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Essays on the macroeconomic impact of heterogeneous banks

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Chapter I: Essays on the macroeconomic impact of heterogeneous banks

Essays on the macroeconomic impact of heterogeneous banks

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1. Introduction

The severe and prolonged recession brought about by the 2007/2008 financial crisis called for a renewed interest, by both the academic profession and policy-makers, into the functioning of the deep mechanics of the financial sector, as well as its interrelations with the real economy. Within the financial sector, interbank markets play a crucial role: they guarantee a stable supply of credit to the real economy by allowing banks in need of liquidity to gather additional funds from institutions with idle resources. However, allocational efficiency of resources comes at some cost. The inter-linkages nested into the interbank market potentially spread idiosyncratic shocks across the whole system, via counterparty risk and other forms of interbank contagion (Heider et al. 2009, Memmel and Sachs 2011). The effect on the aggregate output is therefore unclear ex ante. Do benefits offset costs? Does the efficiency effect prevail on welfare disruption brought about by contagion? In spite of its relevance, surprisingly low attention has been devoted by the macro literature to the analysis of the efficiency/contagion trade-off. Most contributions tend to focus on the adverse effects of interbank contagion, among the exceptions is Goldstein and Paulzner (2004). The perspective of the latter authors is however quite different from the aforementioned macro-literature: they consider a stylised macro-model, where investors can diversify their portfolios by purchasing assets from different countries. Portfolio diversification is welfare improving but, at the same time, creates inter-linkages that spread shocks from one country to the other. The aim of the present work is to investigate the effects of the presence of interbank markets on the main aggregate variables of the economy as a whole, with particular attention to the potential welfare gains that stems from "diversification". We consider a stylised environment, with a financial sector and a real sector. The financial sector is populated by two continua of banks, that interact with each other into an interbank market. The productive sector features a mass of entrepreneurs of two types: risky and safe. Risky entrepreneurs are exposed to idiosyncratic shocks that erode their capital endowments, hence their productive capacity. Such shocks can be tought as failed investments projects that propagate through the interbank market and directly effect safe(r) entrepreneurs. We show that the efficiency effect offsets the contagion effect: the contraction of output arising from the risky cluster of the production sector is more than compensated by the subsequent expansion in the output of the safe cluster. This observed countervailing effect directly stems from the presence of a (well-functioning) interbank market: upon observing the shocks, banks whith surplus liquidity move resources from the - now riskier - interbank market to the safer real investments.

2. Literature Review

Our work contributes to the growing bunch of macro-economic literature labelled *financial frictions* that builds on th eseminal contribution by Bernanke et al. (1999). Broadly speaking, this literature aims at improving the understanding of the real effects of financial variables, by providing a more realistic representations of the behaviour of financial agents and institutions. Within this literature, our contribution relates, in particular, to Gertler and Karadi (2011), Boissay et al. (2016) and Gerali et al. (2010), that consider models where frictions arise essentially from: (i) market power of banks; (ii) moral hazard; and (iii) regulatory constraints. Market power acts as an amplification mechanism of the borrowing constraint of the type devised by Kiyotaki and Moore (1997), whereby the borrowing capacity of productive firms is endogeneously constrained by the market value of collaterals they can provide (Gerali et al. 2010, Iacoviello 2005). Higher mark-ups stemming from banks' profit maximization decision inefficiently contract the supply of credit. The implicit costs generated by the possibility of moral hazard by borrowers reduce, ceteris paribus, the aggregate amount of loans extended to firms (Gertler and Karadi 2011, Boissay et al. 2016). Regulatory constraints, such as capital and reserve requirements, indirectly limit the availability of credit through the anchoring of banks' lending capacity to mandatory leverage ratios. Despite the focus of most papers in the macro-finance literature on shocks to the demand side of credit markets, the latter subset of frictions allows for shocks to originate from the supply side of credit. This seems more in line with the evidence provided by the 2007/2008 crisis (Brunnermeier 2009). Although linkages between individual institutions are shown to potentially spread idiosyncratic shocks across the whole system, e.g. via counterparty risk (Heider et al. 2009), they are seldom explicitly represented in theoretical models. Among the more relevant exceptions are Boissay et al. (2016), Gertler and Karadi (2011), deWalque et al. (2010) and Giri (2018). The first two papers analize the functioning of interbank markets from a purely macro perspective, and abstract from a detailed description of decision making by individual institutions. The latter two, instead, provide a more in-depth investigation into individual decision making, at the cost of a higher stylisation at the aggregate level. In particular, trade on the interbank market is mandated exogeneously via the imposition of a sequential structure, whereby a retailer bank, that lends out money to entrepreneurs, is forced to acquire resources for lending from an upstream bank, which collects deposits from households. As a consequence, the borrower and the lender bank can be seen as the flip-sides of the same coin. In order to provide a more comprehensive analysis of interbank markets, we extend the general setup proposed by Gerali et al. (2010) with a fully-fledged interbank market, that builds on, and extends, deWalque et al. (2010) and Giri (2018). As highlighted by the latter authors, the similarities between our model and the one proposed by Gerali et al. (2010) relates the present work to other two branches of literature. On one hand, there are models with financial intermediaries and a time-varying spread between deposits and lending rates (e.g., Goodfriend and McCallum 2007, Andres and Arce 2008, Christiano, Motto, and Rostagno 2008, Curdia and Woodford 2009, Gilchrist, Ortiz, and Zakrajsek 2009). On the other side, authors have studied the role of equity and bank capital for the transmission of macroeconomic shocks (deWalque, Pierrard, and Rouabah 2008, van den Heuvel 2008, Meh and Moran 2010). To conclude, our is to provide insights of whether an idiosyncratic shock either spreads or fades away through the interbank sector. So, we hope that our contribution will enrich the literature studying propagation through interbank market in which for many works the interbank market is a propagation device for shocks (Heider et al. 2009, Memmel and Sachs 2011) while for others, an idyosincratic shock is absorbed by the system (Steinbacher et al. 2014, Bednarek et al 2015).

3. The model

In this section we present our working strategy. We build up three models: (i) a benchmark model, (ii) a model with a sequential interbank sector and (iii) a model with a financial sector represented by a single bank - no interbank sector. The benchmark model presents a rich setup, where a propagation mechanism of shocks across the real economy arises from the presence of an interbank market. There are two separate productive sectors differentiated by the riskiness of their businesses. The banks differ for their ability to gather deposits from households (one has a deposit constraint) and for the kind of entrepreneurs they lend to. The setup with the sequential interbank sector consider only one intermediate productive sector and only one kind of banks can collect deposits from households. The interbank sector is sequential, because one continuum of banks collect deposits from households and then supply them to the other banks in form of interbank loans. The latter banks are the only one interacting with entrepreneurs. The last model embeds a financial sector à la Gerali (2010), where there is only one continuum of banks gathering deposits from households and providing them to the real economy (entrepreneurs). Exploiting these configurations we aim to understand the inefficiencies (rigidities) brought by the presence of interbank sector, the analogies within different models and then, if possible, study the propagation of an idiosyncratic shock particular of our model. The main goal of the proposed comparison is twofold. First, by comparing the response of the three specifications to standard shocks, such as TFP and banking capital shocks, we verify that the models are indeed comparable. If this is the case, then all the differences observed in the benchmark model must boil down to our representation of the interbank market. Additionally, consistency across the alternative specifications serves as a preliminary robustness check. Second, we show that a more realistic representation of the interbank market induces (positive) spillovers across different sectors of the real economy. To this extent, we introduce an entirely novel type of real shock, in the form of an abrupt loss of capital suffered by risky entrepreneurs. This exogeneous shock can be interpreted as the implicit loss brought about by a failed investment project. We show that the contraction in the demand of credit by risky entrepreneurs, after a shock occurs, induces a corresponding contraction in the demand of interbank loans. Banks with excess liquidity respond to this contraction by expanding their supply of credit to safe entrepreneurs. Overall, the aggregate output remains (almost) stable, yet it shows a mild contraction. A detailed description of the three specifications is provided in the remainder of this section. An in-depth comparison of the results is deferred to section 4.

3.1. Benchmark Model

Our model (BK henceforth) is built following the works of Gerali et al. (2010) and Giri (2018). We consider a stylised economy populated by infinitely lived households and two types (risky and safe) of entrepreneurs, each group having a unit mass. Additionally, the economy features a financial sector populated by two continua of banks and an interbank market. Each continuum of banks is exogenously assigned to one group of entrepreneurs. Households differ from entrepreneurs with respect to their time preferences. In particular, their discount rate (β_h) is higher than entrepreneurs' discount rate (β_e). As a consequence, househoulds are net savers in the economy, and provide funds to the entrepreneurs - that are, therefore, net borrowers. Households consume, work, and save facing a budget constraint. Entrepreneurs of both types consume, ask banks for loans and own firms that produce intermediate goods, but do not work. Entrepreneurs differ in the riskyness of their activity, so that the riskier entrepreneur has to pay back an higher interest on loans received by the banks.

The economy features a simple financial system, with a banking sector supervised by a non-strategic policy making, and an interbank market. The banking sector is populated by two groups of banks: borrowers and lenders. Borrower banks collect deposits from households, provide loans to risky entrepreneurs, exchange resources on the interbank market and are forced to pay a cost for diverging by a mandatory capital requirement. Lender banks, as well, gather deposits from households, provide loans to safe entrepreneurs and provide interbank loans to the borrower banks. We assume an exogeneous matching between entrepreneurs and banks, whereby risky entrepreneurs are allowed to ask for loans only to the borrower banks, whilst safe entrepreneurs interact only with lender banks. For the sake of simplicity, it is convenient to think about safe entrepreneurs as owners of well-estabished businesses that generate stable cash flows over time, and of risky entrepreneurs as innovative firms, subject to an higher degree of idiosyncratic uncertainty. Banks differ also with respect to their ability to gather deposits from households: borrower banks face a deposit constraint, so that the quantity of deposits they can collect is exogeneously limited to a maximum amount. Banks maximise profits subject to a regulatory capital requirement mandated by the supervisor of the financial system. It is worth stressing that, for the sake of exposition, in this setup the difference in risk-taking between the entrepreneurs is represented directly by the interest rate they have to face while asking banks for loans. We follow this way of proceding because the main focus of the work is to investigate the propagation of a shock from one sector to the other. We can think that the borrower banks have a better capacity to evaluate risky business than lender ones, and want to undertake them in order to make more profits. Moreover, they are able to apply an higher interest rate for the loans they provide. In addition, in our specific experiment we design a shock arising only amongst the riskier entrepreneurs. Of course this approach has some drawback and we think that a more precise characterisation of the riskiness of entrepreneurs will be interesting for future research. Along the core of the model presented above, we assume that some agents (called retailers) buy the intermediate goods from entrepreneurs in a competitive market, brand them at a unit cost and sell the differentiated good at a price which includes a mark-up over the purchasing cost; prices are sticky à la Rotemberg (1982), implying the existence of a New Keynesian Phillips curve. As already stated, profits from retailers are redistributed to households. Moreover, fixed-capital creation is subject to some adjustment costs and is carried out by capital-good producers. These agents are introduced as a modelling device for deriving an explicit expression for the price of capital, which enters entrepreneurs' borrowing constraint. Our model is closed by a central bank following a Taylor rule to stabilise the level of both output and inflation, following the one proposed by Gerali et al. (2010). The image below represents the financial flow diagram of the model. For the sake of exposition, in the presentation of the model below we chose to get rid of the indices (i) for the agents acting in perfect competition (households, entrepreneurs, wholesale banks), but we keep them for agents acting in a monopolistic-competitive environment (retailers, and the retailer-branch of borrower bank).

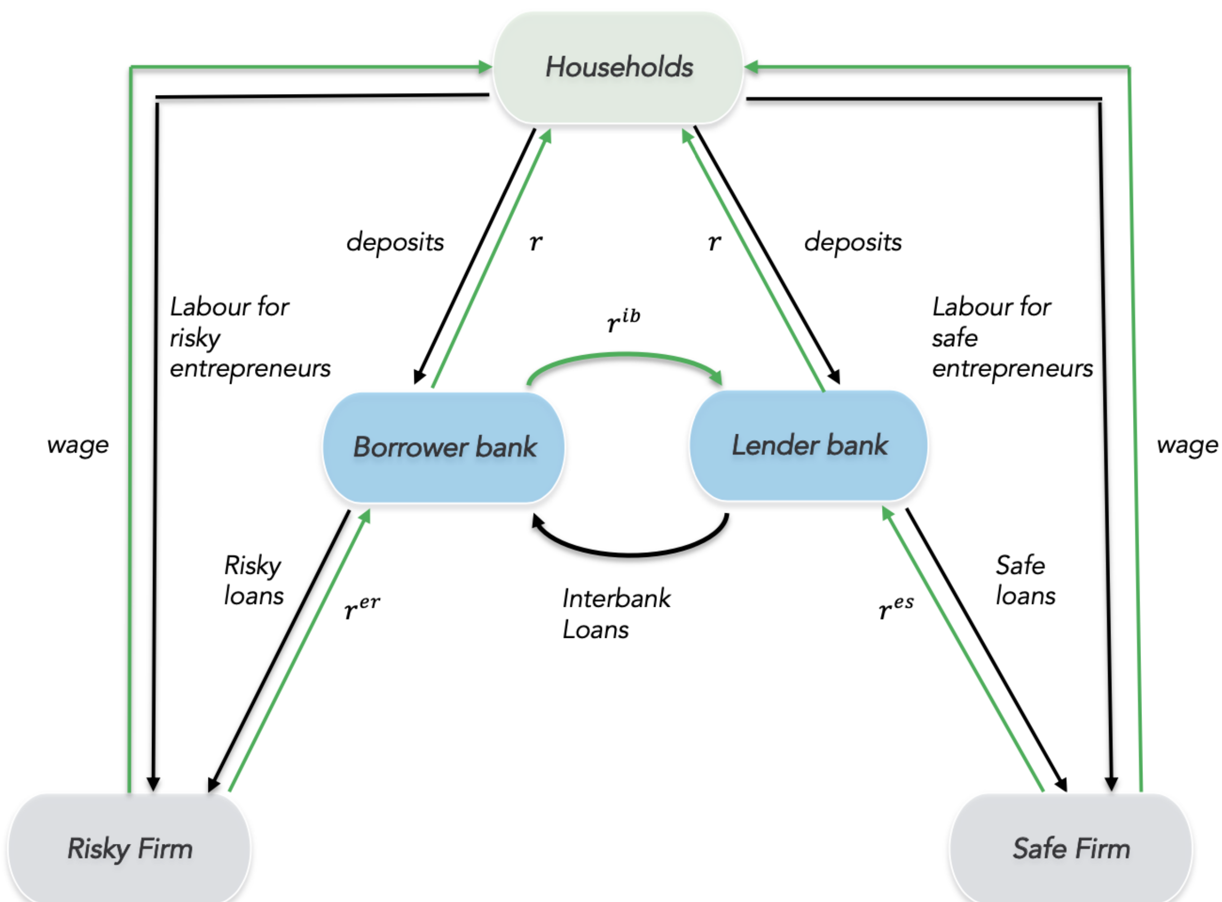

Fig. 1. Financial flow diagram of the model

Note: for sake of exposition the above interest rates are represented only by letters. Rates are: (i) r^{er} interest rate for risky loans; (ii) r^{es} interest rate for risky loans; (iii) r^{ib} interest rate on interbank loans; (iv) r policy rate

3.1.1. Households

There is a continuum of households of unitary mass, uniformly distributed on $[0, 1]$, that maximise their utilities with respect to consumption, labour and deposits. They face the following optimization problem:

$$\max_{c_t^h, n_t^h, d_t^h} E_0 \sum_{t=0}^{\infty} \beta_h^t \left(log\left(c_t^h\right) - \psi \frac{n_t^{h(1+\phi)}}{1+\phi} \right)$$

subject to the sequence of budget constraints:

$$c_t^h + d_t^h \leq w_t^h n_t^h + \frac{(1+r_{t-1})}{\pi_t} d_{t-1}^h + J_t^{lb} + J_t^R \tag{3.1}$$

where c_t^h is the households' consumption, d_t^h are the households' savings, $w_t^h n_t^h$ is the salary per hours worked, r_t is the interest rate on deposits. In this setup, the households are considered to be the owners of both lender banks and retailers, so that J_{t-1}^{lb} are lender banks' profits and J_t^R are profits from retailers' activity. From the F.O.C.s, we obtain the following standard *Euler condition* and *labour supply* of households:

$$\frac{1}{c_t^h} = \beta_h^t E_t \left[\frac{1}{c_{t+1}^h} \frac{(1+r_t)}{\pi_{t+1}} \right] \tag{3.2}$$

$$w_t^h = \psi n_t^{h^\phi} c_t^h \tag{3.3}$$

3.1.2. Entrepreneurs

As already mentioned, the productive sector in the model is composed by two kinds of entrepreneurs: risky and safe. The problem they face is identical, but they feature two main differences: (i) the interest rate on loans they have to pay back is not equal (higher for rikier) and (ii) their relations with the financial sector. In fact, the risky entrepreneurs can borrow funds only from borrower banks, while the safe interact solely with lender banks. We can interpret this difference as a bigger bank willing to undertake more risk in order to gain higher profits, and having a better ability to evaluate these risks having more *skin in the game*. It is important to underline that a proper characterisation of a riskier behaviour from one entrepreneur is not present in this setup. The decision about following this path is that the main focus of the present work is the evaluation of the transmission of risk from one side of the economy to another through the interbank sector. So, we chose to keep the model simpler and to embed the different level of riskiness in the spread between the rates the banks apply to entrepreneurs.

Risky Entrepreneurs choose the optimal quantity of their consumption (c_t^{er}) solving the following standard intertemporal optimisation problem:

$$\max_{c_t^{er}, k_t^{er}, n_t^{er}, l_t^{er}} E_0 \sum_{t=0}^{\infty} \beta_e^t log\left(c_t^{er}\right) \tag{3.4}$$

subject to the following sequence of budget constraints:

$$c_t^{er} + w_t^h n_t^{er} + \frac{(1 + r_{t-1}^{er})}{\pi_t} l_{t-1}^{er} + q_t^k k_t^{er} \leq \frac{y_t^{er}}{x_t} + l_t^{er} + q_t^k (1 - \delta_k) k_{t-1}^{er} \tag{3.5}$$

where c_t^{er} is the entrepreneurs consumption, $w_t^h n_t^{er}$ is the salary they have to pay to workers for the labour they provide, r_t^{er} is the interest rate on loans, l_t^{er} are loans they ask the borrower bank for, k_t^{er} is the quantity of capital they own, y_t^{er} is the production at time t and δ_k is the depreciation rate of capital. The quantity x_t represents the mark-up applied by the retailers on final prices, so that the production in the budget constaint is expressed in real terms. The discount factor β_e^t is common between the two categories of entrepreneurs. The production function is approximated by a Cobb-Douglas: $y_t^{er} = a_t^e {k_{t-1}^{er}}^{\alpha} {n_t^{er}}^{1-\alpha}$. Provided that the entrepreneurs are net borrowers of this stylised economy, they have to face a borrowing constraint of the form:

$$(1 + r_t^{er}) l_t^{er} \leq m_t^{er} E_t (q_{t+1}^k \pi_{t+1} (1 - \delta_k) k_t^{er}) \tag{3.6}$$

where m_t^{er} is the stochastic Loan to Value (LTV) ratio for the collateral, here assumed to be capital. Following this definition, $1 - m_t^{er}$ can be seen as the cost a bank has to bear to exercise its claim on the collaterals - in case the loan would not be payed off. As for the households, from the F.O.C.s we obtain the following consumption and investments *Euler equations* and labour demand for risky entrepreneurs:

$$\frac{1}{c_t^{er}} = \mu_t^{er} (1 + r_t^{er}) + \beta_e E_t \frac{1}{c_{t+1}^{er}} \left(\frac{(1 + r_t^{er})}{\pi_{t+1}} \right) \tag{3.7}$$

$$\frac{1}{c_t^{er}} q_t^k = E_t \{ \mu_t^{er} m_t^{er} q_{t+1}^k \pi_{t+1} (1 - \delta_k) + \beta_e \frac{1}{c_{t+1}^{er}} [q_{t+1}^k (1 - \delta_k) + r_{t+1}^{kr}] \} \tag{3.8}$$

$$\frac{(1-\alpha)y_t^{er}(i)}{n_t^{er}x_t} = w_t \tag{3.9}$$

where μ_t^{er} is the lagrangean multiplier associated to the borrowing constraint and the return on capital is expressed as $r_t^{kr} = \alpha a_t^{er}[k_{t-1}^{er}]^{\alpha-1} n_t^{er1-\alpha}/x$.

Safe Entrepreneurs safe entrepreneurs face an identical intertemporal optimisation problem, defined as:

$$\max_{c_t^{es},k_t^{es},n_t^{es},l_t^{es}} E_0 \sum_{t=0}^{\infty} \beta_e^t log\,(c_t^{es}) \tag{3.10}$$

subject to the sequence of budget constraints:

$$c_t^{es} + w_t^h n_t^{es} + \frac{(1+r_{t-1}^{es})}{\pi_t} l_{t-1}^{es} + q_t^k k_t^{es} \leq \frac{y_t^{es}}{x_t} + l_t^{es} + q_t^k(1-\delta_k)k_{t-1}^{es} \tag{3.11}$$

where c_t^{es} is the safe entrepreneurs consumption, $w_t^h n_t^{es}$ is the salary they have to pay to workers for their labour, r_t^{es} is the interest rate on loans, l_t^{es} are loans they ask the lender bank for, k_t^{es} is the quantity of capital they own, y_t^{es} is the production at time t and δ_k is the depreciation rate of capital. The production function, is approximated by a Cobb-Douglas function of the form: $y_t^{es} = a_t^e {k_{t-1}^{es}}^{\alpha} n_t^{es1-\alpha}$. The borrowing constraint is:

$$(1+r_t^{es})l_t^{es} \leq m_t^{es} E_t(q_{t+1}^k \pi_{t+1}(1-\delta_k)k_t^{es}) \tag{3.12}$$

where m_t^{es} is, as before, the stochastic Loan to Value (LTV) ratio for the collateral. Once again, from the F.O.C.s we obtain the consumpion and investments *Euler equation* and labour demand for safe entrepreneurs:

$$\frac{1}{c_t^{es}} = \mu_t^{es}(1+r_t^{es}) + \beta_e E_t \frac{1}{c_{t+1}^{es}} \left(\frac{(1+r_t^{es})}{\pi_{t+1}} \right) \tag{3.13}$$

$$\frac{1}{c_t^{es}} q_t^k = E_t\{\mu_t^{es} m_t^{es} q_{t+1}^k \pi_{t+1}(1-\delta_k) + \beta_e \frac{1}{c_{t+1}^{es}}[q_{t+1}^k(1-\delta_k) + r_{t+1}^{ks}]\} \tag{3.14}$$

$$\frac{(1-\alpha)y_t^{es}(i)}{n_t^{es}x_t} = w_t \tag{3.15}$$

where μ_t^{es} is the lagrangean multiplier associated to the borrowing constraint and the return on capital is expressed as $r_t^{ks} = \alpha a_t^{es}[k_{t-1}^{es}]^{\alpha-1} n_t^{es1-\alpha}/x$.

