure provides a framework that is able to, simultaneously deal, in the individual utility function, with both positive and negative effects due to the material payoffs of the other players. This provides a generalized setting which allowed us to encompass in an unified setting all the effects evidenced by the experimental literature, exposed in the introduction of the paper, in terms of the outcome of the game. In fact, even considering a very prothotypical and simplified scenario, this setting proved to be capable to describe a wide range of situations. Considering a homogeneous weight distribution (i.e matrix $B = \beta(U - I)$) we characterized families of games both in the case of uniform positive interdependence (altrusitic preferences) and the case of uniform negative interdependence (spiteful preferences) for which volumes of production are coherent with different market forms, ranging fluidly from the monopolistic (as the uniform distribution of weights converges to the 1) to the competitive limit (as it converges to the $-\frac{1}{N-1}$), passing through the classic Cournot oligopoly.

In order to help the economic interpretation of the analytical results we will provide in the next Chapters, we considered a network in which the weigths the single firm places on opponents' material payoffs are the same in order to represent a scenario in which the single firm does not discriminate between its opponents but treats each competitor the same way. We also considered the case of a network in which is the industry to homogeneously behave toward the each single firm, namely the weights the whole industry places on the single firm's material payoffs are the same. Finally, we considered a preference structure in which the firms behave on average the same way, namely the summation of their social weights are the same.

Moreover, in the last part of the Chapter, we showed how the proposed approach provides a reliable framework to work with whose behaviour, with respect to the existence and uniqueness of Nash equilibrium, is in line with the classical oligopoly modelling without interdependence of preferences. Assumption 2, on the inverse demand function, togheter with Assumptions 1,3, on the coefficients of interdependence, allowed us to prove the existence and uniqueness of the Nash equilibrium for families of oligopolies (i.e. oligopolies described by a game in which the utility function is concave) that include classical and relevant examples, such the case of concave oligopolies and the case of isoelastic demand functions. Concerning the admissible distributions of weights, Assumption 1 provides a first restriction on the economically relevant values of coefficients β_{ij} that is not sufficient to guarantee existence and/or uniqueness of the Nash equilibrium of the game with interdependence of preferences. Therefore, we rephrased the problem of studying the existence and uniqueness of the Nash equilibrium into a linear complementarity problem which guarantees the well-posendess of the matrix, associated to the corresponding problem, as it satisfies the condition to be a P-matrix. We showed that if $I + B$ is a P-matrix it bears several interesting properties that allows studying (and characterizing) the family of oligopolies by considering a continuous family of oligopolies with progressively larger but suitable coefficients. In particular, P-matrix assumption is actually a generalization of the $\rho(B) < 1$ condition that it is often imposed in game theory on networks for the exixtence and uniqueness of the Nash equilibrium.

Chapter 3

Analysis of the effects of interdependent preferences

3.1 Introduction

In this chapter we study the role of preference interdependence on the characterization of the Nash equilibrium. Firstly, we characterize the effect of preference interdependence in games in terms of strategic substitutability/complementarity, in order to understand how the network of social interaction alter the degree of strategic interaction between two interdependent firms, and this consequently alters the way a given firm optimally responds to a change in the strategy of one of its opponents. In order to completely understand how best response mechanism of a player affects the equilibrium in the presence of interdependence of preferences we extend the analysis to the n possible degrees of interdependence effects between two given firms, taking into account also n-th order feedback effects on each firm.

We then present the main result of this Chapter. In Proposition 5 we characterize the Nash equilibrium through the two channels of interaction among firms, namely the market interaction and the social interaction. The objective is then to explain the individual role of each of these two channels on the Nash equilibrium. Concerning the social interaction we will make use of some of the elements coming from the theory on networks, such as the centrality measures, and consider their roles on the Nash equilibrium. Similarly, in order to shed some light on the ability the proposed model has to generate an equilibrium coherent with different market forms that range from the

monopolistic to the competitive limit, we will introduce a measure of the degree of competitiveness that is able, for every game Γ, to pin the exact situation represented by the game.
We conclude the Chapter considering again the particular structures introduced in Section 2.3 in light of elements characterizing both the social interaction and the market interaction component encompassed in the result of Proposition 5.

3.2 First order effects

In this Section we study the role of preference interdependence on the resulting properties of the Nash equilibrium [1].

The interdependence of preferences has effect on the strategic interaction among agents. To this end, we recall that the common way to characterize the strategic interaction in games is in terms of strategic substitutability/complementarity. According to [9], we recall that strategy of player j have an effect of strategic substitutability (complementarity) on the strategy of player i if increasing q_j reduces (resp. increases) marginal profits of player i. For regular payoff functions v_i, the kind of strategic interaction (strategic substitutability/complementarity) between the strategies of two firms is then identified by the (negative/positive) sign of the second order cross derivative $\partial^2 v_i/\partial q_i \partial q_j$. In game Γ_0, strategic interaction just depends on the shape of (inverse) demand function that characterizes the market, while in the general case Γ it is significantly affected by network of social interactions.

The goal of the next propositions is to investigate the effect on strategic complementarity/substitutability when preference interdependence is introduced. To fix ideas, let us consider a setting for which game Γ_0 is characterized by strategic substitutability (as, for instance, in the case of a linear demand function).

Proposition 4. *Assume that game Γ_0 is characterized by strategic substitutability. Then, $\partial^2 v_i/\partial q_i \partial q_j$ decreases (increases) due to an increase (a decrease) of coefficient β_{ij}.*

According to the previous proposition, a first effect of preference interdependence is to alter the degree of strategic interaction between two interdependent firms. If firm i is altruistic toward firm j, the strategic substitutability characterizing q_i with respect to q_j in Γ_0 is reinforced, while it is weakened if firm i is spiteful toward firm j, and in this latter case the kind of strategic interaction can possibly turn into strategic complementarity.

[1] The presented results are referred to the Nash equilibrium of any game Γ, and not just to those fulfilling Assumptions of the previous Chapter, which guarantee existence and/or uniqueness of the Nash equilibrium. In this sense, Assumption 1 is fundamental of the proposed model while violating Assumption 3 leads to situations in which the matrix $I + B$ is not invertible, a necessary condition for the characterization of the internal equilibrium of the model. Conversely, Assumption 2, which concerns the demand function, is not mandatory. We will present results referred to the Nash equilibrium of game defined by a demand function that not necessary satisfies Assumption 2 or is not isoelastic.

The economic rationale of this can be understood by looking at the form of the utility function and recalling the subsequent comments. Without interdependent preferences, thanks to the assumption of strategic substitutability, we have that if strategy of a firm j increases this has the effect of reduce marginal profits of firm i. However, if the utility of firm i depends on profits of firm j, we observed that the same marginal utility can be achieved with smaller or larger marginal profits if firm i is altruistic or spiteful toward firm j, respectively.

The first consequence of this is on the way firm i optimally responds to a change in the strategy of firm j. If we assume that for each $\boldsymbol{q}_{-i}$ there exists a unique best response $q_i > 0$ (e.g. if v_i is strictly concave in q_i), we have that if firm i is altruistic toward firm j, an increase in the strategy of firm j reduces the strategic response of firm i, while the opposite occurs for a spiteful behavior. We have the understandable consequence that an altruistic behavior induces a less aggressive interaction, while in the presence of a spiteful behavior the resulting interaction is more aggressive. We exemplify the first order effects for a particular social interaction structure and a linear demand function.

Example 2. (First order effects)
Let consider a market characterized by inverse demand function

$$p(Q) = \max\{a - bQ, 0\} \tag{3.1}$$

populated by 4 firms, whose interdependent preferences are described by matrix

$$B = \begin{bmatrix} 0 & 0.61 & 0 & -0.32 \\ -0.2 & 0 & 0.73 & -0.17 \\ 0.43 & -0.08 & 0 & -0.23 \\ -0.3 & 0.81 & 0 & 0 \end{bmatrix} \tag{3.2}$$

which generates the network graph in Figure 3.1. The utility of the generic firm $i = 1, \ldots, 4$ has the following form

$$\begin{aligned} v_i(\mathbf{q}) = \pi_i(q_i, \mathbf{q_{-i}}) + \beta_{i1}\pi_1(q_1, \mathbf{q_{-1}}) + \beta_{i2}\pi_2(q_2, \mathbf{q_{-2}}) \\ +\beta_{i3}\pi_3(q_3, \mathbf{q_{-3}}) + \beta_{i4}\pi_4(q_4, \mathbf{q_{-4}}) \end{aligned} \tag{3.3}$$

where $\beta_{ii} = 0$. We stress that it is easy to see that matrix B in (3.2) fulfills Assumptions 1,3, while demand function $p(Q)$ fulfills Assumption 2. It is easy to see that the utility function is concave

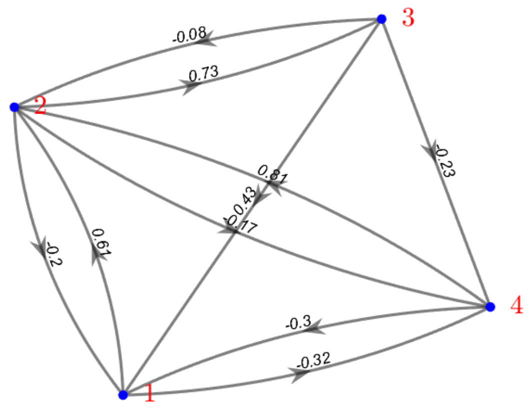


Figure 3.1: Graphical representation of the network described by weight matrix B in (3.2)

and the resulting game has a unique internal Nash equilibrium. In what follows, we assume that we deal with suitable strategies for which the best response is strictly positive. For such reason, we can drop the max function. The utility of firm 1 becomes

$$v_1(\mathbf{q}) = q_1(p(Q) - c) + (0.61q_2 - 0.32q_4)(p(Q) - c) \tag{3.4}$$

The role of weights β_{ij} on the utility is evident. Increasing β_{12} (i.e. increasing the degree of altruism of firm 1 toward firm 2), keeping constant the output quantities, the utility increases. On the contrary, decreasing β_{12}, the utility decreases. Considering, instead, the weight β_{14} we can notice that the effect on the utility is the opposite. Increasing the spitefulness degree, from -0.32 to -0.33, will decrease the value of the utility. Now we can calculate the marginal utility of firm 1 with respect to its decision variable q_1 as follows

$$\frac{\partial v_1(\mathbf{q})}{\partial q_1} = p(Q) - c - bq_1 - 0.61bq_2 + 0.32bq_4 \tag{3.5}$$

The first remark concerns the role of weights β_{ij}, as they act in a opposite way on the marginal utility with respect to the utility v_i. In fact, looking at the marginal utility we can notice how an increase (decrease) in the β_{12} weight will decrease (increase) the marginal utility of firm 1. Contrariwise, an increase (decrease) in the β_{14} weight will increase (decrease) the marginal utility

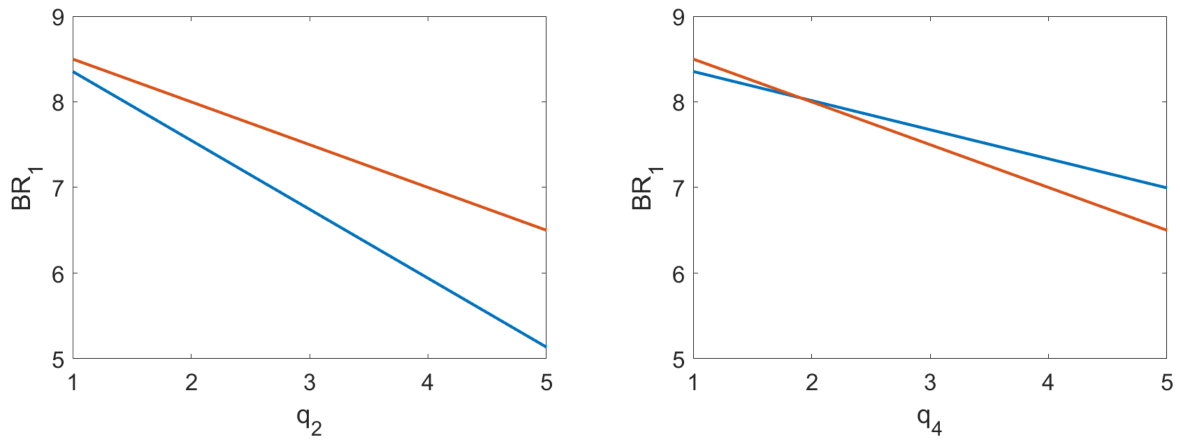


Figure 3.2: Best response with (blue line) and without (red line) preference interdependency: comparing the slope of the blue line and the red line both in 3.2(a) and in 3.2(b) is clear how the interdependent weights modify the degree of strategic substitutability.

of firm 1. In order to determine the optimal strategy that maximizes the utility of a firm, we impose first order condition

$$\frac{\partial v_i(\mathbf{q})}{\partial q_i} = 0 \Longleftrightarrow p(Q) - c - bq_i - \beta_{i2}bq_2 - \beta_{i3}bq_3 - \beta_{i4}bq_4 = 0 \tag{3.6}$$

which implicitly defines the reaction or best response function of firm i with respect to the choices of its other competitors

$$\mathrm{BR}_i(\mathbf{q}_{-i}) = \frac{a-c}{2b} - \frac{q_2(1+\beta_{i2})}{2} - \frac{q_3(1+\beta_{i3})}{2} - \frac{q_4(1+\beta_{i4})}{2} \tag{3.7}$$

So, for the firm 1 the best response function is

$$\mathrm{BR}_1(\mathbf{q}_{-1}) = \frac{a-c}{2b} - \frac{q_2(1+0.61)}{2} - \frac{q_3}{2} - \frac{q_4(1-0.32)}{2} \tag{3.8}$$

First, looking at equation of the best response we can notice how the quantity of the other firms negatively affects the reaction of the firm under consideration. Increasing the strategic decision of the output quantity of one of the competitors decreases the optimal quantity chosen by the firm considered, accordingly to the strategic substitutability (the stronger an opponent plays the lower the player considered will respond) characterizing a Cournotian game with linear demand function (even in the presence of interdependent preferences). Firms are therefore bound (linked) by strategic substitutability, whose degree is indeed affected by the weights describing the network of social interactions. We want to point out how changing the β_{ij} affects the strategic substitutability of pairs of strategies.

In order to show the effects of preference interdependence on strategic substitutability, we focus on firm 1 and we vary the strategies of its opponents, in particular the strategy of firm 2. This means that the interdependence effect we are looking at is that highlighted in red in Figure 3.3(a). Since now, we set $a = 21$, $b = 1$, $c = 1$, while we set initially the output quantities $q_i = 1, \forall i \neq 1$.

We stress that firm's 1 best response is affected by firm's 2 strategy through the two different channels of interaction, namely the market interaction and the social interaction, (see Figure 3.3(b)). In what follows, we want to put in evidence the fraction of the change in the best response of player 1 due to a change in the strategy of player 2 that is solely due to the dependence of preferences of firm 1 on the material payoff of firm 2. For the previous parameters' values we obtain

$$\mathrm{BR}_1(1, 1, 1) = 8.355 \tag{3.9}$$

As a comparison it might be useful to calculate also the best response in the case of game Γ_0 (i.e. when $\beta_{ij} = 0, \forall i, j$). We obtain

$$\mathrm{BR}_1^C(1, 1, 1) = 8.5 \tag{3.10}$$

Now we let q_2 vary slightly, setting $q_2 = 1.5$. In this case the best response for the case with interdependence is

$$\mathrm{BR}_1(1.5, 1, 1) = 7.9525 \tag{3.11}$$

while for the selfish case we have

$$\mathrm{BR}_1^C(1.5, 1, 1) = 8.25 \tag{3.12}$$

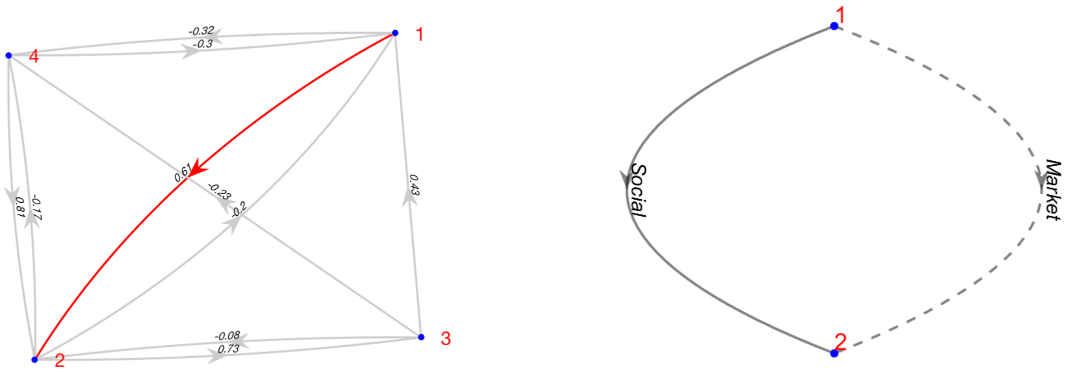
Therefore, we can calculate the variation of the best response after a change in the value of output quantity q_2

$$\Delta_{BR_1} = \mathrm{BR}_1(1.5, 1, 1) - \mathrm{BR}_1(1, 1, 1) = -0.4025 \tag{3.13}$$

while without interdependence of preferences we have

$$\Delta_{BR_1^C} = \mathrm{BR}_1^C(1.5, 1, 1) - \mathrm{BR}_1^C(1, 1, 1) = -0.25 \tag{3.14}$$

We can notice how the variation is different in the two cases, given the same variation in the strategic output q_2. In case of altruistic interdependence the strategic substitutability is amplified



(a) (b)

Figure 3.3: Figure 3.3(a) is the graphical representation of the first order social dependence of firm 2 on firm 1. In Figure 3.3(b) are shown the two components of first order effects: the market (dashed line) and the social (solid line) effect.

and the first order effect on the best response of firm 1 due to the dependence of its utility function on the material payoff of firm 2 can be quantified by

$$FOE_{1,2} = \Delta_{BR_1} - \Delta_{BR_1^C} = -0.1525.$$

In what follows we refer to first order market effect to the outcome due to market interaction among firms, while we refer to first order social effect to the outcome due to the network of social interactions among firms. We stress that since we are most interested in effects due to social interdependence of firms, in what follows we use notation $FOE_{i,j}$ (with $i \neq j$) as the first order effect in the best response of player i due to a change in the strategy of player j that is solely due to the dependence of preferences of firm i on the material payoff of firm j. Now we repeat the previous steps letting q_4 to vary by the same amount and we compare the variation in the best responses of the interdependent case (BR_1) and selfish case (BR_1^C) (i.e. we focus on the interaction described in Figure 3.4 by the highlighted portion of the graph), respectively obtaining

$$\Delta_{BR_1} = \mathrm{BR}_1(1,1,1.5) - \mathrm{BR}_1(1,1,1) = -0.17 \tag{3.15}$$

and

$$\Delta_{BR_1^C} = \mathrm{BR}_1^C(1,1,1.5) - \mathrm{BR}_1^C(1,1,1) = -0.25 \tag{3.16}$$

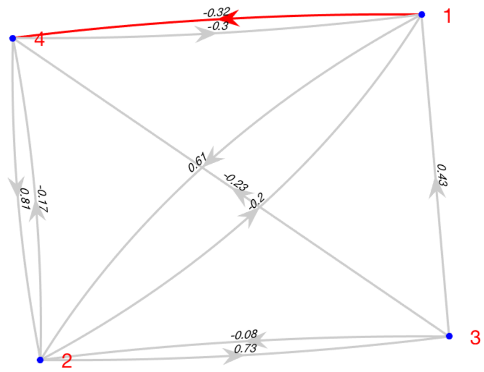


Figure 3.4: Graphical representation of the first order social dependence of firm 4 on firm 1

We notice that this time $\Delta BR_1 < \Delta BR_1^C$. The reason is that firm 1 now acts spitefully towards firm 4. In this case, the first order effect on the best response of firm 1 due to the dependence of its utility function on the material playoff of firm 4 can be quantified by

$$FOE_{1,4} = \Delta_{BR_1} - \Delta_{BR_1^C} = +0.08$$

To summarize, the first order effect is essentially an alteration of the degree of strategic interaction between the firms, increasing the strategic substitutability in the case in which there is altruism and decreasing it in the case of spitefulness (compared to the selfish case). Since the stronger the strategic substitutability effect is the smaller is the best response, we have that altruism acts in a subtractive way on the best response quantity, while spitefulness acts additively.

3.3 High order effects

However, to completely understand how best response mechanism of a player affects the equilibrium in the presence of interdependence of preferences we can not limit to the "direct" effect described in the previous proposition. To show this, we focus on a simple example. Assume that the utility of firm i depends on the profits of firm k and that, in turns, the utility of firm k depends on profits of firm j and that both interdependence effects are of altruistic kind. According to the previous proposition, there is a direct effect on the best response of firm k due to the strategy q_j, which in the particular case leads to a less aggressive reply than without interdependent preferences and the decreasing monotonicity of marginal profits of firm k is bolstered. However, again according to the previous proposition, to such an additional decrease corresponds an additional increase to the marginal profits of firm i. We then have an indirect, second degree interdependence effect between firm i and j, mediated by the interactions involving firm k, which results in a more aggressive response of firm i to the strategy of q_j. The previous reasoning can be repeated considering all the possible couplings of "altruist"-"spiteful" behavior. It easy to see that we have a sort of "rule-of-signs" that allows predicting such second order effect: if we identify "altruism" with "-" (meaning the reduced response to strategies) and "spitefulness" with "+" (meaning the increased response to strategies), two subsequent behaviors lead to a composite behavior that can be identified by the "sign" product of the two starting behaviors. We stress that since $|\beta_{ik}| < 1$ and $|\beta_{kj}| < 1$, the second order effect is reduced with respect to the first order effects induced both by preference interdependence between firm i and firm k and between firm k and firm j, but it can be larger than the first order effect arising from the direct dependence of the utility function of firm i and that of firm j.

Indeed, the previous considerations can be repeated to take into account effects on the response of firm i to the strategy q_j mediated by 2,3,$\ldots, n, \ldots$ firms (i.e. considering a path of length $2, 3, \ldots, n, \ldots$ starting in node i and ending in node j), giving rise to second, third,..., n-th,... order effects. This is particularly relevant when we study the equilibrium of the game, as the overall effect of firms interaction consequent to interdependent preferences will require to take into account every k-th order effects, for any $k \geq 1$.

Note that a change in the strategy of firm i has an effect on the marginal utility of firm i itself, which is due to high order effects of interdependence among firms' preferences, in addition to the obvious direct effect on the utility exerted by the change of the marginal material payoff of firm i. To describe this, assume that the utility function of firm i depends on the material payoff of firm j and vice-versa (i.e. $\beta_{ij} \neq 0$ and $\beta_{ji} \neq 0$) and let us assume a sequence of consecutive choices. The strategy of firm i has a first order effect on the best response of firm j, which, in turns, reflects on the best response of firm i, giving rise to a second order effect. At the equilibrium such "consecutive choices" and the consequent effects simultaneously occur, but this suggests that among high order effects we then have to take into account also n-th order feedback effects on each firm.

We reconsider the setting studied in Example 2 to highlight second order effects.

Example 3. (Second order effects)

The analysis of first order effect is not sufficient to understand how the Nash equilibrium is affected by the introduction of interdependence of preference between players. By definition, a vector of strategies is a Nash equilibrium if the strategy of each player is simultaneously the best response to the strategies of its competitors. The equilibrium is then the result of a combination of best responses which take into consideration the best response of a player to the best response of a second player to the best response of a third one and so on. Namely, to better understand the choice of the equilibrium quantity of a given player we can not stop at the analysis of the direct first order effect on its best response after the variation in the strategic choice of one of its opponents. For instance, we have to consider also the effect of this change on the best response of an intermediary agent who in turn influences the best response of the reference player.

In what follows we give evidence of such second order effect. To better explain the logic, we might consider the interdependent preferences structure as described by matrix 3.2. The goal of next example is to highlight the effects on the best response of player 1 due to a change in the strategy of player 3, taking into account also the effects that such change have on the best response of player 2. The situation we are going to consider is that highlighted in Figure 3.5(a). In particular, we can consider the effect of the variation in the best response of firm 1 after a change in the strategic quantity of firm 3, both in the case of presence of interdependence of preferences

$$\Delta_{BR_1} = \mathrm{BR}_1(q_2, 1.5, q_4) - \mathrm{BR}_1(q_2, 1, q_4) = -0.25 \tag{3.17}$$

and in the case without interdependence of preferences

$$\Delta_{BR_1^C} = \mathrm{BR}_1^C(q_2, 1.5, q_4) - \mathrm{BR}_1^C(q_2, 1, q_4) = -0.25 \tag{3.18}$$

As we notice, the two quantities coincide since the weight which binds the utility of firm 1 to the material payoff of firm 3 is zero ($\beta_{13} = 0$, hence no edge directly connecting firm 1 to firm 3 is highlighted in red in Figure 3.5(a)). A change in the strategy of firm 3 causes only a direct first order market effect on the best response of firm 1, while has a null first order social effect on the best response of firm 1. Namely

$$FOE_{1,3} = \Delta_{BR_1} - \Delta_{BR_1^C} = 0 \tag{3.19}$$

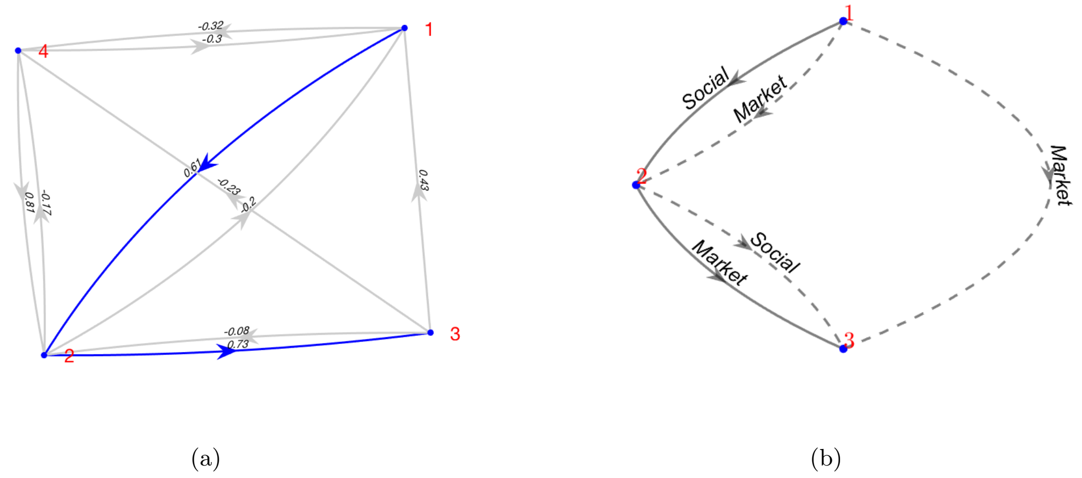


Figure 3.5: Figure 3.5(a) is the graphical representation (in blue) of the second order indirect connection of firm 1 to firm 3, mediated by firm 2. In Figure 3.5(b) are shown the components of the first and second order effects: the market (dashed line) and the social (solid line) effect of interdependence. Notice that firm 3 and firm 1 are bound by the market effect only, since $\beta_{13} = 0$ (for this reason, in Figure 3.5(a), there is no solid line connecting the two firms.)

