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3d SCFTs from S-duality walls

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Declaration of Authorship

The material presented in this dissertation is based on my own research and any result obtained by other authors has been properly acknowledged.

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- **Chapter** 6: I. Garozzo, N. Mekareeya, and M. Sacchi "Duality walls in the 4d $\mathcal{N} = 2$ $SU(N)$ gauge theory with 2N flavours", *JHEP* **1922** (2019) 053 arXiv: 1909.02832 [hep-th]

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- A. Amariti, L. Cassia, I. Garozzo, and N. Mekareeya, "Branes, partition functions and quadratic monopole superpotentials", *Phys. Rev* **D100** (2019) 046001 arXiv: 1901.07559 [hep-th]
- A. Amariti, I. Garozzo, and G. Lo Monaco, "Entropy function from toric geometry", arXiv: 1904.10009 [hep-th]
- I. Garozzo, N. Mekareeya, and M. Sacchi "Symmetry enhancement and duality walls in 5d gauge theories", *JHEP* **06** (2020) 159 arXiv: 2003.07373 [hep-th]

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Contents

Chapter 1

Introduction

Since its beginning, the aim of theoretical physics has been to provide models and theories to describe the Nature within a consistent framework. The intellectual effort of the most brilliant physicist has been devoted to the formulation of theories incorporating as much phenomena as possible starting from the smallest set of assumptions and rules, thus *unifying* seemingly unrelated observations and facts. The spirit is the same of Maxwell, who gave a unified treatment to different aspects of electricity and magnetism, culminating in the set of equations that now bring its name: the *Maxwell's equations*. Phenomenological observations have not been the only guiding principle to build up a given theory, but *elegance* and *beauty* of the model itself, sometimes identified with the mathematical formulation underlying physical concepts, have played prominent role.

Nowadays, the real challenges theoretical physics have to tackle may be identified with the following points: get a quantitative access to the strong coupling regime of quantum field theory, QCD being the most relevant example for the description of fundamental interactions, and find a consistent quantum theory for gravitational interactions, hence a theory of *quantum gravity*, essential in order to understand black holes or the first moments of our Universe. In both respects, string theory is the most exciting framework theoretical physicist have at their disposal at the moment, even though it is fair to admit that still a lot of work have to be done in order to accomplish the aforementioned goals.

One of the most spectacular features that string theory revealed following its first formulations is the existence of *dualities* relating different frames or regimes of the theory. The most prominent example is S-duality, a *strong-weak* duality, which has its historical origin in the of Olive-Montonen duality [2] in the context of non-supersymmetric field theory. Based on the work of Goddard, Nuyts and Olive [3] an *electric-magnetic* duality has been conjectured, under which a given gauge theory is dual to a different theory whose gauge coupling constant is the inverse with respect to the original theory and with the *electric* gauge group exchanged with a *dual, magnetic* group. This essentially generalises the electro-magnetic symmetry of Maxwell's equations once magnetic monopoles are introduced into the game. In fact, what Montonen and Olive conjectured is that each gauge theory in which "electrons" are the basic quantised particles and magnetic monopoles are topological defects has a *dual* frame in which the role of fundamental particle and soliton are exchanged: magnetic monopoles are the fundamental particles and electrons are the solitonic objects.

Despite its original formulation, such a duality has a natural realisation in the context of supersymmetric quantum field theory, the first example being the maximally supersymmetric Yang-Mills theory in four space-time dimensions, *a.k.a* $\mathcal{N} = 4$ SYM [4], and later extended to less supersymmetric set-ups, for instance [5]. The original Montonen-Olive duality for the case of $\mathcal{N} = 4$ SYM is based on the group of fractional linear transformations, $\mathrm{SL}(2,\mathbb{Z})$, which reveals in its full glory once the action on the gauge coupling is extended to include the theta angle. Looking at the holomorphic coupling defined as $\tau = \frac{\theta}{2\pi} + \frac{4\pi i}{g^2}$, the action of the S-duality group can be described as follows

$$\tau \to \frac{a + b\tau}{c + d\tau}, \qquad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2,\mathbb{Z}) \tag{1.0.1}$$

The basic transformations, namely the generators of the $\mathrm{SL}(2,\mathbb{Z})$ duality group, are the the S and T transformations. The former acts as an inversion on the holomorphic coupling

$$S: \quad \tau \to -\frac{1}{\tau}, \tag{1.0.2}$$

while the latter acts as a constant shift

$$T: \quad \tau \to \tau + 1, \tag{1.0.3}$$

and their matrix realisation in terms of elements of $\mathrm{SL}(2,\mathbb{Z})$ are the following

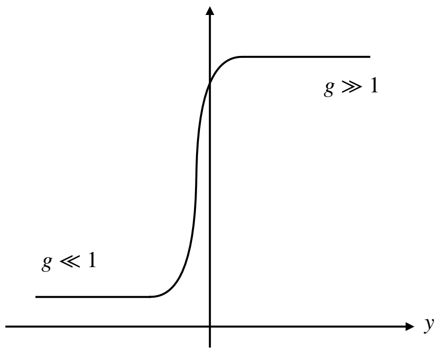
$$S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \qquad T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \tag{1.0.4}$$

satisfying the relations

$$S^2 = -1, \qquad (ST)^3 = 1. \tag{1.0.5}$$

S-duality was shown to play a crucial role in string theory [6], being one of the major advances that made the second superstring revolution to take place. Up to that moment there were five different versions of seemingly unrelated string theories: Type IIA, Type IIB, Type I and the two heterotic $SO(32)$ and $E_8 \times E_8$. Through his work Sen showed that Type IIB string theory with a given coupling constant is mapped to itself once the coupling is inverted and, in the same fashion, Type I is connected to the $SO(32)$ theory. Interconnections among the various string theories like the ones just mentioned then led Witten, in a joint work with Horava [7], to conjecture the existence of M-theory, from which all string theories descend in various limits.

S-duality of $\mathcal{N} = 4$ super-Yang-Mills and its relation to three-dimensional physics via the realisation of the aformentioned duality in Type IIB string theory play a crucial role for the topics analysed in this thesis. The starting point of the discussion is the existence of half-BPS codimension one interfaces in the four dimensional $\mathcal{N} = 4$ SYM theory, called "Janus interface" or "Janus domain wall" [8, 9]. From a holographic perspective, *Janus solutions* of Type IIB supergravity are deformations of the celebrated $\mathrm{AdS}_5 \times S^5$ in which the dilation is allowed to depend on a spatial coordinate and the $F^{(2)}$ form flux vanishes. Janus interfaces, say localised at $y = 0$, where y is a spatial coordinate, cut space-time into two pieces. In each region one has $\mathcal{N} = 4$ SYM with a constant value of the coupling constant, the value being different on the two sides and changing abruptly across the interface. A special case for the spatial dependence of the gauge coupling constant is depicted in the following



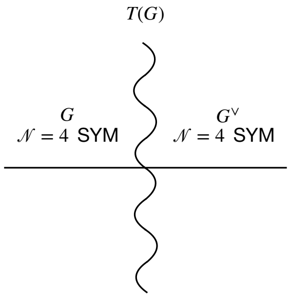
(1.0.6)

meaning that on the left and on the right of the wall a weakly and a strongly coupled $\mathcal{N} = 4$ SYM theory with G gauge group lives respectively. Recall that S-duality is a strong-weak duality, so it possible to *locally* apply it on the strongly coupled side of the interface and get an equivalent, but weakly coupled, theory based on the dual gauge group $G^\vee$, called *Langland* or *GNO* dual. Relevant examples of Langland groups are as follows

G	$G^\vee$
$U(N)$	$U(N)$
$SU(N)$	$SU(N)/\mathbb{Z}_N$
$SO(2N)$	$SO(2N)$
$SO(2N+1)$	$USp(2N)$
$Spin(2N)$	$SO(2N)/\mathbb{Z}_2$
$Spin(2N+1)$	$USp(2N)/\mathbb{Z}_2$
G_2	G_2

(1.0.7)

Gaiotto and Witten analysed the consequences of applying S-duality in the strongly coupled side of space-time in great detail [8] and found that this operation has a "cost": after performing an S-duality transformation the interface supports a non-trivial three-dimensional $\mathcal{N} = 4$ superconformal field theory dubbed as $T(G)$, with $G \times G^\vee$ global symmetry that couples the two SYM theories on the left and right side of the interface. For definiteness, given $SU(N)$ $\mathcal{N} = 4$ SYM theory, the theory on the S-duality wall is $T(SU(N))$ whose global symmetry group is $SU(N) \times SU(N)/\mathbb{Z}_N$. The final situation can be summarised in the following picture



(1.0.8)

where the wiggle line denotes the interface and the boundary theory living on it.

The construction just described makes it evident how S-duality furnishes a natural bridge between the world on four and three dimensional supersymmetric gauge theories, hence it is worth to discuss the latter and spend some words on their properties. In particular, we restrict our attention to a large class of three-dimensional $\mathcal{N} = 4$ theories that can be engineered in Type IIB string theory via *Hanany-Witten* brane systems involving D3, D5 and NS5 branes preserving eight supercharges [10]. The R-symmetry group of three dimensional $\mathcal{N} = 4$ theories is $SO(3)_C \times SO(3)_H \sim SU(2)_C \times SU(2)_H$, realised on the brane system as rotation group acting on the directions $x^{3,4,5}$ and $x^{7,8,9}$. With such a set-up it is possible to construct quiver gauge theories, based on product gauge groups such as $U(N_1) \times U(N_2) \times \cdots \times U(N_k)$, with matter multiplets transforming in the bifundamental representation of the gauge groups and possibly in various representations of the flavour group. The x^6 direction in the brane system can be either taken to be non-compact, in which case linear quiver theory are realised, or also compact, giving rise to circular quivers.

A crucial feature in the study of three dimensional $\mathcal{N} = 4$ theories is the so-called *mirror symmetry* [11], a duality relating pair of theories with non-trivial fixed point. Given a gauge theory, mirror symmetry acts by exchanging two branches of its moduli space of vacua called Higgs and Coulomb branch. The former branch is parametrised by non-trivial vacuum expectation values, *vev*, of the complex scalar in the hypermultiplets, while the latter is parametrised by vevs of the real scalar in the vector multiplets. Due to $\mathcal{N} = 4$ supersymmetry both branches of the moduli space enjoy an hyper-Kähler structure, the complex structure being acted on by the $SU(2)_C$ and $SU(2)_H$ R-symmetry factors. Quantum mechanical effects, generally hard to handle, that arise on the Coulomb branch appear as classical effects on the Higgs branch of the moduli space of vacua of the mirror dual theory. The relevance of mirror symmetry lies on this fact: it allows to "trade" quantum effects for classical ones that are usually computable. Interestingly, this symmetry is realised in string theory [12, 13, 10]. In particular, its Type IIB embedding [10], involves S-duality on Hanany-Witten brane configurations. Such stringy realisation easily allows to construct mirror pairs: starting with a gauge theory realised via an Hanany-Witten system, its mirror dual is obtained via the S-dual of the original brane configuration.

The $T(U(N))$ theory, close cousin of $T(SU(N))$, has the special property of being invariant under mirror symmetry. It enjoys a $U(N) \times U(N)$ global symmetry, one factor acting on the Coulomb and one on the Higgs branch. The nature of the two $U(N)$ symmetry factors is different: the one on the Higgs branch is manifest at the Lagrangian level, while the Coulomb branch one only arises in the deep infrared, thus we refer to it as an *enhanced symmetry*. Despite the different origin of the two $U(N)$ symmetry factors in the global symmetry of the theory, one may consider $T(U(N))$ as a *building block* with $U(N) \times U(N)$ symmetry for constructing quiver theories. In this way one has at hand a sort of *generalised matter* at hand to construct new quiver theories in addition to usual hypermultiplets. The idea is to gauge the global symmetry and to couple it to matter systems. The realisation of this construction leads to the so-called *S-fold theories*; the analysis of various properties of this class of superconformal field theories is the main goal of the present thesis.

One important motivation to study quiver theories with $T(U(N))$ links (with or without non-trivial Chern–Simons levels for the $U(N)$ gauge groups) is because they have interesting holographic duals [14]. The construction involves $\text{AdS}_4 \times K_6$ Type IIB string solutions with monodromies[1] in K_6 in the S-duality group $SL(2,\mathbb{Z})$. These solutions were obtained by quotienting the solutions corresponding to the holographic dual of Janus interfaces in 4d $\mathcal{N} = 4$ SYM [19, 20]. The former type of solutions is referred to as the *S-fold* in [14]. The S-fold solutions can be divided into two classes, known as the *J-fold* and the *S-flip*.

The J-fold solutions are those associated with a monodromy given by an element $J \in SL(2,\mathbb{Z})$ with $\text{tr}\, J > 2$. The corresponding geometry can be constructed by using $\text{AdS}_4 \times S^2 \times S^2 \times \Sigma_2$, where Σ_2 is a non-compact Riemann surface with the topology of a strip. The ends of the strip are then identified with a J-twisted boundary condition. It was shown in [14] that this type of solutions preserve $OSp(4|4)$ symmetry and thus are dual to 3d $\mathcal{N} = 4$ superconformal field theories. The J-fold solutions can, in fact, be obtained as a quotient of a Janus interface solution. As a result, the quiver field theory dual of such a solution contains a component corresponding to such an interface, namely the $T(U(N))$ theory. From the brane perspective, one can introduce a five-dimensional surface implementing the monodromy under the action of J into the brane system. Among the possible choices of the $SL(2,\mathbb{Z})$ elements, we may take the monodromy to be associated with $J_k = -ST^k$ in this case, the corresponding J-fold gives rise to a Chern–Simons level k to one of the $U(N)$ gauge groups.

The S-flip solutions can be discussed in a similar way as for the J-folds. In this case, the $SL(2,\mathbb{Z})$ element implementing the monodromy is taken to be S. Geometrically, we need to perform an exchange of coordinates corresponding to the two S^2 in $\text{AdS}_4 \times S^2 \times S^2 \times \Sigma_2$, together with a flip at

[1]It should be mentioned that a similar solution in AdS_5 was considered in [15, 16], and those in AdS_3 were considered in [17, 18].

the S-interface such that Σ_2 becomes a Möbius strip topologically. Similarly to the J-fold, the insertion of the S-flip into a brane system gives rise to a $T(U(N))$ link between two $U(N)$ gauge groups, where the Chern–Simons levels of those are zero. It was shown in [14] that the S-fold solutions preserve $OSp(3|4)$ and the dual superconformal field theory is expected to have $\mathcal{N} = 3$ supersymmetry.

Hanany-Witten brane systems realising three dimensional $\mathcal{N} = 4$ theories can be enriched including additional objects string theory provides: *orientifold planes* [21]. Orientifold p-planes, denoted as Op, are non-dynamical (at least at weak string coupling) extended objects that can be thought of as the fixed plane of the $\mathbb{Z}_2$ symmetry including both an action on the worldsheet and on spacetime: on the former it acts via parity, while on the latter it reverses the spatial coordinates $x^i \to -x^i$ with $i = p+1, \dots, 9$, and for $p = 2, 3 \bmod 4$ one has to include $(-1)^{F_L}$, where F_L is the left-moving fermion number, to make the combined operation to square to one. Op planes preserve the same supersymmetry as a parallel Dp-brane. A classic review on various aspects of orientifold planes may be found in [22].

Let us discuss in more detail aspects of O-planes that will be needed for a new class of S-fold gauge theories we will introduce momentarily. Op planes come in four variants, at least for $p \geq 5$: Op^+, $\widetilde{Op}^+$, Op^-, $\widetilde{Op}^-$. Each of them allows to realise gauge theories based on *real groups* on the world-volume of a stack of D3 brane on top of the corresponding $O3$ plane. The four variants of O3 planes are distinguished by two discrete $\mathbb{Z}_2$ charges, (b, c), defined as

$$b = \int_{\mathbb{RP}^2} B_{NS}, \qquad c = \int_{\mathbb{RP}^{5-p}} C^{5-p}. \tag{1.0.9}$$

It is useful to summarise the charges of the various O3 planes, the gauge group the realises and, crucial to our applications, their behaviour under the action of S-duality

(b,c)	Op	Charge	Gauge group	S-dual ($p=3$)
$(0,0)$	Op^-	-2^{p-5}	$SO(2N)$	$(0,0)$ Op^-
$(0,1)$	$\widetilde{Op}^-$	$\frac{1}{2} - 2^{p-5}$	$SO(2N+1)$	$(1,0)$ Op^+
$(1,0)$	Op^+	2^{p-5}	$USp(2N)$	$(0,1)$ $\widetilde{Op}^-$
$(1,1)$	$\widetilde{Op}^+$	2^{p-5}	$USp'(2N)$	$(1,1)$ $\widetilde{Op}^+$

(1.0.10)

Observe that the $O3^-$ and the $\widetilde{O3}^+$ are self-dual under S-duality. A comment on the $USp'(2N)$ theory is needed. As a Lie algebra, there is no difference between $USp'(2N)$ and the most familiar $USp(2N)$, the difference is in a global factor when considered as gauge theories: in four dimensions the first has a non-trivial θ-angle set to π, while for the second the θ-angle is vanishing [23].

After this brief discussion on orientifold planes let us come back to S-fold theories. The natural question is whether it is possible to construct such class of superconfomal field theories in the presence of Op planes. The first comment on such a possibility is that, at the moment, there is no class of supergravity solution supporting the construction, differently to the case of *original* S-folds. Nonetheless, from a field theoretical point of view it is still an interesting question to pose. The string theory realisation now involves Hanany-Witten brane set-up in the presence of orientifold planes and S-duality walls. In the absence of O-planes, we discussed the role of the $T(U(N))$ theory as a link corresponding to the S-duality wall inserted in the brane system. In the set-up with O3 on top of D3s the $T(U(N))$ link is replaced by $T(G)$ with $G = SO(2N)$, $SO(2N+1)$, $USp(2N)$, $USp'(2N)$ depending on the type of orientifold. In order for $T(G)$ to be invariant under the S-action, G has to be invariant under S-duality.

For G being $SO(2N)$ and $USp'(2N)$, we propose that the corresponding theory can be realised from a brane construction that contains an intersection between an S-duality wall with the D3 brane segment on top of the orientifold threeplane of types $\mathrm{O3}^-$ and $\widetilde{\mathrm{O3}}^+$ respectively. In other words, the S-fold CFTs of this class can be obtained by inserting an S-duality wall into an appropriate D3 brane segment of the brane systems described in [21]. The mirror theory can be derived by first obtaining the S-dual configuration as discussed in [21], and then insert an S-fold in the position corresponding to the original set-up.

A different extension to this class of theories is realised through a brane system that contains an orientifold fiveplane or its S-dual, which is also known as an ON plane. In which case, the corresponding quiver may contain a hypermultiplet in the antisymmetric (or symmetric) representation, along with fundamental hypermultiplets, under the unitary gauge group, and the mirror quiver may contain a bifurcation [24, 25].

It is interesting that the whole construction, may be extended, in principle, to *exceptional* self-dual gauge groups. Among these, the G_2 case provides an interesting example leading to a completely new class of theories. To the best of our knowledge, the Type IIB brane construction for such theories is not available and mirror theories of this class of models have not been discussed in the literature. In particular, the realised family of quivers contain alternating G_2 and $USp'(4)$ gauge groups, possibly with fundamental flavours under $USp'(4)$. It is then possible to"insert an S-fold" into the G_2 and/or $USp'(4)$ gauge groups in the aforementioned quivers. The mirror theory is also a quiver containing the G_2, $USp'(4)$ and possibly $SO(5)$ gauge groups if the original theory contains fundamental matter under $USp'(4)$.

As it has been discussed when the supergravity realisation of S-fold theories was introduced, the amount of supersymmetry preserved by the superconformal field theory depends on the $SL(2,\mathbb{Z})$ element used to implement the monodromy, and it can be either $\mathcal{N} = 4$ for J-fold solutions or $\mathcal{N} = 3$ for S-flips, in the language of [14]. The techniques employed in that work to unveil the amount of preserved supersymmetry of a given theory are *holographic*, meaning either looking at the supergravity solution itself or studying the large-N partition function. In both cases, the results may be trusted only at large-N, so it is interesting to address a systematic analysis of the IR supersymmetry for S-fold theories at *finite* N. The main technical tool that will be employed to study the supersymmetry possessed in the IR by S-fold theories is the *supersymmetric index*, or *index* for brevity. This allows, in general, to have detailed access to both the global symmetry and supersymmetry of a given theory. The underlying reason is that, for 3d SCFTs, it is possible to put various short multiplets into equivalence classes according to how they contribute to the index [26] (see also [27, 28]). It also allows one to identify the current of the enhanced symmetry. For theories with at least $\mathcal{N} = 3$ supersymmetry, including S-fold SCFTs, the index serves as a rather simple tool to diagnose the presence of the extra-supersymmetry current multiplet, which gives rise to the enhancement of supersymmetry (see *e.g.* [29]). Supersymmetric gauge theories provides plenty of examples of enhancement phenomena, regarding either global symmetries or supersymmetry. The literature on the subject is so vast that it is an hard task to properly cite all the results. Let us however mention the most prominent case of supersymmetry enhancement for three dimensional supersymmetric gauge theories, namely the ABJM theory [30].

Up to now we have only discussed implications for three dimensional supersymmetric gauge theories of the existence of S-duality walls in $\mathcal{N} = 4$ SYM. Nonetheless, the action of S-duality in four dimension has been discussed for less supersymmetric set-up, thus it is conceivable to imagine S-walls for theories other than $\mathcal{N} = 4$ SYM. Specifically, the case of four dimensional $\mathcal{N} = 2$ gauge theories based on $SU(N)$ gauge group with $2N$ massless fundamental hypermultiplets is particularly interesting since it has an exactly marginal gauge coupling with an interesting S-duality group being $SL(2,\mathbb{Z})$ for $N = 2$ [31] and $\Gamma^0(2) \subset SL(2,\mathbb{Z})$ for $N \geq 3$ [32, 33]. This theory can also be realised as a twisted compactification of 6d $(2,0)$ theory of type A_{N-1} on a punctured Riemann surface [34]. The 3d theory associated with the duality wall in this 4d theory can be determined by utilising the AGT correspondence [35, 36], which relates the partition function of the 4d theory on the squashed four-sphere to an observable in the Liouville or Toda theory on the Riemann surface [37, 38]. As pointed out in [39], the partition function of the 3d theory associated with S-duality wall placed along the squashed three-sphere, which is the equator of the aformentioned four-sphere, corresponds to a collection of the duality transformation coefficients of conformal blocks of the Liouville or Toda theory. From such a partition function, one may extract the gauge group and matter content of the 3d theory in question [40, 41, 42]. In fact, this technique has been successfully applied to determine the 3d theory associated with the S-duality wall in the 4d $\mathcal{N} = 2^*$ gauge theory [40]. For the 4d $\mathcal{N} = 2$ $SU(N)$ gauge theory with $2N$ flavours, this method was applied by the authors of [41, 42] (see also [43] for the superconformal index). In [42], the theory associated with the duality wall was then identified as the 3d $\mathcal{N} = 2$ $U(N-1)$ gauge theory with $2N$ flavours, with the R-charges of the chiral fields fixed to certain values. Such a theory will be denoted as $\mathcal{T}_{\mathfrak{M}}$. Later, it was pointed out by the authors of [44] that the superpotential of such a theory should be $W = V_+ + V_-$, where $V_\pm$ are the basic monopole operators of the $U(N-1)$ gauge group. It should be remarked that this approach that is used to identify the 3d theory is different from that used by Gaiotto and Witten [8], mentioned in the previous paragraph. Although the Type IIA brane configuration of the 4d theory [5] and the Type IIB brane configuration of the 3d theory with the monopole superpotential [45, 46, 47, 48, 49] are known, to the best of our knowledge, it is not clear how to identify the latter as the theory associated with the duality wall in the former. Once the theory on the S-duality domain wall has been identified, one may play the same game as with the S-folds and construct various gauge theories where $\mathcal{T}_{\mathfrak{M}}$ is used as component via suitable gaugings of its global symmetry. Thus, the idea is to consider various combinations of a number of duality walls and analyse the properties of the three-dimensional theories obtained in such a way.

The material presented in the thesis is organised as follows:

- **Chapter 2**: We start with a brief summary of basic material on three-dimensional supersymmetric gauge theories. After the supersymmetry algebra and multiplets for $\mathcal{N}=4$ theories have been introduces, we move and introduce monopole operators, playing a fundamental role in the dynamics of three-dimensional gauge theories. Then, the concepts of chiral ring and moduli space of vacua are discussed. A crucial feature of most of the topics we will discuss in the main part of the thesis is mirror symmetry, that is discussed first from a purely field theoretical point of view. Finally, we briefly describe Hanany-Witten brane systems in Type IIB string theory and show how mirror symmetry is embedded in such a context.

- **Chapter 3 [50]**: Starting from this point, the thesis contains original results. In this chapter we address the study of the moduli space of S-fold theories. In section 3.3, we give a brief summary on the S-fold solutions and (p,q) fivebranes, bound states of N5S and D5 branes. In section 3.4, quiver theories corresponding to the brane systems with S-flips are examined. The Higgs and the Coulomb branches of the moduli space are studied using the Hilbert series. We also provide a consistency check of our results against mirror symmetry. In section 3.5, we then consider abelian theories arise from the brane systems with J-folds, along with NS5 and D5 branes. We systematically analyse various branches of the moduli space. In section 3.6, we examine an example of non-abelian theory with $T(U(N))$ links that can be realised on M2-branes on a Calabi–Yau four fold singularity. In this example, we compute the Hilbert series of the moduli space and analyse the contribution from each configuration of magnetic fluxes.

- **Chapter 4 [51]**: In this second chapter we address the study of S-fold theories in the presence of $T(G)$ links different from the $T(U(N))$. In section 4.2, we study the hyperKähler spaces that arise from coupling a nilpotent cone associated with a group G to matter in the fundamental representation of G. Such spaces have some interesting features and this notion turns out to be useful in the later sections because the nilpotent cone arises from the Higgs or Coulomb branch of the $T(G)$ theory. In section 4.3, we investigate quiver theories that arise from brane configuration with an S-fold in the background of the $O5^-$ or the ON^- plane. We provide the consistency conditions for the relative positions between the S-fold and the orientifold plane such that the moduli spaces of theories in question obey the freezing rule and mirror symmetry. In section 4.4 we study various models involving S-folds in the background of the $O3^-$ or the $\widetilde{O3}^+$ planes. The corresponding quivers contain a $T(SO(2N))$ link or a $T(USp'(2N))$ link between gauge nodes. In section 4.5, we propose a new class of mirror pairs involving G_2 gauge nodes, as well as those with $T(G_2)$ link. Finally, in section 4.6, we investigate the quivers that arise from the brane systems with $O5^+$ or its S-dual ON^+. One of the features of the latter is that the quiver contains a "double lace", in the same way as that of the Dynkin diagram of the C_N algebra. Although this part of the quiver does not have a known Lagrangian description, one can still compute the Coulomb branch Hilbert series using the prescription given in [52]. We find that such a Coulomb branch agrees with the Higgs branch with the original theory, and for the theory with an S-fold the former also respects the freezing rule.

- **Chapter 5 [53]**: The main goal of this chapter is to study the supersymmetric index of S-fold theories in order to understand possible global symmetry and supersymmetry enhancement phenomena. In section 5.1.1 the contributions of various superconformal multiplets to the index are discussed. These are the technical tools that are needed for the subsequent analysis. In section 5.2, we discuss S-fold theories with a single gauge group, both in the absence and in the presence of hypermultiplet matters. We also study duality for a theory with two gauge groups and use index to understand the operator map between such theories. In section 5.3, we investigate theories corresponding to two duality walls and with two gauge groups. The addition of fundamental hypermultiplet matter to such theories is discussed in subsection 5.3.1. Finally, in section 5.4, we consider theories with $SU(2)/\mathbb{Z}_2$ gauge group with various Chern–Simons level and use the index to study the discrete $\mathbb{Z}_2$ global symmetry of such theories.

- **Chapter 6 [54]**: In this final chapter we exploit a similar construction of S-fold theories, but with the 3d $\mathcal{T}_{\mathfrak{M}}$ theory arising on the S-duality wall of the 4d $SU(N)$ theory with $2N$ flavours. The chapter is organised as follows. In section 6.1, we introduce the $\mathcal{T}_{\mathfrak{m}}$ theory as the basic building block that will be used to construct the other theories. We discuss first how to couple the 4d fields to $\mathcal{T}_{\mathfrak{m}}$ as well as examine various duality frames. In section 6.2, we present the prescription for gluing many copies of the basic building blocks together as well as propose the prescription for self-gluing. The concept of the "skeleton diagram", which is the analog of the Riemann surface with punctures (used extensively in [1] to construct a large class of theories) and gives rise to a geometric interpretation of the gluing, is introduced in sections 6.1 and 6.2. In section 6.3, we discuss two classes of theories associated with a single wall, whose skeleton diagram contains (1) two external legs and genus one and (2) zero external leg and genus two. The quadralities between such theories are discussed. In sections 6.4 and 6.5, theories associated with two duality walls, using two different types of the basic building block, are constructed and discussed.

- **Chapter 7** : We conclude the thesis with a summary of the topics discussed and with a list of questions that have been left open, that are worth to be addressed in the future.

Chapter 2

$3d$ supersymmetric gauge theories

The topics discussed in this thesis [1], despite their four dimensional original, are based on three-dimensional supersymmetric gauge theories. It is therefore useful to collect some known results that will be used in the rest of the discussion.

The starting point is the superconformal algebra for three dimensional theories with eight supercharges and the structure of matter multiplets. We then move to discuss the Lagrangian for this class of theories. Having the Lagrangian at hand, one may discuss on of the crucial aspects for supersymmetric gauge theories in general, namely the *moduli space of vacua* and its branches, including in particular the Higgs and the Coulomb one. Along the way, the key role of *monopole operators*, one of the main characters for the dynamics of three dimensional gauge theories, will be discussed. As we have already mentioned in the introduction, *mirror symmetry* is an infrared duality which identifies two or more theories that flow to a superconformal fixed points and acts non-trivially on the moduli space of vacua. We will thus discuss the basics of mirror symmetry. Among the three dimensional theories with $\mathcal{N}=4$ supersymmetry, we will pay particular attention to the ones having a realisation in Type IIB string theory via the Hanany-Witten brane systems. After introducing such set-ups, we will discuss how mirror symmetry descends from the action of S-duality on the aformentioned brane construction.

2.1 Basics of $3d$ $\mathcal{N}=4$ gauge theories

The supersymmetry algebra for (2+1)-dimensional $\mathcal{N}=4$ supersymmetric theories is generated by four real supercharges $\mathcal{Q}^A$, with $A=1,\ldots,4$, satisfying

$$\{\mathcal{Q}^A_\alpha, \mathcal{Q}^B_\beta\} = 2\,\sigma^\mu_{\alpha\beta}\delta^{AB}P_\mu + 2\,\epsilon_{\alpha\beta}Z^{[AB]}, \tag{2.1.1}$$

where $\mu=0,1,2$ are Lorentz indices, α,β are spinor indices, σ^μ generates the Clifford algebra and $Z^{[AB]}$ is the antisymmetric matrix of central charges. The supercharges $\mathcal{Q}^A$ transforms in the vector representation **4**, of $SO(4)\simeq SU(2)_H\times SU(2)_C$, the R-symmetry group, which acts as an automorphism of the supersymmetry algebra. In the case of a superconformal theory one has

[1] See also the PhD thesis [55] in which most of the material presented here have already been discussed.

to take the extend the aforementioned algebra to the superconformal one, namely $OSp(4|4)$, whose bosonic subgroup is three-dimensional conformal group $SO(2,3)$ and the R-symmetry group. In the most general case of a theory with $\mathcal{N}$ supersymmetries (namely when the indices $A, B = 1, \ldots \mathcal{N}$), the superalgebra is $OSp(\mathcal{N}|4)$, and the R-symmetry group corresponds to $SO(\mathcal{N})$.

The multiplets needed to construct a three-dimensional $\mathcal{N} = 4$ gauge theory are vector and hypermultiplets, which can be decomposed in terms of $\mathcal{N} = 2$ multiplets. In the latter case, the basic multiplets are the vector and the chiral one, whose component fields are as follows

$$\text{chiral multiplet:} \qquad \Phi = \{\phi, \psi_\alpha, F\}, \tag{2.1.2}$$

$$\text{vector multiplet:} \qquad V_{\mathcal{N}=2} = \{A_\mu, \lambda_\alpha, \sigma, D\}, \tag{2.1.3}$$

where the scalars ϕ and F in the chiral multiplet are both complex, the fermions $\psi_\alpha, \lambda_\alpha$ respectively in the chiral and vector multiplet have two-components, and the vector multiplet scalars σ and D are real. Observe that F and D are auxiliary scalars, and we will see later that their equation of motions are related to the concept of moduli space of vacua. The chiral multiplet transform in a given representation R of the gauge group, while the vector multiplet transform in the adjoint representation.

The $\mathcal{N} = 4$ multiplets, in terms of chiral and $\mathcal{N} = 2$ vector, can be decomposed as follows

$$V_{\mathcal{N}=4} = \underbrace{\{A_\mu, \lambda_\alpha, \sigma, D\}}_{V_{\mathcal{N}=2}} \oplus \underbrace{\{\varphi, \xi_\alpha, F_\varphi\}}_{\chi_{\text{Adj}}}, \tag{2.1.4}$$

$$H = \underbrace{\{\phi, \psi_\alpha, F\}}_{\chi_R} \oplus \underbrace{\{\widetilde{\phi}, \widetilde{\psi}_\alpha, \widetilde{F}\}}_{\chi_{R^*}}, \tag{2.1.5}$$

meaning that the $\mathcal{N} = 4$ vector multiplet decomposes as the sum of an $\mathcal{N} = 2$ vector and a chiral in the adjoint representation, while the hyper contains a chiral in a representation R and one in the complex conjugate representation R^*. It is interesting to observe how the various component fields combine to give representations of the $SU(2)_H \times SU(2)_C$ R-symmetry group:

Multiplet	Fields	$SU(2)_H \times SU(2)_C$
Vector	$\{\sigma, \operatorname{Re}\phi, \operatorname{Im}\phi\}$	$(0,1)$
	$\{\lambda_\alpha, \xi_\alpha\}$	$(\frac{1}{2}, \frac{1}{2})$
	$\{D, \operatorname{Re}F_\Phi, \operatorname{Im}F_\Phi\}$	$(1,0)$
Hyper	$\{\phi^\dagger, \widetilde{\phi}\}$	$(\frac{1}{2}, 0)$
	$\{\psi_\alpha, \widetilde{\psi}_\alpha\}$	$(0, \frac{1}{2})$

(2.1.6)

Having introduced the various multiplets, basics ingredients to construct three-dimensional supersymmetric gauge theories, we now turn to the construction of the action, containing several components. As we did for the multiplets, we will discuss the action in terms of $\mathcal{N} = 2$ supersymmetry which allows for superspace formulation. First of all, the Yang-Mills term for each gauge node contributes with

$$\mathcal{S}_{\text{YM}} = \frac{1}{g^2} \int \mathrm{d}^3 x\, \mathrm{d}^2\theta\, \mathrm{d}^2\overline{\theta}\ \mathrm{Tr} \left(W_\alpha^2 - \Phi^\dagger e^{2V} \Phi \right) + \text{c.c.}. \tag{2.1.7}$$

In the previous expression W_α is the field strength superfield constructed using the $\mathcal{N} = 2$ vector multiplet $V_{\mathcal{N}=2}$, and Φ is the adjoint chiral multiplet in the $\mathcal{N} = 4$ vector. From now on we usually write greek capital letters to denote adjoint chirals in the vector multplets. In writing the various pieced of the action we omit the $\mathcal{N} = 2$ for the vector multiplet, avoiding to clutter the notation. Observe that the gauge gauge coupling for three-dimensional gauge theories have positive dimension, $[g] = \frac{1}{2}$, implying a strong coupling behaviour in the deep infrared, even for abelian theories. This is one of the key aspects of three-dimensional gauge theories: it is possible to find very simple models with an interesting infrared behaviour. In three-dimensions it is possible to construct another type of action for the vector multiplet other than the usual Yang-Mills one, namely the *Chern-Simons* term. The CS action is known for being *topological*, *i.e.* with no propagating degrees of freedom. In the setting of $\mathcal{N} = 4$ gauge theories it is responsible for breaking supersymmetry down to $\mathcal{N} = 3$. The action is parametrised in term of an integer $k \in \mathbb{Z}$ called *level* and reads:

$$\begin{aligned} S_{CS} = \frac{k}{4\pi} \int \mathrm{d}^3 x\ \mathrm{Tr} \left[\epsilon^{\mu\nu\rho} (A_\mu \partial_\nu A_\rho + \frac{2}{3} i A_\mu A_\nu A_\rho) - \lambda\overline{\lambda} + 2D\sigma \right] + \\ - \frac{k}{8\pi} \int \mathrm{d}^3 x\, \mathrm{d}^2\theta\ \mathrm{Tr}(\Phi^2 + \text{c.c.}) \end{aligned} \tag{2.1.8}$$

where the first line is the $\mathcal{N} = 2$ CS term. In particular, observe that the term responsible for breaking the supersymmetry from $\mathcal{N} = 4$ down to $\mathcal{N} = 3$ is the presence of the Φ^2 term in the superpotential breaks the R-symmetry, being charged under the $SU(2)_C$ factor. The R-symmetry group is thus broken down to its diagonal subgroup $SU(2)_H \times SU(2)_C \to SU(2)_{\text{diag}}$. The hypermultiplet kinetic term and its coupling to the vector multiplet are encoded in the action term

$$\mathcal{S}_{\text{hyper}} = - \int \mathrm{d}^3 x\, \mathrm{d}^2\theta\, \mathrm{d}^2\overline{\theta}\, (\phi^\dagger\, e^{2V} \phi + \widetilde{\phi}^\dagger e^{-2V} \widetilde{\phi}). \tag{2.1.9}$$

The $\mathcal{N} = 4$ superpotential is highly constrained and takes the following form

$$\mathcal{S}_{\text{superpot}} = -i\sqrt{2} \int \mathrm{d}^3 x\, \mathrm{d}^2\theta\, \mathrm{d}^2\overline{\theta}\, \widetilde{Q}\, \Phi Q + \text{c.c.}, \tag{2.1.10}$$

where the hypers involved are only those charged under the gauge group.

Let us conclude with two additional terms that may be added to the action to *deform* a gauge theory. However, before doing that, it is necessary to introduce a special feature of three-dimensional theories, namely the existence of a rather special global symmetry: the *topological symmetry*. To introduce it, let us consider for simplicity a $U(1)$ gauge theory. It is possible to construct a current $J_\mu = \frac{1}{4\pi} \epsilon_{\mu\nu\rho} F^{\nu\rho}$, where $F^{\nu\rho}$ is the $U(1)$ field strength, which is automatically conserved due to the Bianchi identity. The topological symmetry acts by shifting the *dual photon*, a periodic scalar[2] a defined as the Hodge dual of the field strength $\partial_\mu a = \epsilon_{\mu\nu\rho}F^{\nu\rho}$. We will discuss more the topological symmetry in the next section when monopole operators will be introduced. It is a general feature of quantum field theory that global symmetries may be described introducing *background* vector multiplets. Such a concept is especially useful to describe the deformation terms we mentioned previously, namely *mass terms* for the hypermultiplets and *Fayet-Iliopoulos (FI) terms*. The former is associated to global non-R symmetries while the latter to the topological symmetry. In detail, mass deformations contain a *real mass*, coming from the $\mathcal{N} = 2$ vector contained in the $\mathcal{N} = 4$ one, and a *complex mass* from the adjoint scalar. Together they form a triplet transforming in the adjoint representation of the $SU(2)_H$ factor of the R-symmetry group. We will not write the action for the mass deformation since it takes the same form of (2.1.9) and (2.1.10). In the case of FI term the three real scalars combine to transform in the adjoint of $SU(2)_C$. The action reads

$$\mathcal{S}_{\text{FI}} = \int \text{d}^3x\, \text{d}^2\theta\, \text{d}^2\overline{\theta}\ \text{Tr}(\Sigma\, V_{\text{FI}}) + \int \text{d}^3x\, \text{d}^2\theta\ \text{Tr}(\Phi\, \Phi_{\text{FI}} + \text{c.c.}), \tag{2.1.11}$$

where V_{FI} and Φ_{FI} are the $\mathcal{N} = 2$ components of the background vector multiplet associated to the topological symmetry and Σ is the *linear multiplet*, whose lowest component is the real scalar in the vector multiplet. Observe that the trace Tr selects the $U(1)$ factor of the gauge group.

2.2 Monopole operators

In the previous section we introduced the basic ingredients to construct a $\mathcal{N} = 4$, also $\mathcal{N} = 3$ in the presence of non-trivial CS terms, gauge theories, namely vector and hypermultiples. However, this is not the end the story. In fact, it is common in quantum field theory that local operators do not have to be described as polynomials in the fundamental fields [56], but they may also include *disorder* or *defect* operators, which are instead introduced to the game by performing the path integral with suitable singular boundary conditions. 't Hooft Monopole operators fall in this class of operators. For the upcoming discussion we will closely follow the very nice review [57]. Let us take for the moment a $U(1)$ gauge group. To introduce a monopole operator at a point x in space one has to integrate over the gauge fields developing a *Dirac monopole singularity* [58]:

$$A^{\pm} \sim \frac{m}{2}(\pm 1 - \cos\theta)\text{d}\phi, \tag{2.2.1}$$

where the expression is given in terms of spherical coordinates centred at the point x where the monopole is inserted, and m is the *magnetic charge*. The plus and minus signs distinguish between the gauge field behaviour in the

[2]The periodicity condition comes from the quantisation condition on the field strength $\int F \in 2\pi\mathbb{Z}$.

North and South hemispheres of the S^2_x surrounding the insertion point. The result is that there is a non-trivial magnetic flux through S^2_x. In the context of supersymmetric gauge theories the singularity (2.2.1) for the gauge field does not preserve any supersymmetry. In order to have a BPS operator, one has to assign similar singular behaviour to the matter fields. For instance, in the case of $\mathcal{N} = 2$ supersymmetry, the real scalar in the vector multiplet σ needs to behave, as $r \to 0$:

$$\sigma \sim \frac{m}{2r}, \tag{2.2.2}$$

and, together with (2.2.1), they preserve the same amount of supersymmetry of an $\mathcal{N} = 2$ chiral multiplet [59]. Given a non-abelian gauge group G with *rank* r, *i.e.* the dimension of its maximal torus, the Dirac singularity is given via the embedding $U(1) \hookrightarrow G$, which allows to define the magnetic charge $m = (m_1, \ldots, m_r)$, $m_i \in \mathbb{Z}$, as an element of the Cartan subalgebra modulo the action of the Weyl group. Concretely, for the case of $G = SU(N)$, modding out the action of the Weyl group $\mathcal{W}_{U(N)} = S^N$, implies that one takes the magnetic charges $m_1 \geq m_2 \geq \cdots \geq m_r > -\infty$, thus $m \in \mathbb{Z}^N / S^N$. The magnetic charge has to satisfy the *Dirac quantisation condition* [60, 61]

$$e^{2\pi i m} = \mathbb{1}_G, \tag{2.2.3}$$

implying that the magnetic charge belongs to the weight lattice of the GNO dual group modulo the action of the Weyl group $\Gamma_{G^\vee} / \mathcal{W}_G$.

A monopole operator with a magnetic charge m breaks the gauge group G down to a *residual* gauge group H_m, defined as the commutant of m in G. Physically, the breaking of the gauge group is due to an *adjoint Higgs mechanism*.

The presentation we gave to introduce monopole operators as disorder operators is based on a modern perspective, however this is not the interpretation that was given when three-dimensional supersymmetric gauge theories was a developing area [62]. Initially, monopole operators were described as follows. Let us consider a $U(1)$ gauge theory. We have already mentioned that it is possible to define a periodic scalar, call it a, the dual photon, and one can combine it with the real scalar in the vector multiplet ϕ into a holomorphic quantity $\Phi = \phi/g^2 + ia$. One then consider the operator

$$V_m \sim e^{m\Phi}, \tag{2.2.4}$$

to be the monopole operator. Observe that such a relation is only valid in a semilclassical picture, namely for large values of the scalar ϕ. Moreover, such description of a monopole operator does not take into account quantum corrections. Another remark is that the process of dualisation defining the periodic scalar a is only known for free abelian vector multiplets.

In the previous section we introduced the concept of topological symmetry for three-dimensional gauge theories. Given a theory based on the gauge group G its topological symmetry is given by the centre $Z(G)$. Monopole operators are the objects that are charged under this symmetry. Take for definiteness $G = U(N)$: its centre, hence its topological symmetry group, is

$U(1)$, and the topological charge of a monopole operator V_m with magnetic charge $m = (m_1, \ldots, m_r)$ is

$$J(V_m) = \sum_{i=1}^{r} m_i. \tag{2.2.5}$$

Monopole operators are not only charged under the topological symmetry, but also under other symmetries, that may either be gauge or global. The charges under discussion receives two type of contribution, one at the *classical* level and one at the *quantum* level. The classical pieces come from mixed Chern-Simons couplings, when these are present. The underlying principle is that in the presence of Chern-Simons terms, a non-trivial magnetic charge induces an electric charge (in this case we refer to the electric charge to indicate either a gauge or a global charge). Denoting with full generality M_A a magnetic flux, that can be for either for the gauge symmetry or other global non-R symmetries, the charge of a monopole operator reads

$$Q_A^{\text{classical}} = -\sum_B k_{AB} M_B, \tag{2.2.6}$$

where k_{AB} is the Chern-Simons level that enters the coupling between the symmetries labelled by A and B. At the quantum level, this expression gets a correction due to integration of fermionic matter, which again, generates new Chern-Simons couplings. The quantum part of the charge is encoded in the following expression

$$Q_A^{\text{quantum}} = -\frac{1}{2} \sum_{\psi_a} Q_A[\psi_a] \Big| \underbrace{\sum_A Q_A[\psi_A] M_A}_{\text{effective mass } m_a^{\text{eff}}(M)} \Big|, \tag{2.2.7}$$

the sum including *all* fermionic fields, both in chiral and vector multiplets. Combining the classical and the quantum contributions it is possible to get the formula for the charge of a monopole operator, which may be written in a compact way

$$Q_A(M) = -\sum_B k_{AB}^{\text{eff}}(M)\, M_B, \tag{2.2.8}$$

with the effective Chern-Simons level is defined as follows

$$k_{AB}^{\text{eff}}(M) = k_{AB} + \frac{1}{2} \sum_{\psi_a} Q_A[\psi_a] Q_B[\psi_b] \, \text{sign}(m_a^{\text{eff}}(M)). \tag{2.2.9}$$

The effective Chern-Simons level cannot take any possible value being constrained by gauge invariance to be integer. This fact implies that the matter content which determines the quantum correction to the effective level puts non-trivial restrictions on the possible *bare* levels k_{AB}.

2.3 Chiral ring

Let us discuss one concept that will be useful and complementary in the upcoming discussion on the moduli space of supersymmetric vacua. Recall that supersymmetric quantum field theories admit a special class of operators which are protected against quantum corrections called *chiral operators*, that we denote as $\mathcal{O}_i$, which are nothing but a set of gauge invariant operators annihilated by half of the supercharges $\overline{\mathcal{Q}}$:

$$\overline{\mathcal{Q}}\,\mathcal{O}_i = 0. \tag{2.3.1}$$

This class of operators enjoys a rather special property

$$\langle \mathcal{O}_1(x_1)\dots O_n(x_n)\rangle = \prod_{i=1}^{n}\langle \mathcal{O}_i\rangle, \tag{2.3.2}$$

namely their correlation functions factorises into products of one-point functions. Such a property immediately descends from the definition of chiral ring operators, namely that their spacetime derivatives vanish in cohomology (in order for this property to hold supersymmetry must be unbroken).

Chiral operators combine to give the structure of a *commutative ring*, the *chiral ring* $\mathcal{R}$ [63]:

$$\mathcal{O}_i\,\mathcal{O}_j = c_{ij}^{k}\mathcal{O}_k + \text{exact terms}, \tag{2.3.3}$$

where the exact terms have to be intended in a cohomological sense with respect to the supercharges $\overline{\mathcal{Q}}$. Expectation values $\langle \mathcal{O}_i\rangle$ of gauge invariant combinations of chiral operators will play a key role in the next section when the moduli space of supersymmetric vacua will be introduced and discussed.

2.4 Moduli space of supersymmetric vacua

One of the most prominent aspects of supersymmetric gauge theories in various dimensions is the possibility to have vacuum solutions to the equations of motion that allows the vacuum expectation values of various scalar fields to form non-trivial spaces, called *moduli space of vacua* [64], of both physical and mathematical interest, especially in the context of algebraic and differential geometry. The relevance from the physical point of view of the concept of moduli space of supersymmetric vacua comes from the fact that it describes the infrared dynamics of a given gauge theory.

For a general supersymmetric theory, a space of vacuum field configurations appears whenever it is possible to find non-trivial configuration of scalar fields that make the scalar potential V to be zero. The expression for V is as follows

$$V = \sum_i |F_i|^2 + \frac{g^2}{2}\sum_a (D^a)^2, \tag{2.4.1}$$

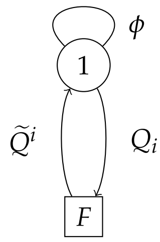
where F_i are the so-called *F-terms* and are expressed in terms of the derivatives of the superpotential with respect to the various chiral fields in the theory

$$F_i = \frac{\partial W}{\partial \phi_i}, \tag{2.4.2}$$

and D_a are the *D-terms*, coming from the integration of the auxiliary D fields in the vector multiplets, and their form depends on the space-time dimension and on the amount of supersymmetry. Since the scalar potential is the sum of squares of F and D-terms, the moduli space of vacua is obtained by putting these two set of expressions to zero giving rise, respectively, to F and D-term equations. Before going on, let us connect this general brief introduction on the moduli space of vacua to the chiral ring. The relation comes from the expectation values of the chiral ring operators $\langle \mathcal{O}_i \rangle$ being holomorphic function on the moduli space of vacua. But there is a bit more: in fact, when the relations coming from the equations of motion for the F and D fields are taken into account, the relation between the chiral ring and the moduli space of vacua is one to one.

Our concern for this discussion is on the structure of the moduli space of vacua for three-dimensional gauge theories with $\mathcal{N} = 3$ and $\mathcal{N} = 4$ supersymmetry, so we will specialise what we have said to such a set-up.

It turns out that the moduli space is made up of two components, or *branches*, called *Higgs* and *Coulomb* branch, depending on which scalar is taking vacuum expectation value (vev). Instead of giving a completely general and abstract presentation we will consider first a simple example, namely the $U(1)$ gauge theory with F hypermultiplets $(Q_i, \widetilde{Q}^i)$ of charge 1, whose $\mathcal{N} = 2$ quiver description is as follows



$$W = \textstyle\sum_i Q_i \phi \widetilde{Q}^i \tag{2.4.3}$$

where we specified the $\mathcal{N} = 2$ superpotential of the theory. The fields in the theory have the following charges

Field	R	$U(1)_{\text{gauge}}$	$SU(F)$
ϕ	1	0	$[0, \ldots, 0]$
Q_i	1/2	+1	$[0, \ldots, 0, 1]$
$\widetilde{Q}^i$	1/2	−1	$[1, 0, \ldots, 0]$

(2.4.4)

where to specify the $SU(F)$ global symmetry we use the Dynkin label associated to the representation. In order to analyse the moduli space of this theory we have to write down the set of F and D-term equations. For the first we take the derivatives of the superpotential with respect to the various chiral fields Q_i, ϕ, $\widetilde{Q}^i$, giving the equations

$$Q_i\phi = 0, \qquad \phi\widetilde{Q}^i = 0, \qquad Q_i\widetilde{Q}^i = 0. \tag{2.4.5}$$

The set of D-term equations instead read

$$\sigma Q_i = 0, \qquad \sigma\widetilde{Q}^i = 0, \qquad Q_i Q^{\dagger j} - \widetilde{Q}^i\widetilde{Q}^\dagger_j = 0, \tag{2.4.6}$$

where σ is the real scalar in the vector multiplet. Now we will try to solve these two sets of equations and, as mentioned before, Higgs and Coulomb branch will arise as possible, and only, solutions.

- **Higgs branch**:

  Consider first the situation in which the scalars that belong to the vector multiplet, ϕ, σ, are set to zero on the vacuum. It is immediate from both F and D-term equations that non-trivial vevs for Q_i, $\widetilde{Q}^i$ are allowed. Nonetheless, the aformentioned VEVs are constrained to satisfy the F and D term equations that are not trivially realised putting $\phi = \sigma = 0$ (we use interchangeably the same letter to denote a scalar field and its vev where it does not cause any confusion), namely

  $$Q_i\widetilde{Q}^i = 0, \qquad Q_i Q^{\dagger j} - \widetilde{Q}^i\widetilde{Q}^\dagger_j = 0. \tag{2.4.7}$$

  Since we want to find gauge invariant combinations of the scalar fields with zero vacuum energy, it turns out that a convenient way to recast in a more clear form such constraints, and hence to parametrise the Higgs branch, is to define a *meson matrix* as follows

  $$M_i^j = Q_i\,\widetilde{Q}^j, \tag{2.4.8}$$

  a gauge invariant operator, since it is made out of the product of two chiral fields with opposite charge with respect to the $U(1)$ gauge group. Thus, the idea is to describe the Higgs branch via a $N \times N$ complex matrix. Nonetheless, such meson operator has to satisfy some conditions, as we are going to describe. First, since M is constructed taking the product of two vectors Q_i and $\widetilde{Q}^j$, it has at most rank one

  $$\text{rank}(M) \leq 1. \tag{2.4.9}$$

  Furthermore, (2.4.7) implies

  $$\text{Tr}(M) = 0, \qquad M^2 = 0. \tag{2.4.10}$$

  Finally, we have found that the Higgs branch of (2.4.3) is parametrised by the set of matrices

  $$\mathcal{M}_{\mathcal{H}} = \{M \in GL(N,\mathbb{C}) \,|\, \text{rank}(M) \leq 1,\, \text{Tr}(M) = 0,\, M^2 = 0\}, \tag{2.4.11}$$

such space of matrices is known in the mathematical literature as the *minimal nilpotent orbit* of $SU(F)$. Another realisation of such a space is obtained exploiting the ADHM construction for the moduli space of instantons. In particular, (2.4.11) describes the moduli space of 1 $SU(F)$ instanton on $\mathbb{C}^2$. Observe that the VEVs for the hypermultiplets have the effect of completely breaking the gauge symmetry on the vacuum. In general, the Higgsing of the gauge group depends on the matter content of the theory. There can be situations in which the hypermultiplets are “not enough” to completely Higgs the gauge group, thus one still has a residual gauge symmetry on the vacuum, in particular one has free vector multiplets [8]. For instance, if we take a $U(N)$ gauge theory with F fundamental hypermultiplets, the group is completely higgsed if $F \geq 2N$. Examples in which there is no complete higgsing have been discussed in [65], where in the case of four dimensional $\mathcal{N} = 2$ theories the $SU(2)$ gauge symmetry is Higgsed down to $U(1)$. The branch of the moduli space of vacua has beed dubbed the *Kibble branch*.

Due to $\mathcal{N} = 4$ supersymmetry, the Higgs branch has the structure of an hyperkäler singular space. The presence of singularities on the Higgs branch, has physical meaning of crucial relevance: the singularity represents the point where all the VEVs are set to zero and coincides with the superconformal point. Physically, the action of the $SU(2)_H$ R-symmetry factor arises in the notion of hyperkäler space as the $SU(2)$ symmetry group acting on the variety itself by rotating the three complex structures. A simple, nonetheless non-trivial example, of an hyperkäler singular space is given by the orbifold $\mathbb{C}^2/\mathbb{Z}^2$. Denote the complex coordinates on $\mathbb{C}^2$ as (z_1, z_2), and consider all the polynomials in such variables, representing the holomorphic ring of $\mathbb{C}^2$. However, we need to take into account the action of the $\mathbb{Z}_2$ orbifold. On the $\mathbb{C}^2$ coordinates there is an action of the non-trivial $\mathbb{Z}_2$ element

$$(z_1, z_2) \xrightarrow{-1\in\mathbb{Z}_2} (-z_1, -z_2). \tag{2.4.12}$$

The holomorphic functions on $\mathbb{C}^2/\mathbb{Z}^2$ are the ones of $\mathbb{C}^2$ which are invariant under the $\mathbb{Z}_2$ action. It is an easy task to recognise that such polynomials are the ones constructed with *even* products of the coordinates (z_1, z_2). It turns out that one may take the combinations

$$X = z_1^2, \qquad Y = z_2^2, \qquad Z = z_1 z_2, \tag{2.4.13}$$

and generate all the $\mathbb{Z}_2$ invariant polynomials. Observe that the three generators X, Y, Z are not independent quantities but satisfy the relation

$$X Y = Z^2. \tag{2.4.14}$$

In the end of this example we may summarise in a rather elegant way that the singular hyperkäler space $\mathbb{C}^2/\mathbb{Z}^2$ is described in terms of three generators X, Y, Z such that $XY = Z^2$. An immediate generalisation of this result is for the orbifold space $\mathbb{C}^2/\mathbb{Z}^n$, whose description involves the same set of generators as before but with a different relation: $XY = Z^n$. We will see how this space appears in the discussion of the Coulomb branch for (2.4.3). The construction we have just described may be given a clear physical interpretation: the generators X, Y, Z represents the gauge invariant combinations of VEVs of scalar fields parametrising the moduli space of vacua, and the relation is reminiscent of the ones coming from F and D term equations, similar to (2.4.10). Another property of the Higgs branch is its *exactness* at the classical level [66]. This property makes the Higgs branch a robust quantity to study and to characterise the infrared dynamics of a supersymmetric gauge theory.

A quantity characterising the Higgs branch of the moduli space is its dimension. Because of the hyperkäler structure the appropriate units to count the dimension are the *quaternionic* ones. To count the dimension of the Higgs branch $\mathcal{M}_{\mathcal{H}}$ one makes use of the Higgs mechanism, namely take into account all the scalars in the hypermultiplet that remains massless. The components that become massive because of the Higgsing procedure have to be subtracted from the total number of scalars. In detail, for a theory with gauge group G with F hypermultiplets in the representation R of the gauge group, one finds

$$\dim_{\mathbb{H}}(\mathcal{M}_{\mathcal{H}}) = \dim(R) \times F - \dim(G). \tag{2.4.15}$$

In the specific case of (2.4.3) one has

$$\dim_{\mathbb{H}}(\mathcal{M}_{\mathcal{H}_{(2.4.3)}}) = F - 1. \tag{2.4.16}$$

The singularity at the origin of the Higgs branch may be *resolved* by turning on non-vanishing mass terms for the hypermultiplets.

- **Coulomb branch**: Opposite to what we have seen for the Higgs branch, let us consider the case in which σ and ϕ have non trivial VEVs while Q_i and $\widetilde{Q}^j$ vanish on the vacuum. This vacuum field configuration automatically satisfies the constraints put by F (2.4.5) and D-terms, (2.4.6). In the case of the Coulomb branch it not immediate to find the appropriate gauge invariant operators conveniently parametrising the moduli space. The name Coulomb branch is reminiscent of the fact that on the vacuum the gauge symmetry is not completely broken but there are still $U(1)$ vector multiplets. To better understand this point let us consider the term in the potential that contains the scalars in the vector multiplets. Schematically, it takes the form

$$V \sim \sum_{i<j} [\Phi^i, \Phi^j]^2, \tag{2.4.17}$$

with $i, j = 1,2,3$, and where we denoted $\Phi^i = (\sigma, \mathrm{Re}\,\phi, \mathrm{Im}\,\phi)$. To achieve the vanishing of the potential V it suffices to have a set of *mutually communting* scalars, meaning that they are actually valued in the maximal torus $U(1)^{\dim(G)}$ of the gauge group. Adjoint Higgs mechanism is the physical process that takes place when going on the Coulomb branch. Thus, starting from a theory with G gauge group, a generic point of its Coulomb branch will have rank(G) massless vector multiplets. Now recall that monopole operators for abelian gauge theories may be construct dualising the gauge field into a periodic scalar and holomorphically combining it with the real scalar in the vector multiplet. On a generic point of the Coulomb branch we are exactly in the situation in which such operators may be constructed. This is just a hint on the fact that monopoles may be the main characters of the Coulomb branch description. It turns out to be exactly the case. We have already introduced this class of operators in section 2.2 and discussed their charges under gauge and global symmetry. In general, given a monopole operator V_m with magnetic charge m, it can be written the m-th power of a *basic* monopole with unit magnetic charge $V_m = (V_1)^m$. This property is related to the fact that a given magnetic lattice hosts only one operator on each site, thus the lattice site corresponding to V_m cannot contain two independent operators, V_m and $(V_1)^m$. For this reason we will consider only basic monopole operators V_+, V_-, the ones with minimal magnetic charge. For the present discussion it is useful to specialise the various charges to the case of theory (2.4.3) in the following table

Field	R	$U(1)_T$	$SU(F)$
ϕ	1	0	$[0,\dots,0]$
V_m	$\frac{F}{2}\lvert m\rvert$	m	$[0,\dots,0]$

(2.4.18)

When we discussed the Higgs branch in the previous paragraph we dealt with a set of *classical* relations satisfied by the meson matrix coming from F and D terms. For the Coulomb branch the situation is dramatically different: there is no relation appearing at the classical level and one has to look at quantum effects. It is conceivable that the generators of the Coulomb branch will be the scalar ϕ in the vector multiplet and the basic monopole operators V_+, V_-. Looking at the table of their charges (2.4.18) it is possible to guess a relation among ϕ, V_+, V_- of the form

$$V_+ V_- = \phi^F. \tag{2.4.19}$$

As a singular space this is nothing but $\mathbb{C}^2/\mathbb{Z}_F$, as anticipated above. Since it does not follow from any classical computation, such as the F-term equations, this relation is dubbed as a *quantum* relation. As it has become clear from the discussion, the Coulomb branch $\mathcal{M}_C$ is not protected against quantum corrections as the Higgs branch. A property the Coulomb branch shares with the Higgs branch is the presence of an hyperkäler structure acting on it, this time the $SU(2)$ isometry being identified with the $SU(2)_C$ factor of the R-symmetry group in the quantum field theory realisation. Again, quaternionic units are appropriate to count the dimension of the Coulomb branch, simply given by the number of abelian vector mutiplets

$$\dim_{\mathbb{H}}(\mathcal{M}_{\mathcal{C}}) = \dim(G). \tag{2.4.20}$$

Similarly to the Higgs branch, the singularity at the origin can be resolved by the presence (when they are admitted) of FI terms.