3.1.3. Banks and the interbank market

Overall, our banking sector is based on Gerali et al. (2010) and Giri (2018). We chose to follow their structure because this framework not only allows to evaluate the effects of shocks hitting the supply side of credit, but seems to better fit our purpose of studying the decision undertaken by a single type of banks to face the shock. Banks that borrow on the interbank market have to obey a balance-sheet identity of the form *loans = deposits + capital + interbank loans* and are required to meet a target capital-to-assets ratio (i.e., the inverse of leverage), exogenously mandated by a regulator. Deviations from this target entail a quadratic cost. Following Gerali et al (2010),

> *the optimal leverage ratio in this context can be thought of as capturing the trade-offs that would arise in the decision of how much own resources to hold, or alternatively as a simple shortcut for studying the implications and costs of regulatory capital requirements.*

Furthermore, we allow for an interbank market where lender banks can provide financial resources to borrower banks in the form of interbank loans. Since borrower banks face a constraint on the deposits they can collect, trade on the interbank market arises spontaneously. Banks differ in their corporate structure. Lender banks have a wholesale branch that set the optimal quantity of loans to safe entrepreneurs and to the interbank market. Borrower banks, instead, are composed by a wholesale branch and a retailer branch. The wholesale sets the optimal quantity of loans and deposits. Both wholesale banks act as price takers under perfect competition. The retail branches of borrower banks act under monopolistic competition. They optimally set a mark-up on the interest rate on loans they extend to risky entrepreneurs. We choose to stick with this setup proposed by Gerali et al. (2010) because it seems reasonable to think that the borrowing banks have market power in setting rates on loans they provide, since the agents they interact with are assumed to be riskier. Since the total amount of loans is set by wholesale branches, retail branches take it as an exogeneous constraint. Moreover, borrower banks are deposit constrained, since they can collect a maximum amount $\overline{D}^{bb} > 0$ of deposits from households. They can, however, obtain additional financial resources by demanding loans to the lender bank on the interbank market. Furthermore, borrower banks can unilaterally decide to repay interbank loans only partially. This is in line with the default mechanism devised by Giri (2018) and DeWalque (2010).

When the banks opt for partial repayment they incur a quadratic reputational cost. The mechanism behind the functioning of our interbank market is the presence of four spreads among interest rates. Specifically, interest rates are tied by the following relation:

$$r^{er} > R^{bb} > r^{ib} > r^{es} > r$$

Where r^{er} is the rate applied by borrower banks on loans to risky entrepreneurs (through their retail branch), R^{bb} is the rate of the wholesale branch, r^{ib} is the rate on the interbank market, r^{es} is the rate to safe entrepreneurs and r is the policy/deposit rate. Several insights come from the above hierarchical relation: (i) there is a mark-up between $r^{er} > R^{bb}$ due to the monopolistic power of deficit bank (retailer branch); (ii) $R^{bb} > r^{ib}$ due to a mark-up given jointly by the presence of regulatory costs and by the possibility for borrower banks to default on interbank loans; (iii) $r^{ib} > r$ with the possibility for the central bank of directly influencing this relation exploitimg its ability to set the policy rate in order to push/depress the interbank activity.

Borrower Banks

As already mentioned, borrower banks are organised according to a corporate structure that features a wholesale and a retail branch. The wholesale branch maximises profits choosing the optimal quantity of loans they provide to firms, deposits they collect from households, the optimal quantity of interbank market loans and the share of interbank default that the bank could decide not to pay back. These banks face a constraint on deposits, so that they are forced to collect resources on the interbank market. The optimization problem faced by the wholesale branch is structured as follows:

$$\max_{L_t^{bb}, IB_t, \sigma_t^{bb}, D_t^{bb}} E_0 \sum_{t=0}^{\infty} \beta_e^t \lambda_t^{er} [\left(1+R_t^{bb}\right) L_t^{bb} - L_{t+1}^{bb}\pi_{t+1} - \left(1+r_t^{ib}\right)\left(1-\sigma_t^{bb}\right) IB_t + \tag{3.16}$$

$$+ IB_{t+1}\pi_{t+1} + \left(K_{t+1}^{b}\pi_{t+1} - K_t^b\right) + D_{t+1}^{bb}\pi_{t+1} - (1+r_t)D_t^{bb} - Adj_t^{kb} - Adj_t^{\sigma}]$$

Where L_t^{bb} are the optimal quantities of loans they provide to retail branch, IB_t is the quantity of interbank loans they ask for to lender banks, D_t^{bb} is the quantity of deposits they collect from the households, K_t^b is the capital of borrowing bank and σ_t^{bb} represents the share of interbank loans the borrowing bank decides not to pay back. Borrower banks face two different types of costs. Costs represented by:

$$Adj_t^{kb} = \frac{\kappa_{kb}}{2}\left(\frac{K_t^b}{L_t^{bb}} - \nu_b\right)^2 K_t^b \tag{3.17}$$

are the adjustment costs due to deviations from the optimal level of leverage. The lower the ratio between bank capital and the total asset, the higher the penalty cost of providing an additional unit of loans to the retail branch. This setup follows Gerali et al. (2010) and ν_b is set to represent the Basel II capital requirement constraint, so that is set at 8%. Borrower banks face also reputational costs as proposed by Dib (2010) and deWalque et al. (2010). The term

$$Adj_t^{\sigma} = \frac{\chi_{bb}}{2}\left(\frac{IB_{t-1}\sigma_{t-1}^{bb}}{\pi_t}\right)^2 \tag{3.18}$$

is the penalty the borrower banks have to pay on loans they not repay. It can be seen as a difficulty to find resources on the interbank market due to "bad reputation" in the following periods. Borrower banks have to face two constraints: a balance sheet constraint and a constraint of the quantity of deposits they can collect;

$$\begin{aligned} L_t^{bb} =& IB_t + K_t^b + D_t^{bb} \\ D_t^{bb} \leq& \overline{D} \end{aligned} \tag{3.19}$$

The problem of the wholesale branch of borrower banks is closed by the capital law of motion:

$$K_t^b \pi_t = (1-\delta_b) K_{t-1}^b + J_{t-1}^{bb} \tag{3.20}$$

From the F.O.C.s, we obtain the following conditions:

$$\begin{aligned} R_t^{bb} =& r_t^{ib} - \sigma_t^{bb}\left(1 + r_t^{ib}\right) - \kappa_{kb}\left(\frac{K_t^b}{L_t^{bb}} - \nu^b\right)\left(\frac{K_t^b}{L_t^{bb}}\right)^2 \\ &+ \beta_e \chi^{bb} E_t\left\{\left(\frac{\sigma_t^{bb}}{\pi_{t+1}}\right)^2 IB_t \frac{\lambda_{t+1}^{er}}{\lambda_t^{er}}\right\} \end{aligned} \tag{3.21}$$

Equation 3.21 shows the relation between the wholesale loan rate of the borrowing bank and the interbank market conditions, taking into account also the adjustments costs this institution has to bear. Specifically, the loan rate is affected by the capital requirement and by the expected value of defaults. If the bank is undercapitalized, the banks transmit the cost it has to pay for diverging by the optimal ratio to borrowers, through the wholesale rate. In the same way we can interpret the contribution of the share of expected interbank defaults on the wholesale interest rate: whenever the bank defaults, the subsequent costs are charged over the interest rate. From the F.O.C.s we obtain also the optimal setting of interbank loans default ratio.

$$\sigma_t^{bb} = E_t \left(\frac{\lambda_t^{er} \left(1 + r_t^{ib}\right) \left(\pi_{t+1}\right)^2}{\lambda_{t+1}^{er} \beta_e \chi^{bb} IB_t} \right) \tag{3.22}$$

Equation 3.22 describes the evolution of the interbank default over time. The default ratio has a positive relation with the interest rate over interbank borrowing. The impact of interbank borrowing is instead negative. The formula above shows that when the interbank rate is high the debt of the borrowing bank is more costly, so it is not convenient to pay back all the amount of interbank loans. On the other side, an high level of interbank loans can be seen as the need of the borrowing bank to gather resources from interbank market, so that an high default ratio will be more expensive due to reputational costs.

$$\gamma_t^{bb} = R_t^{bb} - r_t + \kappa_{kb} \left(\frac{K_t^b}{L_t^{bb}} - \nu_b \right) \left(\frac{K_t^b}{L_t^{bb}} \right)^2 \tag{3.23}$$

The equation 3.23 shows when the borrower banks exhaust their capacity of collecting deposits from households, in other words when the deposits' constraint is binding ($\gamma_t^{bb} > 0$). We can see that it is always convenient for a bank to collect all the deposits it can gather if the spread between the wholesale rate and the deposit rate more than offset the costs of diverging from the mandatory capital-to-assets ratio ν_b. Obviously, the specific case in which the spread has to offset costs is when the bank is exceeding the optimal ratio with an abundance of loans, having $\frac{K_t^b}{L_t^{bb}} - \nu_b < 0$. The total profits of the borrower bank are defined as:

$$J_t^{bb} = r_t^{er} l_t^{bb} + \left(1 + r_t^{ib}\right) \sigma_t^{bb} IB_t - r_t^{ib} IB_t - r_t D_t^{bb} - \sum Adj_t^{bb} \tag{3.24}$$

with $\sum Adj_t^{bb}$ collecting all the costs borrower banks have to face, from the wholesale to the retail branch. Finally, the retailer branches of borrower banks are assumed to act in a monopolistically competitive environment. They optimally set the interest rate on loans to risky entrepreneurs in order to maximise their profits. The problem they face takes the form:

$$\max_{r_t^{er}(j)} E_0 \sum_{t=0}^{\infty} \beta_e^t \lambda_t^{er} \left[r_t^{er}(j) l_t^{bb}(j) - R_t^{bb} L_t^{bb}(j) - Adj_t^{rer} \right] \tag{3.25}$$

subject to the demand for risky loans:

$$l_t^{bb}(j) = \left(\frac{r_t^{er}(j)}{r_t^{er}}\right)^{-\epsilon_t^{er}} l_t^{bb} \tag{3.26}$$

where

$$Adj_t^{rer} = \frac{\kappa_{rer}}{2}\left(\frac{r_t^{er}(j)}{r_{t-1}^{er}(j)} - 1\right)^2 r_t^{er} l_t^{bb} \tag{3.27}$$

From the solution of the above problem, the first order conditions for the retail branches give us a *New Keynesian Phillips Curve* for loan interest rates.

$$\begin{aligned} 1 - \frac{\Lambda_t^{bb}}{\Lambda_t^{bb} - 1} + \frac{\Lambda_t^{bb}}{\Lambda_t^{bb} - 1}\frac{R_t^{bb}}{r_t^{er}} - \kappa_{rer}\left(\frac{r_t^{er}}{r_{t-1}^{er}} - 1\right)\frac{r_t^{er}}{r_{t-1}^{er}} + \\ + \beta_e E_t\left[\frac{\lambda_{t+1}^{er}}{\lambda_t^{er}}\kappa_{rer}\left(\frac{r_{t+1}^{er}}{r_t^{er}} - 1\right)\left(\frac{r_{t+1}^{er}}{r_t^{er}}\right)^2 \frac{l_{t+1}^{bb}}{l_t^{bb}}\right] = 0 \end{aligned} \tag{3.28}$$

where Λ_t^{bb} is the markup of final loans rate on the wholesale rate.

Lender Banks

Lender banks maximize their profits, choosing the optimal quantity of loans they provide to safe firms, and deposits they collect from households. For now, we assume these banks cannot own capital. The balance sheet of the lender bank is summarised by the following relation:

$$IB_t + L_t^{lb} = D_t^{lb}$$

where IB_t are the interbank loans, L_t^{lb} are the loans to the safe entrepreneurs and D_t^{lb} is the quantity of deposits gathereded from the households. Since balance sheet constraint has only deposits on the liabilities side, we can rewright the above relation as $IB_t = s_t D_t^{lb}$, so that the maximization problem a lender bank has to face takes the form:

$$\begin{aligned} \max_{s_t, D_t^{lb}} E_0 \sum_{t=0}^{\infty} \beta_h^t \lambda_t^h [\left(1 + r_t^{ib}\right) s_t D_t^{lb}\left(1 - \sigma_t^{bb}\right) - s_{t+1} D_{t+1}^{lb}\pi_{t+1} + \left(1 + r_t^{es}\right)\left(1 - s_t\right) D_t^{lb} - \\ - \left(1 - s_{t+1}\right) D_t^{lb}\pi_{t+1} - \left(1 + r_t\right) D_t^{lb} + D_{t+1}^{lb}\pi_{t+1} - Adj_t^s] \end{aligned} \tag{3.29}$$

where $s_t D_t^{lb}$ represents the share of interbank lending on total assets. The term Adj_t^s represents the quadratic monitoring costs a lender bank has to face relatively to the quantity of interbank loans it provides relative to its own resources. Namely:

$$Adj_t^s = \frac{\Theta}{2}\left[(s_t - \bar{s})\right]^2 D_t^{lb} \tag{3.30}$$

From the F.O.C.s of the problem above we obtain the optimal ratio of interbank loans for the lender bank:

$$s_t = \bar{s} + \frac{r_t^{ib} - \sigma_t^{bb}\left(1 + r_t^{ib}\right) - r_t^{es}}{\Theta D_t^{lb}} \tag{3.31}$$

with $\bar{s}$ is the steady state quantity of share of own resources deployed for interbank loans. Two main driving forces are in motion here: on one hand, the increase of defaults push up the interbank interest rate. Since the lender bank is risk neutral, higher interest rates represent an incentive to increase the exposition on the interbank market. On the other hand, the increase in defaults negatively affects the amount of interbank lending through the disutility cost. The second relation is given by:

$$r_t^{es} - r_t = -s_t(r_t^{ib} - \sigma_t^{bb}(1 + r_t^{ib}) - r_t^{es}) + \Theta(s_t - \bar{s})^2 \tag{3.32}$$

The equation 3.32 shows the spread between the safe rate and the policy rate. It is easy to see that this spread shrinks if the spread between interbank rate and policy rate widens, since it is more profitable for banks to lend on the interbank market. On the other side, there is a positive relation with the monitoring cost of interbank loans. In fact, when the cost of diverging from the optimal ratio rises, the banks find more convenient to provide loans to safe entrepreneurs, rising the safe rate and enlarging the spread. Lastly, the aggregate profits of lender bank take the following form:

$$J_t^{sb} = r_t^{ib} IB_t + r_t^{es} L_t^{lb} - \left(1 + r_t^{ib}\right)\sigma_t^{bb} IB_t - r_t D_t^{lb} - Adj_t^s \tag{3.33}$$

3.1.4. Capital good producers

Capital goods producers are in perfect competition and buy last period undepreciated capital $(1-\delta)k_{t-1}$ at a price Q_t^k from entrepreneurs (the owners of these firms) and i units of final goods from the retailers at price P_t. The stock of effective capital $\overline{x}$ is then sold back to entrepreneurs at price Q_t^k. Considering that $q_t^k \equiv \frac{Q_t^k}{P_t}$, they solve the following problem:

$$\max_{\bar{x}_t, i_t} E_0 \sum_{t=0}^{\infty} \beta_e^t \lambda_t^{es}\left(q_t^k \Delta\bar{x}_t - i_t\right) \tag{3.34}$$

subject to:

$$\bar{x}_t = \bar{x}_{t-1} + \left[1 - \frac{\kappa_i}{2}\left(\frac{i_t}{i_{t-1}} - 1\right)^2\right] i_t \tag{3.35}$$

with $\Delta\bar{x}_t = k_t - (1-\delta)k_{t-1}$ flow of output.

From the F.O.C.s we obtain the new capital amount produced:

$$k_t = (1-\delta_k)k_{t-1} + \left[1 - \frac{\kappa_i}{2}\left(\frac{i_t}{i_{t-1}} - 1\right)^2\right] i_t \tag{3.36}$$

and the real price of q_t^k is determinde by:

$$1 = q_t^k \left[1 - \frac{\kappa_i}{2}\left(\frac{i_t}{i_{t-1}} - 1\right)^2 - \kappa_i\left(\frac{i}{i_{t-1}} - 1\right)\frac{i_t}{i_{t-1}}\right] + \beta_E E_t\left[\frac{\lambda_{t+1}^{es}}{\lambda_t^{es}} q_{t+1}^k \kappa_i \left(\frac{i_{t+1}}{i_t} - 1\right)\left(\frac{i_{t+1}}{i_t}\right)^2\right] \tag{3.37}$$

3.1.5. Retailers

The model is closed by a retailer sector, with a monopolistic competitive firm wich buy the intermediate goods by entrepreneurs and transform them at a unit cost. The retail goods market is assumed to be monopolistically competitive as in Bernanke, Gertler, and Gilchrist (1999). Retailers' prices are sticky and are indexed to a combination of past and steady-state inflation, with relative weights parameterized by ι_p; if retailers want to change their price beyond what indexation allows, they face a quadratic adjustment cost parameterized by κ_p. Since it is uncommon in financial frictions models to have two productive sectors, considering that the two entrepreneurs are not in competition among themselves and that the main focus of this work is to investigate the dynamics arising from the interbank market, it seems plausible to assume that the price of goods sold by risky entrepreneurs P_t^{er} is equal to the price proposed by safer ones P_t^{er}. Therefore, the problem faced by the retailer takes the form:

$$\max_{P_t(i)} E_0 \sum_{t=0}^{\infty} \beta_h^t \lambda_t^h \left[P_t(i)y_t(i) - P_t^{es}y_t^{es}(i) - P_t^{er}y_t^{er}(i) - \frac{\kappa_p}{2}\left(\frac{P_t(i)}{P_{t-1}(i)} - \pi_{t-1}^{\iota_p}\pi^{1-\iota_p}\right)^2 P_t y_t\right] \tag{3.38}$$

subject to:

$$y_t(i) = \left(\frac{P_t(i)}{P_t}\right)^{-\varepsilon_t^y} y_t \tag{3.39}$$

where ε_t^y is the stochastic demand price elasticity.

Assuming that $P_t^{er} = P_t^{es}$, and remembering that $y_t^{er} + y_t^{es} = y_t$, we obtain the classic Phillips curve of the form:

$$1 - \varepsilon_t^y + \frac{\varepsilon_t^y}{x_t} - \kappa_p \left(\pi_t - \pi_{t-1}^{\iota_p} \pi^{1-\iota_p} \right) \pi_t + \beta_h E_0 \left[\frac{\lambda_{t+1}^h}{\lambda_t^h} \kappa_p (\pi_{t+1} - \pi_t^{\iota_p} \pi^{1-\iota_p}) \pi_{t+1}^2 \frac{y_{t+1}}{y_t} \right] = 0 \quad (3.40)$$

3.1.6. Monetary Policy

We follow Gerali et al. (2010) in setting the Monetary policy exploited by the central bank to set the policy rate. The Taylor rule has the form:

$$(1 + r_t) = (1 + r)^{(1-\phi_R)} (1 + r_{t-1})^{\phi_R} \left(\frac{\pi_t}{\pi} \right)^{\phi_\pi (1-\phi_R)} \left(\frac{y_t}{y_{t-1}} \right)^{\phi_y (1-\phi_R)} \quad (3.41)$$

where ϕ_π is the response of the central bank to the inflation, while ϕ_y is the response to the output growth. Finally, r is the steady state policy rate.

3.2. Alternative specifications and comparison

As already highlited in the previous section, one of the main objectives of the present work is to analyse how a more realistic representation of an interbank market - into an otherwise standard financial-friction model - affects and/or contradicts well-established results. In order to gain as many insights as possible from this exercise, we compare our benchmark model with two alternative specifications: (i) one featuring a banking but no interbank market, (ii) the other presenting a "sequential" interbank sector of the type envisaged by deWalque et al (2010) and Giri (2018). The first model without interbank sector (NIBK henceforth) builds on the simplified version of Gerali et al. (2010) proposed by Gambacorta and Signoretti (2014). The second, featuring the interbank sector, (IBK henceforth) draws upon the work of Giri (2018). As stated above, the main difference between the two models boils down to the way they model the financial sector. The aim of this way of proceeding is twofold. On one hand, we want to understand what are the main insights/advantages of taking an interbank sector into consideration. On the other side, we want to understand what are the gains/costs of introducing some complexity, and from our viewpoint an additional hint of realism, to a sequential interbank sector. In addition, the proposed comparison quantifies the degree of comparability between our model and other, more standards, financial frictions DSGEs. Recall indeed that the aim of this work is (also) to investigate whether, to what extent, and in what direction, the interbank market propagates shocks across the real economy.

The two alternative specifications share with the benchmark model the characterisation of households, entrepreneurs, retailers and capital good producers. Households consume, supply labour and collect deposits. Entrepreneurs consume, produce a wholesale good they sell to retailers, face a borrowing constraint and are net borrowers in the economy. Differently to the benchmark model, the IBK and NIBK models feature a single type of entrepreneurs. On the contrary, the benchmark model features risky and safe entrepreneurs. Furthermore, the IBK and NIBK models differ with respect to their financial sector: while the financial sector of the IBK is made up of a banking sector and an interbank market, in the NIBK the financial sector *coincides* with the banking sector. The NIBK presents a financial intermediary à la Gerali (2010), where the banks are composed by a wholesale and a retail branch. As in the benchmark model, the wholesale branch operates in a perfectly competitive environment, and sets the optimal quantity of deposits the bank can collect as well as the amount of loans the latter can extend. The retail branch operates under monopolistic competition and sets the mark-ups on the interest rate of loans. This design is present also for banks that lend money to entrepreneurs in IBK model. The interbank market analysed in this model is very similar to the one of BK model. However, two differences deserve to be mentioned. First, the lender banks in IBK model are allowed to gather deposits from households but *not* to lend funds to entrepreneurs directly. Its only available investment alternatives are government bonds offered in fixed quantity and with an exogeneous return assumed identical to the policy rate. Second, and crucial, borrower banks can not collect deposits from households, so they must rise liquidity from the lender banks. This peculiar specification implicitely defines a "seuqntial structure" of the interbank market of the type analysed in deWalque (2010) and Giri (2018).

3.3. The transmission channels

To close this section we present shortly what are the main driving forces in financial frictions model. We observe three mechanisms in action: (i) the existence of a collateral channel, (ii) the debt-deflation effect and (iii) the credit-supply channel. First two are quite common in macro-financial literature, while the latter is specific for this type of setup. The existence of a collateral channel is a well-known consequence of the presence of a borrowing constraint and operates via the impact of changes of asset valuations on debtors' balance-sheet conditions. In this literature, this is the main channel able to generate a financial-accelerator effect (Bernanke and Gertler, 1995). The idea is that due to agency problems, some agents in the economy can only borrow a fraction of the value of the assets that they can post as collateral, so that a positive technology shock increases aggregate output as well as asset prices. The increase in asset prices, in turn, pushes up the borrowing limit and induces an increase in lending to those agents for which the constraint was binding. So, the additional available resources can be used to finance more consumption and investment, generating an extra kick to aggregate demand which reinforces the initial rise in output. The *debt-deflation* effect is present in our model because we chose not to index debt-repaiments. In fact, debts are expressed in nominal terms. So, the fall in inflation raises the cost of debt services, further depressing entrepreneurial consumption (Iacoviello 2005). Finally, the credit-supply channel is activated by an exogenously given capital-to-asset ratio (the inverse of a leverage ratio) that influence the supply conditions (i.e. lending spreads) in order to bring this ratio back to the desired level whenever it deviates from it (Gerali et al. 2010).