Furthermore, we can notice $\Delta_{BR_1} < 0$.
We recall that, according to example 2, when we use the notation $FOE_{i,j}$ we mean the first order social effect on the best response of firm i after a change in the strategic choice of firm j.

Then, we calculate the variation of the best response of firm 2 after a change in the value of the strategic quantity q_3, both in the case of interdependence of preferences and without. We obtain, respectively

$$\Delta_{BR_2} = \mathrm{BR}_2(q_1, 1.5, q_4) - \mathrm{BR}_2(q_1, 1, q_4) = -0.4325 \tag{3.20}$$

and

$$\Delta_{BR_2^C} = \mathrm{BR}_2^C(q_1, 1.5, q_4) - \mathrm{BR}_2^C(q_1, 1, q_4) = -0.25 \tag{3.21}$$

In this case, Δ_{BR_2} depends both on the market interaction effect and the first order social effect, since $\beta_{23} = 0.73$, while $\Delta_{BR_2^C}$ only depends on the market interaction effect.
We can notice that $\Delta_{BR_2} < \Delta_{BR_2^C}$, so the first order effect on the best response of firm 2 due solely to the interdependence of its utility function on the material payoff of firm 3 is given by

$$FOE_{2,3} = \Delta_{BR_2} - \Delta_{BR_2^C} = -0.1825 \tag{3.22}$$

Since the firm 2 is bound to firm 3 by a positive coefficient, a change in the strategic quantity of the latter will have the first order social effect to decrease the best response of the former. Namely, firm 2 is altruistic towards firm 3, therefore it reacts to an increase in the choice of its opponent by decreasing its strategic quantity.

Now we want to calculate, in the model with interdependent preferences, the change in the best response of firm 1, after a change in q_3, also taking into account the best response of firm 2. Namely, we are considering in the BR_1 the effects on q_2 of a change in q_3. Hence

$$\Delta_1 = BR_1(BR_2(q_1, 1.5, q_4), 1.5, q_4) - BR_1(BR_2(q_1, 1, q_4), 1, q_4) = 0.098 \tag{3.23}$$

We can reiterate this logic considering the model without interdependency

$$\Delta_1^C = BR_1^C(BR_2^C(q_1, 1.5, q_4), 1.5, q_4) - BR_1^C(BR_2^C(q_1, 1, q_4), 1, q_4) = -0.125 \tag{3.24}$$

Δ_1 considers both the effects due to the market interaction and the, direct and indirect, effects due to the interdependence of preferences. To understand what contributes to make Δ_1 differ from Δ_1^C, we must pay attention to the superimposition of the subsequent market and social effects, as shown if Figure 3.5(b).

If an overlapping of market and social effects was already present in the $FOE_{2,3}$, now we have the superimposition of multiple second order effects of the two kinds, that is

1. the superimposition of two consecutive market effects (dashed subsequent lines connecting firm 2-firm 3 and firm 1-firm 2)

2. a market effect that superimposes to a social effect (solid line connecting firm 2-firm 3 and dashed line connecting firm 1-firm 2)

3. a social effect that superimposes to a market effect (dashed line connecting firm 2-firm 3 and solid line connecting firm 1-firm 2)

4. the superimposition of two social effects (solid subsequent lines connecting firm 2-firm 3 and firm 1-firm 2)

The last effect is what we will consider as the second order social effect of a change in the choice of firm 3 on the best response of firm 1, mediated by the best response of firm 2, namely $SOE_{1,2,3}$. The difference

$$\Delta_1 - \Delta_1^C = 0.223 \tag{3.25}$$

allows us to subtract the first and second order effects referred solely to the market interaction. What we are left with consists of the superimposition of several effects that we want to isolate. First, we should take into account the first order effect on the best response of firm 1 due to the change in q_3, which however in our case is equal to zero ($FOE_{1,3} = 0$). Then, we have to tackle the mixed higher order effects, namely those induced by the change in q_3 on the best response of firm 2, which in turn causes a second order effect on the best response of firm 1. This last effect is partially due to the market interaction between firm 1 and all its opponents and partially due to the interaction firm 1 has with firm 2 through the structure of interdependence of preferences. The former effect is represented by the terms

$$\begin{aligned} BR_1^C(BR_2(q_1, 1.5, q_4), 1, q_4) - BR_1^C(BR_2(q_1, 1, q_4), 1, q_4) - \\ (BR_1^C(BR_2^C(q_1, 1.5, q_4), 1, q_4) - BR_1^C(BR_2^C(q_1, 1, q_4), 1, q_4)) = 0.091 \end{aligned} \tag{3.26}$$

and

$$\begin{aligned} BR_1(BR_2^C(q_1, 1.5, q_4), 1, q_4) - BR_1(BR_2^C(q_1, 1, q_4), 1, q_4) - \\ (BR_1^C(BR_2^C(q_1, 1.5, q_4), 1, q_4) - BR_1^C(BR_2^C(q_1, 1, q_4), 1, q_4)) = 0.076 \end{aligned} \tag{3.27}$$

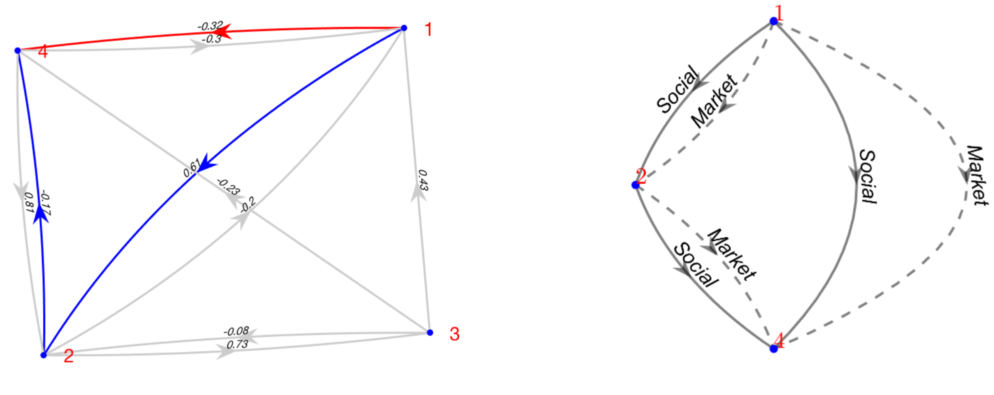
Finally, we obtain the solely second order effect of the interdependence of preferences between firm 1 and firm 3, mediated by firm 2. This effect is quantifiable in

$$\begin{aligned} BR_1((q_2 + BR_2 - BR_2^C), 1, q_4) - BR_1(q_2, 1, q_4) - \\ (BR_1^C(q_2 + BR_2 - BR_2^C), 1.5, q4)) - BR_1^C(q_2, 1, q_4)) = 0.056 \end{aligned} \tag{3.28}$$

We can notice how the following identity is satisfied

$$(3.25) = (3.26) + (3.27) + (3.28) \tag{3.29}$$

Hence, a firm that is positively linked to a second firm, which in turn is positively bound to a third one causes a positive second order effect of the latter on the former. An increase in q_3 causes a negative first order effect on the best response of firm 2. Firm 1, which is positively linked to firm 2 observing a decrease in the best response of the opponent, increases its best response. If we ignore for a while what happens in between and we focus only on the cause-effect chain of a variation of q_3 on the best response of firm 1 we would see that the increase in q_3 increases the best response of firm 1, namely in this specific setting $FOE_{1,3} = 0$ but $SOE_{1,2,3} > 0$.



(a) (b)

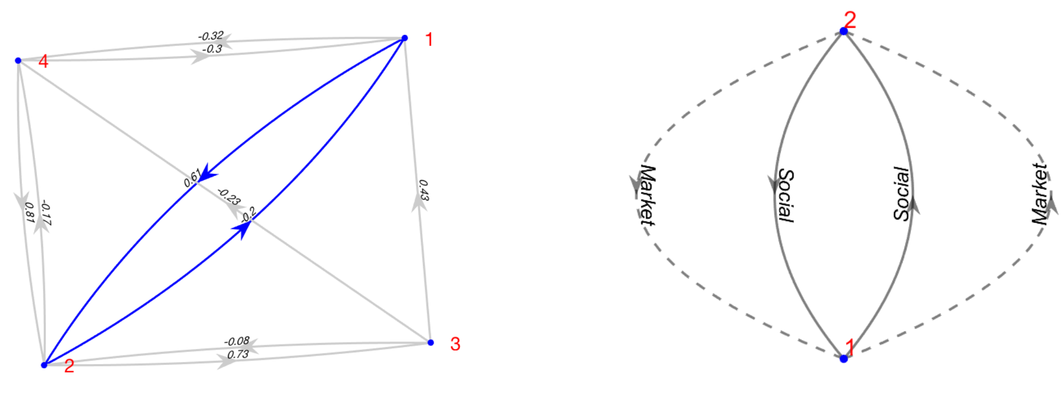
Figure 3.6: Figure 3.6(a) is the graphical representation (in blue) of the second order indirect connection of firm 1 to firm 4, mediated by firm 2. Direct connection of firm 1 to firm 4 is highlighted in red. In Figure 3.6(b) are shown the components of the first and second order effects: the market (dashed line) and the social (solid line) effect of interdependence.

We now consider the second order interdependent effect on the strategic choice of firm 1 after a change in q_4, mediated by the interdependency between firm 2 and firm 4 (see Figure 3.6(a)). Following the previous analysis (all the involved effects are reported in Figure 3.6(b) we may skip the intermediate computation and focus only on the sign of $SOE_{1,2,4}$, that is

$$\begin{aligned} BR_1((q_2 + BR_2 - BR_2^C), q_3, 1) - BR_1(q_2, q_3, 1) - \\ (BR_1^C(q_2 + BR_2 - BR_2^C), q_3, 1.5)) - BR_1^C(q_2, q_3, 1)) = -0.043 \end{aligned} \tag{3.30}$$

In this case, a firm that is positively linked to a second firm, which in turn is negatively bound to a third one causes a negative second order effect of the latter on the former. An increase in q_4 causes a positive first order effect on the best response of firm 2. Firm 1, which is positively linked to firm 2 observing an increase in the best reponse of the opponent, decreases its best response. If we focus only on the cause-effect chain of a variation of q_4 on the best response of firm 1 we would see that the increase in q_4 decreases the best response of firm 1, namely in this specific setting $FOE_{1,4} > 0$ and $SOE_{1,2,4} < 0$.

Finally, we investigate the feedback effect, that is what happens when a given firm decides to change its strategic choice and this decision causes, initially, a first order effect on the best response



(a) (b)

Figure 3.7: Figure 3.7(a) is the graphical representation of the second order feedback connection on firm 1 through firm 2. In Figure 3.7(b) the market (dashed line) and the social (solid line) effect of interdependence.

of an opponent which, ultimetely, causes an other change in the best response of the given firm. For instance, we can compute the first order social effect on q_2 after an increase in q_1

$$FOE_{2,1} = 0.05 \tag{3.31}$$

Therefore, an increase (decrease) in q_1 causes an increase (decrease) in the best response of firm 2, since $\beta_{21} < 0$.

Then, we can calculate the second order social effect

$$SOE_{1,2,1} = -0.015 \tag{3.32}$$

In this case, we can see that an initial increase in q_1 generates a feedback effect that ultimetely causes a decrease in the best response of firm 1, accordingly to $\beta_{12}\beta_{21} > 0$.

3.4 Characterization of Nash equilibria

The previous considerations are crucial to understand the characterization of equilibria in terms of the effects of network structure of social interactions, as shown in the next proposition. We focus on internal equilibria as for boundary equilibria such effects could be hindered or changed by the fact that some firms are actually not active at the equilibrium, as in the case of those having null equilibrium strategies, or because production levels reached the capacity limit. In any case, we stress that the following results could be suitably modified for boundary equilibria.

Proposition 5. *Let $\boldsymbol{q}^*$ be an internal Nash equilibrium for game $\Gamma = (\mathcal{N}, S_i, v_i(q_i, \boldsymbol{q}_{-i}, B))$, and let Q^* be the corresponding aggregate equilibrium output of the industry. Then there exists a vector $\boldsymbol{\xi} \in (0, +\infty)^N$, which just depends on coefficients β_{ij}, such that*

$$\boldsymbol{q}^* = Q^* \boldsymbol{\sigma} = Q^* \frac{\boldsymbol{\xi}}{\mu}, \tag{3.33}$$

with $\mu = \sum_{i=1}^{N} \xi_i$ and where the aggregate equilibrium quantity satisfies

$$Q^* p'(Q^*) = (c - p(Q^*))\mu, \tag{3.34}$$

while vector $\boldsymbol{\xi}$ is implicitly defined by $(I + B)\boldsymbol{\xi} = \boldsymbol{u}$. Vector $\boldsymbol{\xi}$ is defined by

$$\boldsymbol{\xi} = (I + B)^{-1} \boldsymbol{u}, \tag{3.35}$$

in which the i-th component represents a measure of the centrality of the i-th firm in the network described by matrix B. At an internal equilibrium, the utility achieved by each firm is the same, corresponding to $v_i = |Q^* p'(Q^*)|$.

At the equilibrium, each firm realizes profit $\pi_i^* = \sigma_i Q^*(p(Q^*) - c)$, $i = 1, \ldots, N$.

The Nash equilibrium is characterized through the two channels of interaction among firms: the market interaction and the social interaction, whose influences can be identified in both relations (3.33) and (3.34). The effects related to the latter channel are all encompassed in $\boldsymbol{\xi}$ (and, consequently, in μ), which just depends on the network structure of social interaction. The effects related to the former channel are encompassed in Q^*, which however depends both on the inverse demand function and costs (i.e. on the unique elements characterizing the market interaction) and on μ, which is determined by the distribution of weigths β_{ij}. Vector $\boldsymbol{\xi}$ has positive elements, each of which provide a centrality measure of the corresponding firm in the network of social interactions. The centrality of a firm determines its market share, which exactly equals the fraction that the centrality measure of the firm represents with respect to the sum of the centrality measures of all the firms in the industry[2]. Consequently, the centrality measure (and hence the network of social

[2] We stress that if we also take into account firms that are not active at the equilibrium, solving the Nash equilibrium problem we again find a vector $\boldsymbol{\xi}$, in which null elements identifies non active firms.

interactions) determines the ordering of firms with respect to realized profits, as firms with larger centrality measures have higher profits. Ceteris paribus, the more a firm's centrality measure is large, the greater will be its market share and the ordering of firms with respect to their centrality measures provides the ordering of firms with respect to their market share. We stress that since firms are homogeneous in all respects but the distribution of social preferences, without a network of interdependence, all firms would produce the same quantity and would realize the same profits at the equilibrium.

Social preferences of firms determine another key element characterizing the equilibrium, i.e. the scalar μ, which is defined as the sum of the centrality measures of all the firms. It is possible to show (see Lemma 1) that $\mu \in (1, +\infty)$, where the limit $\mu \to 1$ is realized in the monopolistic limit, while $\mu \to +\infty$ is realized in the competitive limit and in the case of oligopoly described by Γ_0 we have $\mu = N$. This means that μ is a scalar index that encompasses the degree of competitiveness that characterizes an oligopoly with interdependent preferences. Note that, besides the limiting games that correspond to the monopolistic and competitive limits, in general we have an infinite set of games, different with respect to the network of interdependent preferences, that are characterized by the same value of μ, i.e. by the same degree of competitiveness.
In the next example we give evidence that, for a given μ, more than a single network structure provides that value of μ.

Example 4. (Network of social interaction and competitiveness)
Let consider the following couple of 5×5 matrices

$$B_1 = \begin{bmatrix} 0 & -0.0926 & 0.3880 & -0.0885 & -0.2069 \\ 0.3221 & 0 & -0.0344 & -0.1526 & -0.1352 \\ -0.0985 & 0.0675 & 0 & 0.0936 & -0.0626 \\ 0.1195 & 0.2360 & -0.1278 & 0 & -0.2278 \\ 0.0054 & 0.1173 & -0.0201 & -0.1025 & 0 \end{bmatrix} \tag{3.36}$$

and

$$B_2 = \begin{bmatrix} 0 & -0.1243 & 0.1736 & -0.1958 & 0.1465 \\ 0.0788 & 0 & 0.3368 & -0.2065 & -0.2090 \\ 0.2129 & -0.0690 & 0 & -0.2456 & 0.1017 \\ 0.1678 & -0.1164 & 0.0298 & 0 & -0.0812 \\ 0.3836 & -0.3188 & -0.3224 & 0.2575 & 0 \end{bmatrix} \tag{3.37}$$

A direct computation provides

$$\tilde{B}_1 = (I + B_1)^{-1} = \begin{bmatrix} 0.9322 & 0.0505 & -0.3374 & 0.1434 & 0.2113 \\ -0.2953 & 0.9242 & 0.1624 & 0.1098 & 0.0990 \\ 0.1155 & -0.0416 & 0.9449 & -0.0785 & 0.0596 \\ -0.0201 & -0.2605 & 0.1261 & 0.9660 & 0.1886 \\ 0.0299 & -0.1362 & 0.0147 & 0.0838 & 1.0078 \end{bmatrix} \tag{3.38}$$

and

$$\tilde{B}_2 = (I + B_2)^{-1} = \begin{bmatrix} 1.0485 & 0.1060 & -0.2525 & 0.1885 & -0.0905 \\ -0.1457 & 1.0588 & -0.2452 & 0.0597 & 0.2724 \\ -0.2406 & 0.0530 & 1.0051 & 0.2205 & -0.0380 \\ -0.2238 & 0.1267 & 0.0115 & 0.9503 & 0.1353 \\ -0.4686 & 0.2813 & 0.3398 & -0.2269 & 1.0745 \end{bmatrix} \tag{3.39}$$

for which we can calculate the column vectors of the centrality measures $\boldsymbol{\xi}$, composed by the row summations, for both matrix B_1

$$\boldsymbol{\xi}^{\mathbf{1}} = \tilde{B}_1 \cdot \boldsymbol{u} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \tag{3.40}$$

and matrix B_2

$$\boldsymbol{\xi^2} = \tilde{B}_2 \cdot \boldsymbol{u} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \tag{3.41}$$

Therefore, the degrees of competitiveness that characterize the two different structures of interdependent preferences coincide

$$\mu_1 = \boldsymbol{u}^T \boldsymbol{\xi^1} = 5 = \mu_2 = \boldsymbol{u}^T \boldsymbol{\xi^2} \tag{3.42}$$

even if the two interdependent structures described by B_1 and B_2 are very different.
Similarly, if we consider the couple of 5×5 matrices

$$B_1 = \begin{bmatrix} 0 & -0.1617 & -0.1663 & -0.0535 & 0.1315 \\ 0.0507 & 0 & -0.2800 & -0.2307 & 0.2099 \\ -0.1621 & -0.0159 & 0 & -0.1684 & 0.0964 \\ 0.0872 & -0.0643 & -0.0913 & 0 & -0.1815 \\ -0.1744 & 0.2839 & -0.1648 & -0.1947 & 0 \end{bmatrix} \tag{3.43}$$

and

$$B_2 = \begin{bmatrix} 0 & 0.0857 & 0.0417 & -0.1661 & -0.2112 \\ -0.1410 & 0 & -0.2785 & 0.1886 & -0.0191 \\ -0.2039 & 0.1150 & 0 & -0.0209 & -0.1402 \\ -0.1143 & 0.3121 & -0.2696 & 0 & -0.1782 \\ -0.2271 & 0.1271 & 0.0982 & -0.2482 & 0 \end{bmatrix} \tag{3.44}$$

direct computation provides

$$\tilde{B}_1 = (I + B_1)^{-1} = \begin{bmatrix} 0.9824 & 0.2188 & 0.2050 & 0.1033 & -0.1761 \\ -0.0651 & 1.0658 & 0.2781 & 0.2510 & -0.1964 \\ 0.1317 & 0.0784 & 1.0436 & 0.1811 & -0.1015 \\ -0.0409 & 0.0114 & 0.1230 & 1.0568 & 0.1829 \\ 0.2036 & -0.2493 & 0.1527 & 0.1824 & 1.0439 \end{bmatrix} \tag{3.45}$$

and

$$\tilde{B}_2 = (I + B_2)^{-1} = \begin{bmatrix} 1.0515 & -0.2036 & -0.0515 & 0.2766 & 0.2602 \\ 0.1853 & 0.9921 & 0.2256 & -0.1353 & 0.0656 \\ 0.2293 & -0.2011 & 0.9539 & 0.1467 & 0.2045 \\ 0.1659 & -0.4335 & 0.1642 & 1.1772 & 0.2595 \\ 0.2339 & -0.2602 & -0.0933 & 0.3578 & 1.0951 \end{bmatrix} \tag{3.46}$$

for which we can calculate the column vectors of the centrality measures $\boldsymbol{\xi}$, composed by the row summations, for both matrix B_1

$$\boldsymbol{\xi}^{\mathbf{1}} = \tilde{B}_1 \cdot \boldsymbol{u} = \begin{bmatrix} 1.333 \\ 1.333 \\ 1.333 \\ 1.333 \\ 1.333 \end{bmatrix} \tag{3.47}$$

and matrix B_2

$$\boldsymbol{\xi}^{\mathbf{2}} = \tilde{B}_2 \cdot \boldsymbol{u} = \begin{bmatrix} 1.333 \\ 1.333 \\ 1.333 \\ 1.333 \\ 1.333 \end{bmatrix} \tag{3.48}$$

Therefore, the degrees of competitiveness that characterize the two different structures of interdependent preferences coincide

$$\mu_1 = \boldsymbol{u}^T \boldsymbol{\xi}^{\mathbf{1}} = 6.667 = \mu_2 = \boldsymbol{u}^T \boldsymbol{\xi}^{\mathbf{2}} \tag{3.49}$$

The two couples of matrices highlight the fact that, in general we may have an infinite family of structures of interdependent preferences characterized by the same value of μ, i.e. by the same degree of competitiveness. In particular, the first couple of matrices, composed by firms that on average are self-interested, realize in an oligopoly described by Γ_0.

When the equilibrium is internal, we have that the market share actually corresponds to the Katz-Bonacich centrality measure $(I - \alpha M)^{-1}\boldsymbol{u}$ associated to the network described by a matrix M. In this case the network is that induced by the structure of social interaction. We stress that the present setting differs from those usually studied in the literature for two aspects. Firstly, in most cases coefficient α is positive, while in the present case we have $\alpha = -1$. Moreover, the network is described by weighted adjacency matrix in which links can be both positive and negative.

It is relevant to understand how firms' interaction determines the centrality measure. From the mathematical viewpoint, this is encompassed in condition (3.35), from which we have that matrix $(I+B)^{-1}$ provides the complete characterization of the corresponding internal equilibrium, for a given inverse demand function p. If we write

$$(I+B)^{-1} = \begin{bmatrix} \tilde{\beta}_{11} & \tilde{\beta}_{12} & \tilde{\beta}_{13} & \cdots & \tilde{\beta}_{1N} \\ \tilde{\beta}_{21} & \tilde{\beta}_{22} & \tilde{\beta}_{23} & \cdots & \tilde{\beta}_{2N} \\ \tilde{\beta}_{31} & \tilde{\beta}_{32} & \tilde{\beta}_{33} & \cdots & \tilde{\beta}_{3N} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \tilde{\beta}_{N1} & \tilde{\beta}_{N2} & \tilde{\beta}_{N3} & \cdots & \tilde{\beta}_{NN} \end{bmatrix},$$

from Proposition 5 we can say that each $\tilde{\beta}_{ij}$ encompasses the *aggregate effect due to any order dependence of social preferences of firm i with respect to firm j*. Recalling the comments following Proposition 4, there is a strategic influence of firm j on firm i due to social interaction not only if the utility of firm i directly depends on the material payoff of firm j, but also as a consequence of any path of length n of subsequently dependent preferences that starts from firm i and ends on firm j. Without interdependence, we indeed have $\tilde{\beta}_{ij} = 0$ for $i \neq j$ and $\tilde{\beta}_{ii} = 1$. So the more $\tilde{\beta}_{ij}$ for $i \neq j$ differs from 0, the greater is the aggregate effect of any order due to the preference interdependence that links firm i to firm j. Similarly, the more $\tilde{\beta}_{ii}$ differs from 1, the greater is the feedback effect of any order on firm i due to the firms' network of social interactions.

Each component ξ_i of the centrality measure $\boldsymbol{\xi}$ is simply the sum of all $\tilde{\beta}_{ij}$, i.e. ξ_i is the *aggregate effect due to any order dependence of social preferences of firm i with respect to the whole industry.* Each $\tilde{\beta}_{ij}$ is then the contribution to the centrality measure of firm i of any order social interaction of firm i with firm j. Note that, independently of the altruistic or spiteful behavior of firm i with respect to firm j, the sign of $\tilde{\beta}_{ij}$ can be positive or negative. This means that, it is in general false that if firm i is, for example, altruistic with respect to firm j, then $\tilde{\beta}_{ij}$ will be negative for sure and this will reduce the centrality of firm i. Weights β_{ij} only accounts for a first order effect of preference interdependence, which is indeed the potentially more relevant one, but aggregating all the effects of the $n > 1$ order indirect dependence the resulting effect can be, in principle, any.

To deepen the role of $(I+B)^{-1}$ and $\boldsymbol{\xi}$ we focus on the case in which the network is such that

$\rho(B) < 1$, so we can use Neumann series expansion of $(I + B)^{-1}$

$$(I+B)^{-1} = \sum_{n=0}^{\infty}(-B)^n = I + (-1)B + B^2 + \ldots + (-1)B^n + \ldots \tag{3.50}$$

from which we have

$$\boldsymbol{\xi} = (I + (-1)\,B + \,B^2 + (-1)\,B^3 + \cdots)\,\boldsymbol{u}, \qquad \xi_i = \sum_{k=1}^{N} \tilde{\beta}_{ik},\; i = 1, \ldots, N. \tag{3.51}$$

The first addend in both (3.50) and (3.51) represents the situation in which we do not have interdependent preferences, i.e. when B is the null matrix. In such case, since firms are identical, we simply have $\boldsymbol{\xi} = \boldsymbol{u}$. All the other terms accounts for effects of increasing orders. In particular, each element $(-B)^n_{ij}$ of each addend in (3.50) represents the effect of order n of firm i with respect to firm j, namely the sum of all the effects exerted by firm j on firm i along any possible path of $n+1$ firms with interdependent preferences starting in i and ending in j. The "$-$" sign in front of B accounts for the effect on the best response of interdependent preferences, which, as a consequence of Proposition 4, is quantified by $-B$. A positive (respectively, negative) coefficient has the direct effect to reduce (respectively, increase) the strategic response so this provides a negative contribution to the market share at the equilibrium.