Before ending the section let us mention that the Lagrangian of a three dimensional $\mathcal{N} = 4$ theory does not contain mixing terms among the scalars in the hypermultiplet and the ones in the vector multiplet, thus the whole moduli space is the product $\mathcal{M}_{\mathcal{H}} \times \mathcal{M}_{\mathcal{C}}$. There may be situations in which also *mixed* branches appear, in which, as suggested by the name, both monopoles and mesons may acquire non trivial VEVs. However, because of the previous argument, one always end up with product of spaces separately parametrised by monopoles and mesons. An example of a discussion of mixed branches for three-dimensional $\mathcal{N} = 4$ theories can be found in [67].

2.4.1 Mirror symmetry

The previous analysis of the moduli space of vacua of three-dimensional $\mathcal{N} = 4$ theories shed light on an interesting fact: one branch is "easy"[3] to study, the Higgs branch, while the other is entirely dictated by quantum corrections, the Coulomb branch. Thus it seems that one may only access the Higgs branch, since in more complicated situation than the one depicted in (2.4.3) it may be harder to find the quantum relations among monopoles and adjoint scalars. This is the point in which mirror symmetry comes at rescue. Originally discovered by Intriligator and Seiberg [11], mirror symmetry is an infrared duality which relates two or more theories that flow to a superconformal fixed point. In more detail, given a gauge theory with an Higgs $\mathcal{M}_{\mathcal{H}}$ and Coulomb branch $\mathcal{M}_{\mathcal{C}}$, the duality conjectures the existence of a *mirror dual* theory whose Higgs and Coulomb branch are exchanged:

$$\mathcal{M}_{\mathcal{H}}^{\text{mirror dual}} = \mathcal{M}_{\mathcal{C}}, \qquad \mathcal{M}_{\mathcal{C}}^{\text{mirror dual}} = \mathcal{M}_{\mathcal{H}}. \tag{2.4.21}$$

As a consequence, mirror symmetry also exchanges the role of the two $SU(2)$ factors of the R-symmetry group and of mass terms, transforming in the adjoint representation of $SU(2)_H$ and trivially under $SU(2)_C$, and FI terms, which instead are in the adjoint of $SU(2)_C$ and in the singlet of $SU(2)_H$. This means that what appears as a classical effect on one duality frame manifest itself as a quantum effect on the other frame, and viceversa. In [11] the authors conjectures several mirror dual pairs starting from gauge theories constructed by Kronheimer [69], whose quiver description corresponds

[3]Note that we refer to the Higgs branch as being easy to analyse only because it requires a *classical* analysis and does not require any quantum corrections. Despite that, it has become clear in recent years, that the Higgs branch of various supersymmetric theories in diverse dimensions display interesting *infinite* coupling phenomena. An example is the so-called *small E_8 instanton transition*, originally discovered in [68].

to the (affine) Dynkin diagram of ADE Lie algebras. However, in the framework of pure quantum field theory it is hard to construct new mirror dual theories. This may be achieved via a Type IIB string theory realisation of three-dimensional gauge theories which automatically incorporates mirror symmetry as an incarnation of S-duality.

2.5 Hanany-Witten brane systems

A large class of three-dimensional $\mathcal{N} = 4$ gauge theories can be realised in Type IIB string theory with systems of D3, D5 and NS5 branes [10] spanning the following space-time directions

	0	1	2	3	4	5	6	7	8	9
D3	×	×	×				×			
NS5	×	×	×	×	×	×				
D5	×	×	×					×	×	×

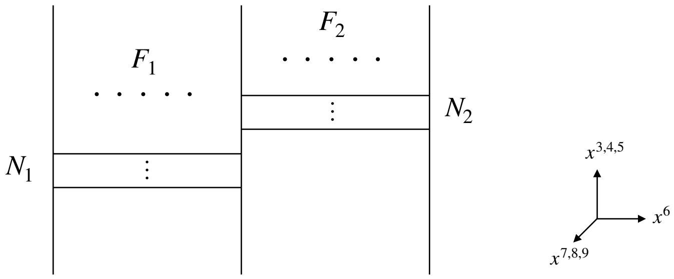
(2.5.1)

where x^6 direction can be taken to be compact or non-compact. The Lorentz group $SO(1,9)$ is broken by the brane configuration to $SO(1,2)$ rotating $x^{0,1,2}$, $SO(3)_C \simeq SU(2)_C$ acting on $x^{3,4,5}$ and $SO(3)_H \simeq SU(2)_H$ acting on $x^{7,8,9}$. To simplify the notation let us define two vectors $\boldsymbol{m} = (x^3, x^4, x^5)$ and $\boldsymbol{w} = (x^7, x^8, x^9)$. The two $SU(2)$ factors introduced as the isometries along the directions $\boldsymbol{m}$ and $\boldsymbol{w}$ combine to give the R-symmetry group of the quantum field theory. In this set-up the three-dimensional gauge theory lives on the worldvolume of D3 branes. Despite being four dimensional extended objects, the D3 branes in the set-up we are discussing, called of Hanany-Witten, extends in a *finite size* direction, x^6. As a consequence the modes in x^6 have to be though as the Kaluza-Klein mode in a circle dimensional reduction, meaning that the effective quantum field theory is three-dimensional.

To identify the quiver theory starting from the brane system we need to know what is the matter content arising from the spectrum of open strings stretching across the various D3 and D5 branes. A single D3 brane suspended between two NS5 brane hosts a $U(1)$ vector multiplet. If one has N D3s, the gauge symmetry is $U(1)^N$ and becomes to $U(N)$ when the D3 branes are coincident. This "classical enhancement" becomes clear when looking at the Chan-Paton factors of the spectrum of open strings stretching among the D3 branes. In a similar fashion, a string that stretches between N coincident D3 and a D5 brane gives rise to an hypermultiplet in the fundamental representation of $U(N)$. The last piece of information we need arises when there are multiple stacks of D3 branes separated by various NS5 branes: in this case string having their ends on two sets of D3 branes and crossing the NS5 separating them gives rise to hypermultiplets in the bi-fundamental representation. For definiteness, if one has N_1 and N_2 D3 branes to the left and to the right of an NS5 the hypermultiplet will transform in the (N_1, N_2), where for simplicity we denoted the representation via its dimension. The gauge coupling of the various vector multiplets can be read from the brane realisation and is given in terms of the distance in the x^6 direction of the NS5 branes:

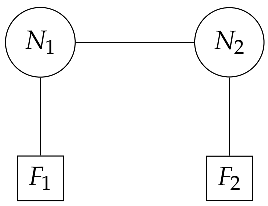
$$\frac{1}{g^2} = |t_1 - t_2|, \tag{2.5.2}$$

where t_1 and t_2 denotes the position in x^6 of the NS5 branes. This suggests that as the NS5 branes gets closer and closer, the gauge theory approaches a strong coupling phase, that become *infinite coupling* in the case of coincident branes. From the general brane set-up it is possible to see that D3 branes have the freedom to move along the $x^{3,4,5}$ directions, spanned by the NS5 branes, and along $x^{7,8,9}$, spanned by D5 branes. These parameters, or *moduli*, encodes the structure of the moduli space of the gauge theory as we will see shortly. Let us try to summarise the rules for constructing a quiver theory from the brane system in a particularly simple set-up



(2.5.3)

that gives rise, with the previous rules, to the following quiver theory



(2.5.4)

Let us now consider how the Coulomb branch of a given gauge theory arises from the brane system perspective. As we said before, a stack of N coincident D3 branes suspended between two NS5 branes give rise to a $U(N)$ vector multiplet. Suppose now we take one of the D3 branes and pull it out from the $N-1$ remaining along the directions $\boldsymbol{m}$. What does it happen to the massless vector multiplet? Denoting as $\boldsymbol{m}_1$ the position of the $N-1$ coincident D3 branes and with $\boldsymbol{m}_2$ the position of the D3 that has been "separated" from the rest, the open string connecting the two sets of branes gais a non-vanishing tension proportional to the displacement $\boldsymbol{m}_1 - \boldsymbol{m}_2$. In terms of the low energy degrees of freedom one gets a massive W-boson. As a consequence, the gauge group that originally was $U(N)$ is broken down to $U(N-1) \times U(1)$. In the most general case where the D3 branes are all separated the gauge group is maximally broken down to the Cartan torus $U(1)^N$. Recall that we have already discussed such a situation when the Coulomb branch of the moduli space was introduces, and it is not a coincidence. What we have seen is in fact the description of the Coulomb branch in terms of the brane set-up realising the gauge theory: the origin of the Coulomb branch corresponds to the case in which the D3 branes are all coincident, while separating them apart in the $\boldsymbol{m}$ direction amounts to go on a generic point of the aforementioned branch where all the scalars in the vector multiplets have non-trivial VEVs. The discussion may be repeated in the same fashion replacing the role of NS5 branes with D5 branes and the directions $\boldsymbol{m}$ with $\boldsymbol{w}$: what we find in this case is the description of the Higgs branch, where massless hypermultiplets in the case of coincident D5 branes gain VEV and allow to explore such branch of the moduli space.

At the end of section 2.4.1 we introduced mirror symmetry and explained that its action on mirror dual theories is to swap their Higgs and Coulomb branches. The Hanany-Witten realisation of three-dimensional $\mathcal{N} = 4$ gauge theories comes equipped with the tools to engineer mirror symmetry directly at the level of the brane set-up. The fundamental ingredient one has to recall is that S-duality acts on Type IIB string theory via the S element of $SL(2, \mathbb{Z})$:

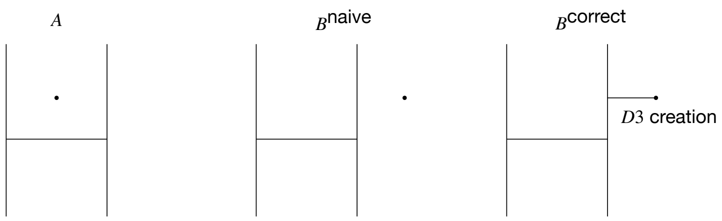
$$S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \tag{2.5.5}$$

and acts non-trivially on the five-branes by exchanging them, while the action on the D3 brane is trivial. However, the S-transformation alone does not correspond to mirror symmetry since one also has to take case of the spatial directions the branes span. Thus, one supplements the action of S-duality with the following rotation R of the spatial coordinates

$$\boldsymbol{m} \to \boldsymbol{w}, \qquad \boldsymbol{w} \to -\boldsymbol{m}, \tag{2.5.6}$$

and trivially acts on the remaining coordinates. The lesson is now the following: starting with a given brane set-up describing a gauge theory, it is possible to construct its mirror dual theory by performing a combined action of S-duality and the rotation R on the brane system. Even though it may sound a simple operation to find the mirror dual configuration, there are some rules one to be respected and non-trivial effect to take into account. The most prominent example being what nowadays goes under the name of *Hanany-Witten transition*. Let us try to explain such effect with an example. Suppose we start with a system of a D3 brane suspended between two NS5 branes and a D5 brane. Denoting the x^6 position of the two NS5 branes respectively with $x^6_{NS5_1}$, $x^6_{NS5_2}$, and x^6_{D5} for the D5 such that $x^6_{NS5_1} < x^6_{D5} < x^6_{NS5_2}$, we ask ourselves what happens if we move the D5 brane along x^6 such that in the end $x^6_{D5} > x^6_{NS5_2}$. Naively one could say that nothing happens and the brane move does not have any "cost". In picture (2.5.7) we depicted the original configuration and called it A, and the naive one after the brane move with

B^{naive}.



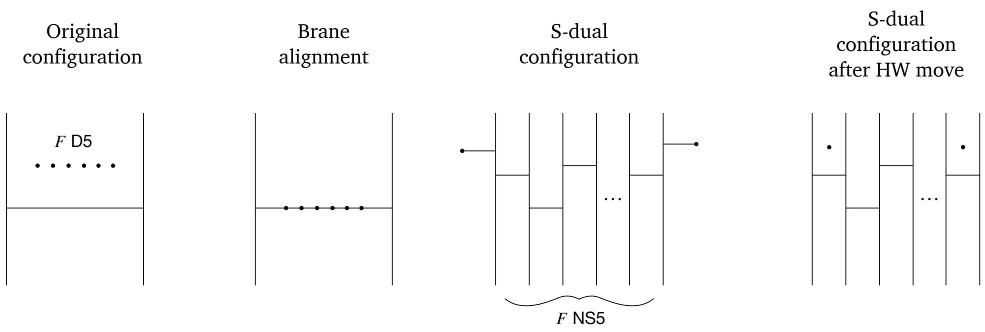
(2.5.7)

However one has to take into account the following fact. In the situation A, when the D5 brane is moved on top of the D3, the mass of the corresponding hypermultiplet proportional to $|\boldsymbol{w}_{D5} - \boldsymbol{w}_{D3}|$ goes to zero, providing a singularity corresponding to the appearance of a massless state. In the naive set-up B^{naive} there is no reason why a singularity should appear. Despite for the class of three-dimensional $\mathcal{N} = 4$ theories the spectrum of BPS particles is allowed to jump, in this specific example there is no reason why the massless hypermultiplet should decay. The problem is resolved conjecturing that the right configuration is B^{correct}, where a D3 brane has been *created* and it connects the D5 and an NS5 brane. Immediately observe that such a D3 brane has not brought any new moduli to the theory. In fact, being the D3 stretched between an N5S and a D5 it does not possess any moduli: both its $\boldsymbol{w}_{D3}$ and $\boldsymbol{m}_{D3}$ are fixed. The paradox of the missing singularity is now resolved noting that a massless hypermultiplets appear in B^{correct} once the original D3 and the new one are aligned along the $\boldsymbol{m}$ directions. It is worth to mention that the transition also holds in the *reverse* way: if we start with B^{correct} configuration and move the D5 brane "inside" the D3 brane with no moduli gets annihilated.

The a priori reason why the Hanany-Witten brane creation/annihilation should take place is related to the conservation of magnetic charge. One can assign a magnetic charge to each type of five brane, either the NS5 of the D5 brane. For definiteness, let us consider an NS5 brane and denote with L_{D5} the number of D5 to its left and with R_{D5} the D5 on the right. In a similar fashion, L_{D3} and R_{D3} denotes the number of D3 branes ending on the NS5 from respectively from the left and from the right. The linking number of the NS5 under consideration reads

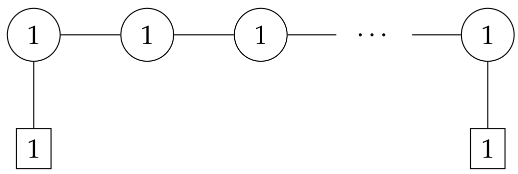
$$L_{\text{NS5}} = \frac{1}{2}(R_{\text{D3}} - L_{\text{D3}}) + (L_{\text{D5}} - R_{\text{D5}}), \tag{2.5.8}$$

and a completely analogue formula holds for the linking number of a given D5. The linking number computes the total magnetic charge of a given five-brane. Such magnetic charge for each five brane is conserved after every brane move, a particular case being represented by the phase transition discussed above. The linking number also constraints a given brane system, which has to satisfy the requirement that the sum of the linking numbers for all the branes involved has to be zero. Other constraints come from unbroken supersymmetry, such as the *S-rule*: given an NS5 brane and a D5 brane, there can be one and only one D3 brane connecting them. In the end have the tools to analyse a given Hanany-Witten brane system, read the associated gauge theory, and finally take the S-dual of the original set-up to read the mirror dual theory. It is pedagogical to end the section with an example. Let us take again the case of a $U(1)$ gauge theory with F flavours. In the following we depict the various steps that have to be performed in order to get the S-dual of the original configuration



(2.5.9)

Using the rules to read the gauge theory from the brane configuration we find that the mirror pair arising from the S-dual configuration reads



(2.5.10)

where the gauge group is $U(1)^{F-1}$. As a final check, we can compute the dimension of Higgs and Coulomb branch of the mirror pair and see that the match under mirror symmetry. The original SQED theory with F flavours has Higgs and Coulomb branch of dimensions:

$$\dim_{\mathbb{H}}\mathcal{H}_{\text{SQED w/ F}} = F-1, \qquad \dim_{\mathbb{H}}\mathcal{C}_{\text{SQED w/ F}} = 1. \tag{2.5.11}$$

For the mirror theory, the dimension of Higgs branch can be counted taking into account the $F-2$ bifundamental hypers plus the 2 coming from the flavour nodes, minus the contribution of the $F-1$ vector mutiplets

$$\dim_{\mathbb{H}}\mathcal{H}_{\text{mirror}} = (F-2)+2-(F-1) = 1, \tag{2.5.12}$$

while for the Coulomb branch it suffices to take the rank of the gauge group:

$$\dim_{\mathbb{H}}\mathcal{C}_{\text{mirror}} = F-1. \tag{2.5.13}$$

Thus, we find, as expected, that the dimension of the Higgs and Coulomb branch across the mirror pairs match

$$\dim_{\mathbb{H}}\mathcal{H}_{\text{SQED w/ F}} = \dim_{\mathbb{H}}\mathcal{C}_{\text{mirror}}, \qquad \dim_{\mathbb{H}}\mathcal{C}_{\text{SQED w/ F}} = \dim_{\mathbb{H}}\mathcal{H}_{\text{mirror}}. \tag{2.5.14}$$

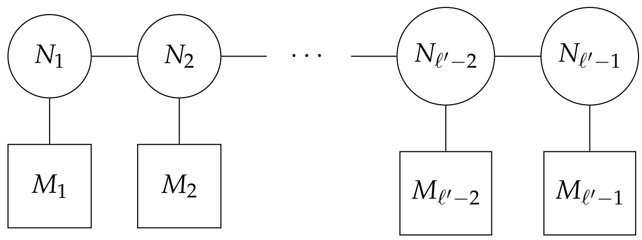
Obviously, matching the dimensions of Higgs and Coulomb branch does not imply that the branches, as singular spaces, they coincide across mirror duality, and usually a more detailed analysis is needed, nonetheless it represents a first good check that the mirror pair has beed correctly identified. We will not go into the detail more in this section as we will see plenty of examples of matching Higgs and Coulomb branch of various mirror dual theories in the main part of the thesis.

Chapter 3

The moduli spaces of S-fold SCFTs

3.1 Linear quivers: $T^{\sigma}_{\rho}(SU(N))$ and its variants

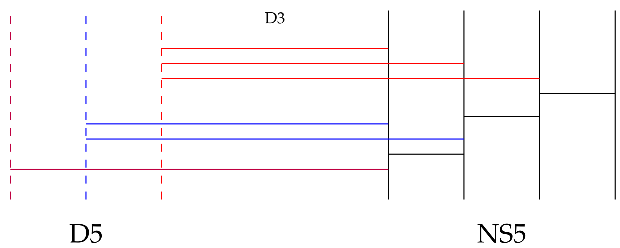
A notable class of linear quivers, realised when the x^6 direction in HW set-up is non-compact, is as follows



(3.1.1)

where a circular node with a label N denotes a $U(N)$ gauge group and a square node with a label M denotes a $U(M)$ flavour symmetry. This class of linear quivers was studied in [8] and each of the theories in this class is represented by $T^{\sigma}_{\rho}(SU(N))$ for some N, with σ and ρ partitions of N.

From the brane perspective, if we move the D5-branes to one side and the NS5-branes to the other side, N is the total number of D3-branes in the middle, σ contains the differences between the number of D3-branes on the left and on the right of each D5-brane, and ρ contains the differences between the number of D3-branes on the left and on the right of each NS5-brane. Let us provide an example for $N = 6$, $\sigma = (3, 2, 1)$ and $\rho = (2^2, 1^2)$:



(3.1.2)

To read off the quiver gauge theory, it is convenient to move the D5-branes inside the NS5-brane intervals as follows:

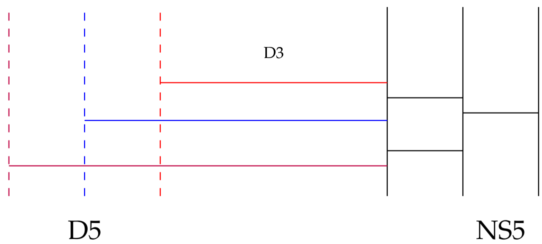
(3.1.3)

Since three dimensional mirror symmetry [70] exchanges D5-brane and NS5-branes [10], it also exchanges σ and ρ. A quiver description of $T^{\sigma}_{\rho}(SU(N))$ for a general σ and ρ can be found in, for example, [71, sec. 2] or [72, sec 2.1].

The $T(SU(N))$ theory. A theory that plays an important role in our analysis is that with $\sigma = \rho = [1^N]$ [1]. Such a theory is denoted by $T(SU(N))$ and its quiver description is

$$\underset{1}{\circ} - \underset{2}{\circ} - \cdots - \underset{N-1}{\circ} - \underset{N}{\square}\,. \tag{3.1.4}$$

As an explicit example, the brane configurations for $T(SU(3))$ are as follows:

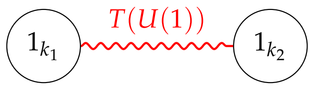


(3.1.5)

In general $T(SU(N))$ is invariant under mirror symmetry. The Higgs and the Coulomb branches of this theory are both isomorphic to the closure of the maximal nilpotent orbit of $SU(N)$ [8], which is denoted by $\mathcal{N}_{SU(N)}$. We can conveniently define $\mathcal{N}_{SU(N)}$ as a set of $N \times N$ complex matrices M such that $\mathrm{tr}(M^p) = 0$, for $p = 1, \dots, N$; the quaternionic dimension of this space is therefore $\frac{1}{2}N(N-1)$. For quiver (3.1.4), the symmetries of the Higgs and Coulomb branch are thus both $SU(N)$; the former is manifest in the Lagrangian (or quiver) description as a flavour symmetry, whereas the latter is not manifest but gets enhanced from the topological symmetry $U(1)^{N-1}$ in the infrared. There is a situation in which it is possible to have manifest Coulomb branch symmetry, namely for the $T(SU(2))$ theory. In fact, there exists an IR duality [41, 74, 75] that relates $T(SU(2))$ to an $SU(2)_1$ gauge theory with four fundamental chiral multiplets, $SU(2) \times SU(2)$ flavour symmetry, which only has $\mathcal{N} = 2$ supersymmetry and no time-reversal symmetry. Thus, it is possible to find a duality frame in which both the Higgs and Coulomb symmetries are manifest, but $\mathcal{N} = 4$ supersymmetry is not.

[1] It has been recently observed that theories specified by non-trivial partitions play a role in understanding the three-dimensional mirror dual of the class A_{2N} of class S theories [73].

The $T(U(N))$ theory. An important variant of the $T(SU(N))$ theory is the $T(U(N))$ theory [8, sec 4.4]. The latter is defined as a product between the $T(SU(N))$ theory and an "almost trivial" $T(U(1))$ theory, where the latter can be characterised as follows. The Coulomb and Higgs branches of $T(U(1))$ are trivial; each of them consists of only one point. Nevertheless, $T(U(1))$ comes with a $U(1) \times U(1)$ background vector multiplet, along with an $\mathcal{N} = 4$ background mixed Chern–Simons term with level 1 between such $U(1)$ vector multiplets. Explicitly, the action for the following quiver



(3.1.6)

in the $\mathcal{N} = 2$ notation is given by (see *e.g.* [76, (4.4)])

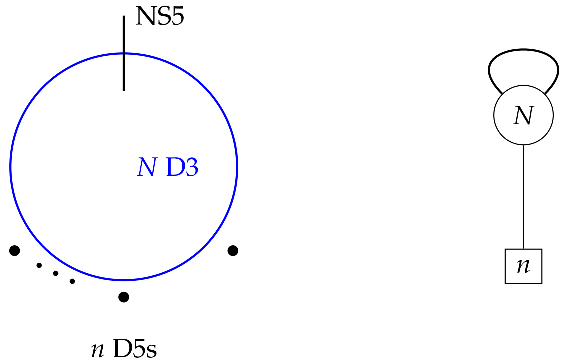
$$\begin{aligned} &\int d^3x d^4\theta \left(\frac{k_1}{4\pi}\Sigma_1 V_1 + \frac{k_2}{4\pi}\Sigma_2 V_2 - \frac{1}{4\pi}\Sigma_1 V_2 - \frac{1}{4\pi}\Sigma_2 V_1 \right) \\ &- \int d^3x d^2\theta \left(\frac{k_1}{4\pi}\Phi_1^2 + \frac{k_2}{4\pi}\Phi_2^2 - \frac{1}{2\pi}\Phi_1\Phi_2 + \text{c.c.} \right) . \end{aligned} \tag{3.1.7}$$

where Σ_i, V_i (with $i = 1, 2$) are, respectively, the $\mathcal{N} = 2$ linear multiplet and vector multiplet of the i-th gauge node, and Φ_i are the $\mathcal{N} = 2$ chiral multiplets of the $\mathcal{N} = 4$ vector multiplets of the i-th gauge group. In the above equation, we highlight the contribution from the mixed Chern–Simons terms due to $T(U(1))$ in blue. We emphasise that the mixed Chern–Simons terms come with the level -1 in our convention for $T(U(1))$. Thus, one may view the $T(U(N))$ theory as having a global symmetry $U(N) \times U(N)$, such that the two $U(1)$ subgroups of each $U(N)$ acts trivially on the theory, and that an $\mathcal{N} = 4$ background mixed Chern–Simons term with level $-N$ is added for the two corresponding $U(1)$ background vector multiplets.

It should be mentioned that there is a close cousin of the $T(U(1))$ theory. This theory is called $\overline{T(U(1))}$ in [77]. This theory can be defined almost in the same way as above, except that the minus signs in the blue terms of (3.1.7) are changed to plus signs. In other words, the level of the mixed Chern–Simons terms is $+1$. One can then define $\overline{T(U(N))}$ theory as a product between $T(SU(N))$ and $\overline{T(U(1))}$. As a consequence, $\overline{T(U(N))}$ has a global symmetry $U(N) \times U(N)$, such that the two $U(1)$ subgroups of each $U(N)$ acts trivially on the theory, and that an $\mathcal{N} = 4$ background mixed Chern–Simons term with level N is added for the two corresponding $U(1)$ background vector multiplets.

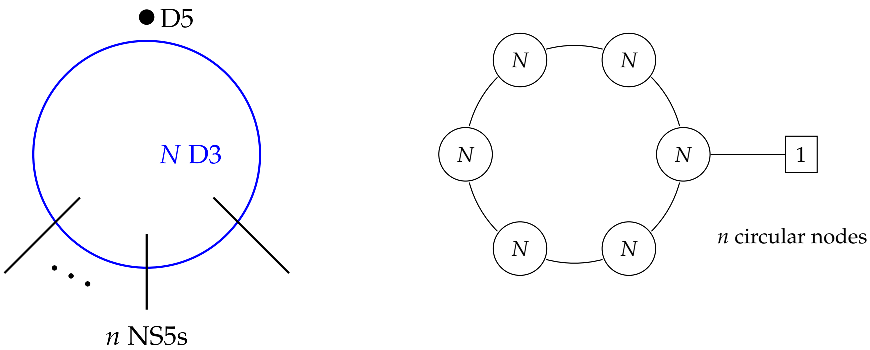
3.2 Compact models

As it has already mentioned before, HW brane set-up allows to realise compact models when the x^6 direction is taken to circular. An example of this is as follows:



(3.2.1)

where the loop around the node denotes a hypermultiplet in the adjoint representation of the $U(N)$ gauge group. The mirror theory can be obtained simply by applying S-duality to the above brane system in the usual way:

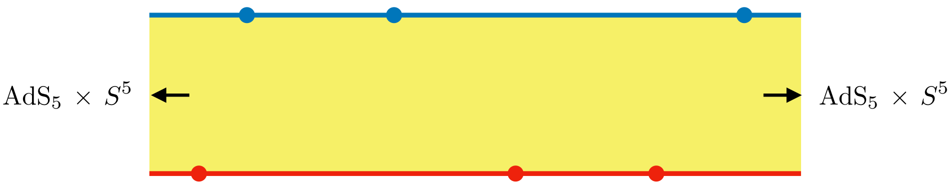


(3.2.2)

3.3 S-fold solutions and their SCFT duals

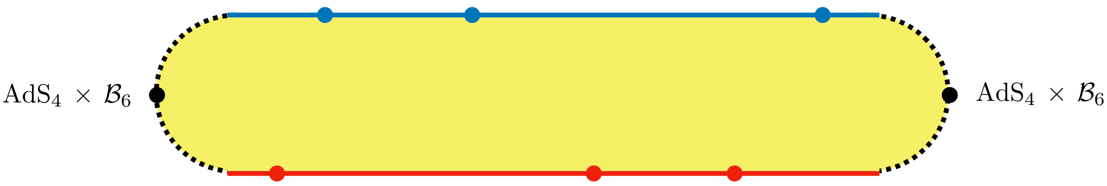
3.3.1 The holographic duals of linear quivers and compact models

Both linear quivers and compact models have known holographic duals in sting theory. Type IIB supergravity solutions have been found in [71, 78]. Historically, these solutions descend from the seminal work [19, 20], where $AdS_4 \times S^2 \times S^2 \times \Sigma_2$ backgrounds have been found, with Σ_2 a non-compact Riemann surface with the topology of infinite strip $\mathbb{R} \times I$ with coordinates (y, x), where I is an interval. The dual field theory is supposed to be four-dimensional SYM with space-dependent coupling constant, since the ten-dimensional metric is actually asymptotically $AdS_5 \times S^5$ in the limit $y \to \infty$. The metric, the dilaton and the fluxes are completely determined in terms of two harmonic functions $\mathcal{A}_i$ on Σ_2. These functions can admit suitable singularities on the boundary of the strip. Those are interpreted as the singularities coming from D5 and NS5 branes, like those presented in example (3.1.5). We illustrate this in figure (3.3.1).



(3.3.1)

Backgrounds dual to 3d $\mathcal{N} = 4$ linear quiver theories can be obtained by picking suitable harmonic functions on Σ_2: specifically, we can make a choice of harmonic functions such that I shrinks to zero as $y \to \pm\infty$. The resulting topology is AdS$_4 \times \mathcal{B}_6$ where $\mathcal{B}_6 \approx S^5 \times I$ is the six-dimensional ball. This is illustrated in (3.3.2).

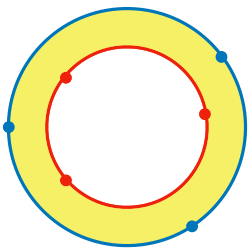


(3.3.2)

Getting holographic duals of 3d $\mathcal{N} = 4$ compact models is more subtle and a quotient procedure is involved. Harmonic functions on Σ_2 can be chosen to have an infinite number of singularities, but in such a way to be periodic along the infinite direction with period T:

$$\mathcal{A}_i(y+T) = \mathcal{A}_i(y)\,.$$

The whole solution is invariant under this translation, being completely determined by $\mathcal{A}_i$. At this stage, we can perform a quotient with respect to "T-symmetry" ending with a configuration where points (x, y) and $(x, y+T)$ of the Riemann surface are identified; we end up with a surface with the topology of the annulus; see figure (3.3.3).



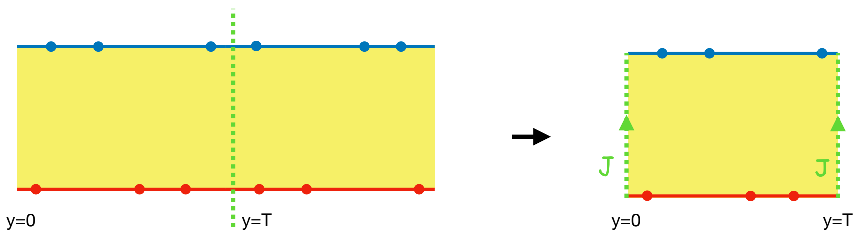
(3.3.3)

3.3.2 J-folds

A more general quotient procedure can, in fact, be implemented. In particular, one may introduce an $SL(2,\mathbb{Z})$ duality-twisted boundary condition [79, 14] upon identifying the two ends of the aforementioned Riemann surface. This can be done as follows. As before, the starting point is a choice of harmonic functions $\mathcal{A}_i$, that completely fixes the physical fields of the solution. For instance, let us focus on the axio-dilaton $\tau = C_0 + i\, e^{-2\phi}$ where C_0 is the potential of the one-form flux F_1 and ϕ is the dilaton. As it is well-known, Type IIB supergravity admits a non-trivial action of $SL(2,\mathbb{Z})$, generating orbits of equivalent solutions; the axio-dilaton is not invariant under this $SL(2,\mathbb{Z})$ action. We can imagine to pick harmonic functions $\mathcal{A}_i$ such:

$$\tau(y+T) = M\,\tau(y) \tag{3.3.4}$$

where M represents the action of $SL(2,\mathbb{Z})$ on the axio-dilaton and we require that similar relations hold for all other fluxes, with an appropriate element of $SL(2,\mathbb{Z})$ acting on them. If such a choice can be performed, we can imagine to quotient with respect to the joint action of $SL(2,\mathbb{Z})$ and translation by T along the non-compact direction y. Points $(x, y+T)$ and (x, y) are again identified; the Riemann surface has a cut along (x, T), passing through the fields undergo an $SL(2,\mathbb{Z})$ transformation. We end up with a Riemann surface with the topology of the annulus and a non-trivial monodromy under $SL(2,\mathbb{Z})$. This is illustrated in (3.3.5).



(3.3.5)

It turns out that such a quotient is related to a particular choice of $SL(2,\mathbb{Z})$ element. Let

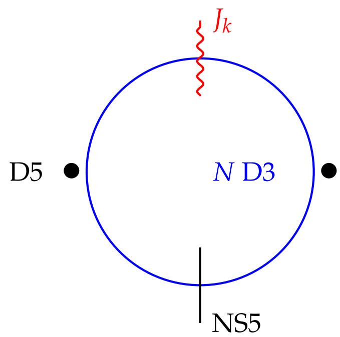
$$S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \qquad T = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}\,, \tag{3.3.6}$$

satisfying $S^2 = -1$ and $(ST)^3 = 1$, be the generators of $SL(2,\mathbb{Z})$. Then the aforementioned quotient can be performed for every element of $SL(2,\mathbb{Z})$ of the form:

$$J_k = -S\,T^k = \begin{pmatrix} k & 1 \\ -1 & 0 \end{pmatrix}\,, \qquad \bar{J}_k = -J_{-k}\,. \tag{3.3.7}$$

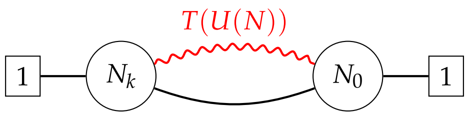
This kind of solutions was studied in the context of abelian theories in [79] and is referred to as the ***J*-fold** in [14]. These are often regarded as non-geometrical, in the sense that we performed a quotient with respect to some symmetry of the theory not descending from isometries of the metric.

The quotient also admits a realisation at the level of brane configurations: it corresponds to a five-dimensional surface implementing the aforementioned monodromy under $SL(2,\mathbb{Z})$ action. As we have seen, Σ_2 has the topology of the annulus, thus corresponding to circular brane configuration with an insertion of J-folds. An example of a brane configuration with a J-fold is as follows:



(3.3.8)

The insertion of the J_k-fold in such a brane system can be viewed as introducing a 3d interface, with a non-trivial $SL(2,\mathbb{Z})$ action J_k, to the 4d $\mathcal{N} = 4$ super-Yang-Mills theory living on the D3-branes on the circle. The theory on such a 3d interface was studied in [8, sec. 8]. This is, in fact, the $T(U(N))$ theory with a Chern–Simons level k for one of the flavour $U(N)$ symmetry, whereas the other $U(N)$ flavour symmetry has Chern–Simons level zero. One can then couple this 3d theory to the theory on the D3-brane on a circle. The $U(N)_k$ and the $U(N)_0$ flavour symmetries[2] are then coupled to the $U(N)_L$ and $U(N)_R$ gauge fields on the left and on the right of the interface, respectively[3]. For instance, the three dimensional quiver theory associated to the brane system (3.3.8) is



(3.3.9)

where N_k and N_0 denotes gauge groups $U(N)$ with Chern–Simons levels k and 0 respectively. We emphasise that there is a mixed CS term with level $-N$ between the two gauge groups. Due to the presence of the $T(U(N))$ theory as a link, this is not a conventional Lagrangian theory, because only one $U(N)$ symmetry is manifest in the Lagrangian description of the $T(U(N))$ theory, whereas the other $U(N)$ symmetry emerges in the infrared[4].

3.3.3 S-flips

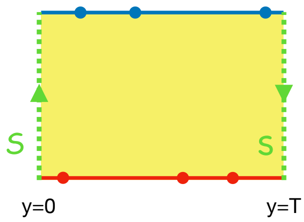
Another type of quotients that is similar to the J-fold is possible. In this case we select the $SL(2,\mathbb{Z})$ element implementing the monodromy to be S.

[2]Unless specified otherwise, we denote the Chern–Simons level as the subscript.

[3]As pointed out in [77, 14], there are two possibilities for coupling the $U(N)$ flavour symmetry to the $U(N)$ gauge field on each side, namely $U(N)_+ = \mathrm{diag}(U(N) \times U(N))$ or $U(N)_- = \mathrm{diag}(U(N) \times U(N)^\dagger)$. For $T(U(N))$, the gauging is chosen to be $U(N)_+$ on both sides, whereas for $\overline{T(U(N))}$, the gauging is chosen to be $U(N)_+$ on one side and $U(N)_-$ on the other side.

[4]It should be mentioned that similar quiver theories, with special unitary gauge groups and $T(SU(N))$ links, were studied in [80, sec. 4.1] and [81, sec. 5.2] in the context of 3d-3d correspondence and the twisted compactification of the 6d $\mathcal{N} = (2,0)$ theory on a torus bundle over S^1.

However, in order to have a desired symmetry of the supergravity solution, we have to perform an exchange of coordinates corresponding to the two S^2 in $AdS_4 \times S^2 \times S^2 \times \Sigma_2$ and a reflection of x coordinate, being identified at the S-interface in an antipodal way, as depicted in (3.3.10).



(3.3.10)

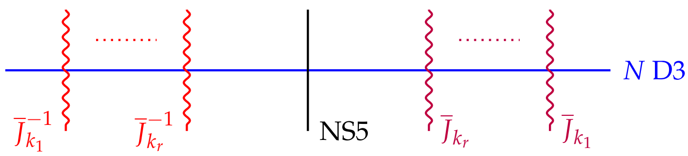
The Riemann surface now has the topology of the Möbius strip. This type of solutions is referred to as an S-**flip** in [14]. Similarly to the J-fold, the S-flip has an avatar at the level of circular brane configuration, as five-dimensional surface passing through the configuration undergoes an $SL(2,\mathbb{Z})$ transformation and a rotation of coordinates such that $(x^{3,4,5}, x^{7,8,9}) \to (x^{7,8,9}, -x^{3,4,5})$. When an S-flip is inserted into a brane system, the corresponding quiver diagram can be obtained in the same way as that with the J-fold, except that the Chern–Simons level is set to zero. An example for this type of configurations is depicted in (3.4.1).

3.3.4 (p,q) fivebranes

Let us now consider (p,q) fivebranes [82, 83], where $(1,0)$ denotes an NS5 brane and $(0,1)$ denotes a D5 brane. For a given ordered pair (p,q), we can write this as

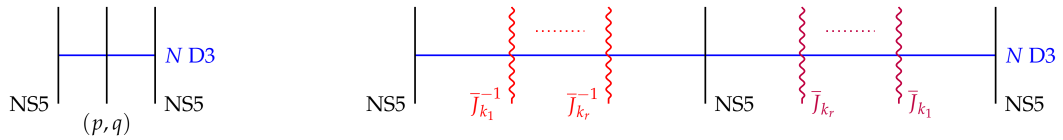
$$(p,q) = \overline{J}_{k_1} \overline{J}_{k_2} \dots \overline{J}_{k_r} (1,0) \tag{3.3.11}$$

for some $k_1, k_2, \dots k_r$. Thus, any (p,q) brane is related to an NS5 brane by an $SL(2,\mathbb{Z})$ transformation. Using this realisation, we can convert a (p,q) brane to an equivalent configuration involving J-folds as follows:



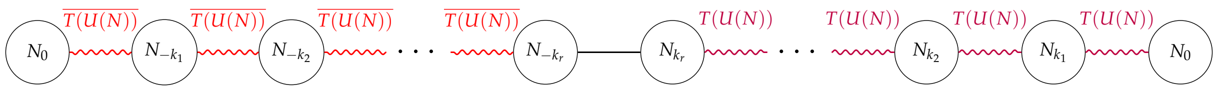
(3.3.12)

From the perspective of the quiver diagram, each $\overline{J}_k$ gives rise to a $T(U(N))$ link with a Chern–Simons level k for the $U(N)$ group on the left, whereas each $\overline{J}_{-k}^{-1}$ gives rise to a $\overline{T(U(N))}$ link with a Chern–Simons level k for the $U(N)$ group on the right. In particular, the corresponding quiver theory for the following $SL(2,\mathbb{Z})$-equivalent brane systems



(3.3.13)

is as follows:



(3.3.14)

This agrees with the description provided in [8, fig. 75] and [77, fig. 6].

3.4 Models with zero Chern–Simons levels

In this section, we consider theories with zero Chern–Simons (CS) levels and with certain links between gauge nodes in the quiver being $T(U(N))$. From the brane perspective, such a theory arises from the Hanany–Witten brane configuration [10], namely a system of D3, NS5 and D5 branes that preserves eight supercharges, with an insertion of S-flips [14]. The presence of an S-flip gives rise to the aforementioned $T(U(N))$ link in the quiver. The moduli space of such quiver theories is studied below. The main result can be summarised as follows.

We find that these theories have two branches of the moduli space, namely the Higgs and the Coulomb branches. Let us first discuss about the Higgs branch. We propose that this is given by the hyperKähler quotient of a product of each component in the quiver by the gauge symmetry. By each component, we mean a bi-fundamental hypermultiplet, a fundamental hypermultiplet and a $T(U(N))$ link that connects two $U(N)$ groups together. The former two can be treated in the usual way as in a Lagrangian theory. whereas each $T(U(N))$ link contributes two copies of the closure of the *maximal nilpotent orbit* of $SU(N)$, denoted by $\mathcal{N}_{SU(N)}$. The reason for latter is two-fold: (1) the Higgs and the Coulomb branches of $T(U(N))$ are both isomorphic to $\mathcal{N}_{SU(N)}$, and (2) in order to realise the two $U(N)$ groups connected by $T(U(N))$, we need two copies of $SU(N)$ subgroups, one arises from the Higgs branch and the other arises from the Coulomb branch of $T(U(N))$.

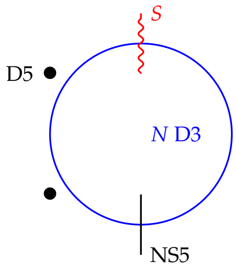
The Coulomb branch is similar to the usual 3d $\mathcal{N} = 4$ gauge theories, but with the following important remark. We propose that the scalars in the vector multiplets of any two gauge nodes that are connected by a $T(U(N))$ link are frozen and do *not* contribute to the Coulomb branch. The other gauge nodes in the quivers still give rise to vector multiplets that contribute to the Coulomb branch. From the brane perspective, this proposal implies that the D3-brane segment between two NS5-branes that is stretched through the S-flip cannot move along the NS5-brane directions (*i.e.* the Coulomb branch directions).

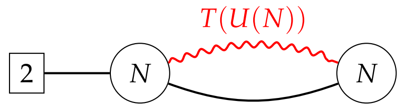
We check that the descriptions of the Higgs and the Coulomb branches mentioned above are consistent with S-duality and mirror symmetry. Given a brane system, say of theory A, we can obtain a brane system of the mirror theory, say theory B, using S-duality. We find that the moduli space of theories A and B are related by mirror symmetry [11, 10]. in the following sense. The Higgs branch (*resp.* Coulomb branch) of theory A computed by using the above proposal is in an agreement with the Coulomb branch (*resp.* Higgs branch) of theory B.

Below we provide examples to demonstrate the above discussion.

3.4.1 Example 1: A flavoured affine A_1 quiver

Let us consider the following brane set-up and the following theory.

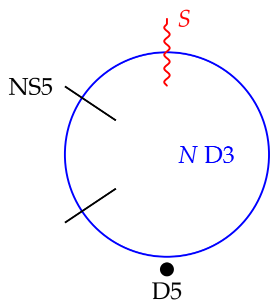


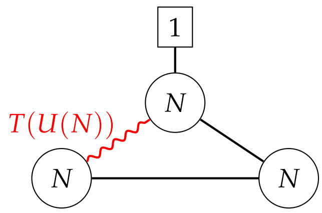


(3.4.1)

where, throughout this section, we denote a gauge group $U(N)$ with zero CS level by a circular node with the label N. The flavour symmetry $U(N_f)$ is denoted by a square node with the label N_f.

The mirror theory can be derived by applying the S-duality to the brane system (3.4.1) which yields





(3.4.2)

The Higgs branches

We claim that the Higgs branch of (3.4.1) is given by

$$\mathcal{H}_{(3.4.1)} = \frac{\mathcal{H}\left([U(2)] - [U(N)_1]\right) \times \mathcal{N}_{SU(N)_1} \times \mathcal{N}_{SU(N)_2} \times \mathcal{H}([U(N)_1] - [U(N)_2])}{U(N)_1 \times U(N)_2}\,, \tag{3.4.3}$$

where $\mathcal{N}_{SU(N)}$ denotes the closure of the maximal nilpotent orbit of $SU(N)$. We shall use shorthand notations $\mathcal{H}$ and $\mathcal{C}$ to stand for the Higgs branch and the Coulomb branch respectively. The quaternionic dimension of (3.4.3) is

$$\dim_{\mathbb{H}} \mathcal{H}_{(3.4.1)} = 2N + 2\left[\frac{1}{2}(N-1)N\right] + N^2 - N^2 - N^2 = N\,. \quad (3.4.4)$$

Similarly, we claim that the Higgs branch of (3.4.2) is

$$\begin{aligned}\mathcal{H}_{(3.4.2)} = \Big[&\mathcal{H}([U(N)_1]-[U(N)_3]) \times \mathcal{H}([U(N)_2]-[U(N)_3]) \times \mathcal{H}\left([U(1)]-[U(N)_2]\right) \\ &\times \mathcal{N}_{SU(N)_1} \times \mathcal{N}_{SU(N)_2}\Big] / \left(U(N)_1 \times U(N)_2 \times U(N)_3\right)\,.\end{aligned} \quad (3.4.5)$$

The dimension of this space is

$$\dim_{\mathbb{H}} \mathcal{H}_{(3.4.2)} = N^2 + N^2 + N + 2\left[\frac{1}{2}(N-1)N\right] - 3N^2 = 0\,. \quad (3.4.6)$$

The Coulomb branches

Since mirror symmetry identifies the Coulomb branch $\mathcal{C}_{(3.4.1)}$ of (3.4.1) with the Higgs branch $\mathcal{H}_{(3.4.2)}$ of (3.4.2), it follows that

$$\dim_{\mathbb{H}} \mathcal{C}_{(3.4.1)} = \dim_{\mathbb{H}} \mathcal{H}_{(3.4.2)} = 0\,, \quad (3.4.7)$$

and hence $\mathcal{C}_{(3.4.1)}$ is trivial. We see that even though the theory (3.4.1) has gauge group $U(N) \times U(N)$, its Coulomb branch is trivial. This is consistent with our proposal: the scalars in the vector multiplets of $U(N) \times U(N)$ gauge group in (3.4.1) are frozen to a particular value, because they are linked by $T(U(N))$. From the brane perspective, this means that the D3-branes do *not* move along the direction of the S-flip, but *get stuck* at a particular position in the $x^{3,4,5}$ directions. On the other hand, since the Higgs branch of (3.4.1) is non-trivial, this means that the D3-branes that align along the direction of the S-fold and NS5-branes can move along the $x^{7,8,9}$ directions.

By the same token,

$$\dim_{\mathbb{H}} \mathcal{C}_{(3.4.2)} = \dim_{\mathbb{H}} \mathcal{H}_{(3.4.1)} = N\,. \quad (3.4.8)$$

We see that even though (3.4.2) has gauge group $U(N) \times U(N) \times U(N)$, its Coulomb branch has dimension N, rather than $3N$ (which is the sum of the ranks of the gauge groups). This is indeed again consistent with our proposal: the scalars of the two $U(N)$ gauge groups connected by $T(U(N))$ are frozen, but those of the remaining $U(N)$ gauge group can acquire VEVs. The latter gauge group has rank N and contributes N to $\dim_{\mathbb{H}} \mathcal{C}_{(3.4.2)}$. From the brane perspective, the D3-brane segment between two NS5 branes that stretch across the S-flip get stuck at a particular position along the $x^{3,4,5}$ directions. On the other hand, the segment that does not stretch across the S-flip can move along the latter.

The Hilbert series

To confirm these statements, we compute the Hilbert series of the Higgs branch of (3.4.1) using the description (3.4.3):[5]

$$\begin{aligned} &H[\mathcal{H}_{(3.4.1)}](t,x) = \\ &\int \mathrm{d}\mu_{U(N)}(\boldsymbol{u}) \int \mathrm{d}\mu_{U(N)}(\boldsymbol{w}) \\ &\times \mathrm{PE}\left[-t^2(u_1+u_2)(u_1^{-1}+u_2^{-1}) - t^2(w_1+w_2)(w_1^{-1}+w_2^{-1})\right] \\ &\times \mathrm{PE}\left[t(x+x^{-1})\left\{\sum_{i=1}^N u_i^{-1} + \sum_{i=1}^N u_i\right\}\right] \\ &\times H[\mathcal{N}_{SU(N)}](t,\boldsymbol{u}) H[\mathcal{N}_{SU(N)}](t,\boldsymbol{w}) \\ &\times \mathrm{PE}\left[\left(\sum_{i=1}^N u_i\right)\left(\sum_{i=1}^N w_i^{-1}\right)t + \left(\sum_{i=1}^N u_i^{-1}\right)\left(\sum_{i=1}^N w_i\right)t\right] , \end{aligned} \tag{3.4.9}$$

where the $U(N)$ Haar measure is given by

$$\int \mathrm{d}\mu_{U(N)}(\boldsymbol{z}) = \left(\prod_{i=1}^N \oint_{|z_i|=1} \frac{\mathrm{d}z_i}{2\pi i z_i}\right) \prod_{1\le i<j\le N} \left(1 - \frac{z_i}{z_j}\right) , \tag{3.4.10}$$

and the Hilbert series of the closure of the maximal orbit of $SU(N)$ is (see [84, (3.4)] and [85]):

$$H[\mathcal{N}_{SU(N)}](t,\boldsymbol{z}) = \left[\prod_{j=2}^N (1-t^{2j})\right] \times \mathrm{PE}\left[t^2 \chi^{SU(N)}_{\mathbf{adj}}(\boldsymbol{z})\right] , \tag{3.4.11}$$

with $\chi^{SU(N)}_{\mathbf{adj}}(\boldsymbol{z})$ the character of the adjoint representation of $SU(N)$:

$$\chi^{SU(N)}_{\mathbf{adj}}(\boldsymbol{z}) = (z_1+z_2)(z_1^{-1}+z_2^{-1}) - 1 . \tag{3.4.12}$$

Let us now explain the contribution of each line in (3.4.9). The first two lines describe the gauging of the symmetry $U(N) \times U(N)$. The second line is the contribution of the fundamental hypermultiplets. The third line is contribution of two copies of $\mathcal{N}_{SU(N)}$; one is the Higgs branch and the other is the Coulomb branch of $T(U(N))$. The last line is the contribution of the bi-fundamental hypermultiplets. Here x is a fugacity for the $SU(2)$ global symmetry.

The integrals in (3.4.9) can be evaluated in an exact manner and yield

$$H[\mathcal{H}_{(3.4.1)}](t,x) = \mathrm{PE}\left[\chi^{SU(2)}_{\mathbf{adj}}(x) \sum_{j=1}^N t^{2j} - \sum_{j=1}^N t^{2N+2j}\right] . \tag{3.4.13}$$

where

$$\chi^{SU(2)}_{\mathbf{adj}}(x) = x^2 + 1 + x^{-2} . \tag{3.4.14}$$

The Higgs branch of (3.4.1) thus has an $SU(2)$ isometry; this is manifest as a flavour symmetry in the quiver. In fact, this Hilbert series is equal to that of the Coulomb

[5]The plethystic exponential (PE) of a multivariate function $f(x_1, x_2, \ldots, x_n)$ such that $f(0,0,\ldots,0) = 0$ is defined as $\mathrm{PE}[f(x_1,x_2,\ldots,x_n)] = \exp\left(\sum_{k=1}^\infty \frac{1}{k} f(x_1^k, x_2^k, \ldots, x_n^k)\right)$.

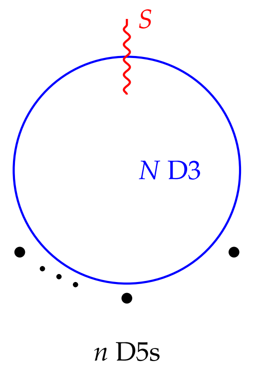
branch of $U(N)$ gauge theory with $2N$ flavours (also known as the $T^{[1^{2N}]}_{[N^2]}(SU(2N))$ theory [8]) [86, (5.6)], where the $U(1)$ topological symmetry gets enhanced to $SU(2)$ at strong coupling :

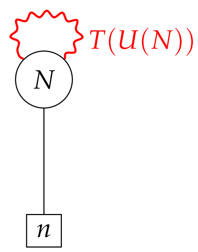
$$\begin{aligned}\mathcal{H}_{(3.4.1)} &= \mathcal{C}\,(U(N) \text{ gauge theory with } 2N \text{ flavours}) \\ &= \mathcal{C}\left(T^{[1^{2N}]}_{[N^2]}(SU(N))\right) \\ &= \text{the intersection between the Slodowy slice} \\ &\quad \text{transverse to the nilpotent orbit associated with } [N,N] \\ &\quad \text{and the nilpotent cone of } SL(2N,\mathbb{C}) \text{ [8]},\end{aligned} \tag{3.4.15}$$

Indeed, we can see an effective $U(N)$ gauge theory with $2N$ flavours from (3.4.2) as follows. Since the two $U(N)$ gauge groups connected by the red line do not contribute to the Coulomb branch, we can effectively think of them as flavour symmetries, and so the $U(N)$ gauge group on the lower right hand corner has effectively $2N$ flavours transformed under it.

3.4.2 Example 3: Quivers with a $T(U(N))$ loop

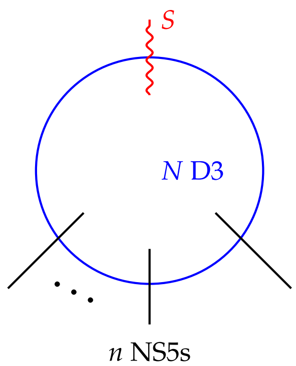
We consider the following brane set-up and the following corresponding theory.

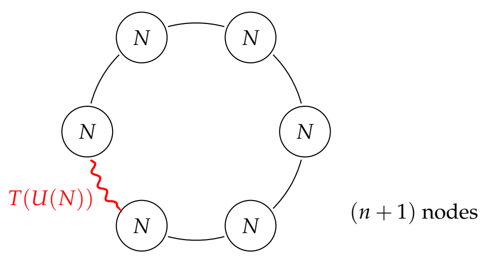




(3.4.16)

The mirror theory can be obtained by applying S-duality to the above system:





(3.4.17)

The Higgs branch of (5.2.1) is given by the following description

$$\mathcal{H}_{(5.2.1)} = \frac{\mathcal{N}_{SU(N)} \times \mathcal{N}_{SU(N)} \times \mathcal{H}\left([U(N)] - [U(n)]\right)}{U(N)}\,. \tag{3.4.18}$$

The quaternionic dimension of which is equal to

$$\dim_{\mathbb{H}} \mathcal{H}_{(5.2.1)} = \left[2 \times \frac{1}{2}(N-1)(N)\right] + nN - N^2 = (n-1)N \,. \tag{3.4.19}$$

Observe that for $n = 1$, the Higgs branch is trivial for any N. On the other hand, the Higgs branch of (3.4.17) is given by the following description

$$\mathcal{H}_{(3.4.17)} = \frac{\mathcal{N}_{SU(N)} \times \mathcal{N}_{SU(N)} \times \mathcal{H}[U(N) - U(N)]^n}{U(N)^{n+1}/U(1)^N} \,, \tag{3.4.20}$$

where we quotiented by $U(N)^{n+1}/U(1)^N$ because at a generic point on the Higgs branch, the gauge symmetry $U(N)^{n+1}$ is not completely broken but it is broken to $U(1)^N$ (see *e.g.* [65]). The dimension of this space is actually zero:

$$\dim_{\mathbb{H}} \mathcal{H}_{(3.4.17)} = \left[2 \times \frac{1}{2}(N-1)(N)\right] + nN^2 - \left[(n+1)N^2 - N\right] = 0 \,. \tag{3.4.21}$$

From mirror symmetry, $\mathcal{C}_{(5.2.1)}$ is identified with $\mathcal{H}_{(3.4.17)}$, and so

$$\dim_{\mathbb{H}} \mathcal{C}_{(5.2.1)} = \dim_{\mathbb{H}} \mathcal{H}_{(3.4.17)} = 0 \,. \tag{3.4.22}$$

This is consistent with our proposal because (5.2.1) has a single circular node that is connected by the $T(U(N))$ link and so it does not contribute to the Coulomb branch dynamics.

On the other hand, it can be checked using the Hilbert series that the Higgs branch $\mathcal{H}_{(5.2.1)}$ is in fact isomorphic to the Coulomb branch of the following quiver[6]

$$\underset{N}{\square} - \underbrace{\underset{N}{\circ} - \cdots - \underset{N}{\circ}}_{(n-1)\text{ nodes}} - \underset{N}{\square} \,. \tag{3.4.23}$$

This quiver can be derived from (3.4.17) using our proposal: since the vector multiplets two gauge nodes linked by $T(U(N))$ in (3.4.17) are frozen, we can take them to be flavour nodes, and quiver (3.4.23) thus follows.

Amusingly, using brane and mirror symmetry (see [87, (2.5)]), we also know that

$$\mathcal{H}_{(5.2.1)} = \mathcal{C}_{(3.4.23)} = \mathcal{H}\left[\underset{1}{\circ} - \underset{2}{\circ} - \cdots - \underset{N-1}{\circ} - \underset{N}{\overset{\square n}{\overset{|}{\circ}}} - \underset{N-1}{\circ} - \cdots - \underset{2}{\circ} - \underset{1}{\circ}\right] \,. \tag{3.4.24}$$

In a special case of or $n = 1$, the quiver on right of the above equation is the star-shaped quiver that is mirror [88] to the S^1 compactification of a clsss S theory of type A_{N-1} associated with a sphere with two maximal and one minimal puncture. The latter is actually a theory of free hypermultiplets. Thus, the spaces in (3.4.24) are zero dimensional; this is in agreement with (3.4.19).

[6]For example, the Hilbert series of the Higgs branch $\mathcal{H}_{(5.2.1)}$ for $N = n = 2$ is precisely the Coulomb branch Hilbert series of 3d $\mathcal{N} = 4$ $U(2)$ gauge theory with 4 flavours. These can be computed similarly as in the preceding subsections.

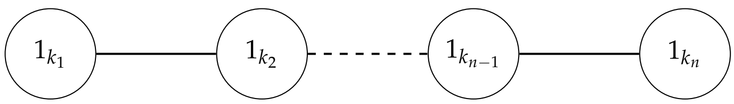
3.5 Abelian theories with non-zero Chern–Simons levels

In this section, we focus on field theories that arise from Hanany–Witten brane configurations, with a single D3-brane on S^1 and with an inclusion of J-folds. These can be represented as abelian quiver theories with non-zero Chern–Simons (CS) levels[7], and $T(U(1))$ connected between quiver nodes. The presence of a $T(U(1))$ link between two quiver nodes gives rise to a mixed CS level between them. In fact, the systems consisting only a D3-brane on the circle and J-folds (but with no D5 and no NS5 brane) were studied in [79]. Such systems give rise to pure CS theories. In order to make the moduli space more interesting, we may also include NS5 and D5 branes in the system. These introduce bi-fundamental and fundamental hypermultiplets into the quiver theory. The moduli space of theories in this section is more sophisticated to analyse than those in section 3.4. This is because the vacuum equations may admit many sets of non-trivial solutions, in which case the moduli space has many branches. Below we systematically analyse such branches, and provide necessary conditions on the CS levels in order to have a non-trivial moduli space.