4. Simulations

In this section we present the simulations and results of a list of shocks to our models. The simulation of two shocks, total factor productivity (TFP) and to banking capital, is performed for all three models. Instead, the shock to the interbank riskiness is exclusive for models which embed the interbank sector: IBK and BK. The TFP shock is a classic productivity shock, while the banking capital shock can be tought as an exogenous and unexpected destruction of banks' capital, as in Gerali et al. (2010). The shock specific to models with an interbank sector is a rise to the quantity of defaults on loans borrower banks has received, introducing more riskiness on the interbank market. Since we want to make the comparison as fair as possible, we set common relations amongst the steady states, calibrating some parameters consequently. In the remainder of this section we present an exercise specific for our model, in which we simulate a capital loss arising from the risky sector.

4.1. Calibration

As mentioned above, some steady state parameters are calibrated in line with Gerali et al. (2010) and some others are calibrated in order to guarantee common relations among different equilibria. In table 1 we report an extensive list of the main parameters and of steady state ratios of our benchmark model. A list of parameters and steady state ratios of the other models is provided in the appendix. We start our numerical setting by imposing a steady state ratio of the Basel II capital requirement equal to 8%. The cost for managing the bank's capital position in line with this optimal ratio is 0.0400. Following this calibration, our steady state default rate for banks is approximately 0.0028, that corresponds to a 1.12% of interbank defaults on a year base. The capital share in the production function and the depreciation rate of physical capital are in line with Gerali et al. (2010) and respectively equal to 0.25 and 0.025. The Loan to value for risky entrepreneur m^{er} is 0.33 while the one faced by the safer, m^{es}, is 0.26. Since we are interested in a steady state where an interbank market is present, we set a spread between the risky and the safe rate of 1.6% and a spread between the interbank rate and the policy rate equal to 1.1%. It is easy to see that, assuming that the steady state capital-to-average ratio (k^b/L^{bb}) is equal to the mandatory regulation (0.08), and setting the spread between the interbank rate and the policy rate to 1.1%, from equation 3.23 a positive spread between the wholesale rate and a policy rate arises naturally, making the lagrange multiplier related with this spread positive as well. Since the multiplier γ^{bb} associated to the deposit constraint of borrower bank is positive, we see that the constraint is binding in steady state, so that banks exhaust all the deposits they can collect.

(i)		
Parameter	Definition	Value
β^p	Patient households discount factor	0.9943
β^e	Entrepreneurs discount factor	0.9750
α	Capital share in the production function	0.2500
δ^k	Depreciation rate of physical capital	0.0250
ϕ	Inverse of Frisch elasticity	1.0000
ν	Basel II capital requirement	0.0800
δ^b	Cost for managing the bank's capital position	0.0400
κ^p	Price stickyness	20.5700
κ^{kb}	Leverage dev. costs	11.4900
Θ	Monitoring costs for default	0.0140
ϕ_R	Taylor rule coefficient on r	0.8000
ϕ_π	Taylor rule coefficient on π	1.3000
ϕ_y	Taylor rule coefficient on y	0.3500
(ii)		
Ratio	Expression	Value
Gross inflation rate	π	1.000
Consumption-to-output ratio	C/Y	0.88
Deposits of borrower banks	D^{bb}/D	0.10
Investment-to-output ratio	I/Y	0.11
Household Income share	$w^h l^h/Y$	0.63
Firms' profit-to-output ratio	J^R/Y	0.17
Banks' capital ratio	K^b/L^{bb}	0.08
Policy rate	r	2%
Safe loans' rate	r^{es}	2%
Spread	$r^{ib} - r$	1.1%
Spread	$r^{er} - r$	1.6%

Table 1: (i) Main parameters (ii) Steady-state ratios

4.2. Technology shock

We start with a technology shock designed, as common, as a perturbation to total factor productivity. Since our model embeds two productive sector, in order to simulate an aggregate technological shock we decide to apply the same perturbation to both agents. Recalling from subsection 3.1.2 that the production functions are defined as:

$$y_t^{er} = a_t^e {k_{t-1}^{er}}^{\alpha} {n_t^{er}}^{1-\alpha}; \quad y_t^{es} = a_t^e {k_{t-1}^{es}}^{\alpha} {n_t^{es}}^{1-\alpha}$$

what we are considering is essentially a shock to the above variable a_t^e, that we assume to follow the $AR(1)$ stochastic process:

$$a_t^e = \rho_a a_{t-1}^e + \eta_t^a$$

in which ρ_a is 0.94 and the variance of η_t^a is calibrated so that the variance of a_t^e equals 1 per cent. The results are reported in Figure 2.

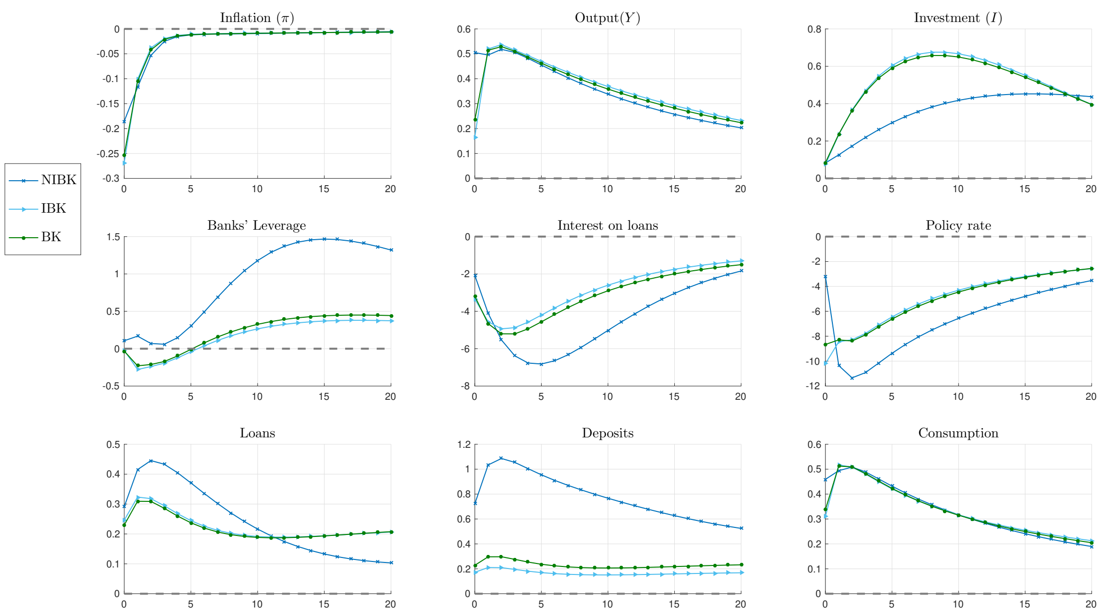

Fig. 2. Transmission of a positive Technology shock

Note: All variables are percentage deviations from the steady state. The blue line with stars is from the model without the interbank market (NIBK). The light-blue line with triangles is from the model with the interbank (IBK). The green line with circles is from the benchmark model (BK).

As we can see, the response of the system to the technological shock is in line with standard results in the literature. Namely, there is an expansion of the output and consumptions, accompanied by an instantaneous decrease in inflation. Investments temporarily rise, and then converge back smootly to their steady state value. As common, the central bank accomodates for the expansion, lowering on impact the policy rate. First thing to underline is that, as mentioned above, these models embed a debt-deflation mechanism. Following Iacoviello (2005), debt deflation plays a role since obligations are not indexed, the fall in inflation raises the cost of debt services, making households better off but further depressing entrepreneurial consumption. Of particular interest are the variables related to the banking sector. Loans to the real economy rise in response to a higher demand for financial resources induced by the increased productivity of factors. As a consequence, interest rates fall due to the relative abundance of financial resources. Following a consistent dynamic, the leverage of banks increases with the quantity of loans extended to firms. While we observe qualitatively the same paths among all three models, quantitatively the dynamic shows differnt outcomes. The economy represented by the model without interbank sector (NIBK) shows a more stable inflation and an higher rise, on impact, of output. The rise in output comes along with a rise in consumption that keeps inflation more stable relative to the other models. This difference is driven by the fact that an interbank sector brings along some dispersion due to the possibility of defaults by borrower banks. Knowing that less (more) resources will be available for the final producers, even though we observe an increase of loans in all models, the central bank acts more (less) aggresively on policy rate. This, in turn, is transmitted to the rate of loans that decreases. In NIBK, the absence of an interbank market allows banks to decrease less, on impact, the final rate that instead decreases more during the life of the shock because there will be less need for resources after first periods. On the other side, the lower interest rate set by the central bank in order to push consumption makes hoseholds less willing to gather deposits, providing less funds to the financial sector. Finally, households are richer in the NIBK framework for a favourable combination of the debt-deflation effect and higher interest rate on deposits. Due to these reasons, labour receive an higher reward in terms of wage, allowing entrepreneurs to invest less. Banking capital remains almost unaffected since lower interest rates drive deficit-banks' profits down. The last observation we point out is about the puzzling response of banks' leverage at the rise of the shock. Initially, the leverage is negative for IBK and BK likely because of the configuration of this variable, since in these settings it takes under consideration only loans provided by the borrower bank, since it is the only one owning capital.

4.3. *Banking capital shock*

The second shock of our exercise hits the financial sector of the models. As in Gerali et al. (2010), we simulate an exogenous and unexpected destruction of banks' capital and study how it spreads through the whole economy. Specifically, the loss hits the banks' directly in its law of motion (eq. 3.20) as presented below:

$$K_t^b \pi_t = (1 - \delta_b) \frac{K_{t-1}^b}{\varepsilon_t^{kb}} + J_{t-1}^{bb}$$

where ϵ_t^{kb} is assumed to follow the $AR(1)$ stochastic process:

$$\varepsilon_t^{kb} = \rho_{kb} \varepsilon_{t-1}^{kb} + \eta_t^{kb}$$

in which ρ_{kb} is 0.81. The results are reported in Figure 2.

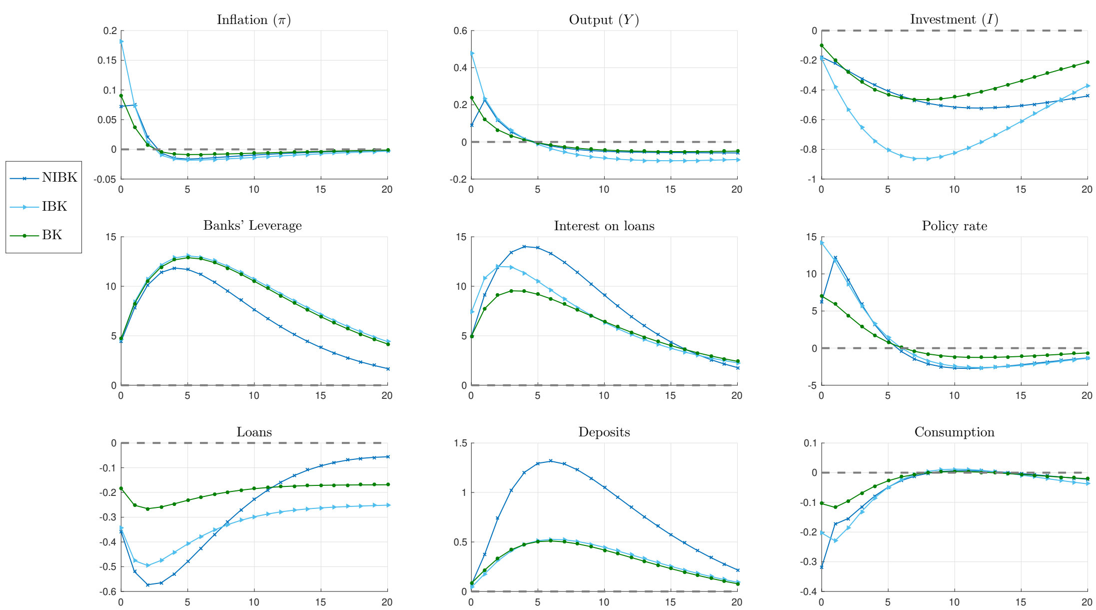

Fig. 3. Transmission of a Banking-capital shock

Note: All variables are percentage deviations from the steady state. The blue line with stars is from the model without the interbank market (NIBK). The light-blue line with triangles is from the model with the interbank (IBK). The green line with circles is from the benchmark model (BK).

As for TFP shock, all the models present the same movement qualitatively speaking, but the persistance and the magnitude of the responses of different variables vary across models. First, it is worth to underline the response of output for all models, which seems strange on a fisrt sight. The reason behind an increase in output after a banking capital shock is that, in all our setups, wages are not sticky but flexible, making the labour supply less rigid. Following the fact that there will be less availability of loans in the economy, entrepreneurs ask for more labour trying to contain a dampening in the level of output. An increased labour demand rises up the level of wages that is followed by a rise in the supply of labour by the households. The peak observed in output for IBK at the impact can be interpreted as a conseuqence of a perceived higher loss in the system due to dispersion of resources of the interbank sector. In fact, since workers understand they will be poorer in the future, they supply more labour right after the shock, driving the output up. The fallen of loans is similar and stronger for NIBK and for IBK, because in BK setup we guarantee a quantity of funds to borrower banks not present in the other economies, represented by the deposits they can gather before the deposit constraint binds. In addition, since the quantity of loans does not plunge the borrower bank is able to contain the mark-up it applies on final loans, less limitating the demand. This transmission mechanism can be seen also observing the pass-through of rate on loans to investments. Since they rise more for IBK, it will be more difficult to obtain loans on impact, depressing the value of the collateral (K). Since it is not convenient for entrepreneurs to invest in capital, the investments fall and wage rises, following the increase in labour demand (as we can see by output in first period). As for the technology shock, the quantity of loans rises faster for the NIBK model, due to the absence of resources dispersion brought by the interbank sector.

4.4. Default shock on the interbank market

The default shock on the interbank market is represented by an higher probability of default on interbank loans by the borrower bank (σ_t^{bb}). In this case, we consider an additive shock to the optimal quantity of loans to default on set by the deficit bank based on Giri (2017). Namely, the shock enters linearly in the equation 3.22:

$$\sigma_t^{bb} = E_t \left(\frac{\lambda_t^{er} \left(1 + r_t^{ib}\right) \left(\pi_{t+1}\right)^2}{\lambda_{t+1}^{er} \beta_e \chi^{bb} IB_t} \right) + \varepsilon_t^{\sigma_{bb}} \tag{4.1}$$

As usual, the shock follows the $AR(1)$ stochastic process:

$$\varepsilon_t^{\sigma_{bb}} = \rho_{\sigma_{bb}} \varepsilon_{t-1}^{\sigma_{bb}} + \eta_t^{\sigma_{bb}}$$

in which $\rho_{\sigma_{bb}}$ is 0.85 and the steady state value of σ_t^{bb} is 0.0028. The obtained impulse responses are reported in Figure 3 below.

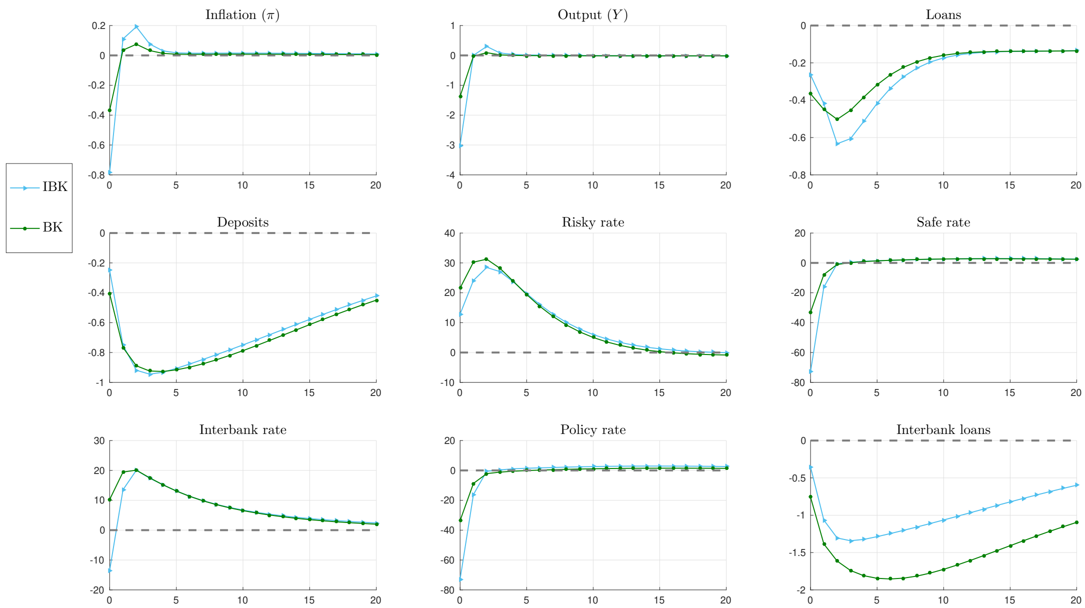

Fig. 4. Transmission of a Bank-riskiness shock

Note: All variables are percentage deviations from the steady state. The light-blue line with triangles is from the model with the interbank (IBK). The green line with circles is from the benchmark model (BK).

As we can see, the common effect of an interbank default shock modifies the composition of the balance sheet of the borrower banks, pushing down their assets by the erosion of interbank loans. These banks, due to the flight-to-quality mechanism (Heider et al. 2015, Caballero and Krishnamurthy 2008) that pushes lender bank to invest more on safe businesses, reduces more the quantity of interbank loans. The combination of a lower supply and an higher riskiness of the interbank market, leads up the interbank rate. While this happens clearly in our BK model, the response of the interbank rate in the IBK model is instead negative on impact. Two mechanism are working here. One is that in our setup the aggregate output is more stable, as resources drain from one sector (risky) to the other one (safe), because lender bank finds interbank loans more dangerous. The wider fluctuation drags down the policy rate, that in turn leads down the interbank rate. On the other side, since the borrower banks in our setup are allowed to collect deposits at a limited extent, after an interbank riskiness shock we observe a stronger decrease in interbank loans that brings the interbank rate up. The increase of interbank rate spreads to the real economy through the rate of loans to entrepreneurs, descouraging the demand of credit and, in turn, the quantity of investments. The banks' leverage move in the opposite direction relative to the technology shock presented above, driven by the fall of loans. It is clear from the IRFs above that our setup with two productive sectors makes the economy less volatile and proner to better face shocks arising from the financial sector. The lender banks face the risk providing more resources to safer businesses in the real economy, making the whole system more stable.

4.5. Idiosincratic shock to riky entrepreneurs' capital

The last shock of our exercise simulates an histantaneous loss of capital by risky entrepreneurs. Since the available capital at the start of time t is the one stockpiled at $t-1$, we simulate a shock directly hitting this quantity. Specifically, the shock takes the form $\frac{k_{t-1}^{er}}{\varepsilon_t^{kr}}$ and enters directly in the riky entrepeurs' budget constraint (eq. 3.5), in the production function and in reteurn on capital relation. As for the above shocks, the capital loss follows an $AR(1)$ stochastic process:

$$\varepsilon_t^{kr} = \rho_{kr}\varepsilon_{t-1}^{kr} + \eta_t^{kr}$$

in which ρ_{kr} is 0.81. Since this shocks hits directly the risky entrepreneurs-borrower banks' side, it can be of some help to compare the results with IRFs from a shock to the riskiness of the interbank system. The obtained impulse responses are reported in Figure 5 below.

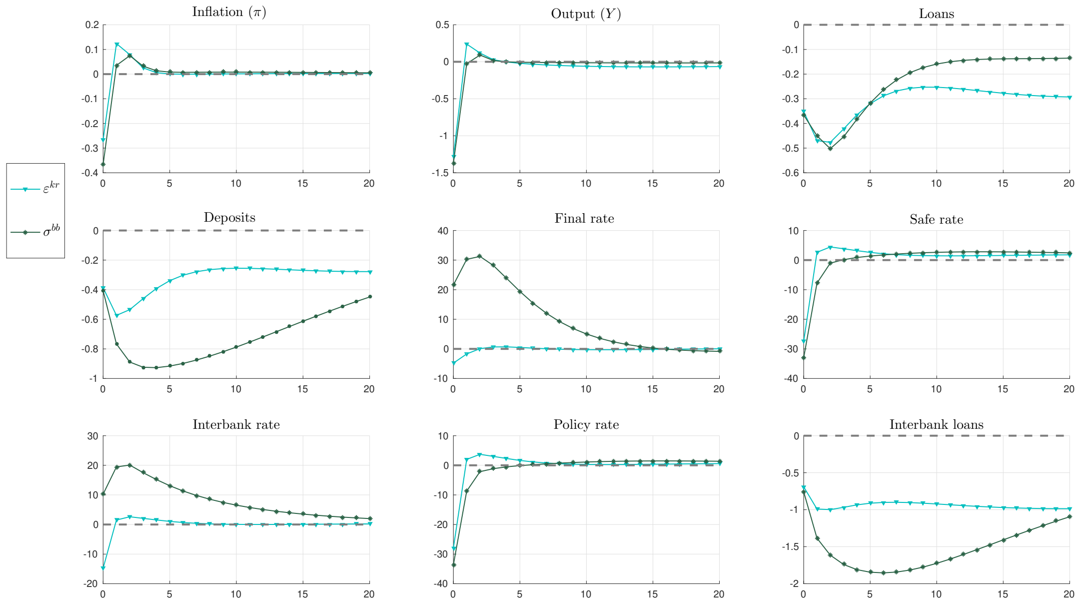

Fig. 5. Transmission of Idiosincratic-capital loss shock

Note: All variables are percentage deviations from the steady state. The light-blue line with triangles is from the shock representing a loss in capital of risky entrepreneurs (ε^{kr}). The dark-green line with circles is from the shock representing a rise in the riskiness of the banking sector (σ^{bb}).

As we can see, the IRFs functions are very similar from a qualitatively viewpoint. This is reasonable since the shock hits the same side of the economy (but not the same actor). However, the forces acting behind are quite different, and show different results on the quantitave side. While the drop seems to be more severe for interbank riskiness shock, on impact, the persistancy of the effects seems to last longer for the capital loss, regarding the main variables of the interbank sector. Specifically, we observe a drop in loans lead by the plunge of demand from risky entrepreneurs. This, in turn reverberates on the interbank loans and interbank rate. Since there is less need of resources from one sector there is, in both cases, a flight to quality effect from one side of the economy to the other. It is worth noticing that, even though the main result provided by the presence of an interbank sector seems to be the absorption of a shock hitting one side of the real economy, there is a mild loss in the aggregate output in comparison with the interbank riskiness shock. So, analysing the above dynamics we can observe that in our stylised economy, on aggregate the effect of an idiosincratic shock fades away. The common dynamic of both these shocks is shown in figure 6. We observe the redistribution effect in action.