The economic meaning of (alternating) signs in front of each term in the Neumann series in (3.50) can be understood recalling the considerations about the role of altruistic and spiteful behaviors on first order and high order effects.

Element $(-B)_{ij}$ represents the direct effect of firm i toward j and it indeed corresponds to $-\beta_{ij}$. Sign "$-$" shows that if firms i is altruistic toward firm j (i.e. $\beta_{ij} > 0$), it will provide a negative (direct) contribution to the centrality of firm i, while if firm i is spiteful toward firm j (i.e. $\beta_{ij} < 0$), it will provide a positive (direct) contribution to the centrality of firm i, while no contribution to the centrality of firm i is provided if firm i is selfish toward firm j (i.e. $\beta_{ij} = 0$).

Similarly, the matrix B^2 collects the indirect second order effects on couples of firms. Element $(B^2)_{ij}$ is obtained adding $\beta_{ik} \cdot \beta_{kj}$, namely adding all the effects due to the dependence of firm i preferences on the choices of firm j mediated by all firms $k = 1, \ldots, N$. If either firm i is selfish with respect to firm k or firm k is selfish with respect to firm j, the second order effect due to the path ikj is indeed null. If the kind of social interaction of firm i toward firm k and of firm k toward firm j is of the same kind (either both altruistic or both spiteful) the resulting contribution to the centrality of agent i with respect to agent j (mediated by agent k) is actually reinforced due to the synergical effect of two consecutive effects of the same kind. Conversely, two consecutive effects of different kinds weaken the contribution to the centrality of firm i.

Each element $\tilde{\beta}_{ij}$ of $(I+B)^{-1}$ is then the result of all the interactions of any order linking firm i to material payoff of firm j. The way $\boldsymbol{\xi}$ changes is completely described in terms of the distribution of $\tilde{\beta}_{ij}$, and this has a strong influence on the way $\boldsymbol{q}^*$ and $\boldsymbol{\pi}^*$ change.

Once more we take into account the scenario considered in Example 2 to show the properties of its Nash equilibrium.

Example 5. (Characterization of Nash equilibrium)

The goal of this example is to show, through a numeric case study, the effects of the structure of preferences interactions on the equilibrium, as analyzed in Proposition 3.

For the sake of the reader, we report again the interdependent preference structure as described by matrix (3.2)

$$B = \begin{bmatrix} 0 & 0.61 & 0 & -0.32 \\ -0.2 & 0 & 0.73 & -0.17 \\ 0.43 & -0.08 & 0 & -0.23 \\ -0.3 & 0.81 & 0 & 0 \end{bmatrix} \tag{3.52}$$

First, we start noting that the spectrum of matrix B is

$$\rho(B) = 0.6173 < 1 \tag{3.53}$$

We then compute the inverse matrix

$$\tilde{B} = (I+B)^{-1} = \begin{bmatrix} 0.7542 & -0.6184 & 0.4514 & 0.2401 \\ 0.3661 & 0.6432 & -0.4695 & 0.1185 \\ -0.3112 & 0.1549 & 0.8869 & 0.1307 \\ -0.0702 & -0.7065 & 0.5157 & 0.9760 \end{bmatrix} \tag{3.54}$$

which allows us to calculate the column vector of the centrality measures

$$\boldsymbol{\xi} = \tilde{B}\boldsymbol{u} = \begin{bmatrix} 0.8273 \\ 0.6582 \\ 0.8614 \\ 0.7150 \end{bmatrix} \tag{3.55}$$

which is a vector of positive elements that determines the ordering with respect to the output levels and profits at the equilibrium. We recall that the heterogeneity in elements of $\boldsymbol{\xi}$ pertains to the presence of the structure of the interdependence of preference, which is the only source of heterogeneity in the model, given that the firms are homogeneous in the production costs.
Then we calculate the index that encompasses the degree of competitiveness that characterizes the model with interdependent preferences

$$\mu = \boldsymbol{u}^T \boldsymbol{\xi} = 3.062 \tag{3.56}$$

from which we can calculate the market share σ, corresponding to the relative measure of centrality of each firm

$$\boldsymbol{\sigma} = \frac{\boldsymbol{\xi}}{\mu} = \begin{bmatrix} 0.2702 \\ 0.2150 \\ 0.2813 \\ 0.2335 \end{bmatrix} \tag{3.57}$$

Solving the system made of the N equations $BR_i(\mathbf{q}^*_{-i}) = q^*_i$ we find the unique Nash equilibrium

$$\boldsymbol{q}^*_i = \begin{bmatrix} 4.073 & 3.241 & 4.241 & 3.521 \end{bmatrix} \tag{3.58}$$

to which corresponds the total equilibrium output produced

$$Q^* = \sum_i^N q^*_i = 15.076 \tag{3.59}$$

We can notice how the following identity is satisfied

$$Q^* p^{'}(Q^*) = (c - p(Q^*))\mu \tag{3.60}$$

We recall that each element β_{ij} of B encompasses the first order effect of preference interdependence between firm i and firm j. Instead, each element $\tilde{\beta}_{ij}$ of $(I + B)^{-1}$ is the result of all the social interactions of any order linking firm i to firm j. Indeed, there is a strategic influence of firm j on firm i not only if the utility of firm i directly depends on the material payoff of firm j, but also as a consequence of any path of length n of subsequently dependent preferences that start from firm i and end on firm j. First order effect described by weight β_{ij} is potentially the more relevant one, but aggregating all the other effects of order $n > 1$ may result, in principle, in a counterintuitive conclusion. We recall that, independently of the altruistic or spiteful behavior of firm i with respect to firm j, the sign of $\tilde{\beta}_{ij}$ can be positive or negative. Moreover, given that each component ξ_i of the centrality index $\boldsymbol{\xi}$ is simply the sum of all $\tilde{\beta}_{ij}$, i.e. ξ_i is the aggregate effect of any order dependence of firm i with respect to the whole industry, then each $\tilde{\beta}_{ij}$ is the contribution to the centrality index of firm i of any order interaction of firm i with firm j.

In general, the signs of elements of matrix $-B$ (i.e. first order effects) are those most relevant for the signs of elements of $\tilde{B}$. However, exceptions are possible. By comparing matrix $\tilde{B}$ and matrix $-B$

$$\tilde{B} - (-B) = \begin{bmatrix} \cdot & -0.0084 & 0.4514 & -0.0799 \\ 0.1661 & \cdot & 0.2605 & -0.0515 \\ 0.1188 & 0.0749 & \cdot & -0.0993 \\ -0.3702 & 0.1035 & 0.5157 & \cdot \end{bmatrix} \tag{3.61}$$

we are able to highlight the contribution of the higher order effects to the equilibrium. First, we notice how it is possible that a null first order effect turns to be positive or negative, such as the case of β_{13}.

Second, we can notice that the sign of $\tilde{\beta}_{41}$ is not the opposite of that of β_{41}. We recall from the previous analysis that, if $\beta_{ij} < 0$ then $FOE_{i,j} > 0$ and this effect is generally the predominant one in $\tilde{\beta}_{ij}$. But this is not the case since $|\beta_{41}^{(2)}| + |\beta_{41}^{(3)}| > |\beta_{41}|$. It is not always true that to a direct spitefulness corresponds an improvement in the centrality measure of a firm.

Entering more into details, weight $\beta_{12} = 0.61$ encompasses an altruistic behaviour on behalf of firm 1 towards firm 2 (in the sense that an increase in the strategic choice of firm 2 will lead firm 1 to decrease its best response). In this case, the weight $\tilde{\beta}_{12} = -0.6184$ confirms that the aggregate effect due to any order dependence of social preferences of firm 1 with respect to firm 2 is to reduce the centrality of firm 1. In this case the contribution of the first order effect to the value of $\tilde{\beta}_{21}$ is the dominant one, as $\tilde{\beta}_{21} \neq -\beta_{12}$. The cumulated $n > 1$ effects just contribute by slightly reinforcing the first order effect by 0.0084. Conversely is the situation involving firm 4 and firm 1. The weight $\beta_{41} = -0.3$ encompasses a spiteful behaviour on behalf of firm 4 towards firm 1 but the aggregate effect due to any order dependence of social preferences of firm 4 with respect to firm 1 is not to increase the centrality of firm 4 but to decrease ($\tilde{\beta}_{41} = -0.0702$) the centrality measure of firm 4. This means that high order indirect effects on the centrality measure of firm 4 due to firm 1 are stronger than first order effect (on aggregate, they amount to 0.3702).

Finally we also draw attention to the dependence of preference of firm 1 on firm 3 material payoff. Although the utility of firm 1 does not depend on the material payoff of firm 3 ($\beta_{13} = 0$), there exists a positive aggregate effect ($\tilde{\beta}_{13} = 0.4514$) due to higher order dependence of social preferences of firm 1 with respect to firm 3 that increase the centrality of firm 1.

Since $\rho(B) = 0.6173 < 1$, in order to quantify the role of higher order social interaction, we can use Neumann series expansion. In particular, each element $(-B)^n_{ij}$ of each addend in Neumann series represents the effect of order n of firm i with respect to firm j, namely the sum of all the effects exerted by firm j on firm i along any possible path of $n+1$ firms with interdependent preferences starting in i and ending in j.
For instance, the matrix B^2 collects the indirect second order effects on couples of firms

$$B^2 = \begin{bmatrix} -0.0260 & -0.2592 & 0.4453 & -0.1037 \\ 0.3649 & -0.3181 & 0 & -0.1039 \\ 0.0850 & 0.0760 & -0.0584 & -0.1240 \\ -0.1620 & -0.1830 & 0.5913 & -0.0417 \end{bmatrix} = [\beta^{(2)}_{ij}] \tag{3.62}$$

where each element $\beta^{(2)}_{ij}$ is obtained adding $\beta_{ik} \cdot \beta_{kj}$, namely adding all the effects due to the dependence of firm i preferences on the choices of firm j mediated by all firms $k = 1, \cdots, N$.
One thing catching the eyes of the reader, is the coefficient $\beta^{(2)}_{23} = 0$. In general, a value $\beta^{(n)}_{ij} = 0$ may depend by the fact that the n order effects binding firm i to firm j are all equal to zero or that such effect cancel out. We recall that B^2 collects the indirect second order effects for couples of firms. Element $\beta^{(2)}_{23}$ is obtained

$$\beta^{(2)}_{23} = \sum_{z=1}^{N} \beta_{2z} \cdot \beta_{z3} = (-0.2 \cdot 0) + (0 \cdot 0.73) + (0.73 \cdot 0) + (-0.17 \cdot 0) = 0, \tag{3.63}$$

i.e. there are no length 2 paths starting from firm 2 and ending in firm 3.
The matrix $-B^3$, instead, collects the indirect third order effects on couples of firms

$$-B^3 = \begin{bmatrix} -0.2744 & 0.1355 & 0.1892 & 0.0500 \\ -0.0948 & -0.1384 & 0.2322 & 0.0627 \\ 0.0031 & 0.0439 & -0.0555 & 0.0267 \\ -0.3034 & 0.1799 & 0.1336 & 0.0530 \end{bmatrix} \tag{3.64}$$

where each element is obtained by $\beta_{ik} \cdot \beta_{kz} \cdot \beta_{zj}$, namely adding all the effects due to the dependence of firm i preferences on the choices of firm j mediated firstly by all firms $k = 1, \cdots, N$ and secondly by all firms $z = 1, \cdots, N$.

Finally, it is worth noting that if we compare the relative centrality index $\boldsymbol{\sigma}$ and the column vector $B\boldsymbol{u}$ (coming from the row summation of each player's coefficients), there is no correspondence between the outgoing degree of social interaction of a firm and its relative centrality in the network. For instance, a more altruistic, on average, firm can be more central in the network with respect to a less altruistic, on average, firm

$$B\boldsymbol{u} = \begin{bmatrix} 0.29 \\ 0.36 \\ 0.12 \\ 0.51 \end{bmatrix} \text{ and } \boldsymbol{\sigma} = \begin{bmatrix} 0.2702 \\ 0.2150 \\ 0.2813 \\ 0.2335 \end{bmatrix} \tag{3.65}$$

For example, firm 4 is more altrustic, on average, than firm 2 ($(B\boldsymbol{u})_4 > (B\boldsymbol{u})_2$) and its relative centrality index is bigger than the one of firm 2 ($\sigma_4 > \sigma_2$). Therefore, firm 4 obtains more profits than firm 2 even if it exerts a more altruistic behaviour on average.

Finally, vector $(I + B)^{-1}\boldsymbol{u}$ accounts for the effects of the heterogeneity in the distribution of weights. Note that in addition to $\boldsymbol{\xi}$, we can introduce vector $\boldsymbol{\chi}^T = \boldsymbol{u}^T(I+B)^{-1}$, namely the vector colleting the *aggregate effect due to any order dependence of social preferences of the industry on firm i*. Vector $\boldsymbol{\chi}$ actually corresponds to the Friedkin-Johnsen centrality measure proposed in [18]. From Proposition 5, we have that vector $\boldsymbol{\chi}$ has no direct influence on the characterization of Nash equilibria. However, in the next Section, we will show that it plays a key role on the way the Nash equilibrium changes as the structure of firms' interactions varies. It is then interesting to point out the differences and the common elements characterizing $\boldsymbol{\xi}$ and $\boldsymbol{\chi}$. Firstly, they are both measures related to the equilibrium and that are consequences of the any order effects that arise in the structure of social interactions. Vector $\boldsymbol{\xi}$ is a consequence of the effects due to the importance that, in its social preferences, each firm gives to the material payoff of its competitors. The market performance of a firm i can result improved or hindered by the direct and indirect influence on the equilibrium choices of the material payoffs of competitors, so vector $\boldsymbol{\xi}$ encompasses the *relevance* that firm i has acquired at the equilibrium thanks to the network of social interactions. Conversely, each element in vector $\boldsymbol{\chi}$ is a consequence of the effects that the corresponding firm exerts, through its material payoff, on the utility of all its competitors. This means that $\boldsymbol{\chi}$ represents the vector of *influences* of each firm in the network of social interactions. We stress that elements of $\boldsymbol{\xi}$ are positive, while those of $\boldsymbol{\chi}$ can be negative.

3.5 Nash equilibria in particular network structures

In this last Section of the Chapter we reconsider the particular structures introduced in Section 2.3 in light of the results of Proposition 5. In most cases, we will focus on the ordering of firms at the equilibrium as a consequence of the weights distributions.

We start studying the case of uniform weights. Thanks to the simplicity of this case, we report the analytical expressions of the elements characterizing the structure of social interaction at the equilibrium and we provide the explicit expression of the equilibrium for two relevant inverse demand functions, namely the (piecewise) linear and the isoelastic ones.

Proposition 6. *Let $B = \beta(U - I)$ where U is the $N \times N$ matrix whose elements are equal to 1, and I is the $N \times N$ identity matrix and let $\frac{-1}{N-1} < \beta < 1$. We then have $\tilde{B} = (I + B)^{-1} = \tilde{\beta}_1 I + \tilde{\beta}_2 U$, where $\tilde{\beta}_1 = \frac{(N-2)\beta+1}{-(N-1)\beta^2+(N-2)+1}$ and $\tilde{\beta}_2 = \frac{-\beta}{-(N-1)\beta^2+(N-2)+1}$, to which corresponds*

$$\xi_i = \frac{1}{(N-1)\beta + 1}, \sigma_i = \frac{1}{N}, \ i = 1, \ldots, N,$$

and

$$\mu = \frac{N}{(N-1)\beta + 1}.$$

If the inverse demand function is $p(Q) = \max\{a - bQ, 0\}$, for marginal cost we have $c < a$ and the capacity limit is suitably close to a/b we have the unique Nash equilibrium is internal and has

$$q_i^* = \frac{(a-c)}{b\,(\beta\,(N-1)+1) + Nb}, \ Q^* = \frac{N\,(a-c)}{b\,(\beta\,(N-1)+1) + Nb}, \ i = 1, \ldots, N$$

If the inverse demand function is $p(Q) = 1/Q$, we have the unique internal Nash equilibrium chatacterized by

$$q_i^* = \frac{(N-1)(1-\beta)}{N^2 c}, \ Q^* = \frac{(N-1)(1-\beta)}{Nc}, \ i = 1, \ldots, N$$

We stress the fact that, if we consider a model without interdependence of preferences ($\beta = 0$) we obtain the exact equilibrium quantities of the model with homogeneous costs function and isoelastic demand function of Puu [34](in the case of a duopoly, see eq. (2.14)) and of Ahmed and Agiza [1](in the case of n competitors, see eq. (2.15))

Indeed, in such a simplified framework, all firms are identical at the equilibrium. However, as already noticed in Section 2.3, such scenario allows us to show that the proposed model can represent all the possible configurations, in terms of competitiveness degree, ranging from the monopolistic limit to the competitive one. We can notice that, for each N, μ is a decreasing function of β and is equal to 1 when $\beta \to 1$ and equal to $+\infty$ when $\beta \to -\frac{1}{(N-1)}$.
When $N = 5$, the matrix $\tilde{B}$ has the following form

$$(I+B)^{-1} = \begin{bmatrix} \tilde{\beta}_1 & \tilde{\beta}_2 & \tilde{\beta}_2 & \tilde{\beta}_2 & \tilde{\beta}_2 \\ \tilde{\beta}_2 & \tilde{\beta}_1 & \tilde{\beta}_2 & \tilde{\beta}_2 & \tilde{\beta}_2 \\ \tilde{\beta}_2 & \tilde{\beta}_2 & \tilde{\beta}_1 & \tilde{\beta}_2 & \tilde{\beta}_2 \\ \tilde{\beta}_2 & \tilde{\beta}_2 & \tilde{\beta}_2 & \tilde{\beta}_1 & \tilde{\beta}_2 \\ \tilde{\beta}_2 & \tilde{\beta}_2 & \tilde{\beta}_2 & \tilde{\beta}_2 & \tilde{\beta}_1 \end{bmatrix} \tag{3.66}$$

with $\tilde{\beta}_1 = \frac{3\beta+1}{-4\beta^2+3\beta+1}$ and $\tilde{\beta}_2 = \frac{-\beta}{-4\beta^2+3\beta+1}$, which provides

$$\xi_i = \frac{1}{4\beta+1} \text{ and } \mu = \frac{5}{4\beta+1} \tag{3.67}$$

The plot of function $\mu(\beta)$ in the case of $N = 5$ is reported in Figure 3.8. The previous example guarantees that there exists at least a network of social interaction for which the (aggregate) equilibrium is characterized by a given $\mu \in (1, +\infty)$.

Now we focus on the case of constant outgoing degrees. The explicit expressions of elements of $(I+B)^{-1}$, $\boldsymbol{\xi}$ and $\boldsymbol{\chi}$ and of μ are quite involved, so are not reported and can be found in Appendix. What is relevant in the present scenario is the ordering at the equilibrium of such quantities. To this end, without loss of generality, we assume that firms are ordered from the most spiteful/least altruistic to the least spiteful/most altruistic one.

Proposition 7. *Let $\beta_{ij} = \beta_i$ for $1 \leq i, j \leq N$, with $i \neq j$ such that the corresponding matrix B satisfies Assumptions 1 and 3. Moreover, assume that $\beta_i \leq \beta_j$ for $1 \leq i < j \leq N$. We then have*

$$\xi_r \geq \xi_s,\ \sigma_r \geq \sigma_s,\ \chi_r \leq \chi_s,\ \textit{for } 1 \leq r < s \leq N.$$

Consequently, we indeed have $q_r^* \geq q_s^*$ for $1 \leq r < s \leq N$.

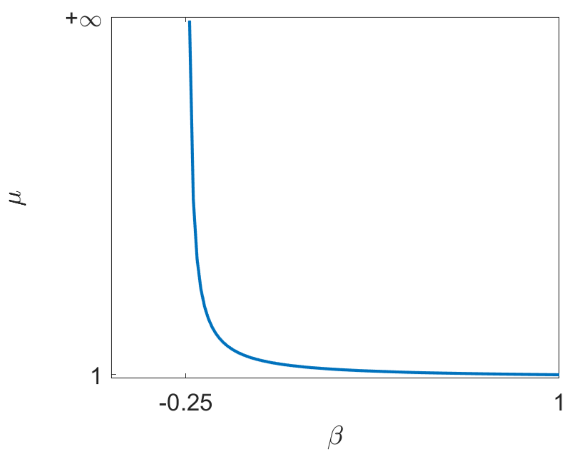


Figure 3.8: μ is a decreasing function of β and is equal to 1 when $\beta \to 1$ and is equal to $+\infty$ when $\beta \to -\frac{1}{(N-1)}$

Proposition 7 shows that more spiteful the firm, the more central in the industry is and therefore the higher is the market share it owns. Moreover, more spiteful the firm is, the less influence it exerts on the competitors.

Now we focus on the case of constant ingoing degrees. Also in this case, we leave the explicit expressions of elements of $(I+B)^{-1}, \boldsymbol{\xi}$ and $\boldsymbol{\chi}$ and of μ to the Appendix and we focus on the ordering of such elements at the equilibrium, assuming again that firms are ordered from that with the smallest ingoing degree to that with the largest one.

Proposition 8. *Let $\beta_{ij} = \beta_j$ for $1 \leq i, j \leq N$, with $i \neq j$ such that the corresponding matrix B satisfies Assumptions 1 and 3. Moreover, assume that $\beta_r \leq \beta_s$ for $1 \leq r < s \leq N$. We then have*

$$\xi_r \leq \xi_s, \ \sigma_r \leq \sigma_s, \ \chi_r \geq \chi_s, \ \textit{for } 1 \leq r < s \leq N.$$

Consequently, we indeed have $q_r^* \leq q_s^*$ for $1 \leq r < s \leq N$.

Proposition 8 shows that the more negatively the firm is taken into account on average in the opponents' utilities, the less central in the industry is and therefore the lower is the market share it owns. Moreover, the more negatively the material payoff of the firm influences, on average, the utility of the competitors, the more influence it exerts on the industry.

Proposition 9. *At the equilibrium firms produce the same amount of good if and only if they have the same centrality index or, equivalently, $\sum_{j=1, i\neq j}^{N} \beta_{ij} = \beta$ for each i*

Proposition 9 precises under which condition we have a "homogeneous" equilibrium scenario, which actually corresponds to the equilibrium scenario of game Γ_0. If vector $B\boldsymbol{u}$, which collects the overall outgoing degrees of social interaction, has identical elements, then the behavior of firms at the equilibrium is homogeneous, independently of any possibile "local" heterogeneity in the distribution of weights β_{ij}. We stress that the previous proposition can not be generalized, in the sense that, as also shown in Example 5, vectors $B\boldsymbol{u}$ do not provide in the general case sufficient information to draw conclusions about the behavior of firms at the equilibrium.

3.6 Conclusions

In this Chapter we studied the role of preference interdependence on the resulting properties of the Nash equilibrium. The first effect of introducing preference interdependence into the model is to alter the degree of strategic interaction between two firms. In such a way an altruistic firm optimally responds to a change in the strategy of one of its opponents in a less aggressive interaction, while the opposite occurs for a spiteful firm. If firm i is altruistic toward firm j, the strategic substitutability characterizing q_i with respect to q_j in Γ_0 is reinforced, while it is weakened if firm i is spiteful toward firm j, and in this latter case the kind of strategic interaction can possibly turn into strategic complementarity.

To completely understand how best response mechanism of a firm affects the equilibrium in the presence of interdependence of preferences the analysis had to be extended to the n possible degrees of interdependence effects between two given firms. For instance, we had to consider also the effect of a change on the best response of any intermediary agents which in turn influences the best response of the reference player. Interestingly enough, we noticed that, although the first order effect is generally the most important on the best response of a given firm, it may happen that high order effects are, instead, the main components of a response, that in same case, can be counterintuitive if looking at the solely structure of interdependence of preferences provided by matrix B. In addition, among the high order effects, we take into account n-th order feedback effects on each firm.

The effect is that the Nash equilibrium is strongly conneted in terms of elements related to the social interaction. With respect to this, a fundamental role is played by the vector $\boldsymbol{\xi}$ and scalar μ. The former one is determined by the distribution of weights β_{ij} and provides the centrality, or *relevance*, measure of each firm in the network of social interactions and ultimately determines the market share of each firm. The latter one is a key element characterizing the aggregate equilibrium and encompasses the degree of competitiveness that characterizes an oligopoly with interdependent preferences.

Through Poposition 5 we showed how the internal Nash equilibrium can be expressed in a simple way in terms of the Bonacich index ($\boldsymbol{\xi}$) and the degree of competitiveness (μ) of the market. In addition, we introduced the vector $\boldsymbol{\chi}$, whose elements quantify the aggregate effect due to any order dependence of social preferences of the industry on a given firm and which represents the level of *influence* a given firm can exert upon the industry. Vector $\boldsymbol{\chi}$ corresponds to the Friedkin-Johnsen centrality measure proposed in [18]. Although vector $\boldsymbol{\chi}$ has no direct influence on the characterization of Nash equilibria, we will show in the next Chapter how it plays a key role on the way the Nash equilibrium changes as the structure of firms' interactions varies.

To this end, we made use of the simplified network structures introduced in Section 2.3 and that, in the last Section of the Chapter, we reconsidered in light of the results of Proposition 5. These particular structures, allowed us to introduce an ordering among the producers at the equilibrium, characteristic that will help the comparative static analysis of the next Chapter.

Chapter 4

Comparative statics of the Nash equilibrium

4.1 Introduction

The previous Chapter have highlighted the crucial role played by the network of social interactions for the characterization of the Nash equilibrium of game Γ. Given the homogeneity in the market components (i.e. demand and cost functions) and recalling the comments on Proposition 5, we have shown how internal equilibrium is actually characterized in terms of elements related to social interaction (the vector of centrality measures $\boldsymbol{\xi}$ and the related measure μ of the degree of competitiveness encompassed in Γ) and elements related to market interaction (the inverse demand function). In particular, the aggregate output level at the equilibrium depends on elements related to both market and social interaction, while the way the industry performance is distributed among firms just depends on the structure of interdependent preferences. Besides these elements, in the previous Chapter we introduced the degree of influence (χ_i) that firm i exerts in the network of social interactions and which corresponds to the Friedkin-Johnsen centrality measure. In this Chapter we will introduce another measure related to the social interaction structure, which was initially proposed by Ballester et al. in [6], that is the intercentrality measure ($\boldsymbol{\rho}$) which identifies in a network the player providing the largest contribution to the aggregate outcome, namely the player who's removal from the network would lead to the largest desruptive effect to the collectivity performance. Although $\boldsymbol{\chi}$ and $\boldsymbol{\rho}$ have no direct influence on the characterization of Nash equilibrium, they play key roles in the comparative statics analysis proposed in this Chapter, i.e. the way the Nash equilibrium changes as the structure of firms' interactions varies. To this end the main element of investigation is the weigth matrix B. The twofold goal of this Chapter is to understand how a change in the social interaction structure of a single player influences the outcome of the player itself and how a change in the interaction structure as a whole affects the collective outcome. In this sense we will focus on the comparative statics of a local change in the preference interdependence and the economic effects on relevant quantities such as the market share of the single firm. In addition, we will take into account the comparative statics of a global change in the preference interdependence and the effect on the industry as a whole in terms of profits.