As a warm-up, we first analyse linear quivers without a $T(U(1))$ link in section 3.5.1. This also serves as a generalise of the analysis in [89] and a complement to the analysis of [90], where here we provide direct analyses of the moduli space from the vacuum equations and compute the Hilbert series. Subsequently in section 3.5.2, we introduce a J-fold in to the brane system. Finally, in section (3.5.3), we add flavours in to the quiver. In the latter, under some conditions, the fundamental hypermultiplets may contribute non-trivially to the moduli space. The analysis for theories with more than one J-fold is more technical and we postpone the discussion to section 3.7.

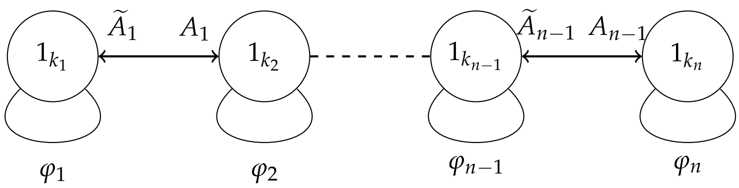
3.5.1 Warm-up: Theories without a J-fold

Before adding a J-fold to the brane systems, it is instructive studying in a systematic way the moduli space of linear quivers without fundamental matter.



(3.5.1)

This is made up of n $U(1)$ gauge nodes with Chern-Simons levels k_i , $i = 1, \dots n$. The i-th node is connected to the $(i-1)$-th one by an hyper-multiplet $(A_i, \widetilde{A}_i)$. In $\mathcal{N} = 2$ language, the quiver appears as:



(3.5.2)

[7] We denote the CS level by a subscript, for example $U(N)_k$ denotes a group $U(N)$ with CS level k. In a quiver node, we abbreviate this as N_k.

with the superpotential

$$W = \sum_{i=1}^{n-1} (\widetilde{A}_i \varphi_i A_i - A_i \varphi_{i+1} \widetilde{A}_i) + \frac{1}{2} \sum_{i=1}^{n} k_i \varphi_i^2 \,. \tag{3.5.3}$$

Due to $\mathcal{N} = 3$ supersymmetry of the theory, we are allowed to collect at the same time both F-terms and D-terms, in such a way we really need to solve a unique set of equations. Let us call $\Phi_i = (\varphi_i \,, \sigma_i)$, $\mu_i = (A_i \, \widetilde{A}_i \,, |A_i|^2 - |\widetilde{A}_i|^2)$: the whole set of F-terms and D-terms now read

$$A_i(\Phi_{i+1} - \Phi_i) \,=\, 0 \,, \quad \widetilde{A}_i(\Phi_{i+1} - \Phi_i) \,=\, 0 \quad i = 1 \,, \ldots \,, n-1 \tag{3.5.4}$$

$$\begin{aligned} k_1 \, \Phi_1 \,&=\, \mu_1 \\ k_i \, \Phi_i \,&=\, \mu_i - \mu_{i-1} \quad i = 2 \,, \ldots \,, n-1 \\ k_n \, \Phi_n \,&=\, -\mu_{n-1} \end{aligned} \tag{3.5.5}$$

Moreover, the R-charge and gauge charges of the monopole operators with flux $(m_1, \ldots, m_n)$ read, respectively:

$$R[V_{(m_1,\ldots,m_n)}] \,=\, \frac{1}{2} \sum_{i=1}^{n-1} |m_{i+1} - m_i| \,, \quad q_i[V_{(m_1,\ldots,m_n)}] \,=\, -k_i \, m_i \tag{3.5.6}$$

where m_i is the magnetic flux of the i-th gauge group.

Cutting the quiver

It is convenient to study the solutions to the vacuum equations according to the vanishing of the VEVs of the bi-fundamental hypermultiplets. In particular, the vacuum equations may admit the solutions in which

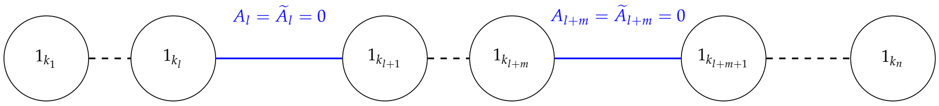
$$\begin{aligned} & A_{l_1} = \widetilde{A}_{l_1} = A_{l_2} = \widetilde{A}_{l_2} = \cdots = A_{l_m} = \widetilde{A}_{l_m} = 0 \,, \text{ for some } l_1 < l_2 < \cdots < l_m \\ & \text{and } \; A_p, \, \widetilde{A}_p \neq 0 \text{ for } p \notin \{l_1, l_2, \ldots, l_m\} \,, \end{aligned} \tag{3.5.7}$$

In which case, the quiver diagram in question is naturally divided into sub-quivers, and we shall henceforth say that the quiver is "cut" at the positions $l_1, l_2, \cdots, l_m$. If the vacuum equations do not admit such a solution, we say that the quiver cannot be cut. As we shall see in explicit examples below, the vacuum equations of certain quivers may admit more than one option of cuts, in which case, each option gives rise to a branch of the moduli space.

In order to determine whether we need to cut the quiver, we can proceed as follows. Suppose that the quiver cannot be cut, *i.e.* all A_i and $\widetilde{A}_i$ are non-zero. This implies that $\Phi_i = \Phi \neq 0$ for all i. If the system of equations (3.5.5) admits a solution in which $\mu_j = 0$ for some j, then our initial assumption that the quiver cannot be cut is contradicted, and we need to cut a quiver somewhere. However, it should be emphasised that if the aforementioned system of equations have a solution in which $\mu_j \neq 0$ for all j, what we can infer is that there is a branch of the moduli space corresponding to no cut; however, there may exist another branch of the moduli space corresponding to a cut in the quiver.

Let us now cut the quiver in question at two positions, namely l and m with $m > l$. This divides the the orginal quiver into three sub-quivers that we will denote as: "left", collecting the nodes first l nodes, "central", collecting the node $l+1, \ldots, l+$

m, and finally "right" encoding the last $n-l-m$ nodes, as depicted below.



(3.5.8)

Below we derive necessary conditions for each sub-quivers to contribute non-trivially to the moduli space.

Let us consider the left sub-quiver. We fix $A_l = \widetilde{A}_l = 0$ and assume that A_i and $\widetilde{A}_i$ are non-vanishing for all $i = 1, 2, \ldots, l$. Then (3.5.85) implies that $\Phi_i = \Phi = (\varphi, \sigma)$ $\forall i = 1, 2, \ldots, l$. The sum of the first l equations in (3.5.5) provides the following constraint

$$\left(\sum_{i=1}^{l} k_i\right) \varphi = A_l \widetilde{A}_l = 0 \,. \tag{3.5.9}$$

Since $\varphi \neq 0$ (otherwise $A_{l-1}\widetilde{A}_{l-1}$ would be zero, contradicting our assumption), we see that a *necessary condition* for the left sub-quiver to contribute non-trivially to the moduli space of vacua is

$$\sum_{i=1}^{l} k_i = 0 \,. \tag{3.5.10}$$

A similar argument also applies for the right sub-quiver. We fix $A_{l+m} = \widetilde{A}_{l+m} = 0$ and assume that A_i and $\widetilde{A}_i$ are non-vanishing for all $i = l+m+1, \ldots, n$. A necessary condition for this sub-quiver to contribute non-trivially to the moduli space is

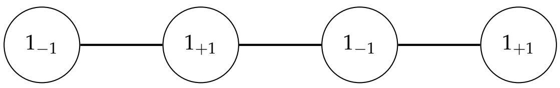
$$\sum_{i=l+m+1}^{n} k_i = 0 \,, \tag{3.5.11}$$

If the central sub-quiver contains a sub-quiver whose CS levels sum to zero, we may cut the former further into smaller sub-quivers. Otherwise, a necessary condition for the central sub-quiver to contribute non-trivially to the moduli space is

$$\sum_{i=l+1}^{l+m} k_i = 0 \,. \tag{3.5.12}$$

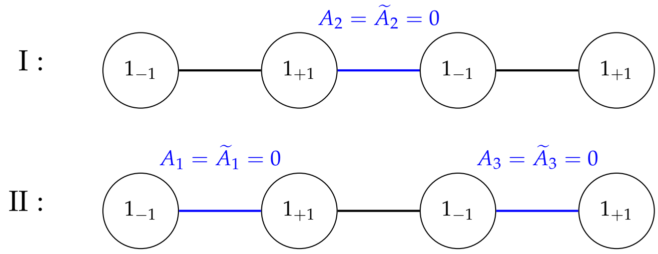
This again follows from the sum of the $(l+1)$-th to the $(l+m)$-th equations in (3.5.5), with $\mu_l = \mu_{l+m} = 0$.

Note that there can be many ways in cutting a given quiver into sub-quivers. Consider the following gauge theory as an example



(3.5.13)

There are two ways in cutting such a quiver in order to obtain a non-trivial moduli space, namely



(3.5.14)

In case I, both left and right sub-quivers contribute non-trivially to the moduli space, whereas in case II, only the central sub-quiver contributes non-trivially. We shall refer to the vacuum spaces corresponding to these two options as *branches* of the moduli space for (3.5.13). We shall go over the detailed computation of the moduli space later.

The Hilbert series

Let us consider quiver (3.5.8) and assume that the left, central and right sub-quivers cannot be cut further. Using (3.5.85), we see that $\sigma_1 = \sigma_2 = \ldots = \sigma_l$, $\sigma_{l+1} = \sigma_{l+2} = \ldots = \sigma_{l+m}$, and $\sigma_{l+m+1} = \ldots = \sigma_n$. In other words, the magnetic fluxes for the monopole operators for all nodes in each sub-quiver are equal:

$$\begin{aligned} m_1 = m_2 = \ldots = m_l &\equiv m_L \,, \\ m_{l+1} = m_{l+2} = \ldots = m_{l+m} &\equiv m_C, \\ m_{l+m+1} = m_{l+m+2} = \ldots = m_n &\equiv m_R \,. \end{aligned} \tag{3.5.15}$$

The R-charge of the monopole operator with the flux $(m_1, \ldots, m_n)$ is therefore

$$R[V_{(m_1,\ldots,m_n)}] = \frac{1}{2}\sum_{i=1}^{n-1} |m_i - m_{i+1}| = \frac{1}{2}\left(|m_L - m_C| + |m_C - m_R|\right) \,. \tag{3.5.16}$$

The Hilbert series can be computed using the same procedure as presented in [89, sec. 4–sec. 6]. The idea is to count the monopole operators dressed by appropriate chiral fields in the theory such that the combination is gauge invariant. The appropriate combination of chiral fields that are used to dress the monopole operators are counted by the baryonic generating function [91].

Let $g_L(t, \boldsymbol{B})$, $g_C(t, \boldsymbol{B})$ and $g_R(t, \boldsymbol{B})$ be baryonic generating functions for the left, central and right sub-quivers, respectively. Then, the Hilbert series for the moduli space for quiver (3.5.8) is given by

$$\begin{aligned} H(t; z_L, z_C, z_R) = \sum_{m_L \in \mathbb{Z}} \sum_{m_C \in \mathbb{Z}} \sum_{m_R \in \mathbb{Z}} & t^{|m_L - m_C| + |m_C - m_R|} z_L^{m_L} z_C^{m_C} z_R^{m_R} \times \\ & g_L(t, \{k_1 m_L, \ldots, k_l m_L\})\, g_C(t, \{k_{l+1} m_C, \ldots, k_{m-1} m_C\}) \times \\ & g_R(t, (k_m m_R, \ldots, k_n m_R\}) \,, \end{aligned} \tag{3.5.17}$$

where $z_{L,C,R}$ are fugacities for the topological symmetries. The first line is the contribution from the monopole operators and the second and third lines are the contribution from an appropriate combination of chiral fields in the quiver that will be used to dress the monopole operators.

Example 1: Quiver (3.5.13)

The two non-trivial cuts depicted in (3.5.14) corresponds to two non-trivial branches of the moduli space.

Branch I. This corresponds to the top diagram in (3.5.14), where the VEVs of A_2 and $\widetilde{A}_2$ are zero, and the VEVs of other bifundamentals are non-zero. The cut splits the quiver (3.5.13) into two sub-quivers, each of which can be identified as the half-ABJM theory[8] [89, sec. 4.1.3]. Let us denote the magnetic fluxes associated with the four nodes of the quiver from left to right by (m_L, m_L, m_R, m_R). The Hilbert series for this branch of the moduli space is then given by

$$\begin{aligned} H^{(\text{I})}_{(3.5.13)}(t; z_1, z_2) &= \sum_{m_L \in \mathbb{Z}} \sum_{m_R \in \mathbb{Z}} t^{|m_L - m_R|} g_{\text{ABJM}/2}(t; m_L) g_{\text{ABJM}/2}(t; m_R) \\ &= \sum_{m_L \in \mathbb{Z}} \sum_{m_R \in \mathbb{Z}} t^{|m_L - m_R|} \frac{t^{|m_L|}}{1-t^2} \frac{t^{|m_R|}}{1-t^2} z_1^{m_L} z_2^{m_R} \\ &= \sum_{m=0}^{\infty} \chi^{SU(3)}_{[m,m]}(z_1, z_2) t^{2m} . \end{aligned} \tag{3.5.18}$$

where $g_{\text{ABJM}/2}(t; B)$ is the baryonic generating function of the half-ABJM theory

$$g_{\text{ABJM}/2}(t; B) = \oint_{|u_1|=1} \frac{\mathrm{d}u_1}{2\pi i u_1^{B+1}} \oint_{|u_2|=1} \frac{\mathrm{d}u_2}{2\pi i u_2^{-B+1}} \text{PE}\left[(u_1 u_2^{-1} + u_1^{-1} u_2) t \right] = \frac{t^{|B|}}{1-t^2} , \tag{3.5.19}$$

and the character of the adjoint representation $[1,1]$ of $SU(3)$ is

$$\chi^{SU(3)}_{[1,1]}(z_1, z_2) = 2 + z_1 z_2 + \frac{1}{z_1 z_2} + z_1 + \frac{1}{z_1} + z_2 + \frac{1}{z_2} . \tag{3.5.20}$$

The last line indicates that this branch is isomorphic to the reduced moduli space of one $SU(3)$ instanton on $\mathbb{C}^2$ [92], or equivalently the closure of the minimal nilpotent orbit of $SU(3)$. The eight generators can be written in terms of a traceless 3×3 matrix as

$$M = \begin{pmatrix} \varphi_L & V_{(1,1,0,0)} & V_{(1,1,1,1)} \\ V_{(-1,-1,0,0)} & \varphi_R & V_{(0,0,1,1)} \\ V_{(-1,-1,-1,-1)} & V_{(0,0,-1,-1)} & -\varphi_L - \varphi_R \end{pmatrix} \tag{3.5.21}$$

where $\varphi_L = \varphi_1 = \varphi_2$ and $\varphi_R = \varphi_3 = \varphi_4$. The Hilbert series indicates that the matrix M satisfies the following conditions [93]:

$$\text{rank}\, M \leq 1 , \qquad M^2 = 0 . \tag{3.5.22}$$

Branch II. This corresponds to the bottom diagram in (3.5.14), where the VEVs of A_1, $\widetilde{A}_1$, A_3 and $\widetilde{A}_3$ are zero, and the VEVs of other bifundamentals are non-zero. In this case, only the central sub-quiver contributes to the computation of the Hilbert series. The magnetic fluxes associated with the four nodes of the quiver from left to right can be written as $(0, m, m, 0)$, with $m \in \mathbb{Z}$, where the zeros follow from the D-term equations. The Hilbert series for this branch of the moduli space is then given

[8] We define the **half-ABJM theory** by a theory with $U(1)_k \times U(1)_{-k}$ gauge symmetry with a single bi-fundamental hypermultiplet.

by

$$\begin{aligned} H^{(\mathrm{II})}_{(3.5.13)}(t;z) &= \sum_{m\in\mathbb{Z}} t^{|0-m|+|m-m|+|m-0|} g_{\mathrm{ABJM}/2}(t;m) z^m \\ &= \sum_{m\in\mathbb{Z}} t^{2|m|} \frac{t^m}{1-t^2} = \mathrm{PE}\left[t^2 + (z+z^{-1})t^3 - t^6\right] . \end{aligned} \tag{3.5.23}$$

This indicates that this branch is isomorphic to $\mathbb{C}^2/\mathbb{Z}_3$. The generators of this moduli space are $V_{(0,1,1,0)}$, $V_{(0,-1,-1,0)}$ and $\varphi \equiv \varphi_2 = \varphi_3$, satisfying the relation

$$V_{(0,1,1,0)} V_{(0,-1,-1,0)} = \varphi^3 . \tag{3.5.24}$$

Branches I and II of (3.5.13) are indeed the Higgs and Coulomb branches of 3d $\mathcal{N} = 4$ $U(1)$ gauge theory with 3 flavours, as pointed out in [94, sec. 4.2]. The brane system of the former can be obtained by applying the $SL(2,\mathbb{Z})$ action T^T to the brane system of the latter.

Example 2: No cut in the quiver (3.5.1)

We assume that A_i and $\widetilde{A}_i$ are non-vanishing for all $i = 1, \ldots, n$, *i.e.* there is no cut in the quiver. In this case, (3.5.85) implies that

$$\Phi_i = \Phi = (\varphi, \sigma) \quad \forall\, i = 1, \ldots, n \tag{3.5.25}$$

As a consequence, the magnetic fluxes are constrained to be all equal $m_1 = m_2 = \ldots = m$. The equations (3.5.5), instead, simply constrain the bilinears μ_i in terms of φ. Summing over the n equations, we obtain the following condition

$$(k_1 + k_2 + \cdots + k_n)\, \Phi = 0 \tag{3.5.26}$$

Note that $\Phi = 0$ would imply $\mu_i = 1\ \forall\ i$ contradicting the initial assumption that all $A_i, \widetilde{A}_i \neq 0$. Thus, as we discuss before, the moduli space is non-trivial if

$$\sum_{i=1}^{n} k_i = 0 \tag{3.5.27}$$

Let us assume (3.5.27) in the subsequent discussion.

The bare monopoles $V_{(m,\ldots,m)}$, with flux $(m, \ldots, m)$, have R-charge $R[V_{(m,\ldots,m)}] = 0$. They need to be dressed in order to make them gauge invariant, because of their gauge charge under the i-th gauge group is $q_i[V_{(m,\ldots,m)}] = -k_i\, m$. Let us define for convenience

$$K_i = \sum_{j=1}^{i} k_j \tag{3.5.28}$$

If $K_i \geq 0$ for all $i = 1, \ldots, n-1$, we can form the following gauge invariant dressed monopole operator:

$$\begin{aligned} \overline{V}_+ &\equiv V_{(1,\ldots,1)}\, A_1^{K_1} A_2^{K_2} \ldots A_{n-1}^{K_{n-1}} , \\ \overline{V}_- &\equiv V_{(-1,\ldots,-1)}\, \widetilde{A}_1^{K_1} \widetilde{A}_2^{K_2} \ldots \widetilde{A}_{n-1}^{K_{n-1}} . \end{aligned} \tag{3.5.29}$$

Note that if $K_j < 0$ for some j, we replace $A_j^{K_j}$ in the first equation by $\widetilde{A}_j^{-K_j}$, and

$\widetilde{A}_j^{K_j}$ in the second equation by $A_j^{-K_j}$. In any case, the R-charges of the above dressed monopole operators are

$$R[\overline{V}_\pm] \;=\; \frac{1}{2}\sum_{i=1}^{n-1} |K_i| = \frac{1}{2}K \tag{3.5.30}$$

with

$$K \equiv \sum_{i=1}^{n-1} |K_i| \,. \tag{3.5.31}$$

The chiral ring is generated by the three operators $\{\varphi\,,\,\overline{V}_+\,,\,\overline{V}_-\}$, statisfying the following relation:

$$\overline{V}_+\,\overline{V}_- \;=\; \varphi^K \,. \tag{3.5.32}$$

Thus, the variety associated to this branch is:

$$\mathbb{C}^2/\mathbb{Z}_K \,. \tag{3.5.33}$$

We can obtain the same result using the Hilbert series. Let us call $\{q_1\,,q_2\,,\ldots\,,\,q_n\}$ the fugacities associated to the n gauge nodes and t the fugacity associated to the R-symmetry. The ingredients entering the Hilbert series are:

- The $n-1$ bifundamental hypermultiplets contribute as:

$$\mathrm{PE}[t(q_1q_2^{-1}+q_1^{-1}q_2)]\,\mathrm{PE}[t(q_2q_3^{-1}+q_2^{-1}q_3)]\ldots\,\mathrm{PE}[t(q_{n-1}q_n^{-1}+q_{n-1}^{-1}q_n)] \tag{3.5.34}$$

- There is also a contribution from φ which gives $\mathrm{PE}[t^2]$.
- The F-terms (3.5.5) impose further $(n-1)$ constraints on the former, after taking into account the condition (3.5.26), which is the overall sum of (3.5.5). These contribute $\mathrm{PE}[-(n-1)t^2]$ to the Hilbert series.

The baryonic generating function is thus:

$$g(t;\boldsymbol{B}) \;= \mathrm{PE}[-(n-1)t^2]\,\mathrm{PE}[t^2]\oint\frac{dq_1}{2\pi i q_1^{1+B_1}}\cdots\oint\frac{dq_n}{2\pi i q_n^{1+B_n}}\prod_{i=1}^{n-1}\mathrm{PE}[t(q_iq_{i+1}^{-1}+q_i^{-1}q_{i+1})] \tag{3.5.35}$$

and can perform a change of variable:

$$\{y_1\,,y_2\,,\ldots\,,\,y_n\} \;=\; \{q_1q_2^{-1}\,,\,q_2q_3^{-1}\;\ldots\,,\,q_{n-1}q_n^{-1}\,,\,q_n\} \tag{3.5.36}$$

Thus, the baryonic function becomes:

$$\mathrm{PE}[-(n-2)t^2]\prod_{i=1}^{n-1}\oint\frac{dy_i}{2\pi i y_i^{1+\widetilde{B}_i}}\,\mathrm{PE}[t(y_i+y_i^{-1})]\oint\frac{dy_n}{2\pi i y_i^{1+\widetilde{B}_n}} \tag{3.5.37}$$

where we defined $\widetilde{B}_i \;=\; \sum_{j=1}^{i} B_j$. The previous integrals are known:

$$\oint\frac{dy_i}{2\pi i y_i^{1+\widetilde{B}_i}}\,\mathrm{PE}[t(y_i+y_i^{-1})] \;=\; \frac{t^{|\widetilde{B}_i|}}{1-t^2}\,, \quad \oint\frac{dy_n}{2\pi i y_i^{1+\widetilde{B}_n}} \;=\; \delta_{\widetilde{B}_n\,,0} \tag{3.5.38}$$

and then the baryonic generating function simplifies to

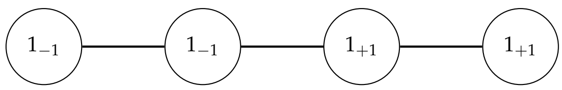
$$g(t;\boldsymbol{B}) = \frac{t^{\sum_{i=1}^{n-1}|\widetilde{B}_i|}}{1-t^2}\delta_{\widetilde{B}_n,0}\,, \text{ with } \widetilde{B}_i = \sum_{j=1}^{i} B_j\,. \tag{3.5.39}$$

Recall that the charge of the monopole operator under the $U(1)_i$ gauge symmetry is $q_i[V_{(m,\ldots,m)}] = -k_i m$. As a consequence, the Hilbert series reads:

$$\begin{aligned} H(t;z) &= \sum_{m\in\mathbb{Z}} g(t;\{k_1 m,\ldots,k_n m\})z^m \\ &= \frac{1}{1-t^2}\sum_{m\in\mathbb{Z}} t^{|m|\sum_{i=1}^{n}|\sum_{j=1}^{i}k_j|}z^m \\ &= \frac{1}{1-t^2}\sum_{m\in\mathbb{Z}} t^{K|m|}z^m \\ &= \mathrm{PE}\left[t^2+(z+z^{-1})t^K - t^{2K}\right]\,, \end{aligned} \tag{3.5.40}$$

where $\widetilde{B}_n$ in (3.5.39) is $m\sum_{i=1}^{n}k_i = 0$ and hence the Kronecker delta gives 1. Here z is the fugacity for the topological symmetry. We obtained exactly the Hilbert series of $\mathbb{C}^2/\mathbb{Z}_K$.

Example. Let us consider the following quiver.

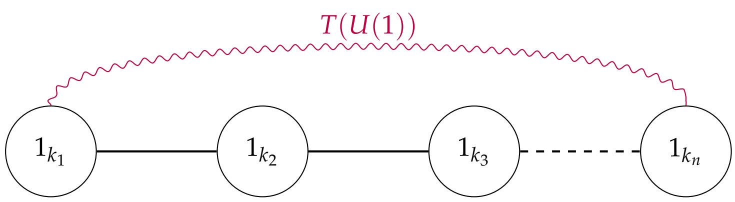


(3.5.41)

This quiver has two non-trivial branches. One corresponds to no cut at all and the other corresponds to the cuts in the first and the third position. As we discussed above, the former branch is isomorphic to $\mathbb{C}^2/\mathbb{Z}_4$. The second branch is the same as that discussed around (3.5.23) and (3.5.24); it is isomorphic to $\mathbb{C}^2/\mathbb{Z}_3$.

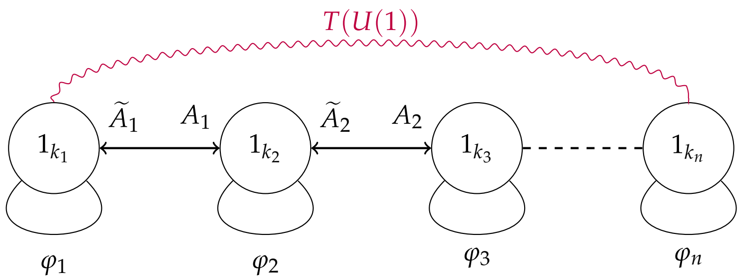
3.5.2 Theories with one J-fold

In this section we want to present the analysis of moduli space of a class of theories dual to a brane configurations with one J-fold and a collection of $(1,k)$ branes. The associated quiver is



(3.5.42)

In the 3d $\mathcal{N} = 2$ notation, this can be rewritten as



(3.5.43)

with the superpotential

$$W = \sum_{i=1}^{n-1}(-\widetilde{A}_i\varphi_i A_i + A_i\varphi_{i+1}\widetilde{A}_i) + \left(\sum_{j=1}^{n}\frac{1}{2}k_j\varphi_j^2\right) - \varphi_1\varphi_n\,. \tag{3.5.44}$$

where we emphasise the contribution from the mixed CS term due to the $T(U(1))$ theory in blue. Let us write $\Phi_i = (\varphi_i\,,\sigma_i)$, $\mu_i = (A_i\,\widetilde{A}_i\,,|A_i|^2 - |\widetilde{A}_i|^2)$. The vacuum equations are

$$A_i(\Phi_{i+1} - \Phi_i) = 0\,, \quad \widetilde{A}_i(\Phi_{i+1} - \Phi_i) = 0 \quad i = 1\,,\ldots\,,n-1 \tag{3.5.45}$$

$$\begin{aligned} k_1\,\Phi_1 - \Phi_n &= \mu_1 \\ k_i\,\Phi_i &= \mu_i - \mu_{i-1} \quad i = 2\,,\ldots\,,n-1 \\ k_n\,\Phi_n - \Phi_1 &= -\mu_{n-1} \end{aligned} \tag{3.5.46}$$

The charges of the monopole operators $V_{(m_1,\ldots,m_n)}$ under the i-th $U(1)$ gauge group are

$$\begin{aligned} q_1[V_{(m_1,\ldots,m_n)}] &= -(k_1\,m_1 - m_n) \\ q_i[V_{(m_1,\ldots,m_n)}] &= -k_i\,m_i\,, \qquad i = 2\,,\ldots\,,n-1 \\ q_n[V_{(m_1,\ldots,m_n)}] &= -(k_n\,m_n - m_1)\,. \end{aligned} \tag{3.5.47}$$

The R-charges of $V_{(m_1,\ldots,m_n)}$ is given by

$$R[V_{(m_1,\ldots,m_n)}] = \frac{1}{2}\sum_{i=1}^{n-1}|m_i - m_{i+1}|\,. \tag{3.5.48}$$

Cutting the quiver

The process of cutting the quiver works similarly as in the previous subsection. However, since there are non-trivial contributions from the $T(U(1))$ theory, some conditions must be modified.

Cutting at one point

Let us consider a case in which $A_l = \widetilde{A}_l = 0$ and other bifundamental hypermultiplets are non-zero. In other words, we cut the quiver precisely at one point where

A_l and $\widetilde{A}_l$ are located. In this case equations (3.5.45) implies

$$\Phi_1 = \cdots = \Phi_l = \Phi = (\varphi\,,\sigma)\,, \quad \Phi_{l+1} = \cdots = \Phi_n = \widetilde{\Phi} = (\widetilde{\varphi}\,,\widetilde{\sigma}) \tag{3.5.49}$$

The system (3.5.46) then becomes:

$$\begin{array}{llll} k_1\Phi - \widetilde{\Phi} = \mu_1\,, & k_2\Phi = \mu_2 - \mu_1\,, & \ldots\,, & k_l\Phi = -\mu_{l-1} \\ k_{l+1}\widetilde{\Phi} = \mu_{l+1}\,, & k_{l+2}\widetilde{\Phi} = \mu_{l+2} - \mu_{l+1}\,, & \ldots\,, & k_n\widetilde{\Phi} - \Phi = -\mu_n \end{array} \tag{3.5.50}$$

The sum of the first l equations and the sum of the remaining $n-l$ ones provide two constraints:

$$\left(\sum_{i=1}^{l} k_i\right)\Phi - \widetilde{\Phi} = 0\,, \quad \left(\sum_{i=l+1}^{n} k_i\right)\widetilde{\Phi} - \Phi = 0 \tag{3.5.51}$$

Since Φ and $\widetilde{\Phi}$ are non-zero (otherwise, this would violate the assumption that A_j and $\widetilde{A}_j$ are non-zero for $j \neq l$), we arrive at the following necessary condition for the existence of a non-trivial solution of the vacuum equation:

$$\left(\sum_{i=1}^{l} k_i\right)\left(\sum_{i=l+1}^{n} k_i\right) = 1 \tag{3.5.52}$$

Since all Chern-Simons levels are integers, the above equation is equivalent to

$$\sum_{i=1}^{l} k_i = \sum_{i=l+1}^{n} k_i = \pm 1 \tag{3.5.53}$$

The system of equations (6.5.2) is now simply solved by $\widetilde{\Phi} = \pm\Phi$. Let us analyse separately the two cases:

- $\Phi = \widetilde{\Phi}$: In this case we choose

$$\sum_{i=1}^{l} k_i = \sum_{i=l+1}^{n} k_i = 1\,. \tag{3.5.54}$$

  This moduli space is parametrised by φ and the two basic dressed monopole operators. Let us define for convenience

$$\begin{aligned} \widetilde{k}_j &= (k_1 - 1\,, k_2\,, \ldots\,, k_{n-1}\,, k_n - 1)\,, \\ \widetilde{K}_i &= \sum_{j=1}^{i} \widetilde{k}_j\,. \end{aligned} \tag{3.5.55}$$

  If $\widetilde{K}_i \geq 0$ for all $i = 1, \ldots, l-1, l+1, \ldots, n-1$, the basic dressed monopole operators are

$$\begin{aligned} \overline{V}_+ &= V_{(1,1,\ldots,1)}\, A_1^{\widetilde{K}_1} \ldots A_{l-1}^{\widetilde{K}_{l-1}}\, A_{l+1}^{\widetilde{K}_{l+1}} \ldots A_{n-1}^{\widetilde{K}_{n-1}} \\ \overline{V}_- &= V_{-(1,1,\ldots,1)}\, \widetilde{A}_1^{\widetilde{K}_1} \ldots \widetilde{A}_{l-1}^{\widetilde{K}_{l-1}}\, \widetilde{A}_{l+1}^{\widetilde{K}_{l+1}} \ldots \widetilde{A}_{n-1}^{\widetilde{K}_{n-1}}\,, \end{aligned} \tag{3.5.56}$$

  If $\widetilde{K}_j < 0$ for some j, we replace $A_j^{\widetilde{K}_j}$ in the first equation by $\widetilde{A}_j^{-\widetilde{K}_j}$, and $\widetilde{A}_j^{\widetilde{K}_j}$ in the second equation by $A_j^{-\widetilde{K}_j}$. In any case, the R-charge of the above dressed monopole operators are

$$R[\overline{V}_{\pm}] = \frac{1}{2} \sum_{\substack{1 \le i \le n-1 \\ i \neq l}} |\widetilde{K}_i| \equiv \frac{1}{2} \widetilde{K} \tag{3.5.57}$$

where the bare monopole operators have R-charge $R[V_{\pm(1,1,\ldots,1)}] = 0$, and we define

$$\widetilde{K} = \sum_{\substack{1 \le i \le n-1 \\ i \neq l}} |\widetilde{K}_i| \ . \tag{3.5.58}$$

Thus, $\overline{V}_{\pm}$ satisfy

$$\overline{V}_+ \overline{V}_- = \varphi^{\widetilde{K}} \ . \tag{3.5.59}$$

This branch of the moduli space is therefore

$$\mathbb{C}^2/\mathbb{Z}_{\widetilde{K}} \ . \tag{3.5.60}$$

Let $g_L(t, \boldsymbol{B})$ and $g_R(t, \boldsymbol{B})$ be baryonic generating functions for the left sub-quiver (containing nodes $1, \ldots, l$) and the right sub-quivers (containing nodes $l+1, \ldots, n$), respectively. Then, the Hilbert series for this case is given by

$$\begin{aligned} H(t; z) = \sum_{m \in \mathbb{Z}} z^m & g_L(t, \{(k_1 - 1)m, k_2 m, \ldots, k_l m\}) \times \\ & g_R(t, \{k_{l+1} m, \ldots, k_{n-1} m, (k-1)m\})(1 - t^2) \ , \end{aligned} \tag{3.5.61}$$

where z is a fugacity for the topological symmetry. Using the expressions for g_L and g_R given by (3.5.39). we obtain

$$\begin{aligned} H(t; z) &= \sum_{m \in \mathbb{Z}} z^m \frac{t^{|m| \sum_{i=1}^{l-1} |\widetilde{K}_i|}}{1 - t^2} \delta_{\sum_{i=1}^{l} k_i, 1} \times \frac{t^{|m| \sum_{i=l+1}^{n-1} |\widetilde{K}_i|}}{1 - t^2} \delta_{\sum_{i=l+1}^{n} k_i, 1} (1 - t^2) \\ &= \begin{cases} \mathrm{PE}\left[t^2 + (z + z^{-1}) t^{\widetilde{K}} - t^{2\widetilde{K}}\right] & \text{if } \sum_{i=1}^{l} k_i = \sum_{i=l+1}^{n} k_i = 1 \\ 0 & \text{otherwise} \ . \end{cases} \end{aligned} \tag{3.5.62}$$

The Hilbert series in the first line in the second equality is indeed that of $\mathbb{C}^2/\mathbb{Z}_{\widetilde{K}}$.

- $\Phi = -\widetilde{\Phi}$: In this case, we choose

$$\sum_{i=1}^{l} k_i = \sum_{i=l+1}^{n} k_i = -1 \ . \tag{3.5.63}$$

The basic monopole operators are $V_{+-} \equiv V_{(1^l, (-1)^{n-l})}$ and $V_{-+} \equiv V_{((-1)^l, 1^{n-l})}$, whose R-symmetry are

$$R[V_{+-}] = R[V_{-+}] = 1 \ . \tag{3.5.64}$$

Let us define for convenience

$$\begin{aligned} \widetilde{k}'_j &= (k_1 + 1 \, , k_2 \, , \ldots , k_{n-1} \, , k_n + 1) \ , \\ \widetilde{K}'_i &= \sum_{j=1}^{i} \widetilde{k}'_j \ . \end{aligned} \tag{3.5.65}$$

For $\widetilde{K}'_i > 0$ for $i = 1, \ldots, l-1$ and $\widetilde{K}'_j < 0$ for $j = l+1, \ldots, n-1$, the basic dressed monopole operators can be written as

$$\begin{aligned} \overline{V}_{+-} &= V_{+-} \, A_1^{\widetilde{K}'_1} \ldots A_{l-1}^{\widetilde{K}'_{l-1}} \, A_{l+1}^{-\widetilde{K}'_{l+1}} \ldots A_{n-1}^{-\widetilde{K}'_{n-1}} \\ \overline{V}_{-+} &= V_{-+} \, \widetilde{A}_1^{\widetilde{K}'_1} \ldots \widetilde{A}_{l-1}^{\widetilde{K}'_{l-1}} \, \widetilde{A}_{l+1}^{-\widetilde{K}'_{l+1}} \ldots \widetilde{A}_{n-1}^{-\widetilde{K}'_{n-1}} \,, \end{aligned} \tag{3.5.66}$$

where it should be noted that in this case $\sum_{i=1}^{l} k_i = \sum_{i=l+1}^{n} k_i = -1$. Similarly as before, $\overline{V}_\pm$ satisfy

$$\overline{V}_{+-} \overline{V}_{-+} = \varphi^{\widetilde{K}'+2} \,, \tag{3.5.67}$$

where we define

$$\widetilde{K}' = \sum_{\substack{1 \leq i \leq n-1 \\ i \neq l}} |\widetilde{K}'_i| \,. \tag{3.5.68}$$

This branch of the moduli space is therefore

$$\mathbb{C}^2 / \mathbb{Z}_{\widetilde{K}'+2} \,. \tag{3.5.69}$$

The Hilbert series for this case is given by

$$\begin{aligned} H(t;z) = \sum_{m \in \mathbb{Z}} t^{|m-(-m)|} z^m g_L(t, \{(k_1+1)m, k_2 m, \ldots, k_l m\}) \times \\ g_R(t, \{-k_{l+1} m, \ldots, -k_{n-1} m, -(k+1)m\})(1-t^2), \end{aligned} \tag{3.5.70}$$

where z is a fugacity for the topological symmetry. Using the expressions for g_L and g_R given by (3.5.39). we obtain

$$\begin{aligned} H(t;z) &= \sum_{m \in \mathbb{Z}} t^{2|m|} z^m \frac{t^{|m| \sum_{i=1}^{l-1} |\widetilde{K}'_i|}}{1-t^2} \delta_{\sum_{i=1}^{l} k_i, -1} \times \frac{t^{|m| \sum_{i=l+1}^{n-1} |\widetilde{K}'_i|}}{1-t^2} \delta_{\sum_{i=l+1}^{n} k_i, -1} (1-t^2) \\ &= \sum_{m \in \mathbb{Z}} z^m \frac{t^{|m|(2+\sum_{i=1}^{l-1} |\widetilde{K}'_i| + \sum_{i=l+1}^{n-1} |\widetilde{K}'_i|)}}{1-t^2} \delta_{\sum_{i=1}^{l} k_i, -1} \delta_{\sum_{i=l+1}^{n} k_i, -1} \\ &= \begin{cases} \mathrm{PE}\left[t^2 + (z+z^{-1}) t^{\widetilde{K}+2} - t^{2(\widetilde{K}+2)}\right] & \text{if } \sum_{i=1}^{l} k_i = \sum_{i=l+1}^{n} k_i = -1 \\ 0 & \text{otherwise} \,. \end{cases} \end{aligned} \tag{3.5.71}$$

The Hilbert series in the first line in the third equality is indeed that of $\mathbb{C}^2/\mathbb{Z}_{\widetilde{K}+2}$.

Cutting at two points

Let us consider a case in which $A_l = \widetilde{A}_l = A_m = \widetilde{A}_m = 0$ (with $m > l$) and other bifundamental hypermultiplets are non-zero. In other words, we cut the quiver precisely at one point where $A_l, \widetilde{A}_l$ and $A_m, \widetilde{A}_m$ are located. This naturally divides the quiver in question into 3 sub-quivers, which we shall refer to as left (L), central (C) and right (R). The central sub-quiver is the same as that is considered in section 3.5.1. In this case equations (3.5.45) implies

$$\begin{aligned} \Phi_1 = \cdots = \Phi_l &= \Phi_L = (\varphi_L \,, \sigma_L) \,, \\ \Phi_{l+1} = \cdots = \Phi_{m-1} &= \Phi_C = (\varphi_C \,, \sigma_C) \,, \\ \Phi_{m+1} = \cdots = \Phi_n &= \Phi_R = (\varphi_R \,, \sigma_R) \,. \end{aligned} \tag{3.5.72}$$

The system (3.5.46) then becomes:

$$\begin{array}{llll} k_1\Phi_L - \Phi_R = \mu_1\,, & k_2\Phi_L = \mu_2 - \mu_1\,, & \ldots, & k_l\Phi_L = -\mu_{l-1} \\ k_{l+1}\Phi_C = \mu_{l+1}\,, & k_{l+2}\Phi_C = \mu_{l+2} - \mu_{l+1}\,, & \ldots, & k_{m-1}\Phi_C = -\mu_{m-1} \\ k_{m+1}\Phi_R = \mu_{m+1}\,, & k_{m+2}\Phi_R = \mu_{m+2} - \mu_{m+1}\,, & \ldots, & k_n\Phi_R - \Phi_L = -\mu_n\,. \end{array} \tag{3.5.73}$$

The sums of the equations in the first, the second and the third lines give

$$\left(\sum_{i=1}^{l} k_i\right)\Phi_L - \Phi_R = 0\,, \quad \left(\sum_{i=l+1}^{m} k_i\right)\Phi_C = 0\,, \quad \left(\sum_{i=m+1}^{n} k_i\right)\Phi_R - \Phi_L = 0\,. \tag{3.5.74}$$

Since Φ_L, Φ_C and Φ_R are non-vanishing (otherwise, this would violate the assumption that A_j and $\widetilde{A}_j$ are non-zero for $j \neq l$), a necessary condition for the existence of a non-trivial solution of the vacuum equation:

$$\sum_{i=1}^{l} k_i = \sum_{i=m+1}^{n} k_i = \pm 1\,, \qquad \sum_{i=l+1}^{m} k_i = 0\,. \tag{3.5.75}$$

Let $g_L(t, \boldsymbol{B})$, $g_C(t, \boldsymbol{B})$ and $g_R(t, \boldsymbol{B})$ be baryonic generating functions for the left, central and right sub-quivers, respectively. Then, the Hilbert series, corresponding to $+$ or $-$ sign in (3.5.75), is

$$\begin{aligned} &H(t; z_L, z_C, z_R) \\ &= \sum_{m_L \in \mathbb{Z}} \sum_{m_C \in \mathbb{Z}} \sum_{m_R \in \mathbb{Z}} t^{|m_L - m_C| + |m_C - m_R|} z_L^{m_L} z_C^{m_C} z_R^{m_R} \times \\ &\quad g_L(t, \{k_1 m_L - m_R, k_2 m_L, \ldots, k_l m_L\})\, g_C(t, \{k_{l+1} m_C, \ldots, k_{m-1} m_C\}) \times \\ &\quad g_R(t, (k_m m_R, \ldots, k_{n-1} m_R, k_n m_R - m_L\})(1 - t^2)\delta_{m_R, \pm m_L}\,, \end{aligned} \tag{3.5.76}$$

where $z_{L,C,R}$ are fugacities for the topological symmetries.

Cutting at more than two points

The above discussion can be easily generalised to the case of cutting the quiver at more than two points. For the moduli space to be non-trivial, the sum of the CS levels in the two sub-quiver that are connected with $T(U(1))$ must be ± 1, and the sum of the CS levels in the other sub-quiver must be zero.

No cutting at all

Assume that A_i and $\widetilde{A}_i$ are non-zero for all i. In this case, a necessary condition for the non-trivial moduli space is

$$\sum_{i=1}^{n} k_i = 2\,. \tag{3.5.77}$$

This again can be obtained from the sum of the equations in (3.5.46), with $\Phi_i = \Phi = (\varphi, \sigma) \neq 0$ (otherwise we would have $\mu_1 = 0$ which contradicts our assumption). The monopole operators $V_{\boldsymbol{m}}$ with fluxes $\boldsymbol{m} = \pm(1, \ldots, 1)$ are not gauge invariant;

however, the following basic dressed monopole operators are gauge invariant

$$\begin{aligned} \overline{V}_+ &= V_{(1,\dots,1)}\, A_1^{\mathcal{K}_1}\, A_2^{\mathcal{K}_2} \,\dots\, A_{n-1}^{\mathcal{K}_{n-1}} \\ \overline{V}_- &= V_{-(1,\dots,1)}\, \widetilde{A}_1^{\mathcal{K}_1}\, \widetilde{A}_2^{\mathcal{K}_2} \,\dots\, \widetilde{A}_{n-1}^{\mathcal{K}_{n-1}}\,, \end{aligned} \tag{3.5.78}$$

for $\mathcal{K}_i \geq 0$ for all $i = 1, \dots, n-1$, where we define

$$\kappa_i \;=\; \{k_1 - 1\,, k_2\,, \dots, k_{n-1}\,, k_n - 1\}\,, \quad \mathcal{K}_i \;=\; \sum_{j=1}^{i} \kappa_j\,. \tag{3.5.79}$$

If $\mathcal{K}_j < 0$ for some j, we replace $A_j^{\mathcal{K}_j}$ by $\widetilde{A}_j^{-\mathcal{K}_j}$ in the first equation and $\widetilde{A}_j^{\mathcal{K}_j}$ by $A_j^{-\mathcal{K}_j}$ in the second equation.

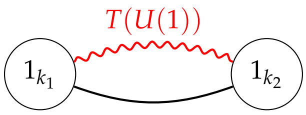
Since the R-charges of $V_{\pm(1,\dots,1)}$ are zero, the R-charges of $\overline{V}_\pm$ are $\frac{1}{2}\sum_{i=1}^{n-1}|\mathcal{K}_i|$. The moduli space is thus generate by the operators $\{\overline{V}_+\ \overline{V}_-\,,\ \varphi\}$ subject to the quantum relation

$$\overline{V}_+\,\overline{V}_- \;=\; \varphi^{\mathcal{K}}\,, \quad \text{with } \mathcal{K} = \sum_{i=1}^{n-1}|\mathcal{K}_i|\,; \tag{3.5.80}$$

this is the algebraic definition of:

$$\mathbb{C}^2/\mathbb{Z}_{\mathcal{K}}\,. \tag{3.5.81}$$

Example. Let us consider the following quiver

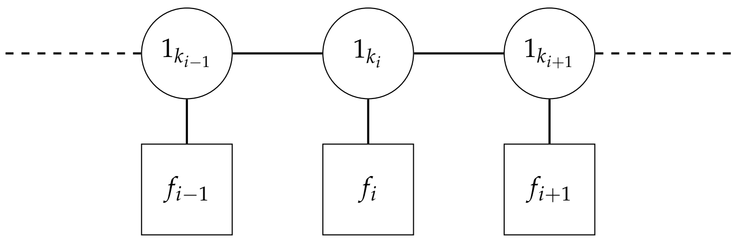


(3.5.82)

It is not possible to introduce a cut to this quiver. As a result, from (3.5.77), it is necessary that $k_1 + k_2 = 2$ for this theory to have a non-trivial moduli space. Let us assume this. Hence $\kappa_i = \{k_1 - 1, k_2 - 1\}$, $\mathcal{K}_i = \{k_1 - 1, k_1 + k_2 - 2 = 0\}$, and so $\mathcal{K} = |k_1 - 1| = |k_2 - 1|$. Therefore the moduli space of this theory is $\mathbb{C}^2/\mathbb{Z}_{|k_1-1|}$.

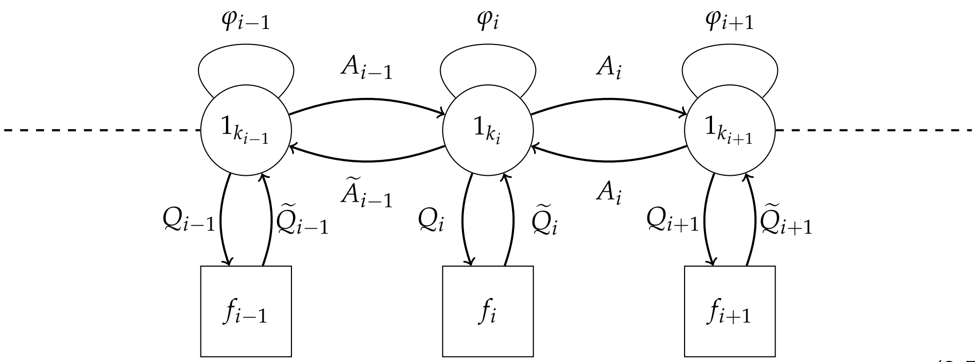
3.5.3 Adding flavours

Let us now add fundamental flavours to the previous discussion.



(3.5.83)

Suppose that there are n gauge groups in total. In the $\mathcal{N} = 2$ notation, this quiver can be written as



(3.5.84)

The vacuum equations read

$$A_{i-1}(\Phi_i - \Phi_{i-1}) = 0\,, \quad A_i(\Phi_{i+1} - \Phi_i) = 0, \tag{3.5.85}$$

also with $A \leftrightarrow \widetilde{A}$,

$$Q_{i-1}\,\Phi_{i-1} = 0\,, \quad Q_i\,\Phi_i = 0\,, \quad Q_{i+1}\,\Phi_{i+1} = 0 \tag{3.5.86}$$

also with $Q \leftrightarrow \widetilde{Q}$, and

$$\begin{aligned} k_{i-1}\Phi_{i-1} &= \mu_{i-1} - \mu_{i-2} + \nu_{i-1} \\ k_i\Phi_i &= \mu_i - \mu_{i-1} + \nu_i \\ k_{i+1}\Phi_{i+1} &= \mu_{i+1} - \mu_i + \nu_{i+1}\,. \end{aligned} \tag{3.5.87}$$

where we define

$$\mu_j = (A_j\widetilde{A}_j,\, |A_j|^2 - |\widetilde{A}_j|^2)\,, \quad \nu_j = (Q_j\widetilde{Q}_j,\, |Q_j|^2 - |\widetilde{Q}_j|^2) \tag{3.5.88}$$

The R-charge of the monopole operators $V_{\boldsymbol{m}}$ with flux $\boldsymbol{m} = (m_1, \ldots, m_n)$ is

$$R[V_{\boldsymbol{m}}] = \frac{1}{2}\left(\sum_{i=1}^{n-1} |m_{i+1} - m_i| + \sum_{i=1}^{n} f_i\,|m_i|\right) \tag{3.5.89}$$

Equation (3.5.86) admits two non-trivial possibilities:

$$\Phi_i = 0 \quad \text{or} \quad Q_i = \widetilde{Q}_i = 0\,. \tag{3.5.90}$$

If we set $Q_i = \widetilde{Q}_i = 0$, the analysis is similar to the linear quiver without flavours. We will instead focus on $\Phi_i = 0$. The remaining constraints in (3.5.85) and (3.5.86) are thus:

$$\begin{aligned} A_{i-1}\,\Phi_{i-1} &= 0\,, \quad A_i\Phi_{i+1} = 0 \\ Q_{i-1}\,\Phi_{i-1} &= 0\,, \quad Q_{i+1}\Phi_{i+1} = 0\,, \end{aligned} \tag{3.5.91}$$

also with $A \leftrightarrow \widetilde{A}$, $Q \leftrightarrow \widetilde{Q}$. Each column of previous set of equations admit two solutions:

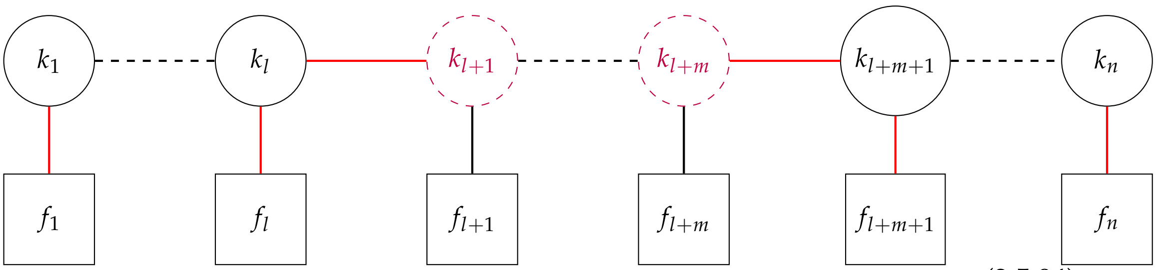
$$\begin{aligned} \Phi_{i-1} &= 0 \quad or \quad \{A_{i-1} = 0\,, Q_{i-1} = 0\} \\ \Phi_{i+1} &= 0 \quad or \quad \{A_{i} \; = 0\,, Q_{i+1} = 0\} \end{aligned} \tag{3.5.92}$$

The case $\{A_{i-1} = 0\,, Q_{i-1} = 0\}$ obviously induce a cut in the quiver and set to zero the adjacent fundamental matter; the same for $\{A_i = 0\,, Q_{i+1} = 0\}$. Let us focus on $\Phi_{i-1} = \Phi_{i+1} = 0$. Now, we have the vacuum equations

$$\begin{aligned} A_{i-2}\,\Phi_{i-2} &= 0\,, \quad A_{i+1}\Phi_{i+2} = 0 \\ Q_{i-2}\,\Phi_{i-2} &= 0\,, \quad Q_{i+2}\Phi_{i+2} = 0 \end{aligned} \tag{3.5.93}$$

Again, the solutions that do not induce a cut are $\Phi_{i+2} = \Phi_{i-2} = 0$ and so on.

The above procedure divides the initial quiver in “Higgs" and “Coulomb" sub-quivers, defined as follows. In the Coulomb one, fundamental matter is set to zero while in Higgs one, all the vector multiplet scalar are set to zero. For instance, we divide the following quiver such that the the first l nodes constitute a Coulomb sub-quiver, the $(l+1)$-th to the $(l+m)$-th nodes constitute a Higgs sub-quiver, and the $(l+m+1)$-th to the (n)-th nodes constitute a Coulomb sub-quiver.



(3.5.94)

where the purple nodes indicate that $\Phi_i = 0$ (with $i = l+1, \ldots, l+m$), and the red lines indicate that $Q_j = \widetilde{Q}_j = 0$ (with $j = 1, \ldots, l, l+m+1, \ldots, n$) and $A_l = \widetilde{A}_l = A_{l+m} = \widetilde{A}_{l+m} = 0$ (we shall discuss about this later). For the sake of readability, in the above diagram, we indicate only the CS level in each circular node and omit the rank, which is 1 for each $U(1)$ gauge group.

Since in the Higgs sub-quiver, $\Phi_i = 0$ for all $i = l+1, \ldots, l+m$; as a consequence, the magnetic flux is set to zero for all gauge nodes in the sub-quiver. Thus, introducing a cut within the Higgs sub-quiver does not produce anything new. For simplicity, we also assume that there is no further cut in the Coulomb branch sub-quiver.

Moreover, a Higgs sub-quiver cannot end with a node without flavours. This can be seen as follows. Suppose, on the contrary, that we cut the quiver at the $(l+m)$-th position, namely set $A_{l+m} = \widetilde{A}_{l+m} = 0$, with $f_{l+m} = 0$. In this case, (3.5.87) implies:

$$k_{l+m}\,\Phi_{l+m} = A_{l+m}\widetilde{A}_{l+m} - A_{l+m-1}\widetilde{A}_{l+m-1} + Q_{l+m}\widetilde{Q}_{l+m}\,. \tag{3.5.95}$$

Since we cut the quiver at the $(l+m)$-th position, $A_{l+m} = \widetilde{A}_{l+m} = 0$. We also have $Q_{l+m} = \widetilde{Q}_{l+m} = 0$ since $f_{l+m} = 0$. Also, $\Phi_{l+m} = 0$ since we are looking at the Higgs sub-quiver. Thus the previous condition becomes:

$$A_{l+m-1}\widetilde{A}_{l+m-1} = 0 \tag{3.5.96}$$

implying a cut at A_{l+m-1}. This procedure must be continued until we have $f_i \neq 0$.

Let us assume that f_{l+1} and f_{l+m} are non-zero. In transiting from the Coulomb sub-quiver to Higgs sub-quiver and vice-versa, we need to introduce a cut at the transition point; this is because from (3.5.85), we have, *e.g.*, $0 = A_l(\Phi_l - \Phi_{l+1}) = A_l\Phi_l$ which indeed implies $A_l = 0$. Indeed we need to set

$$A_l = \widetilde{A}_l = 0\,, \qquad A_{l+m} = \widetilde{A}_{l+m} = 0\,. \tag{3.5.97}$$

In the Higgs sub-quiver, we have the vacuum equation

$$\begin{aligned} A_{l+1}\widetilde{A}_{l+1} + Q_{l+1}\widetilde{Q}_{l+1} &= 0 \\ A_{l+2}\widetilde{A}_{l+2} - A_{l+1}\widetilde{A}_{l+1} + Q_{l+2}\widetilde{Q}_{l+2} &= 0 \\ &\vdots \\ -A_{l+m}\widetilde{A}_{l+m} + Q_{l+m}\widetilde{Q}_{l+m} &= 0\,, \end{aligned} \tag{3.5.98}$$

whereas in the Coulomb sub-quiver, we have

$$\begin{aligned} A_1\widetilde{A}_1 &= k_1\varphi_L \\ A_2\widetilde{A}_2 - A_1\widetilde{A}_1 &= k_2\varphi_L \\ &\vdots \\ -A_l\widetilde{A}_l &= k_l\varphi_L\,, \end{aligned} \tag{3.5.99}$$

and

$$\begin{aligned} A_{l+m+1}\widetilde{A}_{l+m+1} &= k_{l+m+1}\varphi_R \\ A_{l+m+2}\widetilde{A}_{l+m+2} - A_{l+m+1}\widetilde{A}_{l+m+1} &= k_{l+m+2}\varphi_R \\ &\vdots \\ -A_{n-1}\widetilde{A}_{n-1} &= k_n\varphi_R \end{aligned} \tag{3.5.100}$$

The sums of these two sets of equations tell us that necessary conditions for the existence of non-trivial moduli spaces of the Coulomb sub-quivers are

$$\sum_{i=1}^{l} k_i = 0\,, \qquad \sum_{j=l+m+1}^{n} k_j = 0\,. \tag{3.5.101}$$

The gauge charge of the monopole operator $V_{\boldsymbol{m}}$ with flux

$$\boldsymbol{m} = (\underbrace{\mathsf{m}_L, \ldots, \mathsf{m}_L}_{l}, \underbrace{0, \ldots, 0}_{m}, \underbrace{\mathsf{m}_R, \ldots, \mathsf{m}_R}_{n-l-m}) \equiv (\mathsf{m}_L^l, 0^m, \mathsf{m}_R^{n-l-m})\,, \tag{3.5.102}$$

where 0 is the flux for each gauge group in the Higgs sub-quivers and m is the flux for each gauge group in the Coulomb sub-quiver, is

$$\begin{aligned} q_i[V_{\boldsymbol{m}}] &= -k_i\mathsf{m}_L && \text{for } i = 1, \ldots, l\,, \\ q_p[V_{\boldsymbol{m}}] &= 0 && \text{for } p = l+1, \ldots, l+m \\ q_j[V_{\boldsymbol{m}}] &= -k_j\mathsf{m}_R && \text{for } j = l+m+1, \ldots, n \end{aligned} \tag{3.5.103}$$

The R-charge of the monopole operator $V_{\boldsymbol{m}}$ is

$$\begin{aligned} R[V_{\boldsymbol{m}}] &= \frac{1}{2}|\mathsf{m}_L - 0| + \frac{1}{2}\left(|\mathsf{m}_L|\sum_{i=1}^{l} f_i + |\mathsf{m}_R|\sum_{j=l+m+1}^{n} f_j\right) + \frac{1}{2}|0 - \mathsf{m}_R| \\ &\equiv \frac{1}{2}|\mathsf{m}_L|(F_L+1) + \frac{1}{2}|\mathsf{m}_R|(F_R+1)\,, \end{aligned} \tag{3.5.104}$$

where we define $F_{L,R}$ as the total number of flavours in the left and right Coulomb sub-quivers:

$$F_L = \sum_{i=1}^{l} f_i\,, \qquad F_R = \sum_{j=l+m+1}^{n} f_j\,. \tag{3.5.105}$$

The Hilbert series for the Higgs sub-quiver can be written as

$$\begin{aligned} &H_{\text{Higgs}}(t; \boldsymbol{x}^{(l+1)}, \ldots, \boldsymbol{x}^{(l+m)}) \\ &= (1-t^2)^m \prod_{j=l+1}^{l+m} \oint \frac{dq_i}{2\pi i q_j} \text{PE}\left[t\sum_{\alpha=1}^{f_j}\left(q_j(x_\alpha^{(j)})^{-1} + q_j^{-1}(x_\alpha^{(j)})\right)\right] \\ &\quad \prod_{i=l+1}^{l+m-1} \text{PE}[t(q_i\, q_{i+1}^{-1} + q_i^{-1}\, q_{i+1})] \end{aligned} \tag{3.5.106}$$

where the first PE is related to fundamental matter and the second one to bi-fundamental matter; the overall $(1-t^2)^m$ is due to the m F-term constraints. Observe that the Hilbert series of this sub-quiver does not depend on the CS levels. It is also worth noting that (3.5.106) takes the same form as the Higgs branch Hilbert series of 3d $\mathcal{N}=4$ $T_{\boldsymbol{\rho}}^{\boldsymbol{\sigma}}(SU(N))$ theory [8] for some $\boldsymbol{\sigma}$ and $\boldsymbol{\rho}$ [95]; for example, for $m=3$ and $f_{l+1} = f_{l+2} = f_{l+3} = 1$, (3.5.106) is equal to the Higgs branch Hilbert series of $T_{(2^2,1^2)}^{(3,2,1)}(SU(6))$.