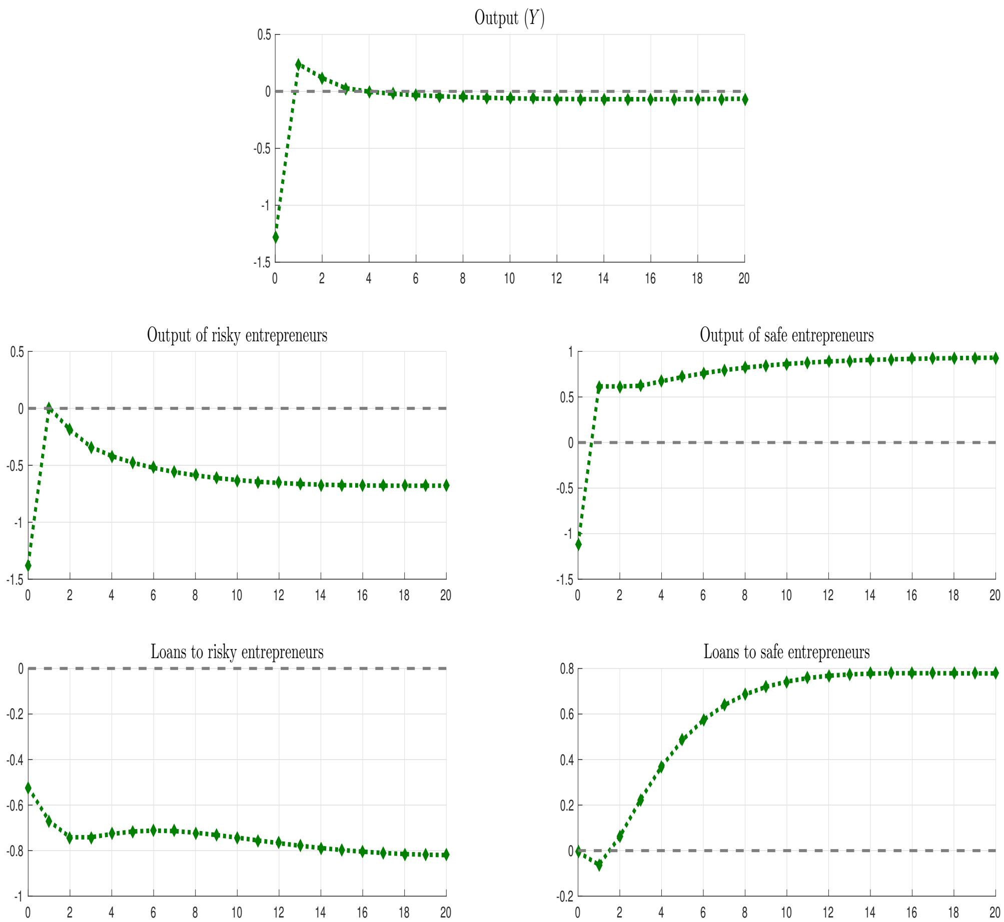

Fig. 6. Idiosincratic-capital loss shock - detail

Note: All variables are percentage deviations from the steady state.

While the output on aggregate is practically stable, we can see that all the resources are fleeing from one side of the economy to the other. Of course, since less resources are available, we observe the same dynamic for loans.

5. Conclusions

In this work we analysed the effects of the presence of an interbank market on the main aggregate economic variables. In addition, particular attention has been devoted to the study of the effects of the presence of an interbank market and, in particular, to the role played by the latter in the propagation of real shocks originating from one side of the productive sector. The analysis presented in the previous sections highlights two main results. First, relatively small amounts of financial resources are diverted from the real sector by the costs tappered from banks as a consequence of their trading activity on the interbank market. In order to offer/acquire liquidity on the interbank market, banks must bear monitoring and reputational costs. Those costs directly subtract resources to their lending activity. As a consequence, we observe that the aggregate output in the presence of an interbank market is slightly lower than the one observed without an interbank market. Second, we show that the interbank market dampens the potentially contractionary effect of negative real shocks, by reallocating financial resources from risky to safe entrepreneurs. We analyse this substitution effect by simulating an entirely novel type of real shock. We simulate an unexpected loss of capital in a productive sector characterised by an high level of riskiness. We show that the negative shock does not propagate through the real economy because, when risky entrepreneurs suffer losses, banks with surplus liquidity stop lending on the interbank market and expand their supply of credit to safe entrepreneurs. In other words, we clearly observe a flight-to-quality effect fostered by the presence of an interbank market. As a consequence, the safe productive sector benefits from the shock suffered by the risky one and expands its output. Overall, the expansionary effects induced by the increased availability of credit to the safe productive sector more than offsets the contractionary effects of the shock suffered from the risky entrepreneurs. The aggregate output remains almost stable. Even though we observe this result on aggregate, studying the main financial variables we saw that the interbank sector remains inefficient for several periods after the shock, and that the whole economy starts to deploy most of resources into one productive sector. The main insight is that well-functioning interbank market improves allocational efficiency by properly redistributing financial resources but, at the same time, it tends to divert such resources from the productive agents that need them most (i.e. from entrepreneurs hit by the shock). We are well aware of the fact that our results are in part driven by some simplification introduced in the model. However, we think that such simplifications might prove useful to start appreciating the non-trivial role played by interbank markets in modern financial sector.

6. Appendix

6.1. *Derivation of the main equations of the model*

6.1.1. *Households*

$$
\begin{aligned}
&\max_{c_t^h, n_t^h, d_t^h} E_0 \sum_{t=0}^{\infty} \beta_h^t \left(log\left(c_t^h\right) - \psi \frac{n_t^{h(1+\phi)}}{1+\phi} \right) \\
&s.t. \quad c_t^h + d_t^h \leq w_t^h n_t^h + \frac{(1+r_{t-1})}{\pi_t} d_{t-1}^h + J_t^{lb} + J_t^R
\end{aligned}
$$

From which the Lagrangean at time t and $t+1$ is:

$$
\begin{aligned}
\mathcal{L} =& \beta_h^t \left(log(c_t^h) - \psi \frac{n_t^{h(1+\phi)}}{1+\phi} \right) \\
&+ \beta_h^t \lambda_t^h (w_t^h n_t^h + \frac{(1+r_{t-1})}{\pi_t} d_{t-1}^h + J_t^{lb} + J_t^r - c_t^h - d_t^h) \\
&+ \beta_h^{t+1} \left(log(c_{t+1}^h) - \psi \frac{{n_{t+1}^h}^{(1+\phi)}}{1+\phi} \right) \\
&+ \beta_h^{t+1} \lambda_{t+1}^h (w_{t+1}^h n_{t+1}^h + \frac{(1+r_t)}{\pi_{t+1}} d_t^h + J_{t+1}^{lb} + J_{t+1}^r - c_{t+1}^h - d_{t+1}^h)
\end{aligned}
$$

F.O.C.s

$$\frac{\partial \mathcal{L}}{\partial c_t^h} = 0 \longrightarrow \lambda_t^h = \frac{1}{c_t^h}$$

$$\frac{\partial \mathcal{L}}{\partial d_t^h} = 0 \longrightarrow -\lambda_t^h + \beta_h E_t \left[\lambda_{t+1}^h \frac{(1+r_t)}{\pi_{t+1}} \right] = 0 \quad \textit{Euler equation}$$

$$\frac{\partial \mathcal{L}}{\partial n_t^h} = 0 \longrightarrow w_t^h = \psi \frac{n_t^{h\phi}}{\lambda_t^h} \quad \textit{labor supply}$$

$$\frac{\partial \mathcal{L}}{\partial \lambda_t^h} = 0 \longrightarrow c_t^h + d_t^h \leq w_t^h n_t^h + \frac{(1+r_{t-1})}{\pi_t} d_{t-1}^h + J_t^{lb} + J_t^r$$

6.2. Risky Entrepreneurs

$$
\begin{aligned}
&\max_{c_t^{er},k_t^{er},n_t^{er},l_t^{er}} E_0 \sum_{t=0}^{\infty} \beta_e^t log\left(c_t^{er}\right) \\
s.t. \quad & c_t^{er} + w_t^h n_t^{er} + \frac{(1+r_{t-1}^{er})}{\pi_t} l_{t-1}^{er} + q_t^k k_t^{er} \leq \frac{y_t^{er}}{x_t} + l_t^{er} + q_t^k (1-\delta) k_{t-1}^{er} \\
&(1+r_t^{er}) l_t^{er} \leq m_t^{er} E_t(q_{t+1}^k \pi_{t+1} (1-\delta) k_t^{er})
\end{aligned}
$$

with:

$$y_t^{er} = a_t^{er} [k_{t-1}^{er}]^{\alpha} n_t^{er\,1-\alpha}$$

From which the Lagrangean at time t and $t+1$ is:

$$
\begin{aligned}
\mathcal{L} =& \beta_e^t log\left(c_t^{er}\right) \\
&+ \beta_e^t \lambda_t^{er} \left(\frac{y_t^{er}}{x_t} + l_t^{er} + q_t^k (1-\delta) k_{t-1}^{er} - c_t^{er} - w_t^h n_t^{er} - \frac{(1+r_{t-1}^{er})}{\pi_t} l_{t-1}^{er} - q_t^k k_t^{er} \right) \\
&+ \beta_e^t \mu_t^{er} (m_t^{er} E_t(q_{t+1}^k \pi_{t+1} (1-\delta) k_t^{er}) - (1+r_t^{er}) l_t^{er}) \\
&+ \beta_e^{t+1} log\left(c_{t+1}^{er}\right) \\
&+ \beta_e^{t+1} \lambda_{t+1}^{er} \left(\frac{y_{t+1}^{er}}{x_{t+1}} + l_{t+1}^{er} + q_{t+1}^k (1-\delta) k_t^{er} - c_{t+1}^{er} - w_{t+1}^h n_{t+1}^{er} - \frac{(1+r_t^{er})}{\pi_{t+1}} l_t^{er} - q_{t+1}^k k_{t+1}^{er} \right) \\
&+ \beta_e^{t+1} \mu_{t+1}^{er} (m_{t+1}^{er} E_{t+1}(q_{t+2}^k \pi_{t+2} (1-\delta) k_{t+1}^{er}) - (1+r_{t+1}^{er}) l_{t+1}^{er})
\end{aligned}
$$

F.O.C.s

$$\frac{\partial \mathcal{L}}{\partial c_t^{er}} = 0 \longrightarrow \lambda_t^{e,r} = \frac{1}{c_t^{er}}$$

$$\frac{\partial \mathcal{L}}{\partial k_t^{er}} = 0 \longrightarrow \lambda_t^{er} q_t^k = E_t\{\mu_t^{er} m_t^{er} q_{t+1}^k \pi_{t+1} (1-\delta) + \beta_e \lambda_{t+1}^{er} [q_{t+1}^k (1-\delta) + r_{t+1}^{kr}]\}$$

with $r_t^{kr} \equiv \alpha a_t^{er} [k_{t-1}^{er}]^{\alpha-1} n_t^{er\,1-\alpha} / x_t$

$$\frac{\partial \mathcal{L}}{\partial n_t^{er}} = 0 \longrightarrow w_t^h = (1-\alpha) \frac{y_t^{er}}{x_t n_t^{er}}$$

$$\frac{\partial \mathcal{L}}{\partial l_t^{er}} = 0 \longrightarrow \lambda_t^{er} = \mu_t^{er}(1 + r_t^{er}) + \beta_e E_t \lambda_{t+1}^{er} \left(\frac{(1 + r_t^{er})}{\pi_{t+1}} \right)$$

$$\frac{\partial \mathcal{L}}{\partial \lambda_t^{er}} = 0 \longrightarrow c_t^{er} + w_t^h n_t^{er} + \frac{(1 + r_{t-1}^{er})}{\pi_t} l_{t-1}^{er} + q_t^k k_t^{er} \leq \frac{y_t^{er}}{x_t} + l_t^{er} + q_t^k (1 - \delta) k_{t-1}^{er}$$

$$\frac{\partial \mathcal{L}}{\partial \mu_t^{er}} = 0 \longrightarrow (1 + r_t^{er}) l_t^{er} \leq m_t^{er} E_t (q_{t+1}^k \pi_{t+1} (1 - \delta) k_t^{er})$$

6.3. Safe Entrepreneurs

$$\max_{c_t^{es}, k_t^{es}, n_t^{es}, l_t^{es}} E_0 \sum_{t=0}^{\infty} \beta_e^t log\,(c_t^{es})$$

$$s.t. \quad c_t^{es} + w_t^h n_t^{es} + \frac{(1 + r_{t-1}^{es})}{\pi_t} l_{t-1}^{es} + q_t^k k_t^{es} \leq \frac{y_t^{es}}{x_t} + l_t^{es} + q_t^k (1 - \delta) k_{t-1}^{es}$$
$$(1 + r_t^{es}) l_t^{es} \leq m_t^{es} E_t (q_{t+1}^k \pi_{t+1} (1 - \delta) k_t^{es})$$

with:

$$y_t^{es} = a_t^{es} [k_{t-1}^{es}]^{\alpha} n_t^{es 1-\alpha}$$

From which the Lagrangean at time t and $t+1$ is:

$$\begin{aligned}
\mathcal{L} =& \beta_e^t log\,(c_t^{es}) \\
&+ \beta_e^t \lambda_t^{es} \left(\frac{y_t^{es}}{x_t} + l_t^{es} + q_t^k (1 - \delta) k_{t-1}^{es} - c_t^{es} - w_t^h n_t^{es} - \frac{(1 + r_{t-1}^{es})}{\pi_t} l_{t-1}^{es} - q_t^k k_t^{es} \right) \\
&+ \beta_e^t \mu_t^{es} (m_t^{es} E_t (q_{t+1}^k \pi_{t+1} (1 - \delta) k_t^{es}) - (1 + r_t^{es}) l_t^{es}) \\
&+ \beta_e^{t+1} log\,(c_{t+1}^{es}) \\
&+ \beta_e^{t+1} \lambda_{t+1}^{es} \left(\frac{y_{t+1}^{es}}{x_{t+1}} + l_{t+1}^{es} + q_{t+1}^k (1 - \delta) k_t^{es} - c_{t+1}^{es} - w_{t+1}^h n_{t+1}^{es} - \frac{(1 + r_t^{es})}{\pi_{t+1}} l_t^{es} - q_{t+1}^k k_{t+1}^{es} \right) \\
&+ \beta_e^{t+1} \mu_{t+1}^{es} (m_{t+1}^{es} E_{t+1} (q_{t+2}^k \pi_{t+2} (1 - \delta) k_{t+1}^{es}) - (1 + r_{t+1}^{es}) l_{t+1}^{es})
\end{aligned}$$

F.O.C.s

$$\frac{\partial \mathcal{L}}{\partial c_t^{es}} = 0 \longrightarrow \lambda_t^{es} = \frac{1}{c_t^{es}}$$

$$\frac{\partial \mathcal{L}}{\partial k_t^{es}} = 0 \longrightarrow \lambda_t^{es} q_t^k = E_t\{\mu_t^{es} m_t^{es} q_{t+1}^k \pi_{t+1}(1-\delta) + \beta_e \lambda_{t+1}^{es}[q_{t+1}^k(1-\delta) + r_{t+1}^{ks}]\}$$

with $r_t^{ks} \equiv \alpha a_t^{es}[k_{t-1}^{es}]^{\alpha-1} n_t^{es1-\alpha}$

$$\frac{\partial \mathcal{L}}{\partial n_t^{es}} = 0 \longrightarrow w_t^h = (1-\alpha)\frac{y_t^{es}}{x_t n_t^{es}}$$

$$\frac{\partial \mathcal{L}}{\partial l_t^{es}} = 0 \longrightarrow \lambda_t^{es} = \mu_t^{es}(1+r_t^{es}) + \beta_e E_t \lambda_{t+1}^{es}\left(\frac{(1+r_t^{es})}{\pi_{t+1}}\right)$$

$$\frac{\partial \mathcal{L}}{\partial \lambda_t^{es}} = 0 \longrightarrow c_t^{es} + w_t^h n_t^{es} + \frac{(1+r_{t-1}^{es})}{\pi_t} l_{t-1}^{es} + q_t^k k_t^{es} \leq \frac{y_t^{es}}{x_t} + l_t^{es} + q_t^k(1-\delta)k_{t-1}^{es}$$

$$\frac{\partial \mathcal{L}}{\partial \mu_t^{es}} = 0 \longrightarrow (1+r_t^{es})l_t^{es} \leq m_t^{es} E_t(q_{t+1}^k \pi_{t+1}(1-\delta)k_t^{es})$$

6.4. Borrower Bank

6.4.1. Wholesale Branch

$$\max_{L_t^{bb}, IB_t, \sigma_t^{bb}, D_t^{bb}} E_0 \sum_{t=0}^{\infty} \beta_e^t \lambda_t^{er}[\left(1+R_t^{bb}\right)L_t^{bb} - L_{t+1}^{bb}\pi_{t+1} - \left(1+r_t^{ib}\right)\left(1-\sigma_t^{bb}\right)IB_t$$
$$+IB_{t+1}\pi_{t+1} + \left(K_{t+1}^{bb}\pi_{t+1} - K_t^{bb}\right) + D_{t+1}^{bb}\pi_{t+1} - (1+r_t)D_t^{bb} - Adj_t^{\kappa b} - Adj_t^{\sigma}]$$
$$s.t. \quad L_t^{bb} = IB_t + K_t^b + D_t^{bb}$$
$$and \quad D_t^{bb} \leq \overline{D}$$

with:

$$Adj_t^{kb} = \frac{\kappa_{kb}}{2}\left(\frac{K_t^b}{L_t^{bb}} - \nu_b\right)^2 K_t^b$$
$$Adj_t^{\sigma} = \frac{\chi_{db}}{2}\left(\frac{IB_{t-1}\sigma_{t-1}^{bb}}{\pi_t}\right)^2$$

from which we obtain the following Lagrangean in t and $t-1$:

$$
\begin{aligned}
\mathcal{L} =&\beta_e^{t-1}\lambda_{t-1}^{er}(\left(1+R_{t-1}^{bb}\right)L_{t-1}^{bb} - L_t^{bb}\pi_t - \left(1+r_{t-1}^{ib}\right)\left(1-\sigma_{t-1}^{bb}\right)IB_{t-1} \\
&+ IB_t\pi_t + \left(K_t^{bb}\pi_t - K_{t-1}^{bb}\right) + D_t^{bb}\pi_t - (1+r_{t-1})D_{t-1}^{bb} - Adj_{t-1}^{\kappa b} - Adj_{t-1}^{\sigma}) \\
&+ \beta_e^{t-1}\lambda_{t-1}^{er}(\lambda_{t-1}^{bb}(IB_{t-1} + K_{t-1}^{b} + D_{t-1}^{bb} - L_{t-1}^{bb})) \\
&+ \beta_e^{t-1}\lambda_{t-1}^{er}(\gamma_{t-1}^{bb}(\overline{D} - D_{t-1}^{bb})) \\
&+ \beta_e^{t}\lambda_{t}^{er}(\left(1+R_{t}^{bb}\right)L_{t}^{bb} - L_{t+1}^{bb}\pi_{t+1} - \left(1+r_{t}^{ib}\right)\left(1-\sigma_{t}^{bb}\right)IB_{t} \\
&+ IB_{t+1}\pi_{t+1} + \left(K_{t+1}^{bb}\pi_{t+1} - K_{t}^{bb}\right) + D_{t+1}^{bb}\pi_{t+1} - (1+r_{t})D_{t}^{bb} - Adj_{t}^{\kappa b} - Adj_{t}^{\sigma}) \\
&+ \beta_e^{t}\lambda_{t}^{er}(\lambda_{t}^{bb}(IB_{t} + K_{t}^{b} + D_{t}^{bb} - L_{t}^{bb})) \\
&+ \beta_e^{t}\lambda_{t}^{er}(\gamma_{t}^{bb}(\overline{D} - D_{t}^{bb}))
\end{aligned}
$$

F.O.C.s

$$\frac{\partial \mathcal{L}}{\partial L_t^{bb}} = 0 \longrightarrow -\beta_e^{t-1}\lambda_{t-1}^{er}\pi_t + \beta_e^t\lambda_t^{er}(1+R_t^{bb}) + \beta_e^t\lambda_t^{er}\kappa_{kb}\left(\frac{K_t^b}{L_t^{bb}} - \nu_b\right)\left(\frac{K_t^b}{L_t^{bb}}\right)^2 - \beta_t\lambda_t^{er}\lambda_t^{bb} = 0 \qquad 1)$$

$$\frac{\partial \mathcal{L}}{\partial IB_t} = 0 \longrightarrow -\beta_e^t\lambda_t^{er}(1+r_t^{ib})(1-\sigma_t^{bb}) - \beta_e^{t+1}\lambda_{t+1}^{er}\chi_{bb}\left(\frac{IB_t\sigma_t^{bb}}{\pi_{t+1}}\right)\left(\frac{\sigma_t^{bb}}{\pi_{t+1}}\right) \qquad 2)$$

$$+ \beta_e^t\lambda_t^{er}\lambda_t^{bb} + \beta_e^{t-1}\lambda_t^{er}\pi_t = 0$$

$$\frac{\partial \mathcal{L}}{\partial \sigma_t^{bb}} = 0 \longrightarrow \beta_e^t\lambda_t^{er}(1+r_t^{ib})IB_t - \beta_e^{t+1}\lambda_{t+1}^{er}\chi_{bb}\left(\frac{IB_t\sigma_t^{bb}}{\pi_{t+1}}\right)\left(\frac{IB_t}{\pi_{t+1}}\right) = 0 \qquad 3)$$

$$\frac{\partial \mathcal{L}}{\partial D_t^{bb}} = 0 \longrightarrow \beta_e^{t-1}\lambda_{t-1}^{er}\pi_t - \beta_e^t\lambda_t^{er}(1+r_t) + \beta_e^t\lambda_t^{er}\lambda_t^{bb} - \beta_e^t\lambda_t^{er}\gamma_t^{bb} = 0 \qquad 4)$$

Now, from 3) we obtain:

$$\sigma_t^{bb} = \frac{\lambda_t^{er}(1+r_t^{ib})\pi_{t+1}^2}{\beta_{er}\lambda_{t+1}^{er}\chi_{bb}IB_t}$$

Then, we can combine 1) and 2):

from 1) $\quad \beta_t\lambda_t^{er}\lambda_t^{bb} = -\beta_e^{t-1}\lambda_{t-1}^{er}\pi_t + \beta_e^t\lambda_t^{er}(1+R_t^{bb}) + \beta_e^t\lambda_t^{er}\kappa_{kb}\left(\frac{K_t^b}{L_t^{bb}} - \nu_b\right)\left(\frac{K_t^b}{L_t^{bb}}\right)^2$