4.2 Intercentrality measure

In the Chapter 2 we have shown that Nash equilibria can be characterized in terms of the Bonacich centrality measure $\boldsymbol{\xi}$, which quantifies the relevance that a firm has from being in the network of social interaction. It indeed depends on the way the preferences of firm i directly depend on the material payoff of its competitors, but it can be significantly altered by the indirect effects of other firms' preferences structure. The vector of centrality measures has a twofold descriptive power. Firstly, the distribution of centrality measures determines the ordering of firms with respect to their market share, describing how much a firm is dominant inside the market. Moreover, aggregating all the ξ_i we are able to quantify the degree of competitiveness, as it indicates where the equilibrium production of the industry places between the monopolistic and the competitive limit.

However, the point of view adopted to determine the Bonacich centrality measure can be reversed. Instead of considering the overall equilibrium effect arising from the way in which firms take into account their competitors in their preferences, we can focus on the overall equilibrium effect from the way in which firms are taken into account by their competitors. This is described by the Friedkin-Johnsen centrality measure $\boldsymbol{\chi}$, which quantifies the influence that a firm has from being part of the network of social interaction. Such measure does not directly determines the equilibrium performance of firms and their ordering with respect to their relevance is in general independent of that with respect to their influence. In fact, both the two extremal situations can occur. If we consider a symmetric network of social interaction, we have that the two orderings coincide, but on the contrary, as shown from some of the examples of Section 2.3, it is even possible that the two orderings are reversed, with most influential firms being those least central.

To summarize, the Bonacich centrality of a firm quantifies the benefits/disadvantages arising from that connections that a firm directly or indirectly has in the network of social interaction and by the (altruistic or spiteful) kind of such interactions. The Friedkin-Johnsen centrality measure quantifies the influence that a firm i exerts on all the other firms as a consequence of the direct or indirect social connections that they have with firm i. It is clear that each measure describes the role of a firm in the network of social interaction from a different point of view. What is the outcome of the combination of both points of view?

In order to answer this question, we draw our attention to the *intercentrality* measure, which was introduced by Ballester et al. in [6]. The intercentrality measure has been proposed to identify the player providing in a network the largest contribution to the aggregate outcome, namely the player who's removal from the network would lead to the largest disruptive effect to the collectivity performance. For a firm i, it is defined as the sum of firm i relevance (i.e. Bonacich centrality) and of i's contributions to the relevance of all the other firms. For a firm $j \neq i$, such contribution can be quantified by supposing to remove player i from the network (i.e. by setting $\beta_{ij} = \beta_{ji} = 0$ for any $j = 1, \dots, N$) and by evaluating the difference between the centrality measure achieved by player j when player i is in the network and when player i is not in the network. Let B_{-i} be the network obtained from B by removing any interaction involving firm i and let $\xi_j(B)$ and $\xi_j(B_{-i})$ be the Bonacich centrality of firm j in networks B and B_{-i}. In the remainder of the Chapter we will restrict to situations in which $(I+B)^{-1}\boldsymbol{u}$ and $(I+B_{-i})^{-1}\boldsymbol{u}$ both consist of nonnegative elements[1]. The intercentrality of a firm $i = 1, \dots N$ is then defined by

$$\rho_i = \xi_i(B) + \sum_{j=1, j\neq i}^{N} (\xi_j(B) - \xi_j(B_{-i})), \tag{4.1}$$

and allows ordering players with respect to the contribution they exert toward the whole set of players. The player with largest ρ_i is usually addressed as the *key player*.

In [6] the intercentrality measure is introduced for a symmetric matrix[2] whose elements correspond to

$$\rho_i = \frac{\xi_i^2}{\tilde{\beta}_{ii}}. \tag{4.2}$$

In the next proposition we provide a new characterization of such measure for a general network described by matrix B.

Proposition 10. *Let B be a matrix that satisfies Assumptions 1 and 3 and for which vectors $(I+B)^{-1}\boldsymbol{u}$ and $(I+B_{-i})^{-1}\boldsymbol{u}$ are nonnegative. Then*

$$\rho_i = \frac{\chi_i \xi_i}{\tilde{\beta}_{ii}}, \ i = 1, \dots N. \tag{4.3}$$

The expression of ρ_i indeed coincides with that (4.2) when B is symmetric, as in such case we have $\chi_i = \xi_i$, namely the relevance of a firm in the network is identical to its influence. The

[1] In the general case, the identities proved along this Chapter are still valid provided that we consider also centrality measures with negative elements. However, in this case the connection between the Bonacich index and the Nash equilibrium no more holds.

[2] Rephrased to the present model, the setting in [6] corresponds to a globally symmetric spiteful scenario (i.e. in which the preferences of each firm negatively depend on the material payoff of all their competitors and $\beta_{ij} = \beta_{ji}$).

expression of ρ_i provided by (4.3) is then in line with that in [6], but by removing the symmetry assumption on matrix B (and hence the identity between relevance and influence of a firm), we obtain a more neat social interpretation of the expression of ρ_i. As in (4.2), from (4.3) it is immediately evident that, ceteris paribus, if a player has a larger Bonacich centrality, it will contribute to a larger extent to the overall centrality of all the players. However, differently from (4.2), in (4.3) it is explicitly specified the role of the influence of player i in the network. The more a player is influential on the collectivity (net of the feedback effect encompassed in $\tilde{\beta}_{ii}$), the more its "achieved centrality" (due to its role in the network) will contribute to the overall centrality of the other players.

We then have that even if a player is central in the network but it is just a few influential, its large centrality will minimally benefit the overall centrality of players. The same occurs when a player has a large influence on the collectivity but it has a small centrality: the resulting contribution to the Bonacich centrality of the collectivity is small. Finally we stress that since χ_i can be also negative, we have that a player can have a negative intercentrality measure. This is perfectly understandable: player i provides a negative contribution to the Bonacich centrality of the collectivity, i.e., on average, the other players would benefit from a removal of the player from the network.

The economic effects are a straightforward consequence of those social: since the intercentrality represents the contribution to the Bonacich centrality of all the firms, which in turns determines the market share and the degree of competitiveness, it is evident that understanding how ρ_i changes is essential to study the way the equilibrium is affected by the social interaction structure.

We show a possible scenario with respect to the distributions of the different measures in the next example.

Example 6. Distribution of centrality, influence and intercentrality

Let consider the 5×5 matrix of negative weights

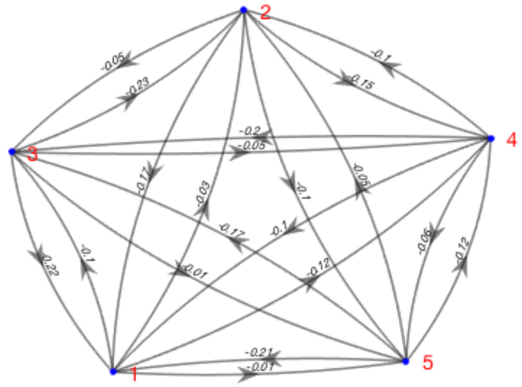


Figure 4.1: Graphical representation of the network described by weight matrix B_1 in (4.4)

$$B = \begin{bmatrix} 0 & -0.03 & -0.1 & -0.12 & -0.01 \\ -0.17 & 0 & -0.05 & -0.15 & -0.1 \\ -0.22 & -0.23 & 0 & -0.05 & -0.01 \\ -0.1 & -0.1 & -0.2 & 0 & -0.06 \\ -0.21 & -0.05 & -0.17 & -0.12 & 0 \end{bmatrix} \tag{4.4}$$

which gives rise to the network shown in Figure 4.1. We report the matrix $(I+B)^{-1}$ of the aggregate effects due to any order dependence of social preferences

$$(I+B)^{-1} = \begin{bmatrix} 1.0674 & 0.0822 & 0.1461 & 0.1513 & 0.0294 \\ 0.2604 & 1.0683 & 0.1432 & 0.2135 & 0.1237 \\ 0.3086 & 0.2741 & 1.0806 & 0.1381 & 0.0496 \\ 0.2134 & 0.1782 & 0.2602 & 1.0758 & 0.0871 \\ 0.3152 & 0.1387 & 0.2528 & 0.1950 & 1.0312 \end{bmatrix} \tag{4.5}$$

We also report the vector of relative centrality measures ($\boldsymbol{\sigma}$), the row vector of the aggregate effects due to any order dependence of social preferences of the industry on each firm ($\boldsymbol{\chi}^T$) and the row vector of intercentralities ($\boldsymbol{\rho}^T$), respectively

$$\boldsymbol{\sigma} = \begin{bmatrix} 0.1662 \\ 0.2036 \\ 0.2083 \\ 0.2043 \\ 0.2176 \end{bmatrix}, \boldsymbol{\chi} = \begin{bmatrix} 2.1651 \\ 1.7414 \\ 1.8829 \\ 1.7737 \\ 1.3210 \end{bmatrix} \text{ and } \boldsymbol{\rho} = \begin{bmatrix} 2.9947 \\ 2.9490 \\ 3.2252 \\ 2.9920 \\ 2.4761 \end{bmatrix} \tag{4.6}$$

We notice that, in general, the ordering of the relative centralities, the ordering of the influences exerted by each firm and the ordering of intercentralities do not necessarily coincides. For instance, given the previous network, firm 5 has the highest centrality measure but it is firm 3 the one with the highest intercentrality measure.

4.3 Comparative statics

The goal of this Section is to investigate how a change in the structure of interdependent preferences (i.e. a change in the weight matrix B) affects the equilibrium. Recalling Proposition 5 and subsequent comments, it is clear how internal equilibria are actually characterized in terms of elements related to social interaction (the vector of centrality measures $\boldsymbol{\xi}$ and the related measure μ of the degree of competitiveness) and elements related to market interaction (the inverse demand function). In particular, the aggregate output level at the equilibrium depends on elements related to both market and social interaction, while the way the industry performance is distributed among firms just depends on the structure of interdependent preferences.

The comparative statics of internal equilibria must then be studied in terms of elements related to $\tilde{B} = (I + B)^{-1}$, to which we indeed have to add the characterization due to the inverse demand function. We start focusing on the role of the social interaction structure. Firstly, we investigate how measures ξ_i, χ_i and ρ_i are affected by an increase of a weight characterizing the social preferences of firm i.

Proposition 11. *Let B be a matrix satisfying Assumptions 1 and 3 and to which corresponds an internal Nash equilibrium and be $1 \leq i, j \leq N$ with $i \neq j$.*
If we linearly increase coefficient β_{ij}, then ξ_i decreases, χ_i increases provided that $\chi_i \tilde{\beta}_{ji} < 0$ and ρ_i decreases provided that $\rho_i > 0$.

In the previous proposition we are assuming that firm i becomes less spiteful or more altruistic toward firm j. The result regarding ξ_i is unambiguous: the Bonacich centrality of firm i always decreases.
The behavior of χ_i is determined by the kind of influence that firm i has toward the overall industry and toward firm j. If they are of the same kind, then the overall influence of firm i toward the industry decreases as the direct influence that firm j has on firm i increases.
Finally, if firm i exerts a positive effect on the centrality of all firms, if it becomes less spiteful or more altruistic toward another firm then such effect will decrease.
We stress that if firm i becomes more spiteful or less altruistic toward firm j, we have the opposite behaviors, so that the centrality of firm i increases, its influence increases provided that firm i has, toward the overall industry and toward firm j, the same kind of influence and finally the negative effect on the centrality of the overall industry will decrease.

In the next propositions we investigate what happens to the share σ_i of a given firm and the degree of competitiveness μ.

Proposition 12. *Let B be a matrix that satisfies Assumptions 1 and 3 and to which corresponds an internal Nash equilibrium and be $1 \leq i, j \leq N$ with $i \neq j$.*
If we linearly increase coefficient β_{ij}, then

$$\sigma_i' = \xi_j(\xi_i \chi_i - \tilde{\beta}_{ii}\mu) \tag{4.7}$$

and the market share σ_i increases provided that

$$\rho_i = \frac{\xi_i \chi_i}{\tilde{\beta}_{ii}} > \mu. \tag{4.8}$$

The first part of Proposition 12 focuses on what happens when the structure of interaction of a given firm i changes, due to an increase in one of the weights through which the preferences of such firm depend on the material payoff of another firm. The main result is encompassed in condition (4.8), which clarifies under what conditions the market share of a given firm increases if its spitefullness decreases or its altruistic behavior becomes more strong. We stress that in (4.8) we find involved all the centrality measures that characterize the outcome of the preference interaction structure at the equilibrium, namely the relevance of firms $\boldsymbol{\xi}$ (and consequently the market share $\boldsymbol{\sigma}$ and the degree of competitiveness μ), the influence $\boldsymbol{\chi}$ and the intercentrality $\boldsymbol{\rho}$. The behavior of the market share of a given firm on increasing β_{ij} basically depends on a comparison of such measures through a simple relation.

From condition[3](4.8), we can infer that if the overall influence degree is negative ($\chi_i < 0$), then the effect of an increase of altruism (or a decrease in spitefulness) will result in a decrease of the market share of firm i inside the market.

Conversely, if $\chi_i > 0$, then the market share of firm i can improve as β_{ij} increases.

Condition (4.8) is very clear: increasing β_{ij} can result in a strengthening of the position of firm i in the market only provided that its overall contribution to the equilibrium of the industry is large enough. Moreover, the threshold at which this occurs is larger as the degree of competitiveness is higher. This means that if the aggregate equilibrium of game Γ is suitably close to the monopolistic limit, it is more likely that the market share of firm i improves through an increase in the degree of altruism (or a decrease in the degree of spitefullness)

From (4.8), the joint effect of the overall influence and of the relevance of firm i has to be suitably large. Ceteris paribus, it is more likely for a firm with a large (relative) centrality measure than for a firm with a small (relative) centrality measure to have an improvement in the equilibrium performance with a more altruistic behavior. We stress that condition (4.8) is independent on j, namely the increase of a weight that defines the social preferences of firm i can be toward any firm.

Conversely, the centrality measure of firm j determines the speed of increase of the market share of firm i, as evident from (4.7), in which ξ_j is a multiplicative coefficient of the positive term within brackets. In the opposite situation, i.e. when condition (4.8) is violated, the role of the centrality measure ξ_j conversely has a negative effect on the change of the market share of firm i. In fact, in such case we have that the market share of firm i decreases faster as the firm is more central.
To summarize we can say that the more the overall industry gives relevance at the equilibrium to firm i through the network of social interdependency of preferences, the more an increase of altruism (or a decrease of spitefulness) can be convenient to firm i, and vice-versa. In fact, in this latter case, if condition (4.8) is violated, an increase in the market share σ_i realizes if firm i reduces any weight describing its network of social interdependences, i.e. if it becomes less altruistic or more spiteful.

Even if in the model under consideration the distribution of weights is kept exogenous and the case in which firms can decide or change their social preferences is not under investigation, the previous considerations open interesting considerations in view of a possible endogenization and

[3]We recall that elements of ξ, the value of μ and the diagonal elements of matrix $\tilde{B}$ are always positive.

evolution of coefficients β_{ij}. Proposition 12 shows that a firm can improve its performance at the equilibrium if it changes its social preferences in the following way: when the degree χ_i with which the industry, as a whole, takes into account its performance is sufficiently high, the firm can improve its performance by increasing the weight it places on the material payoff of its competitors, while an improvement is obtained by reducing β_{ij} when χ_i is low or even negative. It is easy to read in the previous considerations a first, very prototypical and stylized, justification for a "tit-for-tat" dynamical way to adjust social preferences.

Now we investigate the effect that an increase of the weight another firm places on the material payoff of firm i has on the equilibrium performance of firm i.

Proposition 13. *Let B be a matrix that satisfies Assumptions 1 and 3 and to which corresponds an internal Nash equilibrium and be $1 \leq i, j \leq N$ with $i \neq j$.*
If we linearly increase coefficient β_{ji}, then

$$\sigma_i' = \xi_i(\xi_i \chi_j - \tilde{\beta}_{ij}\mu) \tag{4.9}$$

and the market share σ_i increases provided that

$$\chi_j \xi_i > \frac{\tilde{\beta}_{ij}}{\mu}. \tag{4.10}$$

The condition (4.10) under which σ_i increases is structurally very similar to that related to the first part of the proposition, and again results in a comparison of χ_j, ξ_i and μ. Hoverer, in this case, the discriminant is how much influential is firm j in the network of social interactions. The more firm j is influential, the more the weight that such firm gives to the material payoff of firm i will positively affect the equilibrium performance of firm i.

In line with (4.8), for the validity of condition (4.10), also in the present case the greater is the centrality measure of firm i, the smaller is the level of influence that must characterize firm j. Finally, in line with (4.7), from (4.9) we have that the greater is the centrality measure of firm i, the faster will increase the market share of firm i when (4.10) holds.

To summarize we can say that the more the overall industry gives relevance at the equilibrium to firm j through the network of social interdependency of preferences, the more an increase of altruism (or a decrease of spitefulness) of firm j toward firm i can be convenient to firm i, and vice-versa. In fact, once more, in this latter case, if condition (4.10) is violated, an increase in the market share σ_i realizes if firm j reduces the weight through which it is linked to firm i, i.e. if firm

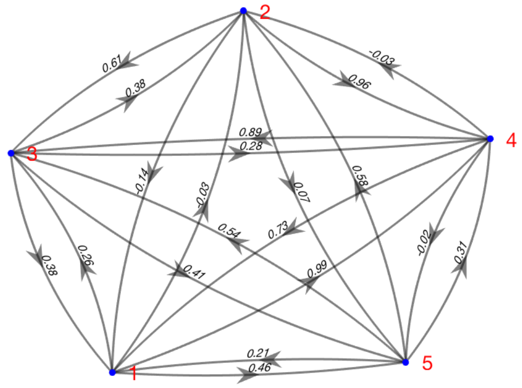


Figure 4.2: Graphical representation of the network described by weight matrix B in (4.11)

j becomes less altruistic or more spiteful toward firm i. We deepen the description of the results of Proposition 13 in the next example.

Example 7. Comparative statics: a general case

Let consider the following 5×5 matrix

$$B = \begin{bmatrix} 0 & -0.03 & 0.26 & 0.99 & 0.46 \\ -0.14 & 0 & 0.61 & 0.96 & 0.07 \\ 0.38 & 0.38 & 0 & 0.28 & 0.41 \\ 0.73 & -0.03 & 0.89 & 0 & -0.02 \\ 0.21 & 0.58 & 0.54 & 0.31 & 0 \end{bmatrix} \tag{4.11}$$

which gives rise to the network shown in Figure 4.2 We report the matrix that encompasses the aggregate effects due to any order dependence of social preferences, namely

$$(I+B)^{-1} = \begin{bmatrix} 2.4124 & 0.2617 & 2.9176 & -2.7191 & -2.3786 \\ 1.7755 & 1.1569 & 2.5497 & -2.9615 & -2.0023 \\ -1.0866 & -0.3358 & 0.1532 & 1.2049 & 0.4846 \\ -0.7550 & 0.1308 & -2.2194 & 1.8450 & 1.2850 \\ -0.7156 & -0.5852 & -1.4862 & 1.0661 & 2.0008 \end{bmatrix} \tag{4.12}$$

We also report the column vector of the absolute centrality measures ($\boldsymbol{\xi}$) and of the relative centrality measures ($\boldsymbol{\sigma}$), respectively

$$\boldsymbol{\xi} = \begin{bmatrix} 0.4940 \\ 0.5182 \\ 0.4204 \\ 0.2864 \\ 0.2799 \end{bmatrix} \text{ and } \boldsymbol{\sigma} = \begin{bmatrix} 0.2471 \\ 0.2593 \\ 0.2103 \\ 0.1433 \\ 0.1400 \end{bmatrix} \tag{4.13}$$

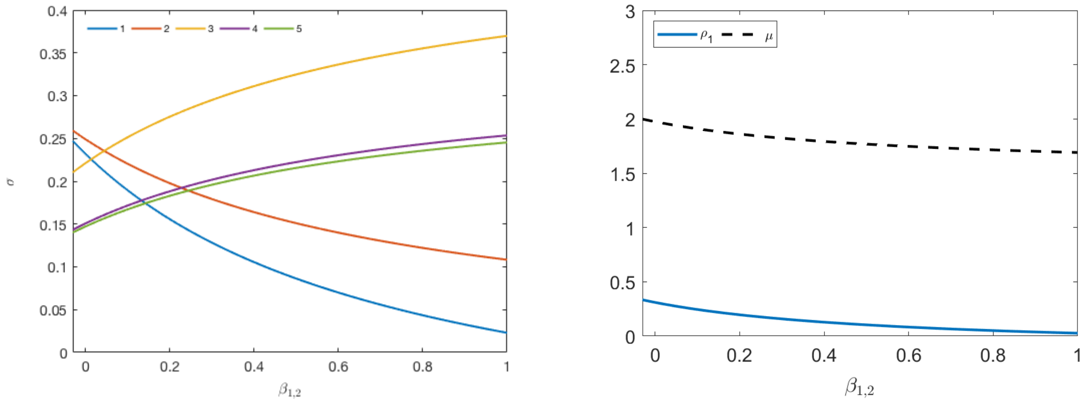
Finally, we report the row vector ($\boldsymbol{\chi}^T$) of aggregate effects due to any order dependence of social preferences of all the firms in the industry on each competitor

$$\boldsymbol{\chi}^T = \begin{bmatrix} 1.6307 & 0.6284 & 1.9148 & -1.5646 & -0.6105 \end{bmatrix} \tag{4.14}$$

The goal of this example is to show the possible behaviors of market share σ_i of a firm when the firm gives more relevance to the material payoff of one of its competitors, and when one of its competitors increases the relevance given to the i-th firm material payoff.

In general, the relative centrality measure of firm i ($\sigma_i = \frac{\xi_i}{\sum_{i=1}^N \xi_i}$) decreases if any of its social weights β_{ij}, with $j \neq i$, tends towards 1. In other words, acting more altruistically towards any of its opponents tends to disadvantage the firm in terms of centrality. Conversely, if any of its opponent $j \neq i$ tends to increase the weight of the material payoff of firm i into its utility ($\beta_{ji}, j \neq i$), firm i's centrality measure will increase.

As an example of this, we let coefficient β_{12} increase in the interval $[-0.03, 1]$ to see what effect this may have, initially, on the relative centrality measure of firm 1 (blue line) and then on all its opponents, *in primis* on firm 2 (red line). The reason we chose to increase this particular coefficient is that it allows for a greater interval of variation compared to the other coefficients of firm



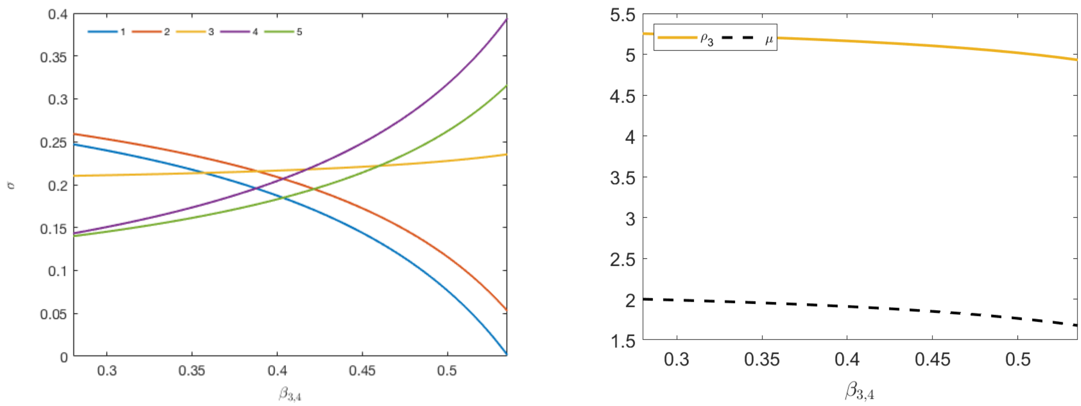
(a) (b)

Figure 4.3: Figure 4.3(a) shows firms' market shares given the variation in the coefficient β_{12}. In Figure 4.3(b) we report the monotonicity relation in terms of ρ_1 and increasing values of the coefficient β_{12} (blue line). Figure 4.3(b) also reports the graph of the degree of competitiveness μ for increasing values of the coefficient β_{12} (dark dashed line).

1. Figure 4.3(a) shows that an increase in the altruism of firm 1 towards one of its opponents (in this case firm 2) has the direct effect to monotonically decrease the market share of firm 1 but also to decrease the market share of firm 2. We then let coefficient β_{34} increase in the interval $[0.28, 1]$ to see what effect this may have, initially, on the relative centrality measure of firm 3 (yellow line) and then on all its opponents, *in primis* on firm 4 (purple line). As before, we chose β_{34} since it allows for a greater interval of variation compared to the other coefficients of firm 3.

We notice that an increase in the altruism of firm 3 towards firm 4 has the direct effect to monotonically increase the market share of firm 3 and the indirect effect to increase the market share of firm 4. Particularly interesting is the fact that for values of β_{34} in the interval $[0.387, 0.417]$ firm 3, which for $\beta_{34} = 0.28$ realized less profits than firm 1 and firm 2, is the firm with the highest profits, in the network.