Let us now focus on the Coulomb sub-quiver. The analysis is very similar to that described in the case without flavours, discussed earlier. We emphasise that even if all the fundamental matter is set to zero, it still contributes to the dimension of the monopole operators. For example, if there is no cut in the left and right Coulomb sub-quivers in (3.5.94), the baryonic generating function of each of these Coulomb sub-quivers are similar to (3.5.40):

$$G_{\text{Coulomb}}^{L,R}(t;m) = \frac{1}{1-t^2} t^{|m|K_{L,R}} \tag{3.5.107}$$

where

$$K_L = \sum_{i=1}^{l} |\sum_{j=1}^{i} k_j|\,, \qquad K_R = \sum_{i=l+m+1}^{n} |\sum_{j=l+m+1}^{i} k_j|\,. \tag{3.5.108}$$

The total Hilbert series of (3.5.94) is therefore

$$\begin{aligned} H(t;\boldsymbol{x}) &= H_{\text{Higgs}}(t;\{\boldsymbol{x}^{(i)}\}) \sum_{\mathsf{m}_L\in\mathbb{Z}}\sum_{\mathsf{m}_R\in\mathbb{Z}} t^{(F_L+1)|\mathsf{m}_L| + (F_R+1)|\mathsf{m}_R|} z_L^{\mathsf{m}_L} z_R^{\mathsf{m}_R} \\ &\qquad \times G_{\text{Coulomb}}^{L}(t;\mathsf{m}_L) G_{\text{Coulomb}}^{R}(t;\mathsf{m}_R) \\ &= H_{\text{Higgs}}(t;\{\boldsymbol{x}^{(\alpha)}\}) \left[\sum_{\mathsf{m}_L\in\mathbb{Z}} \frac{1}{1-t^2} t^{(F_L+K_L+1)|\mathsf{m}_L|} z_L^{\mathsf{m}_L}\right] (L \leftrightarrow R) \\ &= H_{\text{Higgs}}(t;\{\boldsymbol{x}^{(\alpha)}\}) H[\mathbb{C}^2/\mathbb{Z}_{F_L+K_L+1}](t,z_L) H[\mathbb{C}^2/\mathbb{Z}_{F_R+K_R+1}](t,z_R) \end{aligned} \tag{3.5.109}$$

where

$$H[\mathbb{C}^2/\mathbb{Z}_{F_L+K_L+1}](t, z_L) = t^2 + (z_L + z_L^{-1})t^{F_L+K_L+1} - t^{2(F_L+K_L+1)} . \tag{3.5.110}$$

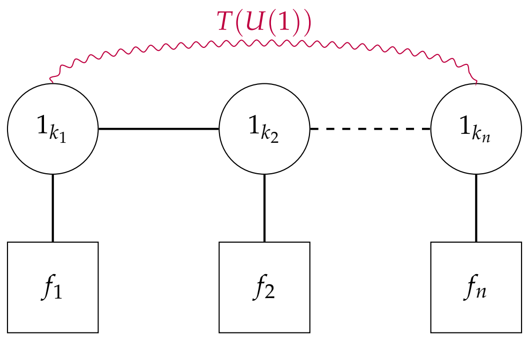
and the same for $(L \leftrightarrow R)$. The moduli space of quiver (3.5.94) is therefore

$$(\mathbb{C}^2/\mathbb{Z}_{F_L+K_L+1}) \times \mathcal{M}_{\text{Higgs}} \times (\mathbb{C}^2/\mathbb{Z}_{F_R+K_R+1}) , \tag{3.5.111}$$

where $\mathcal{M}_{\text{Higgs}}$ denotes the moduli space of the Higgs sub-quiver, which is isomorphic to the Higgs branch moduli space of $T^{\sigma}_{\rho}(SU(N))$ for some appropriate N, σ and ρ.

3.5.4 Adding flavour with one J-fold

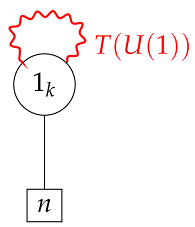
Now we want to study the branches of a theory with one J-fold and fundamental matter:



(3.5.112)

If all Φ_i (with $i = 1, \ldots, n$) are set to zero and the presence of the $T(U(1))$ link does not affect the moduli space, the analysis is the same as that discussed in the previous subsection. On the other hand, if all Q_i and $\widetilde{Q}_i$ are set to zero, the analysis is similar to that discussed in section 3.5.2; one needs to take into account of the contribution from the fundamental matter to the R-charge of the monopole operator.

Example. Let us consider a simple example with a $U(1)_k$ gauge group, one $T(U(1))$ link and n flavours.



(3.5.113)

It is not possible to introduce a cut to this quiver. $T(U(1))$ is an almost empty theory; it contributes the CS level -2 to the $U(1)$ gauge group, so effectively the CS level is $k - 2$.

$$W = \widetilde{Q}_i \varphi Q^i + \frac{1}{2}(k-2)\varphi^2 , \quad i = 1, \ldots, n . \tag{3.5.114}$$

We have the F-term equations:

$$\widetilde{Q}_i Q^i + (k-2)\varphi = 0 , \qquad \widetilde{Q}_i \varphi = 0 , \qquad \varphi Q^i = 0 . \tag{3.5.115}$$

The vacuum equations involving the real scalar field σ in the vector multiplet is

$$Q^i\sigma = \sigma\widetilde{Q}_i = 0\,. \tag{3.5.116}$$

The D-term equation reads

$$(Q^\dagger)_i Q^i - \widetilde{Q}_i(\widetilde{Q}^\dagger)^i = (k-2)\sigma\,. \tag{3.5.117}$$

If $k = 2$, the superpotential and the moduli space are the same as that of 3d $\mathcal{N} = 4$ $U(1)$ gauge theory with n flavours. The F-term with respect to ϕ implies that $\widetilde{Q}_i Q^i = 0$. The Higgs branch is generating by the mesons $M^i_j = Q^i\widetilde{Q}_j$; this meson matrix has rank at most 1 and subject to the matrix relation $M^2 = 0$, which follows from the F-term. Thus, the Higgs branch is isomorphic to the closure of the minimal nilpotent orbit of $SU(n)$. On the other hand, the Coulomb branch of this theory is $\mathbb{C}^2/\mathbb{Z}_n$; this is generated by the monopole operators V_+ and V_-, carrying the topological charges ± 1 and R-charges $\frac{1}{2}n$, subject to the relation $V_+V_- = \varphi^n$. Note that for $n = 1$ and $k = 2$, this theory has no Higgs branch and its Coulomb branch is isomorphic to $\mathbb{C}^2$.

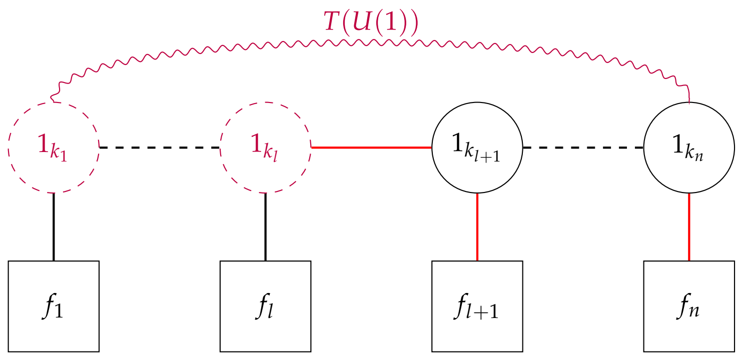
Let us now suppose that $k \neq 2$. If (φ, σ) is non-zero, (3.5.115) and (3.5.116) implies Q^i and $\widetilde{Q}_i$ are zero, but this is in contradiction with the D-term. Hence $(\varphi, \sigma) = 0$ and the Coulomb branch is trivial in this case. However, there is still the Higgs branch generated by $M^i_j = Q^i\widetilde{Q}_j$. As before, this meson matrix has rank at most 1 and subject to the matrix relation $M^2 = 0$ (since $\widetilde{Q}_i Q^i = 0$). The Higgs branch is therefore isomorphic to the closure of the minimal nilpotent orbit of $SU(n)$. Note that for $n = 1$ and $k \neq 2$, this theory has a trivial moduli space.

The case with one cut

For simplicity, let us first focus on the case of *precisely one cut*. In this case we have two sub-quivers, left and right, connected by the $T(U(1))$ link. We have three possibilities:

- Both the sub-quivers are in the Coulomb sector: this require the usual analysis as in section 3.5.2.
- Both the sub-quivers are in the Higgs sector: all Φ_i are set to zero and the T-link does not affect the moduli space.
- One is a Higgs sub-quiver (say, the left one) and the other is a Coulomb sub-quiver (say, the right one).

The last case is the interesting one.



(3.5.118)

where the dashed circles mean that their vector multiplet scalars are zero, and the red lines mean that the hypermultiplets are set to zero:

$$\Phi_1 = \Phi_2 = \ldots = \Phi_l = 0\,, \tag{3.5.119}$$

The first set of vacuum equations are

$$A_j(\Phi_{j+1} - \Phi_j) = \widetilde{A}_j(\Phi_{j+1} - \Phi_j) = 0\,, \quad j = 1, \ldots, n-1 \tag{3.5.120}$$

As a consequence, we see that

$$\begin{aligned} &\Phi_{l+1} = \Phi_{l+2} = \ldots = \Phi_n = \Phi = (\varphi, \sigma) \\ &A_l = \widetilde{A}_l = 0\,, \end{aligned} \tag{3.5.121}$$

The latter set of equations say that we need to introduce a cut in transiting from the Higgs sub-quiver to the Coulomb sub-quiver and vice-versa. The other vacuum equations are

$$\begin{aligned} A_1\widetilde{A}_1 + Q_1\widetilde{Q}_1 &= -\varphi \\ A_2\widetilde{A}_2 - A_1\widetilde{A}_1 + Q_2\widetilde{Q}_2 &= 0 \\ A_3\widetilde{A}_3 - A_2\widetilde{A}_2 + Q_3\widetilde{Q}_3 &= 0 \\ &\vdots \\ A_l\widetilde{A}_l - A_{l-1}\widetilde{A}_{l-1} + Q_l\widetilde{Q}_l &= 0 \end{aligned} \tag{3.5.122}$$

and

$$\begin{aligned} A_{l+1}\widetilde{A}_{l+1} - A_l\widetilde{A}_l &= k_{l+1}\varphi \\ A_{l+2}\widetilde{A}_{l+2} - A_{l+1}\widetilde{A}_{l+1} &= k_{l+2}\varphi \\ &\vdots \\ -A_{n-1}\widetilde{A}_{n-1} &= k_n\varphi \end{aligned} \tag{3.5.123}$$

where the contribution from the $T(U(1))$ link is denoted in blue. We denote the vanishing terms in grey in (3.5.122) and (3.5.123). The sum of (3.5.122) gives

$$\varphi = -\sum_{i=1}^{l} Q_i\widetilde{Q}_i\,. \tag{3.5.124}$$

Moreover, a necessary condition for a non-trivial moduli space for the Coulomb sub-quiver can be determined by summing (3.5.123) and requiring that $\varphi \neq 0$:

$$\sum_{i=l+1}^{n} k_i = 0\,. \tag{3.5.125}$$

The gauge charge of the monopole operator $V_{\boldsymbol{m}}$ with flux

$$\boldsymbol{m} = (\underbrace{0, \ldots, 0}_{l}, \underbrace{m, \ldots, m}_{n-l}) \equiv (0^l, m^{n-l})\,, \tag{3.5.126}$$

where 0 is the flux for each gauge group in the Higgs sub-quiver and m is the flux for each gauge group in the Coulomb sub-quiver, is

$$\begin{aligned} q_1[V_{\boldsymbol{m}}] &= m\,, \quad q_j[V_{\boldsymbol{m}}] \;=\; 0 \;\text{ for } j=2,\ldots,l\,, \\ q_p[V_{\boldsymbol{m}}] &= -k_p m \;\text{ for } p=l+1,\ldots,n\,. \end{aligned} \tag{3.5.127}$$

The R-charge of the monopole operator $V_{\boldsymbol{m}}$ is

$$R[V_{\boldsymbol{m}}] = \frac{1}{2}|m-0| + \frac{1}{2}|m| \sum_{i=l+1}^{n} f_i \equiv \frac{1}{2}|m|(F_C+1)\,, \tag{3.5.128}$$

where we define F_C as the total number of flavours in the Coulomb sub-quiver:

$$F_C = \sum_{i=l+1}^{n} f_i\,. \tag{3.5.129}$$

We can construct the dressed monopole operators that are gauge invariant as follows.

$$\begin{aligned} \overline{V}^{(\alpha)}_{+} &= V_{(0^l,1^{n-l})}(\widetilde{Q}_\alpha \widetilde{A}_{\alpha-1}\widetilde{A}_{\alpha-2}\cdots\widetilde{A}_1)\left(A_{l+1}^{K_{l+1}}\,A_{l+2}^{K_{l+2}}\cdots A_{n-1}^{K_{n-1}}\right) \\ \overline{V}^{(\alpha)}_{-} &= V_{(0^l,1^{n-l})}(A_1 A_2\ldots A_{\alpha-1}Q_\alpha)\left(\widetilde{A}_{l+1}^{K_{l+1}}\,\widetilde{A}_{l+2}^{K_{l+2}}\ldots\widetilde{A}_{n-1}^{K_{n-1}}\right)\,. \end{aligned} \tag{3.5.130}$$

where $\alpha = 1,\ldots,l$ and

$$K_i \;=\; \sum_{p=l+1}^{i} k_p\,, \;\;\text{for } i=l+1,\ldots,n\,. \tag{3.5.131}$$

Note that if $K_j<0$ for some j, we replace $A_j^{K_j}$ in the first equation by $\widetilde{A}_j^{-K_j}$, and $\widetilde{A}_j^{K_j}$ in the second equation by $A_j^{-K_j}$. The R-charges of $V^{(\alpha)}_{\pm}$ are

$$R[V^{(\alpha)}_{\pm}] = \frac{1}{2}\left[(F_C+1)+\alpha+\sum_{p=l+1}^{n-1}|K_p|\right] = \frac{1}{2}\left[(F_C+1)+\alpha+K\right]\,, \tag{3.5.132}$$

with

$$K \equiv \sum_{p=l+1}^{n-1} |K_p|\,. \tag{3.5.133}$$

As in the preceding subsection, if $f_l = 0$ (which means $Q_l = \widetilde{Q}_l = 0$), then the Higgs sub-quiver cannot end at the l-th position because from (3.5.122) we have $A_{l-1}\widetilde{A}_{l-1} = 0$, *i.e.* we need to introduce a cut at the $(l-1)$-th position. However, if $f_1 = 0$ (which means $Q_1 = \widetilde{Q}_1 = 0$), the Higgs sub-quiver still can end at the 1st position because $A_1\widetilde{A}_1 = -\varphi$.

The Hilbert series of quiver (3.5.118) can be obtained as follows. The baryonic generating function for the Higgs sub-quiver is

$$
\begin{aligned}
&G_{\text{Higgs}}(t; \boldsymbol{x}^{(1)}, \ldots, \boldsymbol{x}^{(l)}; m) \\
&= (1-t^2)^l \oint \frac{dq_1}{2\pi i q_1^{1+m}} \, \mathrm{PE}\left[t \sum_{\alpha=1}^{f_1} \left(q_1 (x_\alpha^{(1)})^{-1} + q_1^{-1} x_\alpha^{(1)} \right) \right] \times \\
&\prod_{j=2}^{l} \oint \frac{dq_j}{2\pi i q_j} \, \mathrm{PE}\left[t \sum_{\alpha=1}^{f_j} \left(q_j (x_\alpha^{(j)})^{-1} + q_j^{-1} x_\alpha^{(j)} \right) \right] \prod_{i=1}^{l-1} \mathrm{PE}[t(q_i \, q_{i+1}^{-1} + q_i^{-1} \, q_{i+1})] \,,
\end{aligned}
\tag{3.5.134}
$$

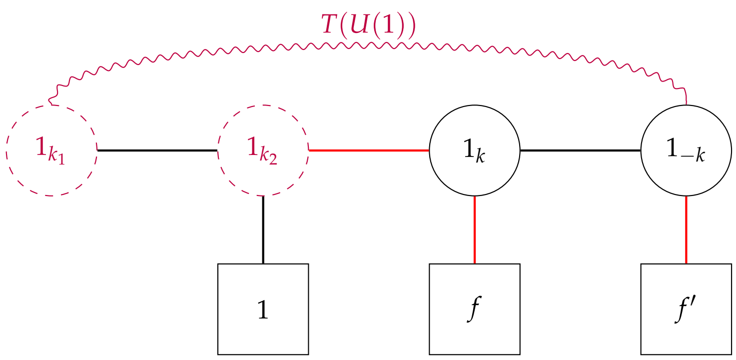
where we indicated m in blue to emphasise that this is due to the presence of the $T(U(1))$ link. The baryonic generating for the Coulomb sub-quiver is similar to (3.5.40):

$$
G_{\text{Coulomb}}(t; m) = \frac{1}{1-t^2} t^{|m|K} \,. \tag{3.5.135}
$$

The total Hilbert series of (3.5.118) is therefore

$$
\begin{aligned}
&H(t; \{\boldsymbol{x}^{(i)}\}, z; m) \\
&= \sum_{m \in \mathbb{Z}} t^{(F_C+1)|m|} G_{\text{Higgs}}(t; m) G_{\text{Coulomb}}(t; m) z^m \\
&= \sum_{m \in \mathbb{Z}} t^{(K+F_C+1)|m|} z^m (1-t^2)^{l-1} \oint \frac{dq_1}{2\pi i q_1^{1+m}} \, \mathrm{PE}\left[t \sum_{\alpha=1}^{f_1} \left(q_1 (x_\alpha^{(1)})^{-1} + q_1^{-1} x_\alpha^{(1)} \right) \right] \times \\
&\prod_{j=2}^{l} \oint \frac{dq_j}{2\pi i q_j} \, \mathrm{PE}\left[t \sum_{\alpha=1}^{f_j} \left(q_j (x_\alpha^{(j)})^{-1} + q_j^{-1} x_\alpha^{(j)} \right) \right] \prod_{i=1}^{l-1} \mathrm{PE}[t(q_i \, q_{i+1}^{-1} + q_i^{-1} \, q_{i+1})] \,.
\end{aligned}
\tag{3.5.136}
$$

Example. Let us consider the following quiver



(3.5.137)

Assume that $k \geq 0$. In this case, we have $K = k$ and $F_C = f + f'$. The Hilbert series is then

$$
\begin{aligned}
H_{(3.5.137)}(t; x^{(2)}) &= \sum_{m \in \mathbb{Z}} t^{(k+F_C+1)|m|} z^m (1-t^2) \oint \frac{dq_1}{2\pi i q_1^{1+m}} \oint \frac{dq_2}{2\pi i q_2} \\
&\quad \mathrm{PE}\left[t \left(q_2 (x^{(2)})^{-1} + q_2^{-1} x^{(2)} \right) \right] \mathrm{PE}[t(q_1 \, q_2^{-1} + q_1^{-1} \, q_2)] \\
&= \mathrm{PE}\left[t^2 + \left(x^{(2)} z^{-1} + (x^{(2)})^{-1} z \right) t^{3+k+F_C} - t^{2(3+k+F_C)} \right] \,.
\end{aligned}
\tag{3.5.138}
$$

Hence, the moduli space of this quiver is $\mathbb{C}^2/\mathbb{Z}_{3+k+F_C}$. It is generated by φ and $\overline{V}^{(2)}_{\pm}$, where

$$\overline{V}^{(2)}_{+} = V_{(0,0,1,1)}\widetilde{Q}_2\widetilde{A}_1 A_3^k\,, \quad \overline{V}^{(2)}_{-} = V_{(0,0,-1,-1)}A_1 Q_2 \widetilde{A}_3^k\,, \tag{3.5.139}$$

subject to the relation

$$\overline{V}^{(2)}_{+}\overline{V}^{(2)}_{-} = \varphi^{3+k+F_C}\,. \tag{3.5.140}$$

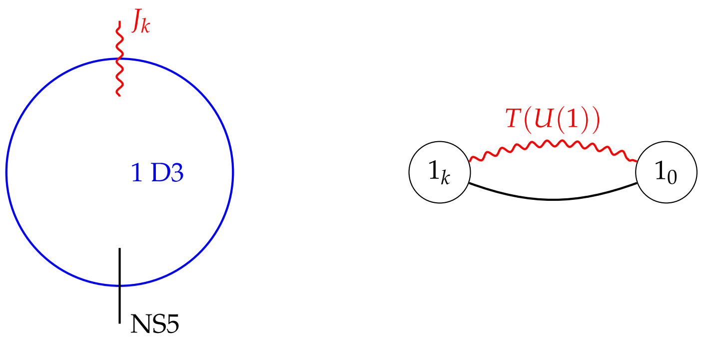
The case with more than one cuts

In this case, the original quiver is divided into many sub-quivers. The parts that are not connected to $T(U(1))$ can be analysed as in section 3.5.3, and the parts that are connected to $T(U(1))$ can be analysed as in section 3.5.4.

3.5.5 More examples

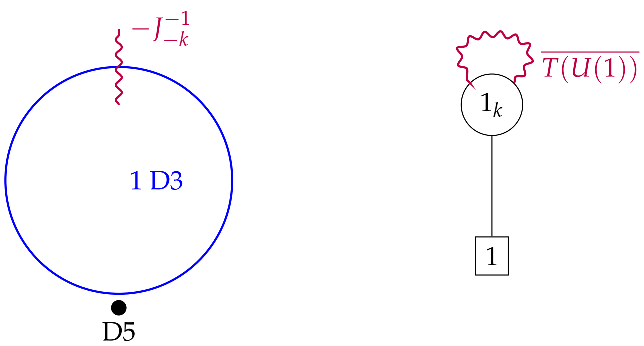
One J_k fold and one NS5 or D5-brane

Let us consider the following model:



(3.5.141)

Upon applying S-duality to the above system, we obtain

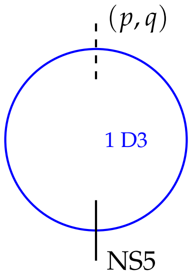


(3.5.142)

Both of these models are analysed in detail around (3.5.82) and (3.5.113), respectively. The moduli spaces these model are non-trivial if and only if $k = 2$. In which case, they are isomorphic to $\mathbb{C}^2$.

One (p, q)-brane and one NS5-brane

The techniques that we introduced in the section 3.5 are particularly useful to study in a systematic way the moduli space of quiver gauge theories associated to (p, q)-brane systems. Let us consider for instance the following brane system

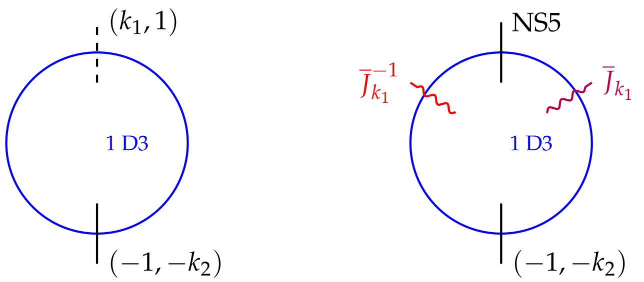


(3.5.143)

For simplicity, let us take (p, q) to be the following value: $(p, q) = \overline{J}_{k_3}\overline{J}_{k_2}\overline{J}_{k_1}(1, 0)$, so that

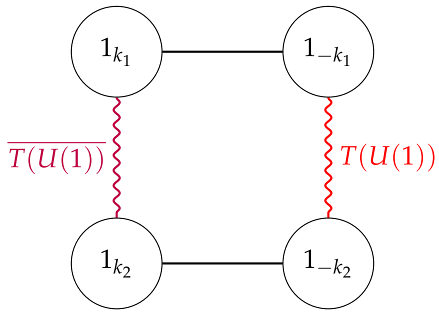
$$p = k_1k_2k_3 - k_1 - k_3 \,, \qquad q = k_1k_2 - 1 \,. \tag{3.5.144}$$

Performing a duality transformation, $\overline{J}_{k_2}^{-1}\overline{J}_{k_3}^{-1}$, we can study the following $SL(2, \mathbb{Z})$ equivalent problem:



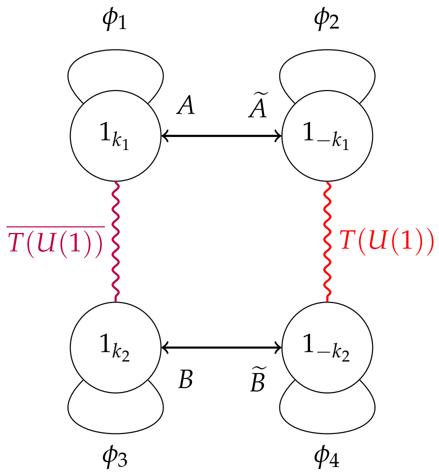
(3.5.145)

The associated quiver is



(3.5.146)

In $\mathcal{N} = 2$ language, this can be written as



(3.5.147)

The vacuum equations are

$$\begin{aligned} &A(\varphi_1 - \varphi_2) \;=\; 0 \;=\; \widetilde{A}(\varphi_1 - \varphi_2)\,, && B(\varphi_3 - \varphi_4) \;=\; 0 \;=\; \widetilde{B}(\varphi_3 - \varphi_4) \\ &k_1\varphi_1 - \varphi_3 \;=\; A\,\widetilde{A}\,, && k_2\varphi_3 - \varphi_1 \;=\; B\,\widetilde{B} \\ &-k_1\varphi_2 + \varphi_4 \;=\; -\,A\,\widetilde{A}\,, && -k_2\varphi_4 + \varphi_2 \;=\; -\,B\,\widetilde{B}\,. \end{aligned} \tag{3.5.148}$$

where we emphasised the contributions due to the mixed CS levels in blue. We have two branches as will be analysed as follow.

Branch I: $A\widetilde{A} \neq 0$ and $B\widetilde{B} \neq 0$

In this case the F-terms implies:

$$\varphi_1 \;=\; \varphi_2 \;=\; \varphi\,, \qquad \varphi_3 \;=\; \varphi_4 \;=\; \widetilde{\varphi}\,; \tag{3.5.149}$$

moreover, two constraints are still present, fixing $\varphi\,,\,\widetilde{\varphi}$ in terms of the mesons:

$$k_1\varphi - \widetilde{\varphi} \;=\; A\widetilde{A}\,, \quad k_2\widetilde{\varphi} - \varphi \;=\; B\widetilde{B}\,. \tag{3.5.150}$$

An analogous analysis of the D-terms can be performed. The flux $\boldsymbol{m}$ for the monopole operator $V_{\boldsymbol{m}}$ takes the form

$$\boldsymbol{m} = (m, m, \widetilde{m}, \widetilde{m})\,. \tag{3.5.151}$$

The gauge charges and the R-charges of $V_{\boldsymbol{m}}$ are

$$\begin{aligned} q_1[V_{\boldsymbol{m}}] \;&=\; -q_2[V_{\boldsymbol{m}}] \;=\; -(k_1 m - \widetilde{m})\,, \\ q_3[V_{\boldsymbol{m}}] \;&=\; -q_4[V_{\boldsymbol{m}}] \;=\; -(k_2\widetilde{m} - m)\,. \end{aligned} \tag{3.5.152}$$

and

$$R[V_{\boldsymbol{m}}] = 0\,. \tag{3.5.153}$$

Let us now determine the moduli space and compute the Hilbert series of this theory. The baryonic generating function is given by

$$\begin{aligned} G(t;B,\widetilde{B}) &= \left(\prod_{i=1}^{4}\oint \frac{dq_i}{2\pi i q_i}\right)\frac{1}{q_1^B q_2^{-B} q_3^{\widetilde{B}} q_4^{-\widetilde{B}}}\,\mathrm{PE}[t(q_1q_2^{-1}+q_2q_1^{-1})]\,\mathrm{PE}[t(q_3q_4^{-1}+q_4q_3^{-1})] \\ &= g_{\mathrm{ABJM}/2}(t;B)\,g_{\mathrm{ABJM}/2}(t;\widetilde{B})\,. \end{aligned} \tag{3.5.154}$$

where

$$g_{\mathrm{ABJM}/2}(t;B) = \frac{t^{|B|}}{1-t^2}\,. \tag{3.5.155}$$

The Hilbert series of (3.5.147) is thus:

$$\begin{aligned} H_{(3.5.147)}(t,z) &= \sum_{m\in\mathbb{Z}}\sum_{\widetilde{m}\in\mathbb{Z}} z^{m+\widetilde{m}} g_{\mathrm{ABJM}/2}(t;k_1m-\widetilde{m})g_{\mathrm{ABJM}/2}(t;k_2\widetilde{m}-m) \\ &= \sum_{m\in\mathbb{Z}}\sum_{\widetilde{m}\in\mathbb{Z}} z^{m+\widetilde{m}}\frac{t^{|k_1m-\widetilde{m}|}}{1-t^2}\frac{t^{|k_2\widetilde{m}-m|}}{1-t^2}\,. \end{aligned} \tag{3.5.156}$$

This turns out to be equal to

$$\begin{aligned} H_{(3.5.147)}(t,z) &= \frac{1}{k_1k_2-1}\sum_{j=1}^{k_1k_2-1}\frac{1}{(1-t\,u_j)(1-t\,w_j)}\frac{1}{(1-t/u_j)(1-t/w_j)} \\ &= H[\mathbb{C}^4/\Gamma(k_1,k_1k_2-1)](t,z)\,, \end{aligned} \tag{3.5.157}$$

where

$$u_j = z^{\frac{k_1+1}{k_1k_2-1}}\,e^{j\frac{2\pi i k_1}{k_1k_2-1}}\,, \qquad w_j = z^{\frac{k_2+1}{k_1k_2-1}}\,e^{j\frac{2\pi i}{k_1k_2-1}}\,. \tag{3.5.158}$$

This is the Molien formula for the Hilbert series of $\mathbb{C}^4/\Gamma(\mathsf{p},\mathsf{q})$ [96], with $\mathsf{p}=k_1$ and $\mathsf{q}=k_1k_2-1$, where $\Gamma(\mathsf{p},\mathsf{q})$ is a discrete group acting on the four complex coordinate of $\mathbb{C}^4$ as:

$$\Gamma(\mathsf{p},\mathsf{q}):\ \ (z_1\,,z_2\,,z_3\,,z_4)\ \to\ (z_1e^{\frac{2\pi i\mathsf{p}}{\mathsf{q}}}\,,z_2e^{\frac{2\pi i}{\mathsf{q}}}\,,z_3e^{-\frac{2\pi i\mathsf{p}}{\mathsf{q}}}\,,z_4e^{-\frac{2\pi i}{\mathsf{q}}})\,. \tag{3.5.159}$$

This is in agreement with [97, 90].

Branch II: $A\widetilde{A} = 0$ or $B\widetilde{B} = 0$

The second branch appears when we set one of the bi-fundamental hypers to zero, say $A\widetilde{A} = 0$. In this case, (3.5.148) implies again that:

$$\varphi_1 = \varphi_2 = \varphi\,, \qquad \varphi_3 = \varphi_4 = \widetilde{\varphi}. \tag{3.5.160}$$

Moreover, we have[9]:

$$k_1\varphi = \widetilde{\varphi}\,, \quad k_2\widetilde{\varphi} - \varphi = B\widetilde{B}\,. \tag{3.5.161}$$

Because of $\mathcal{N}=3$ supersymmetry of the problem, the real scalar in the vector multiplet satisfies the same equation as the complex scalar in the vector multiplet. As a consequence, the flux $\boldsymbol{m} = (m,m,\widetilde{m},\widetilde{m})$ of the monopole operator $V_{\boldsymbol{m}}$ has to

[9] A special case is $k_1 = k_2 = \pm 1$. In this case $B\widetilde{B} = 0$ and we are left with φ and the basic monopole operators. The corresponding moduli space is thus simply $\mathbb{C}^2$.

satisfy

$$k_1\, m \;=\; \widetilde{m} \tag{3.5.162}$$

The gauge charges of V_m are

$$\begin{aligned} q_1[V_m] \;&=\; -q_2[V_m] \;=\; -(k_1 m - \widetilde{m}) = 0\,, \\ q_3[V_m] \;&=\; -q_4[V_m] \;=\; -(k_2 \widetilde{m} - m) = -(k_1 k_2 - 1) m\,. \end{aligned} \tag{3.5.163}$$

The R-charge of V_m is $R[V_m] = 0$. The gauge invariant dressed monopole operators are

$$\overline{V}_+ \;=\; V_{(1,1,k_1,k_1)}\, B^{k_1 k_2 - 1}\,, \quad \overline{V}_- \;=\; V_{(-1,-1,-k_1,-k_1)}\, \widetilde{B}^{k_1 k_2 - 1}\,, \tag{3.5.164}$$

for $k_1 k_2 - 1 > 0$. If $k_1 k_2 - 1 < 0$, we replace $B^{k_1 k_2 - 1}$ by $\widetilde{B}^{-(k_1 k_2 - 1)}$ and $\widetilde{B}^{k_1 k_2 - 1}$ by $B^{-(k_1 k_2 - 1)}$ in the above equations. They carry R-charges $R[\overline{V}_\pm] = \frac{|k_1 k_2 - 1|}{2}$. Since $(k_1 k_2 - 1)\varphi = B\widetilde{B}$, we see that these dressed monopole operators satisfy the quantum relation

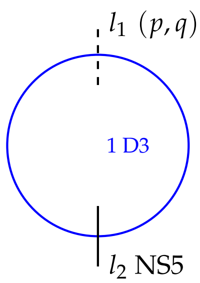
$$\overline{V}_+\, \overline{V}_- \;=\; \varphi^{|k_1 k_2 - 1|}\,. \tag{3.5.165}$$

Hence the moduli space is $\mathbb{C}^2/\mathbb{Z}_{|k_1 k_2 - 1|}$.

Note that (3.5.162) implies that the magnetic lattice given by $\widetilde{m}$ jumps by a multiple of k_1, since $m \in \mathbb{Z}$. If we further require that the magnetic lattice do not jump, we can impose a further condition that $k_1 = \pm 1$. In this case, the brane system contains a $(\pm 1, 1)$-brane and a $(-1, -k_2)$-brane. Applying $T^{\mp 1}$ to this system, $(\pm 1, 1)$ becomes $(\pm 1, 0)$, and $(-1, k_2)$ becomes $(-1, -k_2 \mp 1)$. This gives rise to the ABJM theory with CS level $k_2 - 1$ and $-k_2 + 1$. Indeed, Branch I (which is $\mathbb{C}^4/\mathbb{Z}_{|k_2 - 1|}$) and Branch II (which is $\mathbb{C}^2/\mathbb{Z}_{|k_2 - 1|}$) are the geometric branch of the ABJM theory and the moduli space of the half-ABJM theory, respectively.

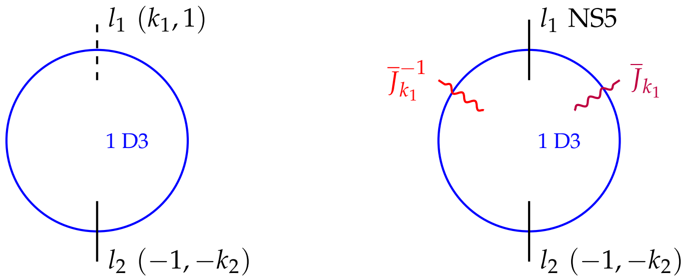
Multiple (p, q) and NS5-branes

An interesting generalisation of the example we presented in the previous subsection is the following brane configuration:



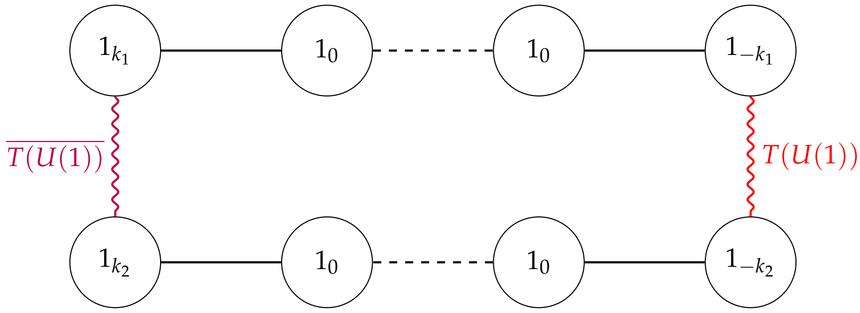
(3.5.166)

As before, let us take for simplicity $(p,q) = \overline{J}_{k_3}\overline{J}_{k_2}\overline{J}_{k_1}(1,0)$. Performing a transformation, $\overline{J}_{k_2}^{-1}\overline{J}_{k_3}^{-1}$, we can study the following $SL(2,\mathbb{Z})$ equivalent systems:



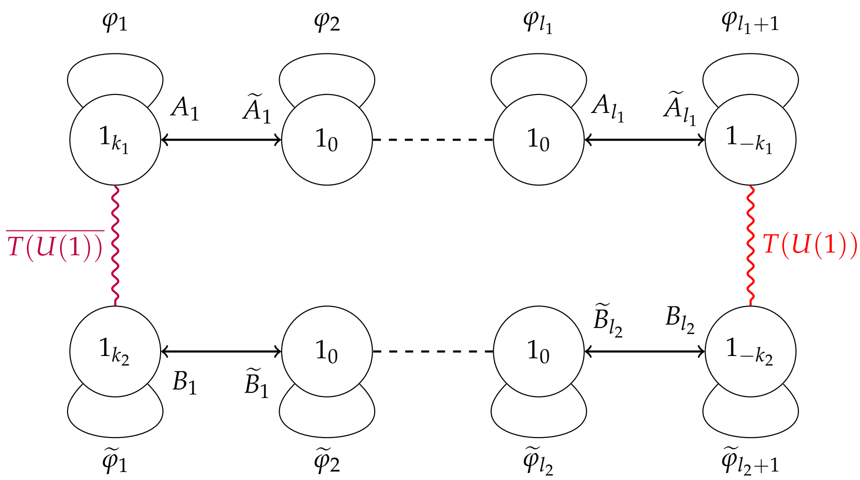
(3.5.167)

The quiver associated with the brane system on the right is



(3.5.168)

where the numbers of gauge nodes are $l_1 + 1$ and $l_2 + 1$ on the upper and the lower sides of the quiver, respectively. In the $\mathcal{N} = 2$ notation, this can be written as



(3.5.169)

The vacuum equations are

$$
\begin{aligned}
&A(\varphi_i - \varphi_{i+1}) = 0 \;\; i = 1,\ldots,l_1\,, && B(\widetilde{\varphi}_i - \widetilde{\varphi}_{i+1}) = 0 \;\; i = 1,\ldots,l_2\\
&k_1\varphi_1 - \widetilde{\varphi}_1 = A_1\,\widetilde{A}_1\,, && k_2\widetilde{\varphi}_1 - \varphi_1 = B_1\,\widetilde{B}_1\\
&0 = A_i\,\widetilde{A}_i - A_{i-1}\,\widetilde{A}_{i-1} \;\; i = 2,\ldots,l_1\,, && 0 = B_i\,\widetilde{B}_i - B_{i-1}\,\widetilde{B}_{i-1} \;\; i = 2,\ldots,l_2\,,\\
&-k_1\varphi_{l_1+1} + \widetilde{\varphi}_{l_2+1} = -A_{l_1}\,\widetilde{A}_{l_1}\,, && -k_2\varphi_{l_2+1} + \widetilde{\varphi}_{l_1+1} = -B_{l_2}\,\widetilde{B}_{l_2}\,,
\end{aligned}
\tag{3.5.170}
$$

where we highlighted in blue the contributions from the mixed CS terms due to $T(U(1))$ and $\overline{T(U(1))}$. We focus on the geometric branch, corresponding to the case

$\varphi_i = \varphi$ for all $i = 1 \dots l_1 + 1$ and $\widetilde{\varphi}_i = \widetilde{\varphi}$ for all $i = 1 \dots l_2 + 1$. Imposing these conditions, we are left with the following constraints of the mesons:

$$\begin{aligned} k_1\varphi - \widetilde{\varphi} &= A_1\widetilde{A}_1\,, \quad A_{i+1}\widetilde{A}_{i+1} - A_i\widetilde{A}_i = 0\,, \quad -k_1\varphi + \widetilde{\varphi} = -A_{l_1}\widetilde{A}_{l_1} \\ k_2\widetilde{\varphi} - \varphi &= B_1\widetilde{B}_1\,, \quad B_{i+1}\widetilde{B}_{i+1} - B_i\widetilde{B}_i = 0\,, \quad -k_1\widetilde{\varphi} + \varphi = -B_{l_1}\widetilde{B}_{l_1} \end{aligned} \tag{3.5.171}$$

Let us consider the monopole operator $V_{\boldsymbol{m}}$ with flux

$$\boldsymbol{m} = (\underbrace{m,\dots,m}_{l_1+1},\underbrace{\widetilde{m},\dots,\widetilde{m}}_{l_2+1}) = (m^{l_1+1},\widetilde{m}^{l_2+1})\,. \tag{3.5.172}$$

The R-charge of $V_{\boldsymbol{m}}$ is zero:

$$R[V_{\boldsymbol{m}}] = 0\,. \tag{3.5.173}$$

and the gauge charges are

$$\begin{aligned} &q_1[V_{\boldsymbol{m}}] = -(k_1\,m - \widetilde{m})\,, \quad q_2[V_{\boldsymbol{m}}] = 0\,, \quad \dots\,, \quad q_{l_1}[V] = 0\,, \quad q_{l_1+1}[V_{\boldsymbol{m}}] = (k_1\,m - \widetilde{m})\,, \\ &q_{\widetilde{1}}[V_{\boldsymbol{m}}] = -(k_2\,\widetilde{m} - \widetilde{m}) \quad q_{\widetilde{2}}[V_{\boldsymbol{m}}] = 0\,, \quad \dots\,, \quad q_{\widetilde{l_1}}[V_{\boldsymbol{m}}] = 0\,, \quad \widetilde{q}_{\widetilde{l_1+1}}[V_{\boldsymbol{m}}] = (k_2\,\widetilde{m} - m)\,. \end{aligned} \tag{3.5.174}$$

Now we have all the ingredients in order to compute the baryonic generic function:

$$\begin{aligned} g(t;B,\widetilde{B}) &= \mathrm{PE}[-t^2]^{l_1-1}\,\mathrm{PE}[-t^2]^{l_2-1} \oint \frac{dq_1 dq_2 \dots dq_{l_1+1}}{(2\pi i)^{l_1+1} q_1^{1+B} q_2 \dots q_{l_1} q_{l_1+1}^{1-B}} \\ &\quad \oint \frac{d\widetilde{q}_1 d\widetilde{q}_2 \dots d\widetilde{q}_{l_1+1}}{(2\pi i)^{l_2+1} \widetilde{q}_1^{1+\widetilde{B}} \widetilde{q}_2 \dots \widetilde{q}_{l_1} \widetilde{q}_{l_1+1}^{1-\widetilde{B}}} \prod_{i=1}^{l_1} \mathrm{PE}[t(q_i q_{i+1}^{-1} + q_i^{-1} q_{i+1})] \\ &\quad \prod_{j=1}^{l_2} \mathrm{PE}[t(q_j q_{j+1}^{-1} + q_i^{-1} q_{j+1})] \\ &= g^{\mathrm{ABJM}/2}(t;l_1\,B)\, g^{\mathrm{ABJM}/2}(t;l_2\,\widetilde{B}) = \frac{t^{l_1|B| + l_2|\widetilde{B}|}}{(1-t^2)^2} \end{aligned} \tag{3.5.175}$$

The Hilbert series of the geometric branch of (3.5.168) is then

$$\begin{aligned} H_{(3.5.168)}(t,z) &= g(t;k_1 m - \widetilde{m}, k_2\widetilde{m} - m) \\ &= \frac{1}{(1-t^2)^2} \sum_{m\in\mathbb{Z}} \sum_{\widetilde{m}\in\mathbb{Z}} z^{m+\widetilde{m}}\, t^{l_1|k_1 m - \widetilde{m}| + l_2|k_2\widetilde{m} - m|}\,. \end{aligned} \tag{3.5.176}$$

Note that for $l_1 = l_2 = 1$ we recover (3.5.156) as expected.

In *some* cases, the geometric branch of (3.5.168) turns out to be isomorphic to $(\mathbb{C}^2/\mathbb{Z}_{l_1} \times \mathbb{C}^2/\mathbb{Z}_{l_2})/\Gamma[k_1\,,\,k_1\,k_2 - 1]$, where the action of $\Gamma[k_1, k_1k_2 - 1]$ being

$$\Gamma[k_1, k_1k_2 - 1] : (z_1, z_2; \widetilde{z}_1, \widetilde{z}_2) \to (\omega z_1, \omega^{-1} z_2;\ \widetilde{\omega}\widetilde{z}_1, \widetilde{\omega}^{-1}\widetilde{z}_2)\,, \tag{3.5.177}$$

with $\omega = e^{2\pi i \frac{k_1}{k_1k_2-1}}$ and $\widetilde{\omega} = e^{2\pi i \frac{1}{k_1k_2-1}}$; and (z_1, z_2) and $(\widetilde{z}_1, \widetilde{z}_2)$ are the coordinates of $\mathbb{C}^2/\mathbb{Z}_{l_1}$ and $\mathbb{C}^2/\mathbb{Z}_{l_2}$ respectively. For example, when $\{k_1 = 2\,, k_2 = 3\,,\, l_1 = l_2 = 2\}$ we have

$$\begin{aligned} H_{(3.5.168)}(t, z=1) &= \frac{1 - t^2 + 4t^6 - t^{10} + t^{12}}{(1-t^2)^3(1-t^{10})} \\ &= H[(\mathbb{C}^2/\mathbb{Z}_2 \times \mathbb{C}^2/\mathbb{Z}_2)/\Gamma[2\,,\,5]](t, z=1)\,. \end{aligned} \tag{3.5.178}$$

and when $\{k_1 = 2, k_2 = 2, l_1 = 5, l_2 = 1\}$, we have

$$
\begin{aligned}
H_{(3.5.168)}(t, z=1) &= \frac{(1-t+t^4-t^7+t^8)(1+t^2+t^3+t^6+t^7+t^9)}{(1-t)\,(1-t^2)\,(1-t^3)\,(1-t^{15})} \\
&= H[(\mathbb{C}^2/\mathbb{Z}_5 \times \mathbb{C}^2)/\Gamma[2,3]](t, z=1)\,.
\end{aligned} \tag{3.5.179}
$$

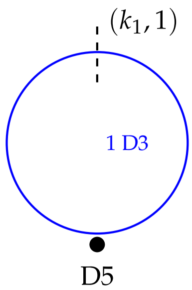
Other cases can be more complicated. For example, for $\{k_1 = 2, k_2 = 3, l_1 = 1, l_2 = 3\}$, we find that

$$
\begin{aligned}
H_{(3.5.168)}(t, z=1) &= \frac{1}{(1-t)^2\,(1-t^5)\,(1-t^{15})} \times (1-t+t^2)(1-t+t^2-2t^3 \\
&\quad + 2t^4 + t^5 + 2t^6 - 3t^8 + \text{palindrome up to } t^{16}) \\
&= H[(\mathbb{C}^2 \times \mathbb{C}^2/\mathbb{Z}_3)/\widehat{\Gamma}](t, z=1)\,,
\end{aligned} \tag{3.5.180}
$$

where the orbifold $\widehat{\Gamma}$ acts as (3.5.177) but with $\omega = e^{2\pi i \frac{2}{5}}$ and $\widetilde{\omega} = e^{2\pi i \frac{3}{5}} = \omega^{-1}$.

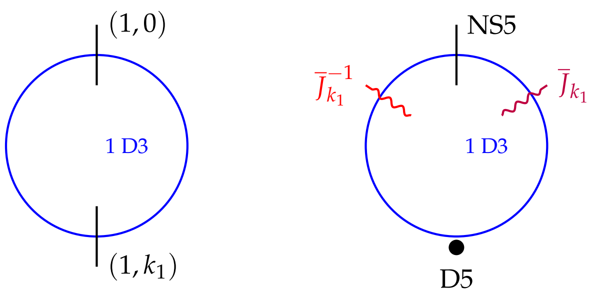
One (p, q)-brane and one D5-brane

Let us consider an example of one (p, q)-brane and one D5-brane. In particular, let us assume that $(p, q) = \bar{J}_{k_1}(1, 0) = (k_1, 1)$:



(3.5.181)

One may apply $SL(2, \mathbb{Z})$ action to this configuration and obtain the following configurations:

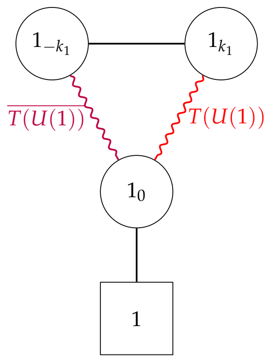


(3.5.182)

where the first configuration is obtained by applying a $\bar{J}_{k_1}^{-1}$ duality transformation to (3.5.181) and using the fact that $\bar{J}_{k_1}^{-1}(0,1) = (1, k_1)$, and for the second configuration we use the fact that $\bar{J}_{k_1}(1,0) = (k_1, 1)$, so we recover the original set-up (3.5.181).

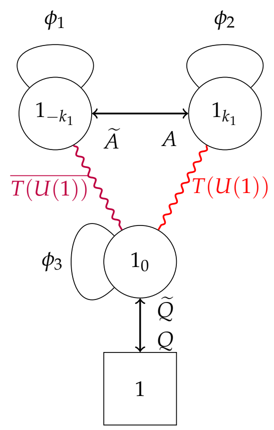
The brane configuration on the left in (3.5.182) is that of the ABJM theory with CS level $(k_1, -k_1)$. Thus, we expect that the moduli space of the field theories associated with these brane configurations has two branches, namely (1) $\mathbb{C}^4/\mathbb{Z}_{|k_1|}$, which is the geometric moduli space of the ABJM theory, and (2) $\mathbb{C}^2/\mathbb{Z}_{|k_1|}$, which is the moduli space of the half-ABJM theory, where a pair of bi-fundamental chiral multiplets of the ABJM theory is set to zero.

Let us derive these moduli spaces for the theory associated with the configuration on the right in (3.5.182). The quiver diagram is given by



(3.5.183)

In the $\mathcal{N} = 2$ notation, this quiver can be rewritten as



(3.5.184)

The vacuum equations are

$$\begin{aligned} &A(\varphi_1-\varphi_2) = 0 = \widetilde{A}(\varphi_1-\varphi_2)\,, && Q\,\varphi_3 = 0 = \widetilde{Q}\,\varphi_3 \\ &-k_1\varphi_1 - \varphi_3 = A\,\widetilde{A}\,, && Q\widetilde{Q} = 0 \\ &k_1\varphi_2 + \varphi_3 = -\,A\,\widetilde{A}\,, && \end{aligned} \tag{3.5.185}$$

where we indicate the contributions from the mixed CS terms due to $T(U(1))$ and $\overline{T(U(1))}$ in blue. Let us assume that A and $\widetilde{A}$ are non-zero. Therefore $\varphi_1 = \varphi_2 = \varphi$ (and the corresponding magnetic fluxes are set equal: $m_1\,,= m_2 = m$). Thus, we have two branches: (1) $Q = \widetilde{Q} = 0$, and (2) $\varphi_3 = 0$.

Branch I: $Q = \widetilde{Q} = 0$

The moduli space is parametrised by $A\widetilde{A}\,,\ \varphi\,,\ \varphi_3$ and the monopole operators, with the following constraint from the vacuum equations:

$$-\,k_1\varphi - \varphi_3 = A\,\widetilde{A}\,. \tag{3.5.186}$$

The monopole operator $V_{\boldsymbol{m}}$, with flux $\boldsymbol{m} = (m, m, m_3)$, carries gauge and R charges:

$$q_1[V_{\boldsymbol{m}}] = -q_2[V_{\boldsymbol{m}}] = k_1 m - m_3\,, \quad q_3[V_{\boldsymbol{m}}] = 0\,, \quad R[V_{\boldsymbol{m}}] = \frac{1}{2}\,|m_3| \tag{3.5.187}$$

where we stress that $q_3[V_{\boldsymbol{m}}] = 0$ since $T(U(1))$ and $\overline{T(U(1))}$ contribute m and $-m$ respectively, and the non-trivial contribution to the R-charge is due to the presence of the flavour. The baryonic generating function is given by

$$\begin{aligned} g(t;B) &= \frac{1}{1-t^2} \oint \frac{dq_1}{2\pi i\, q_1^{1+B}} \frac{dq_2}{2\pi i\, q_2^{1-B}} \frac{dq_3}{2\pi i\, q_3} \, \mathrm{PE}[t(q_1^{-1} q_2 + q_1 q_2^{-1})] = \\ &= \frac{1}{1-t^2} g^{\mathrm{ABJM}/2}(t,B) = \frac{t^{|B|}}{(1-t^2)^2} \end{aligned} \tag{3.5.188}$$

where the overall $(1-t^2)^{-1}$ is due to the the fact that only one among φ and φ_3 gets fixed. The Hilbert series is thus given by

$$\begin{aligned} H_{\mathrm{I}}(t,z) &= \sum_{m=\infty}^{+\infty} \sum_{m_3=\infty}^{+\infty} z^{m+m_3}\, t^{|m_3|} g(t; -k_1 m - m_3) = \\ &= \sum_{m=\infty}^{+\infty} \sum_{m_3=\infty}^{+\infty} z^{m+m_3} \frac{t^{|m_3|+|k_1 m + m_3|}}{(1-t^2)^2} \, . \end{aligned} \tag{3.5.189}$$

This turns out to be equal to the following Hilbert series of $\mathbb{C}^4/\mathbb{Z}_{|k_1|}$:

$$\begin{aligned} H_{\mathrm{I}}(t,z) &= \frac{1}{|k_1|} \sum_{j=1}^{|k_1|} \frac{1}{(1-t\,w^j)^2(1-t\,/\,w^j)^2} \, , \quad w = z\, e^{\frac{2\pi i}{|k_1|}} \, , \\ &= H[\mathbb{C}^4/\mathbb{Z}_{|k_1|}](t,z) \, . \end{aligned} \tag{3.5.190}$$

This is in agreement with the geometric branch of the ABJM theory.

Branch II: $\varphi_3 = 0$

In this case, the vacuum equations imply that $Q\widetilde{Q} = 0$. The moduli space is generated by $\varphi = -\frac{1}{k} A\widetilde{A}$ and the dressed monopole operators $\overline{V}_+ = V_{(1,1,0)} A^{k_1}$ and $\overline{V}_- = V_{(-1,-1,0)} \widetilde{A}^{k_1}$ if $k_1 > 0$. If $k_1 < 0$, we simply change A^{k_1} to $\widetilde{A}^{-k_1}$ and $\widetilde{A}^{k_1}$ to A^{-k_1} in these equations. These dressed monopole operators satisfy the quantum relation

$$\overline{V}_+ \overline{V}_- = \varphi^{|k_1|} \, . \tag{3.5.191}$$

Hence, this branch is isomorphic to $\mathbb{C}^2/\,\mathbb{Z}_{|k_1|}$, which is the moduli space of the half-ABJM theory.

3.5.6 Comments on abelian theories with zero Chern–Simons levels

Let us now revisit abelian theories with zero CS levels, namely those studied in section 3.4 with $N = 1$, from the point of view of this section.

One can start by taking simple examples: comparing (5.2.1) to (3.5.113). We set $N = 1$ and $n = 1$ in the former and set $k = 0$ In the latter. Indeed, as we discussed below (3.5.113), such theory has a trivial Coulomb branch, because the scalar in the vector multiplets are set to zero by the vacuum equations. This is perfectly consistent with the proposal in section 3.4, namely the scalar fields in the vector multiplet of the gauge nodes that are connected by $T(U(N))$ are frozen. Moreover, from (3.4.19), we see that when $n = 1$ the Higgs branch is also trivial; this is also in accordance with the analysis below (3.5.113), where the meson vanishes. Hence the two approaches, one presented in section 3.4 and the other presented in this section, yield the same results. The same result can be derived easily for the mirror theory (3.4.17), with $N = 1$ and $n = 1$, and (3.5.82) with $k_1 = k_2 = 0$.

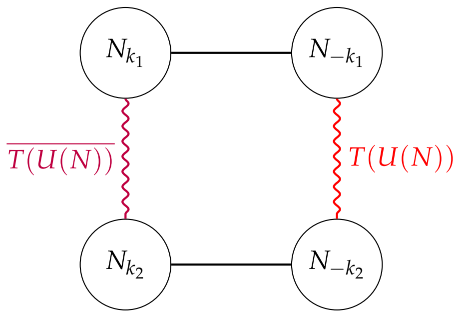
This analysis can be generalised to other models discussed in this section. When we set all CS levels to zero, the vacuum equations set the scalars in the vector multiplets corresponding to the gauge groups that are connected by $T(U(1))$ to zero. Other parts of the quiver may still contribute non-trivially to the moduli space.

3.6 Non-abelian theories with non-zero Chern–Simons levels

In this section, we focus on non-abelian quiver theories that contain $T(U(N))$ and/or $\overline{T(U(N))}$ theories as edges of the quiver. In terms of a brane system, these theories involve multiple D3-branes, along with J-folds and possibly with other types of branes. In contrast to the abelian case, we do not have a general prescription of computing the Hilbert series of the geometric branch of non-abelian theories. Nevertheless, for theories that arise from N M2-branes probing Calabi-Yau 4-fold singularities, we expect that the geometric branch is the N-fold symmetric product of such a Calabi-Yau 4-fold. In such cases, we can analyse the Hilbert series for each configuration of magnetic fluxes. Let us demonstrate this in the following example.

One $(k, 1)$ and one $(1, k')$ brane

Let us consider the generalisation of (3.5.146) for non-abelian gauge groups.



(3.6.1)

In section 3.5.5, we see that the geometric branch of the moduli space for the abelian theory ($N = 1$) is a Calabi-Yau 4-fold (this is referred to as Branch I in that section); the latter is identified to be $\mathbb{C}^4/\Gamma(k_1, k_1k_2 - 1)$. For a general N, we expect that the geometric branch of (3.6.1) is the N-th fold symmetric product of $\mathbb{C}^4/\Gamma(k_1, k_1k_2 - 1)$, namely $\mathrm{Sym}^N\left(\mathbb{C}^4/\Gamma(k_1, k_1k_2 - 1)\right)$.

Let us focus on $N = 2$ in the following discussion. The Hilbert series of $\mathrm{Sym}^2\left(\mathbb{C}^4/\Gamma(k_1, k_1k_2 - 1)\right)$ is given by

$$H_{(3.6.1),\,N=2}(t, z) = \frac{1}{2}\left[H_{(3.5.147)}(t, z)^2 + H_{(3.5.147)}(t^2, z^2)\right] , \qquad (3.6.2)$$

where $H_{(3.5.147)}(t, z)$ is given by (3.5.156). This computation can be split into five different cases depending on the fluxes and the residual gauge symmetries.