Plugging this into 2) we obtain:

$$\beta_e^t \lambda_t^{er}(1+R_t^{bb}) + \beta_e^t \lambda_t^{er} \kappa_{kb} \left(\frac{K_t^b}{L_t^{bb}} - \nu_b\right)\left(\frac{K_t^b}{L_t^{bb}}\right)^2 - \beta_e^{t-1}\lambda_t^{er}\pi_t + \beta_e^{t-1}\lambda_t^{er}\pi_t$$
$$- \beta_e^t \lambda_t^{er}(1+r_t^{ib})(1-\sigma_t^{bb}) - \beta_e^{t+1} E_t \lambda_{t+1}^{er}\chi_{bb}\left(\frac{IB_t\sigma_t^{bb}}{\pi_{t+1}}\right)\left(\frac{\sigma_t^{bb}}{\pi_{t+1}}\right) = 0$$
$$\longrightarrow R_t^{bb} = r_t^{ib} - \sigma_t^{bb}(1+r_t^{ib}) - \kappa_{kb}\left(\frac{K_t^b}{L_t^{bb}} - \nu_b\right)\left(\frac{K_t^b}{L_t^{bb}}\right)^2 + \beta_e \chi_{bb} E_t\left[\left(\frac{\sigma_t^{bb}}{\pi_{t+1}}\right)^2 IB_t \frac{\lambda_{t+1}^{er}}{\lambda_t^{er}}\right]$$

Combining 1) and 4):

Starting from 1):

$$\beta_t \lambda_t^{er}\lambda_t^{bb} = -\beta_e^{t-1}\lambda_{t-1}^{er}\pi_t + \beta_e^t\lambda_t^{er}(1+R_t^{bb}) + \beta_e^t\lambda_t^{er}\kappa_{kb}\left(\frac{K_t^b}{L_t^{bb}} - \nu_b\right)\left(\frac{K_t^b}{L_t^{bb}}\right)^2$$

Plugging this into 4):

$$\beta_e^{t-1}\lambda_{t-1}^{er}\pi_t - \beta_e^t\lambda_t^{er}(1+r_t) - \beta_e^{t-1}\lambda_{t-1}^{er}\pi_t + \beta_e^t\lambda_t^{er}(1+R_t^{bb}) + \beta_e^t\lambda_t^{er}\kappa_{kb}\left(\frac{K_t^b}{L_t^{bb}} - \nu_b\right)\left(\frac{K_t^b}{L_t^{bb}}\right)^2 - \beta_e^t\lambda_t^{er}\gamma_t^{bb} = 0$$

from which we obtain:

$$\gamma_t^{bb} = R_t^{bb} - r_t + \kappa_{kb}\left(\frac{K_t^b}{L_t^{bb}} - \nu_b\right)\left(\frac{K_t^b}{L_t^{bb}}\right)^2$$

6.4.2. Retail Branch

$$\max_{r_t^{er}} E_0 \sum_{t=0}^{\infty} \beta_e^t \lambda_t^{er}\left[r_t^{er} l_t^{bb}(j) - R_t^{bb} L_t^{bb}(j) - Adj_t^{rer}\right]$$
$$\text{s.t} \quad l_t^{bb}(j) = \left(\frac{r_t^{er}(j)}{r_t^{er}}\right)^{-\epsilon_t^{er}} l_t^{bb}$$

where

$$Adj_t^{rer} = \frac{\kappa_{rer}}{2}\left(\frac{r_t^{er}(j)}{r_{t-1}^{er}(j)} - 1\right)^2 r_t^{er} l_t^{bb}$$

Now, substituting the downword sloping demand curve into the objective function, and considering times t and $t+1$ we obtain:

$$\Pi = \beta_e^t \lambda_t^{er} \left[r_t^{er}(j) \left(\frac{r_t^{er}(j)}{r_t^{er}} \right)^{-\varepsilon_t^{er}} l_t^{bb} - R_t^{bb} \left(\frac{r_t^{er}(j)}{r_t^{er}} \right)^{-\varepsilon_t^{er}} l_t^{bb} - \frac{\kappa_{rer}}{2} \left(\frac{r_t^{er}(j)}{r_{t-1}^{er}(j)} - 1 \right)^2 r_t^{er} l_t^{bb} \right]$$
$$+ \beta_e^{t+1} E_0 \lambda_{t+1}^{er} \left[r_{t+1}^{er}(j) \left(\frac{r_{t+1}^{er}(j)}{r_{t+1}^{er}} \right)^{-\varepsilon_{t+1}^{er}} l_{t+1}^{bb} - R_{t+1}^{bb} \left(\frac{r_{t+1}^{er}(j)}{r_{t+1}^{er}} \right)^{-\varepsilon_{t+1}^{er}} l_{t+1}^{bb} - \frac{\kappa_{rer}}{2} \left(\frac{r_{t+1}^{er}(j)}{r_t^{er}(j)} - 1 \right)^2 r_{t+1}^{er} l_{t+1}^{bb} \right]$$

Deriving for $r_t^{er}(j)$ we obtain:

$$\frac{\partial \Pi}{\partial r_t^{er}(j)} = 0 \rightarrow \beta_e^t \lambda_t^{er} \left[\left(\frac{r_t^{er}(j)}{r_t^{er}} \right)^{-\varepsilon_t^{er}} l_t^{bb} - \varepsilon_t^{er} \left(\frac{r_t^{er}(j)}{r_t^{er}} \right)^{-\varepsilon_t^{er}-1} \left(\frac{r_t^{er}(j)}{r_t^{er}} \right) l_t^{bb} + \varepsilon_t^{er} \frac{R_t^{bb}}{r_t^{er}} \left(\frac{r_t^{er}(j)}{r_t^{er}} \right)^{-\varepsilon_t^{er}-1} l_t^{bb} \right.$$
$$\left. - \kappa_{rer} \left(\frac{r_t^{er}(j)}{r_{t-1}^{er}(j)} - 1 \right) \frac{1}{r_{t-1}^{er}(j)} r_t^{er} l_t^{bb} \right]$$
$$+ \beta_e^{t+1} E_0 \lambda_{t+1}^{er} \left[\kappa_{rer} \left(\frac{r_{t+1}^{er}(j)}{r_t^{er}(j)} - 1 \right) \frac{r_{t+1}^{er}(j)}{r_t^{er}(j)^2} r_{t+1}^{er} l_{t+1}^{bb} \right] = 0$$

Considering that in symmetric equilibrium we have $r_t^{er}(j) = r_t^{er}$ and remembering that $\frac{R_t^{bb}}{r_t^{er}} = \frac{1}{mk_t^{rer}}$ dividing all members for l_t^{bb} we obtain:

$$\beta_e^t \lambda_t^{er} \left[1 - \varepsilon_t^{er} + \frac{\varepsilon_t^{er}}{mk_t^{rer}} - \kappa_{rer} \left(\frac{r_t^{er}}{r_{t-1}^{er}} - 1 \right) \frac{r_t^{er}}{r_{t-1}^{er}} \right] + \beta_e^{t+1} E_0 \lambda_{t+1}^{er} \left[\kappa_{rer} \left(\frac{r_{t+1}^{er}}{r_t^{er}} - 1 \right) \left(\frac{r_{t+1}^{er}}{r_t^{er}} \right)^2 \frac{l_{t+1}^{bb}}{l_t^{bb}} \right] = 0$$

To conclude, we divide both members for $\beta_e^t \lambda_t^{er}$ and we end up with the following "Taylor rule" for loan rates:

$$1 - \varepsilon_t^{er} + \frac{\varepsilon_t^{er}}{mk_t^{rer}} - \kappa_{rer} \left(\frac{r_t^{er}}{r_{t-1}^{er}} - 1 \right) \frac{r_t^{er}}{r_{t-1}^{er}} + \beta_e E_0 \frac{\lambda_{t+1}^{e}}{\lambda_{t+1}^{er}} \left[\kappa_{rer} \left(\frac{r_{t+1}^{er}}{r_t^{er}} - 1 \right) \left(\frac{r_{t+1}^{er}}{r_t^{er}} \right)^2 \frac{l_{t+1}^{bb}}{l_t^{bb}} \right] = 0$$

N.B.: we can rewrite the equation above exploiting the relation between the elasticity of substitution and the markup given their frictionless relation.

Starting from the fact that without frictions, the markup is given by:

$$
\begin{aligned}
mk_t =& \frac{\varepsilon_t}{\varepsilon_t - 1} \\
\rightarrow& \frac{\varepsilon_t - 1}{\varepsilon_t} = \frac{1}{mk_t} \\
\rightarrow& 1 - \frac{1}{\varepsilon_t} = \frac{1}{mk_t} \\
\rightarrow& 1 - \frac{1}{mk_t} = \frac{1}{\varepsilon_t} \\
\rightarrow& \frac{mk_t - 1}{mk_t} = \frac{1}{\varepsilon} \\
\rightarrow& \varepsilon_t = \frac{mk_t}{mk_t - 1}
\end{aligned}
$$

so that the equation above can be rewritten as:

$$
1 - \frac{mk_t^{rer}}{mk_t^{rer} - 1} + \frac{mk_t^{rer}}{mk_t^{rer} - 1} \frac{R_t^{bb}}{r_t^{er}} - \kappa_{bb} \left(\frac{r_t^{e,r}}{r_{t-1}^{e,r}} - 1 \right) \frac{r_t^{e,r}}{r_{t-1}^{e,r}} + \beta_e E_0 \frac{\lambda_{t+1}^e}{\lambda_{t+1}^e} \left[\kappa_{bb} \left(\frac{r_{t+1}^{e,r}}{r_t^{e,r}} - 1 \right) \frac{r_{t+1}^{e,r}}{r_t^{e,r}}^2 \frac{l_{t+1}^{bb}}{l_t^{bb}} \right] = 0
$$

6.5. Lender Bank

Starting from the fact that a lender bank faces the following balance sheet constraint:

$$
IB_t + L_t^{lb} = D_t^{lb}
$$

Since on the liabilities side there are only deposits, we can consider the following relation:

$$
s_t D_t^{lb} = IB_t
$$

We can rewrite the problem a lender bank has to face adding a new variable of choice. By now, the lender bank is able to choose also the optimal quantity of deposits it can collect:

$$
\begin{aligned}
\max_{s_t, D_t^{lb}} E_0 \sum_{t=0}^{\infty} \beta_h^t \lambda_t^h & \left[\left(1 + r_t^{ib}\right) s_t D_t^{lb} \left(1 - \sigma_t^{bb}\right) - s_{t+1} D_{t+1}^{lb} \pi_{t+1} + \left(1 + r_t^{es}\right) \left(1 - s_t\right) D_t^{lb} \right. \\
& \left. - \left(1 - s_{t+1}\right) D_{t+1}^{lb} \pi_{t+1} - \left(1 + r_t\right) D_t^{lb} + D_{t+1}^{lb} \pi_{t+1} - Adj_t^s \right]
\end{aligned}
$$

with

$$
Adj_t^s = \frac{\Theta}{2} \left[\left(s_t - \bar{s}\right) \right]^2 D_t^{lb}
$$

The Lagrangean associated to the problem above, evaluated in $t-1$ and t, is:

$$
\begin{aligned}
\mathcal{L} =& \beta_h^{t-1}\lambda_{t-1}^h[\left(1+r_{t-1}^{ib}\right)s_{t-1}D_{t-1}^{lb}\left(1-\sigma_{t-1}^{bb}\right) - s_t D_t^{lb}\pi_t + \left(1+r_{t-1}^{es}\right)\left(1-s_{t-1}\right)D_{t-1}^{lb} \\
&- \left(1-s_t\right)D_t^{lb}\pi_t - \left(1+r_{t-1}\right)D_{t-1}^{lb} + D_t^{lb}\pi_t - Adj_{t-1}^s] \\
&+ \beta_h^t\lambda_t^h[\left(1+r_t^{ib}\right)s_t D_t^{lb}\left(1-\sigma_t^{bb}\right) - s_{t+1}D_{t+1}^{lb}\pi_{t+1} + \left(1+r_t^{es}\right)\left(1-s_t\right)D_t^{lb} \\
&- \left(1-s_{t+1}\right)D_{t+1}^{lb}\pi_{t+1} - \left(1+r_t\right)D_t^{lb} + D_{t+1}^{lb}\pi_{t+1} - Adj_t^s]
\end{aligned}
$$

F.O.C.s

$$
\begin{aligned}
\frac{\partial \mathcal{L}}{\partial s_t} = 0 &\longrightarrow \beta_h^t\lambda_t^h(1+r_t^{ib})D_t^{lb}(1-\sigma_t^{bb}) - (1+r_t^{es})D_t^{lb}\beta_h^t\lambda_t^h - \Theta(s_t-\overline{s})D_t^{lb}\beta_h^t\lambda_t^h = 0 \\
&\longrightarrow 1-\sigma_t^{bb}+r_t^{ib}-r_t^{ib}\sigma_t^{bb}-1-r_t^{es}-\Theta D_t^{lb}s_t+\Theta D_t^{lb}\overline{s} = 0 \\
&\longrightarrow s_t = \overline{s} + \frac{r_t^{ib}-(1+r_t^{ib})\sigma_t^{bb}-r_t^{es}}{\Theta D_t^{lb}}
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial \mathcal{L}}{\partial D_t^{lb}} = 0 &\longrightarrow \beta_h^t\lambda_t^h(1+r_t^{ib})s_t(1-\sigma_t^d) + \beta_h^t\lambda_t^h\left(1+r_t^{e,s}\right)\left(1-s_t\right) - \beta_h^t\lambda_t^h\left(1+r_t\right) \\
&- \beta_h^t\lambda_t^h\Theta(s_t-\overline{s})^2 = 0 \\
&\longrightarrow (1+r_t^{ib})s_t(1-\sigma_t^d) + (1+r_t^{e,s})\left(1-s_t\right) - (1+r_t) \\
&- \Theta(s_t-\overline{s})^2 = 0 \\
&\longrightarrow s_t + s_t r_t^{ib} - \sigma_t^d s_t - s_t\sigma_t^d r_t^{ib} + 1 + r_t^{e,s} - s_t - r_t^{e,s}s_t - 1 - r_t - \Theta(s_t-\overline{s})^2 = 0 \\
&\longrightarrow r_t^{e,s} - r_t = -s_t(r_t^{ib} - \sigma_t^d(1+r_t^{ib}) - r_t^{e,s}) + \Theta(s_t-\overline{s})^2
\end{aligned}
$$

6.6. Capital goods Producers

The problem a capital good producer faces takes the following form:

$$
\max_{\bar{x}_t, i_t} E_0 \sum_{t=0}^{\infty} \beta_e^t \lambda_t^{es}\left(q_t^k \Delta\bar{x}_t - i_t\right)
$$

$$
\text{s.t.:} \quad \bar{x}_t = \bar{x}_{t-1} + \left[1 - \frac{\kappa_i}{2}\left(\frac{i_t}{i_{t-1}} - 1\right)^2\right] i_t
$$

with $\Delta\bar{x}_t = k_t - (1-\delta_k)k_{t-1}$

Remembering that capital goods producers buy last period undepreciated capital (1 −

$\delta)k_{t-1}$ at a price Q_t^k from entrepreneurs (the owners of these firms) and i units of final goods from the retailers at price P_t. The stock of effective capital $\overline{x}$ is then sold back to entrepreneurs at price Q_t^k. Considering that $q_t^k \equiv \frac{Q_t^k}{P_t}$. Knowing that in t the effective capital $\bar{x}$ is equal to the capital produced k_t, so that the problem can be rewritten as:

$$\begin{aligned}
&\max_{\bar{x}_t, i_t} E_0 \sum_{t=0}^{\infty} \beta_e^t \lambda_t^{es} \left(q_t^k (\bar{x}_t - (1-\delta_k)k_{t-1}) - i_t \right) \\
&\text{s.t.:} \quad \bar{x}_t = \quad \bar{x}_{t-1} + \left[1 - \frac{\kappa_i}{2} \left(\frac{i_t}{i_{t-1}} - 1 \right)^2 \right] i_t
\end{aligned}$$

Plus, we can rewrite $\bar{x}_t$ as $\bar{x}_{t-1} + k_t - (1-\delta_k)k_{t-1} \rightarrow \bar{x}_{t-1} + x_t - (1-\delta_k)k_{t-1}$ so that the Lagrangean takes the form:

$$\begin{aligned}
\mathcal{L} =& \beta_e^t \lambda_t^{es} \left(q_t^k (\bar{x}_t - (1-\delta_k)k_{t-1}) - i_t \right) \\
&+ \beta_e^t \lambda_t^{es} \lambda_t^{kp} \left(\quad \bar{x}_{t-1} + \left[1 - \frac{\kappa_i}{2} \left(\frac{i_t}{i_{t-1}} - 1 \right)^2 \right] i_t - \bar{x}_{t-1} - \bar{x}_t + (1-\delta)k_{t-1} \right) \\
&+ \beta_e^{t+1} \lambda_{t+1}^{es} \left(q_{t+1}^k (\bar{x}_{t+1} - (1-\delta_k)k_t) - i_{t+1} \right) \\
&+ \beta_e^{t+1} \lambda_{t+1}^{es} \lambda_t^{kp} \left(\quad \bar{x}_t + \left[1 - \frac{\kappa_i}{2} \left(\frac{i_{t+1}}{i_t} - 1 \right)^2 \right] i_{t+1} - \bar{x}_t - \bar{x}_{t+1} + (1-\delta)k_t \right)
\end{aligned}$$

So, the F.O.C.s are:

$$\begin{aligned}
\frac{\partial \mathcal{L}}{\partial \bar{x}_t} &= 0 \longrightarrow \beta_e^t \lambda_t^{es} q_t^k - \beta_e^t \lambda_t^{es} \lambda_t^{kp} = 0 \\
&\quad \lambda_t^{kp} = q_t^k \\
\frac{\partial \mathcal{L}}{\partial i_t} &= 0 \longrightarrow -\beta_e^t \lambda_t^{es} + \beta_e^t \lambda_t^{es} \lambda_t^{kp} \left(1 - \kappa_i \left(\frac{i_t}{i_{t-1}} - 1 \right) \left(\frac{i_t}{i_{t-1}} \right) - \frac{\kappa_i}{2} \left(\frac{i_t}{i_{t-1}} - 1 \right)^2 \right) \\
&\quad + \beta_e^{t+1} E_0 \lambda_{t+1}^{es} \lambda_{t+1}^{kp} \kappa_i \left(\frac{i_{t+1}}{i_t} - 1 \right) \left(\frac{i_{t+1}}{i_t} \right)^2 = 0
\end{aligned}$$

After substituting the multiplier from the first F.O.C. and dividing all members for $\beta_e^t E_0 \lambda_t^{es}$ we obtain:

$$1 = q_t^k \left(1 - \kappa_i \left(\frac{i_t}{i_{t-1}} - 1\right)\left(\frac{i_t}{i_{t-1}}\right) - \frac{\kappa_i}{2}\left(\frac{i_t}{i_{t-1}} - 1\right)^2\right)$$

$$+ \beta_e E_0 q_{t+1}^k \frac{\lambda_{t+1}^{es}}{\lambda_t^{es}} \kappa_i \left(\frac{i_{t+1}}{i_t} - 1\right)\left(\frac{i_{t+1}}{i_t}\right)^2$$

$$\frac{\partial \mathcal{L}}{\partial \lambda_t^{kp}} = 0 \longrightarrow \bar{x}_t = \bar{x}_{t-1} + \left[1 - \frac{\kappa_i}{2}\left(\frac{i_t}{i_{t-1}} - 1\right)^2\right] i_t$$

$$\longrightarrow \Delta \bar{x}_t = \left[1 - \frac{\kappa_i}{2}\left(\frac{i_t}{i_{t-1}} - 1\right)^2\right] i_t$$

but since we know that $\Delta \bar{x}_t = k_t - (1 - \delta_k) k_{t-1}$ we obtain:

$$\longrightarrow k_t = (1 - \delta_k) k_{t-1} + \left[1 - \frac{\kappa_i}{2}\left(\frac{i_t}{i_{t-1}} - 1\right)^2\right] i_t$$

6.7. Retailers

$$E_0 \sum_{t=0}^{\infty} \beta_h^t \lambda_t^h \left[P_t(i) y_t(i) - P_t^{es} y_t^{es}(i) - P_t^{er} y_t^{er}(i) - \frac{\kappa_p}{2}\left(\frac{P_t(j)}{P_{t-1}(j)} - 1\right)^2 P_t y_t\right]$$

if we assume $P_t^r = P_t^s \equiv P_t^w$, and remembering that $y_t^{e,r}(i) + y_t^{e,s}(i) = y_t(i)$, we obtain:

$$E_0 \sum_{t=0}^{\infty} \beta_h^t \lambda_t^h \left[P_t(i) y_t(i) - P_t^w y_t(i) - \frac{\kappa_p}{2}\left(\frac{P_t(i)}{P_{t-1}(i)} - 1\right)^2 P_t y_t\right]$$

so that retailers have to face the following problem:

$$\max_{P_t(i)} E_0 \sum_{t=0}^{\infty} \beta_h^t \lambda_t^h \left[P_t(i) y_t(i) - P_t^w y_t(i) - \frac{\kappa_p}{2}\left(\frac{P_t(i)}{P_{t-1}(i)} - 1\right)^2 P_t y_t\right]$$

$$s.t. \;\; y_t(i) = \left(\frac{P_t(i)}{P_t}\right)^{-\varepsilon_t^y} y_t$$

Now, substituting the downword sloping demand curve into the objective function, and considering times t and $t+1$ we obtain:

$$\beta_h^t \lambda_t^h \left[P_t(i)\left(\frac{P_t(i)}{P_t}\right)^{-\varepsilon_t^y} y_t - P_t^w \left(\frac{P_t(i)}{P_t}\right)^{-\varepsilon_t^y} y_t - \frac{\kappa_p}{2}\left(\frac{P_t(i)}{P_{t-1}(i)} - 1\right)^2 P_t y_t\right] +$$

$$+\beta_h^{t+1} \lambda_{t+1}^h \left[P_{t+1}(i)\left(\frac{P_{t+1}(i)}{P_{t+1}}\right)^{-\varepsilon_{t+1}^y} y_{t+1} - P_{t+1}^w \left(\frac{P_{t+1}(i)}{P_{t+1}}\right)^{-\varepsilon_{t+1}^y} y_{t+1} - \frac{\kappa_p}{2}\left(\frac{P_{t+1}(i)}{P_t(i)} - 1\right)^2 P_{t+1} y_{t+1}\right]$$

Differentiating for $P_t(i)$ we obtain:

$$
\begin{aligned}
&\beta_h^t \lambda_t^h \Bigg[\left(\frac{P_t(i)}{P_t}\right)^{-\varepsilon_t^y} y_t - \varepsilon_t^y \left(\frac{P_t(i)}{P_t}\right)^{-\varepsilon_t^y - 1} \left(\frac{P_t(i)}{P_t}\right) y_t + \varepsilon_t^y \frac{P_t^w}{P_t} \left(\frac{P_t(i)}{P_t}\right)^{-\varepsilon_t^y - 1} y_t \\
&- \kappa_p \left(\frac{P_t(i)}{P_{t-1}(i)} - 1\right) \frac{1}{P_{t-1}(i)} P_t y_t \Bigg] \\
&+ \beta_h^{t+1} E_0 \lambda_{t+1}^h \left[\kappa_p \left(\frac{P_{t+1}(i)}{P_t(i)} - 1\right) \frac{P_{t+1}(i)}{P_t(i)^2} P_{t+1} y_{t+1} \right] = 0
\end{aligned}
$$