If $\chi_i > 0$, there exists the possibility that an increase in the altruistic level of firm i towards some of its opponent has the effect to increase the relative centrality of firm i in the network. The higher χ_i is the smaller the initial market share firm i needs to own in order to have a positive effect due to an increase in one of its weights β_{ij}. The more central in the network is firm i, the more is probable that, even for small but positive level of altruism exerted by the industry at the



(a) (b)

Figure 4.4: Figure 4.4(a) shows firms' market shares given the variation in the coefficient β_{34}. In Figure 4.4(b) we report the monotonicity relation in terms of ρ_3 and increasing values of the coefficient β_{34} (yellow line). Figure 4.4(b) also reports the graph of the degree of competitiveness μ for increasing values of the coefficient β_{34} (dark dashed line).

aggregate level towards i, an increase of the level of altruism towards one of its opponents have the effect to increase its centrality in the network. Conversely, the smaller firm i's relative centrality is the more probable, even for high level of aggregate altruism of the industry towards i (χ_i), the effect of a decrease in the centrality of firm i (σ_i) is.
Given the low degree ($\chi_1 = 1.18$), with which the industry as a whole takes into account the performance of firm 3, the increase of its evaluation of the material payoff of firm 2 in its utility has the effect to decrease its centrality in the network. Given the low influential role played by firm 1 in the network, an increase in the coefficient β_{12} also negatively affect the equilibrium performance of firm 2. We stress the fact that the main contribution to the centrality of firm 1 is due to the feedback effect $\tilde{\beta}_{11}$. Instead, given the high degree χ_3, with which the industry as a whole takes into account the performance of firm 3, the increase of its evaluation of the material payoff of firm 4 in its utility has the effect to increase its centrality in the network. Given the influential role played by firm 3 in the network, an increase in the coefficient β_{34} also positively affects the equilibrium performance of firm 4. We stress that in this last scenario the feedback effect on firm 3 is very small ($\tilde{\beta}_{33} \approx 0.15$) .

To show how the role of influence and centrality are both essential to understand comparative statics of the characterization of the market share at the Nash equilibrium, we study the following structures for which the effect of a change in coefficients β_{ij} is unambiguous. Recalling Propositions 7 and 8, when the ordering of firms with respect to influence is the reversed one of that with respect to the centrality.

Proposition 14. *Let B be a matrix that satisfies Assumptions 1 and 3. Assume $\beta_{ij} = \beta_i$ for $i = 1, \ldots, N$ and $i \neq j$ and that q^* is an internal Nash equilibrium. Then σ_i decreases if β_i increases and increases if $\beta_j, j \neq i$ increases.*

Conversely, assume $\beta_{ij} = \beta_j$ for $i = 1, \ldots, N$ and $i \neq j$ and that q^* is an internal Nash equilibrium. Then σ_i increases if β_j increases and decreases if $\beta_i, j \neq i$ increases.

In the former scenario depicted in Proposition 14 we have that if a firm becomes more altruistic or less spiteful, its market share always decreases. This in particular also holds for the most central firm: the reason is that, recalling Proposition 7, it is also the least influential, and the joint effect of them is too small. In the latter scenario, the situation is instead the opposite one.

The next example focuses on a particular situation of the scenario investigated in Proposition 14.

Example 8. Comparative statics of scenarios in Proposition 14
Let consider the 5×5 matrix

$$B = \begin{bmatrix} 0 & -0.17 & -0.17 & -0.17 & -0.17 \\ -0.13 & 0 & -0.13 & -0.13 & -0.13 \\ -0.04 & -0.04 & 0 & -0.04 & -0.04 \\ 0.15 & 0.15 & 0.15 & 0 & 0.15 \\ 0.16 & 0.16 & 0.16 & 0.16 & 0 \end{bmatrix} \tag{4.15}$$

which gives rise to the network shown in Figure 4.5 The main feature of matrix B is that each firm i evaluates the opponents' material payoff the same way in its own utility, either positively or negatively.
We report the matrix $(I + B)^{-1}$ of the aggregate effects due to any order dependence of social

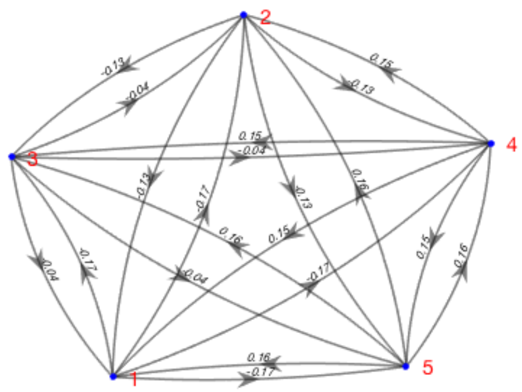


Figure 4.5: Graphical representation of the network described by weight matrix B in (4.15)

preferences

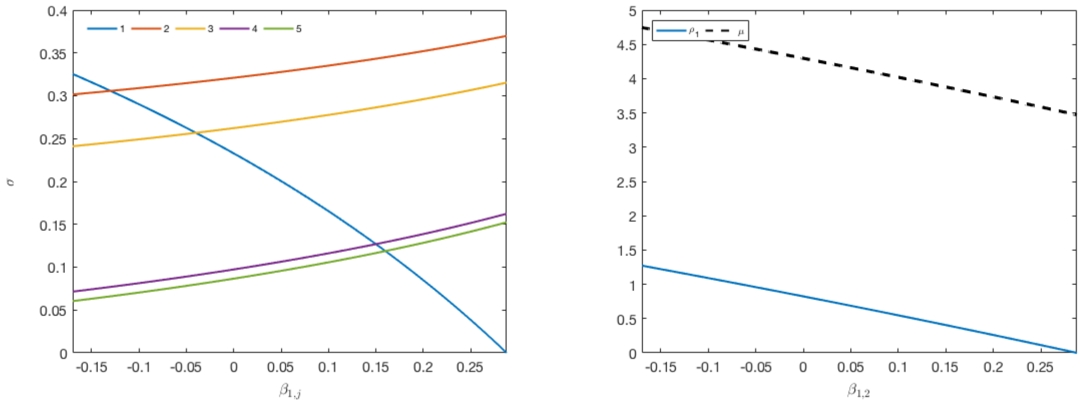
$$(I+B)^{-1} = \begin{bmatrix} 0.9710 & 0.1204 & 0.1308 & 0.1600 & 0.1619 \\ 0.0921 & 0.9803 & 0.1036 & 0.1267 & 0.1282 \\ 0.0308 & 0.0319 & 0.9962 & 0.0424 & 0.0429 \\ -0.1412 & -0.1462 & -0.1589 & 0.9821 & -0.1967 \\ -0.1524 & -0.1578 & -0.1715 & -0.2098 & 0.9782 \end{bmatrix} \tag{4.16}$$

We compute the column vector of the centrality measures $\boldsymbol{\xi}$ and the column vector of relative centrality measures $\boldsymbol{\sigma}$, respectively

$$\boldsymbol{\xi} = \begin{bmatrix} 1.5441 \\ 1.4308 \\ 1.1440 \\ 0.3391 \\ 0.2867 \end{bmatrix} \text{ and } \boldsymbol{\sigma} = \begin{bmatrix} 0.3254 \\ 0.3016 \\ 0.2411 \\ 0.0715 \\ 0.0604 \end{bmatrix} \tag{4.17}$$

and the row vector of the aggregate effects due to any order dependence of social preferences of the industry on each firm $i = 1, 2, \cdots, N$ made by the column summations of matrix $\tilde{B}$

$$\boldsymbol{\chi}^T = \begin{bmatrix} 0.8002 & 0.8285 & 0.9002 & 1.1014 & 1.1145 \end{bmatrix} \tag{4.18}$$



(a) (b)

Figure 4.6: Figure 4.6(a) shows firms' market shares given the variation in coefficient β_{1j}, with $j \neq i$. In Figure 4.6(b) we report the monotonicity relation in terms of ρ_1 and increasing values of the coefficient $\beta_{1,j}$ (blue line). Figure 4.6(b) also reports the graph of the degree of competitiveness μ for increasing values of the coefficient $\beta_{1,j}$ (dark dashed line).

We let coefficients β_{1j}, with $j \neq i$ to increase in the interval $[-0.0725, 0.3]$ to see what effect this may have, initially, on the relative centrality measure of firm 1 (blue line) and then on all its opponents. Firm 1 represents an interesting case since it is, *a priori*, the most central firm in the network ($\xi_1 = \max(\boldsymbol{\xi}) = 0.2871$), given the highest overall outgoing degree of spitefulness in the network ($(B\boldsymbol{u})_1 = \max((B\boldsymbol{u})) = -0.29$), but is the less influential firm in the network ($\chi_1 = \min(\boldsymbol{\chi}) = 0.8002$). The blue line in Figure 4.6(a) shows that a linear increase in the level of altruism exerted by firm 1 towards all firms (β_{1j}) has the direct effect to decrease its centrality in the network and therefore the market share. Note that firm 1 is initially spiteful, and as β_{1j} increases, it becomes less and less spiteful, turning into an altruistic firm on $(0.17, 0.3)$. The market share lost by firm 1 is then redistributed among all its opponents whose profits firm 1 evaluates in its utility the same way.

The interpretation of the inequality $\chi_1 > \frac{\tilde{\beta}_{ii}}{\sigma_i}$ can be facilitated by the introduction of another measure coming from the literature on network. Among the several specification of the centrality measure, we made use of the Bonacich index, namely each row summation of the $\tilde{B}$ matrix, which provides the relevance of each firm in the network, and so its market share. Besides this measure, the Friedkin-Johnsen index is a well known measure of the influence exerted by an agent in the network and corresponds to row vector coming from each column summation of the $\tilde{B}$ matrix.

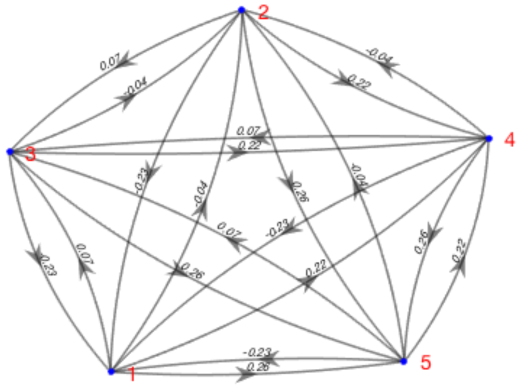


Figure 4.7: Graphical representation of the network described by weight matrix B in (4.19)

In our case the Friedkin-Johnsen index corresponds to vector $\boldsymbol{\chi}$ and measures how much a firm i influences its opponents, throughout the aggregation of quantities referred to the n-th order effects the profits of firm i exert on the preferences of the competitors.

We then consider the 5×5 matrix

$$B = \begin{bmatrix} 0 & -0.04 & 0.07 & 0.22 & 0.26 \\ -0.23 & 0 & 0.07 & 0.22 & 0.26 \\ -0.23 & -0.04 & 0 & 0.22 & 0.26 \\ -0.23 & -0.04 & 0.07 & 0 & 0.26 \\ -0.23 & -0.04 & 0.07 & 0.22 & 0 \end{bmatrix} \tag{4.19}$$

which gives rise to the network shown in Figure 4.7 The main feature of matrix B is that firm i's material payoff are considered with the same weight in each of its opponents' utility, either positively or negatively.

We report the matrix $(I + B)^{-1}$ of the aggregate effects due to any order dependence of social preferences

$$(I+B)^{-1} = \begin{bmatrix} 0.9155 & 0.0211 & -0.0413 & -0.1546 & -0.1926 \\ 0.1212 & 0.9865 & -0.0488 & -0.1828 & -0.2278 \\ 0.1356 & 0.0279 & 1.0207 & -0.2045 & -0.2547 \\ 0.1616 & 0.0332 & -0.0651 & 1.0383 & -0.3037 \\ 0.1704 & 0.0350 & -0.0686 & -0.2570 & 1.0312 \end{bmatrix} \tag{4.20}$$

We compute the column vector of the centrality measures $\boldsymbol{\xi}$ and of market shares $\boldsymbol{\sigma}$

$$\boldsymbol{\xi} = \begin{bmatrix} 0.5481 \\ 0.6483 \\ 0.7250 \\ 0.8644 \\ 0.9111 \end{bmatrix} \text{ and } \boldsymbol{\sigma} = \begin{bmatrix} 0.1483 \\ 0.1754 \\ 0.1961 \\ 0.2338 \\ 0.2465 \end{bmatrix} \tag{4.21}$$

and the row vector of the aggregate effects due to any order dependence of social preferences of the industry on each firm $i = 1, 2, \cdots, N$ made by the column summations of matrix $\tilde{B}$

$$\boldsymbol{\chi}^T = \begin{bmatrix} 1.5043 & 1.1037 & 0.7970 & 0.2394 & 0.0525 \end{bmatrix} \tag{4.22}$$

The $\boldsymbol{\sigma}$ vector shows that firm 1 is the least central while is firm 5 the one to own the largest market share. Looking at the $\boldsymbol{\chi}$ vector, we notice that firm 1 is the most influential while firm 5 is the least one in the network.

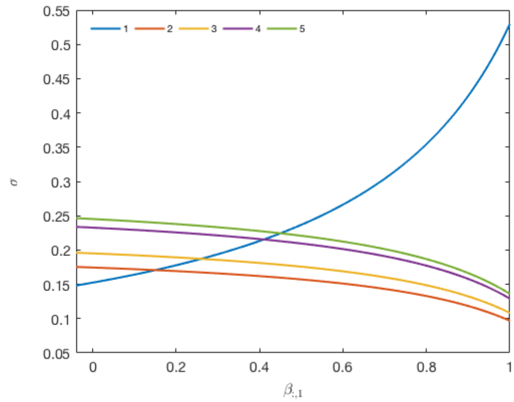
As shown in Figure 4.8(a) by increasing homogeneously the influence of firm 1 over its opponents the market share also increases, to the point that firm 1, from being the least powerful oligopolist in the market, becomes the most central.

The previous pattern is confirmed by looking at firm 5 situation shown in Figure 4.9(a). Increasing the influence of firm 5 into its opponents' utility preserves its leadership in the network.

In general, it is not possible to have monotonicity results in a completely heterogeneous structure. The unique situation is that in which all firms are spiteful, as shown in the next proposition.

Proposition 15. *Assume that all firms are spiteful or selfish with respect to all the other firms, then if $|B_1| \geq |B_2|$ we have $\xi_1 \geq \xi_2$ and $\chi_1 \geq \chi_2$.*

The following example reports a situation described by Proposition 15.



(a)

Figure 4.8: Figure 4.8(a) shows firms' market shares given the variation in coefficients β_{j1}, with $j \neq i$.

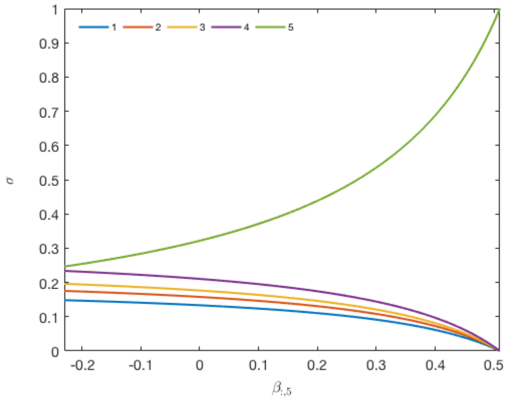
Example 9. Comparison of oligopolies with spiteful firms

Let consider the 5×5 matrix of negative weights

$$B_1 = \begin{bmatrix} 0 & -0.04 & -0.11 & -0.13 & -0.02 \\ -0.18 & 0 & -0.06 & -0.16 & -0.11 \\ -0.23 & -0.24 & 0 & -0.06 & -0.02 \\ -0.22 & -0.11 & -0.13 & 0 & -0.17 \\ -0.21 & -0.05 & -0.17 & -0.12 & 0 \end{bmatrix} \tag{4.23}$$

which gives rise to the network shown in Figure 4.10 The main feature of matrix B_1 is that it describes a scenario in which each firm is spiteful toward each competitor, i.e. each coefficient $\beta_{ij}^{(1)} < 0$, with $i \neq j$, and each firm i evaluates negatively each opponent j's material payoff in its utility.

We report the matrix $(I + B_1)^{-1}$ of the aggregate effects due to any order dependence of social



(a)

Figure 4.9: Figure 4.9(a) shows firms' market shares given the variation in coefficients β_{j5}, with $j \neq i$.

preferences

$$(I + B_1)^{-1} = \begin{bmatrix} 1.1105 & 0.1067 & 0.1634 & 0.1794 & 0.0677 \\ 0.3231 & 1.0874 & 0.1619 & 0.2462 & 0.1712 \\ 0.3634 & 0.3011 & 1.0960 & 0.1722 & 0.0916 \\ 0.3879 & 0.2082 & 0.2400 & 1.1254 & 0.2268 \\ 0.3577 & 0.1530 & 0.2575 & 0.2143 & 1.0656 \end{bmatrix} \tag{4.24}$$

We compute the column vector of the centrality measures $\boldsymbol{\xi}$ and the vector of relative centrality measures $\boldsymbol{\sigma}$, respectively

$$\boldsymbol{\xi}_1 = \begin{bmatrix} 1.6277 \\ 1.9898 \\ 2.0242 \\ 2.1883 \\ 2.0480 \end{bmatrix} \text{ and } \boldsymbol{\sigma}_1 = \begin{bmatrix} 0.1648 \\ 0.2014 \\ 0.2049 \\ 0.2215 \\ 0.2073 \end{bmatrix} \tag{4.25}$$

and the row vector of the aggregate effects due to any order dependence of social preferences of the industry on each firm $i = 1, 2, \cdots, N$ made by the column summations of matrix $\tilde{B}_1$

$$\boldsymbol{\chi}_1^T = \begin{bmatrix} 2.5425 & 1.8565 & 1.9188 & 1.9374 & 1.6228 \end{bmatrix} \tag{4.26}$$

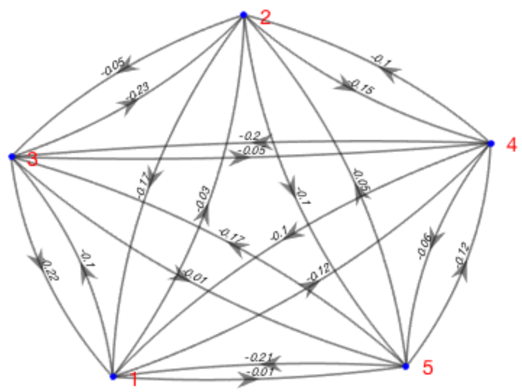


Figure 4.10: Graphical representation of the network described by weight matrix B_1 in (4.23)

We now consider another 5×5 matrix of negative weights

$$B_2 = \begin{bmatrix} 0 & -0.03 & -0.1 & -0.12 & -0.01 \\ -0.17 & 0 & -0.05 & -0.15 & -0.1 \\ -0.22 & -0.23 & 0 & -0.05 & -0.01 \\ -0.21 & -0.1 & -0.12 & 0 & -0.16 \\ -0.21 & -0.05 & -0.17 & -0.12 & 0 \end{bmatrix} \tag{4.27}$$

which gives rise to the network shown in Figure 4.11 The main feature of matrix B_2 is that each coefficient $|\beta_{ij}^{(2)}| \leq |\beta_{ij}^{(1)}|$, $\forall i, j$, so that in the second network a firm i is less or equally spiteful toward firm j than in the first network, for any $i \neq j$.
We report the matrix $(I + B_2)^{-1}$ of the aggregate effects due to any order dependence of social

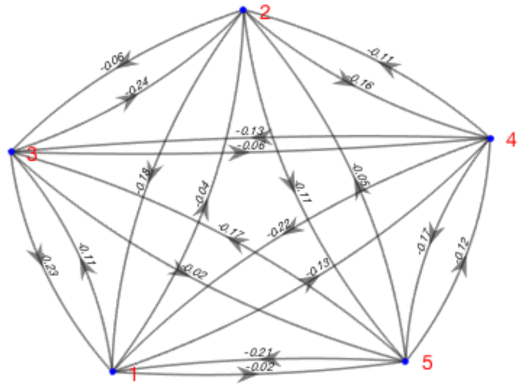


Figure 4.11: Graphical representation of the network described by weight matrix B_2 in (4.27)

preferences

$$(I + B_2)^{-1} = \begin{bmatrix} 1.0867 & 0.0824 & 0.1391 & 0.1552 & 0.0453 \\ 0.2876 & 1.0685 & 0.1333 & 0.219 & 0.1461 \\ 0.3262 & 0.2742 & 1.0742 & 0.1417 & 0.0641 \\ 0.3505 & 0.1793 & 0.2105 & 1.1035 & 0.2001 \\ 0.3401 & 0.1389 & 0.2438 & 0.2 & 1.0517 \end{bmatrix} \tag{4.28}$$

We compute the column vector of the centrality measures $\boldsymbol{\xi}$, composed by the row summations, for matrix B_2, and vector of relative centrality measures $\boldsymbol{\sigma}$, respectively

$$\boldsymbol{\xi}_2 = \begin{bmatrix} 1.5087 \\ 1.8545 \\ 1.8804 \\ 2.0438 \\ 1.9745 \end{bmatrix} \text{ and } \boldsymbol{\sigma}_2 = \begin{bmatrix} 0.1629 \\ 0.2002 \\ 0.2030 \\ 0.2207 \\ 0.2132 \end{bmatrix} \tag{4.29}$$

and the row vector of the aggregate effects due to any order dependence of social preferences of the industry on each firm $i = 1, 2, \cdots, N$ made by the column summations of matrix B_2

$$\boldsymbol{\chi}_2 = \begin{bmatrix} 2.3911 & 1.7432 & 1.8008 & 1.8193 & 1.5073 \end{bmatrix} \tag{4.30}$$

We notice that given that each coefficient $|\beta_{ij}^{(2)}| \leq |\beta_{ij}^{(1)}|$, $\forall i, j$, then $\boldsymbol{\xi}_2 \leq \boldsymbol{\xi}_1$.
From the comparison between $\tilde{B}_1$ and $\tilde{B}_2$, we notice that $\tilde{\beta}_{ij}^{(2)} \leq \tilde{\beta}_{ij}^{(1)}$, for any i, j.
Interestingly enough, we notice that even if $\beta_{5,j}^{(1)} = \beta_{5,j}^{(2)}$, for any $j = 1, 2, \cdots, N$, we have that $\boldsymbol{\sigma}_2 > \boldsymbol{\sigma}_1$, for $i = 5$.

In the last part of this Section we move from the study of the effects on a single individual of the change of a single weight to the investigation of the effects on the collectivity of the change in the collective behavior. To this end, we firstly need to focus on what happens on the degree of competitiveness as the social preference structure changes.

Proposition 16. *Let B be a matrix that satisfies Assumptions 1 and 3 and to which corresponds an internal Nash equilibrium and be $1 \leq i, j \leq N$ with $i \neq j$.*
If a given β_{ij} linearly increases, the degree of competitiveness μ increases provided that $\chi_j < 0$, or, equivalently, if $\rho_j < 0$. If all coefficients β_{ij} linearly increase, the degree of competitiveness μ decreases.

From Proposition 16 we have that if the positive influence of the material payoff of a firm j on the preferences of another firm increases (or if the negative influence reduces), the degree of competitiveness increases if $\chi_j < 0$. As predictable, when all the players become more altruistic or less spiteful, the degree of competitiveness decreases.

Now, the main question is: when does a *collective* change of the social preference structure is beneficial to *every* firm? This is clarified in the next proposition for two relevant inverse demand functions.

Proposition 17. *Let B be a matrix that satisfies Assumptions 1 and 3 and to which corresponds an internal Nash equilibrium and let $p(Q) = \max\{a - bQ, 0\}$ or $p(Q) = 1/Q$. Then as all β_{ij} linearly increase, the profits of all firms simultaneously increase provided that the distribution of firms with respect to centrality is suitably close to the a uniform distribution.*

The previous proposition shows that a beneficial effect in terms of the achieved profits is possible provided that firms are suitably "homogeneous" in terms of their relevance at the equilibrium. We deepen the investigation through the next example.

Example 10. Collective effects of a collective increase of altruism

Let consider the 5×5 matrix

$$B = \begin{bmatrix} 0 & -0.1467 & -0.0061 & -0.0858 & 0.0086 \\ -0.1323 & 0 & -0.0272 & 0.1520 & -0.2224 \\ 0.1391 & -0.1852 & 0 & -0.1602 & -0.0236 \\ -0.0472 & -0.1801 & -0.2298 & 0 & 0.2271 \\ 0.2231 & -0.2648 & -0.1599 & -0.0284 & 0 \end{bmatrix} \tag{4.31}$$

that describes a scenario in which there is homogeneity in the firms' centralities, i.e. the row summations are all equals.

Let us consider a perturbation matrix B_0, given by

$$B_0 = \begin{bmatrix} 0 & 0.0879 & -0.0043 & 0.0854 & 0.0870 \\ 0.0580 & 0 & 0.0909 & 0.0530 & 0.0285 \\ 0.0394 & -0.0076 & 0 & 0.0689 & 0.0956 \\ 0.0931 & 0.0759 & 0.0689 & 0 & 0.0579 \\ 0.0483 & 0.0247 & 0.0657 & 0.0087 & 0 \end{bmatrix} \tag{4.32}$$

whose network is represented in Figure 4.12

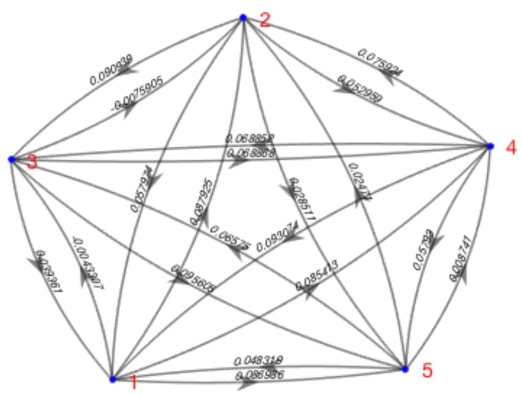


Figure 4.12: Graphical representation of the network described by weight matrix B_0 in (4.32)

The goal is to study the behaviour of profits of the firms in the network described by $B+\alpha B_0+\beta(U-I)$ where $\alpha \geq 0$ and β ranges from 0 to a suitable maximum value. Matrix $B+\alpha B_0$ is an initial network consisting in a perturbation of the homogeneous scenario given by B. The larger is α, the greater is the degree of heterogeneity encompassed in $B+\alpha B_0$. Moreover, we stress that heterogeneity also increases as β grows up.