1. The magnetic fluxes for the two nodes on the upper edge are both (m, m), and the magnetic flux for the two nodes on the lower edge are both (n, n). In this case, the residual gauge symmetry is $U(2) \times U(2) \times U(2) \times U(2)$. The Hilbert series in this case can be computed as a second rank symmetric product of the abelian case (which is a product of two half-ABJM theories). The result is

$$
\begin{aligned}
H^{(1)}_{N=2}(t, z) = \frac{1}{2} \sum_{m,n \in \mathbb{Z}} \Big[& g^{\text{ABJM}/2}(t; k_1 m - n)^2 g^{\text{ABJM}/2}(t; k_2 n - m)^2 \\
& + g^{\text{ABJM}/2}(t^2; k_1 m - n) g^{\text{ABJM}/2}(t^2; k_2 n - m) \Big] z^{2(m+n)} ,
\end{aligned}
\tag{3.6.3}
$$

where the terms indicated in blue are due to the mixed CS terms due to the presence of $T(U(2))$ and $\overline{T(U(2))}$ and

$$
g^{\text{ABJM}/2}(t; B) = \frac{t^{|B|}}{1 - t^2} .
\tag{3.6.4}
$$

Let us report the unrefined Hilbert series, for $k_1 = 1$ and $k_2 = 2$, for this case up to order t^{12}:

$$
\begin{aligned}
H^{(1)}_{N=2,\boldsymbol{k}=(1,2)}(t, z = 1) = 1 + 6t^2 + 22t^4 + 62t^6 + 147t^8 \\
+ 308t^{10} + 588t^{12} + \ldots .
\end{aligned}
\tag{3.6.5}
$$

In fact, we can also compute (3.6.5) using the Molien integration [91] as follows:

$$
\begin{aligned}
& H^{(1)}_{N=2,\boldsymbol{k}=(1,2)}(t, z = 1) \\
& = \oint_{|z_1|=1} \frac{\mathrm{d}z_1}{2\pi i z_1} \cdots \oint_{|z_4|=1} \frac{\mathrm{d}z_4}{2\pi i z_4} \oint_{|q_1|=1} \frac{\mathrm{d}q_1}{2\pi i q_1} \oint_{|q_2|=1} \frac{\mathrm{d}q_2}{2\pi i q_2} \times \\
& \left(\prod_{j=1}^{4} H[\mathbb{C}^2/\mathbb{Z}_2](t, z_j) \right) \mathrm{PE} \Big[(z_1 + z_1^{-1})(z_2 + z_2^{-2})(q_1 + q_1^{-1}) t \\
& \quad + (z_3 + z_3^{-1})(z_4 + z_4^{-2})(q_2 + q_2^{-1}) t \\
& \quad - (z_1^2 + 1 + z_1^{-2}) t^2 - (z_3^2 + 1 + z_3^{-2}) t^2 + t^4 - t^8 \Big] .
\end{aligned}
\tag{3.6.6}
$$

We have checked that (3.6.6) agrees with (3.6.5) up to order t^{20}. Here $z_1, \ldots, z_4$ are fugacities for the gauge groups $SU(2)_{1,2,3,4}$ that are subgroups of $U(2)_{1,2,3,4}$ gauge groups corresponding to top left, top right, bottom left and bottom right nodes in (3.6.1) respectively. The fugacities q_1 and q_2 corresponds to the two diagonal $U(1)$ gauge groups that are subgroups of $\mathrm{diag}(U(2)_1 \times U(2)_2)$ and $\mathrm{diag}(U(2)_3 \times U(2)_4)$ of (3.6.1) respectively. $H[\mathbb{C}^2/\mathbb{Z}_2](t, z)$ denotes the Hilbert series of the space $\mathbb{C}^2/\mathbb{Z}_2$, which is the Higgs and the Coulomb branches of $T(U(2))$ and $\overline{T(U(2))}$, and its expression is given by

$$
H[\mathbb{C}^2/\mathbb{Z}_2](t, z) = \mathrm{PE} \left[(z^2 + 1 + z^{-2}) t^2 - t^4 \right] .
\tag{3.6.7}
$$

The first and the second terms in the PE denote the contributions from the bifundamental hypermultiplets under $U(2) \times U(2)$. The last line of (3.6.6) deserves some comments. For a theory with Lagrangian, these terms would represent the contribution from the F-terms. In this case, however, $T(U(2))$ and

$\overline{T(U(2))}$ do not have a manifest Lagrangian description in the quiver. Nevertheless, such terms can still be interpreted as "effective F-terms", where at t^2 there are relations that transform in the adjoint representations of $\mathrm{diag}(SU(2)_1 \times SU(2)_2)$ and $\mathrm{diag}(SU(2)_3 \times SU(2)_4)$. There is also a relation at order t^4 and a syzygy (relation among the relations) at order t^8.[10]

2. The magnetic fluxes for the two nodes on the upper edge are both (m_1, m_2), with $m_1 > m_2$, and the magnetic flux for the two nodes on the lower edge are both (n, n). In this case, each of the $U(2)$ gauge groups on the upper edge is broken to $U(1)^2$. Each of the $U(2)$ gauge groups on the lower edge remains unbroken. In this case, $T(U(2))$ is expected to become $T(U(1))^2$ (and similarly $\overline{T(U(2))}$ becomes $\overline{T(U(1))}^2$). The Hilbert series in this case is given by

$$\begin{aligned} H^{(2)}_{N=2}(t,z) = \sum_{m_1>m_2}\sum_{n\in\mathbb{Z}} & g^{\mathrm{ABJM}/2}(t;k_1m_1-n)g^{\mathrm{ABJM}/2}(t;k_1m_2-n) \\ & g^{\mathrm{ABJM}/2}(t;k_2n-m_1)g^{\mathrm{ABJM}/2}(t;k_2n-m_1)z^{m_1+m_2+2n}\ . \end{aligned} \tag{3.6.8}$$

As an example, for $k_1 = 1$ and $k_2 = 2$, the unrefined Hilbert series up to t^{12} is

$$H^{(2)}_{N=2,\boldsymbol{k}=(1,2)}(t,z=1) = 4t^2 + 33t^4 + 148t^6 + 483t^8 + 1288t^{10} + 2982t^{12} + \dots\ . \tag{3.6.9}$$

3. The magnetic fluxes for the two nodes on the upper edge are both (m, m) and the magnetic flux for the two nodes on the lower edge are both (n_1, n_2), with $n_1 > n_2$. In this case, each of the $U(2)$ gauge groups on the lower edge is broken to $U(1)^2$. Each of the $U(2)$ gauge groups on the upper edge remains unbroken. $T(U(2))$ is expected to become $T(U(1))^2$, and similarly $\overline{T(U(2))}$ becomes $\overline{T(U(1))}^2$. The Hilbert series in this case is given by

$$\begin{aligned} H^{(3)}_{N=2}(t,z) = \sum_{n_1>n_2}\sum_{m\in\mathbb{Z}} & g^{\mathrm{ABJM}/2}(t;k_1m-n_1)g^{\mathrm{ABJM}/2}(t;k_1m-n_2) \\ & g^{\mathrm{ABJM}/2}(t;k_2n_1-m)g^{\mathrm{ABJM}/2}(t;k_2n_2-m)z^{2m+n_1+n_2}\ . \end{aligned} \tag{3.6.10}$$

[10] It is instructive to compare this to the following example. Let us consider a 3d $\mathcal{N} = 4$ gauge theory with $U(2) \times U(2)$ gauge group with two bi-fundamental hypermultiplets. This quiver is an A_1 affine Dynkin diagram, so it arises from two M2-branes probing $\mathbb{C}^2/\mathbb{Z}_2$ singularity. We expect the geometric branch of this theory to be $\mathrm{Sym}^2(\mathbb{C}^2/\mathbb{Z}_2)$. The Hilbert series of which can be computed from the Molien integral:

$$\begin{aligned} H(t,x) = & \oint_{|z_1|=1} \frac{dz_1}{2\pi i z_1}\left(\frac{1-z_1^2}{z_1}\right)\oint_{|z_2|=1}\frac{dz_2}{2\pi i z_2}\left(\frac{1-z_2^2}{z_2}\right)\oint_{|q|=1}\frac{dq}{2\pi i q} \\ & \times \mathrm{PE}\left[(z_1+z^{-1})(z_2+z^{-2})(q+q^{-1})(x+x^{-1}) - (z_1^2+1+z_1^{-2}+1)t^2 - t^4\right]\ . \end{aligned}$$

This is indeed equal to $H[\mathrm{Sym}^2(\mathbb{C}^2/\mathbb{Z}_2)](t,x) = \frac{1}{2}\left[H[\mathbb{C}^2/\mathbb{Z}_2](t,x)^2 + H[\mathbb{C}^2/\mathbb{Z}_2](t^2,x^2)\right]$. The first term in the PE is the contribution from the bi-fundamental hypermultiplets. Since on the generic point on the moduli space $U(2) \times U(2)$ is not completely broken, but it is broken to the diagonal subgroup $\mathrm{diag}(U(2) \times U(2))$. The second term indicates the F-terms in such a diagonal subgroup. The last term $-t^4$ is there due to the fact that the F-flat moduli space is not a complete intersection because of the unbroken gauge symmetry on the moduli space (see the detailed discussion in [65]).

As an example, for $k_1 = 1$ and $k_2 = 2$, the unrefined Hilbert series up to t^{12} is

$$\begin{aligned} H^{(3)}_{N=2,\mathbf{k}=(1,2)}(t,z=1) &= 6t^3 + 34t^5 + 15t^6 + 114t^7 + 76t^8 + 322t^9 \\ &\quad + 234t^{10} + 778t^{11} + 609t^{12} + \dots \, . \end{aligned} \tag{3.6.11}$$

4. The magnetic fluxes for the two nodes on the upper edge are both (m_1, m_2), with $m_1 > m_2$. and the magnetic flux for the two nodes on the lower edge are both (n_1, n_2), with $n_1 > n_2$. In this case, each of the $U(2)$ gauge groups in the quiver is broken to $U(1)^2$. $T(U(2))$ becomes $T(U(1))^2$, and similarly $\overline{T(U(2))}$ becomes $\overline{T(U(1))}^2$. The Hilbert series in this case is given by

$$\begin{aligned} H^{(4)}_{N=2}(t,z) = \sum_{n_1>n_2}\sum_{m_1>m_2} & g^{\mathrm{ABJM}/2}(t;k_1m_1-n_1)g^{\mathrm{ABJM}/2}(t;k_1m_2-n_2) \\ & g^{\mathrm{ABJM}/2}(t;k_2n_1-m_1)g^{\mathrm{ABJM}/2}(t;k_2n_2-m_2)z^{m_1+m_2+n_1+n_2} \, . \end{aligned} \tag{3.6.12}$$

As an example, for $k_1 = 1$ and $k_2 = 2$, the unrefined Hilbert series up to t^{12} is

$$\begin{aligned} H^{(4)}_{N=2,\mathbf{k}=(1,2)}(t,z=1) &= 4t + 10t^2 + 54t^3 + 115t^4 + 350t^5 + 643t^6 + \\ &\quad + 1520t^7 + 2505t^8 + 5076t^9 + \\ &\quad + 7771t^{10} + 14142t^{11} + 20501t^{12} + \dots \, . \end{aligned} \tag{3.6.13}$$

5. The magnetic fluxes for the two nodes on the upper edge are both (m_1, m_2), with $m_1 < m_2$. and the magnetic flux for the two nodes on the lower edge are both (n_1, n_2), with $n_1 > n_2$. The discussion is very similar to the previous case. The Hilbert series in this case is given by

$$\begin{aligned} H^{(5)}_{N=2}(t,z) = \sum_{n_1>n_2}\sum_{m_1<m_2} & g^{\mathrm{ABJM}/2}(t;k_1m_1-n_1)g^{\mathrm{ABJM}/2}(t;k_1m_2-n_2) \\ & g^{\mathrm{ABJM}/2}(t;k_2n_1-m_1)g^{\mathrm{ABJM}/2}(t;k_2n_2-m_2)z^{m_1+m_2+n_1+n_2} \, . \end{aligned} \tag{3.6.14}$$

As an example, for $k_1 = 1$ and $k_2 = 2$, the unrefined Hilbert series up to t^{12} is

$$\begin{aligned} H^{(5)}_{N=2,\mathbf{k}=(1,2)}(t,z=1) &= 12t^5 + 82t^7 + 24t^8 + 322t^9 + 151t^{10} \\ &\quad + 992t^{11} + 556t^{12} + \dots \, . \end{aligned} \tag{3.6.15}$$

Indeed, the Hilbert series $H_{(3.6.1),\, N=2}(t,z)$ given by (3.6.2) is then equal to the sum of the contributions from these five cases:

$$H_{(3.6.1),\, N=2}(t,z) = \sum_{i=1}^{5} H^{(i)}_{N=2,\mathbf{k}=(1,2)}(t,z) \, . \tag{3.6.16}$$

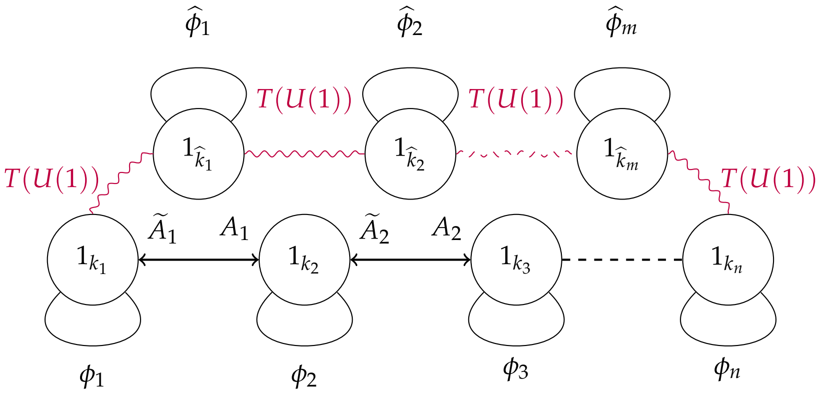
For $k_1 = 1$ and $k_2 = 2$, we have the unrefined Hilbert series

$$\begin{aligned} H_{(3.6.1),\, N=2,\mathbf{k}=(1,2)}(t,z=1) &= 1 + 4t + 20t^2 + 60t^3 + 170t^4 + 396t^5 \\ &\quad + 868t^6 + 1716t^7 + 3235t^8 + 5720t^9 \\ &\quad + 9752t^{10} + 15912t^{11} + 25236t^{12} + \dots \, . \end{aligned} \tag{3.6.17}$$

This is indeed an unrefined Hilbert series of $\mathrm{Sym}^2(\mathbb{C}^4)$.

3.7 Theories with multiple consecutive J-folds

In this section, we generalise our discussion to theories dual to brane system containing $(m+1)$ consecutive J-folds.



(3.7.1)

The vacuum equations are

$$A_i(\Phi_{i+1} - \Phi_i) = 0\,, \qquad i = 1, \dots, n-1 \tag{3.7.2}$$

$$\begin{aligned} k_1\,\Phi_1 - \widehat{\Phi}_1 &= \mu_1 \\ k_i\Phi_i &= \mu_i - \mu_{i-1} \qquad i = 2\dots n-1 \\ k_n\,\Phi_n - \widehat{\Phi}_m &= \mu_{n-1} \\ \widehat{k}_1\widehat{\Phi}_1 - \Phi_1 - \widehat{\Phi}_2 &= 0 \\ \widehat{k}_i\widehat{\Phi}_i - \widehat{\Phi}_{i+1} - \widehat{\Phi}_{i-1} &= 0 \qquad i = 2, \dots, m-1 \\ \widehat{k}_m\widehat{\Phi}_m - \widehat{\Phi}_{m-1} - \Phi_n &= 0 \end{aligned} \tag{3.7.3}$$

As in the preceding subsection, we analyse the solution of these equations according to the VEVs of bi-fundamental fields that are set to zero (*i.e.* the cuts in the quiver).

No cut in the quiver

Let us first focus on the solution in which A_i and $\widetilde{A}_i$ are non-zero for all $i = 1, \dots, n-1$. Equations (3.7.2) are solved as usual imposing $\Phi_1 = \Phi_2 = \dots = \Phi_n = \Phi = (\varphi, \sigma)$. The sum of the first three equations in (3.7.3) gives

$$\left(\sum_{i=1}^{n} k_i\right)\Phi - \widehat{\Phi}_1 - \widehat{\Phi}_m = 0 \tag{3.7.4}$$

This consistency equation must be added to set of equation formed by the last three in (3.7.3). Calling

$$K_n = \sum_{i=1}^{n} k_i \tag{3.7.5}$$

the above system of equations can be written in a compact way as:

$$M_{CS} \begin{pmatrix} \Phi \\ \widehat{\Phi}_1 \\ \widehat{\Phi}_2 \\ \vdots \\ \widehat{\Phi}_m \end{pmatrix} = 0 \tag{3.7.6}$$

where we define the matrix M_{CS} as

$$M_{CS} = \begin{pmatrix} K_n & -1 & 0 & 0 & 0 & \dots & -1 \\ -1 & \widehat{k}_1 & -1 & 0 & 0 & \dots & 0 \\ 0 & -1 & \widehat{k}_2 & -1 & 0 & \dots & 0 \\ 0 & 0 & -1 & \widehat{k}_3 & -1 & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ -1 & 0 & 0 & 0 & \dots & -1 & \widehat{k}_m \end{pmatrix} \tag{3.7.7}$$

Since we assumed that all A_i and $\widetilde{A}_i$ are non-zero, we require (3.7.7) to have a non-trivial solution; this is the case if and only if

$$\det M_{CS} = 0 \tag{3.7.8}$$

This is a necessary condition for the existence of a non-trivial moduli space.

The magnetic flux has to be of the form

$$\mathfrak{m} = (\underbrace{\mathsf{m}, \dots, \mathsf{m}}_{n \text{ times}}, \widehat{\mathsf{m}}_1, \dots, \widehat{\mathsf{m}}_m) \equiv (\mathsf{m}^n, \widehat{\boldsymbol{m}}) \, . \tag{3.7.9}$$

Then, (3.7.7) implies that this must satisfy the following condition:

$$M_{CS} \mathfrak{m}^T = 0 \, . \tag{3.7.10}$$

In particular,

$$K_n \mathsf{m} - \widehat{\mathsf{m}}_1 - \widehat{\mathsf{m}}_m = 0 \tag{3.7.11}$$

The gauge charges of the monopole operator $V_{\mathfrak{m}}$ are

$$\begin{aligned} q_1[V_{\mathfrak{m}}] &= -(k_1 \mathsf{m} - \widehat{\mathsf{m}}_1) \\ q_i[V_{\mathfrak{m}}] &= -k_i \mathsf{m} \, , \qquad i = 2, \dots, n-1 \\ q_n[V_{\mathfrak{m}}] &= -(k_n \mathsf{m} - \widehat{\mathsf{m}}_m) \\ q_{\widehat{1}}[V_{\mathfrak{m}}] &= -(\widehat{k}_1 \widehat{\mathsf{m}}_1 - \mathsf{m} - \widehat{\mathsf{m}}_2) \\ q_{\widehat{i}}[V_{\mathfrak{m}}] &= -(\widehat{k}_i \widehat{\mathsf{m}}_i - \widehat{\mathsf{m}}_{i+1} - \widehat{\mathsf{m}}_{i-1}) \, , \qquad i = 2, \dots, m-1 \\ q_{\widehat{m}}[V_{\mathfrak{m}}] &= -(\widehat{k}_m \widehat{\mathsf{m}}_m - \widehat{\mathsf{m}}_{m-1} - \mathsf{m}) \, . \end{aligned} \tag{3.7.12}$$

Let us now compute gauge invariant dressed monopole operators. The last three sets of equations, setting to zero, constitute m equations in total; they give a unique solution for $\widehat{\boldsymbol{m}} = (\widehat{\mathsf{m}}_1, \dots, \widehat{\mathsf{m}}_m)$ in terms of the flux m. We denote such a solution by $\widehat{\boldsymbol{m}}^*(\mathsf{m})$. It should be emphasised that m, $\widehat{\mathsf{m}}_i^*$ (with $i = 1, \dots, m$), and the CS level K_n, must be integers. Such integrality and equations (3.7.8), (3.7.11) put a constraint on the possible values of $(\widehat{k}_1, \dots, \widehat{k}_m)$, as well as their relation to K_n, in order to obtain a non-trivial moduli space. Note also that $\widehat{\boldsymbol{m}}^*(1) + \widehat{\boldsymbol{m}}^*(-1) = 0$.

For example, in the case of three J-folds ($m = 2$), we have $\widehat{\mathsf{m}}_1^*(\mathsf{m}) = \frac{\widehat{k}_2+1}{\widehat{k}_1\widehat{k}_2-1}\mathsf{m}$ and $\widehat{\mathsf{m}}_2^*(\mathsf{m}) = \frac{\widehat{k}_1+1}{\widehat{k}_1\widehat{k}_2-1}\mathsf{m}$. From (3.7.11), we obtain $K_n = \frac{\widehat{k}_1+\widehat{k}_2+2}{\widehat{k}_1\widehat{k}_2-1}$. The integrality of K_n, $\widehat{m}_1^*(m)$ and $\widehat{m}_2^*(m)$ puts constraints on the values of $\widehat{k}_1$ and $\widehat{k}_2$:

$$\frac{\widehat{k}_1 + \widehat{k}_2 + 2}{\widehat{k}_1\widehat{k}_2 - 1} \in \mathbb{Z}\,, \quad \frac{\widehat{k}_2 + 1}{\widehat{k}_1\widehat{k}_2 - 1} \in \mathbb{Z}\,, \quad \frac{\widehat{k}_1 + 1}{\widehat{k}_1\widehat{k}_2 - 1} \in \mathbb{Z}\,. \tag{3.7.13}$$

Since $\mathsf{m} \in \mathbb{Z}$, we see that the magnetic lattices given by $\widehat{\mathsf{m}}_1^*(\mathsf{m})$ and $\widehat{\mathsf{m}}_2^*(\mathsf{m})$ "jump" by multiples of $\frac{\widehat{k}_2+1}{\widehat{k}_1\widehat{k}_2-1}$ and $\frac{\widehat{k}_1+1}{\widehat{k}_1\widehat{k}_2-1}$ respectively. If we further require that $\widehat{\mathsf{m}}_1^*(\mathsf{m}) = \widehat{\mathsf{m}}_2^*(\mathsf{m}) = \mathsf{m}$ (*i.e.* there is no such jump), we have $\widehat{k}_1 = \widehat{k}_2 = K_n = 2$, assuming that both $\widehat{k}_1$ and $\widehat{k}_2$ are non-zero.

For convenience, let us define

$$\kappa_i = \{k_1 - \widehat{\mathsf{m}}_1^*(1)\,, k_2\,, \ldots, k_{n-1}\,, k_n - \widehat{\mathsf{m}}_m^*(1)\}\,, \quad \mathcal{K}_i = \sum_{j=1}^{n} \kappa_j\,. \tag{3.7.14}$$

For $\mathcal{K}_i > 0$ for all $i = 1, \ldots, n-1$, the basic gauge invariant dressed monopole operators are

$$\begin{aligned} \overline{V}_+ &= V_{(1^n, \widehat{\boldsymbol{m}}^*(1))} A_1^{\mathcal{K}_1} A_2^{\mathcal{K}_2} \cdots A_{n-1}^{\mathcal{K}_{n-1}} \\ \overline{V}_- &= V_{((-1)^n, -\widehat{\boldsymbol{m}}^*(1))} \widetilde{A}_1^{\mathcal{K}_1} \widetilde{A}_2^{\mathcal{K}_2} \cdots \widetilde{A}_{n-1}^{\mathcal{K}_{n-1}}\,. \end{aligned} \tag{3.7.15}$$

If $\mathcal{K}_j < 0$ for some j, we replace $A_j^{\mathcal{K}_j}$ by $\widetilde{A}_j^{-\mathcal{K}_j}$ in the first equation and $\widetilde{A}_j^{\mathcal{K}_j}$ by $A_j^{-\mathcal{K}_j}$ in the second equation. Since the R-charges of $V_{((\pm 1)^n, \pm\widehat{\boldsymbol{m}}^*(1))}$ are zero, we have

$$R[\overline{V}_+] = \frac{1}{2}\sum_{i=1}^{n-1} |\mathcal{K}_i| \equiv \frac{1}{2}\mathcal{K}\,, \qquad \mathcal{K} = \sum_{i=1}^{n-1} |\mathcal{K}_i|\,. \tag{3.7.16}$$

The generators of the moduli space are $\varphi, \overline{V}_\pm$ subject to the quantum relation

$$\overline{V}_+ \overline{V}_- = \varphi^{\mathcal{K}}\,. \tag{3.7.17}$$

The moduli space is indeed $\mathbb{C}^2/\mathbb{Z}_{\mathcal{K}}$. We emphasise that the dependence of $\mathcal{K}$ on $\widehat{k}_1, \ldots, \widehat{k}_m$ is due to $\widehat{\mathsf{m}}_1^*(1)$.

One cut in the quiver

Let us analyse the case $A_l = \widetilde{A}_l = 0$, *i.e.* the quiver is cut at the position l. Equations (3.7.2) implies $\Phi_1 = \Phi_2 = \cdots = \Phi_l = \Phi$ and $\Phi_{l+1} = \Phi_{l+2} = \cdots = \Phi_n = \widetilde{\Phi}$. The sums of the first l equations and the last $n-l$ ones in the first three sets of equations in (3.7.3) imply that

$$\begin{aligned} (k_1 + k_2 + \cdots + k_l)\Phi - \widehat{\Phi}_1 &= 0 \\ (k_{l+1} + k_{l+2} + \cdots + k_n)\widetilde{\Phi} - \widehat{\Phi}_m &= 0 \end{aligned} \tag{3.7.18}$$

These two condition must be supplemented by the last three sets of equations (3.7.3) constraining $\widehat{\Phi}_i$ $i = 1 \ldots m$ These can be put in a matrix form. Calling

$$\sum_{i=1}^{l} k_i = K\,, \quad \sum_{i=l+1}^{n} k_i = \widetilde{K} \tag{3.7.19}$$

we have

$$M_{CS}\begin{pmatrix}\Phi\\ \widehat{\Phi}_1\\ \vdots\\ \widehat{\Phi}_m\\ \widetilde{\Phi}\end{pmatrix} = 0 \tag{3.7.20}$$

where

$$M_{CS} = \begin{pmatrix} K & -1 & 0 & 0 & 0 & \ldots & 0 & 0\\ -1 & \widehat{k}_1 & -1 & 0 & 0 & \ldots & 0 & 0\\ 0 & -1 & \widehat{k}_2 & -1 & 0 & \ldots & 0 & 0\\ 0 & 0 & -1 & \widehat{k}_3 & -1 & \ddots & 0 & 0\\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots\\ 0 & 0 & 0 & 0 & \ldots & -1 & \widehat{k}_m & -1\\ 0 & 0 & 0 & 0 & \ldots & 0 & -1 & \widetilde{K}\end{pmatrix} \tag{3.7.21}$$

A necessary condition for the existence of the non-trivial moduli space is

$$\det\, M_{CS} = 0\,. \tag{3.7.22}$$

The magnetic flux has to be of the form

$$\mathfrak{m} = (\underbrace{\mathsf{m},\ldots,\mathsf{m}}_{l\text{ times}},\underbrace{\widetilde{\mathsf{m}},\ldots,\widetilde{\mathsf{m}}}_{n-l\text{ times}},\widehat{\mathsf{m}}_1,\ldots,\widehat{\mathsf{m}}_m) \equiv (\mathsf{m}^l,\widetilde{\mathsf{m}}^{n-l},\widehat{\boldsymbol{m}})\,. \tag{3.7.23}$$

Then, (3.7.7) implies that this must satisfy the following condition:

$$M_{CS}\mathfrak{m}^T = 0\,. \tag{3.7.24}$$

In particular, it follows from (3.7.18) that

$$\begin{aligned}\widehat{\mathsf{m}}_1 &= (k_1 + k_2 + \cdots + k_l)\mathsf{m} = K\mathsf{m}\\ \widehat{\mathsf{m}}_m &= (k_{l+1} + k_{l+2} + \cdots + k_n)\widetilde{\mathsf{m}} = \widetilde{K}\widetilde{\mathsf{m}}\,.\end{aligned} \tag{3.7.25}$$

The gauge charges of the monopole operator $V_{\mathfrak{m}}$ are

$$\begin{aligned} q_1[V_{\mathfrak{m}}] &= -(k_1\mathsf{m} - \widehat{\mathsf{m}}_1)\\ q_i[V_{\mathfrak{m}}] &= -k_i\mathsf{m}\,, \qquad i = 2,\ldots,l\\ q_j[V_{\mathfrak{m}}] &= -k_j\widetilde{\mathsf{m}}\,, \qquad j = l+1,\ldots,n\\ q_n[V_{\mathfrak{m}}] &= -(k_n\widetilde{\mathsf{m}} - \widehat{\mathsf{m}}_m)\\ q_{\widehat{1}}[V_{\mathfrak{m}}] &= -(\widehat{k}_1\widehat{\mathsf{m}}_1 - \mathsf{m} - \widehat{\mathsf{m}}_2)\\ q_{\widehat{i}}[V_{\mathfrak{m}}] &= -(\widehat{k}_i\widehat{\mathsf{m}}_i - \widehat{\mathsf{m}}_{i+1} - \widehat{\mathsf{m}}_{i-1})\,, \qquad i = 2\,,\ldots,\, m-1\\ q_{\widehat{m}}[V_{\mathfrak{m}}] &= -(\widehat{k}_m\widehat{\mathsf{m}}_m - \widehat{\mathsf{m}}_{m-1} - \widetilde{\mathsf{m}})\,.\end{aligned} \tag{3.7.26}$$

Let us now compute gauge invariant dressed monopole operators. The last three sets of equations, setting to zero, constitute m equations in total; they give a unique solution for $\widehat{\boldsymbol{m}} = (\widehat{\mathsf{m}}_1, \ldots, \widehat{\mathsf{m}}_m)$ in terms of the fluxes m and $\widetilde{m}$. We denote such a solution by $\widehat{\boldsymbol{m}}^*(\mathsf{m}, \widetilde{\mathsf{m}})$. The integrality of such a solution, together with (3.7.24) and in particular (3.7.25), put restrictions on the relation between K, $\widetilde{K}$ and $\widehat{k}_i$ (with $i = 1, \ldots, m$).

For example, for the case of three J-folds ($m = 2$), solving the last three sets of equations gives

$$\widehat{\mathsf{m}}_1^* = \frac{\mathsf{m}\widehat{k}_2 + \widetilde{\mathsf{m}}}{\widehat{k}_1\widehat{k}_2 - 1} , \qquad \widehat{\mathsf{m}}_2^* = \frac{\mathsf{m} + \widetilde{\mathsf{m}}\widehat{k}_1}{\widehat{k}_1\widehat{k}_2 - 1} \tag{3.7.27}$$

Using (3.7.25) we have

$$\mathsf{m} = -\frac{\widetilde{\mathsf{m}}}{K + \widehat{k}_2 - K\widehat{k}_1\widehat{k}_2} , \qquad \mathsf{m} = -\widetilde{\mathsf{m}}(\widetilde{K} + \widehat{k}_1 - \widetilde{K}\widehat{k}_1\widehat{k}_2) \tag{3.7.28}$$

Suppose that we look for a solution in which m and $\widetilde{\mathsf{m}}$ are non-zero. The integrality of K, $\widetilde{K}$, $\widehat{k}_{1,2}$ implies that

$$K\widehat{k}_1\widehat{k}_2 - (K + \widehat{k}_2) = \widetilde{K}\widehat{k}_1\widehat{k}_2 - (\widetilde{K} + \widehat{k}_1) = \pm 1 \, . \tag{3.7.29}$$

The choice $+1$ sets $\mathsf{m} = \widetilde{\mathsf{m}}$, whereas the choice -1 sets $\mathsf{m} = -\widetilde{\mathsf{m}}$. Using these with (3.7.27), we also obtain the constriants on $\widehat{k}_1$ and $\widehat{k}_2$, namely

$$\frac{\widehat{k}_1 \pm 1}{\widehat{k}_1\widehat{k}_2 - 1} , \ \frac{\widehat{k}_2 \pm 1}{\widehat{k}_1\widehat{k}_2 - 1} \in \mathbb{Z} \, . \tag{3.7.30}$$

Since $\mathsf{m}, \widetilde{\mathsf{m}} \in \mathbb{Z}$, we see that the magnetic lattices given by $\widehat{\mathsf{m}}_1^*$ and $\widehat{\mathsf{m}}_2^*$ "jump" by multiples of $\frac{\widehat{k}_2 \pm 1}{\widehat{k}_1\widehat{k}_2 - 1}$ and $\frac{\widehat{k}_1 \pm 1}{\widehat{k}_1\widehat{k}_2 - 1}$ respectively. If we further require that $\widehat{\mathsf{m}}_1^* = \widehat{\mathsf{m}}_2^* = \mathsf{m}$ (*i.e.* there is no such jump), we have $\widehat{k}_1 = \widehat{k}_2 = K = \widetilde{K} = \pm 2$, assuming that both $\widehat{k}_1$ and $\widehat{k}_2$ are non-zero.

This can easily be generalised to an arbitrary number of J-folds. The generalisation of (3.7.29) is

$$\mathrm{minor}_{1,1}\, M_{CS} = \mathrm{minor}_{m+1,m+1}\, M_{CS} = \pm 1 \tag{3.7.31}$$

These two choices correspond to $\mathsf{m} = \pm\widetilde{\mathsf{m}}$. The integrality of $\widehat{\boldsymbol{m}}^*(\mathsf{m}, \widetilde{\mathsf{m}})$ and $\widehat{\boldsymbol{m}}^*(\mathsf{m}, -\widetilde{\mathsf{m}})$ impose further constraints on $\widehat{k}_j$. The analysis of the moduli space is similar to that presented after (3.5.53).

Two or more cuts in the quiver

The analysis is similar to that of presented around (3.5.74). For the case of two cuts, the quiver is divided into the left, central and right sub-quivers. The analysis for the central part is presented in section 3.5.1, whereas those for the left and right sub-quivers are as presented above for the one cut case. One can repeat this procedure for the case with more than two cuts.

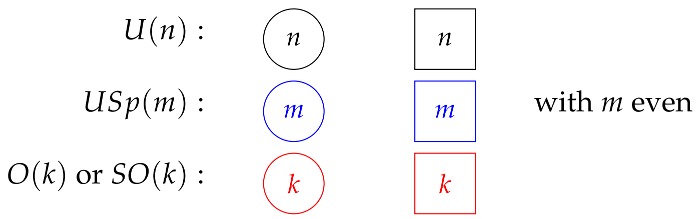
Chapter 4

Variations on S-fold CFTs

4.1 Notations and conventions

Let us state the notations and conventions that will be adopted.

Gauge and global symmetries. In a quiver diagram, we denote the 3d $\mathcal{N} = 4$ vector multiplet in a given gauge group by a circular node, and a flavour symmetry by a rectangular node. A black node with a label n denotes the symmetry group $U(n)$, a blue node with an even label m denotes the symmetry group $USp(m)$, and a red node with a label k denotes the symetry group $O(k)$ or $SO(k)$.



(4.1.1)

We shall be explicit whenever we would like to emphasise whether the group is $O(k)$ or $SO(k)$ We will also deal with the group known as $USp'(2M)$, arising in the worldvolume M physical D3 branes on the $\widetilde{\text{O3}}^+$ plane [23]. Note that under S-duality, $USp'(2M)$ transforms into itself. This is in contrast to the group $USp(2M)$, arising in the worldvolume M physical D3 branes on the O3$^+$ plane, where under the S-duality transforms into $SO(2M+1)$. We denote the algebra corresponding to $USp'(M)$, with M even, in the quiver diagram by a blue node with the label M'. In the case that the brane configuration does not give a clear indication whether the group is $USp(M)$ or $USp'(M)$, we simply denote the label in the corresponding blue node by M.

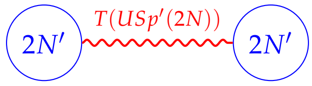
Brane configurations. The brane systems involved in the constructions include D3, D5, NS5 branes, possibly with orientifold planes, that preserve eight supercharges [10, 24, 25, 21, 81]. Each type of branes spans the following directions:

	0	1	2	3	4	5	6	7	8	9
D3, O3	×	×	×				×			
NS5, O5	×	×	×	×	×	×				
D5	×	×	×					×	×	×

(4.1.2)

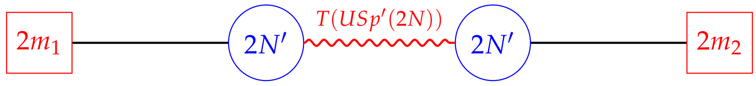
The x^6 direction can be taken to be compact or non-compact.

The $T(G)$ theory. In the following, we also study the 3d $\mathcal{N}=4$ superconformal theory, known as $T(G)$, arising from a half BPS domain wall in the 4d $\mathcal{N}=4$ super-Yang-Mills theory with gauge group G that is self-dual under S-duality [8]. We focus on $G = U(N)$, $SO(2N)$, $USp'(2N)$, G_2. The quiver descriptions for $T(U(N))$ and $T(SO(2N))$ are given in [8], whereas that for $T(USp'(2N))$ are given by [72, sec. 2.5]. The $T(G)$ theory has a global symmetry $G \times G$. The Higgs and the Coulomb branches are both equal to the nilpotent cones $\mathcal{N}_g$, where g is the Lie algebra associated with the group G. We denote the theory $T(G)$ by a wiggly red line connecting two nodes, both labelled by G. As an example, the diagram below denotes the $T(USp'(2N))$ theory, with the global symmetry $USp'(2N) \times USp'(2N)$ being gauged:



(4.1.3)

Furthermore, we can couple this theory to half-hypermultiplets in the fundamental representations of such $USp'(2N)$ gauge groups. For example, if we have m_1 and m_2 flavours of fundamental hypermultiplets under the left and the right gauge groups of (4.1.3) respectively, the corresponding flavour symmetry algebras are $so(2m_1)$ and $so(2m_2)$, and the quiver diagram reads



(4.1.4)

Let us examine the case in which G is not self-dual under S-duality. This can lead to a certain issue in constructing quivers that admit a consistent brane configuration and we shall not consider them. However, for the sake of completeness, let us discuss such an issue explicitly using an example. First of all, $T(G)$ may not be an appropriate notation to use, but one may also need to provide the information regarding the partitions $\boldsymbol{\sigma}$ and $\boldsymbol{\rho}$ of G and its dual group $G^\vee$ to specify the theory $T^\sigma_\rho(G)$ [8]. For example, when $G = USp(4)$, $\boldsymbol{\sigma}$ is a C-partition[1] of 4 whereas $\boldsymbol{\rho}$ is a B-partition[2] of 5, associated with the dual group $G^\vee = SO(5)$ [98, 99, 72]. Given this example, one may would like to consider $T^{[1^4]}_{[1^5]}(USp(4))$, which has a global symmetry $G \times G^\vee = USp(4) \times SO(5)$. Suppose that we couple this theory to matter as in (4.1.4), where we replace the wiggly line (T-link) by $T^{[1^4]}_{[1^5]}(USp(4))$ with $USp(4)$ gauge group on the left and $SO(5)$ gauge group on the right. Then, the m_1 and m_2 fundamental flavours in (4.1.4) give rise to a $SO(2m_1) \times USp(2m_2)$ flavour symmetry, and such half-hypermultiplets transform under the representation $\frac{1}{2}(\mathbf{4};\mathbf{2m_1})$ of $USp(4) \times SO(2m_1)$ and $\frac{1}{2}(\mathbf{5};\mathbf{2m_2})$ of $SO(5) \times USp(2m_2)$, respectively. One may take $m_1 = m_2 = 2$ and form a circular quiver with alternating SO/USp gauge groups with equal rank by gauging both flavour symmetries and couple them to bifundamental matter. Specifically, let us consider a circular quiver with the $USp(4) \times SO(4) \times USp(4) \times SO(5)$ gauge group, where the first $USp(4)$ and the last $SO(5)$ are connected by the T-link and other groups are connected by bifundamental half-hypermultiplets. One cannot realise this theory using a Type IIB brane configuration with an orientifold threeplane and an "S-fold" in a simple way for the following reason. Note that the first $USp(4)$ and the last $SO(5)$ connected by

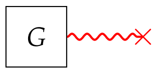
[1] A C-partition is a non- increasing sequence of integers where all the odd parts appear an even number of times.

[2] A B-partition is a non-increasing sequence of integers where all the even parts appear an even number of times.

the T-link must be associated with O3$^+$ and $\widetilde{\text{O3}}^-$ respectively, and as the O3 plane crosses a half-NS5 brane it changes sign. Starting from the left $USp(4)$ as we go through the sequence of the gauge groups to the right, we obtain the sequence of the associated O3 plane to be $(\text{O3}^+, \text{O3}^-, \text{O3}^+, \text{O3}^-)$. However, this is in contradiction to the fact that the $SO(5)$ gauge group must be associated with $\widetilde{\text{O3}}^-$, and not O3$^-$. Due to this reason, we shall not further discuss the case of a T-link associated with a non-self-dual group G.

4.2 Coupling hypermultiplets to a nilpotent cone

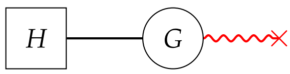
In this section we study the hyperKähler space that arises from coupling hypermultiplets or half-hypermultiplets to nilpotent cone $\mathcal{N}_g$ of the Lie algebra g associated with a gauge group G. We start from the nilpotent cone of g, and denote this geometrical object by



(4.2.1)

Note that a subgroup of G may acts trivially on $\mathcal{N}_g$. For example, we may take G to be $U(N)$; since the symmetry of the corresponding nilpotent cone is really $SU(n)$, the $U(1)$ subgroup of $G = U(N)$ acts trivially on the nilpotent cone.

The symmetry G can be gauged and can then be coupled to hypermultiplets or half-hypermultiplets, which give rise to a flavour symmetry H. We denote the resulting theory by the quiver diagram:



(4.2.2)

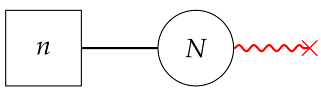
The hyperKähler quotient $\mathcal{H}_{(4.2.2)}$ associated with this diagram is

$$\mathcal{H}_{(4.2.2)} = \frac{\mathcal{H}\left([H]-[G]\right) \times \mathcal{N}_g}{G} \tag{4.2.3}$$

where $\mathcal{H}\left([H]-[G]\right)$ denotes the Higgs branch of quiver $[H]-[G]$. We emphasise that we do not interpret (4.2.2) as a field theory by itself. Instead, we regard it as a notation that can be conveniently used to denote the hyperKähler quotient (4.2.3).

4.2.1 $G = U(N)$ and $H = U(n)/U(1)$

We take $G = U(N)$ and couple n flavours of hypermutiplets to G:



(4.2.4)

The hyperKähler quotient associated with this diagram is

$$\mathcal{H}_{(4.2.4)} = \frac{\mathcal{H}\left([U(n)]-[U(N)]\right) \times \mathcal{N}_{su(N)}}{U(N)} \tag{4.2.5}$$

where $\mathcal{H}\left([U(n)]-[U(N)]\right)$ denotes the Higgs branch of the quiver $[U(n)]-[U(N)]$. The quaternionic dimension is

$$\dim_{\mathbb{H}} \mathcal{H}_{(4.2.4)} = \frac{1}{2}N(N-1) + nN - N^2 \,. \tag{4.2.6}$$

The flavour symmetry in this case is $H = U(n)/U(1)$, whose algebra is $h = su(n)$.

For $N = 1$, $\mathcal{N}_{su(N)}$ is trivial. The quotient (4.2.5) becomes the Higgs branch of the $U(1)$ gauge theory with n flavours. $\mathcal{H}_{(4.2.4)}$, therefore, turns out to be the closure of the minimal nilpotent orbit of $su(n)$, denoted by $\overline{\mathcal{O}}_{(2,1^{n-2})}$ [85, 100]. This space is also isomorphic to the Higgs branch of the $T_{(n-1,1)}(SU(n))$ theory of [8], and is also isomorphic to the reduced moduli space of one $su(n)$ instanton on $\mathbb{C}^2$. It is precisely $n-1$ quaternionic dimensional.

For $N = 2$, it turns out that $\mathcal{H}_{(4.2.4)}$ is the closure $\overline{\mathcal{O}}_{(3,1^{n-3})}$ of the orbit $(3, 1^{n-3})$ of $su(n)$. This is isomorphic to the Higgs branch of the $T_{(n-2,1^2)}(SU(n))$ theory, namely that of the quiver $[U(n)] - (U(2)) - (U(1))$. The quaternionic dimension of this is precisely $2n-3$. This is indeed in agreement with (4.2.6).

For a general N, such that $n \geq N+1$, we see that $\mathcal{H}_{(4.2.4)}$ is in fact

$$\mathcal{H}_{(4.2.4)} = \overline{\mathcal{O}}_{(N+1,1^{n-N-1})} , \tag{4.2.7}$$

and in the special case of $n = N$, we have the nilpotent cone of $su(N)$:

$$\mathcal{H}_{(4.2.4)}|_{n=N} = \overline{\mathcal{O}}_{(N)} = \mathcal{N}_{su(N)} . \tag{4.2.8}$$

One way to verify this proposition is to compute the Hilbert series of $\mathcal{H}_{(4.2.4)}$. This is given by[3]

$$\begin{aligned} H[\mathcal{H}_{(4.2.4)}](t; \boldsymbol{x}) = \int \mathrm{d}\mu_{SU(N)}(\boldsymbol{z}) \oint_{|q|=1} \frac{dq}{2\pi i q} \, \mathrm{PE} \Big[\chi^{su(N)}_{[1,0,\ldots,0]}(\boldsymbol{x}) \chi^{su(N)}_{[0,\ldots,0,1]}(\boldsymbol{z}) q^{-1} t \\ + \chi^{su(N)}_{[0,\ldots,0,1]}(\boldsymbol{x}) \chi^{su(N)}_{[1,0,\ldots,0]}(\boldsymbol{z}) q - \chi^{su(N)}_{[1,0,\ldots,0,1]} t^2 \Big] H[\mathcal{N}_{su(N)}](t, \boldsymbol{z}) \end{aligned} \tag{4.2.9}$$

where $\boldsymbol{x}$ denotes the flavour fugacities of $su(N)$ and $\mathrm{d}\mu_{SU(N)}(\boldsymbol{z})$ denotes the Haar measure of $SU(N)$. We refer the reader to the detail of the characters and the Haar measures in [101]. The Hilbert series of the nilpotent cone of $su(N)$ was computed in [84] and is given by

$$H[\mathcal{N}_{su(N)}](t, \boldsymbol{z}) = \mathrm{PE} \left[\chi^{su(N)}_{[1,0,\cdots,0,1]}(\boldsymbol{z}) t^2 - \sum_{p=2}^{N} t^{2p} \right] . \tag{4.2.10}$$

The Hilbert series (4.2.9) can then be used to checked against the results presented in [85]. In this way, the required nilpotent orbits in (4.2.7) and (4.2.8) can be identified. This technique can also be applied to other gauge groups, as will be discussed in the subsequent subsections. For the sake of brevity of the presentation, we shall not go through further details.

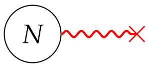
We remark that for $n \geq 2N+1$, the hyperKähler space (4.2.7) is isomorphic the Higgs branch of the $T_{(n-N,1^N)}(SU(n))$ theory[4], which corresponds to the quiver [8]:

$$T_{(n-N,1^N)}(SU(n)) : \quad [U(n)] - (U(N)) - (U(N-1)) - \cdots - (U(1)) . \tag{4.2.11}$$

[3]The plethystic exponential (PE) of a multivariate function $f(x_1, x_2, \ldots, x_n)$ such that $f(0, 0, \ldots, 0) = 0$ is defined as $\mathrm{PE}[f(x_1, x_2, \ldots, x_n)] = \exp\left(\sum_{k=1}^{\infty} \frac{1}{k} f(x_1^k, x_2^k, \ldots, x_n^k) \right)$.

[4]The partition $(n-N, 1^N)$ is indeed the transpose of the partition $(N+1, 1^{n-N-1})$ in (4.2.7).

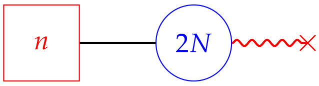
Note that quiver (4.2.4) can be obtained from (4.2.11) simply by replacing the wiggly line by the quiver tail as follows:



$$\longrightarrow \quad (U(N)) - (U(N-1)) - \cdots - (U(1))\,. \tag{4.2.12}$$

4.2.2 $G = USp(2N)$ and $H = O(n)$ or $SO(n)$

We take $G = USp(2N)$ and couple n half-hypermultiplets to G:



(4.2.13)

The corresponding hyperKähler quotient is

$$\mathcal{H}_{(4.2.13)} = \frac{\mathcal{H}\left([SO(n)] - [USp(2N)]\right) \times \mathcal{N}_{usp(2N)}}{USp(2N)}\,. \tag{4.2.14}$$

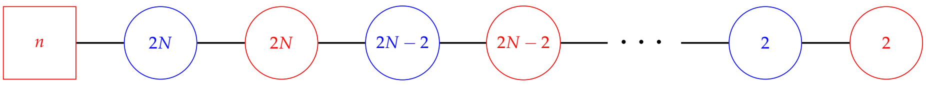
The dimension of this space is

$$\begin{aligned} \dim_{\mathbb{H}} \mathcal{H}_{(4.2.13)} &= nN + \frac{1}{2}\left[\frac{1}{2}(2N)(2N+1) - N\right] - \frac{1}{2}(2N)(2N+1) \\ &= N(n-N-1)\,. \end{aligned} \tag{4.2.15}$$

For $n \geq 2N+1$, the hyperKähler quotient (4.2.14) turns out to be isomorphic to the closure of the nilpotent orbit $(2N+1, 1^{n-(2N+1)})$ of $so(n)$:

$$\mathcal{H}_{(4.2.13)} = \overline{\mathcal{O}}_{(2N+1,1^{n-(2N+1)})}\,. \tag{4.2.16}$$

For even n, say $n = 2m$, this is isomorphic to the Higgs branch of $T_\rho(SO(n))$, with $\rho = (n-2N-1, 1^{2N+1})$,[5] whose quiver description is



(4.2.18)

For odd n, say $n = 2m+1$, this is isomorphic to the Higgs branch of $T_\rho(SO(n))$, with $\rho = (n-2N-1, 2, 1^{2N-2})$ if $n > 2N+1$ and $\rho = (1^{2N})$ if $n = 2N+1$,[6] whose

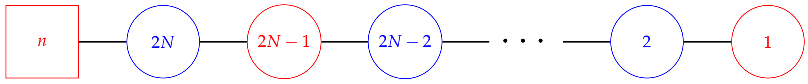
[5]Note that the partition $\rho = (n-2N-1, 1^{2N+1})$ can be obtained from the partition $\lambda = (2N+1, 1^{n-(2N+1)})$ of (4.2.16) by first computing the transpose of λ, and then performing the D-collapse. For example, for $N = 2$ and $m = 4$ (or $n = 8$),

$$\lambda = (5, 1^4) \quad \xrightarrow{\text{transpose}} \quad (4, 1^4) \quad \xrightarrow{D\text{-coll.}} \quad \rho = (3, 1^5)\,. \tag{4.2.17}$$

[6]Note that the partition $\rho = (n-2N-1, 2, 1^{2N-2})$ can be obtained from the partition $\lambda = (2N+1, 1^{n-(2N+1)})$ of (4.2.16) by first computing the transpose of λ, subtracting 1 from the last entry, and then performing the C-collapse. For example, for $N = 3$ and $m = 4$ (or $n = 9$),

$$\lambda = (7, 1^2) \quad \xrightarrow{\text{transpose}} \quad (3, 1^6) \quad \longrightarrow \quad (3, 1^5) \quad \xrightarrow{C\text{-coll.}} \quad (2^2, 1^4)\,. \tag{4.2.19}$$

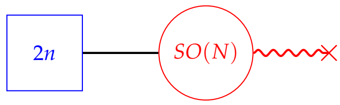
quiver description is



(4.2.20)

4.2.3 $G = SO(N)$ or $O(N)$ and $H = USp(2n)$

Let us first take $G = SO(N)$ and take $H = USp(2n)$.



(4.2.21)

This diagram defines the hyperKähler quotient

$$\mathcal{H}_{(4.2.21)} = \frac{\mathcal{H}\left([USp(2n)] - [SO(N)]\right) \times \mathcal{N}_{so(N)}}{SO(N)} \,. \tag{4.2.22}$$

The quaternionic dimension of this quotient is

$$\dim_{\mathbb{H}} \mathcal{H}_{(4.2.21)} = \begin{cases} m(2n-m)\,, & N = 2m \\ m(2n-m-1)+n\,, & N = 2m+1 \end{cases} \,. \tag{4.2.23}$$

It is interesting to examine (4.2.22) for a few special cases. For $N = 2n$ or $N = 2n+1$ or $N = 2n-1$, we find that (4.2.22) is in fact the nilpotent cone $\mathcal{N}_{usp(2n)}$ of $usp(2n)$, whose quaternionic dimension is n^2:

$$\mathcal{H}_{(4.2.21)}|_{N=2n} = \mathcal{H}_{(4.2.21)}|_{N=2n\pm1} = \mathcal{N}_{usp(2n)} \,. \tag{4.2.24}$$

This statement can be checked using the Hilbert series:

$$\begin{aligned} H[\mathcal{H}_{(4.2.21)}](t;\boldsymbol{x}) &= \int d\mu_{SO(N)}(\boldsymbol{z}) \,\mathrm{PE}\Big[\chi^{C_n}_{[1,0,\dots,0]}(\boldsymbol{x})\chi^{so(N)}_{[1,0,\dots,0]}(\boldsymbol{z})t \\ &\qquad - \chi^{so(N)}_{[0,1,0,\dots,0]}(\boldsymbol{z})t^2\Big] H[\mathcal{N}_{so(N)}](t,\boldsymbol{z}) \\ &= \mathrm{PE}\left[\chi^{C_n}_{[2,0,\dots,0]}(\boldsymbol{x})t^2 - \sum_{j=1}^{n} t^{4j}\right] \,, \text{ if } N = 2n \text{ or } 2n\pm1\,. \end{aligned} \tag{4.2.25}$$

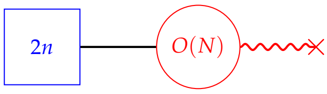
where the Haar measure and the relevant characters are given in [101]. The last line is indeed the Hilbert series of the nilpotent cone $\mathcal{N}_{usp(2n)}$ [85].

It is important to note that the quotient (4.2.22) is not the closure of a nilpotent orbit in general. For example, let us take $n = 4$ and $N = 3$, *i.e.* $G = SO(3)$ and $H = USp(8)$. The Hilbert series takes the form

$$H[\mathcal{H}_{(4.2.21)}|_{n=4,N=3}](t;\boldsymbol{x}) = \mathrm{PE}\left[\chi^{C_4}_{[2,0,0,0]}t^2 + (\chi^{C_4}_{[0,0,1,0]} + \chi^{C_4}_{[1,0,0,0]})t^3 - t^4 + \dots\right] \,. \tag{4.2.26}$$

Observe that there are generators with $SU(2)_R$-spin $3/2$ in the third rank antisymmetric representation $\wedge^3[1,0,0,0] = [0,0,1,0] + [1,0,0,0]$ of $USp(8)$. These should be identified as "baryons". Using Namikawa's theorem [102], which states that all generators of the closure of a nilpotent orbit must have $SU(2)_R$-spin 1 (see also [103]), we conclude that $\mathcal{H}_{(4.2.21)}|_{n=4,N=3}$ is not the closure of a nilpotent orbit. In general, these baryons can be removed by taking gauge group to be $O(N)$, instead of $SO(N)$. The reason is because the $O(N)$ group does not have an epsilon tensor as an invariant tensor, whereas the $SO(N)$ group has one.

Let us now take $G = O(N)$ and take $H = USp(2n)$:



(4.2.27)

This diagram defines the hyperKähler quotient

$$\mathcal{H}_{(4.2.27)} = \frac{\mathcal{H}\left([USp(2n)] - [O(N)]\right) \times \mathcal{N}_{so(N)}}{O(N)} \,. \tag{4.2.28}$$

The dimension of this hyperKähler space is the same as (4.2.23). This quotient turns out to be isomorphic to the closure of the following nilpotent orbit of $usp(2n)$:

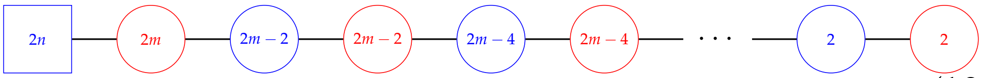
$$\mathcal{H}_{(4.2.27)} = \begin{cases} \overline{\mathcal{O}}_{(N,2,1^{2n-N-2})} & N \text{ even} \\ \overline{\mathcal{O}}_{(N+1,1^{2n-N-1})} & N \text{ odd} \end{cases} \,. \tag{4.2.29}$$

In the special case where $N = 2n$, $N = 2n - 1$ or $N = 2n + 1$, we have

$$\mathcal{H}_{(4.2.27)}|_{N=2n} = \mathcal{H}_{(4.2.27)}|_{N=2n\pm1} = \overline{\mathcal{O}}_{(2n)} = \mathcal{N}_{usp(2n)} \,, \tag{4.2.30}$$

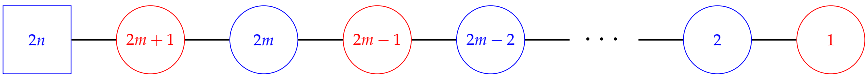
which is the same as (4.2.24).

For even $N = 2m$, $\mathcal{H}_{(4.2.27)}$ is isomorphic to the Higgs branch of $T_\rho(USp(2n))$ theory, with $\rho = (2n - N + 1, 1^N)$, whose quiver description is



(4.2.31)

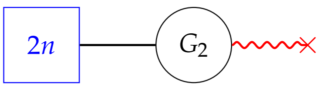
On the other hand, for odd $N = 2m + 1$, $\mathcal{H}_{(4.2.27)}$ is isomorphic to the Higgs branch of $T_\rho(USp'(2n))$ theory, with $\rho = (2n - N + 1, 1^{N-1})$, whose quiver description is



(4.2.32)

4.2.4 $G = G_2$ and $H = USp(2n)$

We take $G = G_2$ and $H = USp(2n)$:



(4.2.33)

This diagram defines the hyperKähler quotient

$$\mathcal{H}_{(4.2.33)} = \frac{\mathcal{H}\left([USp(2n)] - [G_2]\right) \times \mathcal{N}_{g_2}}{G_2} \,. \tag{4.2.34}$$

For $n \geq 2$, the quaternionic dimension of this space is

$$\dim_{\mathbb{H}} \mathcal{H}_{(4.2.33)} = 7n + \frac{1}{2}(14 - 2) - 14 = 7n - 8 \,, \tag{4.2.35}$$

and the Hilbert series of (4.2.34) is given by

$$H[\mathcal{H}_{(4.2.34)}](t,\boldsymbol{x}) = \int \mathrm{d}\mu_{G_2}(\boldsymbol{z})\,\mathrm{PE}\left[\chi^{G_2}_{[1,0]}(\boldsymbol{z})\chi^{usp(2n)}_{[1,0,\dots,0]}(\boldsymbol{x})t - \chi^{G_2}_{[0,1]}(\boldsymbol{z})t^2\right] H[\mathcal{N}_{\mathfrak{g}_2}](t,\boldsymbol{z})\,, \tag{4.2.36}$$

where the relevant characters and the Haar measure is given in [101], and the Hilbert series of the nilpotent cone of G_2 can be obtained from [104, Table 4]. The special case of $n=2$ is particularly simple. The corresponding space is a complete intersection whose Hilbert series is

$$H[\mathcal{H}_{(4.2.33)}|_{n=2}](t;x_1,x_2) = \mathrm{PE}\left[\chi^{C_2}_{[2,0]}(x_1,x_2)t^2 + \chi^{C_2}_{[1,0]}(x_1,x_2)t^3 - t^8 - t^{12}\right]\,. \tag{4.2.37}$$

Note that $\mathcal{H}_{(4.2.33)}$ is not the closure of a nilpotent orbit, due to the existence of a generator at $SU(2)_R$-spin $3/2$ and Namikawa's theorem.

The case of $n=1$ needs to be treated separately, since (4.2.35) becomes negative. We claim that

$$\mathcal{H}_{(4.2.33)}|_{n=1} = \mathbb{C}^2/\mathbb{Z}_2 = \mathcal{N}_{su(2)}\,. \tag{4.2.38}$$

The reason is as follows. Let us denote by Q^i_a the half-hypermultiplets in the fundamental representation of the G_2 gauge group[7], where $i,j,k = 1,2$ are the $USp(2)$ flavour indices and $a,b,c,d = 1,\dots,7$ are the G_2 gauge indices. Let us also denote by X_{ab} the generators of the nilpotent cone of G_2. Transforming in the adjoint representation of G_2, X_{ab} is an antisymmetric matrix satisfying[8]

$$f^{abc}X_{ab} = 0\,; \tag{4.2.39}$$

this is because $\wedge^2[1,0] = [0,1]+[1,0]$. Moreover, being the generators of the nilpotent cone, X_{ab} satisfy

$$\mathrm{tr}(X^2) = \delta^{ad}\delta^{bc}X_{ab}X_{cd} = 0\,, \qquad \mathrm{tr}(X^6) = 0\,. \tag{4.2.40}$$

The moment map equations for G_2 read

$$\epsilon_{ij}Q^i_aQ^j_b = X_{ab}\,. \tag{4.2.41}$$

The generators of (4.2.34), for $n=1$, are

$$M^{ij} = \delta_{ab}Q^i_aQ^i_b \tag{4.2.42}$$

transforming in the adjoint representation of $USp(2)$. Note that baryons vanish:

$$f^{abc}Q^i_aQ^j_bQ^k_c = 0\,, \qquad \widetilde{f}^{abcd}Q^i_aQ^j_bQ^k_cQ^l_d = 0\,, \tag{4.2.43}$$

because $i,j,k,l = 1,2$. Other gauge invariant combinations also vanish; for example, $X_{ab}Q^i_aQ^j_b$ has one independent component and it vanishes thanks to (4.2.40) and

[7] The three independent invariant tensors for G_2 can be taken as (1) the Kronecker delta δ^{ab}, (2) the third-rank antisymmetric tensor f^{abc} and (3) the fourth-rank antisymmetric tensor $\widetilde{f}^{abcd}$. See *e.g.* [105] for more details.

[8] Using the identity $f^{[abc}f^{cde]} = \widetilde{f}^{abde}$ (see [105, (A.13)]), it follows immediately from this relation that $\widetilde{f}^{abde}X_{ab}X_{de} = 0$.

(4.2.41). Furthermore, the square of M vanishes:

$$\epsilon_{il}\epsilon_{jk}M^{ij}M^{kl} = (\epsilon_{il}Q^i_a Q^l_b)(\epsilon_{jk}Q^j_a Q^k_b) \overset{(4.2.41)}{=} \mathrm{tr}(X^2) \overset{(4.2.40)}{=} 0\,. \tag{4.2.44}$$

Therefore, we reach the conclusion (4.2.38).

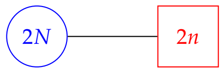
4.3 Models with orientifold fiveplanes

In this section, we consider models that arise from brane systems involving an S-fold and orientifold 5-planes. For the latter, we focus on the case of the $O5^-$ plane and postpone to discussion about the $O5^+$ plane to section 4.6. In the absence of the S-fold, such models and the corresponding mirror theories were studied in detailed in [25, 8]. We start this section by reviewing the latter and then discuss the insertion of an S-fold in the subsequent subsections.

4.3.1 The cases without an S-fold

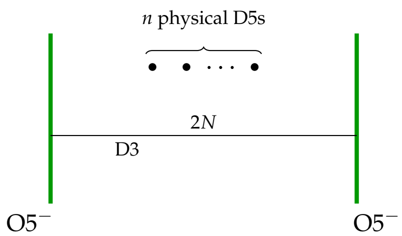
We consider three types of models, depending on the presence of NS5 branes and their positions relative to each $O5^-$ plane [25].