Considering that in symmetric equilibrium we have $P_t(i) = P_t$ and remembering that $\frac{P_t^w}{P_t} = \frac{1}{x_t}$ dividing all members for y_t we obtain:

$$
\beta_h^t \lambda_t^h \left[1 - \varepsilon_t^y + \frac{\varepsilon_t^y}{x_t} - \kappa_p \left(\pi_t - 1\right) \pi_t \right] + \beta_h^{t+1} E_0 \lambda_{t+1}^h \left[\kappa_p (\pi_{t+1} - 1) \pi_{t+1}^2 \frac{y_{t+1}}{y_t} \right] = 0
$$

So, dividing both members for $\beta_h^t \lambda_t^p$ we end up with:

$$
1 - \varepsilon_t^y + \frac{\varepsilon_t^y}{x_t} - \kappa_p \left(\pi_t - 1\right) \pi_t + \beta_h E_0 \left[\frac{\lambda_{t+1}^h}{\lambda_t^h} \kappa_p (\pi_{t+1} - 1) \pi_{t+1}^2 \frac{y_{t+1}}{y_t} \right] = 0
$$

6.8. ***Main equations of the models***

6.8.1. *Main equations of the Benchmark Model (BK)*

$Household \rightarrow c_t^h, d_t^h, w_t, n_t^h, \lambda_t^h;$
$Risky\ \ Entrepreneur \rightarrow c_t^{er}, n_t^{er}, k_t^{er}, l_t^{er}, y_t^{er}, r_t^{kr}, \lambda_t^{er}, \mu_t^{er};$
$Safe\ \ Entrepreneur \rightarrow c_t^{es}, n_t^{es}, k_t^{es}, l_t^{es}, y_t^{es}, r_t^{ks}, \lambda_t^{es}, \mu_t^{es};$
$Capital\ \ goods\ \ producers \rightarrow q_t^k, i_t;$
$Retailers \rightarrow J_t^R, \pi_t, x_t;$
$Borrower\ \ Bank \rightarrow L_t^{bb}, IB_t, K_t^b, D_t^{bb}, \sigma_t^{bb}, l_t^{bb}, J_t^{bb}, \gamma_t^{bb};$
$Lender\ \ Bank \rightarrow D_t^{lb}, L_t^{lb}, s_t, J_t^{lb};$
$Rates \rightarrow R_t^{bb}, r_t^{er}, r_t^{es}, r_t^{ib}, r_t;$
$Aggregation \rightarrow k_t; y_t; c_t$

$$\lambda_t^h = \frac{1}{c_t^h} \tag{6.1}$$

$$-\lambda_t^h + \beta_h E_t \left[\lambda_{t+1}^h \frac{(1+r_t)}{\pi_{t+1}} \right] = 0 \tag{6.2}$$

$$w_t^h = \frac{\psi n_t^{h\,\phi}}{\lambda_t^h} \tag{6.3}$$

$$c_t^h + d_t^h \leq w_t^h n_t^h + \frac{(1+r_{t-1})}{\pi_t} d_{t-1}^h + J_t^{lb} + J_t^R \tag{6.4}$$

$$\lambda_t^{er} = \frac{1}{c_t^{er}} \tag{6.5}$$

$$\lambda_t^{er} q_t^k = E_t \{ \mu_t^{er} m_t^{er} q_{t+1}^k \pi_{t+1} (1-\delta_k) + \beta_e \lambda_{t+1}^{er} [q_{t+1}^k (1-\delta_k) + r_{t+1}^{kr}] \} \tag{6.6}$$

$$y_t^{er} = a_t^{er} k_{t-1}^{er\ \alpha} n_t^{er\,1-\alpha} \tag{6.7}$$

$$r_t^{kr} = \alpha a_t^{er} [k_{t-1}^{er}]^{\alpha-1} n_t^{er\,1-\alpha} / x \tag{6.8}$$

$$w_t^h = (1-\alpha) \frac{y_t^{er}}{x_t n_t^{er}} \tag{6.9}$$

$$\lambda_t^{er} = \mu_t^{er}(1 + r_t^{er}) + \beta_e E_t \lambda_{t+1}^{er} \left(\frac{(1 + r_t^{er})}{\pi_{t+1}} \right) \tag{6.10}$$

$$c_t^{er} + w_t^h n_t^{er} + \frac{(1 + r_{t-1}^{er})}{\pi_t} l_{t-1}^{er} + q_t^k k_t^{er} \leq \frac{y_t^{er}}{x_t} + l_t^{er} + q_t^k (1 - \delta_k) k_{t-1}^{er} \tag{6.11}$$

$$(1 + r_t^{er}) l_t^{er} \leq m_t^{er} E_t (q_{t+1}^k \pi_{t+1} (1 - \delta_k) k_t^{er}) \tag{6.12}$$

$$\lambda_t^{es} = \frac{1}{c_t^{es}} \tag{6.13}$$

$$\lambda_t^{es} q_t^k = E_t \{ \mu_t^{es} m_t^{es} q_{t+1}^k \pi_{t+1} (1 - \delta_k) + \beta_e \lambda_{t+1}^{es} [q_{t+1}^k (1 - \delta_k) + r_{t+1}^{ks}] \} \tag{6.14}$$

$$y_t^{es} = a_t^{es} {k_{t-1}^{es}}^{\alpha} n_t^{es 1-\alpha} \tag{6.15}$$

$$r_t^{ks} = \alpha a_t^{es} [k_{t-1}^{es}]^{\alpha - 1} n_t^{es 1-\alpha} / x \tag{6.16}$$

$$w_t^h = (1 - \alpha) \frac{y_t^{es}}{x_t n_t^{es}} \tag{6.17}$$

$$\lambda_t^{es} = \mu_t^{es}(1 + r_t^{es}) + \beta_e E_t \lambda_{t+1}^{es} \left(\frac{(1 + r_t^{es})}{\pi_{t+1}} \right) \tag{6.18}$$

$$c_t^{es} + w_t^h n_t^{es} + \frac{(1 + r_{t-1}^{es})}{\pi_t} l_{t-1}^{es} + q_t^k k_t^{es} \leq \frac{y_t^{es}}{x_t} + l_t^{es} + q_t^k (1 - \delta_k) k_{t-1}^{es} \tag{6.19}$$

$$(1 + r_t^{es}) l_t^{es} \leq m_t^{es} E_t (q_{t+1}^k \pi_{t+1} (1 - \delta_k) k_t^{es}) \tag{6.20}$$

$$k_t = (1 - \delta_k) k_{t-1} + \left[1 - \frac{\kappa_i}{2} \left(\frac{i_t}{i_{t-1}} - 1 \right)^2 \right] i_t \tag{6.21}$$

$$\begin{aligned} 1 =& q_t^k \left(1 - \kappa_i \left(\frac{i_t}{i_{t-1}} - 1 \right) \left(\frac{i_t}{i_{t-1}} \right) - \frac{\kappa_i}{2} \left(\frac{i_t}{i_{t-1}} - 1 \right)^2 \right) \\ &+ \beta_e E_0 q_{t+1}^k \frac{\lambda_{t+1}^{es}}{\lambda_t^{es}} \kappa_i \left(\frac{i_{t+1}}{i_t} - 1 \right) \left(\frac{i_{t+1}}{i_t} \right)^2 \end{aligned} \tag{6.22}$$

$$\sigma_t^d = \frac{\lambda_t^{e,r}(1+r_t^{ib})\pi_{t+1}^2}{\beta_{e,r}\lambda_{t+1}^{e,r}\chi IB_t} \tag{6.23}$$

$$R_t^{bb} = r_t^{ib} - \sigma_t^d(1+r_t^{ib}) - \kappa_{kb}\left(\frac{K_t^b}{L_t^{bb}} - \nu_b\right)\left(\frac{K_t^b}{L_t^{bb}}\right)^2 + \beta_e \chi E_t\left[\left(\frac{\sigma_t^d}{\pi_{t+1}}\right)^2 IB_t \frac{\lambda_{t+1}^{e,r}}{\lambda_t^{e,r}}\right] \tag{6.24}$$

$$\gamma_t^{bb} = R_t^{bb} - r_t + \kappa_{kb}\left(\frac{K_t^b}{L_t^{bb}} - \upsilon_b\right)\left(\frac{K_t^b}{L_t^{bb}}\right)^2 \tag{6.25}$$

$$L_t^{bb} = IB_t + K_t^b + D_t^{bb} \tag{6.26}$$

$$D_t^{bb} = \overline{D} \tag{6.27}$$

$$K_t^b = (1-\delta^b)K_{t-1}^b + (1-\omega^b)J_{t-1}^{bb} \tag{6.28}$$

$$1 - \varepsilon_t^{er} + \frac{\varepsilon_t^{er}}{mk_t^{rer}} - \kappa_{bb}\left(\frac{r_t^{e,r}}{r_{t-1}^{e,r}} - 1\right)\frac{r_t^{e,r}}{r_{t-1}^{e,r}} + \beta_e E_0 \frac{\lambda_{t+1}^{e,r}}{\lambda_t^{e,r}}\left[\kappa_{bb}\left(\frac{r_{t+1}^{e,r}}{r_t^{e,r}} - 1\right){\frac{r_{t+1}^{e,r}}{r_t^{e,r}}}^2 \frac{l_{t+1}^{bb}}{l_t^{bb}}\right] = 0 \tag{6.29}$$

$$J_t^{bb} = r_t^{e,r} l_t^{bb} + \left(1+r_t^{ib}\right)\sigma_t^d IB_t - r_t^{ib} IB_t - r_t D_t^{bb} - Adj_t^{db} \tag{6.30}$$

$$s_t = \overline{s} + \frac{r_t^{ib} - (1+r_t^{ib})\sigma_t^d - r_t^{e,s}}{\Theta D_t^{lb}} \tag{6.31}$$

$$r_t^{e,s} - r_t = -s_t(r_t^{ib} - \sigma_t^d(1+r_t^{ib}) - r_t^{e,s}) + \Theta(s_t - \overline{s})^2 \tag{6.32}$$

$$IB_t + L_t^{lb} = D_t^{lb} \tag{6.33}$$

$$s_t D_t^{lb} = IB_t \tag{6.34}$$

$$J_t^{sb} = r_t^{ib} IB_t + r_t^{ls} L_t^{lb} - \left(1+r_t^{ib}\right)\sigma_t^d IB_t - r_t D_t^{lb} - Adj_t^m \tag{6.35}$$

$$J_t^R = y_t - \frac{y_t}{x_t} - \kappa_p(\pi_t - 1)^2 y_t \tag{6.36}$$

$$1 - \varepsilon_t^y + \frac{\varepsilon_t^y}{x_t} - \kappa_p\left(\pi_t - 1\right)\pi_t + \beta_h E_0\left[\frac{\lambda_{t+1}^h}{\lambda_t^h}\kappa_p(\pi_{t+1} - 1)\pi_{t+1}^2\frac{y_{t+1}}{y_t}\right] = 0 \tag{6.37}$$

$$(1 + r_t) = (1 + r)^{(1-\phi_R)}\left(1 + r_{t-1}\right)^{\phi_R}\left(\frac{\pi_t}{\pi}\right)^{\phi_\pi(1-\phi_R)}\left(\frac{y_t}{y_{t-1}}\right)^{\phi_y(1-\phi_R)} \tag{6.38}$$

$$c_t = c_t^h + c_t^{e,r} + c_t^{e,s} \tag{6.39}$$

$$d_t^h = D_t^{bb} + D_t^{lb} \tag{6.40}$$

$$l_t^{bb} = L_t^{bb} \tag{6.41}$$

$$l_t^{bb} = l_t^{e,r} \tag{6.42}$$

$$L_t^{lb} = l_t^{e,s} \tag{6.43}$$

$$k_t = k_t^{e,r} + k_t^{e,s} \tag{6.44}$$

$$n_t^h = n_t^{e,r} + n_t^{e,s} \tag{6.45}$$

$$y_t = y_t^{e,r} + y_t^{e,s} \tag{6.46}$$

6.8.2. Main equations of the Model with Interbank sector (IBK)

$Household \rightarrow c_t^P, d_t^P, w_t, n_t^P;$
$Firm \rightarrow c_t^E, n_t^E, k_t^E, l_t^{EE}, y_t^E, r_t^k, x_t;$
$Capital\ \ goods\ \ producers \rightarrow q_t^k, I_t;$
$Retailers \rightarrow J_t^R, \pi_t;$
$Deficit\ \ Bank \rightarrow B_t, IB_t, K_t^b, \sigma_t^b, b_t^E, J_t^{bb};$
$Surplus\ \ Bank \rightarrow D_t, GB_t, s_t, J_t^{lb};$
$Rates \rightarrow R_t^b, r_t^e, r_t^{ib}, r_t;$
$Aggregation \rightarrow K_t; Y_t; C_t, Y_t^{SS}$

Households

$$c_t^P + d_t^P \leq w_t n_t^P + \frac{(1+r_{t-1})}{\pi_t} d_{t-1}^P + J_t^R + J_t^{lb} \tag{6.47}$$

$$\frac{1}{c_t^P} = E_t \frac{\beta_p (1+r_t)}{c_{t+1}^P \pi_{t+1}} \tag{6.48}$$

$$\psi n_t^{P\phi} = \frac{w_t}{c_t^p} \tag{6.49}$$

Firms

$$c_t^E + \frac{\left(1+r_{t-1}^e\right)}{\pi_t} l_{t-1}^{EE} + w_t n_t^E + q_t^k k_t^E \leq \frac{y_t^E}{x_t} + l_t^{EE} + q_t^k \left(1-\delta^k\right) k_{t-1}^E \tag{6.50}$$

$$l_t^{EE} \leq \frac{m^E q_{t+1}^K k_t^E (1-\delta^k) \pi_{t+1}}{1+r_t^e} \tag{6.51}$$

$$y_t^E = A_t^E (k_{t-1}^E)^{\alpha} (n_t^E)^{1-\alpha} \tag{6.52}$$

$$r_t^k \equiv \alpha \frac{A_t^E (k_{t-1}^E)^{\alpha-1} (n_t^E)^{1-\alpha}}{x_t} \tag{6.53}$$

$$\frac{\beta_E \left[q_{t+1}^k (1-\delta^k) + r_t^k - (1+r_t^e)\, \chi_t\right]}{c_{t+1}^E} = \frac{q_t^k - \chi_t}{c_t^E} \tag{6.54}$$

$$\frac{(1-\alpha) y_t^E}{n_t^E x_t} = w_t \tag{6.55}$$

Borrower Banks

$$L_t = IB_t + K_t^b \tag{6.56}$$

$$L_t = l_t^E \tag{6.57}$$

$$R_t^b = r_t^{ib} - \sigma_t^b \left(1 + r_t^{ib}\right) - k_{kb} \left(\frac{K_t^b}{L_t} - \nu^b\right) \left(\frac{K_t^b}{L_t}\right)^2 + \beta^e \chi^{bb} E_t \left\{ \left(\frac{\sigma_t^b}{\pi_{t+1}}\right)^2 IB_t \frac{c_{t+1}^E}{c_t^E} \right\} \tag{6.58}$$

$$\sigma_t^b = E_t \left(\frac{c_t^E \left(1 + r_t^{ib}\right) \left(\pi_{t+1}\right)^2}{\beta^e c_{t+1}^E \chi^{bb} IB_t} \right) \tag{6.59}$$

$$J_t^{bb} = r_t^e l_t^E + \left(1 + r_t^{ib}\right) \sigma_t^b IB_t - Adj_t^{bb} \tag{6.60}$$

$$K_t^b \pi = (1 - \delta^b) K_{t-1}^b + J_{t-1}^{bb} \tag{6.61}$$

$$1 - \frac{\Lambda_t^{bn}}{\Lambda_t^{bn} - 1} + \frac{R_t^b}{r_t^e} \frac{\Lambda_t^{bn}}{\Lambda_t^{bn} - 1} - k_{bn} \left(\frac{r_t^e}{r_{t-1}^e} - 1\right) \frac{r_t^e}{r_{t-1}^e} + \beta_E E_t \left[\frac{c_t^E}{c_{t+1}^E} k_{bn} \left(\frac{r_{t+1}^e}{r_t^e} - 1\right) \left(\frac{r_{t+1}^e}{r_t^e}\right)^2 \frac{l_{t+1}^E}{l_t^E} \right] = 0 \tag{6.62}$$

Lender Banks

$$IB_t + GB_t = D_t \tag{6.63}$$

$$IB_t = s_t D_t \tag{6.64}$$

$$s_t = \bar{s} + \frac{r_t^{ib} - \sigma_t^b \left(1 + r_t^{ib}\right) - r_t}{\Theta D_t} \tag{6.65}$$

$$J_t^{lb} = r_t^{ib} IB_t + r_t GB_t - \left(1 + r_t^{ib}\right) \sigma_t^b IB_t - r_t D_t - Adj_t^{lb} \tag{6.66}$$

Capital goods firms

$$1 = q_t^k \left[1 - \frac{\kappa^i}{2}\left(\frac{I_t}{I_{t-1}} - 1\right)^2 - \kappa^i \left(\frac{I_t}{I_{t-1}} - 1\right)\frac{I_t}{I_{t-1}}\right] + \beta_E E_t \left[\frac{c_t^E}{c_{t+1}^E} q_{t+1}^k \left(\frac{I_{t+1}}{I_t} - 1\right)\left(\frac{I_{t+1}}{I_t}\right)^2\right] \tag{6.67}$$

$$K_t = \left(1 - \delta^k\right) K_{t-1} + \left[1 - \frac{\kappa^i}{2}\left(\frac{I_t}{I_{t-1}} - 1\right)^2\right] I_t \tag{6.68}$$

Retailers

$$J_t^R = Y_t - \frac{Y_t}{x_t} - \kappa_p(\pi_t - 1)^2 Y_t \tag{6.69}$$

$$1 - \frac{mk_t^y}{mk_t^y - 1} + \frac{mk_t^y}{mk_t^y - 1} mc_t^E - \kappa_p\left(\pi_t - 1\right)\pi_t + \beta_P E_t \left[\frac{c_t^P}{c_{t+1}^P}\kappa_p\left(\pi_{t+1} - 1\right)\pi_{t+1}\frac{Y_{t+1}}{Y_t}\right] = 0 \tag{6.70}$$

Aggregation and clearing

$$Y_t^{ss} = C_t + q_t^k(K_t - (1 - \delta^k)K_{t-1}) + \frac{\delta^b K_{t-1}^b}{\pi_t} \tag{6.71}$$

$$n_t^E = n_t^P \tag{6.72}$$

$$C_t = c_t^E + c_t^P \tag{6.73}$$

$$l_t^e = l_t^{EE} \tag{6.74}$$

$$D_t = d_t^P \tag{6.75}$$

$$K_t = k_t^E \tag{6.76}$$

$$Y_t = y_t^E \tag{6.77}$$

$$GB_t = \overline{GB} \tag{6.78}$$

Central Bank (Taylor rule)

$$(1+r_t) = (1+r)^{(1-\phi_R)} (1+r_{t-1})^{\phi_R} \left(\frac{\pi_t}{\pi}\right)^{\phi_\pi(1-\phi_R)} \left(\frac{Y_t}{Y_{t-1}}\right)^{\phi_y(1-\phi_R)} \tag{6.79}$$

With:

$$\chi_t \equiv m^E q_{t+1}^k \left(1-\delta^k\right) / 1 + r_t^E$$

$$Adj_t^{kb} = \frac{k_{kb}}{2} \left(\frac{K_t^b}{L_t} - \nu_b\right)^2 K_t^b$$

$$Adj_t^{\sigma} = \frac{\chi_{bb}}{2} \left(\frac{IB_{t-1}\sigma_{t-1}^b}{\pi_t}\right)^2$$

$$Adj_t^{db} = Adj_t^{kb} + Adj_t^{\sigma} + \kappa_{kb} \left(\frac{r_t^{be}}{r_{t-1}^{be}} - 1\right)^2 r_t^{be} l_t^e$$

$$Adj_t^{lb} = \frac{\Theta}{2} \left[(s_t - \bar{s}) D_t\right]^2$$

6.8.3. Main equations of the Model without the Interbank sector (NIBK)

$Household \rightarrow c_t^P, d_t^P, w_t, n_t^P;$
$Firm \rightarrow c_t^E, n_t^E, k_t^E, l_t^{EE}, y_t^E, r_t^k, x_t;$
$Capital\ \ goods\ \ producers \rightarrow q_t^k, I_t;$
$Retailers \rightarrow J_t^R, \pi_t;$
$Banks \rightarrow L_t, K_t^b, l_t^E, J_t^b, D_t;$
$Rates \rightarrow R_t^b, r_t^e, r_t;$
$Aggregation \rightarrow K_t; Y_t; C_t, Y_t^{SS}$

Households

$$c_t^P + d_t^P \leq w_t n_t^P + \frac{(1+r_{t-1})}{\pi_t} d_{t-1}^P + J_t^R + J_t^{lb} \tag{6.80}$$

$$\frac{1}{c_t^P} = E_t \frac{\beta_p (1+r_t)}{c_{t+1}^P \pi_{t+1}} \tag{6.81}$$

$$\psi n_t^{P\phi} = \frac{w_t}{c_t^p} \tag{6.82}$$

Firms

$$c_t^E + \frac{\left(1+r_{t-1}^e\right)}{\pi_t} l_{t-1}^{EE} + w_t n_t^E + q_t^k k_t^E \leq \frac{y_t^E}{x_t} + l_t^{EE} + q_t^k \left(1-\delta^k\right) k_{t-1}^E \tag{6.83}$$

$$l_t^{EE} \leq \frac{m^E q_{t+1}^K k_t^E (1-\delta^k) \pi_{t+1}}{1+r_t^e} \tag{6.84}$$

$$y_t^E = A_t^E (k_{t-1}^E)^{\alpha} (n_t^E)^{1-\alpha} \tag{6.85}$$

$$r_t^k \equiv \alpha \frac{A_t^E (k_{t-1}^E)^{\alpha-1} (n_t^E)^{1-\alpha}}{x_t} \tag{6.86}$$

$$\frac{\beta_E \left[q_{t+1}^k (1-\delta^k) + r_t^k - (1+r_t^e)\,\chi_t\right]}{c_{t+1}^E} = \frac{q_t^k - \chi_t}{c_t^E} \tag{6.87}$$

$$\frac{(1-\alpha) y_t^E}{n_t^E x_t} = w_t \tag{6.88}$$

Banks

$$L_t = D_t + K_t^b \tag{6.89}$$

$$L_t = l_t^E \tag{6.90}$$

$$R_t^b = r_t - k_{kb}\left(\frac{K_t^b}{L_t} - \nu^b\right)\left(\frac{K_t^b}{L_t}\right)^2 \tag{6.91}$$

$$J_t^b = r_t^e l_t^E + rD_t - \frac{\kappa_{kb}}{2}\left(\frac{K_t^b}{L_t} - \nu^b\right)^2 K_t^b \tag{6.92}$$

$$K_t^b \pi = (1-\delta^b)K_{t-1}^b + J_{t-1}^{bb} \tag{6.93}$$

$$1 - \frac{\Lambda_t^{bn}}{\Lambda_t^{bn}-1} + \frac{R_t^b}{r_t^e}\frac{\Lambda_t^{bn}}{\Lambda_t^{bn}-1} - k_{bn}\left(\frac{r_t^e}{r_{t-1}^e} - 1\right)\frac{r_t^e}{r_{t-1}^e} + \beta_E E_t\left[\frac{c_t^E}{c_{t+1}^E} k_{bn}\left(\frac{r_{t+1}^e}{r_t^e} - 1\right)\left(\frac{r_{t+1}^e}{r_t^e}\right)^2 \frac{l_{t+1}^E}{l_t^E}\right] = 0 \tag{6.94}$$