We report the matrix $(I+B+\alpha B_0)^{-1}$ of the aggregate effects due to any order dependence of social preferences both for values of $\alpha \approx 0.1$ and $\alpha \approx 0.5$

$$(I+B+0.1B_0)^{-1} = \begin{bmatrix} 1.0274 & 0.0380 & -0.0015 & -0.0093 & -0.0881 \\ 0.0283 & 1.0282 & -0.0747 & -0.2123 & 0.2625 \\ -0.1561 & 0.1787 & 0.9921 & 0.0532 & -0.0370 \\ 0.0123 & 0.0616 & 0.1296 & 0.9937 & -0.2818 \\ -0.2865 & 0.2546 & 0.0784 & -0.0239 & 1.0779 \end{bmatrix} \tag{4.33}$$

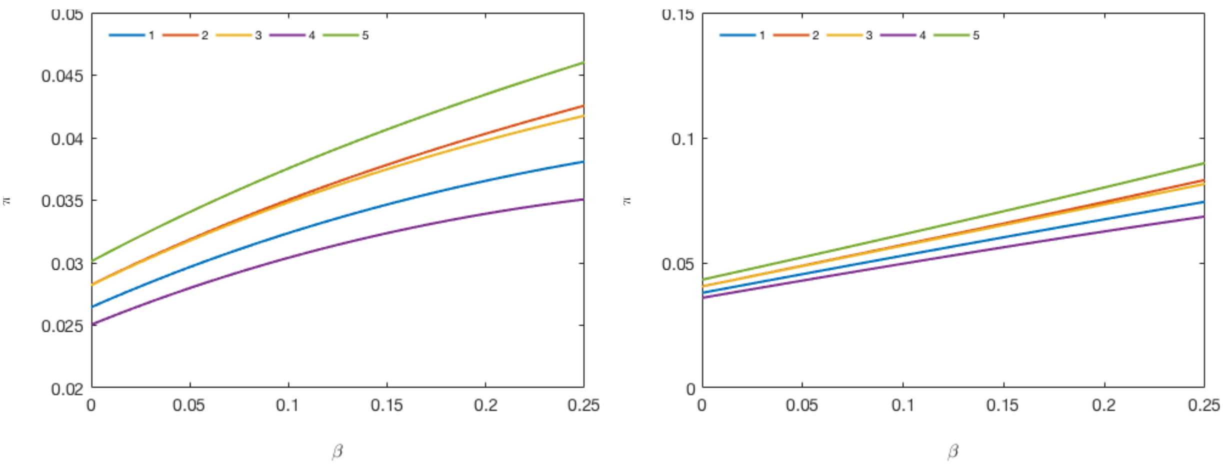
and

$$(I+B+0.5B_0)^{-1} = \begin{bmatrix} 1.3765 & -0.3380 & 0.3257 & -0.3799 & -0.5892 \\ -0.1079 & 1.0563 & -0.5071 & -0.3181 & 0.5270 \\ -0.1637 & 0.2730 & 0.9412 & -0.2279 & -0.2156 \\ -0.2129 & -0.2370 & 0.0550 & 1.1698 & -0.5539 \\ -0.6241 & 0.2638 & -0.3827 & 0.1523 & 1.3931 \end{bmatrix} \tag{4.34}$$

We report the vector of centrality measures ($\boldsymbol{\xi}$), for values of $\alpha \approx 0.1$ and $\alpha \approx 0.5$, respectively

$$\boldsymbol{\xi}_{0.1} = \begin{bmatrix} 0.9666 \\ 1.0320 \\ 1.0309 \\ 0.9154 \\ 1.1005 \end{bmatrix}, \ \boldsymbol{\xi}_{0.5} = \begin{bmatrix} 0.3952 \\ 0.6502 \\ 0.6071 \\ 0.2210 \\ 0.8023 \end{bmatrix} \tag{4.35}$$

We notice that the values in $\boldsymbol{\xi}_{0.1}$ are suitably close, representing an initial situation in which the firms own almost the same market share, while the values in $\boldsymbol{\xi}_{0.5}$ are much more sparse. We run the experiment for both the linear demand (for parameters' values of $a=2$, $b=1$) and the isoelastic demand function. In both cases we set marginal costs $c=1$.



(a) (b)

Figure 4.13: Profits for each firm in the network described in Figure 4.12 for increasing values of β (i.e. increasing the degree of altruism) given a linear demand function (in Figure 4.13(a)) and given an isoelastic demand function (in Figure 4.13(b)) for value of the perturbation parameter $\alpha \approx 0.1$

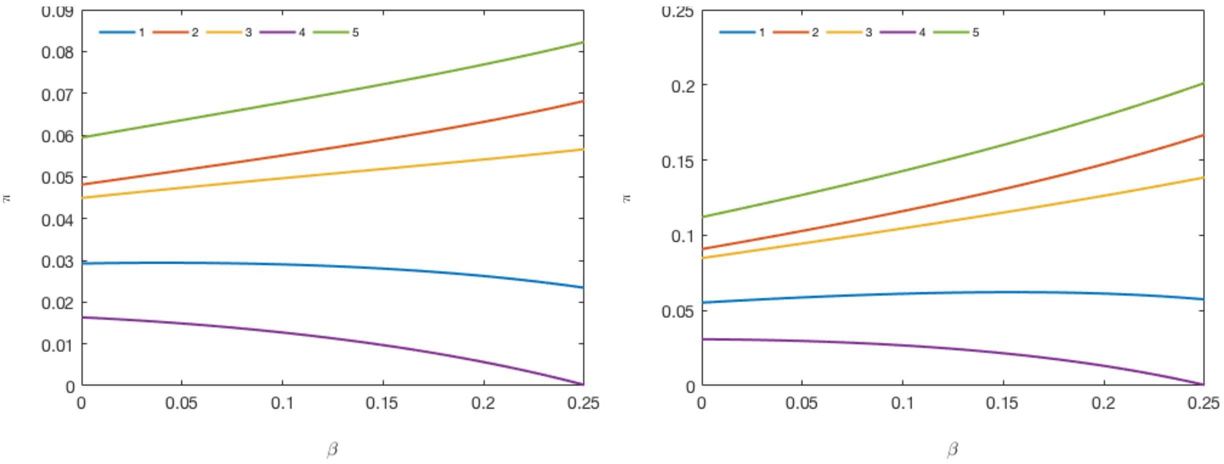
Proposition 17 shows that linearly increasing each coefficient β_{ij} of the matrix B, the profits of all firms simultaneously increase provided that the distribution of firms with respect to centrality is suitably close to the a uniform distribution and matrix B satisfies 1-3.

In Figure 4.13(a) and Figure 4.13(b) we report achieved profits for the scenario with the perturbation paramenter $\alpha = 0.1$. We notice how all firms' profits are increasing as β increases.

Conversely, if the initial distribution of centralities is too heterogeneous, there is no chance for a firm with a low centrality to increase its profits by increasing its altruism toward the industry, as evident from Figure 4.14(a) and Figure 4.14(b), related to the case with $\alpha \approx 0.5$

4.4 Conclusions

The comparative statics analysis provided in this Chapter has highlighted the crucial role the elements, characterizing the social interaction structure, play on the equilibrium. Among these elements we focused on vector of centrality measures $\boldsymbol{\xi}$, known in the literature as the Bonacich centrality index, on the degree of competitiveness of the market μ, the vector of the degree of influence $\boldsymbol{\chi}$ that a single firm exerts in the network of social interactions, known in the literature



(a) (b)

Figure 4.14: Profits for each firm in the network described in Figure 4.12 for increasing values of β (i.e. increasing the degree of altruism) given a linear demand function (in Figure 4.14(a)) and given an isoelastic demand function (in Figure 4.14(b)) for value of the perturbation parameter $\alpha \approx 0.5$

as the Friedkin-Johnsen centrality measure, and the vector of intercentrality measure $\boldsymbol{\rho}$ which identifies in a network the player providing the largest contribution to the aggregate outcome. The contribution of each element to the equilibrium, in terms of market share and/or profits, has been studied in order to understand the reasonableness of an individualistic behaviour or a collusive one. For instance, a given firm i can improve its individual performance, in terms of market share, at the equilibrium by increasing the weight it places on the material payoff of its competitors if the degree χ_i is sufficiently high or by reducing β_{ij} when χ_i is low or even negative. Even if in the proposed model the distribution of weights is kept exogenous and the case in which firms can decide or change their social preferences is not under investigation, it is easy to read in the previous considerations a first, very prothotipical and stylized, justification for a "tit-for-tat" dynamical way to adjust social preferences in order to achieve higher market share. Moreover, we investigated the effects on the industry of the change in the collective behavior. Comparative statics shows that a beneficial effect in terms of the achieved profits is possible provided that firms are suitably "homogeneous" in terms of their relevance at the equilibrium.

To conclude, the importance of the local analysis provided in this Chapter is to understand and answer to the question of what drives, in the first stages of the game, the individual partecipant in Friedman laboratory experiment. That is, given the interaction structure of the game, how it is possible for a player to individually increase its performance. Simultaneously, the global analysis can explain why, from a certain stage of Friedman's laboratory experiment, emerges a decisional behaviour that is fundamentally collective, passing from a heuristic that tries to improve the performance of the single player to a heuristic that tries to improve the performances of all the players, simultaneously. Friedman himself concludes that, in a triopoly context, the change from a individualistic heuristic to a collective one is facilitated when players interact, in groups or as a whole, in a more homogeneous social structure. In line with Friedman's conclusions, we showed that an increase in the collective performance, in terms of profits, is possible when the industry acts more altruistically, given that the interaction structure is characterized by less heterogeneity as possible between the players.

Chapter 5

Further perspective

The aim of this thesis has been to analyze the presence of a structure of social interedependent preferences in a Cournot oligopoly. To do this, we have introduced a game in which the network of interactions reflects on the utility functions of firms. As discussed, the first straightforward effect of considering interdependent preferences is that we can identify an additional channel of interaction among firms, along with the usual market interaction. Such latter channel is the unique one that is present in the classical Cournot game Γ_0 and establishes a "global", market related, form of interaction among all firms, mediated by the common inverse price function through the aggregate output level. Interdependent preferences establish another, possibly local or even one-to-one form of interaction, described by the distribution of coefficients β_{ij}.

The proposed modelling approach proved to be suitable to extend the results about existence and uniqueness of the Nash equilibrium for Cournotian oligopoly models without interdependent preferences.

Then we studied the role of preference interdependence on the resulting properties of the Nash equilibrium of any game Γ, in terms of strategic substitutability/complementarity.

We characterized the Nash equilibrium through the two channels of interaction among firms: the market interaction and the social interaction, whose influences can be identified in both relations (3.33) and (3.34). We stress that since firms are homogeneous in all respects but the distribution of social preferences, without a network of interdependence, all firms would produce the same quantity and would realize the same profits at the equilibrium. What emerged from the analysis is then the primacy of the social sphere over the economic one in particular economic structures (i.e.

Cournotian oligopolies) with interdependent preferences.

The characterization of the equilibrium, the resulting degree of competitiveness arising from the interdependence of social preferences and the comparative statics can be all expressed in terms of measures that describe the network properties. Bonacich centrality measure $\boldsymbol{\xi}$ determines the relevance of firms in the network and consequently the equilibrium market share and the degree of competitiveness μ. The Friedkin-Johnsen vector $\boldsymbol{\chi}$ that represents the degree of influence a firm exerts in the network of social interactions plays a key role on the way the Nash equilibrium changes as the structure of firms' interactions varies, i.e. plays a key role in the comparative statics analysis. The joint role of firms' relevance and influence can be understood through another measure related to the social interaction structure that is the intercentrality measure $\boldsymbol{\rho}$, identifying the player providing the largest contribution to the aggregate outcome. Comparative statics allowed us to pursue a twofold intent: to understand both how a change in the social interaction structure of a single player influences the outcome of the player itself, in terms of increased/decreased market share, and how a change in the interaction structure as a whole affects the collective outcome, in terms of increased/decreased profits.

The present results are at the basis of several possible research strands. The first one has the aim to test the robusteness of the conclusions. To this end, we aim at investigating the primacy of the social aspects also in other classes of games with preference interdependence such as the prisoner's dilemma, tragedy of commons and contest games. Moreover, the interest is to test if the general conclusion is robust in terms of symmetry of the game. In the analysis performed in this thesis we assumed a symmetric structure on the side of the market, namely we assumed players facing the same demand function and using the same tecnology. The twofold goal is to let heterogeneity enter the market sphere and study how the primacy of social aspects is affected by the heterogeneity.

A second research strand will aim at introducing dynamical aspects into the model. Even if in the proposed model the distribution of weights has been kept exogenous and the case in which firms can decide or change their social preferences was not under investigation, the results of the analysis opened interesting considerations in view of a possible endogenization and evolution of coefficients β_{ij}.

Differently from Friedman, we may, explicitly, allow an additional degree of freedom in the players' decision dimension (a part from choosing the optimal quantity) which is the choice of the coefficients of social interdependence. Rather than a change in the heuristics as proposed by Friedman, we may allow that the bounded rational players also select the appropriate, more successful coefficients, and consequently play their best response. Similarly to Sethi and Somanathan we may provide an evolutionary theory for the survival of agents that are not self-interested as an aspect of preference interdependence in a less general class of games, i.e. Cournotian oligopoly games. A limitation in the model of Sethi and Somanathan is that there is not an explicit endogenous dynamic evolution of the coefficients of interdependence. We aim to overcome this limitation and simultaneously propose a model with the same descriptive power the one from Friedman.

The analysis provided has been able to shed some light on the question of what drives, in both the first and the final stages of the game, the individual partecipant in Friedman laboratory experiment. That is, given the interaction structure of the game, how it is possible for a player to individually increase its performance. Simultaneously, the global analysis can provide elements to explain why, from a certain stage of Friedman's laboratory experiment, emerges a decisional behaviour that is fundamentally collective, passing from a heuristic that tries to improve the performance of the single player to a heuristic that tries to improve the performances of all the players, simultaneously. In line with Friedman's conclusions, we showed that an increase in the collective performance, in terms of profits, is possible when the industry acts more altruistically, given that the interaction structure is characterized by less heterogeneity as possible between the players.

It is theoretically interesting both to look at the single instant picture of a particular interdependent structure and to look at the general dynamics of the system of players as it diverges from a self-interested structure to better understand how a specific behaviour on behalf of the players can influence a particular equilibrium to emerge. The analysis conducted at the static level fully fit into the debate concerning the survival, in an evolutionary context, of both altruistic and spiteful behaviours. Moreover, the preliminary results on the dynamic analysis of out-of-equilibrium aspects of the game, confirmed us the necessity of the deep investigation provided in the static part, for two reasons. First, when the dynamical model loses its stability, it is due to the loss of stability of the Nash equilibrium, which is the steady state of the dynamical system itself. Second, once the dynamical model loses its stability, the consequent out-of-equilibrium, chaotic and oscillatory, dynamics lie in the neighborhood of the Nash equilibrium. However, the complete understanding of the dynamical phenomena requires a dedicated investigation we plan to tackle in the next future.

Appendix

Proof of Proposition 1. The utility function is $v_i = q_i(p(q_i + Q_{-i}) - c) + \beta \sum_{j=1, i \neq j}^{N} (q_j(p(q_i + Q_{-i}) - c))$ from which we have that marginal utility is

$$\frac{\partial v_i}{\partial q_i} = p(Q) - c + q_i p'(Q) + \beta Q_{-i} p'(Q) \tag{5.1}$$

Since we are considering the aggregated output level at an internal equilibrium $\boldsymbol{q}^*$, first order condition necessarily requires $\partial_{q_i} v_i = 0$, Thanks to this and summing the right hand side of (5.1) for $i = 1$ to N we obtain

$$Qp'(Q) - \frac{N}{(N-1)\beta + 1} \cdot (c - p(Q)) = 0,$$

By assumption, if $\beta > 0$ the previous equality is solved by $Q^*(\beta) > 0$ for any $\beta > 0$. We have that $\lim_{\beta \to 1^-} Q^*(\beta)$ is then the solution to $Qp'(Q) = c - p(Q)$, which is exactly the output level of a monopoly with inverse demand function $p(Q)$. Conversely, if $\beta < 0$, we have that $\lim_{\beta \to -\frac{1}{N-1}} Q^*(\beta)$ is then the solution to $c = p(Q)$, which is exactly the output level of a competitive market in which the inverse function is $p(Q)$. $\square$

Proof of Proposition 2. We start noting that Assumption 2 guarantees the concavity of the utility function of each player. The existence of a Nash equilibrium is then a consequence of Nikaido-Isoda Theorem (see e.g. [17]) for more details). Now assume that for equilibria there hold $q_i < L_i$ for all $i \in \mathcal{N}$.

We find the best response function of the i-th firm, for a given vector of strategies $\boldsymbol{q}_{-i}$. In principle, we have to distinguish three cases:

a) $\partial_{q_i} v_i(0) \leq 0$: since v_i is strictly concave on $[0, L]$, in this case it is also strictly decreasing it attains its maximum at $q_i = 0$;

b) $\partial_{q_i} v_i(L_i) \geq 0$: in this case the concavity of v_i guarantees that v_i is strictly increasing and hence it attains its maximum at $q_i = L_i$;

c) in the remaining situations Assumption 2 guarantees existence and uniqueness of a solution to equation $p'(z_i+Q_{-i})z_i+p(z_i+Q_{-i})-c+\sum_{j=1}^N \beta_{ij}p'(z_i+Q_{-i})z_j=0$, since the right hand side is strictly decreasing on $(0,L)$, positive for $z_i \to 0^+$ and negative for $z_i \to L^-$.

We then have

$$\mathrm{BR}_i(\boldsymbol{q}_{-i}) = \begin{cases} 0 & \text{if} \quad \partial_{q_i} v_i(0) \leq 0 \\ L_i & \text{if} \quad \partial_{q_i} v_i(L_i) \geq 0 \\ z_i & \quad\text{otherwise} \end{cases} \tag{5.2}$$

Any equilibrium with $q_i < L_i$ for all $i \in \mathcal{N}$ must fulfill

$$q_i \frac{\partial v_i}{\partial q_i} = q_i \left(p'(q_i+Q_{-i})q_i + p(q_i+Q_{-i}) - c + \sum_{j\neq i}^N \beta_{ij} p'(Q) q_j \right) = 0 \tag{5.3}$$

and

$$\frac{\partial v_i}{\partial q_i} = p'(Q)q_i + p(Q) - c + \sum_{j\neq i}^N \beta_{ij} p'(Q) q_j \leq 0 \tag{5.4}$$

Conditions (5.3) and (5.4) can be equivalently rewritten as

$$\begin{cases} \boldsymbol{q} \geq 0 \\ \boldsymbol{q}^T \left(\boldsymbol{q} + \frac{p(Q)-c}{p'(Q)} \boldsymbol{u} + B\boldsymbol{q} \right) = 0 \\ \boldsymbol{q} + \frac{p(Q)-c}{p'(Q)} \boldsymbol{u} + B\boldsymbol{q} \geq 0 \end{cases}$$

Let us introduce $y = -\frac{p(Q)-c}{p'(Q)}$, so the previous system is

$$\begin{cases} \boldsymbol{q} \geq 0 \\ \boldsymbol{q}^T (\boldsymbol{q} - y\boldsymbol{u} + B\boldsymbol{q}) = 0 \\ \boldsymbol{q} - y\boldsymbol{u} + B\boldsymbol{q} \geq 0 \\ y = -\frac{p(Q)-c}{p'(Q)} \end{cases}$$

Note that the first three conditions describe a linear complementarity problem, in which the pattern of solution $\boldsymbol{q}$ (i.e. the position of null vs. non-null components) is independent of y. Thanks to Assumption 3, for each $y > 0$, there exists a unique solution q (different from the null vector) to such problem in which we have either $q_i > 0$ or $q_i = 0$ for each $i = 1, \ldots, N$. This rules out the possibility to have multiple equilibria with $q_i < L_i$ for all $i \in \mathcal{N}$.

If in addition $q_i > 0$, we have that the solution can be written as $\boldsymbol{q} = -\frac{p(Q)-c}{p'(Q)}(I+B)^{-1}\boldsymbol{u} = -\frac{p(Q)-c}{p'(Q)}\boldsymbol{\xi}$. Multiplying both sides by $\boldsymbol{u}^T$ we have $Q = -\frac{p(Q)-c}{p'(Q)} \sum_{j=1}^N \xi_i$, which, combined with the relation for $\boldsymbol{q}$, allows concluding. $\square$

Lemma 1. *Let B fulfill Assumptions 3 and assume that $\boldsymbol{\xi} = (I+B)^{-1}\boldsymbol{u}$ is componentwise non-negative. Then $\boldsymbol{u}^T(I+B)^{-1}\boldsymbol{u} > 1$*

Proof. Let N be the size of B. Thanks to Assumptions 3 we have that $I+B$ is invertible and thanks to Assumption 1, we have that $\rho(I+B) = K < N$, so $\rho\left(\frac{I+B}{N}\right) < 1$ and we can use series expansion

$$(I+B)^{-1} = \sum_{n=0}^{+\infty}\left(I - \frac{I+B}{N}\right)^n \frac{1}{N}$$

We then have

$$\begin{aligned}
\boldsymbol{u}^T(I+B)^{-1}\boldsymbol{u} &= \frac{\boldsymbol{u}^T}{N}\sum_{n=0}^{+\infty}\left(I - \frac{(I+B)}{N}\right)^n \boldsymbol{u} \\
&= 1 + \frac{\boldsymbol{u}^T}{N}\sum_{n=1}^{+\infty}\left(I - \frac{(I+B)}{N}\right)^n \boldsymbol{u} \\
&= 1 + \frac{\boldsymbol{u}^T}{N}\left(I - \frac{(I+B)}{N}\right)\sum_{n=0}^{+\infty}\left(I - \frac{(I+B)}{N}\right)^n \boldsymbol{u} \\
&= 1 + \boldsymbol{u}^T\left(I - \frac{(I+B)}{N}\right)\boldsymbol{\xi} = 1 + \boldsymbol{u}^T\left(I\left(1 - \frac{1}{N}\right) - \frac{B}{N}\right)\boldsymbol{\xi}
\end{aligned}$$

Elements in $\boldsymbol{u}^T\left(I\left(1-\frac{1}{N}\right) - \frac{B}{N}\right)$ are given by

$$1 - \frac{1}{N} - \sum_{i=1, j\neq i}^{N}\frac{\beta_{ij}}{N} > 1 - \frac{1}{N} - \sum_{i=1, j\neq i}^{N}\frac{1}{N} > 0$$

so $\boldsymbol{u}^T\left(I\left(1-\frac{1}{N}\right) - \frac{B}{N}\right)\xi > 0$. This allows concluding. □

Proof of Proposition 3. Utility function is

$$v_i = q_i\left(\frac{1}{Q} - c\right) + \sum_{i=1}^{N}\beta_{ij}q_j\left(\frac{1}{Q} - c\right) = \frac{1}{Q}\left(q_i - cQ + \sum_{i=1}^{N}\beta_{ij}q_j(1 - cQ)\right)$$

The null vector can not be the Nash equilibrium, as p is not defined for $Q = 0$. It is easy to see that a Nash equilibrium can not have more than $N-2$ null components. Without loss of generality, let us assume that $q_i = 0$ for $i > 2$, so that we have

$$v_i = q_i\left(\frac{1}{q_1+q_2} - c\right) + \beta_{i,-i}q_{-i}\left(\frac{1}{q_1+q_2} - c\right),\ i = 1, 2$$

and

$$\frac{\partial v_1}{\partial q_1} = \frac{-cq_1^2 - 2cq_1q_2 - \beta_{12}q_2 + q_2 - cq_2^2}{(q_1+q_2)^2}, \qquad \frac{\partial v_2}{\partial q_2} = -\frac{\beta_{21}q_1 - q_1 + cq_1^2 + cq_2^2 + 2cq_1q_2}{(q_1+q_2)^2}$$

If $q_2 > 0$, we have two possibilities: v_1 is strictly decreasing if $1 - \beta_{12} - cq_2 < 0$ or it is concave and unimodal. In the first case, the best response is $q_1 = 0$, but the best response to $q_1 = 0$ can not be $q_2 > 0$ (utility function v_2 is strictly decreasing in this case). So we necessarily need that q_1 and q_2 are strictly positive at the equilibrium.

In the general case, marginal utility is

$$\frac{\partial v_i}{\partial q_i} = -\frac{q_i}{Q^2} + \frac{1}{Q} - c - \sum_{j=1}^{N} \beta_{ij} \frac{q_j}{Q^2} = \frac{1}{Q^2}\left(-cQ^2 + Q - q_i - \sum_{j=1}^{N} \beta_{ij} q_j\right)$$

so its sign is determined by the sign of the second degree polynomial $\partial_{q_i} v_i(q_i) = -cq_i^2 - 2cq_i Q_{-i} - cQ_{-i}^2 + \sum_{j=1}^{N}(1 - \beta_{ij})q_j$, which represents a concave parabola, strictly decreasing for $q_i \geq 0$. We then have two possibilities for the best response

- $\mathrm{BR}_i(q_{-i}) = 0$, in which case we necessarily have $\partial_{q_i} v_i(0) \leq 0$
- $\mathrm{BR}_i(q_{-i}) > 0$

so at a Nash equilibrium q we must have a couple of relations similar to (5.3) and (5.4), so we can again write the equilibrium condition as

$$\begin{cases} \boldsymbol{q} \geq 0 \\ \boldsymbol{q}^T \left(\boldsymbol{q} + \frac{p(Q)-c}{p'(Q)}\boldsymbol{u} + B\boldsymbol{q}\right) = 0 \\ \boldsymbol{q} + \frac{p(Q)-c}{p'(Q)}\boldsymbol{u} + B\boldsymbol{q} \geq 0 \end{cases}$$

Let us introduce $y = -\frac{p(Q)-c}{p'(Q)} = Q - cQ^2$, so the previous system is

$$\begin{cases} \boldsymbol{q} \geq 0 \\ \boldsymbol{q}^T(\boldsymbol{q} - y\boldsymbol{u} + B\boldsymbol{q}) = 0 \\ \boldsymbol{q} - y\boldsymbol{u} + B\boldsymbol{q} \geq 0 \\ y = Q - cQ^2 \end{cases}$$

Note that the first three conditions describe a linear complementarity problem, in which the pattern of solution q (i.e. the position of null vs. non-null components) is independent of y. Thanks to Assumption 3, for each y, there exists a unique solution $\boldsymbol{q}$ (different from the null vector) to such problem in which we have either $q_i > 0$ or $q_i = 0$ for each $i = 1, \ldots, N$. However, $y > 0$, as otherwise $\boldsymbol{q}^T(\boldsymbol{q} - y\boldsymbol{u} + B\boldsymbol{q}) = 0$ would have the unique null solution, which is not consistent with $y < 0$ and would provide $Q = 0$, which is impossible as p is not defined at $Q = 0$.

Let $\tilde{B}$ be a matrix in which the ith row and column are made by null elements if $q_i = 0$ while the remaining elements are those of B. Note that $I + B$ is a P matrix and hence it is invertible. Let $\tilde{\boldsymbol{u}}$ be a vector in which the ith element is null if $q_i = 0$ while the remaining elements are equal to 1.