The $USp(2N)$ gauge theory with n flavours. The quiver diagram is



(4.3.1)

The brane system for this quiver is

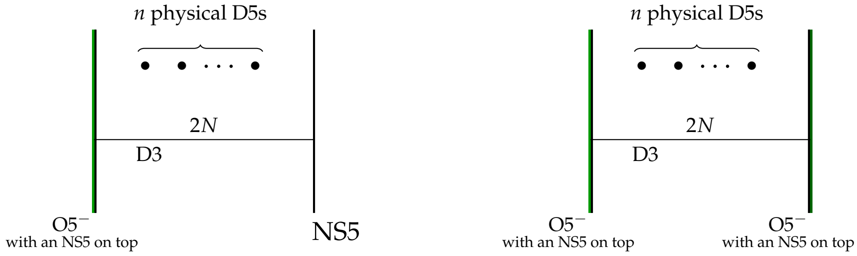


(4.3.2)

The $U(2N)$ gauge theory with one or two rank-two antisymmetric hypermultiplets and n flavours in the fundamental representation. The quiver diagrams are

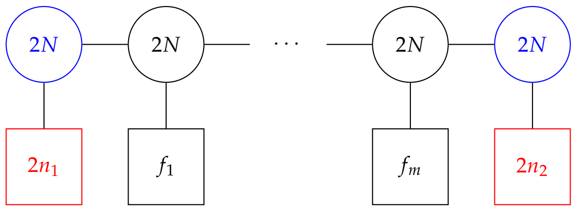
(4.3.3)

The brane systems for the cases with one adjoint and two adjoints are, respectively, as follows:



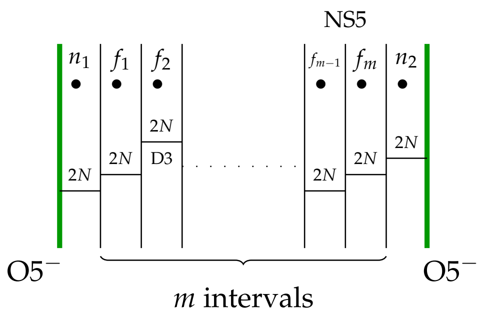
(4.3.4)

The $USp(2N) \times U(2N)^m \times USp(2N)$ gauge theory with $(n_1, f_1, \ldots, f_m, n_2)$ flavours in the fundamental representations under each gauge group. The quiver diagram is



(4.3.5)

The brane system for this quiver is

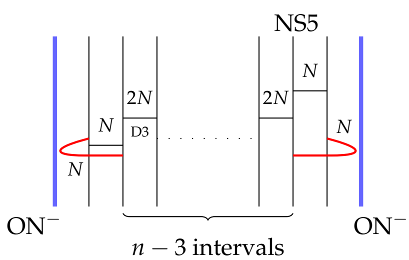


(4.3.6)

where each black dot with a label k denotes k physical D5 branes, and each black vertical line denotes a physical NS5 brane.

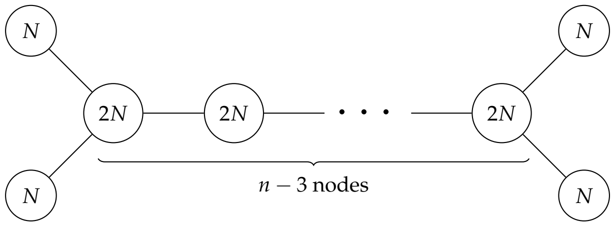
Let us now discuss their mirror theories and the corresponding brane configurations. Under the S-duality, each NS5 brane becomes a D5 brane and vice-versa, and an O5$^-$ plane becomes an ON$^-$ plane. The following results can be obtained [25].

A mirror of (4.3.1). The brane system for this is



(4.3.7)

Each of the left and the right boundaries contains an ON$^-$ plane, which is an S-dual of the O5$^-$ plane. The combination of an ON$^-$ plane and one NS5 brane is also known as ON^0 and was studied in detail in [24, 106]. The way that the D3-branes stretch between two NS5 branes at each boundary is depicted in red. The corresponding theory can be represented by the following quiver diagram:

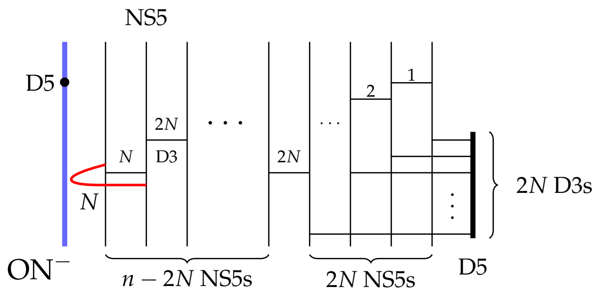


(4.3.8)

This is indeed the affine Dynkin diagram of the D_n algebra [24].

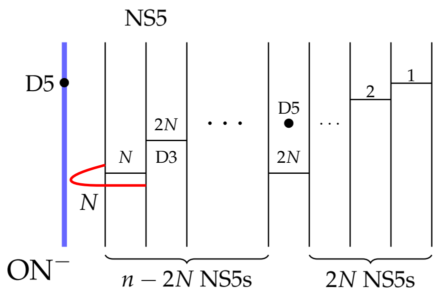
Mirrors of (4.3.3). We consider two cases as follows:

1. The case of one antisymmetric hypermultiplet. In this case the brane configuration of the mirror theory is



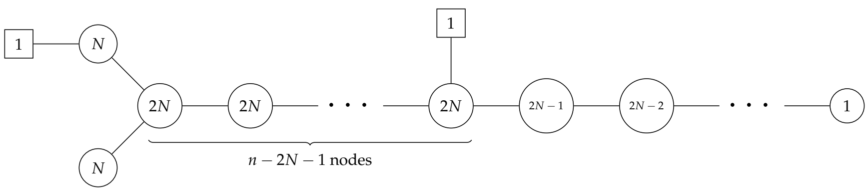
(4.3.9)

One can then move the rightmost D5 brane into the interval and obtain



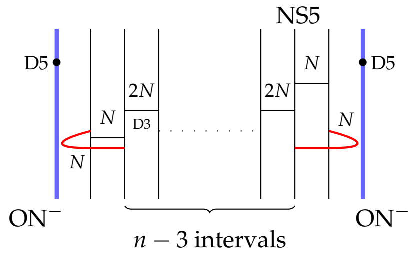
(4.3.10)

Hence the corresponding quiver is



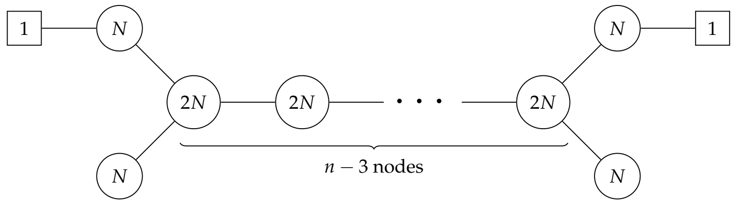
(4.3.11)

2. The case of two antisymmetric hypermultiplets. In this case the brane configuration of the mirror theory is



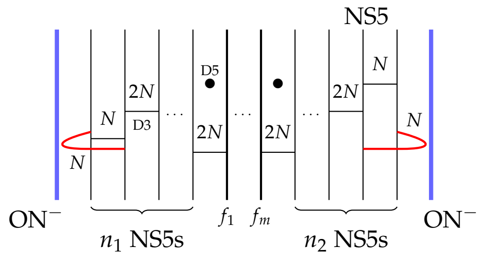
(4.3.12)

The corresponding quiver theory is



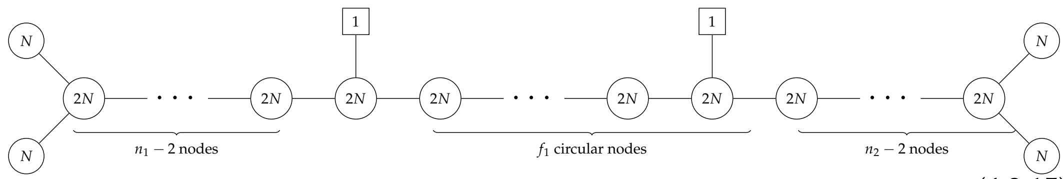
(4.3.13)

A mirror of (4.3.5). The brane construction is



(4.3.14)

where the boldface vertical line labelled by f_j (with $j = 1, \ldots, m$) denotes a set of f_j NS5 branes, with $2N$ D3 branes stretching between two successive NS5 branes. Note that there is also one D5 brane at the interval between each set. For simplicity, let us present the quiver for the case of $m = 1$:



(4.3.15)

This can be easily generalised to the case of $m > 1$ by simply repeating the part under the second brace with $f_2, f_3, \ldots, f_m$ in a consecutive manner.

4.3.2 The cases with an S-fold

In this subsection, we insert an S-fold into a brane interval of the aforementioned configurations. In general, the resulting quiver theory contains a $T(U(N))$ link connecting two gauge nodes corresponding to the interval where we put the S-fold. The mirror configuration can simply be obtained by inserting the S-fold in the same position in the S-dual brane configuration. In the following, the moduli spaces of such a theory and its mirror are analysed in detail.

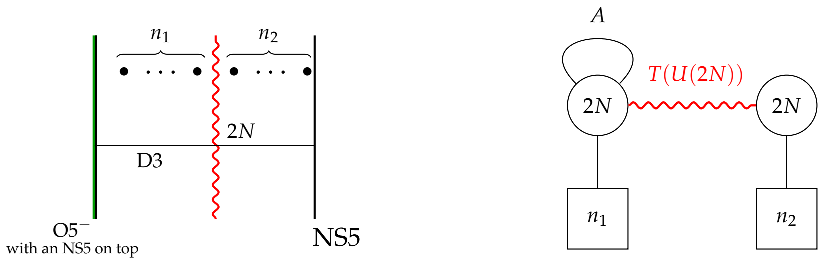
We make the following important observation. The Higgs (*resp.* Coulomb) branch of a given theory gets exchanged with the Coulomb (*resp.* Higgs) branch of the mirror theory in a "regular way", provided that

1. the S-fold is not inserted "too close" to the orientifold plane; and
2. the S-fold is not inserted in the "quiver tail", arising from a set of D3 branes connecting a D5 brane with distinct NS5 branes.

Subsequently, we shall give more precise statements for these two points using various examples. In other words, we use mirror symmetry as a tool to indicate the consistency of the insertion of an S-fold to the brane system with an orientifold fiveplane.

Models with one or two antisymmetric hypermultiplets

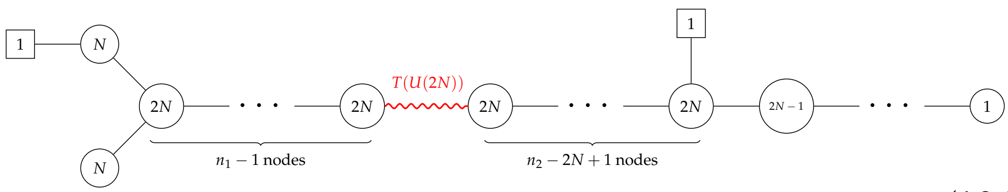
In this subsection, we focus on the models with one antisymmetric hypermultiplet for definiteness. The case for two antisymmetric hypermultiplets can be treated almost in the same way. Let us insert an S-fold in the left diagram in (4.3.4) such that there are n_1 physical D5 branes on the left of the S-fold and there are n_2 physical D5 branes on the right. The resulting theory is



(4.3.16)

The case in which $n_1 \geq 2$ and $n_2 \geq 2N$

The corresponding mirror theory is



(4.3.17)

The condition $n_1 \geq 2$, $n_2 \geq 2N$ ensures that the $T(U(2N))$ link in the mirror theory (4.3.17) stay between the first $U(2N)$ gauge node and the $U(2N)$ gauge node with 1 flavour.

The Higgs branch of theory (4.3.16) has dimension

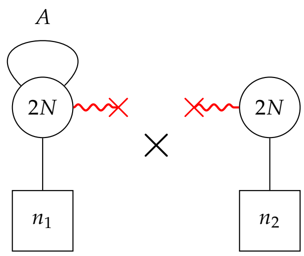
$$\begin{aligned}\dim_{\mathbb{H}} \mathcal{H}_{(4.3.16)} &= 2Nn_1 + \frac{1}{2}2N(2N-1) + 2\cdot\frac{1}{2}(4N^2-2N) + 2Nn_2 \\ &\quad - 4N^2 - 4N^2 \\ &= N(2n_1 + 2n_2 - 2N - 3),\end{aligned} \tag{4.3.18}$$

while the Coulomb branch is empty because there are only two gauge nodes connected by a $T(U(2N))$-link

$$\dim_{\mathbb{H}} \mathcal{C}_{(4.3.16)} = 0. \tag{4.3.19}$$

Since the moduli space of $T(U(2N))$ contains the Higgs and Coulomb branches, each of which is isomorphic to the nilpotent cone of $SU(2N)$, it follows that the Higgs branch of (4.3.16) also splits into a product of two hyperKähler spaces which can be written in the notation of section 4.2 as

$$\mathcal{H}_{(4.3.16)} = \qquad (4.3.20)$$



The symmetry of $\mathcal{H}_{(4.3.16)}$ is $U(n_1) \times (U(n_2)/U(1))$, coming from the first and second factors respectively. According to (4.2.7) and below, the hyperKähler space corresponding to the second factor is identified with $\overline{\mathcal{O}}_{(2N+1,1^{n_2-2N-1})}$ for $n_2 \geq 2N+1$ and $\overline{\mathcal{O}}_{(2N)}$ for $n_2 = 2N$.

The mirror theory (4.3.17) has the following Coulomb branch dimension

$$\begin{aligned} \dim_{\mathbb{H}} \mathcal{C}_{(4.3.17)} &= N + N + (2N)(n_1 + n_2 - 2N - 2) + \sum_{i=1}^{2N-1} i \\ &= N(2n_1 + 2n_2 - 2N - 3), \end{aligned} \qquad (4.3.21)$$

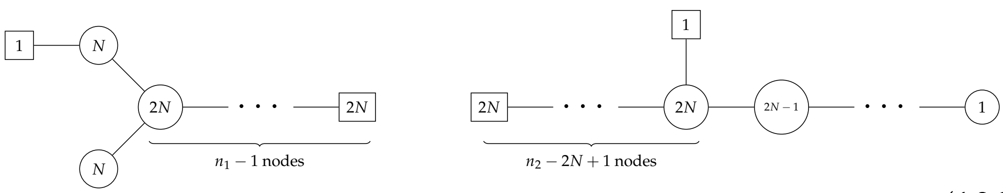
while the Higgs branch has dimension

$$\begin{aligned} \dim_{\mathbb{H}} \mathcal{H}_{(4.3.17)} =& N + 4N^2 + 4N^2(n_1 + n_2 - 2N - 1 - 1) + (4N^2 - 2N) \\ & + 2N + \sum_{i=1}^{2N-1} i(i+1) - 2N^2 - 4N^2(n_1 + n_2 - 2N) \\ & - \sum_{i=1}^{2N-1} i^2 = 0 \end{aligned} \qquad (4.3.22)$$

Indeed, we find an agreement for the dimensions of the Higgs and Coulomb branches under mirror symmetry, namely

$$\dim_{\mathbb{H}} \mathcal{C}_{(4.3.16)} = \dim_{\mathbb{H}} \mathcal{H}_{(4.3.17)}, \qquad \dim_{\mathbb{H}} \mathcal{C}_{(4.3.17)} = \dim_{\mathbb{H}} \mathcal{H}_{(4.3.16)}. \qquad (4.3.23)$$

It should be pointed out the the Coulomb branch of (4.3.17) is also a product of two hyperKähler spaces. The reason is that the nodes that are connected by the $T(U(2N))$ link do not contribute to the Coulomb branch and hence can be taken as flavours nodes. Therefore, the Coulomb branch of (4.3.17) is the product of the Coulomb branches of the following theories:

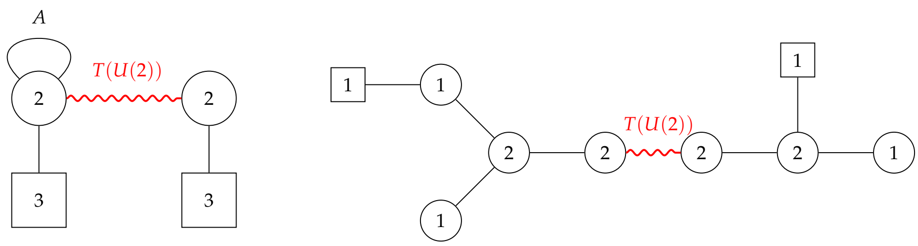


(4.3.24)

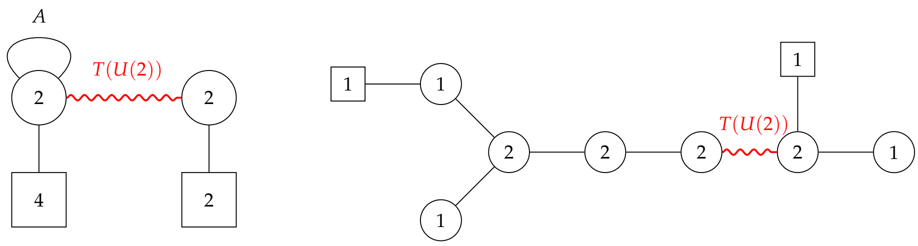
Under mirror symmetry, each of the factor in the product (4.3.20) is mapped to the Coulomb brach of each of the above quiver. Let us examine the symmetry of the

Coulomb branch using the technique of [8]. In the left quiver, all balanced gauge nodes form a Dynkin diagram of A_{n_1-1}; together with the top left node which is overbalanced, these give rise to the global symmetry algebra $A_{n_1-1} \times u(1)$, corresponding to $U(n_1)$. In the right quiver, all gauge nodes are balanced; these give rise to the symmetry algebra A_{n_2-1}, corresponding to $U(n_2)/U(1)$. This is agreement of the symmetry of the Higgs branch $\mathcal{H}_{(4.3.16)}$.

It is worth commenting on the distribution of the flavours in theory (4.3.16). It is clear from the computation of the dimension of the Higgs branch (4.3.18) that one can change n_1 and n_2 keeping their sum $n = n_1 + n_2$ constant, without changing the dimension of the Higgs branch. However, as can be clearly seen from (4.3.20), the structure of the Higgs branch depends on n_1 and n_2. In addition, modifying the distribution of the flavour will change the position of the $T(U(2N))$ link in the mirror theory (4.3.17). Let us focus the case of $N = 1$ with $n_1 = 3$, $n_2 = 3$ and $n_1 = 4$, $n_2 = 2$. The theories and their mirrors are



(4.3.25)



(4.3.26)

As explained in (4.3.20), the Higgs branch of the left diagram in each case splits into a product of two hyperKähler spaces. According to (4.2.8), the second factor in each line is the Hilbert series for the closure of the nilpotent orbit $\overline{\mathcal{O}}_{(3)}$ and $\overline{\mathcal{O}}_{(2)}$, coincident with the Higgs branch of the theories $T(SU(3))$ and $T(SU(2))$ respectively. The unrefined Hilbert series for the first factor is

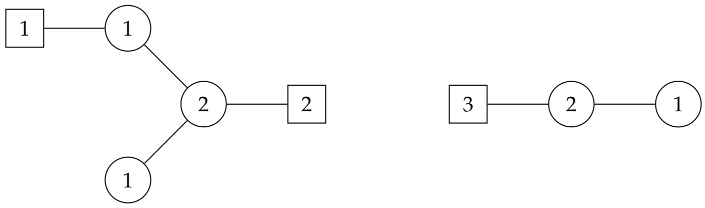
$$\begin{aligned}
&\oint_{|z|=1} \frac{dz}{2\pi i z}(1-z^2) \oint_{|q|=1} \frac{dq}{2\pi i q} \, \mathrm{PE}\left[n_1(z+z^{-1})(q+q^{-1}) \right. \\
&\qquad \left. + (q^2+q^{-2})t + (z^2+1+z^{-2})t^2 - t^4 - (z^2+1+z^{-2}+1)t^2 \right] \\
&\qquad \times \mathrm{PE}\left[(z^2+1+z^{-2})t^2 - t^4 \right] .
\end{aligned} \tag{4.3.27}$$

We therefore arrive at the following results:

$$\begin{aligned}
H[\mathcal{H}^{n_1=3,n_2=3}_{(4.3.16)}] &= \mathrm{PE}\left[9t^2+6t^3-t^4-6t^5-10t^6+\ldots\right] \mathrm{PE}\left[8t^2-t^4-t^6\right], \\
H[\mathcal{H}^{n_1=4,n_2=2}_{(4.3.16)}] &= \mathrm{PE}\left[16t^2+12t^3-t^4-32t^5-54t^6+\ldots\right] \mathrm{PE}\left[3t^2-t^4\right],
\end{aligned} \tag{4.3.28}$$

These indicate that the symmetry of the Higgs branch is $U(n_1) \times (U(n_2)/U(1))$.

Of course, the above Hilbert series can also be obtained from the Coulomb branch of the corresponding mirror theory. As an example, as stated in (4.3.24), for $n_1 = 4, n_2 = 2$, the Coulomb branch of the right quiver of (4.3.26) is a product of the Coulomb branches of the following theories:



(4.3.29)

The Coulomb branch Hilbert series of the left quiver can be computed as follows:

$$\begin{aligned} &\sum_{a_1 \geq a_2 > -\infty} \sum_{m \in \mathbb{Z}} \sum_{n \in \mathbb{Z}} t^{2\Delta(\boldsymbol{a},m,n)} P_{U(2)}(t,\boldsymbol{a}) P_{U(1)}(t,m) P_{U(1)}(t,n) \\ &= \mathrm{PE}\left[16t^2 + 20t^3 - 12t^5 - 32t^6 + \ldots\right], \end{aligned} \tag{4.3.30}$$

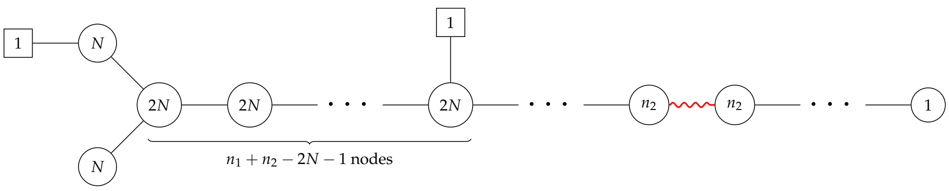
with $\boldsymbol{a} = (a_1, a_2)$,

$$\begin{aligned} \Delta(\boldsymbol{a},m,n) = \Delta_{U(2)-U(1)}(\boldsymbol{a},m) + \Delta_{U(2)-U(1)}(\boldsymbol{a},n) + \Delta_{U(2)-U(2)}(\boldsymbol{a},0) \\ + \Delta_{U(1)-U(1)}(m,0) - \Delta^{\mathrm{vec}}_{U(2)}(\boldsymbol{a}) \end{aligned} \tag{4.3.31}$$

and all of the other notations are defined in (4.6.10). This is indeed equal to the first factor in the first line of (4.3.28). The right quiver in (4.3.29) is the $T(SU(3))$ theory whose Coulomb and Higgs branch Hilbert series is equal to the second factor in the first line of (4.3.28).

Issues regarding S-folding the quiver tail

Let us consider the case in which $n_2 < 2N$. In this case, in the mirror theory (4.3.11), the T-link appears on right of the $U(2N)$ node that is attached with one flavour. Let us suppose that the T-link connects two $U(n_2)$ gauge nodes where $1 \leq n_2 \leq 2N - 1$.



(4.3.32)

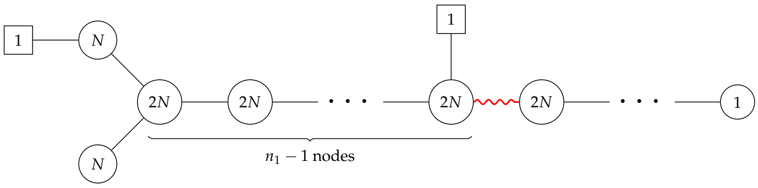
The Higgs branch dimension of such theory is

$$\dim_{\mathbb{H}} \mathcal{H}_{(4.3.32)} = \dim_{\mathbb{H}} \mathcal{H}_{(4.3.11)} + (n_2^2 - n_2) - n_2^2 = 2N - n_2 \,. \tag{4.3.33}$$

Observe that this is non-zero for $1 \leq n_2 \leq 2N - 1$. However, as in (4.3.19), we have $\dim_{\mathbb{H}} \mathcal{C}_{(4.3.16)} = 0$ for any n_2, since the two gauge nodes are connected by a T-link. Hence, this is inconsistent with mirror symmetry, based on our assumption that the gauge nodes connected by a T-link do not contribute to the Coulomb branch. One possible explanation of this inconsistency is that, in the presence of the S-fold, when move the D5 brane into the interval between NS5 branes, as depicted in (4.3.9), such a D5 brane has to cross the S-fold. Since S-fold can be regarded as the duality wall, the aforementioned D5 brane turns into an NS5 brane, with fractional D3 branes ending on it. In this sense, the mirror theory is not (4.3.32).

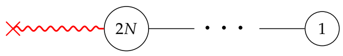
We shall see in section 4.3.3 that, from the perspective of the duality frames in which the quivers do not contain a T-link, the range $1 \leq n_2 \leq 2N-1$ corresponds, in one frame, to a problematic quiver and, in the other frame, to a theory which is not "good" in the sense of [8].

Now let us consider the following possibility:



(4.3.34)

In the brane picture (4.3.10), this corresponds to putting the S-fold just next to the right of the D5 brane located in the the $(2N)$-th interval from the right. This also corresponds to taking $n_2 = 2N$. As before, the Higgs branch of this theory is expected to be a product of two hyperKähler spaces, with one factor being



(4.3.35)

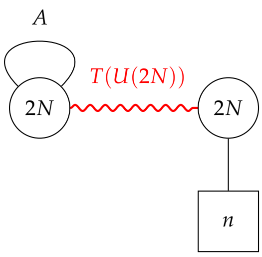
The Higgs branch dimension turns out to be negative if one assume that all gauge groups are completely broken:

$$\frac{1}{2}(4N^2 - 2N) + \frac{1}{2}(2N-1)(2N) - (2N)^2 = -2N \,. \tag{4.3.36}$$

Since the case of $n_2 = 2N$ has been discussed earlier, we should not use (4.3.34) to describe this case and we shall not explore this possibility further.

Issues regarding putting the S-fold "too close" to the orientifold plane

Consider the model with one rank-two antisymmetric hypermultiplet where we put an S-fold next to the $O5^-$ plane in the left diagram of (4.3.4). In this case we have $n_1 = 0$ and $n_2 = n$ (with $n \geq 2N$). The corresponding quiver diagram is

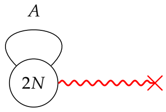


(4.3.37)

The dimension of the Higgs branch is

$$\begin{aligned} \dim_{\mathbb{H}} \mathcal{H}_{(4.3.37)} &= \frac{1}{2}(2N)(2N-1) + (4N^2 - 2N) + 2Nn - 4N^2 - 4N^2 \\ &= 2Nn - 2N^2 - 3N \,, \end{aligned} \tag{4.3.38}$$

assuming that the gauge symmetry is completely broken. For a given N, this is positive for a sufficiently large n. However, it is also worth pointing out that if we split the above Higgs branch into a product as in (4.3.20), we see that the first factor



(4.3.39)

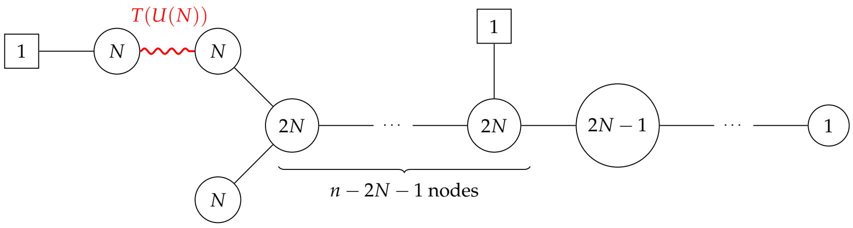
has a negative dimension, provided that the gauge symmetry $U(2N)$ is completely broken:

$$\frac{1}{2}(4N^2 - 2N) + \frac{1}{2}(2N)(2N-1) - (2N)^2 = -2N\,. \tag{4.3.40}$$

Since both gauge nodes are connected by the T-link, we expect that

$$\dim_{\mathbb{H}} \mathcal{C}_{(4.3.37)} = 0 \tag{4.3.41}$$

The putative mirror theory can be obtained by inserting an S-fold next to the ON^- plane in (4.3.10). The corresponding quiver is



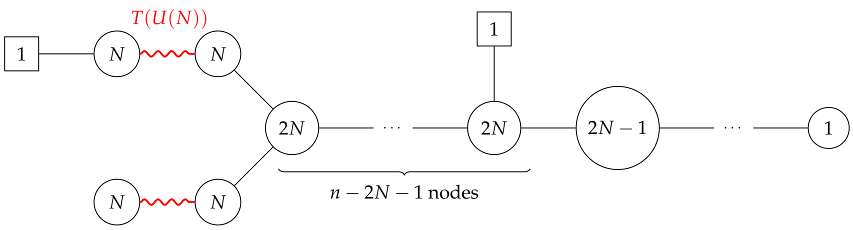
(4.3.42)

The Higgs and Coulomb branch dimensions read

$$\begin{aligned}
\dim_{\mathbb{H}} \mathcal{C}_{(4.3.42)} &= N + 2N(n-2N-1) + \sum_{i=1}^{2N-1} i = 2Nn - 2N^2 - 2N\,, \\
\dim_{\mathbb{H}} \mathcal{H}_{(4.3.42)} &= N + (N^2 - N) + 2N^2 + 2N^2 + 4N^2(n-2N-2) \\
&\qquad + 2N + \sum_{i=1}^{2N-1} i(i+1) - N^2 - N^2 - N^2 \\
&\qquad - 4N^2(n-2N-1) - \sum_{i=1}^{2N-1} i(i+1) \\
&= N\,.
\end{aligned} \tag{4.3.43}$$

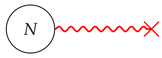
We see that these are inconsistent with mirror symmetry, if we assume that the gauge symmetry is completely broken and that the circular nodes that are connected by a T-link do not contribute to the Coulomb branch. We see that these assumptions are violated or (4.3.42) is not a mirror theory of (4.3.37) if we insert the S-fold next to the orientifold plane.

A similar issue also happens if we take $n_1 = 1$ and $n_2 = n - 1$ (with $n - 1 \geq 2N$). In which case, the putative mirror theory looks like



(4.3.44)

Upon computing the Higgs branch of this theory, the lower left part contributes a factor:



(4.3.45)

Assuming that the gauge symmetry is completely broken, we obtain a negative Higgs branch dimension:

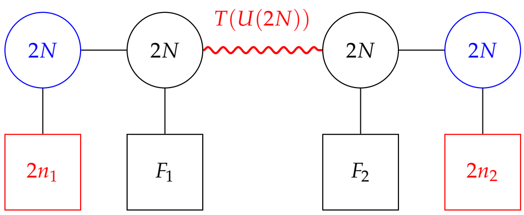
$$\frac{1}{2}(N^2 - N) - N^2 = -\frac{1}{2}N(N+1)\,. \tag{4.3.46}$$

This, again, confirms the statement that under the aforementioned assumptions, the S-fold cannot be inserted "too close" to the orientifold plane ($n_1 \geq 2$). In other words, in order for the S-fold to co-exist with an orientifold fiveplane, it must be "shielded" by a sufficient number of fivebranes.

Similarly to the preceding case of $1 \leq n_2 \leq 2N-1$, we can also see the issue regarding $0 \leq n_1 \leq 1$ from the perspective of the duality frames in which the quivers do not contain a T-link in section 4.3.3.

S-folding the $USp(2N) \times U(2N) \times USp(2N)$ gauge theory

Let us consider the following theory:



(4.3.47)

The brane construction for this is given by (4.3.6), with $m = 1$ and with an S-fold inserted in the interval labelled by f_1. The S-fold partitions f_1 D5 branes into F_1 and F_2 D5 branes on the left and on the right of the S-fold, respectively. The dimension of the Higgs branch of this theory reads

$$\begin{aligned}\dim_{\mathbb{H}} \mathcal{H}_{(4.3.47)} &= 2Nn_1 + 4N^2 + 2NF_1 + (4N^2 - 2N) + 2NF_2 + 4N^2 \\ &\quad + 2Nn_2 - N(2N+1) - 4N^2 - 4N^2 - N(2N+1) \\ &= 2N(F_1 + F_2 + n_1 + n_2 - 2)\,,\end{aligned} \tag{4.3.48}$$

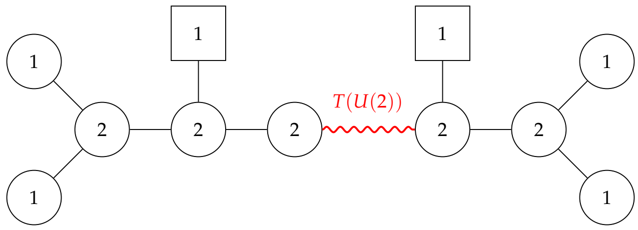
and, for the Coulomb branch, we find

$$\dim_{\mathbb{H}} \mathcal{C}_{(4.3.47)} = 2N. \tag{4.3.49}$$

We remark that it is not possible to insert an S-fold in the interval labelled by n_1 in the diagram (4.3.6). The reason is that such a brane interval corresponds to the gauge group $USp(2N)$, and not $USp'(2N)$. We do not have the notion of a $T(USp(2N))$ link since $USp(2N)$ is not invariant under the S-duality. This supports the point we made earlier that the S-fold cannot be inserted "too close" to the orientifold plane; it must be "shielded" by a sufficient numbers of fivebranes.

In order to obtain the mirror configuration, we can insert an S-fold anywhere between two D5-branes denoted by the black dots in (4.3.14). (Recall that $m = 1$ in this case.) In terms of the quiver, this means that we can put the T-link anywhere in between the two $(2N)$-nodes attached by one flavour. For example, for $N = 1$,

$n_1 = n_2 = 3$, $F_1 = 1$ and $F_2 = 0$, the mirror theory is



(4.3.50)

In order to compute the dimensions of Higgs and Coulomb branches of the mirror theory we can simply start with the corresponding non S-folded theory and observe that inserting a T-link implies the following:

- For the Higgs branch, we need to add the dimension of the $T(U(2N))$ link, that in this case gives $4N^2 - 2N$ and subtract the gauging of the extra $U(2N)$, hence we subtract $4N^2$; in total we find that

$$\begin{aligned} \dim_{\mathbb{H}} \mathcal{H}_{\text{mirr of (4.3.47)}} &= \dim_{\mathbb{H}} \mathcal{H}_{(4.3.15)} + (4N^2 - 2N) - 4N^2 \\ &= \dim_{\mathbb{H}} \mathcal{H}_{(4.3.15)} - 2N \\ &= (N + 2N + N) - 2N = 2N\,. \end{aligned} \tag{4.3.51}$$

- For the Coulomb branch, the result of inserting an S-fold is to add one gauge node and then consider that the ones connected by the T-link are frozen, so in total we have

$$\begin{aligned} \dim_{\mathbb{H}} \mathcal{C}_{\text{mirr of (4.3.47)}} &= \dim_{\mathbb{H}} \mathcal{C}_{(4.3.15)} - 2N \\ &= 2N(F_1 + F_2 + n_1 + n_2 - 2)\,, \text{ with } f_1 = F_1 + F_2\,. \end{aligned} \tag{4.3.52}$$

These are in agreement with mirror symmetry.

In the above example of $N = 1$, $n_1 = n_2 = 3$, $f_1 = 1$ and $f_2 = 0$, one can compute the Hilbert series for (4.3.47) and its mirror (4.3.50). The unrefined results are

$$\begin{aligned} H[\mathcal{H}_{(4.3.47)}] &= H[\mathcal{C}_{(4.3.50)}] \\ &= \text{PE}\left[16t^2 + 12t^3 - 15t^4 - 40t^5 + 19t^6 + \ldots\right] \times \\ &\quad\ \text{PE}\left[15t^2 - 16t^4 + 35t^6 + \ldots\right], \end{aligned} \tag{4.3.53}$$

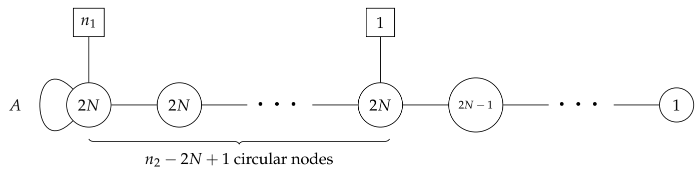
and

$$\begin{aligned} H[\mathcal{C}_{(4.3.47)}] &= H[\mathcal{H}_{(4.3.50)}] \\ &= H[\mathcal{C}_{USp(2)\text{ with 5 flv}}]^2 = \text{PE}\left[t^4 + t^6 + t^8 + \ldots\right]^2. \end{aligned} \tag{4.3.54}$$

The above results deserve some explanations. In (4.3.50), the Coulomb branch symmetry can be seen from the after taking the two $U(2)$ gauge groups connected by the T-link to be two separate flavour symmetries. The left part gives an $SU(4) \times U(1)$ symmetry due to the fact that the balanced nodes form an A_3 Dynkin diagram and that there is one overbalanced node (namely, the $U(2)$ node that is attached to one flavour). The right part gives an $SU(4)$ symmetry due to the fact that the balanced nodes form an A_3 Dynkin diagram [8]. The Coulomb branch of (4.3.47) is identified with a product of two copies of the Coulomb branch of $USp(2)$ gauge theory with 5 flavours due to the following reason. The nodes connected by the T-link do not contribute to the Coulomb branch and therefore each of the left and the right parts contains the $USp(2)$ gauge theory with $2N + n_1 = 2 + 3 = 5$ flavours.

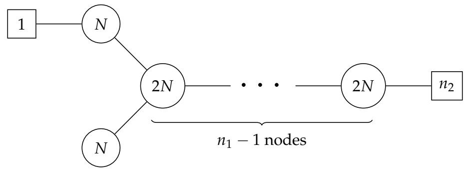
4.3.3 Duality with theories without an S-fold

For theories with one orientifold fiveplane and an S-fold, one can move the S-fold away from the brane system to infinity. For example, in (4.3.16), this corresponds to moving the S-fold to $+\infty$ in x^6-direction on the right of the brane diagram. Each time the S-fold crosses an NS5 brane (*resp.* a D5 brane), it is transformed into a D5 brane (*resp.* an NS5 brane) by S-duality. Specifically, in doing so, the n_2 D5 branes in (4.3.16) turn into n_2 NS5 brane and the rightmost NS5 brane turns into a D5 brane. The corresponding quiver can be obtained in a similar way to that of (4.3.17):



(4.3.55)

The mirror quiver of (4.3.55) is



(4.3.56)

Observe that (4.3.55) and (4.3.56) no longer contain a T-link. We expect that (4.3.16) and (4.3.17) are dual to (4.3.55) and (4.3.56).

We see that if the value of n_2 falls in the interval $1 \leq n_2 \leq 2N - 1$, quiver (4.3.55) is problematic and (4.3.56) is not a good theory in the sense of [8], due to the fact the last $U(2N)$ gauge node does not have a sufficiently large number of flavours. A similar situation occurs when $0 \leq n_1 \leq 1$.

Note that the procedure of moving the S-fold away from the brane system to infinity can be applied, in general, to "non-compact" models and obtain four theories that are dual to each other. It would be interesting to study such a duality further, for example, by computing and matching the three sphere partition functions of such theories in a similar fashion to [14] (see also [107]).

For theories with two orientifold fiveplanes, such as the models with two antisymmetric hypermultiplets as well as the model discussed in (4.3.2), the notion of "infinity" in the x^6 direction no longer makes sense. This is due to the action of the two orientifold fiveplanes that defines the boundaries of the brane system and, in this sense, the model should be regarded as "compact", as discussed in [25]. Hence, in this case, we do not expect to have a duality with non-S-fold theories as in the preceding case[9].

[9] If one really insists to move the S-fold crossing one of the $O5^-$ planes, the resulting brane system consists of an $O5^-$ plane on one side and an ON^- plane on the other side. Although such a system has not been much explored in the literature, one may try to write down the gauge theory corresponding to such boundary conditions and study its properties such as the three sphere partition function.

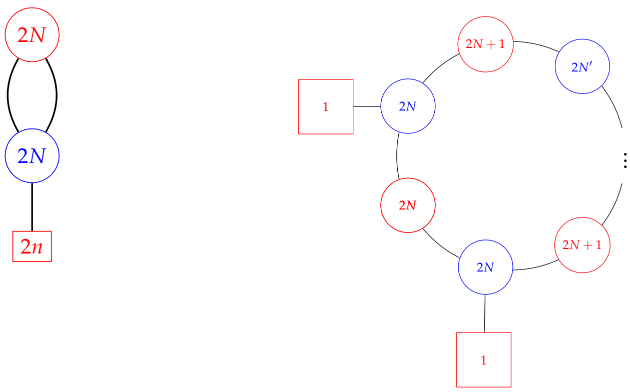
4.4 Models with an orientifold threeplane

In this section, we discuss circular quivers whose brane configurations contain an orientifold threeplane. We first review the brane configurations for theories without an S-fold and their mirrors. Subsequently, we introduce an S-fold to such brane systems. We only focus the cases in which the types of the orientifold planes on both sides of the S-fold are $(\mathrm{O3}^-, \mathrm{O3}^-)$ or $(\widetilde{\mathrm{O3}}^+, \widetilde{\mathrm{O3}}^+)$. These correspond to the presence of an $T(SO(2N))$ link or an $T(USp'(2N))$ link in the quiver. Using the same argument as in [14], we expect that the theories with an S-fold has $\mathcal{N} = 3$ supersymmetry. This is because, when the global symmetry $G \times G$ of $T(G)$ is gauged in the circular quiver (so that both of them act on the Coulomb branch), the two $SU(2)$ factors in the original $SU(2) \times SU(2)$ R-symmetry of $T(G)$ are identified to a single $SU(2)$ R-symmetry, corresponding to $\mathcal{N} = 3$ supersymmetry.

4.4.1 The cases without an S-fold

In this subsection, we summarise brane constructions for the elliptic models with alternating orthogonal and symplectic gauge groups, in the absence of the S-fold. Such brane configurations and their S-duals were studied extensively in [21] (see also [108] for a related discussion). For brevity of the discussion, we shall not go through the detail on how to obtain the S-dual configurations but simply state the results. The following quiver diagrams and their brane configurations will turn out to be useful for the discussion in the subsequent subsections.

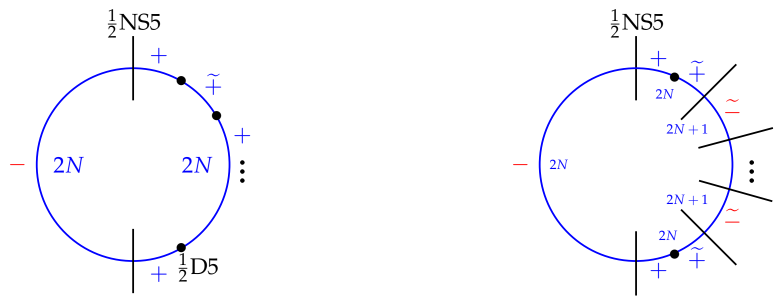
The $SO(2N) \times USp(2N)$ gauge theory with two bifundamentals and n flavours for $USp(2N)$ and its mirror. Their quivers are



One red $(2N)$ node + two blue $(2N)$ nodes
with a half-flavour each, and alternating
$(n-2)$ blue $(2N')$ nodes with no flavour
+ $(n-1)$ red $(2N+1)$ nodes with no flavour

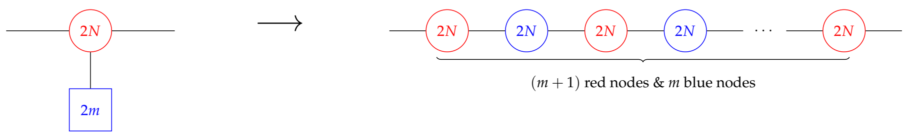
(4.4.1)

Their brane configurations are, respectively, given by [21, Fig. 23]:



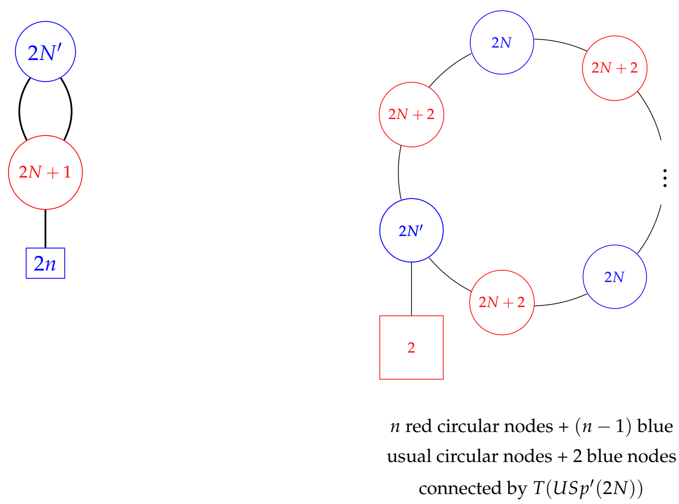
(4.4.2)

where in the left diagram we have n half-D5 branes, and in the right diagram we have n half-NS5 branes. Here and subsequently, we denote in blue the number of half-D3 branes at each interval between two succesive half-NS5 branes. Note that one may also add flavours (say, m flavours, or equivalently a blue rectangular node with label $2m$) to the $SO(2N)$ gauge group in the left diagram of (4.4.1), the resulting mirror quiver can be obtained from the right diagram of (4.4.1) by simply replacing the $(2N)$ red node by a series of alternating $m+1$ red $(2N)$ nodes and m blue $(2N)$ nodes:



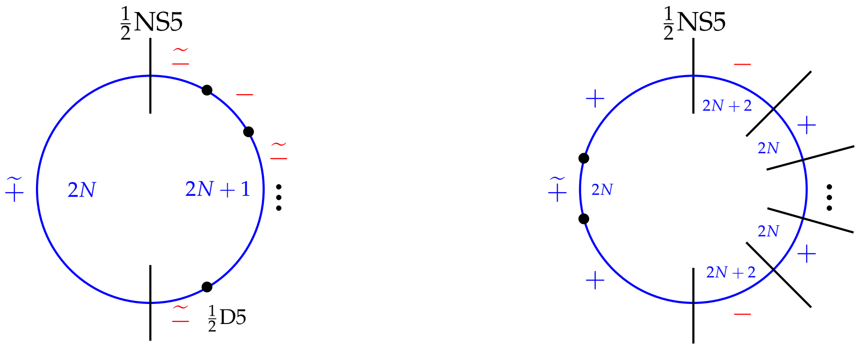
(4.4.3)

The $USp'(2N) \times SO(2N+1)$ gauge theory with two bifundamentals and n flavours for $SO(2N+1)$ and its mirror. Their quivers are



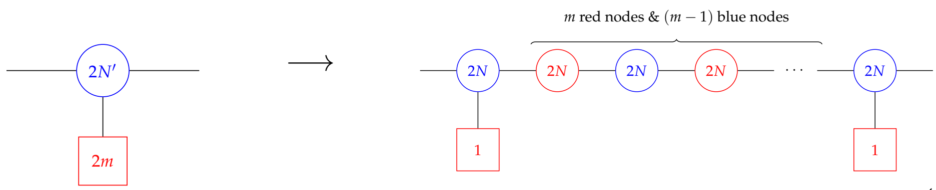
(4.4.4)

The corresponding brane configurations are respectively given by [21, Fig. 29]:



(4.4.5)

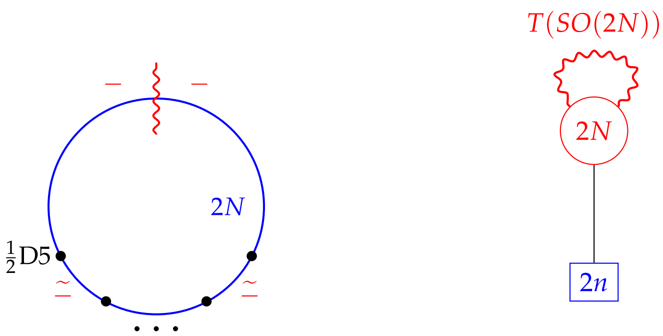
where in the left diagrams there are $2n$ half-D5 branes, and on the right diagram there are $2n$ half NS5 branes. One may also add flavours (say, m flavours or equivalently a red square node with label $2m$) to the $USp'(2m)$ gauge group in the left diagram of (4.4.4), the resulting mirror quiver can be obtained from the right diagram of (4.4.4) by making the following replacement:



(4.4.6)

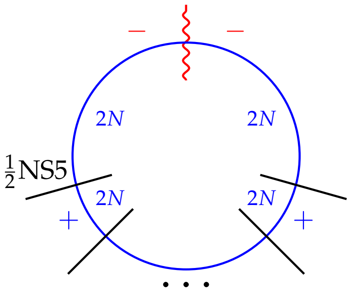
4.4.2 Quiver with a $T(SO(2N))$ loop

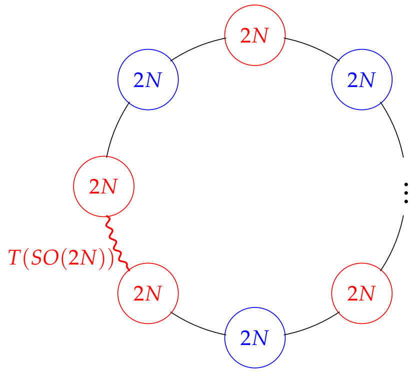
We start by examining the following brane configuration and the corresponding quiver:



(4.4.7)

where in the left diagram the red wriggly denotes the S-fold and there are $2n$ half D5 branes. In order to obtain the mirror theory, we apply S-duality to the above brane system. The result is



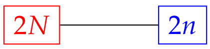


n blue circular nodes + $(n-1)$ red usual circular nodes + 2 red nodes connected by $T(SO(2N))$

(4.4.8)

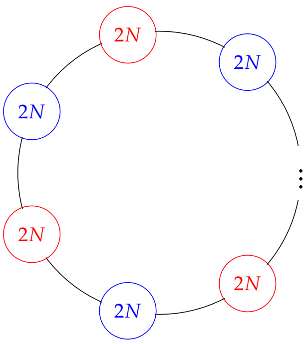
where in the left diagram there are $2n$ half-NS5 branes.

In the absence of the S-fold, quivers (4.4.7) and (4.4.8) reduce to conventional Lagrangian theories that are related to each other by mirror symmetry. In particular, (4.4.7) reduces to a theory of free $4Nn$ half-hypermultiplets, namely



(4.4.9)

and quiver (4.4.8) reduces to



$2n$ alternating red/blue circular nodes

(4.4.10)

where the two $SO(2N)$ gauge groups that were connected by $T(SO(2N))$ merged into a single $SO(2N)$ circular node. It can be checked that the Higgs branch dimension of (4.4.10) is indeed zero:

$$(2n)(2N^2) - n\left[\frac{1}{2}(2N)(2N-1)\right] - n\left[\frac{1}{2}(2N)(2N+1)\right] = 0\,, \qquad (4.4.11)$$

and the quaternionic dimension of the Coulomb branch of (4.4.10) is $2Nn$. These are in agreement with mirror symmetry.

Theory (4.4.7)

The Higgs branch of this theory is given by the hyperKähler quotient:

$$\mathcal{H}_{(4.4.7)} = \frac{\mathcal{N}_{so(2N)} \times \mathcal{N}_{so(2N)} \times \mathcal{H}\left([S/O(2N)] - [USp(2n)]\right)}{S/O(2N)}\,. \qquad (4.4.12)$$

where the notation S/O means that we may take the gauge group to be $SO(2N)$ or $O(2N)$. The dimension of this space is

$$\dim_{\mathbb{H}} \mathcal{H}_{(4.4.7)} = \left[\frac{1}{2}(2N)(2N-1) - N\right] + 2Nn - \frac{1}{2}(2N)(2N-1) = (2n-1)N \,. \tag{4.4.13}$$

Since the circular nodes that are connected by $T(SO(2N))$ do not contribute to the Coulomb branch, it follows that the Coulomb branch of (4.4.7) is trivial:

$$\dim_{\mathbb{H}} \mathcal{C}_{(4.4.7)} = 0 \,. \tag{4.4.14}$$

Let us now discuss certain interesting special cases below.

The Higgs branch of (4.4.7) for $N = 1, 2$

For $N = 1$, since $\mathcal{N}_{so(2)}$ is trivial, it follows that $\mathcal{H}_{(4.4.7)}$ is the Higgs branch of the 3d $\mathcal{N} = 4$ $S/O(2)$ gauge theory with n flavours. If the gauge group is taken to be $O(2)$, $\mathcal{H}_{(4.4.7)}$ is isomorphic to the closure of the minimal nilpotent orbit of $usp(2n)$. On the other hand, if the gauge group is taken to be $SO(2)$, $\mathcal{H}_{(4.4.7)}$ turns out to be isomorphic to the closure of the minimal nilpotent orbit of $su(2n)$. The reason is because the generators of the moduli space with $SU(2)_R$-spin 1 are mesons and baryons; they transform in the representation $[2,0,\dots,0] + [0,1,0,\dots,0]$ of $usp(2n)$. This representation combines into the adjoint representation $[1,0,\dots,0,1]$ of $su(2n)$.

For $N = 2$, let us denote the fundamental half-hypermultiplets by Q^i_a with $i,j,k,l = 1,\dots,2n$ and $a,b,c,d = 1,2,3,4$, and the generators of $\mathcal{N}_{so(4)}$ by a rank-two antisymmetric tensor X_{ab}. We find that for the $O(4)$ gauge group, the generators of the Higgs branch are as follows:

- The mesons $M^{ij} = Q^i_a Q^j_b \delta^{ab}$, with $SU(2)_R$-spin 1, transforming in the adjoint representation $[2,0,\dots,0]$ of $usp(2n)$.
- The combinations $Q^i_a Q^j_b X_{ab}$, with $SU(2)_R$-spin 2, transforming in the adjoint representation $[0,1,0,\dots,0]$ of $usp(2n)$.

For the $SO(4)$ gauge group, we have, in addition to the above, the following generators of the Higgs branch:

- The baryons $B^{ijkl} = \epsilon^{abcd} Q^i_a Q^j_b Q^k_c Q^l_d$, with $SU(2)_R$-spin 2, transforming in the adjoint representation $[0,0,0,1,0,\dots,0] + [0,1,0,\dots,0]$ of $usp(2n)$.
- The combinations $\epsilon^{abcd} Q^i_a Q^j_b X_{cd}$, with $SU(2)_R$-spin 2, transforming in the adjoint representation $[0,1,0,\dots,0]$ of $usp(2n)$.
- The $USp(2n)$ flavour singlet $\epsilon^{abcd} X_{ab} X_{cd}$, with $SU(2)_R$-spin 2.

The Higgs branch of (4.4.7) for $n = 1$

In this case, it does not matter whether we take the gauge group to be $SO(2N)$ or $O(2N)$, the Higgs branch is the same. The corresponding Hilbert series is

$$H[\mathcal{H}_{(4.4.7)}|_{n=1}] = \mathrm{PE}\left[\chi^{su(2)}_{[2]}(x) \sum_{j=0}^{N-1} t^{4j+2} - \sum_{l=N}^{2N-1} t^{4l}\right] \,. \tag{4.4.15}$$

Indeed, for $N = n = 1$, we recover the nilpotent cone of $su(2)$, which is isomorphic to $\mathbb{C}^2/\mathbb{Z}_2$.

Theory (4.4.8)

Since the nodes that are connected by $T(SO(2N))$ do not contribute to the Coulomb branch, it follows that the dimension of the Coulomb branch is

$$\mathcal{C}_{(4.4.8)} = (2n-1)N\,. \tag{4.4.16}$$

Note, however, that quiver (4.4.8) is a "bad" theory in the sense of [8], due to the fact that each $USp(2N)$ gauge group has $2N$ flavours. Nevertheless, we shall analyse the case of $n = 1$ and general N in detail below. In which case, we shall see that the result is consistent with mirror symmetry.

The computation of the Higgs branch dimension of (4.4.8) indicates that the gauge symmetry is not completely broken at a generic point of the Higgs branch. Indeed, if we assume (incorrectly) that the gauge symmetry is completely broken, we would obtain the $\dim_{\mathbb{H}} \mathcal{H}_{(4.4.8)}$ to be

$$(4.4.11) + \left[\frac{1}{2}(2N)(2N-1) - N\right] - \frac{1}{2}(2N)(2N-1) = -N\,. \tag{4.4.17}$$

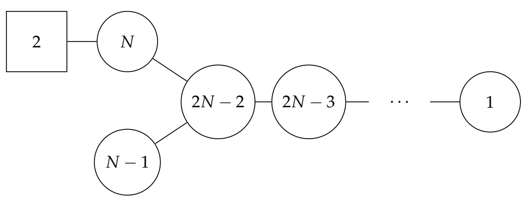
We conjecture that the $SO(2N) \times SO(2N)$ gauge group connected by $T(SO(2N))$ is broken to $SO(2)^N$, whose dimension is N. This statement can be checked explicitly in the case of $N = 1$, where $T(SO(2))$ is trivial. Taking into account such an unbroken symmetry, we obtain $\dim_{\mathbb{H}} \mathcal{H}_{(4.4.8)} = 0$, which is in agreement with the Coulomb branch of (4.4.7).

The special case of $n = 1$

In this case, the Coulomb branch of (4.4.8) is equal to that of the $USp(2N)$ gauge theory with $2N$ flavours. As pointed out in [109], the most singular locus of the Coulomb branch consists of two points, related by a $\mathbb{Z}_2$ global symmetry. The infrared theory at any of these two points is an interacting SCFT, which we denote by $\mathcal{T}_N$.

For $n = N = 1$, the corresponding singularity is of an A_1 type [110], and the corresponding SCFT is $\mathcal{T}_2 = T(SU(2))$ whose Higgs/Coulomb branch is a nilpotent cone of $su(2)$; this is indeed in agreement with the Higgs branch of (4.4.7). The situation here is the same as that described on Page 30 of [109], namely mirror symmetry is realized locally at each of the two singular points. The Higgs branch of (4.4.7) has one component, whereas the the Coulomb branch of (4.4.8) (for $N = n = 1$) splits into two components, each of which is isomorphic to the former.

For $n = 1$ and $N > 1$, the mirror theory of $\mathcal{T}_N$ is described by the following quiver [111]:



(4.4.18)

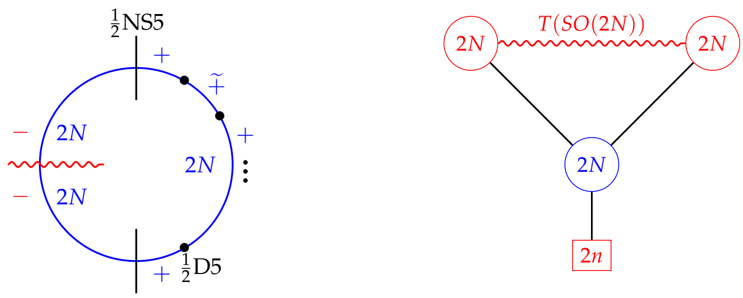
By mirror symmetry, the Coulomb branch of $\mathcal{T}_N$ is equal to the Higgs branch of (4.4.18), whose Hilbert series is given by [109, (D.11)]:

$$H[\mathcal{C}_{\mathcal{T}_N}](t,x) = H[\mathcal{H}_{(4.4.18)}](t,x) = \mathrm{PE}\left[\chi^{su(2)}_{[2]}(x)\sum_{j=0}^{N-1} t^{4j+2} - \sum_{l=N}^{2N-1} t^{4l}\right] . \qquad (4.4.19)$$

This is perfectly in agreement with (4.4.15).

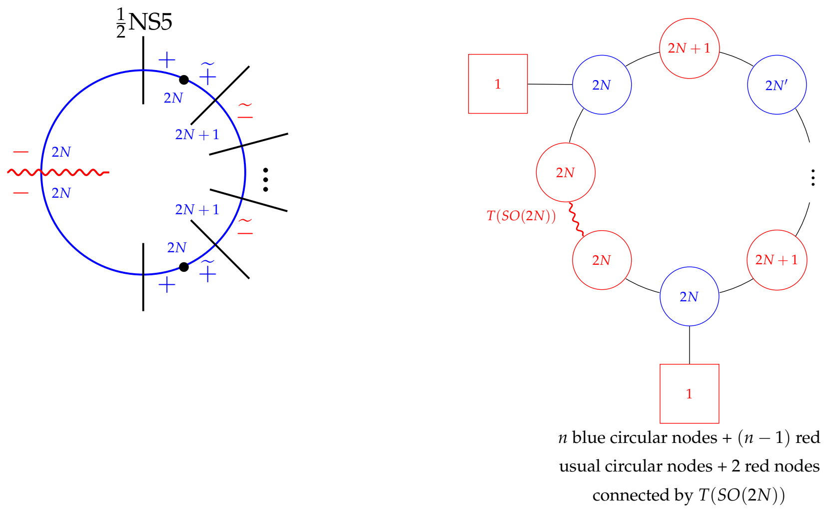
4.4.3 Quivers with a $T(SO(2N))$ link or a $T(USp'(2N))$ link

Let us insert an S-fold in the brane interval marked by red minus sign ($-$) in each brane set-up in (4.4.2). This leads to the presence of $T(SO(2N))$ link in the corresponding quiver diagram. In particular, the insertion of an S-fold in the left diagram of (4.4.2) leads to the following configuration:



(4.4.20)

The mirror theory can be obtained from the S-dual configuration of the above, or simply inserting an S-fold to the left interval of the right diagram in (4.4.2). The result is



(4.4.21)

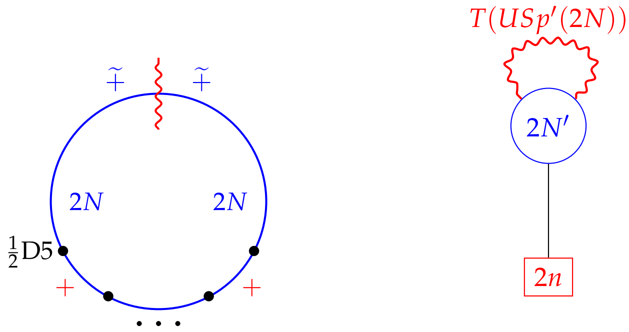
where the number of half-NS5 branes is $2n$. Note that for $n = 1$, the theory is self-mirror.