Capital goods firms

$$1 = q_t^k\left[1 - \frac{\kappa^i}{2}\left(\frac{I_t}{I_{t-1}} - 1\right)^2 - \kappa^i\left(\frac{I_t}{I_{t-1}} - 1\right)\frac{I_t}{I_{t-1}}\right] + \beta_E E_t\left[\frac{c_t^E}{c_{t+1}^E} q_{t+1}^k\left(\frac{I_{t+1}}{I_t} - 1\right)\left(\frac{I_{t+1}}{I_t}\right)^2\right] \tag{6.95}$$

$$K_t = \left(1-\delta^k\right)K_{t-1} + \left[1 - \frac{\kappa^i}{2}\left(\frac{I_t}{I_{t-1}} - 1\right)^2\right] I_t \tag{6.96}$$

Retailers

$$J_t^R = Y_t - \frac{Y_t}{x_t} - \kappa_p(\pi_t - 1)^2 Y_t \tag{6.97}$$

$$1 - \frac{mk_t^y}{mk_t^y - 1} + \frac{mk_t^y}{mk_t^y - 1} mc_t^E - \kappa_p\left(\pi_t - 1\right)\pi_t + \beta_P E_t\left[\frac{c_t^P}{c_{t+1}^P}\kappa_p\left(\pi_{t+1} - 1\right)\pi_{t+1}\frac{Y_{t+1}}{Y_t}\right] = 0 \tag{6.98}$$

Aggregation and clearing

$$Y_t^{ss} = C_t + q_t^k(K_t - (1-\delta^k)K_{t-1}) + \frac{\delta^b K_{t-1}^b}{\pi_t} \tag{6.99}$$

$$n_t^E = n_t^P \tag{6.100}$$

$$C_t = c_t^E + c_t^P \tag{6.101}$$

$$l_t^e = l_t^{EE} \tag{6.102}$$

$$D_t = d_t^P \tag{6.103}$$

$$K_t = k_t^E \tag{6.104}$$

$$Y_t = y_t^E \tag{6.105}$$

Central Bank (Taylor rule)

$$(1 + r_t) = (1 + r)^{(1-\phi_R)} (1 + r_{t-1})^{\phi_R} \left(\frac{\pi_t}{\pi}\right)^{\phi_\pi (1-\phi_R)} \left(\frac{Y_t}{Y_{t-1}}\right)^{\phi_y (1-\phi_R)} \tag{6.106}$$

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Chapter II: Optimal Lock-Down with Endogenous Compliance

Optimal Lock-Down with Endogenous Compliance

Davide Bosco - Luca Portoghese

1. Introduction

The afterward of the COVID-19 has been characterised by a wide prodcution of scientific works. Many economists have tried to provide their contribution, commonly starting from epidemiologic models widely used to predict the course of an epidemic (Ferguson 2020). From our point of view, two aspects have not been already properly inspected, hence they require a bit more of attention. Specifically, as far as we know, the economic models tipically used to study the Corona-virus effects did not take into account i) the possibility to provide workers forced into lockdown with a subsidy, in order to mitigate the economic loss that comes along with the lockdown, and ii) the different levels of people's compliance to the lockdown policy and how to influence it. The aim of this paper is to shed some light on this two aspects. We try to evaluate the effectiveness of a common lockdown policy combined with a subsidy in influencing the behaviour of people in respecting the confinement.

As already pointed out by Alvarez et al. (2020), a lockdown policy cannot be completely effective, since there will always be a mass of people that will not respect the lockdown. While in their setup they consider this dispersion as exogeneous, we try to embedded it in our setup with an endogenous mechanism. The purpose is twofold. On one side, nesting this mechanism inside our model allows us to take into account a moltitude of different factors that could contribute to determine the behaviour of different agents, i.e. the inner mortality of the virus, the cost to stay home, the heterogeneous response of individual characteristics to the virus. Taking all these aspects under consideration can be helpful to the planner in the preparation of context-specific solutions to make the containment-policy more efficient facing different scenarios. On the other side, our endogeneous setup consider a cost-to-stay home as a possible factor of influence. This cost can be interpreted as a subsidy-policy deployed by the government, allowing us to evaluate the effects of this additional tool to fight the pandemic. Different levels of subsidy could modify the extent of compliance exerted by the individuals, helping to be more effective in containing the losses, both economic and of lives.

In addition, we depart from the majority of other models with focus on the pandemic since we endogenise the probability of getting infected, while in other models is generally taken as exogeneous (valuable exceptions are Eichbaum et al. (2020) and Jones et al. (2020)). As already mentioned above, this endogeneous probability takes into account several aspects, so that we are able to simulate a wider variety of possible scenarios relative to the ones commonly shown by a classic SIR model. This possibility can potentially help to i) better undertsand different dynamics of a pandemic and ii) design strategies to specificly face different situations.

Several economic models regarding COVID-19 focus their attention on the lockdown-policy as the only instrument to fight the spread of the disease and evaluate the economic loss due to this policy. Among others, Alvarez et al. (2020) study a lockdown planning problem under SIR dynamics. The only available policy-tool for the planner is a linear lockdown technology. They find that the congestion externality plays an important role in shaping the policy response since in their setup the mortality rate increases as the number of infected increased, reproducing the stress that the virus has put on the health system.

Following this work, Eichenbaum et al. (2020) extend the classic SIR model proposed by Kermack and McKendrick (1927) to study the equilibrium interaction between economic decisions and epidemic dynamics. Combining a DSGE and a SIR model, they reproduce a lockdown policy as a tax on consumption and overline how the cut back on consumption and work is able to both reduce the death toll of the pandemic and to exacerbate its effects in terms of economic losses. We share with this model and with the one proposed by Callum et al. (2020) an endogeneous determination of the probability to get infected.

Finally, the work proposed by Acemoglu et al. (2020) studies the effects of targeted lockdowns in a multi-group SIR model, where different age-groups ("young", "middle-aged", "old") face different probabilities of infection, hospitalisation and mortality. They found that a targeted lockdown is able to better mitigate both the total number of death and the economic loss relative to a uniformly distributed linear containment policy.

We think that our work can find its place in the branch of COVID-19 literature concerning the optimal-containment policy with the aforementioned works. In addition, it can contribute to enrich the incipient economics-epidemiological literature. For example, among the others, Atkeson (2020) and Stock (2020) provide an introduction to the SIR framework and its implications for COVID-19 in the US. Fernández-Villaverde and Jones (2020) fit a standard SIR model to multiple regions (countries, states and cities) and uses the model to infer unobservables (such as nnumber of recovered) and create forecasts. Closer to our paper, a number of recent papers started incorporating economic trade-offs and conducting optimal policy analysis within the SIR framework (e.g. Rowthorn and Toxvaerd, 2020, Eichenbaum, Rebelo and Trabandt 2020, Alvarez, Argente and Lippi 2020, Jones, Philippon and Venkateswaran, 2020, Farboodi, Jarosch and Shimer, 2020 and Garriga et al., 2020).

2. The Model

We enrich a classic SIR model as proposed by (McKallum ()) with an endogeneous mechanism of the propagation of the disease which stems from the level of compliance of different individuals. We evaluate the problem in discrete time, quite uncommon for this class of models, following the approach proposed by Rebelo et al. (2020) and Callum et al. (2020) among the others. Our setup is characterised by a continuum of agents who differ for their sensitivity to the virus. As common in this literature, they are divided in four groups labelled as their statut relative to the virus. Namely, an agent in our model can be, at each time t, i) *Susceptible*, ii) *Infected*, iii) *Recovered* or iv) *Dead*. We close the model with a planner who wants to minimise both the economic and life losses due to the virus. In order to do so, the optimal planner can i) set an optimal level of lockdown policy and ii) provide the population with a subsidy, helping them to bear the pandemic. For the design of our planner we stick to the formulation proposed by Alvarez et al. (2020).

2.1. Actors of the model

Citizens are uniformly distributed over $[0, 1]$ and indexed by i, and the unit interval is endowed with the *Lebesgue* measure. After the appearance of a new pathogen, an infectious disease begins to spread across the population. We indicate with $t = 0$ the date at which the policy-maker becomes aware of the ongoing infection. For the sake of simplicity, throughout the paper we use the term i) *contagiousness* to indicate the probability that a non-infected individual contracts the disease after being exposed to the pathogen; the term ii) *mortality* to indicate the probability of death of an infected individual; and the term iii) *susceptibility* to indicate the probability that a single, non-infected individual becomes infected. It worth highlighting that while contagiousness and mortality are intrinsic, exogenous characteristics of the disease, susceptibility directly depends on the behavior of citizens both at the individual and at the aggregate level. As a consequence, the latter is endogenous. The contagiousness of the disease is assumed identical across citizens, and it is parameterized by the exogenous variable $\beta > 0$, that we assume to be common knowledge. Mortality is heterogeneous across citizens, for it is affected by both the specific characteristics of the disease and of the infected individual – e.g. her age and/or the presence of preexisting chronic pathologies. We refer to the mortality induced by the disease per se as the intrinsic component of mortality, and indicate it with $\theta \in \mathbb{R}$. The intrinsic component θ is identical across all citizens. Conversely, we refer to the mortality induced by specific characteristics of the infected individual as the idiosyncratic component mortality, for the sake of simplicity summarized into a unidimensional statistic $x^i \in \mathbb{R}$ defined as:

$$x^i = \bar{x} + \varepsilon^i \tag{2.1}$$

for all $i \in [0,1]$, where $\overline{x}$ is easily interpreted as the average/median idiosyncratic mortality [1], and with ε^i a noise variable defined as

$$\varepsilon^i \sim \mathcal{N}\left(0, \sigma_\varepsilon^2\right) \tag{2.2}$$

for all $i \in [0,1]$. Both mortality variables θ and $\tilde{x}$ are independent and common knowledge. Moreover, they are independent from noise terms ε^i, that are assumed *i.i.d.* and constant over time. The intrinsic component x^i is inherently heterogeneous in the cross-section of citizens. In particular, we assume that higher values of x^i indicate an higher post-infection mortality, ceteris paribus. Intrinsic and idiosyncratic mortality are mapped into an individual probability of death post-infection $\Pi^D(\theta, x^i)$ via a monotone function $\Pi^D : \mathbb{R}^2 \longmapsto [0,1]$ defined as:

$$\Pi^D\left(\theta, x^i\right) = \Phi\left(\theta + \alpha x^i\right) \tag{2.3}$$

where $\Phi(\bullet)$ is the normal standard CDF, $\theta \in \mathbb{R}$ is the intrinsic mortality of the disease, $x^i \in \mathbb{R}$ is the idiosyncratic component of mortality, and $\alpha > 0$ is a scaling coefficient that determines the relative importance of individual characteristics in determining the probability of death of infected individuals. In the context of the recent SARS-CoV-19 pandemics, x^i can be interpreted as a summary statistic that (strictly) increases in the age of the individual. Notice that, since both θ and x^i are normally distributed, then the individual probability of death post-infection $\Pi^D(\theta, x^i)$ follows *approximatively* a Beta distribution.

[1] Indicate with $\tilde{x}$ the average idiosyncratic mortality. Due to the continuum-player nature of the static game under inspection, and since the mass of players is normalized to one, average $\tilde{x}$ is defined as

$$\tilde{x} = \int_0^1 x^i \mathrm{d}i$$

Given expression (2.1), the above integral can be rewritten as

$$\tilde{x} = \int_0^1 \left(\bar{x} + \varepsilon^i\right) \mathrm{d}i = \bar{x} + \int_0^1 \varepsilon^i \mathrm{d}i$$

It is immediate to check from expression (2.2) that $\int_0^1 \varepsilon^i \mathrm{d}i = 0$ almost surely, so that $\tilde{x} = \overline{x}$ by construction.

When the new infectious disease is discovered at the initial date $t = 0$, the policy-maker can (try to) contain the infection by calling for a general lock-down, whereby social interactions are partially or completely forbidden. In order to guarantee the regular functioning of essential activities, the lock-down can be imposed only to a fraction L of citizens, with $0 < L \leq \overline{L}$ and $\overline{L} < 1$. A lock-down L imposes a strict quarantine at home to all citizens with $i \in [0, L] \subset [0, 1]$.

2.2. *The Static Game*

2.2.1. *Actions and Payoffs*

The policy-maker has a limited capacity to enforce the lock-down, so that the actual mass of quarantined individual – hence, the effectiveness of the lock-down as a device to contain the infection – arises from the decentralized decision-making of single citizens. To simplify the analysis, we assume that each citizen $i \in [0, L]$ subject to lock-down faces a simple, binary choice between being compliant or not with the prescriptions of the policy-maker. In other words, each citizens is called to decide whether or not to stay at home in voluntary quarantine. Moreover, citizens are not required to commit in advance to a plan of actions, hence individual binary choices are repeated at every date t. Indicate with $a_t^i = 1$ the choice to comply at date t of the $i - th$ generic citizen, and with $a_t^i = 0$ the choice not to comply [2]. The total instantaneous share $A_t \in [0, 1]$ of citizens subject to lock-down that comply at date t can be defined as:

$$A_t = \int_0^L a_t^i \mathrm{d}i \tag{2.4}$$

We assume further that all citizens $i \in [L, 1]$ that are not subject to the lock-down are not quarantined, so that the total mass NQ_t of citizens that are *not* quarantined at date t can be written as [3]:

$$NQ_t = 1 - A_t L_t \tag{2.5}$$

Assume that citizens use the following monotone strategy to choose their action:

$$a_t^i = 1 \Leftrightarrow x_t^i \geq x_t^*, \tag{2.6}$$

[2]Therefore, the action space $\mathcal{A} = \{0, 1\}$ is symmetric across citizens and time-invariant.

[3]Since we assume that the mass $1 - L$ of citizens not subject to lock-down is not quarantined, from expression (2.4) we have that the aggregate mass Q_t of quarantined citizens can simply be defined as $Q_t = A_t L$. Given the normalization $N = 1$ of the total mass of citizens, it is straightforward to check that $NQ_t = 1 - Q_t$.

so that the mass of compliant citizens (2.4) can be written as:

$$A_t = Pr(x_t^i \geq x_t^*) = \Phi\left(\frac{\bar{x} - x_t^*}{\sigma_\varepsilon}\right). \tag{2.7}$$

Substituting (2.7) into (2.5) the mass of people who interact (M_t), represented by not quarantined citizens can be written as:

$$M_t = NQ_t = 1 - L_t\left[\Phi\left(\frac{\bar{x} - x_t^*}{\sigma_\varepsilon}\right)\right]. \tag{2.8}$$

Now, in our setup the amount of infected individuals is divided in two categories: Symptomatics (SY) and Asymptomatics (ASY). Hence, the total amount of infected at each time t is:

$$I_t = SY_t + ASY_t, \quad \forall t \in \{0, 1, 2, \ldots\} \tag{2.9}$$

where $SY_t = (1-p)I_t$ and $ASY_t = pI_t$, with the coefficient $p \in (0,1)$ common knowledge and costant over time. All symptomatic citizens are recognized as infected – hence infective - and immediately quarantined, so that they are not allowed to spread the infection. Conversely, asymptomatic individuals are never recognized as infected. Since asymptomatic individuals are not quarantined, they spread the contagion if either i) they are not subject to lock-down, or ii) they are subject to lock-down but not compliant. The spread of the infection is therefore governed by the evolution in time of the mass of asymptomatic-infected individuals.

Rearranging 1.1, we obtain the portion of asymptomatics depending on symptomatics:

$$ASY_t = SY_t\left(\frac{p}{1-p}\right). \tag{2.10}$$

At each time, the number of symptomatic individulas is communicated, so that it is possible for each agent to know the probability to meet an infected who does not show symptoms, since these two informations are related (p is known). The total mass of susceptible people at time t is defined as:

$$S_t = N_t - I_t \tag{2.11}$$

and people who are not forced to stay in quarantine (NFQ_t) - and so free to circulate and spread the virus - are:

$$NFQ_t = \underbrace{(N_t - I_t)}_{S_t} + ASY_t \tag{2.12}$$

so that we are able to derive the probability to intercact with an infected citizen, defined as the ratio between the asymptomatic people (that are able too circulate if do not comply) and the total amount of people not forced to stay under quarantine. Namely:

$$\Lambda_t = \frac{pI_t}{S_t + pI_t} = \frac{pI_t}{(N_t - I_t) + pI_t} = \frac{pI_t}{N_t - (1-p)I_t} \tag{2.13}$$

Hence, the probability to contract the disease if not compliant (Σ_t) as:

$$\Sigma_t = \beta M_t \Lambda_t = \beta \left(\frac{pI_t}{S_t + I_t}\right)\left[1 - L_t\left(\Phi\left(\frac{\bar{x} - x_t^*}{\sigma_\varepsilon}\right)\right)\right], \tag{2.14}$$

and a mass of potential infected is:

$$P_t = S_t \Sigma_t = S_t \beta M_t \Lambda_t = (N_t - I_t)\,\beta \left(\frac{pI_t}{S_t + I_t}\right)\left[1 - L_t\left(\Phi\left(\frac{\bar{x} - x_t^*}{\sigma_\varepsilon}\right)\right)\right], \tag{2.15}$$

Indicate with $I_t \in [0,1]$ the (potentially unobserved) aggregate mass of infected citizens at date t. Each citizen's instantaneous probability to become infected depends both on the constant contagiousness β of the disease, and on the number of interactions of the citizen with other, potentially infected individuals. When compliant, a citizen is quarantined, so that the number of interactions is zero. As a consequence, so it is her individual probability of contagion. When not compliant, a citizen interacts with all other non-quarantined individuals. We can express the individual, action-contingent probability of contagion $\Pi_t^C(a_t^i, A_t)$ at any date t as

$$\Pi_t^C\left(a_t^i, A_t\right) = \left(1 - a_t^i\right)\left[\beta \Lambda_t S_t \left(1 - A_t L\right)\right] \tag{2.16}$$

where, once again, a_t is the action of the $i-th$ citizen at date t, $\beta > 0$ is the exogenous contagiousness of the disease, Λ_t is the (endogenous) instantaneous probability to interact with an infected individual, and $1 - A_t L$ is the total mass of non-quarantined citizens at date t. When a citizen becomes infected, her idiosyncratic risks of death is quantified by the probability $\Pi^D(\theta, x)$. We assume that death entails a private cost $d > 0$ arbitrarily large but *finite*, so that the individual cost of infection $K(\theta, x_i)$ can be defined as

$$K\left(\theta, x^i\right) = d\Pi^D\left(\theta, x^i\right) \tag{2.17}$$

for every citizen $i \in [0,1]$, where $d > 0$ is the private cost of death and $\Pi^D(\theta, x^i)$ is the idiosyncratic probability of death post-infection defined in (2.3).

Direct inspection of (2.16) immediately reveals that the probability of infection of a non-compliant citizen *strictly decreases* in the total mass of quarantined individuals. This property is quite intuitive: if the mass of compliant citizens is large, then the risk associated to individual defection is low. However, *strategic substitutability* governs individual decision making. If the level of lockdown is high, an individual may think that the number of people outside is low, so she could misperceive the risk of meeting asymptomatics and get ting infected.

Actions are mapped into payoffs via a utility function $u : \{0,1\} \times \mathbb{R}^2 \times [0,1] \longmapsto \mathbb{R}$ common to all citizens [4]. Compliant citizens – i.e. citizens that opted for action $a_t^i = 1$ – bear a sunk cost $c > 0$, assumed identical across citizens for the sake of simplicity. The cost of compliance c quantifies the instantaneous welfare loss suffered by the citizens as a consequence of the limitations imposed by the quarantine. Furthermore, recall from (2.6) that the probability of contagion is zero for compliant citizens. Therefore, the instantaneous (individual) utility from compliance $u_t^i(a_t^i = 1)$ can be defined as

$$u_t^i \left(a_t^i = 1 \right) = -c \tag{2.18}$$

for every $i \in [0, L]$ and every $t \in \{0, 1, 2, \dots\}$. Note that the payoff is not risky, hence the ex post utility from compliance $u_t^i(a_t^i = 1)$ coincides with the *ex ante* expected utility from compliance $\mathbb{E}[u_t^i(a_t^i = 1)|\mathcal{I}_t^i] \equiv EU_t^i(a^i = 1)$. Non-compliance entails no fixed cost to the citizen. However, the probability of infection of non-compliant citizens is positive, hence their expected cost of infection is nonzero, i.e.

$$u_t^i \left(a_t^i = 0, A_t, x^i \right) = -\Pi_t^C \left(a_t^i, A_t \right) K \left(\theta, x^i \right) \tag{2.19}$$

where Π_t^C is the action-contingent probability of infection defined in (2.16), and K is the individual cost of infection defined in (2.7). Note that many of the payoff-relevant elements of (2.19) are unknown to the citizen at the moment she chooses her course of action. Therefore, all instantaneous decisions are driven by the comparison between the cost of compliance $-c$ and the *expected* payoff from non-compliance $EU(a_t = 0, A_t, x^i)$. Properly substituting in expression (2.19) for Π_t^j with $j = \{C, D\}$ and K, as defined in (2.16), (2.3) and (2.17) respectively, we can define the expected payoff from non compliance as

[4] In words, at any date $t \in \{0, 1, 2, \dots\}$, the instantaneous payoff of the generic $i - th$ citizen depends on her action $a_t^i \in \{0, 1\}$, on both the intrinsic and the idiosyncratic mortality post-infection $\Pi(\theta, x^i)$, and on the mass of citizens that are not quarantined.

$$EU\left(a_t^i = 0, A_t, x^i\right) = -d\beta\mathbb{E}\left[\Lambda_t S_t\left(1 - A_t L\right)\Phi\left(\theta + \alpha x^i\right)|\mathcal{H}_t^i\right] \tag{2.20}$$

where $\mathcal{H}_t^i$ is the information set of the $i - th$ citizen at date t.

Hence, the threshold above which the mass of citizens that will be compliant to the lockdown policy set by the planner, following the optimal strategy defined in (2.6), is the that solve the following individual indifference condition between being compliant or not:

$$u_t^i\left(a_t^i = 0, A_t, x^i\right) = u_t^i\left(a_t^i = 1, A_t, x^i\right) \tag{2.21}$$

from which we obtain:

$$-c = -d\beta\mathbb{E}\left[\Lambda_t S_t\left(1 - A_t L\right)\Phi\left(\theta + \alpha x^i\right)|\mathcal{H}_t^i\right]. \tag{2.22}$$

Rearranging the above expression and remembreing that $\Lambda_t = \frac{pI_t}{N_t-(1-p)I_t}$ we obtain the following expression:

$$\frac{c}{d\beta}\left(\frac{S_t + ASY_t}{S_t ASY_t}\right) = \Phi\left(\theta + \alpha x_t^*\right)\left(1 - L\Phi\left(\frac{\bar{x} - x_t^*}{\sigma_\varepsilon}\right)\right). \tag{2.23}$$

For proper values of the ratio $\frac{c}{d}$, it is possible to find a value of the threshold satisfying (2.23). The mass of citizens who are compliant to the lockdown policy at time t will be given by all $x^i \leq x_t^*$. In figure 1 we show both the scenarion where a threshold is obtained or not. It worth noticing that the it is possible to obtain the case represented in the right table in figure 1 even when the number of infected peolpe is small, so that the probability the probability of getting infected is too low. In formula, this can be seen as:

$$\lim_{\Lambda\to 0} u_t^i(a_t^i = 0) = 0, \tag{2.24}$$

making the strategy to comply strictly dominated ($u_t^i(a_t^i = 0) > u_t^i(a_t^i = 1)$)[5].