We indeed have

$$\begin{cases} \boldsymbol{u} - y\tilde{\boldsymbol{u}} + \tilde{B}\boldsymbol{u} = 0 \\ y = Q - cQ^2 \end{cases} \Leftrightarrow \begin{cases} \boldsymbol{q} = y(I + \tilde{B})^{-1}\tilde{\boldsymbol{u}} \\ y = Q - cQ^2 \end{cases}$$

from which $Q = y\boldsymbol{u}^T(I + \tilde{B})\tilde{\boldsymbol{u}} = y\mu$. The last equation can be rewritten as

$$\frac{Q}{\mu} = Q - cQ^2 \Leftrightarrow \mu = \frac{1}{1 - cQ}$$

since $Q \neq 0$. The previous equation has a unique solution since $\mu > 1$. This follows from Lemma 1 noting that $\mu = \boldsymbol{u}^T(I + \tilde{B})^{-1}\tilde{\boldsymbol{u}} = \hat{\boldsymbol{u}}^T(I + \hat{B})^{-1}\hat{\boldsymbol{u}}$ in which $\hat{B}$ is the submatrix obtained from $\tilde{B}$ by removing all the rows/columns for which $q_i = 0$ and $\hat{\boldsymbol{u}}$ is a constant unitary vector with as many elements as the non-null components in $\boldsymbol{q}$. $\square$

Proof of Prop. 4. Assumption about strategic substitutability in Γ_0 guarantees that $\partial^2\pi_r/\partial q_r q_s = q_r p'(Q) + p''(Q) < 0$ for any r, s. The degree of strategic interaction between i and j is given by

$$\frac{\partial^2 v_i}{\partial q_i q_j} = p'(Q) + q_i p''(Q) + \sum_{r=1, r\neq j}^{N} \beta_{ir} q_r p''(Q) + \beta_{ij}(q_j p'(Q) + p''(Q)),$$

which, as a consequence of strategic substitutability in Γ_0, negatively depends on β_{ij}. This concludes the proof. $\square$

Proof of Prop. 5. At an internal equilibrium q first order condition must hold, so we have

$$p'(Q)q_i + p(Q) - c + \sum_{j=1}^{N} \beta_{ij} p'(Q) q_j = 0, \tag{5.5}$$

which, in vector form, can be rewritten as $p'(Q)(I + B)\boldsymbol{q} + (p(Q) - c)\boldsymbol{u} = 0$. Setting $y = -(p(Q) - c)/p'(Q)$, the last system becomes

$$\begin{cases} (I + B)\frac{\boldsymbol{q}}{y} = \boldsymbol{u} \\ y = -\frac{p(Q) - c}{p'(Q)} \end{cases}$$

Since the game has solution q, the former vector equation has at least a solution that can be written as $\frac{\boldsymbol{q}}{y} = (I + B)^{+}\boldsymbol{u} + [I - (I + B)^{+}(I + B)]\boldsymbol{z} = \boldsymbol{\xi}$, where A^{+} is the Moore-Penrose inverse and $\boldsymbol{z}$ is an arbitrary vector (in the particular case of an invertible matrix $I + B$ we obtain (3.35)).

Left multiplying both sides by $\boldsymbol{u}^T$ we immediately obtain (3.34) and then, using $y = Q/(\boldsymbol{u}^T\boldsymbol{\xi})$, we find (3.33). Profits immediately follow. □

Proof of Propositions 6, 7 and 8. We start proving Proposition 7, so let B such that $\beta_{ij} = \beta_i$ for $i \neq j, i, j \in \mathcal{N}$. We can write $B = D + \boldsymbol{b}\boldsymbol{u}^T$, where D and $\boldsymbol{b}$ are respectively a diagonal matrix in which $d_{ii} = 1 - \beta_i$ for $i \in \mathcal{N}$ and a vector with $b_i = \beta_i$ for $i \in \mathcal{N}$. Thanks to the Sherman-Morrison formula we can write $(I + B)^{-1} = D^{-1} - \frac{D^{-1}\boldsymbol{b}\boldsymbol{u}^T D^{-1}}{1+\boldsymbol{u}^T D^{-1}\boldsymbol{b}}$. It is easy to see that the elements of $D^{-1}\boldsymbol{b}\boldsymbol{u}^T D^{-1}$ are given by $a_{ij} = \frac{\beta_i}{(1-\beta_i)(1-\beta_j)}$ while $1 + \boldsymbol{u}^T D^{-1}\boldsymbol{b} = 1 + \sum_{j=1}^N \frac{\beta_j}{1-\beta_j}$.

Note that thanks to Assumption 1 we have

$$1 + \sum_{j=1}^{N} \frac{\beta_j}{1-\beta_j} > 1 - \sum_{j=1}^{N} \frac{1}{N} = 0,$$

so $I + B$ is invertible and the Sherman-Morrison formula can be applied.

The generic elements of $\tilde{B}$ are then

$$\tilde{\beta}_{ii} = \frac{1}{1-\beta_i} - \frac{\frac{\beta_i}{(1-\beta_i)(1-\beta_j)}}{1 + \sum_{k=1}^N \frac{\beta_k}{1-\beta_k}}, \quad \tilde{\beta}_{ij} = -\frac{\frac{\beta_i}{(1-\beta_i)(1-\beta_j)}}{1 + \sum_{k=1}^N \frac{\beta_k}{1-\beta_k}}, i \neq j \tag{5.6}$$

We have $\boldsymbol{\xi} = D^{-1}\boldsymbol{u} - \frac{D^{-1}\boldsymbol{b}\boldsymbol{u}^T D^{-1}\boldsymbol{u}}{1+\boldsymbol{u}^T D^{-1}\boldsymbol{b}}$ and $\boldsymbol{u}^T\boldsymbol{\xi} = \boldsymbol{u}^T D^{-1}\boldsymbol{u} - \frac{\boldsymbol{u}^T D^{-1}\boldsymbol{b}\boldsymbol{u}^T D^{-1}\boldsymbol{u}}{1+\boldsymbol{u}^T D^{-1}\boldsymbol{b}}$.

The generic component of the centrality index results

$$\begin{aligned}
\xi_i &= \frac{1}{1-\beta_i} - \frac{\sum_{j=1}^{N} \frac{\beta_i}{(1-\beta_i)(1-\beta_j)}}{1 + \sum_{k=1}^{N} \frac{\beta_k}{1-\beta_k}} = \frac{1}{1-\beta_i} - \frac{\frac{\beta_i}{1-\beta_i} \sum_{j=1}^{N} \frac{1}{1-\beta_j}}{1 + \sum_{k=1}^{N} \frac{\beta_k}{1-\beta_k}} \\
&= \frac{1}{1-\beta_i} \left(\frac{1 + \sum_{k=1}^{N} \frac{\beta_k}{1-\beta_k} - \beta_i \sum_{k=1}^{N} \frac{1}{1-\beta_k}}{1 + \sum_{k=1}^{N} \frac{\beta_k}{1-\beta_k}} \right) \\
&= \frac{1}{1-\beta_i} \left(\frac{1 - N + (1-\beta_i) \sum_{k=1}^{N} \frac{1}{1-\beta_k}}{1 + \sum_{k=1}^{N} \frac{\beta_k}{1-\beta_k}} \right)
\end{aligned}$$

Noting that

$$
\begin{aligned}
\frac{\partial \xi_i}{\partial \beta_i} &= \frac{1}{(1-\beta_i)^2} - \frac{\left(\sum_{k=1}^{N} \frac{1}{(1-\beta_i)(1-\beta_k)} + \frac{\beta_i}{(1-\beta_i)^2(1-\beta_k)} + \frac{\beta_i}{(1-\beta_i)^3}\right)\left(1+\sum_{k=1}^{N} \frac{\beta_k}{1-\beta_k}\right)}{\left(1+\sum_{k=1}^{N} \frac{\beta_k}{1-\beta_k}\right)^2} \\
&+ \frac{\sum_{k=1}^{N} \frac{\beta_i}{(1-\beta_i)(1-\beta_k)} \cdot \frac{1}{(1-\beta_i)^2}}{\left(1+\sum_{k=1}^{N} \frac{\beta_k}{1-\beta_k}\right)^2} \\
&= \frac{1}{(1-\beta_i)^2} - \frac{\left(\frac{1}{(1-\beta_i)^2}\sum_{k=1}^{N} \frac{1}{1-\beta_k} + \frac{\beta_i}{1-\beta_i}\right)\left(1+\sum_{k=1}^{N} \frac{\beta_k}{1-\beta_k}\right) - \frac{\beta_i}{(1-\beta_i)^3}\sum_{k=1}^{N} \frac{1}{(1-\beta_k)}}{\left(1+\sum_{k=1}^{N} \frac{\beta_k}{1-\beta_k}\right)^2} \\
&= \frac{\left(1+\sum_{k=1}^{N} \frac{\beta_k}{1-\beta_k}\right)^2 - \left(\sum_{k=1}^{N} \frac{1}{1-\beta_k} + \frac{\beta_i}{1-\beta_i}\right)\left(1+\sum_{k=1}^{N} \frac{\beta_k}{1-\beta_k}\right) + \frac{\beta_i}{1-\beta_i}\sum_{j=1}^{N} \frac{1}{1-\beta_k}}{(1-\beta_i)^2\left(1+\sum_{k=1}^{N} \frac{\beta_k}{1-\beta_k}\right)^2} \\
&= \frac{\left[1-N-\frac{\beta_i}{1-\beta_i}\right]\left(1+\sum_{k=1}^{N} \frac{\beta_k}{1-\beta_k}\right) + \frac{\beta_i}{1-\beta_i}\sum_{k=1}^{N} \frac{1}{(1-\beta_k)}}{(1-\beta_i)^2\left(1+\sum_{k=1}^{N} \frac{\beta_k}{1-\beta_k}\right)^2} \\
&= \frac{(1-N)\left(1+\sum_{k=1}^{N} \frac{\beta_k}{1-\beta_k}\right) - \frac{\beta_i}{1-\beta_i}\left(1+\sum_{k=1}^{N} \frac{\beta_k}{1-\beta_k}\right) + \frac{\beta_i}{1-\beta_i}\sum_{k=1}^{N} \frac{1}{(1-\beta_k)}}{(1-\beta_i)^2\left(1+\sum_{k=1}^{N} \frac{\beta_k}{1-\beta_k}\right)^2} \\
&= \frac{(1-N)\left(1+\sum_{k=1}^{N} \frac{\beta_k}{1-\beta_k} - \frac{\beta_i}{1-\beta_i}\right)}{(1-\beta_i)^2\left(1+\sum_{k=1}^{N} \frac{\beta_k}{1-\beta_k}\right)^2} < 0
\end{aligned}
$$

we can conclude the ordering of elements in vector $\boldsymbol{\xi}$.

The influence vector $\boldsymbol{\chi}$ is given by $\boldsymbol{\chi}^T = \boldsymbol{u}^T D^{-1} - \frac{\boldsymbol{u}^T D^{-1}\boldsymbol{b}\boldsymbol{u}^T D^{-1}}{1+\boldsymbol{u}^T D^{-1}\boldsymbol{b}}$

Moreover,

$$
\begin{aligned}
\mu = u^T\xi &= \sum_{k=1}^{N}\frac{1}{1-\beta_i} - \frac{\displaystyle\sum_{k=1}^{N}\sum_{j=1}^{N}\frac{\beta_k}{(1-\beta_k)(1-\beta_j)}}{1+\displaystyle\sum_{k=1}^{N}\frac{\beta_k}{1-\beta_k}} \\
&= \sum_{k=1}^{N}\left(\frac{1}{1-\beta_k}\right) - \frac{\left(\displaystyle\sum_{k=1}^{N}\frac{\beta_k}{1-\beta_k}\right)\left(\displaystyle\sum_{k=1}^{N}\frac{1}{1-\beta_k}\right)}{1+\displaystyle\sum_{k=1}^{N}\frac{\beta_k}{1-\beta_k}} \\
&= \sum_{k=1}^{N}\left(\frac{1}{1-\beta_k}\right) - \left(1-\frac{1}{1+\displaystyle\sum_{k=1}^{N}\frac{\beta_k}{1-\beta_k}}\right)\left(\sum_{k=1}^{N}\frac{1}{1-\beta_k}\right) \\
&= \frac{\displaystyle\sum_{k=1}^{N}\frac{1}{1-\beta_k}}{1+\displaystyle\sum_{k=1}^{N}\frac{\beta_k}{1-\beta_k}} = \frac{\displaystyle\sum_{k=1}^{N}\frac{1-\beta_k+\beta_k}{1-\beta_k}}{1+\displaystyle\sum_{k=1}^{N}\frac{\beta_k}{1-\beta_k}} \\
&= \frac{N+\displaystyle\sum_{k=1}^{N}\frac{\beta_k}{1-\beta_k}}{1+\displaystyle\sum_{k=1}^{N}\frac{\beta_k}{1-\beta_k}}
\end{aligned}
$$

Similarly we have $\chi = \boldsymbol{u}^T D^{-1} - \frac{\boldsymbol{u}^T D^{-1}\boldsymbol{b}\boldsymbol{u}^T D^{-1}}{1+\boldsymbol{u}^T D^{-1}\boldsymbol{b}}$ so

$$
\begin{aligned}
\chi_i &= \frac{1}{1-\beta_i} - \frac{\displaystyle\sum_{k=1}^{N}\frac{\beta_k}{(1-\beta_k)(1-\beta_j)}}{1+\displaystyle\sum_{k=1}^{N}\frac{\beta_k}{1-\beta_k}} \\
&= \frac{1}{1-\beta_i} - \frac{\displaystyle\frac{1}{1-\beta_j}\sum_{k=1}^{N}\frac{\beta_k}{(1-\beta_k)}}{1+\displaystyle\sum_{k=1}^{N}\frac{\beta_k}{1-\beta_k}} \\
&= \frac{1}{1-\beta_i} - \frac{1}{1-\beta_j}\left(1-\frac{1}{1+\displaystyle\sum_{k=1}^{N}\frac{\beta_k}{1-\beta_k}}\right)
\end{aligned}
$$

Noting that

$$\frac{\partial \chi_i}{\partial \beta_i} = \frac{1}{(1-\beta_i)^2} + \frac{1}{1-\beta_j} \cdot \frac{1}{\left(1 + \sum_{k=1}^{N} \frac{\beta_k}{1-\beta_k}\right)^2} \cdot \frac{1}{(1-\beta_i)^2} > 0$$

we can conclude the ordering of elements in vector $\boldsymbol{\chi}$.

We use the previous results to prove Proposition 6. The expressions for $\tilde{\beta}_{ii} = \tilde{\beta}_1$ and $\tilde{\beta}_{ij} = \tilde{\beta}_2$ can be easily found setting $\beta_i = \beta$ in (5.6). The elements of $\boldsymbol{\xi}$ are obtained by computing $\xi_i = \tilde{\beta}_1 + (N-1)\tilde{\beta}_2$, from which it is straightforward to compute $\sigma_i = \xi_i / N\xi_i = 1/N$ and $\mu = N\xi_i$. When the Nash equilibrium is internal and the demand is linear, from (3.34) we have that Q^* must fulfill

$$-bQ^* = (c - a + bQ^*)\mu = (c - a + bQ^*)\frac{N}{(N-1)\beta + 1}$$

while if $p(Q) = 1/Q$, from (3.34) we have that Q^* must fulfill

$$-\frac{1}{Q^*} = \left(1 - \frac{1}{Q^*}\right)\frac{N}{(N-1)\beta + 1}$$

Solving by Q^* and then setting $q^* = Q^*/N$ provides the equilibrium values.

The proof of Proposition 8 is obtained by noting that it is simply the transposed case of that in Proposition 7, so we simply have to swap $\boldsymbol{\xi}$ and $\boldsymbol{\chi}$. □

Lemma 2. *Let B satisfying Assumption 3 for which $(I+B)^{-1}\boldsymbol{u}$ has positive elements and let B_{-i} the matrix obtained setting $\beta_{ij} = 0$ and $\beta_{ji=0}$ for any $j = 1, \ldots, N$. We have*

$$\tilde{\beta}_{ji}\tilde{\beta}_{ik} = \tilde{\beta}_{ii}(\tilde{\beta}_{jk} - \tilde{\beta}_{-i,jk}),\ j \neq k \tag{5.7}$$

where $\tilde{\beta}_{-i,ij}$ are the elements of $(I+B_{-i})^{-1}$.

Proof. The proof is essentially the same as that of Lemma 1 in [6]. We have that through series expansion we can write

$$(I+B)^{-1} = \sum_{n=0}^{+\infty}\left(I - \frac{I+B}{N}\right)^n \frac{1}{N},\ (I+B_{-i})^{-1} = \sum_{n=0}^{+\infty}\left(I - \frac{I+B_{-i}}{N}\right)^n \frac{1}{N}$$

Let $M = I - \frac{I+B}{N}$ and $M_{-i} = I - \frac{I+B_{-i}}{N}$. For a matrix A we denote by $a_{rs}^{[n]}$ the summation of all the weighted paths of length n starting from r and ending in s, corresponding to the elements of A^n. Note that comparing $m_{rs}^{[n]}$ and $m_{-i,rs}^{[n]}$, we have that if a path does not cross node i it provides the same contribution to both $m_{rs}^{[n]}$ and $m_{-i,rs}^{[n]}$, so in $m_{rs}^{[n]} - m_{-i,rs}^{[n]}$ we just have contributions due to paths that pass through node i. Let us denote them by $m_{r(i)s}^{[n]}$

$$\begin{aligned} N\tilde{\beta}_{ii}(N\tilde{\beta}_{jk} - N\tilde{\beta}_{-i,jk}) &= N^2 \sum_{n=0}^{+\infty} \sum_{a+b=n} m_{ii}^{[a]}(m_{jk}^{[b]} - m_{-i,jk}^{[b]}) \\ &= N^2 \sum_{n=0}^{+\infty} \sum_{a+b=n} m_{ii}^{[a]} m_{j(i)k}^{[b]} = N^2 \sum_{n=0}^{+\infty} \sum_{a'+b'=n} m_{ji}^{[a']} m_{ik}^{[b']} = N\tilde{\beta}_{ji} N\tilde{\beta}_{ik} \end{aligned}$$

which allows concluding. □

Proof of Proposition 10. The proof is essentially the same as that of Theorem 3 in [6]. From the definition of ρ_i in (4.1) we can write

$$\rho_i = \xi_i + \sum_{j=1, j\neq i}^{N} \sum_{k=1}^{N} (\tilde{\beta}_{jk} - \tilde{\beta}_{-i,jk}) = \xi_i + \sum_{j=1, j\neq i}^{N} \sum_{k=1}^{N} \frac{\tilde{\beta}_{ji}\tilde{\beta}_{ik}}{\tilde{\beta}_{ii}}$$

in which we used (5.7). We have

$$\rho_i = \xi_i + \sum_{j=1, j\neq i}^{N} \frac{\tilde{\beta}_{ji} \sum_{k=1}^{N} \tilde{\beta}_{ik}}{\tilde{\beta}_{ii}} = \xi_i + \sum_{j=1, j\neq i}^{N} \frac{\tilde{\beta}_{ji}\xi_i}{\tilde{\beta}_{ii}} = \xi_i \left(1 + \sum_{j=1, j\neq i}^{N} \frac{\tilde{\beta}_{ji}}{\tilde{\beta}_{ii}}\right)$$

which allows concluding. □

Proof of Proposition 11. Let $A_z = I + B + zE$ and $A = I + B$, where E is a matrix in which the unique non-null element is $(E)_{ij} = 1$. Let $\boldsymbol{e}_i$ the i-th vector of the euclidean basis of $\mathbb{R}^N$. We recall that

$$\frac{dA_z^{-1}}{dz}(0) = -A^{-1}\frac{dA}{dz}(0)A^{-1} = -A^{-1}EA^{-1}$$

In what follows we drop the evaluation at $z = 0$: we implicitly mean that all the involved functions depend on z and are evaluated at $z = 0$.

We have $\xi_i = \boldsymbol{e}_i^T A^{-1}\boldsymbol{u}$, so $\partial\xi_i/\partial z = -\boldsymbol{e}_i^T A^{-1}EA^{-1}\boldsymbol{u} = -\tilde{\beta}_{ii}\boldsymbol{e}_j^T\boldsymbol{\xi} = -\tilde{\beta}_{ii}\xi_j$. Since both $\tilde{\beta}_{ii}$ and ξ_j are positive, we conclude $\partial\xi_i/\partial z < 0$.

We have $\chi_i = \boldsymbol{u}^T A^{-1}\boldsymbol{e}_i$, so $\partial\chi_i/\partial z = -\boldsymbol{u}^T A^{-1}EA^{-1}\boldsymbol{e}_i = -\chi_i\tilde{\beta}_{ji}$, which allows concluding.

We have $\rho_i = (\boldsymbol{e}_i^T A^{-1}\boldsymbol{u}\boldsymbol{e}_i^T A^{-1}\boldsymbol{u})/(\boldsymbol{e}_i^T A^{-1}\boldsymbol{e}_i)$, so

$$\begin{aligned} \frac{\partial\rho_i}{\partial z} &= \frac{(-\boldsymbol{e}_i^T A^{-1}EA^{-1}\boldsymbol{u}\boldsymbol{e}_i^T A^{-1}\boldsymbol{u} - \boldsymbol{e}_i^T A^{-1}\boldsymbol{u}\boldsymbol{u}^T A^{-1}EA^{-1}\boldsymbol{e}_i)\boldsymbol{e}_i^T A^{-1}\boldsymbol{e}_i + \boldsymbol{e}_i^T A^{-1}\boldsymbol{u}\boldsymbol{e}_i^T A^{-1}\boldsymbol{u}\boldsymbol{e}_i^T A^{-1}EA^{-1}\boldsymbol{e}_i}{(\boldsymbol{e}_i^T A^{-1}\boldsymbol{e}_i)^2} \\ &= \frac{(-\tilde{\beta}_{ii}\xi_j\chi_i - \xi_i\chi_i\tilde{\beta}_{ji})\tilde{\beta}_{ii} + \xi_i\chi_i\tilde{\beta}_{ji}\tilde{\beta}_{ii}}{(\tilde{\beta}_{ii})^2} = -\frac{\xi_j\chi_i}{\tilde{\beta}_{ii}} \end{aligned}$$

□

Proof of Propositions 12,12 and 16. Let $A_z = I + B + zE$ and $A = I + B$.

Without loss of generality, we can focus on what happens to component 1. Let $\boldsymbol{e}_1$ the first vector of the euclidean basis of $\mathbb{R}^N$. We recall that

$$\frac{dA_z^{-1}}{dz} = -A_z^{-1}\frac{dA_z}{dz}A_z^{-1} = -A_z^{-1}EA_z^{-1} \Leftrightarrow \frac{dA^{-1}}{dz} = -A^{-1}EA^{-1}$$

From

$$\bar{\xi}_1(z) = \frac{\xi_i(z)}{\sum_{k=1}^N \xi_k(z)} = \frac{\boldsymbol{e}_1^T A_z^{-1}\boldsymbol{u}}{\boldsymbol{u}^T A_z \boldsymbol{u}}$$

we have

$$\frac{d\bar{\xi}_1}{dz}(0) = \frac{-\boldsymbol{e}_1^T A^{-1}EA^{-1}\boldsymbol{u}\boldsymbol{u}^T A^{-1}\boldsymbol{u} + \boldsymbol{e}_1^T A^{-1}\boldsymbol{u}\boldsymbol{u}^T A^{-1}EA^{-1}\boldsymbol{u}}{(\boldsymbol{u}^T A^{-1}\boldsymbol{u})^2} \tag{5.8}$$

We consider the case in which the unique non-null element is $(E)_{ij} = 1$, with $i \neq j$. It is easy to see that $A^{-1}\boldsymbol{u} = \boldsymbol{\xi}(0)$, $\boldsymbol{e}_1^T A^{-1}\boldsymbol{u}\boldsymbol{u}^T = \xi_1(0)\boldsymbol{u}^T$, $\boldsymbol{e}_1^T A^{-1}E = \tilde{\beta}_{1i}\boldsymbol{e}_j^T$ and $\boldsymbol{u}^T A^{-1}\boldsymbol{u} = \mu$, so we have that the numerator of the previous expression can be rewritten as (we drop evaluation at $z = 0$)

$$-\tilde{\beta}_{1i}\boldsymbol{e}_j^T\boldsymbol{\xi}\mu + \xi_1\boldsymbol{u}^T A^{-1}E\boldsymbol{\xi} = -\tilde{\beta}_{1i}\xi_j\mu + \xi_1\boldsymbol{u}^T A^{-1}E\xi_j$$

Since $\boldsymbol{u}^T A^{-1}\boldsymbol{e}_i = \chi_i$, we have that the right hand side in the last expression can be rewritten as

$$-\tilde{\beta}_{1i}\xi_j\mu + \xi_1\chi_i\xi_j \tag{5.9}$$

from which we can obtain the corresponding conditions.

The case of $E = U - I$ can be obtained by summing all terms in (5.9). We have

$$\begin{aligned}
\sum_{i=1}^N \sum_{j=1, j\neq i}^N (-\tilde{\beta}_{1i}\mu + \xi_1\chi_i)\xi_j &= \sum_{i=1}^N (-\tilde{\beta}_{1i}\mu + \xi_1\chi_i)(\mu - \xi_i) \\
&= \sum_{i=1}^N -\tilde{\beta}_{1i}\mu^2 + \xi_1\chi_i\mu + \tilde{\beta}_{1i}\mu\xi_i - \xi_1\chi_i\xi_i \\
&= -\xi_1\mu^2 + \xi_1\mu^2 + \tilde{\boldsymbol{\beta}}_1^T\boldsymbol{\xi}\mu - \xi_1\boldsymbol{\chi}^T\boldsymbol{\xi}
\end{aligned}$$

which allows concluding.

For the comparative statics on μ, we have

$$\frac{d\mu}{dz} = -\boldsymbol{u}^T A^{-1}EA^{-1}\boldsymbol{u}$$

If the unique non-null element of E is $(E)_{ij} = 1$, we have

$$\frac{d\mu}{dz} = -\boldsymbol{\chi}_j\boldsymbol{\xi}_i$$

If $E = U - I$, we have

$$\frac{d\mu}{dz} = -\boldsymbol{\chi}^T(\mu\boldsymbol{u} - \boldsymbol{\xi}) = \boldsymbol{\chi}^T\boldsymbol{\xi} - \mu^2$$

Noting that both the sum of the elements of $\boldsymbol{\chi}$ and $\boldsymbol{\xi}$ provides μ we can conclude that the previous difference is always negative. $\square$

Proof of Proposition 14. For both cases considered in the current proposition, we have already computed the values of ξ_i and μ in the proof of Propositions 7 and 8, so we refer to it for the related expressions.