Theory (4.4.20)

The Higgs branch of (4.4.20) is described by the hyperKähler quotient

$$
\begin{aligned}
\mathcal{H}_{(4.4.20)} &= \Big(\mathcal{N}_{so(2N)} \times \mathcal{H}([SO(2N)] - [USp(2N)]) \times \mathcal{N}_{so(2N)} \times \mathcal{H}(SO(2N)] - [USp(2N)]) \times \\
&\qquad \mathcal{H}([USp(2N)] - [SO(2n)])\Big) \,/\, (SO(2N) \times SO(2N) \times USp(2N)) \\
&= \frac{\mathcal{N}_{usp(2N)} \times \mathcal{N}_{usp(2N)} \times \mathcal{H}([USp(2N)] - [SO(2n)])}{USp(2N)} \,,
\end{aligned}
\tag{4.4.22}
$$

where we have used (4.2.24) to obtain the last line. We remark that both red circular nodes can be chosen to be either $SO(2N)$ or $O(2N)$ and the results for both options are the same, thanks to the equality between (4.2.24) and (4.2.30). Moreover, the hyperKähler quotient in the last line of (4.4.22) suggests the equality between (4.4.22) and the Higgs branch of the following theory:



(4.4.23)

where the blue circular node is a $USp'(2N)$ gauge group. In other words, we have the following equality of the Higgs branch between two different gauge theories:

$$
\mathcal{H}_{(4.4.20)} = \mathcal{H}_{(4.4.23)} \,. \tag{4.4.24}
$$

The quaternionic dimension of (4.4.22) is

$$
\begin{aligned}
\dim_{\mathbb{H}} \mathcal{H}_{(4.4.20)} &= \left[\frac{1}{2}(2N)(2N-1) - N\right] + 2(4N^2) + 2Nn \\
&\qquad - \left[2 \times \frac{1}{2}(2N)(2N-1)\right] - \frac{1}{2}(2N)(2N+1) \\
&= (2n-1)N \,.
\end{aligned}
\tag{4.4.25}
$$

Since the nodes that are connected by $T(SO(N))$ does not contribute to the Coulomb branch of the theory, the Coulomb branch of (4.4.20) is isomorphic to the Coulomb branch of the 3d $\mathcal{N}=4$ $USp(2N)$ gauge theory with $2N+n$ flavours, whose Hilbert series is given by [86, (5.14)]. Its quaternionic dimension is

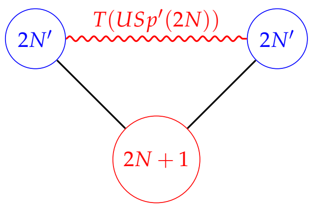
$$
\dim_{\mathbb{H}} \mathcal{C}_{(4.4.20)} = N \,. \tag{4.4.26}
$$

Example: $n = 1$**.** The theory is self-mirror. One can check that the Hilbert series of the quotient (4.4.34) is indeed equal to the Coulomb branch of $USp(2N)$ gauge theory with $2N+1$ flavours [86, (5.14)], which is

$$\mathrm{PE}\left[\sum_{j=1}^{2N} t^{2j} + \sum_{j=1}^{N} t^{4j} - \sum_{j=1}^{N} t^{4j+4N}\right] . \tag{4.4.27}$$

Note that for $N = n = 1$, we have $\mathbb{C}^2/\mathbb{Z}_4$, as expected from the Coulomb branch of $USp(2)$ with 3 flavours.

There is another way to check that theory (4.4.20) for $n = 1$ (and a general N) is self-mirror. We can easily compute a mirror theory of (4.4.23), with $n = 1$, by applying S-duality to the brane system; see (4.4.31). The result is



(4.4.28)

The Coulomb branch of this theory is isomorphic to that of 3d $\mathcal{N} = 4$ $SO(2N+1)$ gauge theory with $2N$ flavours, whose Hilbert series is given in [86, (5.18)]. However, as pointed out in that reference, this turns out to be isomorphic to the Coulomb branch of the $USp(2N)$ gauge theory with $2N+1$ flavours, whose Hilbert series is given by (4.4.27). We thus establish the self-duality of (4.4.20) for $n = 1$.

Theory (4.4.21)

The Higgs branch dimension of (4.4.21) is

$$\begin{aligned} \dim_{\mathbb{H}} \mathcal{H}_{(4.4.21)} &= (2)(2N^2) + (2n-2)N(2N+1) + \left[\frac{1}{2}(2N)(2N-1) - N\right] \\ &\quad + N + N - n\left[\frac{1}{2}(2N)(2N+1)\right] - 2\left[\frac{1}{2}(2N)(2N-1)\right] \\ &\quad - (n-1)\left[\frac{1}{2}(2N+1)(2N)\right] \\ &= N\,. \end{aligned} \tag{4.4.29}$$

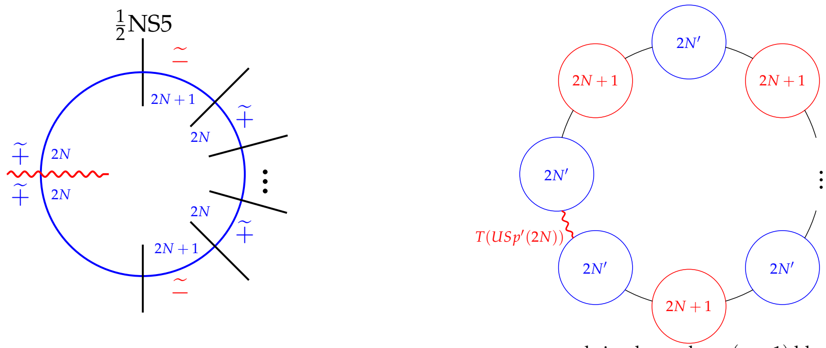
The Coulomb branch dimension of (4.4.21) is equal to the total rank of the gauge groups that are not connected by $T(SO(2N))$:

$$\dim_{\mathbb{H}} \mathcal{C}_{(4.4.21)} = (2n-1)N\,. \tag{4.4.30}$$

These agree with the dimensions of the Coulomb and the Higgs branches of (4.4.20).

Similarly to the previous discussion, the red circular nodes that are connected by $T(SO(2N))$ can be taken as $O(2N)$ or $SO(2N)$ without affecting the Higgs branch moduli space of (4.4.21). Moreover, we find that this applies to other red circular nodes in the quiver, namely the choice between $O(2N+1)$ and $SO(2N+1)$ does not change the Higgs branch of the theory. This can be checked directly using the Hilbert series.

It is worth pointing out that there is another gauge theory that gives the same Coulomb branch as (4.4.20). This is the mirror theory of (4.4.23) which is given by



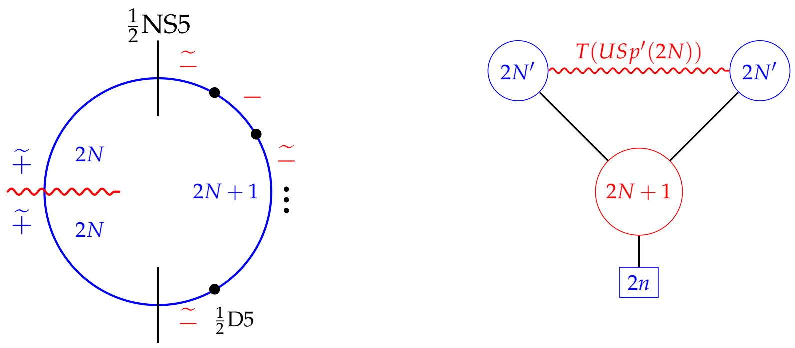
n red circular nodes + $(n-1)$ blue usual circular nodes + 2 blue nodes connected by $T(USp'(2N))$

(4.4.31)

where the number of half-NS5 branes is $2n$. We expect that the Coulomb branch of (4.4.31) has to be equal to the Coulomb branch of (4.4.21). This can be seen as follows. Let us focus on (4.4.31). Note that the two blue circular nodes that are connected by $T(USp'(2N))$ do not contribute to the Coulomb branch computation, so we can take them to be two flavour nodes that are not connected. As pointed out below [86, (5.18)], the Coulomb branch of the $SO(2N+1)$ gauge theory with $2N$ flavours is the same as that of Coulomb branch of the $USp(2N)$ gauge theory with $2N+1$ flavours. We can apply this fact to every node in quiver (4.4.31) and see that the resulting quiver has the same Coulomb branch as that of (4.4.21).

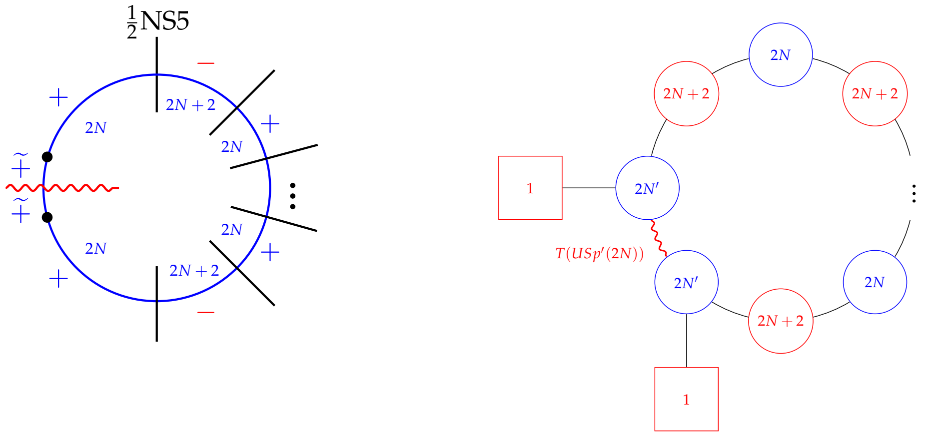
4.4.4 More quivers with a $T(USp'(2N))$ link

Let us insert an S-fold in the interval labelled by $\widetilde{+}$ in each of the diagram in (4.4.5). Doing so in the left diagram yields the following theory:



(4.4.32)

On the other hand, inserting an S-fold to the right diagram yields the mirror configuration:



n red circular nodes + $(n-1)$ blue usual circular nodes + 2 blue nodes connected by $T(USp'(2N))$

(4.4.33)

Theory (4.4.32)

The Higgs branch of (4.4.32) is described by the hyperKähler quotient

$$\mathcal{H}_{(4.4.32)} = \frac{\mathcal{N}_{so(2N+1)} \times \mathcal{N}_{so(2N+1)} \times \mathcal{H}([SO(2N+1)] - [USp(2n)])}{SO(2N+1)}\,, \tag{4.4.34}$$

where we have used (4.2.14) and (4.2.16) (with $n = 2N+1$). The dimension of this is

$$\begin{aligned} \dim_{\mathbb{H}} \mathcal{H}_{(4.4.32)} &= \left[\frac{1}{2}(2N+1)(2N) - N\right] + (2N+1)n - \frac{1}{2}(2N+1)(2N) \\ &= 2nN + n - N\,. \end{aligned} \tag{4.4.35}$$

A special case of $N = n = 1$ is particularly simple. The corresponding Higgs branch is a complete intersection with the Hilbert series

$$H[\mathcal{H}_{(4.4.20)}|_{N=n=1}](t;x) = \mathrm{PE}\left[\chi^{su(2)}_{[2]}(x)t^2 + \chi^{su(2)}_{[1]}(x)t^3 - t^8\right]\,. \tag{4.4.36}$$

The Coulomb branch of (4.4.32) is isomorphic to that of the 3d $\mathcal{N} = 4$ $SO(2N+1)$ gauge theory with $2N+n$ flavours, whose Hilbert series is given by [86, (5.18)]. Note that this is also equal to that of the Coulomb branch of the $USp(2N)$ gauge theory with $2N+n+1$ flavours.

Theory (4.4.33)

The quaternionic dimension of the Coulomb branch of this theory is

$$\dim_{\mathbb{H}} \mathcal{C}_{(4.4.33)} = n(N+1) + (n-1)N = 2nN + n - N\,. \tag{4.4.37}$$

This matches with the dimension of the Higgs branch of (4.4.32). It should be noted that (4.4.33) is a "bad" theory in the sense of [8], due to the fact that each $SO(2N+2)$

gauge group effectively has $2N$ flavours. Nevertheless, we shall analyse the case of $N = n = 1$ below.

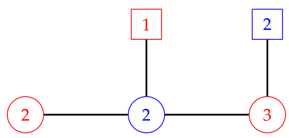
Let us now turn to the Higgs branch. In the absence of the S-fold, it was pointed out below [21, (7.1)] that the gauge symmetry is not completely broken at a generic point on the Higgs branch, but is broken to n copies of $SO(2)$. We conjecture that this still holds for (4.4.33). Indeed, if we assume that this is true, we obtain the quaternionic dimension of the Higgs branch to be

$$\begin{aligned} &\dim_{\mathbb{H}} \mathcal{H}_{(4.4.33)} \\ &= \left[\frac{1}{2}(2N)(2N+1) - N\right] + N + N + (2n)N(2N+2) \\ &- (n)\left[\frac{1}{2}(2N+2)(2N+1)\right] - (n-1+2)\left[\frac{1}{2}(2N)(2N+1)\right] + n \\ &= N\,, \end{aligned} \tag{4.4.38}$$

where n in the second line is there due to the unbroken symmetry $SO(2)^n$ at a generic point of the Higgs branch. This is in agreement with the dimension of the Coulomb branch of (4.4.32), and is indeed consistent with mirror symmetry.

The special case of $N = n = 1$

In this case, the Coulomb branch of (4.4.33) is equal to that of the $SO(4)$ gauge theory with 2 flavours (which has a $USp(4)$ flavour symmetry). Although the latter is a bad theory, there is a mirror theory which has a "good" Lagrangian description. The latter is denoted by $T^{(2,1^2)}(USp(4))$, whose quiver description is (see [72, Table 2]):



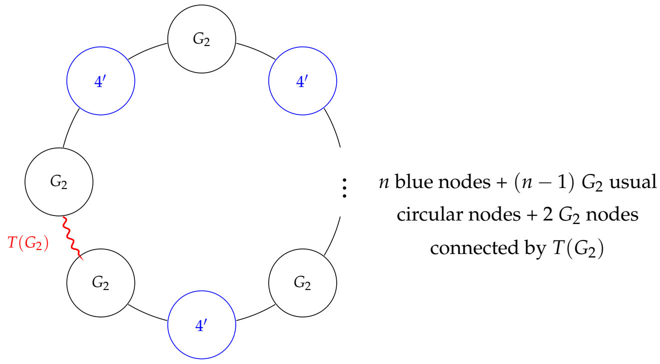
(4.4.39)

where each red circular node should be taken as an SO gauge group. The Higgs branch Hilbert series of (4.4.39) is indeed in agreement with (4.4.36), consistent with mirror symmetry.

4.5 Models with the exceptional group G_2

4.5.1 Self-mirror models with a $T(G_2)$ link

In this section, we turn to models with a $T(G_2)$ link connecting between two G_2 gauge groups. We do not have the Type IIB brane construction for such theories. Nevertheless, it is still possible to make some interesting statements regarding the moduli space. We consider the following quiver:



(4.5.1)

Note that every gauge group in the quiver has the same rank, in the same way as the preceding sections. The Higgs branch dimension of this quiver is

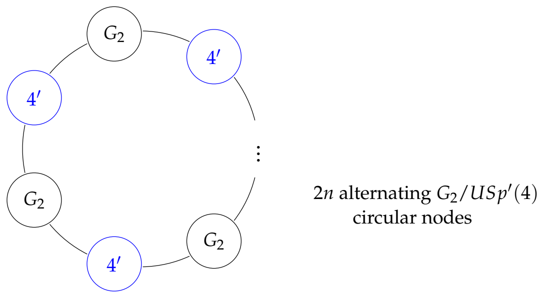
$$\dim_{\mathbb{H}} \mathcal{H}_{(4.5.1)} = (14-2) + \frac{1}{2}(2n)(4)(7) - 10n - 14(n-1+2) = 2(2n-1)\,. \quad (4.5.2)$$

On the other hand, the Coulomb branch dimension of this quiver is

$$\dim_{\mathbb{H}} \mathcal{C}_{(4.5.1)} = 2(2n-1)\,. \quad (4.5.3)$$

Observe that the dimensions of the Higgs and Coulomb branches are equal. Indeed, we claim that quiver (4.5.1) if **self-mirror**. We shall consider some special cases and compute the Hilbert series to support this statement below.

In the absence of S-fold, the two G_2 gauge groups merge into a single gauge group and quiver (4.5.1) reduces to



(4.5.4)

It can also be checked that the Higgs and Coulomb branch dimensions of this quiver are equal:

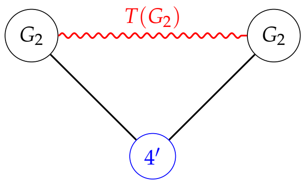
$$\dim_{\mathbb{H}} \mathcal{H}_{(4.5.4)} = \dim_{\mathbb{H}} \mathcal{C}_{(4.5.4)} = 4n\,. \quad (4.5.5)$$

Again, we claim that quiver (4.5.4) is also self-mirror. Indeed, one can check using the Hilbert series (say for $n = 1, 2$), in a similar way as that will be presented below, that the Higgs and Coulomb branches of (4.5.4) are equal.

Since we do not know the brane configurations for (4.5.1) and (4.5.4), we cannot definitely confirm if the gauge nodes labelled by 4 is really $USp(4)$ or $USp'(4)$. Nevertheless, we conjecture that such gauge nodes are $USp'(4)$, due to the fact that we can perform an "S-folding" and obtain another quiver which is self-dual. The latter is depicted in (4.5.15) and will be discussed in detail in the next subsection.

The case of $n = 1$

In this case, (4.5.1) reduces to the following quiver:



(4.5.6)

The Higgs branch Hilbert series can be computed as

$$H[\mathcal{H}_{(4.5.6)}](t) = \int \mathrm{d}\mu_{USp(4)}(\boldsymbol{z}) \left\{ H[\mathcal{H}_{(4.2.33)}](t;\boldsymbol{z}) \right\}^2 \mathrm{PE}\left[-\chi^{C_2}_{[2,0]}(\boldsymbol{z}) t^2 \right] , \quad (4.5.7)$$

where $\boldsymbol{z} = (z_1, z_2)$ and $H[\mathcal{H}_{(4.2.33)}](t;\boldsymbol{z})$ is given by (4.2.37). The integration yields

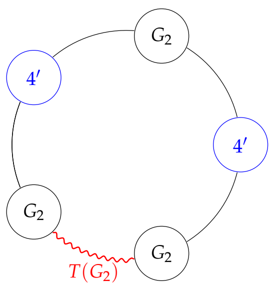
$$H[\mathcal{H}_{(4.5.6)}](t) = \mathrm{PE}\left[t^4 + t^6 + 2t^8 + t^{10} + t^{12} - t^{20} - t^{24} \right] . \quad (4.5.8)$$

This is the Coulomb branch Hilbert series of 3d $\mathcal{N} = 4$ $USp(4)$ gauge theory with 7 flavours [86, (5.14)]. On the other hand, since the vector multiplet of the G_2 gauge groups connected by $T(G_2)$ do not contribute to the Coulomb branch, the Coulomb branch of (4.5.6) is also isomorphic to the Coulomb branch of 3d $\mathcal{N} = 4$ $USp(4)$ gauge theory with 7 flavours.

We see that the Higgs and the Coulomb branches of (4.5.6) are equal to each other. We thus expect that theory (4.5.6) is self-mirror.

The case of $n = 2$

In this case, (4.5.1) reduces to the following quiver:



(4.5.9)

The Higgs branch Hilbert series can be computed similarly as before:

$$\begin{aligned} H[\mathcal{H}_{(4.5.9)}](t) = \int \mathrm{d}\mu_{USp(4)}(\boldsymbol{u}) \int \mathrm{d}\mu_{USp(4)}(\boldsymbol{v}) \int \mathrm{d}\mu_{G_2}(\boldsymbol{w}) \times \\ H[\mathcal{H}_{(4.2.33)}](t;\boldsymbol{u}) H[\mathcal{H}_{(4.2.33)}](t;\boldsymbol{v}) \, \mathrm{PE}\left[\chi^{C_2}_{[1,0]}(\boldsymbol{u}) \chi^{G_2}_{[1,0]}(\boldsymbol{w}) + \boldsymbol{u} \leftrightarrow \boldsymbol{v} \right] \\ \mathrm{PE}\left[-\chi^{C_2}_{[2,0]}(\boldsymbol{u}) t^2 - \chi^{C_2}_{[2,0]}(\boldsymbol{v}) t^2 - \chi^{G_2}_{[0,1]}(\boldsymbol{w}) t^2 \right] . \end{aligned} \quad (4.5.10)$$

The Coulomb branch Hilbert series can be computed as if the two G_2 symmetries that are connected by $T(G_2)$ becomes two separated flavour nodes:

$$H[\mathcal{C}_{(4.5.9)}](t) = \sum_{n_1,n_2\geq 0} \sum_{a_1\geq a_2\geq 0} \sum_{b_1\geq b_2\geq 0} t^{2\Delta(\boldsymbol{n},\boldsymbol{a},\boldsymbol{b})} P_{G_2}(t;\boldsymbol{n}) P_{C_2}(t;\boldsymbol{a}) P_{C_2}(t;\boldsymbol{b}) \tag{4.5.11}$$

where $\boldsymbol{n} = (n_1, n_2)$ are the fluxes of the G_2 gauge group, $\boldsymbol{a} = (a_1, a_2)$ and $\boldsymbol{b} = (b_1, b_2)$ are the fluxes for the two $USp(4)$ gauge groups. Here $\Delta(\boldsymbol{n},\boldsymbol{a},\boldsymbol{b})$ is the dimension of the monopole operator:

$$\begin{aligned}
\Delta(\boldsymbol{n},\boldsymbol{a},\boldsymbol{b}) &= \Delta^{\text{hyp}}_{G_2-C_2}(\boldsymbol{0},\boldsymbol{a}) + \Delta^{\text{hyp}}_{G_2-C_2}(\boldsymbol{0},\boldsymbol{b}) + \Delta^{\text{hyp}}_{G_2-C_2}(\boldsymbol{n},\boldsymbol{a}) + \Delta^{\text{hyp}}_{G_2-C_2}(\boldsymbol{n},\boldsymbol{b}) \\
&\quad - \Delta^{\text{vec}}_{G_2}(\boldsymbol{n}) - \Delta^{\text{vec}}_{C_2}(\boldsymbol{a}) - \Delta^{\text{vec}}_{C_2}(\boldsymbol{b}) \\
2\Delta^{\text{hyp}}_{G_2-C_2}(\boldsymbol{n},\boldsymbol{a}) &= \frac{1}{2}\sum_{\pm}\sum_{i=1}^{2} \Big[|n_1 \pm a_i| + |n_1+n_2 \pm a_i| + |2n_1+n_2 \pm a_i| + \\
&\qquad + (n_1 \to -n_1,\, n_2 \to -n_2) + |\pm a_i| \Big] \\
\Delta^{\text{vec}}_{G_2}(\boldsymbol{n}) &= |n_1| + |n_2| + |n_1+n_2| + |2n_1+n_2| + |3n_1+n_2| + |3n_1+2n_2| \\
\Delta^{\text{vec}}_{C_2}(\boldsymbol{a}) &= |2a_1| + |2a_2| + |a_1+a_2| + |a_1-a_2| \,.
\end{aligned} \tag{4.5.12}$$

The dressing factors $P_{C_2}(t;\boldsymbol{a})$ and $P_{G_2}(t;\boldsymbol{n})$ are given by [86, (A.8), (5.27)]:

$$\begin{aligned}
P_{C_2}(t;a_1,a_2) &= \begin{cases} (1-t^2)^{-2} & a_1 > a_2 > 0 \\ (1-t^2)^{-1}(1-t^4)^{-1} & a_1 > a_2 = 0 \text{ or } a_1 = a_2 > 0 \\ (1-t^4)^{-1}(1-t^8)^{-1} & a_1 = a_2 = 0 \end{cases} \\
P_{G_2}(t;n_1,n_2) &= \begin{cases} (1-t^2)^{-2} & n_1 > n_2 > 0 \\ (1-t^2)^{-1}(1-t^4)^{-1} & n_1 = 0, n_2 > 0 \text{ or } n_1 > 0, n_2 = 0 \\ (1-t^4)^{-1}(1-t^{12})^{-1} & n_1 = n_2 = 0 \end{cases} .
\end{aligned} \tag{4.5.13}$$

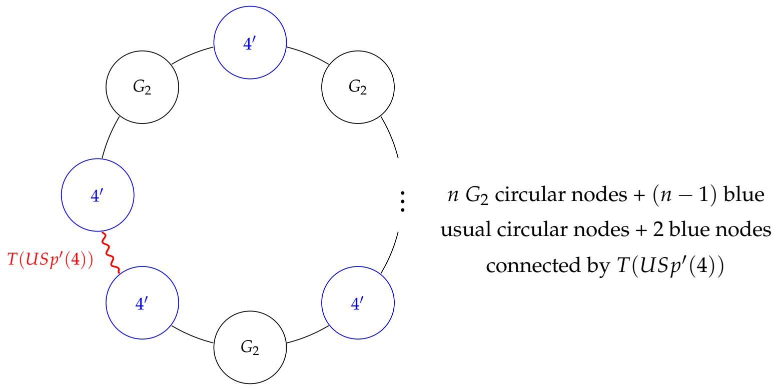
Upon calculating the integrals and the summations, we check up to order t^8 that the Higgs branch and the Coulomb branch Hilbert series are equal to each other:

$$H[\mathcal{H}_{(4.5.9)}](t) = H[\mathcal{C}_{(4.5.9)}](t) = \text{PE}\left[4t^4 + 5t^6 + 10t^8 + \ldots\right] . \tag{4.5.14}$$

This again supports our claim that (4.5.9) is self-mirror.

4.5.2 Self-mirror models with a $T(USp'(4))$ link

We can obtain another variation of (4.5.1) by simply S-folding one of the $USp'(4)$ gauge nodes in (4.5.4). The result is



(4.5.15)

The dimension of the Higgs branch is indeed equal to that of the Coulomb branch:

$$\dim_{\mathbb{H}} \mathcal{H}_{(4.5.15)} = \dim_{\mathbb{H}} \mathcal{C}_{(4.5.15)} = 2(2n-1) \, . \tag{4.5.16}$$

We claim that (4.5.15) is also self-mirror for any $n \geq 1$. One can indeed check, for example in the cases of $n = 1, 2$, that the Higgs and the Coulomb branch Hilbert series are equal, in the same way as presented in the preceding subsection. As an example, for $n = 1$, these are equal to the Coulomb branch Hilbert series of the G_2 gauge theory with 4 flavours [86, (5.28)]:

$$H[\mathcal{H}_{(4.5.15)}|_{n=1}] = H[\mathcal{C}_{(4.5.15)}|_{n=1}] = \mathrm{PE}\left[2t^4 + t^6 + t^8 + t^{10} + 2t^{12} + \ldots\right] \, . \tag{4.5.17}$$

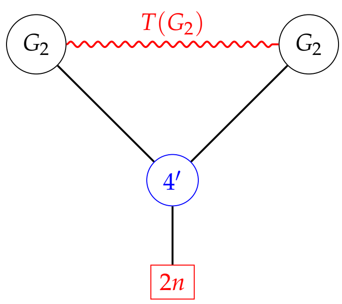
We finally remark that since we can perform an "S-folding" at any blue node, this confirms that each blue node labelled by 4 is indeed $USp'(4)$.

4.5.3 More mirror pairs by adding flavours

In this subsection, we add fundamental flavours to the self-mirror models discussed earlier and obtain mirror pairs that are not self-dual.

Models with a $T(G_2)$ link

Let us start the discussion by adding n flavours to the $USp'(4)$ gauge group in (4.5.6). This yields

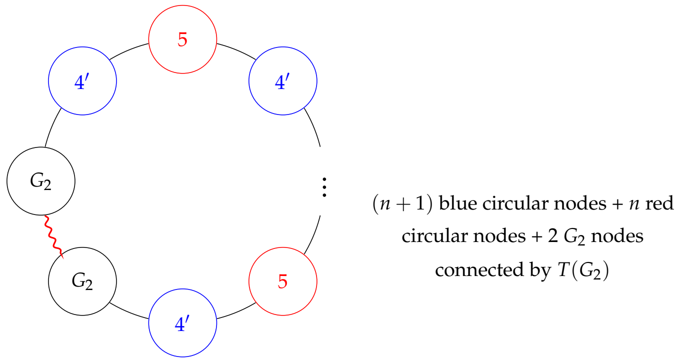


(4.5.18)

where the flavour symmetry is $SO(2n)$. The dimensions of the Higgs and Coulomb branches of this quiver are

$$\dim_{\mathbb{H}} \mathcal{H}_{(4.5.18)} = 4n + 2\,, \qquad \dim_{\mathbb{H}} \mathcal{C}_{(4.5.18)} = 2\,. \tag{4.5.19}$$

We propose that (4.5.18) is mirror dual to



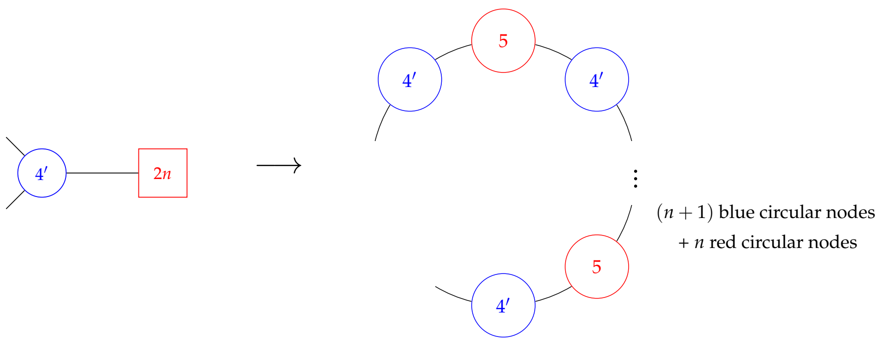
(4.5.20)

The Higgs branch dimension of this model is

$$\begin{aligned} \dim_{\mathbb{H}} \mathcal{H}_{(4.5.20)} &= (14-2) + 2\left(\frac{1}{2}\times 7 \times 4\right) + 10(2n) \\ &\qquad - 14 - 14 - 10(n+1) - 10n \\ &= 2\,. \end{aligned} \tag{4.5.21}$$

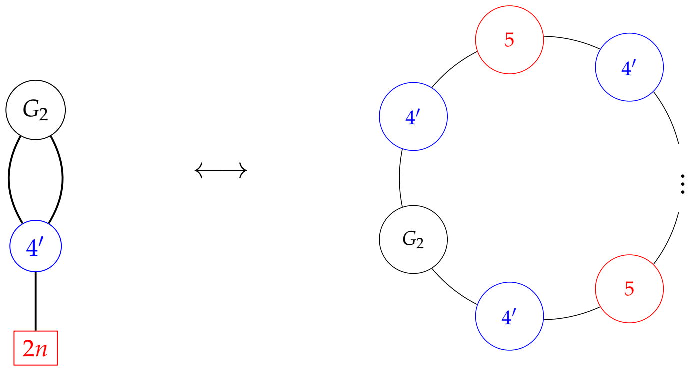
and the Coulomb branch dimension of this is $\dim_{\mathbb{H}} \mathcal{C}_{(4.5.20)} = 2(2n+1)$. This is consistent with mirror symmetry. We shall soon match the Higgs/Coulomb branch Hilbert series of (4.5.18) with the Coulomb/Higgs branch Hilbert series of (4.5.20) for $n = 1$.

Although we do not have a brane construction for (4.5.20) due to the presence of the G_2 gauge groups, the part of the quiver that contains alternating $USp'(4)/SO(5)$ gauge groups could be "realised" by a series of brane segments involving alternating $\widetilde{\text{O3}}^+/\widetilde{\text{O3}}^-$ across NS5 branes. In other words, starting from (4.5.18), the mirror theory (4.5.20) can be obtained by making the following replacement:



(4.5.22)

In the absence of the S-fold, the two G_2 gauge groups that were connected by $T(G_2)$ merge into a single one. We thus obtain the mirror pair between the following elliptic models:



(4.5.23)

$(n+1)$ blue circular nodes +
n red circular nodes + 1 G_2 node

The case of $n = 1$

Let us first focus on (4.5.18). The Higgs branch Hilbert series can be computed simply by putting the term $\mathrm{PE}[(x+x^{-1})\chi^{C_2}_{[1,0]}(z)t]$ in the integrand of (4.5.7), where x is the $SO(2)$ flavour fugacity. Performing the integral, we obtain (after setting $x = 1$):

$$H\left[\mathcal{H}_{(4.5.18)}|_{n=1}\right](t; x=1) = 1 + t^2 + 9t^4 + 15t^6 + 60t^8 + 113t^{10} + \ldots . \qquad (4.5.24)$$

The Coulomb branch Hilbert series for (4.5.18) is equal to that of the 3d $\mathcal{N} = 4$ $USp(4)$ gauge theory with $7+1 = 8$ flavours. The latter is given by

$$H\left[\mathcal{C}_{(4.5.18)}|_{n=1}\right](t) = \mathrm{PE}\left[t^4 + 2t^8 + t^{10} + t^{12} + t^{14} - t^{24} - t^{28}\right] . \qquad (4.5.25)$$

Let us now turn to (4.5.20). The Higgs branch Hilbert series is given by (4.5.10) with the following replacement:

$$\int d\mu_{G_2}(\boldsymbol{w}) \to \int d\mu_{SO(5)}(\boldsymbol{w}) , \quad \chi^{G_2}_{[1,0]}(\boldsymbol{w}) \to \chi^{B_2}_{[1,0]}(\boldsymbol{w}) , \quad \chi^{G_2}_{[0,1]}(\boldsymbol{w}) \to \chi^{B_2}_{[0,2]}(\boldsymbol{w}) . \qquad (4.5.26)$$

We checked that the result of this agrees with (4.5.25) up to order t^{10}. The Coulomb branch Hilbert series of (4.5.20) can be obtained in a similar way from (4.5.11) with the following replacement:

$$\begin{aligned} \Delta^{\mathrm{vec}}_{G_2}(\boldsymbol{n}) &\to \Delta^{\mathrm{vec}}_{B_2}(\boldsymbol{n}) = |n_1| + |n_2| + |n_1+n_2| + |n_1-n_2| \\ \Delta^{\mathrm{hyp}}_{G_2-C_2}(\boldsymbol{n}, \boldsymbol{a} \text{ or } \boldsymbol{b}) &\to \Delta^{\mathrm{hyp}}_{B_2-C_2}(\boldsymbol{n}, \boldsymbol{a} \text{ or } \boldsymbol{b}) \\ P_{G_2}(t;\boldsymbol{n}) &\to P_{C_2}(t;\boldsymbol{n}) \end{aligned} \qquad (4.5.27)$$

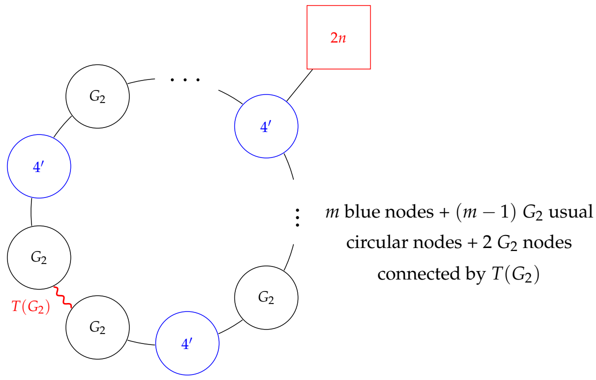
with

$$\Delta^{\mathrm{hyp}}_{B_2-C_2}(\boldsymbol{n}, \boldsymbol{a}) = \frac{1}{2} \times \frac{1}{2} \sum_{s_1,s_2=0}^{1} \sum_{j=1}^{2} \left[|(-1)^{s_2} a_j| + \sum_{i=1}^{2} |(-1)^{s_1} n_i + (-1)^{s_2} a_j| \right] . \qquad (4.5.28)$$

Again, we checked that the result of this agrees with (4.5.24) up to order t^{10}.

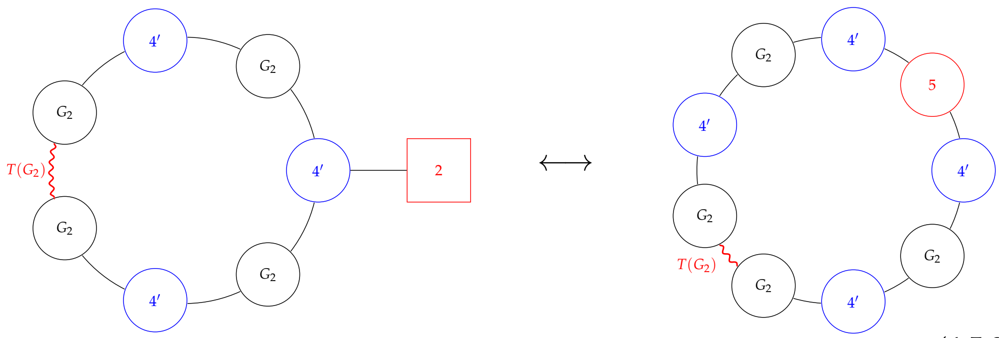
Generalisation of (4.5.18) to a polygon with flavours added

We can generalise (4.5.18) to a polygon consisting of alternating $G_2/USp'(4)$ gauge groups, with n flavours added to one of the $USp'(4)$ gauge group. This is depicted below.



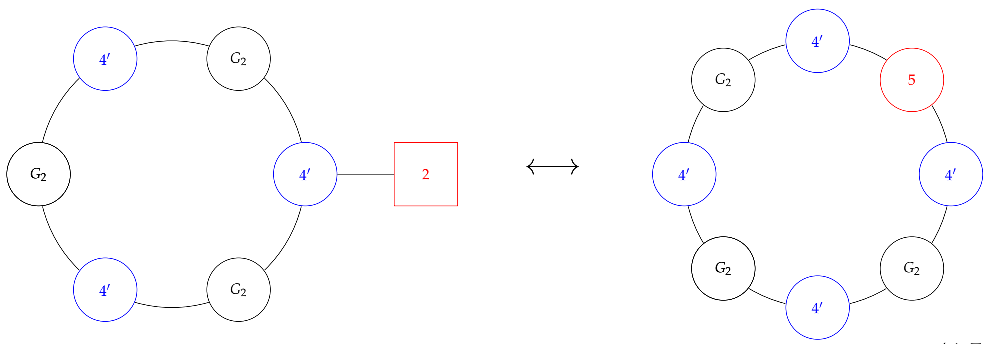
(4.5.29)

The mirror theory can simply be obtain by applying the replacement rule (4.5.22). For example, we have the following mirror pair



(4.5.30)

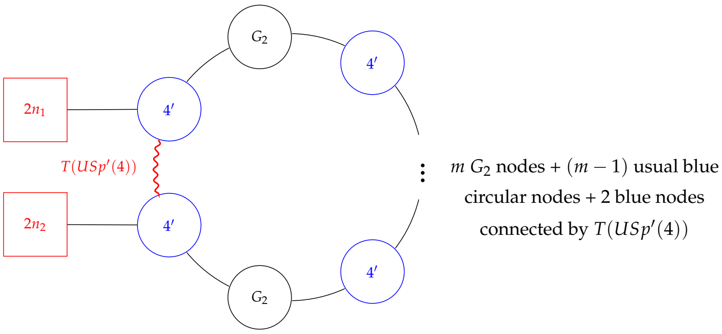
As emphasised before, as a by-product, one may obtain a mirror pair between the usual field theories, without an S-fold, by simply merging the two G_2 nodes that are connected by $T(G_2)$. The replacement rule described in (4.5.22) still applies. As an example, (4.5.30) becomes



(4.5.31)

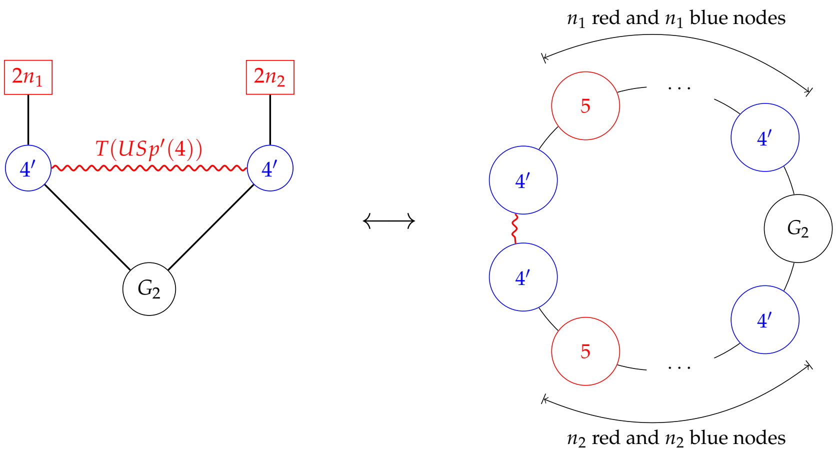
Models with a $T(USp'(4))$ link

Instead of S-folding the G_2 node as in (4.5.29), we can S-fold the $USp'(4)$ node and obtain



(4.5.32)

The mirror theory of this quiver can be obtained by applying the replacement rule (4.5.22), with one of the external legs on each side being a T-link. As an example, we have the following mirror pair:



(4.5.33)

Yet another generalisation one can possibly consider is to add flavour to any of the $4'$-node that is not connected by the T-link in (4.5.32). The mirror theory can simply be obtained, again, by applying the replacement rule given by (4.5.22).

As emphasised before, as a by-product, one may obtain a mirror pair between the usual field theories, without an S-fold, by simply merging the two $USp'(4)$ nodes that are connected by $T(USp'(4))$.

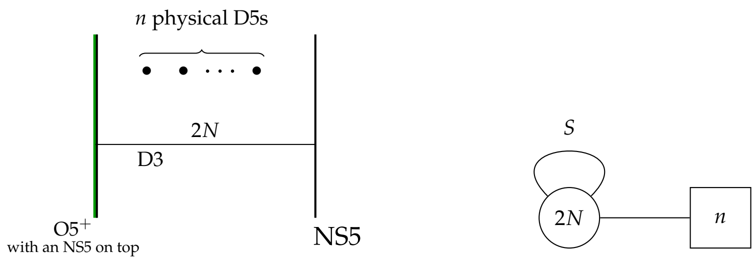
4.6 Models with an O5$^+$ plane

In this last section, we analyse models with O5$^+$ plane. In particular, we focus on a theory with one symmetric hypermultiplet and its mirror theory. One of the important features is that the mirror theory does not admit a conventional Lagrangian description. Nevertheless, we can represent it by a quiver diagram with a "multiple-lace", in the same sense of the Dynkin diagram of the C_N algebra [112, 52]. As pointed out in [52], it is possible to compute the Coulomb branch Hilbert series of such a mirror theory with the multiple-lace, and equate the result with the Higgs branch Hilbert series of the original theory with one symmetric hypermultiplet.

The point is to demonstrate that one may insert an S-fold into the brane system of the original theory and the corresponding mirror configuration, and still obtain a consistent mirror theory. Again, one can compute the Coulomb branch Hilbert series of the latter and match it with the Higgs branch Hilbert series of the former.

Models without an S-fold

We start by looking at the following theory:

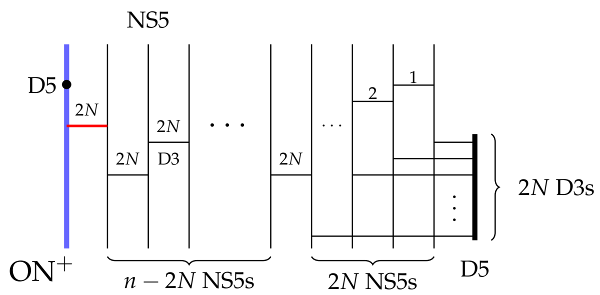


(4.6.1)

The presence of the O5$^+$ plane gives rise to a rank-two symmetric hypermultiplet. The Higgs and Coulomb branch dimensions for theory are as follows

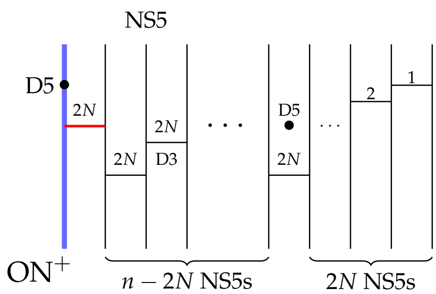
$$
\begin{aligned}
\dim_{\mathbb{H}} \mathcal{C}_{(4.6.1)} &= 2N \,, \\
\dim_{\mathbb{H}} \mathcal{H}_{(4.6.1)} &= \frac{1}{2} 2N(2N+1) + 2Nn - 4N^2 = 2Nn - 2N^2 + N \,.
\end{aligned}
\tag{4.6.2}
$$

Applying S-duality to the brane system (4.6.1) we get



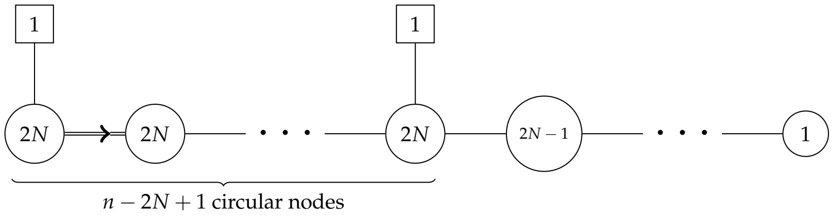
(4.6.3)

and, after moving the rightmost D5 brane into the brane interval, we arrive at



(4.6.4)

The corresponding quiver theory associated to this system is [112, 52]



(4.6.5)

The presence of the ON$^+$ plane gives rise to the double lace at the left end. This part of the quiver does not have a known Lagrangian description. However, as explained in [52], the part of the quiver that corresponds to a double lace, whose arrow goes from the gauge group $U(N_1)$ to $U(N_2)$, contributes to the dimension of the monopole operator as

$$\Delta_{(U(N_1))\Rightarrow(U(N_2))}(\boldsymbol{m}^{(1)}, \boldsymbol{m}^{(2)}) = \frac{1}{2}\sum_{i=1}^{N_1}\sum_{j=1}^{N_2} |2m_i^{(1)} - m_j^{(2)}|, \tag{4.6.6}$$

where $\boldsymbol{m}^{(1)}$ and $\boldsymbol{m}^{(2)}$ are the magnetic fluxes associated with the gauge groups $U(N_1)$ and $U(N_2)$ respectively. For (4.6.5), the Coulomb branch has dimension

$$\dim_{\mathbb{H}} \mathcal{C}_{(4.6.5)} = 2N(n-2N+1) + \sum_{i=1}^{2N-1} i = 2Nn - 2N^2 + N\,, \tag{4.6.7}$$

equal to the Higgs branch dimension of (4.6.1), as expected from mirror symmetry. Note that we have assumed that the two gauge nodes connected by the double lace contribute as the others. Since we do not have information about matter associated with the double lace, we cannot compute the Higgs branch dimension of (4.6.5) using the quiver description.

Let us consider a specific example by choosing $N = 1$ and $n = 4$. The unrefined Higgs branch Hilbert series of (4.6.1) is

$$\begin{aligned} H[\mathcal{H}_{(4.6.1)}] &= \oint_{|z|=1} \frac{dz}{2\pi i z}(1-z^2) \oint_{|q|=1} \frac{dq}{2\pi i q}\, \mathrm{PE}\Big[4(z+z^{-1})(q+q^{-1}) \\ &\qquad + (z^2+1+z^{-2})(q^2+q^{-2})t - (z^2+1+z^{-2}+1)t^2\Big] \\ &= \mathrm{PE}\left[16t^2 + 20t^3 - 12t^5 - 32t^6 + \ldots\right]. \end{aligned} \tag{4.6.8}$$

For the mirror theory (4.6.5) the unrefined Coulomb branch Hilbert series can be computed in the same way as described in [52]. The result is

$$\begin{aligned} H[\mathcal{C}_{(4.6.5)}] = &\sum_{m_1^{(1)} \geq m_2^{(1)} > -\infty} \ \sum_{m_1^{(2)} \geq m_2^{(2)} > -\infty} \ \sum_{m_1^{(3)} \geq m_2^{(3)} > -\infty} \ \sum_{m \in \mathbb{Z}} t^{2\Delta(\boldsymbol{m}^{(1)}, \boldsymbol{m}^{(2)}, \boldsymbol{m}^{(3)}, m)} \\ &\times P_{U(2)}(t, \boldsymbol{m}^{(1)}) P_{U(2)}(t, \boldsymbol{m}^{(2)}) P_{U(2)}(t, \boldsymbol{m}^{(3)}) P_{U(1)}(t, m) \\ = &\ \mathrm{PE}\left[16t^2 + 20t^3 - 12t^5 - 32t^6 + \ldots\right], \end{aligned} \tag{4.6.9}$$

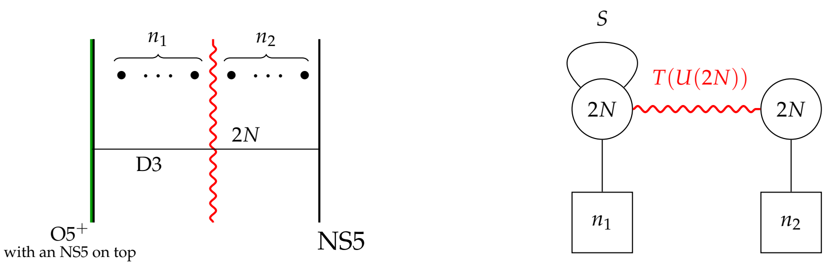
where $\boldsymbol{m}^{(i)} = (m_1^{(i)}, m_2^{(i)})$ for $i = 1, 2, 3$ and we define

$$\begin{aligned}
\Delta(\boldsymbol{m}^{(1)}, \boldsymbol{m}^{(2)}, \boldsymbol{m}^{(3)}, m) &= \Delta_{U(2) \Rightarrow U(2)}(\boldsymbol{m}^{(1)}, \boldsymbol{m}^{(2)}) + \Delta_{U(2)-U(2)}(\boldsymbol{m}^{(2)}, \boldsymbol{m}^{(3)}) \\
&\quad + \Delta_{U(2)-U(1)}(\boldsymbol{m}^{(3)}, m) + \Delta_{U(2)-U(1)}(\boldsymbol{m}^{(1)}, 0) \\
&\quad + \Delta_{U(2)-U(1)}(\boldsymbol{m}^{(3)}, 0) - \sum_{i=1}^{3} \Delta^{\text{vec}}_{U(2)}(\boldsymbol{m}^{(i)}) \\
2\Delta_{U(N_1) \Rightarrow U(N_2)}(\boldsymbol{m}, \boldsymbol{n}) &= \sum_{i=1}^{N_1} \sum_{j=1}^{N_2} |2m_i - n_j| \\
2\Delta_{U(N_1) - U(N_2)}(\boldsymbol{m}, \boldsymbol{n}) &= \sum_{i=1}^{N_1} \sum_{j=1}^{N_2} |m_i - n_j| \\
\Delta^{\text{vec}}_{U(2)}(\boldsymbol{m}) &= |m_1 - m_2| \\
P_{U(2)}(t; m_1, m_2) &= \begin{cases} (1-t^2)^{-2}\,, & m_1 \neq m_2 \\ (1-t^2)^{-1}(1-t^4)^{-1}\,, & m_1 = m_2 \end{cases} \\
P_{U(1)}(t; m) &= (1-t^2)^{-1}\,.
\end{aligned} \tag{4.6.10}$$

The two Hilbert series are equal as expected.

The case with an S-fold

One can insist with the insertion of an S-fold also for theories involving an O5$^+$ plane. The brane configuration and the quiver theory are as follows

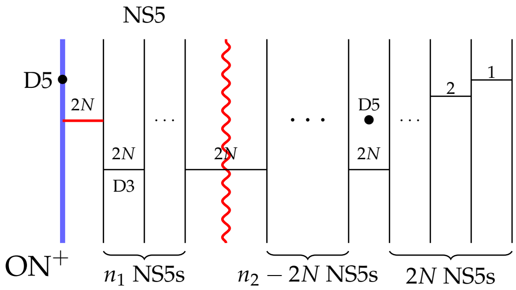


(4.6.11)

This theory has Coulomb and Higgs branches with the following dimensions

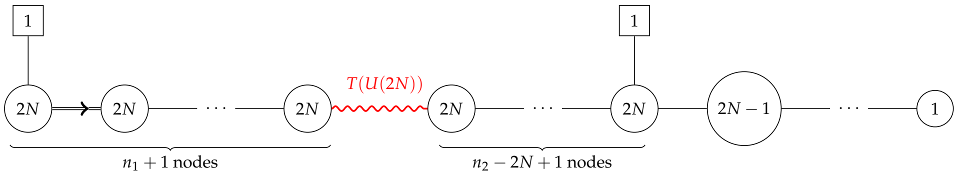
$$\begin{aligned}
\dim_{\mathbb{H}} \mathcal{C}_{(4.6.11)} &= 0, \\
\dim_{\mathbb{H}} \mathcal{H}_{(4.6.11)} &= \dim_{\mathbb{H}} \mathcal{H}_{(4.6.1)}|_{n=n_1+n_2} + (4N^2 - 2N) - 4N^2 \\
&= 2N(n_1 + n_2) - 2N^2 - N\,,
\end{aligned} \tag{4.6.12}$$

where the first line follows from the fact that the two circular nodes are connected by the T-link and hence do not contribute to the Coulomb branch. The brane system we get after applying S-duality is



(4.6.13)

whose associated gauge theory reads



(4.6.14)

The Coulomb branch dimension of this theory reads

$$\dim_{\mathbb{H}} \mathcal{C}_{(4.6.14)} = 2N(n_1 + 1 + n_2 - 2N + 1 - 2) + \sum_{i=1}^{2N-1} i = 2N(n_1 + n_2) - 2N^2 - N \,, \tag{4.6.15}$$

which equal to (4.6.12).

Let us consider the example of $N = 1$, $n_1 = 2$ and $n_2 = 2$. The Higgs branch of (4.6.11) splits into a product of two hyperKähler spaces as usual. The right part gives the nilpotent cone of $su(2)$ (which is isomorphic to $\mathbb{C}^2/\mathbb{Z}_2$), as pointed out in (4.2.8); the corresponding unrefined Hilbert series is $\mathrm{PE}[3t^2 - t^4]$. The left part contributes to the Hilbert series as

$$\begin{aligned} &\oint_{|z|=1} \frac{dz}{2\pi i z}(1-z^2) \oint_{|q|=1} \frac{dq}{2\pi i q} \mathrm{PE}\Big[2(z+z^{-1})(q+q^{-1}) \\ &\quad + (z^2+1+z^{-2})(q^2+q^{-2})t + (z^2+1+z^{-2})t^2 - t^4 \\ &\quad - (z^2+1+z^{-2}+1)t^2\Big] = \mathrm{PE}\left[4t^2 + 6t^3 + 4t^4 + \ldots\right]. \end{aligned} \tag{4.6.16}$$

Hence the Higgs branch Hilbert series of (4.6.11) is

$$H[\mathcal{H}_{(4.6.11)}] = \mathrm{PE}\left[4t^2 + 6t^3 + 4t^4 + \ldots\right] \mathrm{PE}\left[3t^2 - t^4\right]. \tag{4.6.17}$$

The Coulomb branch Hilbert series of (4.6.14), with $N = 1$, $n_1 = 2$ and $n_2 = 2$, can be obtained by taking the circular nodes connected by the T-link to be separated flavour nodes. Hence, the quiver splits into two parts. The right part contributes as the $U(1)$ gauge theory with 2 flavours, whose Coulomb branch is $\mathbb{C}^2/\mathbb{Z}_2$. The Coulomb branch Hilbert series of the left part can be computed in a similar way as (4.6.9). The result is therefore

$$H[\mathcal{C}_{(4.6.14)}] = \mathrm{PE}\left[4t^2 + 6t^3 + 4t^4 + \ldots\right] \mathrm{PE}\left[3t^2 - t^4\right]. \tag{4.6.18}$$

This is equal to the Higgs branch Hilbert series of (4.6.11) and is, therefore, consistent with mirror symmetry.

Chapter 5

Supersymmetric Indices of $3d$ S-fold SCFTs

5.1 A brief review of the 3d supersymmetric index

In this section, we briefly review the 3d supersymmetric index, which we shall refer to as the *index* for brievity. This is the supersymmetric partition function on $S^2 \times S^1$. It is defined as a trace over states on $S^2 \times \mathbb{R}$ [113, 114, 115, 116, 117, 118] (we also use the same notation as [119, 120]):

$$\mathcal{I}(x, \boldsymbol{\mu}) = \mathrm{Tr}\left[(-1)^{2J_3} x^{\Delta+J_3} \prod_i \mu_i^{T_i}\right], \tag{5.1.1}$$

where Δ is the energy in units of the S^2 radius (for superconformal field theories, Δ is related to the conformal dimension), J_3 is the Cartan generator of the Lorentz $SO(3)$ isometry of S^2, and T_i are charges under non-R global symmetries. The index only receives contributions from the states that satisfy:

$$\Delta - R - J_3 = 0, \tag{5.1.2}$$

where R is the R-charge. As a partition function on $S^2 \times S^1$, localisation implies that the index receives contributions only from BPS configurations, and it can be written in the following compact way:

$$\mathcal{I}(x; \{\boldsymbol{\mu}, \boldsymbol{n}\}) = \sum_{\boldsymbol{m}} \frac{1}{|\mathcal{W}_{\boldsymbol{m}}|} \int \frac{d\boldsymbol{z}}{2\pi i \boldsymbol{z}} Z_{\text{cl}}\, Z_{\text{vec}}\, Z_{\text{mat}}, \tag{5.1.3}$$

where we denoted by $\boldsymbol{z}$ the fugacities parameterising the maximal torus of the gauge group, and by $\boldsymbol{m}$ the corresponding GNO magnetic fluxes on S^2. Here $|\mathcal{W}_{\boldsymbol{m}}|$ is the dimension of the Weyl group of the residual gauge symmetry in the monopole background labelled by the configuration of magnetic fluxes $\boldsymbol{m}$. We also use $\{\boldsymbol{\mu}, \boldsymbol{n}\}$ to denote possible fugacities and fluxes for the background vector multiplets associated with global symmetries, respectively. As usual in localisation computations, the index receives contributions from the non-exact terms of the classical action and from the 1-loop corrections, and each term in the above equation can be described as follows.

Z_{cl}: The classical contribution is associated to Chern-Simons and BF interactions only. Denoting with k the CS level and with ω and $\mathfrak{n}$ the fugacity and the background flux for the topological symmetry, the classical contribution takes the form

$$Z_{\rm cl} = \prod_{i=1}^{{\rm rk}G} \omega^{m_i} z_i^{k\, m_i + \mathfrak{n}} \,, \tag{5.1.4}$$

where rk$G$ is the rank of the gauge group G.

$\boldsymbol{Z}_{\mathbf{vec}}$: This is the contribution of the $\mathcal{N}=2$ vector multiplet in the theory:

$$Z_{\rm vec} = \prod_{\alpha\in\mathfrak{g}} x^{-\frac{|\alpha(\boldsymbol{m})|}{2}} \left(1-(-1)^{\alpha(\boldsymbol{m})} \boldsymbol{z}^{\alpha} x^{|\alpha(\boldsymbol{m})|}\right) \tag{5.1.5}$$

where α are roots in the gauge algebra $\mathfrak{g}$.

$\boldsymbol{Z}_{\mathbf{mat}}$: The term encoding the matter fields in the theory enters as the product of the contributions of each $\mathcal{N}=2$ chiral field χ, transforming in some representation $\mathcal{R}$ and $\mathcal{R}_F$ of the gauge and the flavour symmetry respectively. Denoting by r_χ the R-charge of χ, its contribution to the index is of the form

$$\begin{aligned} Z_\chi = \prod_{\rho\in\mathcal{R}} \prod_{\widetilde{\rho}\in\mathcal{R}_F} & \left(\boldsymbol{z}^{\rho}\, \boldsymbol{\mu}^{\widetilde{\rho}}\, x^{r_\chi - 1}\right)^{-\frac{|\rho(\boldsymbol{m})+\widetilde{\rho}(\boldsymbol{n})|}{2}} \times \\ & \times \frac{\left((-1)^{\rho(\boldsymbol{m})+\widetilde{\rho}(\boldsymbol{n})}\, \boldsymbol{z}^{-\rho}\, \boldsymbol{\mu}^{-\widetilde{\rho}}\, x^{2-r_\chi+|\rho(\boldsymbol{m})+\widetilde{\rho}(\boldsymbol{n})|}; x^2\right)_\infty}{\left((-1)^{\rho(\boldsymbol{m})+\widetilde{\rho}(\boldsymbol{n})}\, \boldsymbol{z}^{\rho}\, \boldsymbol{\mu}^{\widetilde{\rho}}\, x^{r_\chi+|\rho(\boldsymbol{m})+\widetilde{\rho}(\boldsymbol{n})|}; x^2\right)_\infty} \,, \end{aligned} \tag{5.1.6}$$

where ρ and $\widetilde{\rho}$ are the weights of $\mathcal{R}$ and $\mathcal{R}_F$ respectively.