[5] More clearly, when $\lim_{\Lambda\to 0} u_t^i(a_t^i = 0) = 0$ we have that $u_t^i(a_t^i = 0) = 0$ while $u_t^i(a_t^i = 1) = -c$ is the same in each scenario. In this scenario, the cost to stay home is always higher of the cost of getting infected, making the *staying-home* strategy strictly dominated. Hence, the entire mass of people does not comply with the lockdown policy

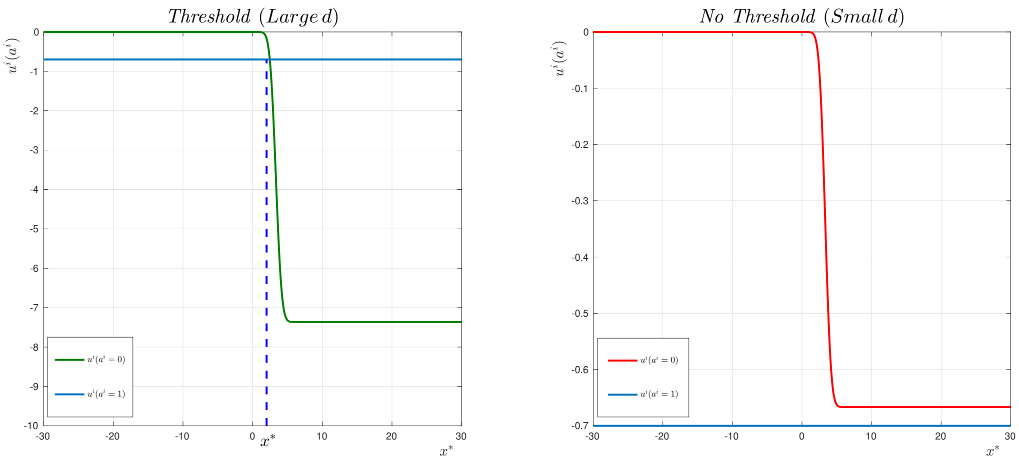

Fig. 1. *On the left panel the value of d is high enough to guarantee the presence of a threshold satisfying the indifference contagion. It is clear that for values of $x^i \leq x^*$ we have $u^i(a^i = 0) \leq u^i(a^i = 1)$, so that the optimal strategy for an individual is not to be compliant. For values of $x^i \geq x^*$ the otherwise is true. On the right panel, the value of d does not guarantee a threshold, and it is always the case $u^i(a^i = 0) < u^i(a^i = 1)$. In this case, respecting the lockdown is a strictly dominated strategi, so all no one is compliant.*

2.3. *Propagation of the contagion*

Following the mechanism described above, at the beginning of each period a mass of citizens is still infected from the previous time. Among them, a mass of asymptomatics is present and spread the virus. The width of the contagion is determined by the actions undertaken by citizens, who decide whether to comply or not with the lockdown. It worth noting that individuals consider the action of the others and the probability to meet an asymptomatic individual when the decide their optimal action, but they are not able to determine if they were infected and asymptomatic in previous times. To be more precise, they know the probability to meet an asymptomatic, but they consider the sickness to last only for one period, so they do not consider the cases in which they could have got sicked at time $t-2$ or time $t-1$ in their evaluation problem. So, every period an individual consider her optimal decision as if she was susceptible, even though this could not be the case. This assumption is a bit stronger, but since as for now we are not taking into account the presence of a private texting device seems not too unreasonable. That said, the real dynamic of infected people follows a path quite similar to the one present in the SIR models. Specifically, the path of infected at time $t+1$ is:

$$I_{t+1} = I_t + NI_t - ND_t - NR_t, \tag{2.25}$$

where I_t are the infected at previous time t, NI_t are the new infeted at time t, ND_t are the death people due to the virus and NR_t are the ones who recovered instead, both defined in t. It is important to notice, that the share of death and recovered people in our setup do not participate to the propagation of the virus, they are helpful to show the dynamic of the population. At the beginning of time t, a portion of infected from time $t-1$ is asymptomatic and a share of asymptomatic people from $t-1$ is still present (people without symptoms who neither died nor recovered yet). The infected individuals at the beginning of time t spread the virus and determine the new infected (NI_t) of period t. Among these new infected, a share p does not present any symptom. Thus, we have the total number of asymptomatic individuals at time t:

$$ASY_{t+1} = pNI_t + p(I_t - NR_t - ND_t), \tag{2.26}$$

with p percentage of infected people who do not show symptoms, exogeneously defined. The evolution of the mass of new infected people at time t requests a particular attention. When there are *enough* infected citizens free to circulate, there exists a threshold solving the individuals' indifference condition described in section 2.2. So that, people decide either to comply or not observing the action of the other people. Thus, the mass of new infected is given by:

$$NI_t = \beta \Lambda_t S_t \left(1 - A_t L_t\right) = \frac{\beta S_t ASY_t}{S_t + ASY_t} \left(1 - L_t \Phi\left(\frac{\bar{x} - x_t^*}{\sigma_\varepsilon}\right)\right)^6 \tag{2.27}$$

where β and A_t are respectively the constant contagiousness of the disease and the total instantaneous share of citizens subject to lock-down that comply at date t. Nevertheless, a scenario in which the share of infected people is too small is possible. In this case, since the number of infected people does not have any effect on the actions of individuals, since the probability to contract the virus too low, *being compliant* is a *strictly dominated* strategy, so that the mass of not compliant people in this case is equal to 1 - no one stays home.

Once again, in order to be as clearer as possible, the number of new infected individuals at the end of time t will be the number of infected at the beginning of $t+1$. In our model

[6] When $\Lambda \longrightarrow 0$ the strategy to comply is strictly dominated, as shown by equation (2.24). In this case, the number of new infected in t is determined as:

$$NI_t = \beta S_t \frac{ASY_t}{S_t + ASY_t}.$$

we keep track of whole infected people in the law of motion of asymptomatics.

The percentage of susceptible people at time $t+1$ follows a quite common path exploited in SIR setup. The amount on new susceptible is given by:

$$S_{t+1} = S_t - NI_t \tag{2.28}$$

The number of recovered people at time $t+1$ is the number of recovered people at time t plus the number of infected people who just recovered (NR_t):

$$R_{t+1} = R_t + NR_t \tag{2.29}$$

The number of new recovered people at time t is given by:

$$NR_t = \pi_r I_t \tag{2.30}$$

where π_r is exogeneously set in our setup.

Finally, the number of deceased people at time $t+1$ is the number of deceased people at time t plus the number of new deaths (ND_t):

$$D_{t+1} = D_t + ND_t \tag{2.31}$$

where ND_t is given by:

$$ND_t = \pi_t^d(x_t^*) I_t \tag{2.32}$$

The probability of death is endogeneously set in our setup and it is a function of the threshold (x^*). It is sensitive to the mass of people who are compliant. The ratio behind it is that people who are compliant cannot die in our setup, since they are not susceptible to the virus. So, the probability of death need to be the average mortality among people who do not respect the lockdown. Since, as mentioned above, the idiosincratic mortality of our population follows a normal distribution, each time we have to reconsider the average mortality of the mass of people who decide to go out. In order to do so, each time the mortality rate $\pi_t^d(x_t^*)$ will be given by:

$$\pi_t^d(x_t^*) = E[\Phi(\theta + \alpha x) | x^i \leq x_t^*] = \frac{1}{\sigma_\varepsilon} \int_{-\infty}^{x_t^*} \Phi(\theta + \alpha x^i) \frac{\phi(\frac{x^i - \mu}{\sigma_\varepsilon})}{\Phi(\frac{x_t^* - \mu}{\sigma_\varepsilon})} dx^7 \tag{2.33}$$

Total population, N_{t+1}, evolves according to:

[7] As already explained, when $\Lambda \longrightarrow 0$ the entire mass of people is not compliant, so that our mortality is

$$N_{t+1} = N_t - ND_t \tag{2.34}$$

2.4. Optimal planner

As for now, we design the problem of the planner as proposed by Alvarez et al. (2020). Specifically, we assume that each agent alive produces w units of output, when she is not in lockdown. Agents are assumed to live forever and the only cause of death is the virus. The planner need to set the optimal lockdown policy minimising the present value of the following flow of costs:

$$\sum_{t=0}^{\infty} \gamma^t \left((S_t + I_t)L_t w + ND_t \left[\chi + \frac{w}{r} \right] \right) \tag{2.35}$$

The first part of the equation represents the economic loss due to people potentially in lockdown, if this would have been completely effective. The cost of fatalities due to the infection is given by the present value of output that they would have produced, w/r, as well as by the extra cost of death, χ. Of course, what matters for the problem is the magnitude $\frac{w}{r} + \chi$, which has the same units as the value of a statistical life. In the above formula ND_t is the number of new deaths as represented in (2.22) and γ^t is the discount factor on future disutilities. It worth noticing that the planner in our setup is not able to clearly see the optimal decisions of each individual, so she takes her policy decision evaluating the aggregate movements of population.

3. Simulations and results

In this section we simulate and solve our model numerically, exploiting an algorithm drawn upon the one proposed by Eichenbaum et al. (2020). In the first part of this section we analyse the main mechanism behind our model. In order to do so, we do an exercise of comparative statics, keeping constant the lockdown policy and analysing the dynamic for different values of the cost to stay home (c). We perform this exercise for two values of lockdown policy, simulating a high lockdown policy ($L = 0.8$) and a loose regime ($L = 0.1$). Our model is simulated for 80 periods.

given by the average mortality of the whole population. Namely, we have:

$$\pi_t^d = \Phi \left(\frac{\theta + \alpha \overline{x}}{\sqrt{1 + \alpha^2 \sigma_\varepsilon^2}} \right)$$

We set an appropriate parametrization of the parameters describing the distribution of our population. A list of our parameters is provided in Table 1. The initial population is normalized to one. The number of people that are initially infected I_0 is 0.001. The value of individual production w is normalised to one, and the *value of statistical life* is set equal to 20, following the parametrization proposed by Alvarez et al. (2020). Considering our parameters, θ is -5, meaning that the virus is not very mortal *per se*. Nevertheless, we consider it to be quite contagious, with $\beta = 1.5$.

(i) Parameter	Value	Definition
α	1.5	Weight of individual characteristics on mortality
θ	-5	Virus' intrinsic Mortality
$\overline{x}$	2	Average idiosyncratic mortality
σ_ε	10	Variance
β	1.5	Contagiousness of the disease
d	100	Individual cost of death
p	0.8	Share of asymptomatics on infected people
w	1	Unit of output produced by each individual
vsl	20	Value of staistical life
(ii) Initial Value	**Value**	**Definition**
N_0	1	Initial Population
I_0	0.0008	Mass of initial infected
S_0	0.9992	Mass of initial susceptible
D_0	0	Mass of initial death
R_0	0	Mass of initial recored

Table 1: *(i) Main parameters (ii) Initial values of individuals' groups*

3.1. Comparative statics

3.1.1. $L = 0.8$

We start our exercise considering a strict lockdown policy ($L = 0.8$). The cost of stay home assumes the following values: $c = [0.50, 2.33, 4.17, 6]$. We can consider a lower cost to stay home as a higher *subsidy* provided by the state. In figure 2 is reported the dynamic of infected individuals, while figure 3 shows the path of population and the share of people who are compliant in each period.

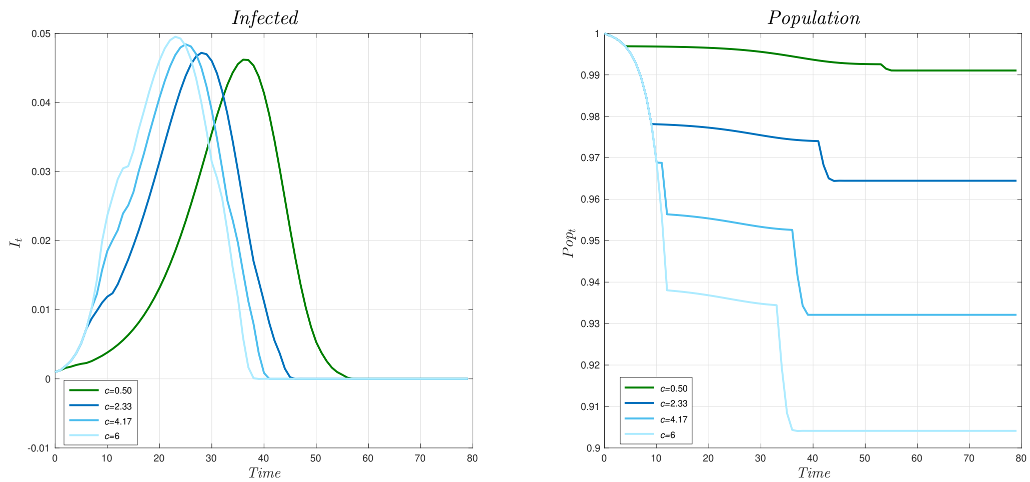

Fig. 2. *In the left table is reported the dynamic of infected individuals, while in the right panel we observe the dynamic of the population. Number of periods in our simulation = 80.*

As we can see in figure 2, an higher *subsidy* (lower c) is able to lower the speed of the process, theoretically allowing the government to buy more time and develop more actions against the Virus (for instance increasing the capacity of the health system). Nevertheless, we do not observe a significant difference concerning the amount of total infected. The effect on the population is more evident, since the wedge between the remaining population determined by the upper and lower values of c is around 10 %. A lower c is more effective in containing the disease, as at the end of our simulation the 90% of the population is still alive. Interestingly enough, the dynamic in place in our model is clearly shown in figure 2. In fact, with a lower c, people are generally more compliant with the lockdown policy, curbing the number of deaths. In the first table of figure 2 we observe that people stay highly compliant for long time (approximately until period 52), and the curve of deaths flattens consequently. The number of deaths starts to increase again once people start not to respect the lockdown (right after time 53). Thus, from our model it seems that a subsidy provided by the state, deployed to help people to bear the economic loss due to the Covid-19 and the relative lockdown, can supply some relief and nudge individuals to stay home, containing the relative effect of the virus on lives losses.

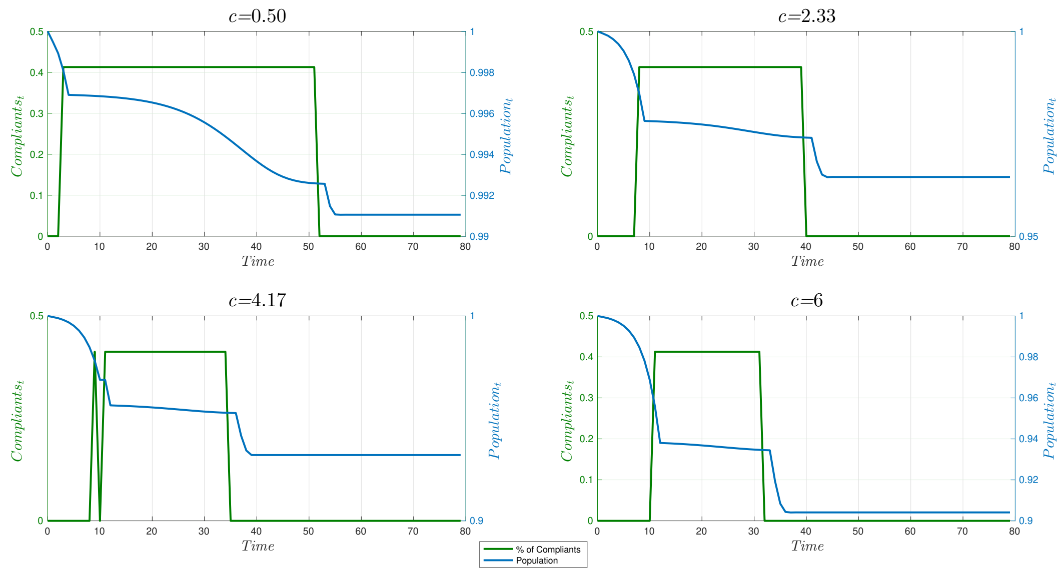

Fig. 3. *In these tables are shown together the paths of compliance and population for different levels of c but with constant lockdown policy ($L = 0.8$). Number of periods in our simulation = 80.*

3.1.2. $L = 0.1$

For this second scenario we simulate a lockdown equal to 0.1, in order to reproduce a very loose policy and almost a total freedom of movement. Nevertheless, we repeat the experiment of the above section and we study the dynamic of the model keeping the same levels of c. Figure 1 shows, as above, the dynamics of infected people (Table 1) and population (Table 2). Interestingly, with $L = 0.1$ we obtain a classic dynamic proposed by several SIR models. Concerning the comparison with the results obtained with a stricter policy, we observe a higher peak of infected, since now the number reach the 15% of people at its highest value. Yet, in our setup the infected people seem to fade away faster relative to common results shown in this literature. Even the number of remaining population can appear quite puzzling, since the number of total deaths for the highest value of c is lower relative to the same value in the scenario proposed above with a stricter lockdown ($L = 0.8$). This result since to be driven by the parametrization of our setup, since as shown in figure 5 our endogeneous mechanism does not seem to play an important role in this simulation. We think that a differ calibration of the parameters coul lead to more common dynamic, so a deeper analysis in this direction is needed.

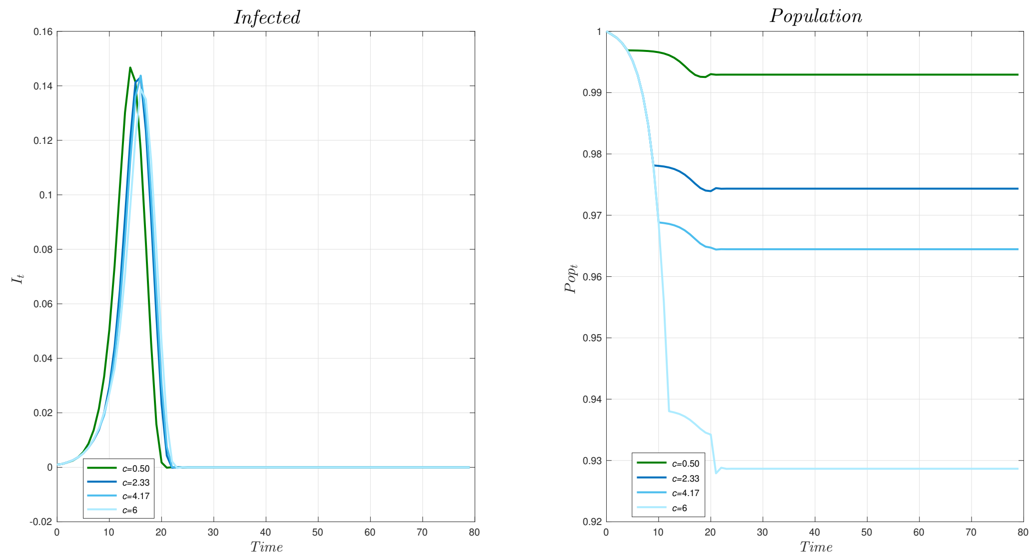

Fig. 4. *In the left table is reported the dynamic of infected individuals, while in the right panel we observe the dynamic of the population. Number of periods in our simulation = 80.*

In figure 5 we observe the aforementioned dynamic. For each value of c people seem to comply only for very limited time. Furthermore, the maximum number number of compliants is never higher than the 20%. Yet, since the value of potential lockdown is very low (only 10%), it seems plausible. Nevertheless, even in this scenario we see the curbing effect due to compliance, since the decline in population flatten in correspondence with every start of "compliance" periods.

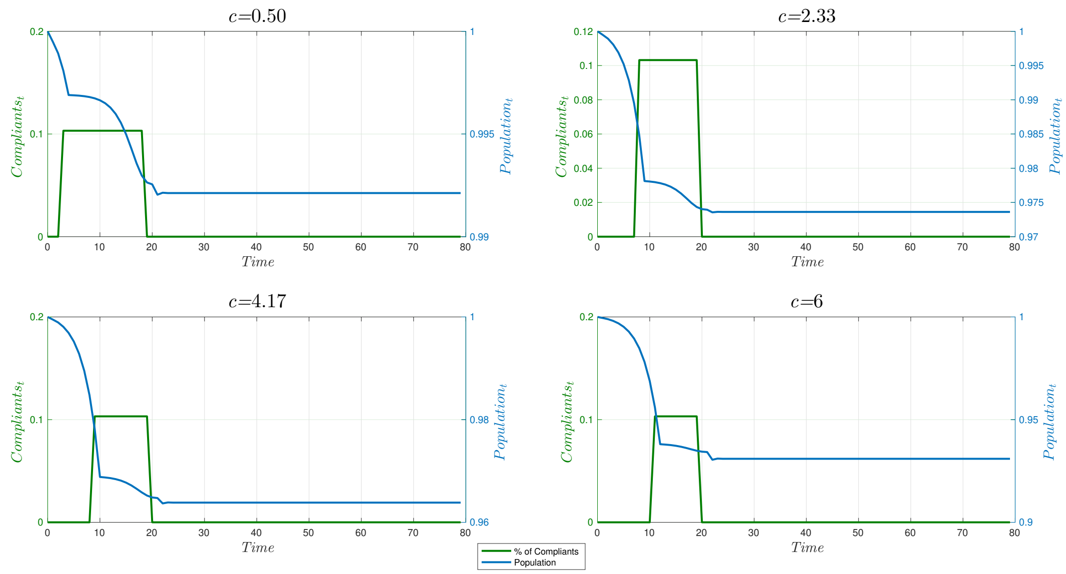

Fig. 5. *In these tables are shown together the paths of compliance and population for different levels of c but with constant lockdown policy (L = 0.8). Number of periods in our simulation = 80.*

4. Conclusions

Preliminary results reported in the section above show that the a proper subsidy policy can help in containing the spread of the disease, making individuals more willing to accept the lockdown. An economic subsidy can provide them with some relief in bearing the pandemic. As we have seen, during times with high level of compliance population stops to drop and the curve flatten. In addition, our mechanism seems able to simulate different scenarios returning results in line with classic SIR models when the lockdown policy is loose ($L = 0.1$). We are aware that a deeper inspection of this scenario is needed, in order to obtain a smoother (and more realistic) dynamic.

Furthermore, as for now the planner does not play any role in our work, but next speps will be i) to calculate the optimal lockdown policy endogeneously set by the planner, ii) evaluate diffrent sets of parameters in order to replicate several (and more articulate) scenarios and finally iii) try to endogenise the decision of the subsidy c, introducing some cost of implementation.

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