We study the first scenario, in which B is such that $\beta_{ij} = \beta_i$ for $i \neq j, i, j \in \mathcal{N}$. The goal is to study the sign of

$$\frac{\partial\left(\frac{\xi_i}{\sum_{k=1}^N \xi_k}\right)}{\partial\beta_i} = \frac{\frac{\partial\xi_i}{\partial\beta_i}\left(\sum_{k=1}^N \xi_k\right) - \xi_i \sum_{k=1}^N \frac{\partial\xi_k}{\partial\beta_i}}{\left(\sum_{k=1}^N \xi_k\right)^2}$$

We have

$$\begin{aligned}
\frac{\partial\xi_i}{\partial\beta_j} &= \frac{-\displaystyle\sum_{k=1}^N \frac{\beta_i}{(1-\beta_i)(1-\beta_k)}}{1+\displaystyle\sum_{k=1}^N \frac{\beta_k}{1-\beta_k}} \\
&= -\frac{\dfrac{\beta_i}{(1-\beta_i)(1-\beta_j)^2}\left(1+\displaystyle\sum_{k=1}^N \frac{\beta_k}{1-\beta_k}\right) - \dfrac{\beta_i}{(1-\beta_i)(1-\beta_j)^2}\displaystyle\sum_{k=1}^N \frac{1}{1-\beta_k}}{\left(1+\displaystyle\sum_{k=1}^N \frac{\beta_k}{1-\beta_k}\right)^2} \\
&= \frac{-\dfrac{\beta_i}{(1-\beta_i)(1-\beta_j)^2}\left[\left(1+\displaystyle\sum_{k=1}^N \frac{\beta_k}{1-\beta_k}\right) - \displaystyle\sum_{k=1}^N \frac{1}{1-\beta_k}\right]}{\left(1+\displaystyle\sum_{k=1}^N \frac{\beta_k}{1-\beta_k}\right)^2} = \\
&= \frac{\dfrac{\beta_i}{(1-\beta_i)(1-\beta_j)^2}(N-1)}{\left(1+\displaystyle\sum_{k=1}^N \frac{\beta_k}{1-\beta_k}\right)^2} > 0
\end{aligned}$$

As a consequence we can compute

$$
\begin{aligned}
\sum_{k=1}^{N} \frac{\partial \xi_k}{\partial \beta_i} &= \frac{\frac{1}{(1-\beta_i)^2}\left(1+\sum_{k=1}^{N}\frac{\beta_k}{1-\beta_k}\right)-\frac{1}{(1-\beta_i)^2}\sum_{k=1}^{N}\frac{1}{1-\beta_k}}{\left(1+\sum_{k=1}^{N}\frac{\beta_k}{1-\beta_k}\right)^2} \\
&= \frac{\frac{1}{(1-\beta_i)^2}\left(1+\sum_{k=1}^{N}\frac{\beta_k}{1-\beta_k}-\sum_{k=1}^{N}\frac{1}{1-\beta_k}\right)}{\left(1+\sum_{k=1}^{N}\frac{\beta_k}{1-\beta_k}\right)^2} \\
&= \frac{(1-N)}{(1-\beta_i)^2\left(1+\sum_{k=1}^{N}\frac{\beta_k}{1-\beta_k}\right)^2}
\end{aligned}
$$

The derivative of the relative centrality index of firm i with respect to β_i is then

$$
\frac{\partial\left(\frac{\xi_i}{\sum_{k=1}^{N}\xi_k}\right)}{\partial \beta_i} = \frac{X}{Y}
$$

where

$$\begin{aligned}
X &= \frac{(1-N)\left(1+\sum_{k=1}^{N}\frac{\beta_k}{1-\beta_k}-\frac{\beta_i}{1-\beta_i}\right)}{(1-\beta_i)^2\left(1+\sum_{k=1}^{N}\frac{\beta_k}{1-\beta_k}\right)^2}\cdot\frac{\sum_{k=1}^{N}\frac{1}{1-\beta_k}}{1+\sum_{k=1}^{N}\frac{\beta_k}{1-\beta_k}} \\
&\quad -\left(\frac{1}{1-\beta_i}-\frac{\frac{\beta_i}{1-\beta_i}\sum_{j=1}^{N}\frac{1}{1-\beta_j}}{1+\sum_{k=1}^{N}\frac{\beta_k}{1-\beta_k}}\right)\frac{(1-N)}{(1-\beta_i)^2\left(1+\sum_{k=1}^{N}\frac{\beta_k}{1-\beta_k}\right)^2} \\
&= \frac{(1-N)}{(1-\beta_i)^2\left(1+\sum_{k=1}^{N}\frac{\beta_k}{1-\beta_k}\right)^2}\cdot\left[\frac{\left(1+\sum_{k=1}^{N}\frac{\beta_k}{1-\beta_k}-\frac{\beta_i}{1-\beta_i}\right)\sum_{k=1}^{N}\frac{1}{1-\beta_k}}{1+\sum_{k=1}^{N}\frac{\beta_k}{1-\beta_k}}\right. \\
&\quad \left. -\left(\frac{1}{1-\beta_i}-\frac{\frac{\beta_i}{1-\beta_i}\sum_{j=1}^{N}\frac{1}{1-\beta_j}}{1+\sum_{k=1}^{N}\frac{\beta_k}{1-\beta_k}}\right)\right]
\end{aligned}$$

in which the first factor is negative and

$$Y=\left(\frac{\sum_{k=1}^{N}\frac{1}{1-\beta_k}}{1+\sum_{k=1}^{N}\frac{\beta_k}{1-\beta_k}}\right)^2>0 \tag{5.10}$$

The sign of the fraction is then provided by

$$
\begin{aligned}
&-\frac{\left(1+\sum_{k=1}^{N}\frac{\beta_k}{1-\beta_k}-\frac{\beta_i}{1-\beta_i}\right)\sum_{k=1}^{N}\frac{1}{1-\beta_k}}{1+\sum_{k=1}^{N}\frac{\beta_k}{1-\beta_k}}+\left(\tfrac{1}{1-\beta_i}-\frac{\tfrac{\beta_i}{1-\beta_i}\sum_{k=1}^{N}\frac{1}{1-\beta_k}}{1+\sum_{k=1}^{N}\frac{\beta_k}{1-\beta_k}}\right)\\
&=-\frac{\tfrac{\beta_i}{1-\beta_i}\sum_{k=1}^{N}\frac{1}{1-\beta_k}}{1+\sum_{k=1}^{N}\frac{\beta_k}{1-\beta_k}}+\tfrac{1}{1-\beta_i}-\frac{\left(1+\sum_{k=1}^{N}\frac{\beta_k}{1-\beta_k}-\frac{\beta_i}{1-\beta_i}\right)\sum_{k=1}^{N}\frac{1}{1-\beta_k}}{1+\sum_{k=1}^{N}\frac{\beta_k}{1-\beta_k}}\\
&=\frac{\sum_{k=1}^{N}\frac{1}{1-\beta_k}}{1+\sum_{k=1}^{N}\frac{\beta_k}{1-\beta_k}}\left(-\tfrac{\beta_i}{1-\beta_i}-1-\sum_{k=1}^{N}\frac{\beta_k}{1-\beta_k}+\frac{\beta_i}{1-\beta_i}\right)\Bigg)+\tfrac{1}{1-\beta_i}\\
&=\frac{\sum_{k=1}^{N}\frac{1}{1-\beta_k}}{1+\sum_{k=1}^{N}\frac{\beta_k}{1-\beta_k}}\left(-1-\sum_{k=1}^{N}\frac{\beta_k}{1-\beta_k}\right)+\tfrac{1}{1-\beta_i}\\
&=\tfrac{1}{1-\beta_i}-\sum_{k=1}^{N}\frac{1}{1-\beta_k}<0
\end{aligned}
$$

Similarly, the derivative of the relative centrality index of firm i with respect to β_j is

$$
\frac{\partial \frac{\xi_i}{\sum_{k=1}^{N}\xi_k}}{\partial \beta_j}=\frac{X}{Y}
$$

where Y is defined by (5.10) while

$$\begin{aligned}
X &= \frac{\frac{\beta_i}{(1-\beta_i)(1-\beta_j)^2}(N-1)}{\left(1+\sum_{k=1}^{N}\frac{\beta_k}{1-\beta_k}\right)^2} \cdot \frac{\sum_{k=1}^{N}\frac{1}{1-\beta_k}}{1+\sum_{k=1}^{N}\frac{\beta_k}{1-\beta_k}} \\
&\quad - \left(\frac{1}{1-\beta_i} - \frac{\frac{\beta_i}{1-\beta_i}\sum_{k=1}^{N}\frac{1}{1-\beta_k}}{1+\sum_{k=1}^{N}\frac{\beta_k}{1-\beta_k}}\right) \frac{(1-N)}{(1-\beta_j)^2\left(1+\sum_{k=1}^{N}\frac{\beta_k}{1-\beta_k}\right)^2} \\
&= \frac{\frac{1}{(1-\beta_j)^2}(N-1)}{\left(1+\sum_{k=1}^{N}\frac{\beta_k}{1-\beta_k}\right)^2} \left[\frac{\beta_i}{1-\beta_i} \cdot \frac{\sum_{k=1}^{N}\frac{1}{1-\beta_k}}{1+\sum_{k=1}^{N}\frac{\beta_k}{1-\beta_k}} \right. \\
&\quad \left. + \left(\frac{1}{1-\beta_i} - \frac{\frac{\beta_i}{1-\beta_i}\sum_{k=1}^{N}\frac{1}{1-\beta_k}}{1+\sum_{k=1}^{N}\frac{\beta_k}{1-\beta_k}}\right)\right]
\end{aligned}$$

Simplifying X/Y we have

$$\frac{\partial \frac{\xi_i}{\sum_{k=1}^{N}\xi_k}}{\partial \beta_j} = \frac{N-1}{(1-\beta_j)^2(1-\beta_i)\left(\sum_{k=1}^{N}\frac{1}{1-\beta_k}\right)^2} > 0$$

Now we consider the second part of the proposition, in which B is such that $\beta_{ij} = \beta_j$ for $i \neq j, i, j \in \mathcal{N}$. Note that we can write $C = B^T = D + \boldsymbol{u}\boldsymbol{b}^T$, where D and $\boldsymbol{b}$ are respectively a diagonal matrix in which $d_{ii} = 1 - \beta_i$ for $i \in \mathcal{N}$ and a vector with $b_i = \beta_i$ for $i \in \mathcal{N}$. We indeed have $(I+C)^{-1} = ((I+B)^{-1})^T$ and applying again Sherman-Morrison formula we can write $(I+C)^{-1} = D^{-1} - \frac{D^{-1}\boldsymbol{u}\boldsymbol{b}^T D^{-1}}{1+\boldsymbol{u}^T D^{-1}\boldsymbol{b}}$. It is easy to see that the elements of $D^{-1}\boldsymbol{b}\boldsymbol{u}^T D^{-1}$ are given by $a_{ij} = \frac{\beta_j}{(1-\beta_i)(1-\beta_j)}$ while $1 + \boldsymbol{b}^T D^{-1}\boldsymbol{u} = 1 + b^T D^{-1}\boldsymbol{u} = 1 + \sum_{j=1}^{N}\frac{\beta_j}{1-\beta_j}$

We have $\boldsymbol{\xi} = D^{-1}\boldsymbol{u} - \frac{D^{-1}\boldsymbol{b}\boldsymbol{u}^T D^{-1}\boldsymbol{u}}{1+\boldsymbol{u}^T D^{-1}\boldsymbol{b}}$ and $u^T\boldsymbol{\xi} = \boldsymbol{u}^T(I+C)^{-1}\boldsymbol{u} = \boldsymbol{u}^T((I+B)^{-1})^T\boldsymbol{u} = \boldsymbol{u}^T((I+$

$B)^{-1})\boldsymbol{u}$

$$
\begin{aligned}
\xi_i &= \frac{\frac{1}{1-\beta_i} - \sum_{k=1}^{N} \frac{\beta_k}{(1-\beta_i)(1-\beta_k)}}{\left(1 + \sum_{k=1}^{N} \frac{\beta_k}{1-\beta_k}\right)} \\
&= \frac{1}{1-\beta_i} - \frac{\frac{1}{1-\beta_i} \sum_{k=1}^{N} \frac{\beta_k}{1-\beta_k}}{1 + \sum_{k=1}^{N} \frac{\beta_k}{1-\beta_k}} \\
&= \frac{1}{1-\beta_i} \left(1 - \frac{\sum_{k=1}^{N} \frac{\beta_k}{1-\beta_k}}{1 + \sum_{k=1}^{N} \frac{\beta_k}{1-\beta_k}} \right) \\
&= \frac{1}{1-\beta_i} \cdot \frac{1}{1 + \sum_{k=1}^{N} \frac{\beta_k}{1-\beta_k}}
\end{aligned}
$$

We note that $\boldsymbol{u}^T \boldsymbol{\xi}$ provides the same result for B and B^T. We have

$$
\begin{aligned}
\frac{\partial \xi_i}{\partial \beta_i} &= \frac{1}{(1-\beta_i)^2} \cdot \frac{1}{1 + \sum_{k=1}^{N} \frac{\beta_k}{1-\beta_k}} - \frac{1}{1-\beta_i} \cdot \frac{\frac{1}{(1-\beta_i)^2}}{\left(1 + \sum_{k=1}^{N} \frac{\beta_k}{1-\beta_k}\right)^2} \\
&= \frac{1}{(1-\beta_i)^2 \left(1 + \sum_{k=1}^{N} \frac{\beta_k}{1-\beta_k}\right)} \cdot \left(1 - \frac{1}{1-\beta_i} \cdot \frac{1}{1 + \sum_{k=1}^{N} \frac{\beta_k}{1-\beta_k}} \right) \\
&= \frac{(1-\beta_i)\left(1 + \sum_{k=1}^{N} \frac{\beta_k}{1-\beta_k}\right) - 1}{(1-\beta_i)^3 \left(1 + \sum_{k=1}^{N} \frac{\beta_k}{1-\beta_k}\right)^2}
\end{aligned}
$$

and

$$
\frac{\partial \xi_i}{\partial \beta_j} = -\frac{1}{1-\beta_i} \cdot \frac{\frac{1}{(1-\beta_j)^2}}{\left(1 + \sum_{k=1}^{N} \frac{\beta_k}{1-\beta_k}\right)^2}
$$

which allows writing

$$
\frac{\partial \xi_j}{\partial \beta_i} = -\frac{1}{1-\beta_j} \cdot \frac{\frac{1}{(1-\beta_i)^2}}{\left(1 + \sum_{k=1}^{N} \frac{\beta_k}{1-\beta_k}\right)^2}
$$

We need to evaluate the sign of

$$\frac{\partial\left(\frac{\xi_i}{\sum_{k=1}^{N}\xi_k}\right)}{\partial\beta_i} = \frac{\frac{\partial\xi_i}{\partial\beta_i}\left(\sum_{k=1}^{N}\xi_k\right) - \xi_i\sum_{k=1}^{N}\frac{\partial\xi_k}{\partial\beta_i}}{\left(\sum_{k=1}^{N}\xi_k\right)^2} = \frac{X}{Y}$$

where Y is provided by (5.10). We have

$$\begin{aligned}
\sum_{k=1}^{N}\frac{\partial\xi_k}{\partial\beta_i} &= -\frac{1}{(1-\beta_i)^2\left(1+\sum_{k=1}^{N}\frac{\beta_k}{1-\beta_k}\right)^2}\sum_{k=1,k\neq i}^{N}\frac{1}{1-\beta_k} \\
&\quad + \frac{1}{(1-\beta_i)^2\left(1+\sum_{k=1}^{N}\frac{\beta_k}{1-\beta_k}\right)}\cdot\left(1-\frac{1}{1-\beta_i}\cdot\frac{1}{1+\sum_{k=1}^{N}\frac{\beta_k}{1-\beta_k}}\right) \\
&= -\frac{1}{(1-\beta_i)^2\left(1+\sum_{k=1}^{N}\frac{\beta_k}{1-\beta_k}\right)^2}\sum_{k=1}^{N}\frac{1}{1-\beta_k} + \frac{1}{(1-\beta_i)^3\left(1+\sum_{k=1}^{N}\frac{\beta_k}{1-\beta_k}\right)^2} \\
&\quad + \frac{(1-\beta_i)\left(1+\sum_{k=1}^{N}\frac{\beta_k}{1-\beta_k}\right)-1}{(1-\beta_i)^3\left(1+\sum_{k=1}^{N}\frac{\beta_k}{1-\beta_k}\right)^2} \\
&= -\frac{1}{(1-\beta_i)^2\left(1+\sum_{k=1}^{N}\frac{\beta_k}{1-\beta_k}\right)^2}\sum_{k=1}^{N}\frac{1}{1-\beta_k} + \frac{1}{(1-\beta_i)^2\left(1+\sum_{k=1}^{N}\frac{\beta_k}{1-\beta_k}\right)} \\
&= \frac{1}{(1-\beta_i)^2\left(1+\sum_{k=1}^{N}\frac{\beta_k}{1-\beta_k}\right)}\left[-\frac{\sum_{k=1}^{N}\frac{1}{1-\beta_k}}{1+\sum_{k=1}^{N}\frac{\beta_k}{1-\beta_k}}+1\right]
\end{aligned}$$

which allows obtaining

$$
\begin{aligned}
X &= \frac{(1-\beta_i)\left(1+\sum_{k=1}^{N}\frac{\beta_k}{1-\beta_k}\right)-1}{(1-\beta_i)^3\left(1+\sum_{k=1}^{N}\frac{\beta_k}{1-\beta_k}\right)^2}\cdot\frac{\sum_{k=1}^{N}\frac{1}{1-\beta_k}}{1+\sum_{k=1}^{N}\frac{\beta_k}{1-\beta_k}} \\
&\quad -\frac{1}{1-\beta_i}\cdot\frac{1}{1+\sum_{k=1}^{N}\frac{\beta_k}{1-\beta_k}}\cdot\frac{1}{(1-\beta_i)^2\left(1+\sum_{k=1}^{N}\frac{\beta_k}{1-\beta_k}\right)}\left[-\frac{\sum_{k=1}^{N}\frac{1}{1-\beta_k}}{1+\sum_{k=1}^{N}\frac{\beta_k}{1-\beta_k}}+1\right] \\
&= \frac{(1-\beta_i)\left(1+\sum_{k=1}^{N}\frac{\beta_k}{1-\beta_k}\right)-1}{(1-\beta_i)^3\left(1+\sum_{k=1}^{N}\frac{\beta_k}{1-\beta_k}\right)^3}\cdot\left(\sum_{k=1}^{N}\frac{1}{1-\beta_k}\right) \\
&\quad -\frac{1}{(1-\beta_i)^3\left(1+\sum_{k=1}^{N}\frac{\beta_k}{1-\beta_k}\right)^2}\left[-\frac{\sum_{k=1}^{N}\frac{1}{1-\beta_k}}{1+\sum_{k=1}^{N}\frac{\beta_k}{1-\beta_k}}+1\right] \\
&= \frac{\left((1-\beta_i)\left(1+\sum_{k=1}^{N}\frac{\beta_k}{1-\beta_k}\right)-1\right)\cdot\left(\sum_{k=1}^{N}\frac{1}{1-\beta_k}\right)-\left[-\frac{\sum_{k=1}^{N}\frac{1}{1-\beta_k}}{1+\sum_{k=1}^{N}\frac{\beta_k}{1-\beta_k}}+1\right]\left(1+\sum_{k=1}^{N}\frac{\beta_k}{1-\beta_k}\right)}{(1-\beta_i)^3\left(1+\sum_{k=1}^{N}\frac{\beta_k}{1-\beta_k}\right)^3} \\
&= \frac{(1-\beta_i)\left(1+\sum_{k=1}^{N}\frac{\beta_k}{1-\beta_k}\right)\left(\sum_{k=1}^{N}\frac{1}{1-\beta_k}\right)-\left(\sum_{k=1}^{N}\frac{1}{1-\beta_k}\right)-\left(1+\sum_{k=1}^{N}\frac{\beta_k}{1-\beta_k}\right)+\sum_{k=1}^{N}\frac{1}{1-\beta_k}}{(1-\beta_i)^3\left(1+\sum_{k=1}^{N}\frac{\beta_k}{1-\beta_k}\right)^3} \\
&= \frac{(1-\beta_i)\left(\sum_{k=1}^{N}\frac{1}{1-\beta_k}\right)-1}{(1-\beta_i)^3\left(1+\sum_{k=1}^{N}\frac{\beta_k}{1-\beta_k}\right)^2} \\
&= \frac{\sum_{k=1,\neq i}^{N}\frac{1}{1-\beta_k}}{(1-\beta_i)^3\left(1+\sum_{k=1}^{N}\frac{\beta_k}{1-\beta_k}\right)^2} > 0
\end{aligned}
$$

Similarly, we have

$$\frac{\partial \frac{\xi_i}{\sum_{k=1}^{N} \xi_k}}{\partial \beta_j} = \frac{\frac{\partial \xi_i}{\partial \beta_j}\left(\sum_{k=1}^{N} \xi_k\right) - \xi_j \sum_{k=1}^{N} \frac{\partial \xi_k}{\partial \beta_i}}{\left(\sum_{k=1}^{N} \xi_k\right)^2} = \frac{X}{Y}$$

where Y is provided by (5.10) and

$$\begin{aligned}
X &= -\tfrac{1}{1-\beta_j} \cdot \frac{\frac{1}{(1-\beta_i)^2}}{\left(1+\sum_{k=1}^{N} \frac{\beta_k}{1-\beta_k}\right)^2} \cdot \frac{\sum_{k=1}^{N} \frac{1}{1-\beta_k}}{1+\sum_{k=1}^{N} \frac{\beta_k}{1-\beta_k}} \\
&\quad -\tfrac{1}{1-\beta_j} \cdot \frac{1}{1+\sum_{k=1}^{N} \frac{\beta_k}{1-\beta_k}} \frac{1}{(1-\beta_i)^2\left(1+\sum_{k=1}^{N} \frac{\beta_k}{1-\beta_k}\right)} \left[-\frac{\sum_{k=1}^{N} \frac{1}{1-\beta_k}}{1+\sum_{k=1}^{N} \frac{\beta_k}{1-\beta_k}} + 1\right] \\
&= -\tfrac{1}{1-\beta_j} \cdot \frac{\frac{1}{(1-\beta_i)^2}}{\left(1+\sum_{k=1}^{N} \frac{\beta_k}{1-\beta_k}\right)^3} \cdot \left(\sum_{k=1}^{N} \frac{1}{1-\beta_k}\right) \\
&\quad -\frac{1}{(1-\beta_j)(1-\beta_i)^2\left(1+\sum_{k=1}^{N} \frac{\beta_k}{1-\beta_k}\right)^2} \left[-\frac{\sum_{k=1}^{N} \frac{1}{1-\beta_k}}{1+\sum_{k=1}^{N} \frac{\beta_k}{1-\beta_k}} + 1\right] \\
&= -\frac{\sum_{k=1}^{N} \frac{1}{1-\beta_k}}{(1-\beta_j)(1-\beta_i)^2\left(1+\sum_{k=1}^{N} \frac{\beta_k}{1-\beta_k}\right)^3} + \frac{\sum_{k=1}^{N} \frac{1}{1-\beta_k}}{(1-\beta_j)(1-\beta_i)^2\left(1+\sum_{k=1}^{N} \frac{\beta_k}{1-\beta_k}\right)^3} \\
&\quad -\frac{1}{(1-\beta_j)(1-\beta_i)^2\left(1+\sum_{k=1}^{N} \frac{\beta_k}{1-\beta_k}\right)^2} < 0
\end{aligned}$$

□

Proof of Proposition 17. Let $p(Q) = a - bQ$. Using (3.34), the aggregate equilibrium is

$$Q^* = \frac{\mu(a-c)}{b(\mu+1)}$$

so profits can be written as

$$\pi_i = \frac{\xi_i(a-c)^2}{b(\mu+1)^2}$$

Assume that $B = \bar{B} + Z + \beta E$, where $E = U - I$, $\bar{B}$ is matrix with off-diagonal constant elements and Z is an hollow matrix whose off-diagonal elements describe the departure of elements of matrix B from the homogeneous weights distribution in $\bar{B}$. We assume that $\bar{B} = \bar{\beta}(U - I)$ is chosen in such a way the elements of Z have zero mean.

Computing the derivative of profits with respect to β we have

$$\pi_i' = \frac{(a-c)^2(\xi_i' + \mu\xi_i' - 2\xi_i\mu')}{b(\mu+1)^3}$$

The monotonicity of π_i' is determined by the sign of $\xi_i' + \mu\xi_i' - 2\xi_i\mu'$. Let $\bar{\boldsymbol{\xi}} = (I + \bar{B} + \beta E)^{-1}\boldsymbol{u}, \bar{\mu} = \boldsymbol{u}^T\bar{\boldsymbol{\xi}}, \boldsymbol{\varepsilon} = \boldsymbol{\xi} - \bar{\boldsymbol{\xi}}$ and $\mu_\varepsilon = \mu - \bar{\mu}$, we have

$$\begin{aligned}
\xi_i' + \mu\xi_i' - 2\xi_i\mu' &= (\bar{\xi} + \varepsilon_i)' + (\bar{\mu} + \mu_\varepsilon)(\bar{\xi} + \varepsilon_i)' - 2(\bar{\xi} + \varepsilon_i)(\bar{\mu} + \mu_\varepsilon)' \\
&= \bar{\xi}' + \bar{\mu}\bar{\xi}' - 2\bar{\xi}\bar{\mu}' + \varepsilon_i' + \mu_\varepsilon\bar{\xi}' + \bar{\mu}\varepsilon_i' + \mu_\varepsilon\varepsilon_i' \\
&\quad -2(\bar{\xi}\mu_\varepsilon' + \varepsilon_i\bar{\mu}' + \varepsilon_i\mu_\varepsilon')
\end{aligned}$$

We know that

$$\bar{\xi} = \frac{1}{(N-1)\bar{\beta}+1}, \; \bar{\mu} = \frac{N}{(N-1)\bar{\beta}+1}$$

from which

$$\bar{\xi}' = -\frac{N-1}{((N-1)\bar{\beta}+1)^2}, \; \bar{\mu} = -\frac{N(N-1)}{((N-1)\bar{\beta}+1)}$$

and hence

$$\bar{\xi}' + \bar{\mu}\bar{\xi}' - 2\bar{\xi}\bar{\mu}' = \frac{(N-1)^2(1-\bar{\beta})}{((N-1)\bar{\beta}+1)^3}$$

To have $\xi_i' + \mu\xi_i' - 2\xi_i\mu' > 0$ we then need

$$\left|\varepsilon_i' + \mu_\varepsilon\bar{\xi}' + \bar{\mu}\varepsilon_i' + \mu_\varepsilon\varepsilon_i' - 2(\bar{\xi}\mu_\varepsilon' + \varepsilon_i\bar{\mu}' + \varepsilon_i\mu_\varepsilon')\right| < \frac{(N-1)^2(1-\bar{\beta})}{((N-1)\bar{\beta}+1)^3}$$

We note that all the elements related to ε and their derivatives depend with continuity on the elements of Z, so, provided that Z is suitably small in some norm, the previous inequality holds.

Now let $p(Q) = 1/Q$. Using (3.34), the aggregate equilibrium is

$$Q^* = \frac{\mu - 1}{c\mu}$$

so profits can be written as

$$\pi_i = \frac{\xi_i}{\mu^2}$$

The derivative of π_i with respect to β is then

$$\pi_i' = \frac{\mu(\xi_i'\mu - 2\xi_i\mu')}{\mu^4}$$

Repeating the last part of the proof for the linear case we obtain a similar conclusion. $\square$

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