Let us discuss some examples that will be used later. The $T(U(1))$ theory is an almost empty theory, containing only the mixed CS coupling between two $U(1)$ background vector multiplets; its index is

$$\mathcal{I}_{T(U(1))}(\{\mu, n\}, \{\tau, p\}) = \tau^n \mu^p \,. \tag{5.1.7}$$

Next, we consider 3d $\mathcal{N}=4$ $U(1)$ gauge theory with 2 flavours, whose SCFT is known as $T(SU(2))$. The index of this theory is

$$\begin{aligned} & \mathcal{I}_{T(SU(2))}(\{\boldsymbol{\mu}, \boldsymbol{n}\}, \{\boldsymbol{\tau}, \boldsymbol{p}\}) \\ & = \sum_{m\in\mathbb{Z}} \left(\frac{\tau_1}{\tau_2}\right)^m \oint \frac{dz}{2\pi i z} z^{n_1-n_2} \prod_{a=1}^{2} x^{\frac{|m-p_a|}{2}} \frac{\left((-1)^{m-p_a} z^{\mp 1} \mu_a^{\pm 1} x^{3/2+|m-p_a|}; x^2\right)_\infty}{\left((-1)^{m-p_a} z^{\pm 1} \mu_a^{\mp 1} x^{1/2+|m-p_a|}; x^2\right)_\infty} \,, \end{aligned} \tag{5.1.8}$$

with the conditions $\mu_1\mu_2 = \tau_1\tau_2 = 1$ and $n_1+n_2 = p_1+p_2 = 0$ being imposed.

Another important example is the index for $T(U(2))$:

$$\begin{aligned} & \mathcal{I}_{T(U(2))}(\{\boldsymbol{\mu}, \boldsymbol{n}\}, \{\boldsymbol{\tau}, \boldsymbol{p}\}) \\ & = \left[\prod_{i=1}^{2} \mathcal{I}_{T(U(1))}(\{\mu_i, n_i\}, \{\tau_i, p_i\})\right] \times \mathcal{I}_{T(SU(2))}(\{\boldsymbol{\mu}, \boldsymbol{n}\}, \{\boldsymbol{\tau}, \boldsymbol{p}\}) \,, \end{aligned} \tag{5.1.9}$$

where in this expression there is no need to impose the constraints on $\{\boldsymbol{\mu}, \boldsymbol{n}\}, \{\boldsymbol{\tau}, \boldsymbol{p}\}$ as for $T(SU(2))$. Hence we may regard $\{\boldsymbol{\mu}, \boldsymbol{n}\}$ as fugacities and fluxes for the flavour $U(2)$ symmetry, and $\{\boldsymbol{\tau}, \boldsymbol{p}\}$ as fugacities and fluxes for the enhanced $U(2)$ topological symmetry. The fact that $T(U(2))$ is a self-mirror theory can be translated into the invariance of $\mathcal{I}_{T(U(2))}(\{\boldsymbol{\mu}, \boldsymbol{n}\}, \{\boldsymbol{\tau}, \boldsymbol{p}\})$ under the simultaneous exchange $\boldsymbol{\mu} \leftrightarrow \boldsymbol{\tau}$, $\boldsymbol{n} \leftrightarrow \boldsymbol{p}$. For our purpose, we turn off background magnetic fluxes.

5.1.1 Superconformal multiplets and the index

Let us now focus on 3d superconformal field theories (SCFTs). The index keeps track of the short multiplets, up to recombination. This feature makes the reconstruction of the whole content of short multiplets from the index an extremely hard task. Nevertheless, one may classify the equivalence classes of the multiplets according to their contribution to the index; see [27, 28] for 4d SCFTs, and [26] for 3d SCFTs. For this purpose, it is convenient to set the background magnetic fluxes to zero and expand the index as a power series in x

$$\mathcal{I}(x, \{\boldsymbol{\mu}, \boldsymbol{n} = 0\}) \;=\; \sum_{p=0}^{\infty} \chi_p(\boldsymbol{\mu})\, x^p \tag{5.1.10}$$

where $\chi_p(\boldsymbol{\mu})$ is the character of a certain representation of the global symmetry of the theory. As demonstrated in [26], one can study the contribution of superconformal multiplets to each order of x in the power series.

Since the shortening conditions for 3d superconformal algebras have been classified [121, 122] (we follow the notation of [122]), one can extract a lot of useful information about the SCFT in question using the power series of the index. Indeed this approach has proved successful, in the context of 3d $\mathcal{N} = 2$ gauge theories, for the study of global symmetry enhancement (see *e.g.* [123, 26, 124, 125]) and supersymmetry enhancement (see *e.g.* [29, 126]). We adopt this approach to study enhancement of supersymmetry and other global symmetries in the context of 3d S-fold SCFTs. More recent investigation of $\mathcal{N} = 2$ preserving exactly marginal deformations for 3d S-fold SCFTs has been presented in [127].

As pointed out in [29], it is useful to define the modified index as follows:

$$\widetilde{\mathcal{I}}(x, \{\boldsymbol{\mu}, \boldsymbol{n} = 0\}) = (1 - x^2)\,[\mathcal{I}(x, \{\boldsymbol{\mu}, \boldsymbol{n} = 0\}) - 1] \tag{5.1.11}$$

Note that all of the terms up to order x^2 in the modified index $\widetilde{\mathcal{I}}$ are equal to those in the original index $\mathcal{I}$ with the same power. As discussed in [26], the $\mathcal{N} = 2$ multiplets that can non-trivially contribute to the modified index at order x^p for $p \leq 2$ are as follows:

Multiplet	Contribution to the modified index	Comment
$A_2\overline{B}_1[0]^{(1/2)}_{1/2}$	$+x^{1/2}$	free fields
$B_1\overline{A}_2[0]^{(-1/2)}_{1/2}$	$-x^{3/2}$	free fields
$L\overline{B}_1[0]^{(1)}_{1}$	$+x$	relevant operators
$L\overline{B}_1[0]^{(2)}_{2}$	$+x^2$	marginal operators
$A_2\overline{A}_2[0]^{(0)}_{1}$	$-x^2$	conserved currents

(5.1.12)

Indeed, as pointed out in [26, 125] (see also [28]), the coefficient of x^2 in the index counts the number of marginal operators minus the number of conserved currents.

Since our S-fold SCFTs has at least $\mathcal{N} = 3$ supersymmetry, we shall work with $\mathcal{N} = 3$ superconformal multiplets. The ones that are relevant to us are tabulated below, along with the decomposition rules into $\mathcal{N} = 2$ superconformal multiplets

[29].

Type	$\mathcal{N}=3$ multiplet	Decomposition into $\mathcal{N}=2$ multiplets
Flavour current	$B_1[0]_1^{(2)}$	$L\overline{B}_1[0]_1^{(1)} + B_1\overline{L}[0]_{-1}^{(1)} + A_2\overline{A}_2[0]_1^{(0)}$
Extra SUSY-current	$A_2[0]_1^{(0)}$	$A_2\overline{A}_2[0]_1^{(0)} + A_1\overline{A}_1[1]_{3/2}^{(0)}$
Stress tensor	$A_1[1]_{3/2}^{(0)}$	$A_1\overline{A}_1[1]_{3/2}^{(0)} + A_1\overline{A}_1[2]_2^{(0)}$

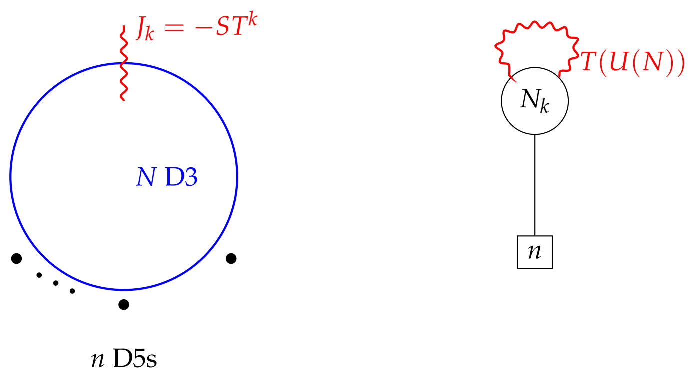
(5.1.13)

From the point of view of $\mathcal{N}=2$ supersymmetry, the presence of the extra SUSY-current multiplet leads to an enhanced flavour symmetry. The latter then combines with the $SO(2)$ R-symmetry of the original $\mathcal{N}=2$ supersymmetry to become an R-symmetry of the theory with higher supersymmetry. We discuss in an example below (5.2.4).

In [29], the author also provided certain conditions on the index regarding supersymmetry enhancement from $\mathcal{N}=3$. Let us denoted by a_p the coefficient of x^p in the unrefined modified index $\widetilde{\mathcal{I}}(x, \{\boldsymbol{\mu} = (1, \dots, 1), \boldsymbol{n} = 0\})$, where μ_i are set to 1 for all i. A sufficient condition for supersymmetry enhancement states that if $-a_2 > a_1$, then supersymmetry is enhanced from $\mathcal{N}=3$ to $\mathcal{N}=3-a_1-a_2$. The explanation of this condition is as follows. As it can be seen from tables (5.1.12) and (5.1.13), $(-a_2)$ is the number of flavour current multiplets plus the number of extra-SUSY current multiplets, and a_1 is the number of flavour current multiplets. The quantity $(-a_2)-a_1$ is therefore the number of extra-SUSY current multiplets that give rise to the supersymmetry enhancement. Furthermore, the author of [29] also discussed necessary conditions for supersymmetry enhancement to $\mathcal{N}=4$ and $\mathcal{N}=5$. For enhancement to $\mathcal{N}=4$, one must have $a_1+a_2+2 \geq 0$ and a_1 equal to the dimension of the flavour symmetry. For enhancement to $\mathcal{N}=5$, one must have $a_1=1$, $a_2 \geq -3$ and a_p even for non-integer p. We emphasise, however, that if one uses the refined index (*i.e.* not setting the fugacities μ_i to unity), one may get more information regarding the presence of the extra SUSY-current multiplets (and hence supersymmetry enhancement), because such a contribution to the index may get cancelled in the unrefined version by the one coming from marginal operators. We discuss this point in detail in the main text.

5.2 A single $U(N)_k$ gauge group with a T-link and n flavours

In this section, we consider the following theory:



(5.2.1)

In [14], the following statements are proposed:

1. For $k = 0$, the SCFT has $\mathcal{N} = 3$ supersymmetry.
2. For $k \geq 3$ and $n = 0$, the SCFT has $\mathcal{N} = 4$ supersymmetry. This statement was confirmed at large N using the corresponding supergravity solutions and the computation of the three sphere partition function in the large N limit.

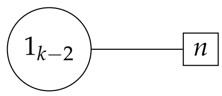
In the following we compute the superconformal index at low rank N and small values of n. Whenever possible, we deduce the amount of supersymmetry of the SCFT from the index.

5.2.1 The abelian case: $N = 1$

The moduli space of this theory was analysed in [50, sec. 4.4]. Recall that the $T(U(1))$ is almost an empty theory, with only a prescription for how coupling external gauge fields A_1 and A_2, which is the supersymmetric completion of the following CS coupling [8]

$$-\frac{1}{2\pi}\int A_1 \wedge dA_2\,. \tag{5.2.2}$$

In (5.2.1), we identify the $U(1)$ gauge fields A_1 and A_2 to a single one, and hence the above equation gives rise to a CS level -2 to the $U(1)$ gauge group. In other words, quiver (5.2.1), with $N = 1$, can be identify with the following theory



(5.2.3)

where we emphasise that this theory no longer contains a T-link.

As an immediate consequence, for $k = 2$, this theory is simply a 3d $\mathcal{N} = 4$ $U(1)$ gauge theory with n flavours. For $k = 2$ and $n = 1$, this is dual to a theory of a free hypermultiplet.

Another interesting case is when $k = 1$ and $n = 1$, which is equivalent to having 3d $\mathcal{N} = 3$ $U(1)_{-1}$ gauge theory with 1 flavour. The index in this case reads

$$\begin{aligned}\mathcal{I}_{(5.2.1),\,N=1,k=1,n=1}(x;\omega) &= \mathcal{I}_{U(1)_{\pm 1}\text{ with 1 flavour}}(x;\omega)\\ &= 1 + x - x^2\left(\omega + \omega^{-1} + 1\right) + x^3\left(\omega + \omega^{-1} + 2\right)\\ &\quad - x^4\left(\omega + \omega^{-1} + 2\right) + x^5 + \dots\,.\end{aligned} \tag{5.2.4}$$

where ω denotes the topological fugacity. The modified index of this theory is

$$(1 - x^2)\left[\mathcal{I}_{(5.2.1),\,N=1,k=-1}(x;\omega) - 1\right] = x - x^2\left(\omega + \omega^{-1} + 1\right) + \dots\,. \tag{5.2.5}$$

We expect the enhancement of supersymmetry from $\mathcal{N} = 3$ to $\mathcal{N} = 5$ due to the following argument[1]. The presence of the term $+x$ indicates that there must be an

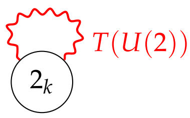
[1]Upon setting $\omega = 1$, we obtain the unrefined modified index $x - 3x^2 + 3x^3 - x^4 - 3x^5 + \dots$. Denoting the coefficient of x^k by a_k, we see that $(-a_2) = 3 > a_1 = 1$. Therefore according to [29, sec. 4.3], it is expected that supersymmetry gets enhanced from $\mathcal{N} = 3$ to $\mathcal{N} = 3 - a_1 - a_2 = 5$. Moreover, since $a_1 = 1$, $a_2 = -3 \geq -3$ and $a_p = 0$ (which is even) for all non-integers p, the necessary condition in [29, sec. 4.3] for having $\mathcal{N} = 5$ supersymmetry is satisfied.

$\mathcal{N}=3$ flavour current multiplet $B_1[0]_1^{(2)}$, which gives rise to the $\mathcal{N}=2$ multiplet $L\overline{B}_1[0]_1^{(1)}$ contributing $+x$ and the $\mathcal{N}=2$ multiplet $A_2\overline{A}_2[0]_1^{(0)}$ contributing $-x^2$. Since the coefficient of x^2 counts the number of marginal operators minus the number of conserved currents [26, 125] (see also [28]), there must be two extra conserved currents associated with the terms $-\left(\omega+\omega^{-1}\right)x^2$. Such extra conserved currents come from two $\mathcal{N}=3$ extra SUSY-current multiplets $A_2[0]_1^{(0)}$, one carries fugacity ω and the other carries fugacity ω^{-1}.

It is also instructive to consider the above argument from the perspective of $\mathcal{N}=2$ supersymmetry, whose R-symmetry is $SO(2)_R$ to begin with. At the level of the Lagrangian, the theory has $U(1)\cong SO(2)$ topological symmetry. However, from the x^2 terms in (5.2.4), we see that this global symmetry is enhanced to $SO(3)$. This symmetry then combines with the original $SO(2)_R$ symmetry to become $SO(5)$ R-symmetry of the theory with $\mathcal{N}=5$ supersymmetry.

5.2.2 $U(2)_k$ gauge group and no flavour

We focus on the following quiver



(5.2.6)

We remark that the theory (5.2.6) can be also be represented as $T(U(2))/U(2)_k^{\text{diag}}$, where the diagonal subgroup $U(2)^{\text{diag}}$ of the symmetry $U(2)\times U(2)$ of $T(U(2))$ is gauged with CS level k. Nevertheless, we find that the index of such a theory does not depend on the fugacity associated with the topological symmetry, and it is equal to $T(SU(2))/SU(2)_k^{\text{diag}}$, where the diagonal subgroup $SU(2)^{\text{diag}}$ is gauged with CS level k.

In fact, the theory $T(SU(2))/SU(2)_k^{\text{diag}}$ was studied in a series of papers [80, 81, 124, 126], mainly in the context of the 3d-3d correspondence. In particular, it was pointed out in [126] that for $k=3$, $T(SU(2))/SU(2)_3^{\text{diag}}$ is a product of two identical 3d $\mathcal{N}=4$ SCFTs. Such an SCFT admits a 3d $\mathcal{N}=2$ Lagrangian in terms of the $U(1)_{-3/2}$ gauge theory with 1 chiral multiplet carrying gauge charge $+1$ (denoted by $\mathcal{T}_{-3/2,1}$), where it turns out that supersymmetry of this theory gets enhanced to $\mathcal{N}=4$ in the infrared.

In addition to the case of $|k|=3$, we find that the supersymmetry gets enhanced for all k such that $|k|\geq 4$. We summarise the results in the following table.

CS level	Index	Type of J_k	Comment
$\lvert k\rvert \geq 4$	(5.2.7)	hyperbolic	
$\lvert k\rvert = 3$	(5.2.10)	hyperbolic	Studied in [126], a product of two $\mathcal{N}=4$ SCFTs
$\lvert k\rvert = 2$	diverges	parabolic	
$\lvert k\rvert = 1$	1	elliptic	
$k=0$	1	elliptic	

We emphasise the cases whose indices indicate supersymmetry enhancement in yellow. In the following, we discuss the detail of each case.

For $|k|\geq 4$, J_k is hyperbolic. We find that the index reads

$$\mathcal{I}_{(5.2.6),\,N=2,\,|k|\geq 4}(x)=1-x^2+2x^3-x^4+\dots\,. \tag{5.2.7}$$

where, for each k such that $|k| \geq 4$, the indices differ at order of x greater than 4. For example, up to order x^8, the indices are as follows:

$$\begin{array}{ll} |k| = 4 & 1 - x^2 + 2x^3 - x^4 - 4x^5 + 10x^6 - 10x^7 + 8x^8 + \dots \\ |k| = 5 & 1 - x^2 + 2x^3 - x^4 - 2x^5 + 6x^6 - 8x^7 + 4x^8 + \dots \\ |k| = 6 & 1 - x^2 + 2x^3 - x^4 - 2x^5 + 6x^6 - 8x^7 + 6x^8 + \dots \end{array} \tag{5.2.8}$$

The modified index is

$$(1 - x^2)\left[\mathcal{I}_{(5.2.6),\, N=2,\, |k|\geq 4}(x) - 1\right] = -x^2 + 2x^3 + \dots \,. \tag{5.2.9}$$

The fact that the coefficient of x vanishes implies that we have no $\mathcal{N} = 3$ flavour current multiplet $B_1[0]_1^{(2)}$. The term $-x^2$ indicates the presence of the $\mathcal{N} = 3$ extra SUSY-current multiplet $A_2[0]_1^{(0)}$. We thus conclude that the supersymmetry gets enhanced from $\mathcal{N} = 3$ to $\mathcal{N} = 4$ when $|k| \geq 4$.

For $|k| = 3$, the index reads

$$\mathcal{I}_{(5.2.6),\, N=2,\, |k|=3}(x) = 1 - 2x^2 + 4x^3 - 3x^4 + \dots \,. \tag{5.2.10}$$

According to [126], this is equal to the square of the index of $\mathcal{T}_{-3/2,1}$. In the notation we adopted, the index of $\mathcal{T}_{-3/2,1}$ reads

$$\mathcal{I}_{\mathcal{T}_{-3/2,1}}(x; w) = 1 - x^2 + \left(w + w^{-1}\right) x^3 - 2x^4 + \dots \,. \tag{5.2.11}$$

where w is the topological fugacity. Indeed, we find that

$$\left[\mathcal{I}_{\mathcal{T}_{-3/2,1}}(x; w = 1)\right]^2 = \mathcal{I}_{(5.2.6),\, N=2,\, k=-3}(x) \,. \tag{5.2.12}$$

The modified index corresponding to (5.2.10) reads

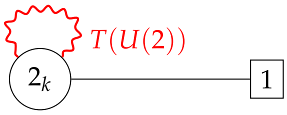
$$(1 - x^2)\left[\mathcal{I}_{(5.2.6),\, N=2,\, |k|=3}(x) - 1\right] = -2x^2 + 4x^3 - x^4 \dots \,. \tag{5.2.13}$$

Let us denote the coefficient of x^p by a_p. Naively, from the condition $-a_2 = 2 > a_1 = 0$ discussed in [29], one might expect that supersymmetry gets enhanced to $\mathcal{N} = 3 - a_1 - a_2 = 5$. However, this cannot be true, for the reason that the $\mathcal{N} = 5$ stress tensor multiplet in the representation $[1, 0]$ of $SO(5)$ decomposes into one $\mathcal{N} = 2$ multiplet $L\overline{B}_1[0]_1^{(1)}$, which contributes $a_1 = 1$ [29, (B.25)] (but here we have $a_1 = 0$). Since this theory is a product of two copies of $\mathcal{T}_{-3/2,1}$, which has enhanced $\mathcal{N} = 4$ supersymmetry, there are two copies of the $\mathcal{N} = 3$ extra SUSY-current multiplet $A_2[0]_{\Delta=1}^{(0)}$. This is consistent with the fact that the modified index has $a_1 = 0$ and $a_2 = -2$.

For the theory with $|k| = 2$ (J_2 is parabolic), the index diverges, and so we have a "bad" theory in the sense of [8]. For $|k| = 1$ and $k = 0$ (J_k is elliptic in these cases), we find that the index is equal to unity.

5.2.3 Adding one flavour ($n = 1$) to the $U(2)_k$ gauge group

We now consider the following theory



(5.2.14)

Let us summarise the results in the following table.

CS level	Index	Type of J_k	Comment
$k = -2$	(5.2.23)	parabolic	
$k = -1$	(5.2.22)	elliptic	
$k = 0$	(5.2.22)	elliptic	
$k = 1$	(5.2.15)	elliptic	
$k = 2$	(5.2.17)	parabolic	A free hyper $\times$ an $\mathcal{N} = 4$ SCFT
$\lvert k \rvert \geq 3$	(5.2.22)	hyperbolic	

where we emphasise the cases that have supersymmetry enhancement in yellow.

For $k = 1$, we find that the index reads

$$\begin{aligned}\mathcal{I}_{(5.2.14),k=1}(x;\omega) = 1 + x + x^2\left[1-(1+\omega+\omega^{-1})\right] - x^3\left(\omega+\omega^{-1}\right) \\ + x^4(4+\omega^2+\omega^{-2}+3\omega+3\omega^{-1}) + \dots\,,\end{aligned} \tag{5.2.15}$$

where ω is the topological fugacity. From the above expression, we find that the modified index is as follows:

$$(1-x^2)\left[\mathcal{I}_{(5.2.14),k=1}(x;\omega) - 1\right] = x + x^2\left[1-(1+\omega+\omega^{-1})\right] + \dots\,. \tag{5.2.16}$$

From this, one can see the enhancement of supersymmetry from $\mathcal{N} = 3$ to $\mathcal{N} = 5$ as follows. The presence of the term $+x$ indicates that there must be an $\mathcal{N} = 3$ flavour current multiplet $B_1[0]_1^{(2)}$, which gives rise to the $\mathcal{N} = 2$ multiplet $L\overline{B}_1[0]_1^{(1)}$ contributing $+x$ and the $\mathcal{N} = 2$ multiplet $A_2\overline{A}_2[0]_1^{(0)}$ contributing $-x^2$. Since the coefficient of x^2 counts the number of marginal operators minus the number of conserved currents [26, 125] (see also [28]), there must be an $\mathcal{N} = 2$ marginal operator (in the multiplet $L\overline{B}_1[0]_2^{(2)}$) contributing $+x^2$ to cancel the aforementioned contribution $-x^2$, and there must be two extra conserved currents associated with the terms $-\left(\omega+\omega^{-1}\right)x^2$. The latter can only come from two copies of the $\mathcal{N} = 3$ extra SUSY-current multiplet $A_2[0]_1^{(0)}$, carrying the global symmetry associated with ω and ω^{-1}. (This gives rise to two copies of $\mathcal{N} = 2$ $A_2\overline{A}_2[0]_1^{(0)}$ multiplet contributing the term $-\left(\omega+\omega^{-1}\right)x^2$.) The presence of such a multiplet leads to the enhancement of supersymmetry from $\mathcal{N} = 3$ to $\mathcal{N} = 5$[2].

[2] We remark that one has to use the sufficient condition stated in [29, sec. 4.3] with great care. Upon setting $\omega = 1$ in the modified index, we obtain $x - 2x^2$. Denoting the coefficient of x^k by a_k, we see that $-a_2 = 2 > a_1 = 1$, and from [29], one might naively expect that supersymmetry gets enhanced to $\mathcal{N} = 3 - a_1 - a_2 = 4$, because we have only $(-a_2) - a_1 = 1$ extra SUSY-current multiplet. The unrefinement of the index is misleading here, because we in fact have two extra SUSY-current multiplets carrying the global fugacities ω and ω^{-1}, and these cannot be cancelled with -1 at order x^2 in the index. The reason for us to write $x^2\left[1-(1+\omega+\omega^{-1})\right]$ is to show explicitly that the contribution -1 of the conserved

For $k = 2$, the index reads

$$\begin{aligned}
&\mathcal{I}_{(5.2.14),\,k=2}(x; w) \\
&= 1 + \left(w + \frac{1}{w}\right) x^{\frac{1}{2}} + \left(2w^2 + \frac{2}{w^2} + 2\right) x + \left(2w^3 + \frac{2}{w^3} + 2w + \frac{2}{w}\right) x^{\frac{3}{2}} \\
&\quad + \left(3w^4 + \frac{3}{w^4} + 2w^2 + \frac{2}{w^2} + 1\right) x^2 + \ldots
\end{aligned} \tag{5.2.17}$$

The term $x^{1/2}$ indicates that this theory contains a free part due to the fact that the R-charge of the basic monopole operators hits the unitary bound. The above index can be rewritten as

$$\mathcal{I}_{(5.2.14),\,k=2}(x; w) = \mathcal{I}_{\text{free}}(x; w) \times \mathcal{I}_{\text{SCFT}}^{(5.2.14),\,k=2}(x; w) \tag{5.2.18}$$

where the index of a free hypermultiplet is given by

$$\mathcal{I}_{\text{free}}(x; w) = \frac{(x^{2-\frac{1}{2}} w; x^2)_\infty}{(x^{\frac{1}{2}} w^{-1}; x^2)_\infty} \frac{(x^{2-\frac{1}{2}} w^{-1}; x^2)_\infty}{(x^{\frac{1}{2}} w; x^2)_\infty} \tag{5.2.19}$$

and the index of the interacting SCFT part is

$$\begin{aligned}
&\mathcal{I}_{\text{SCFT}}^{(5.2.14),\,k=2}(x; w) \\
&= 1 + x\left(w^2 + \frac{1}{w^2} + 1\right) + x^2\left(w^4 + \frac{1}{w^4} - 1\right) + x^{5/2}\left(-w - \frac{1}{w}\right) + \ldots \\
&= 1 + x\chi_{[2]}^{SU(2)}(w) + x^2\left[\chi_{[4]}^{SU(2)}(w) - (\chi_{[2]}^{SU(2)}(w) + \chi_{[0]}^{SU(2)}(w))\right] \\
&\quad - x^{\frac{5}{2}}\chi_{[2]}^{SU(2)}(w) + \ldots\,,
\end{aligned} \tag{5.2.20}$$

with the unrefinement

$$\mathcal{I}_{\text{SCFT}}^{(5.2.14),\,k=2}(x; w = 1) = 1 + 3x + x^2 - 2x^{5/2} + 4x^3 + 4x^{7/2} + 3x^4 + \ldots\,. \tag{5.2.21}$$

As can be seen from (5.2.20), the interacting SCFT has enhanced $\mathcal{N} = 4$ supersymmetry. The argument is similar to the one used before. The term $+x\chi_{[2]}^{SU(2)}(w)$ indicates that the theory has an $SU(2)$ flavour symmetry. Indeed, there is an $\mathcal{N} = 3$ flavour current multiplet $B_1[0]_1^{(2)}$ transforming in the adjoint representation [2] of this symmetry; this gives rise to the $\mathcal{N} = 2$ multiplet $L\overline{B}_1[0]_1^{(1)}$ contributing $+x\chi_{[2]}^{SU(2)}(w)$ and the $\mathcal{N} = 2$ multiplet $A_2\overline{A}_2[0]_1^{(0)}$ contributing $-x^2\chi_{[2]}^{SU(2)}(w)$. The term $+x^2\chi_{[4]}^{SU(2)}(w)$ corresponds to the $\mathcal{N} = 2$ marginal operator[3] in the multiplet $L\overline{B}_1[0]_2^{(2)}$. It can be clearly seen that there is another conserved current corresponding to the term $-x^2\chi_{[0]}^{SU(2)}(w)$. Indeed, the latter comes from the $\mathcal{N} = 3$ extra SUSY-current multiplet $A_2[0]_1^{(0)}$ in the trivial representation [0] of $SU(2)$; this gives rise to an $\mathcal{N} = 2$

current has to be cancelled with the contribution +1 from the marginal operator, which is neutral under the symmetry associated with ω. Note that since $a_1 = 1$, $a_2 = -2 \geq -3$ and $a_p = 0$ (which is even) for all non-integers p, the necessary condition in [29, sec. 4.3] for having $\mathcal{N} = 5$ supersymmetry is satisfied.

[3] It should be noted that the 2nd symmetric power of [2] is $\text{Sym}^2[2] = [4] + [0]$. The representation [4], appearing at order x^2 of the index, is a part of this symmetric power.

conserved current multiplet $A_2\overline{A}_2[0]_1^{(0)}$ contributing the term $-x^2\chi_{[0]}^{SU(2)}(w)$. The existence of the extra SUSY-current multiplet indicates that there is an enhancement of supersymmetry from $\mathcal{N}=3$ to $\mathcal{N}=4$.

For $k=0$, -1 and $|k|\geq 3$, we find that the index reads

$$1+x+0x^2+\dots\,. \tag{5.2.22}$$

The term $+x$ indicates that there must be an $\mathcal{N}=3$ flavour current multiplet $B_1[0]_1^{(2)}$, which gives rise to the $\mathcal{N}=2$ multiplet $L\overline{B}_1[0]_1^{(1)}$ contributing $+x$ and the $\mathcal{N}=2$ multiplet $A_2\overline{A}_2[0]_1^{(0)}$ contributing $-x^2$. Hence the theory has a $U(1)$ flavour symmetry. The fact that the term x^2 vanishes implies that there is an $\mathcal{N}=2$ marginal operator in the multiplet $L\overline{B}_1[0]_2^{(2)}$, contributing $+x^2$, which cancels the aforementioned $-x^2$ term. Hence, in this case, there is no signal of the existence of the extra SUSY-current multiplet, *i.e.* we cannot deduce the enhancement of supersymmetry.

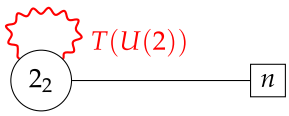
For $k=-2$, we find that the index reads

$$1+2x+x^2+8x^4+\dots\,. \tag{5.2.23}$$

There are two $\mathcal{N}=3$ flavour current multiplet $B_1[0]_1^{(2)}$ which gives rise to two copies of $\mathcal{N}=2$ multiplets $L\overline{B}_1[0]_1^{(1)}$ contributing $+2x$ and two copies of $\mathcal{N}=2$ multiplets $A_2\overline{A}_2[0]_1^{(0)}$ contributing $-2x^2$. Hence the theory has a $U(1)^2$ flavour symmetry. We may construct three $\mathcal{N}=2$ marginal operators by taking a symmetric product of two relevant operators in the $L\overline{B}_1[0]_1^{(1)}$ multiplets. Their contribution $+3x^2$ cancels the aforementioned $-2x^2$ and yields $+x^2$. There is no signal of the existence of the extra SUSY-current multiplet, *i.e.* we cannot deduce the enhancement of supersymmetry.

5.2.4 Adding n flavours to the $U(2)_2$ gauge group

In this section, we add an arbitrary number of flavours to the parabolic case[4], namely



(5.2.24)

When the number of flavours is one (*i.e.* $n=1$), we have seen from (5.2.18) that the theory factorises into a product of the theory of a free hypermultiplet and an interacting SCFT with enhanced $\mathcal{N}=4$ supersymmetry. For $n\geq 2$, the index does not exhibit explicitly the presence of the extra SUSY-current multiplet. Nevertheless, as we demonstrate below, the theory still has interesting physics that is bares certain resemblance to the 3d $\mathcal{N}=4$ $U(1)$ gauge theory with n flavours, such as the properties of monopole operators.

[4]Here $J_2=-ST^2$ is a parabolic element of $SL(2,\mathbb{Z})$. It is related to T^{-1} by the following similarity transformation: $(TST)J_2(TST)^{-1}=T^{-1}$. However, we emphasise that, when fundamental flavours are added as in (5.2.24), the theory is different from $U(2)_{-1}$ with n flavours. This can be seen clearly from the indices. For example, for $n=1$, the index for $U(2)_{-1}$ with 1 flavour is 1 but (5.2.18) is non-trivial.

For concreteness, let us first consider the case of $n = 2$. The index reads[5]

$$\begin{aligned}
&\mathcal{I}_{(5.2.24),\, n=2}(x;\omega,y) \\
&= 1 + x\left[\chi^{SU(2)}_{[2]}(\omega) + \chi^{SU(2)}_{[2]}(y)\right] + x^2\Big[\Big(1 + 2\chi^{SU(2)}_{[4]}(\omega) + \chi^{SU(2)}_{[4]}(y) \\
&\quad + \chi^{SU(2)}_{[2]}(\omega)\chi^{SU(2)}_{[2]}(y) + \chi^{SU(2)}_{[2]}(y)\Big) - \Big(\chi^{SU(2)}_{[2]}(\omega) + \chi^{SU(2)}_{[2]}(y)\Big)\Big] + \dots ,
\end{aligned} \tag{5.2.25}$$

with the unrefinement

$$\mathcal{I}_{(5.2.24),\, n=2}(x;\omega=1,y=1) = 1 + 6x + 22x^2 + 18x^3 + 29x^4 + \dots . \tag{5.2.26}$$

where the topological fugacity is denoted by $w = \omega^2$. We see that the $U(1)$ topological symmetry gets enhanced to $SU(2)$. This phenomenon also occurs for 3d $\mathcal{N} = 4$ $U(1)$ gauge theory with 2 flavours, whose index is

$$\begin{aligned}
&\mathcal{I}_{T(SU(2))}(x;\omega,y) \\
&= 1 + x\left[\chi^{SU(2)}_{[2]}(\omega) + \chi^{SU(2)}_{[2]}(y)\right] \\
&\quad + x^2\left[\chi^{SU(2)}_{[4]}(\omega) + \chi^{SU(2)}_{[4]}(y) - \Big(\chi^{SU(2)}_{[2]}(\omega) + \chi^{SU(2)}_{[2]}(y) + 1\Big)\right] + \dots
\end{aligned} \tag{5.2.27}$$

with the unrefinement

$$\mathcal{I}_{T(SU(2))}(x;\omega=1,y=1) = 1 + 6x + 3x^2 + 6x^3 + 17x^4 + \dots , \tag{5.2.28}$$

For $n = 3$, we find that the index of (5.2.24) reads

$$\begin{aligned}
&\mathcal{I}_{(5.2.24),\, n=3}(x;w,y) \\
&= 1 + x\left[1 + \chi^{SU(3)}_{[1,1]}(\boldsymbol{y})\right] + x^{\frac{3}{2}}(w + w^{-1}) \\
&\quad + x^2\left[\chi^{SU(3)}_{[2,2]}(y) + 2\chi^{SU(3)}_{[1,1]}(\boldsymbol{y}) + 1\right] + \dots ,
\end{aligned} \tag{5.2.29}$$

with the unrefinement

$$\begin{aligned}
\mathcal{I}_{(5.2.24),\, n=3}(x;w=1,\boldsymbol{y}=(1,1)) &= 1 + 9x + 2x^{\frac{3}{2}} + 44x^2 + 18x^{\frac{5}{2}} \\
&\quad + 117x^3 + 34x^{\frac{7}{2}} + 188x^4 + \dots ,
\end{aligned} \tag{5.2.30}$$

where w the topological fugacity and $\boldsymbol{y}$ the $SU(3)$ flavour fugacities. Again, this bares some similarity with the $U(1)$ gauge theory with 3 flavours, whose index is

$$\begin{aligned}
\mathcal{I}_{T_{(2,1)}(SU(3))}(x;w,\boldsymbol{y}) &= 1 + x\left[1 + \chi^{SU(3)}_{[1,1]}(\boldsymbol{y})\right] + x^{\frac{3}{2}}(w + w^{-1}) \\
&\quad - x^2\left[\chi^{SU(3)}_{[2,2]}(\boldsymbol{y}) - (1 + \chi^{SU(3)}_{[1,1]}(\boldsymbol{y}))\right] + \dots ,
\end{aligned} \tag{5.2.31}$$

[5] The symmetric product of the representation $[2;0] + [0;2]$ of $SU(2) \times SU(2)$ is $2[0;0] + [4;0] + [0;4] + [2;2]$. The representation in the first bracket of order x^2 (*i.e.* those with plus signs) can be written as $\mathrm{Sym}^2([2;0] + [0;2]) + [4;0] + [0;2] - [0;0]$. In the same way as in footnote 3, one singlet in the decomposition of the symmetric power does not participate in the index; this explains the term $-[0;0]$. Moreover, it is worth pointing out that, in this case, there are extra representations that are not contained in the symmetric product, namely $[4;0]$ and $[0;2]$.

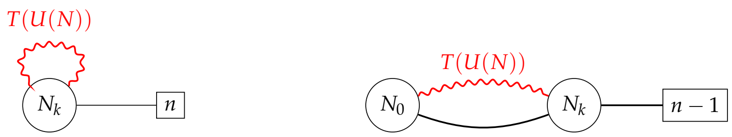
with the unrefinement

$$\mathcal{I}_{T_{(2,1)}(SU(3))}(x; w = 1, \boldsymbol{y} = (1,1)) = 1 + 9x + 2x^{\frac{3}{2}} + 18x^2 + 21x^3 + 54x^4 + \dots, \tag{5.2.32}$$

We observe that for a general n, (5.2.24) has a global symmetry $SU(n) \times U(1)$, where the $U(1)$ is the topological symmetry, which is enhanced to $SU(2)$ for $n = 2$. Moreover, the terms $x^{\frac{n}{2}}(w + w^{-1})$ indicate that theory (5.2.24) contains the basic monopole operators $V_{\pm(1,0)}$ (with flux $\pm(1,0)$ under the $U(2)$ gauge group), carrying R-charge $\frac{n}{2}$, similar to $V_{\pm 1}$ in the $U(1)$ gauge theory with n flavours. Moreover, with CS level $k = 2$, these basic monopole operators are gauge neutral, so they are gauge invariant themselves without any dressing by a chiral field in the fundamental hypermultiplet[6]. A non-trivial physical implication is that the contribution of the T-link cancel the contribution of the non-abelian vector multiplet in the R-charge of the monopole operator.

5.2.5 Duality with theories with two gauge groups

Here we examine the duality between the following theories



(5.2.33)

This duality can be seen from the brane system by moving one of the D5-brane across the J-fold and, thereby, turning it into an NS5 brane. For general values of N and k, both theories have a global symmetry $U(n)$. However, as can be seen from the indices, they arise from different origins in the quiver description.

Let us take, for example, $N = 2$, $k = 2$ and $n = 3$. The index of the left quiver is given by (5.2.29). The index of the right quiver reads

$$\begin{aligned} 1 &+ x\left[2 + (w_1 + w_1^{-1})\chi^{SU(2)}_{[1]}(\tilde{y}) + \chi^{SU(2)}_{[2]}(\tilde{y})\right] + x^{\frac{3}{2}}\left(w_1 w_2 + \frac{1}{w_1 w_2}\right) \\ &+ x^2\left[4 + (w_1^2 + 3 + w_1^{-2})\chi^{SU(2)}_{[2]}(\tilde{y}) + (w_1 + w_1^{-1})\left(\chi^{SU(2)}_{[3]}(\tilde{y}) + 2\chi^{SU(2)}_{[1]}(\tilde{y})\right)\right. \\ &\left. + \chi^{SU(2)}_{[4]}(\tilde{y})\right] + \dots . \end{aligned} \tag{5.2.34}$$

where w_1 and w_2 are the topological fugacities associated with the left and right nodes, and we denote the flavour fugacities by $\tilde{y}$. This expression can be rewritten in the way that the $SU(3)$ symmetry is manifest by setting

$$w_1 = y_1^{-\frac{3}{2}}, \qquad \tilde{y} = y_1^{-\frac{1}{2}} y_2, \tag{5.2.35}$$

upon which we recover the expression (5.2.29).

From the coefficient of x, we see that the mesons in the adjoint representation $[1,1]$ of $SU(3)$ of the left quiver in (5.2.33) are mapped to the following operators of the right quiver in (5.2.33):

[6]Note that this statement does not hold when the CS level is not equal to 2, and in order to form a gauge invariant combination, the monopole operators need to be dressed by chiral fields in the fundamental hypermultiplet.

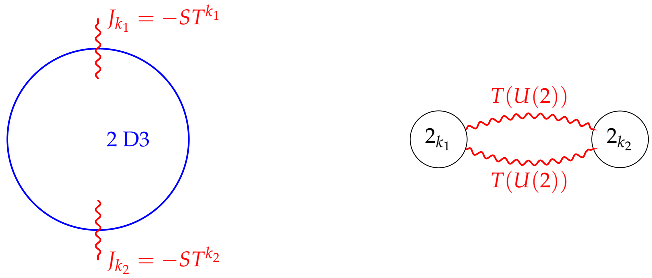
1. the mesons in the adjoint representation [2] of the $SU(2)$ flavour symmetry;
2. the dressed monopole operators in the fundamental representation [1] of $SU(2)$ and carrying topological charges ± 1 under the left node[7]; and
3. the trace of the adjoint chiral field associated with the left node.

Moreover, by comparing the terms at order $x^{\frac{3}{2}}$ in (5.2.29) and (5.2.34), we see that the basic monopole operators $V_{\pm}$, carrying topological charges ± 1, in the left quivers are mapped to the basic monopole operators $V_{\pm(1,1)}$, carrying topological charges $\pm(1,1)$, in the right quivers.

These statements can be easily generalised to other values of k and n.

5.3 $U(2)_{k_1} \times U(2)_{k_2}$ with two T-links

In this subsection we consider the following theory



(5.3.1)

The three sphere partition function as well as the supergravity solution corresponding to $U(N)_{k_1} \times U(N)_{k_2}$ gauge group (*i.e.* N D3 branes), in the large N limit, were studied in [14]. In such a reference, the CS levels were restricted such that $\mathrm{tr}(\pm J_{k_1} J_{k_2}) > 2$, equivalently $\pm(k_1 k_2 - 2) > 2$, where the sign $\pm$ is chosen such that the trace is greater than 2. In which case, $J_{k_1} J_{k_2}$ is a hyperbolic element of $SL(2, \mathbb{Z})$, and the theory was predicted to have $\mathcal{N} = 4$ supersymmetry in the large N limit. Here, instead, we focus on the superconformal indices and supersymmetry enhancement when the gauge group is taken to be $U(2)_{k_1} \times U(2)_{k_2}$ for general values of k_1 and k_2.

Note that if one of k_1 or k_2 is 1, say $k_1 = 1$, we have $J_1 J_{k_2} = STST^{k_2}$. This is related by a T-similarity transformation to $T J_1 J_{k_2} T^{-1} = T(STSTT^{k_2-1})T^{-1} = -ST^{k_2-2} = J_{k_2-2}$, where have used the identity $TSTST = -S$ (see also [14, footnote 19]). In other words, the two duality walls J_1 and J_{k_2} can be reduced to a single duality wall J_{k_2-2} (assuming that there are no NS5 and D5 branes). Henceforth, we shall not consider such a possibility in the absence of hypermultiplet matter.

In general, we observe that whenever $J_{k_1} J_{k_2}$ is a *parabolic* element of $SL(2, \mathbb{Z})$, *i.e.* $|\,\mathrm{tr}(J_{k_1} J_{k_2})| = |k_1 k_2 - 2| = 2$ or equivalently $k_1 k_2 = 0$ or 4, the index diverges and the theory is “bad” in the sense of [8]. In which case, we cannot deduce the low energy behaviour of the theory from its quiver description.

We observe that the index of (5.3.1) does not depend on the fugacities associated with the topological symmetries. Similarly to section 5.2.2, the gauge group in (5.3.1) can be taken to be $SU(2)_{k_1} \times SU(2)_{k_2}$ and this yields the same index.

Let us now take $k_1 = 2$ and examine various values of k_2 as follows.

[7] This is similar to the dressed monopole operators (5.5.18) in the abelian theory.

CS levels (k_1, k_2)	Index	Type of $J_{k_1} J_{k_2}$	Comment
$(2, 5)$	$1 + x^4 + \ldots$	hyperbolic	
$(2, 4)$	$1 + x^4 + \ldots$	hyperbolic	
$(2, 3)$	$1 - x^2 + 2x^3 - x^4 + \ldots$	hyperbolic	New, SUSY enhancement
$(2, 2)$	diverges	parabolic	
$(2, 1)$	1	elliptic	Same as (5.2.6), $k = 0$
$(2, 0)$	diverges	parabolic	
$(2, -1)$	(5.2.7)	hyperbolic	Same as (5.2.6), $k = \pm 4$
$(2, -2)$	$1 + x^4 + \ldots$	hyperbolic	
$(2, -3)$	$1 + x^4 + \ldots$	hyperbolic	
$(2, -4)$	$1 + x^4 + \ldots$	hyperbolic	

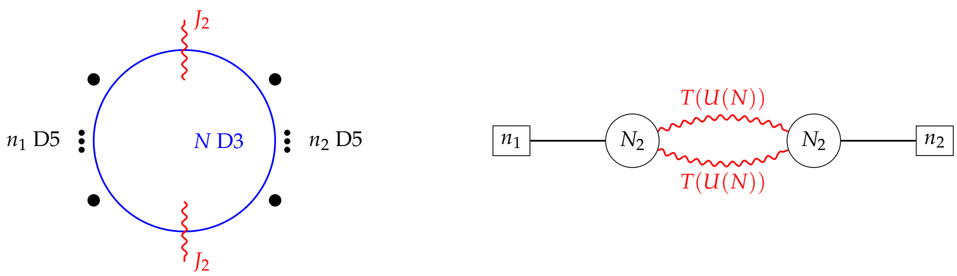
The cases whose indices exhibit supersymmetry enhancement are emphasised in yellow. The indices for the cases not highlighted in yellow do not signalise the presence of extra SUSY-current multiplets. The CS levels $(k_1, k_2) = (2, 3)$ gives a new SCFT with enhanced $\mathcal{N} = 4$ supersymetry, whereas the case with $(k_1, k_2) = (2, -1)$ is the same as theory (5.2.6) with $k = -4$, which also has supersymmetry enhancement to $\mathcal{N} = 4$.

For $k_1 = 3$, we find a similar pattern, as tabulated below. Unfortunately, the cases that has supersymmetry enhancement, namely $(k_1, k_2) = (3, 2)$ and $(3, -1)$, are identical with certain theories that have been discussed before.

CS levels (k_1, k_2)	Index	Type of $J_{k_1} J_{k_2}$	Comment
$(3, 4)$	$1 + x^4 + \ldots$	hyperbolic	
$(3, 3)$	$1 + 2x^4 + \ldots$	hyperbolic	
$(3, 2)$	$1 - x^2 + 2x^3 + \ldots$	hyperbolic	Same as $(k_1, k_2) = (2, 3)$
$(3, 1)$	1	elliptic	Same as (5.2.6), $k = \pm 1$
$(3, 0)$	diverges	parabolic	
$(3, -1)$	(5.2.7)	hyperbolic	Same as (5.2.6), $k = \pm 5$
$(3, -2)$	$1 + x^4 + \ldots$	hyperbolic	

5.3.1 Adding flavours to the parabolic case

In this section, we add fundamental flavours to either or both nodes in the parabolic case. For definiteness, we consider the theory involving two J_2 duality walls[8] and a collection of D5 branes arranged in the following way:

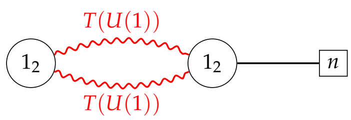


(5.3.2)

[8]Similarly to the remark in footnote 4, even though J_2^2 is related to T^{-2} by a similarity transformation in $SL(2, \mathbb{Z})$, upon adding hypermultiplet matter, the theory becomes non-trivial.

and focus on the cases of $N = 1$ and $N = 2$. Such theories have interesting physical properties as we shall describe below.

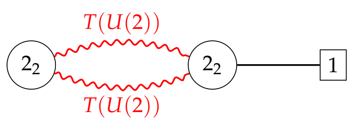
Let us first discuss the abelian case. Since this theory admits a conventional Lagrangian description, we can easily analyse this theory along the line of [50]. The detailed analysis is provided in section 5.5. We find that whenever fundamental hypermultiplets are added to the quiver associated with parabolic J-folds, an interesting branch of the moduli space arises, mainly due to the presence of the gauge neutral monopole (or dressed monopole) operators. In particular, for quiver (5.3.2) with $N = 1$, we find that there are two branches of the moduli space. One can be identified as the Higgs branch and the other can be identified as the Coulomb branch, both of which are hyperKähler cones. This feature is very similar to that of general 3d $\mathcal{N} = 4$ gauge theories. The Higgs branch is isomorphic to a product of the closures of the minimal nilpotent orbits $\overline{\mathcal{O}}^{SU(n_1)}_{\text{min}} \times \overline{\mathcal{O}}^{SU(n_2)}_{\text{min}}$, where each factor is generated by the mesons constructed using the chiral multiplets in each fundamental hypermultiplet; see (5.5.7). The Coulomb branch is isomorphic to $\mathbb{C}^2/\mathbb{Z}_{n_1+n_2}$, which is generated by the monopole operators $V_{\pm(1,1)}$ with fluxes $\pm(1,1)$ and the complex scalar in the vector multiplet; see (5.5.11). For a general n_1 and n_2, this theory has a global symmetry $\left(\frac{U(n_1)\times U(n_2)}{U(1)}\right) \times U(1)$, where the former factor denotes the flavour symmetry coming from the fundamental hypermultiplets and latter $U(1)$ denotes the topological symmetry. For the special case of $n_1 + n_2 = 2$, the $U(1)$ topological symmetry gets enhanced to $SU(2)$, which is also an isometry of the Coulomb branch $\mathbb{C}^2/\mathbb{Z}_2$. Interestingly, if we set one of n_1 or n_2 to zero, say



(5.3.3)

This theory turns out to be the same as quiver (5.2.1) with $N = 1$, $k = 2$, which is identical to 3d $\mathcal{N} = 4$ $U(1)$ gauge theory with n flavours (*i.e.* the $T_{(n,n-1)}(SU(n))$ theory [8]). One can indeed check that the moduli spaces and the indices of the two theories are equal. Such an identification indicates that when $n_1 = 0$ (or $n_2 = 0$), the two J_2 duality walls can be "collapsed" into one, and the gauge node in (5.3.3) that is not flavoured can be removed such that the T-link becomes a loop around the other gauge node. We remark that this statement only holds for the abelian case; we will see that for $N = 2$ this is no longer true.

Quiver (5.3.2) with $N > 1$ still bares the same features as in the abelian ($N = 1$) theory. In general, the index of (5.3.2) contains the terms $x^{\frac{1}{2}(n_1+n_2)}(w_1 w_2 + w_1^{-1} w_2^{-1})$, which indicates that there are gauge invariants monopole operators $V_{\pm(1,0,\ldots,0;1,0,\ldots,0)}$, with fluxes $\pm(1,0,\ldots,0)$ under each of the $U(N)$ gauge group, carrying R-charge $\frac{1}{2}(n_1 + n_2)$. Again, for $n_1 + n_2 = 2$, the $U(1)$ topological symmetry gets enhanced to $SU(2)$. Furthermore, when $n_1 + n_2 = 1$, *i.e.* $(n_1, n_2) = (1, 0)$ or $(0, 1)$, such monopole operators decouple as a free hypermultiplet (this is similar to the one flavour case discussed in section 5.2.3). Let us consider, in particular, the case of $N = 2$, $n_1 = 0$ and $n_2 = 1$:



(5.3.4)

Indeed the index can be written as

$$\mathcal{I}_{(5.3.4)} = \mathcal{I}_{\text{free}}(x;w) \times \mathcal{I}^{(5.3.4)}_{\text{SCFT}}(x;w) \tag{5.3.5}$$

where we define w as the product of the topological fugacities associated with the two gauge groups: $w = w_1 w_2$. The index of the free hypermultiplet $\mathcal{I}_{\text{free}}(x;w)$ is given by (5.2.19), and the index for the interacting SCFT is

$$\begin{aligned}\mathcal{I}^{(5.3.4)}_{\text{SCFT}}(x;w) = 1 + x\chi^{SU(2)}_{[2]}(w) + x^2\left[\chi^{SU(2)}_{[4]}(w) - \chi^{SU(2)}_{[2]}(w)\right] \\ - x^{\frac{5}{2}}\chi^{SU(2)}_{[2]}(w) + \dots .\end{aligned} \tag{5.3.6}$$

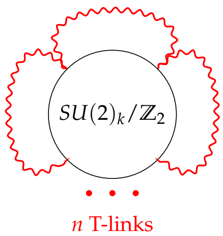
with the unrefinement

$$\mathcal{I}^{(5.3.4)}_{\text{SCFT}}(x;w=1) = 1 + 3x + 2x^2 - 2x^{5/2} - 4x^3 + \dots \tag{5.3.7}$$

The interacting SCFT has a flavour symmetry $SU(2)$. Notice that the index of the SCFT (5.3.6) is different from (5.2.20). (Hence, we cannot collapse two J_2 duality walls into one as in the abelian case.) In particular, while (5.2.20) exhibits the presence of the extra-SUSY current multiplet, (5.3.6) does not.

5.4 Gauge group $SU(2)_k/\mathbb{Z}_2$

In this section we consider a theory with a single $SU(2)_k/\mathbb{Z}_2$ gauge node with n T-links attached to it.



(5.4.1)

Although the brane configuration for $n > 1$ is not known, we demonstrate below that such theories have interesting properties from the field theoretic perspective.

The indices involving $SU(N)/\mathbb{Z}_N$ gauge group for theories in 3d were discussed in [125, 128][9]. Here, we write down the expression of the index for (5.4.1), analogous to those presented in [125]:[10]

$$\begin{aligned}\mathcal{I}_{(5.4.1)}(x;g) = \frac{1}{2}\sum_{l=0}^{1} g^l \sum_{m\in\mathbb{Z}+\frac{l}{2}} \oint \frac{dz}{2\pi i z} x^{-2|m|} \prod_{\pm}\left(1-(-1)^{2m} z^{\pm 2} x^{2|m|}\right) \\ \times z^{2km}\left[\widehat{\mathcal{I}}_{T(SU(2))}(x;\{z,m\},\{z,m\})\right]^n ,\end{aligned} \tag{5.4.2}$$

[9]Note the index for 3d gauge theories can be obtained as the limit of the lens space index for 4d gauge theories [129]. As discussed extensively in [130], the latter is sensitive to the global structure of the gauge group.

[10]Since $SU(2)/\mathbb{Z}_2$ is isomorphic to $SO(3)$, one can also compute the index for $SO(3)$ gauge group using the formulae described in [120, sec. 6.1]. Note that the normalisation for the CS level for $SO(3)$ is such that $SO(3)_k = SU(2)_{2k}/\mathbb{Z}_2$, and that the fugacity ζ for the $\mathbb{Z}_2^{\mathcal{M}}$ symmetry in [120, sec. 6.1] is identified with the fugacity g here.

where g is a fugacity for the global $\mathbb{Z}_2$ symmetry which takes values 1 or -1, and $\widehat{\mathcal{I}}_{U(1)\text{ with 2 flv}}$ is the index for the $U(1)$ gauge theory with 2 flavours such that the sum over the gauge flux is properly quantised:

$$\begin{aligned}&\widehat{\mathcal{I}}_{T(SU(2))}(x;\{\mu,p\},\{\tau,n\})\\&=\sum_{m\in\mathbb{Z}+\frac{1}{2}(p\bmod 2)}\tau^{2m}\oint\frac{\mathrm{d}z}{2\pi iz}z^{2n}x^{\frac{|m-p|}{2}}\frac{((-1)^{m-p}z^{\mp1}\mu^{\pm1}x^{3/2+|m-p|};x^2)_\infty}{((-1)^{m-p}z^{\pm1}\mu^{\mp1}x^{1/2+|m-p|};x^2)_\infty}\times\\&\quad x^{\frac{|m+p|}{2}}\frac{((-1)^{m+p}z^{\mp1}\mu^{\mp1}x^{3/2+|m+p|};x^2)_\infty}{((-1)^{m+p}z^{\pm1}\mu^{\pm1}x^{1/2+|m+p|};x^2)_\infty}\,.\end{aligned}\tag{5.4.3}$$

We may obtain the result for the gauge group $SU(2)_k$, instead of $SU(2)_k/\mathbb{Z}_2$, by gauging the $\mathbb{Z}_2$ global symmetry associated with g:

$$\mathcal{I}_{(5.4.1)\text{ with }SU(2)_k\text{ gauge group}}=\frac{1}{2}\left[\mathcal{I}_{(5.4.1)}(x;g=1)+\mathcal{I}_{(5.4.1)}(x;g=-1)\right]\,.\tag{5.4.4}$$

We find that for $k\in\mathbb{Z}$ and $n=1$, the index is the same as that of the theory with the same k presented in section 5.2.2. However, the result becomes more interesting when both n and k is even, since the index depends on the fugacity g. This indicates the presence of the operators carrying a non-trivial charge under the $\mathbb{Z}_2$ discrete symmetry which are gauge invariant monopole operators. Let us focus on the case of $n=2$. We provide some examples in the following table.

CS level	Index
$k=0$	$1+2gx+4x^2-4x^3+(9+8g)x^4+\dots$
$\lvert k\rvert=2$	$1+(2-g)x^2-(4+4g)x^3+(9+8g)x^4+\dots$
$\lvert k\rvert=4$	$1+gx+3x^2-(4+2g)x^3+(8+8g)x^4+\dots$
$\lvert k\rvert=6$	$1+(2+g)x^2-(4+4g)x^3+(7+4g)x^4+\dots$
$\lvert k\rvert=8$	$1+2x^2-4x^3+(7-4g)x^4+\dots$

The case of $|k|=2$ is highlighted in yellow to indicate that the index exhibits supersymmetry enhancement. Since the coefficient of x is zero, the theory has no flavour current. The term $-g$ at order x^2 indicates the presence of an extra SUSY-current multiplet, acted non-trivially by the $\mathbb{Z}_2$ global symmetry. For this reason, we conclude that supersymmetry is enhanced to $\mathcal{N}=4$. We emphasise that it is important to refine the index with respect to g in order to see such a multiplet. On the other hand, the indices for the other values of k do not exhibit the existence of the extra SUSY-current multiplet. The same is also true if we gauge the $\mathbb{Z}_2$ global symmetry as described in (5.4.4).

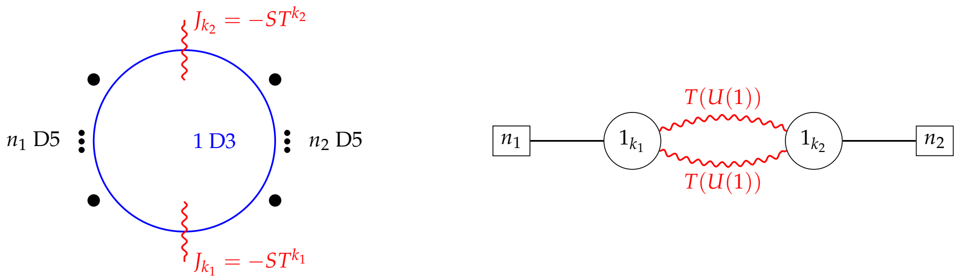
Finally, let us consider the case in which k is half-odd-integral and $n=1$. For $|k|\geq\frac{1}{2}$, we find that the indices are different from those theories that have been considered in earlier, and so it seems to us that these theories are new. Moreover, they exhibit the presence of an extra SUSY-current multiplet, which leads to the conclusion that supersymmetry is enhanced to $\mathcal{N}=4$. We tabulate the indices for a few values of half-odd-integral CS levels below.

CS level	Index
$\lvert k\rvert=1/2$	$1-x^2+2x^3-x^4-2x^5+6x^6+\dots$
$\lvert k\rvert=3/2$	$1-x^2+2x^3-2x^5+6x^6+\dots$
$\lvert k\rvert=5/2$	$1-x^2+2x^3-2x^4-2x^5+6x^6+\dots$

5.5 Moduli space of flavoured abelian parabolic J-fold theories

In this section we study the moduli space of a class of parabolic J-fold theories, in the presence of the hypermultiplet fundamental matter. We focus on the models with abelian gauge group, since the Lagrangian description is available. More general detailed discussions can be found in [50].

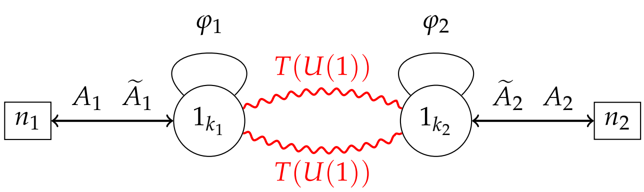
For definiteness, let us focus on the following model with $U(1)_{k_1} \times U(1)_{k_2}$ gauge group:



(5.5.1)

For the moment we allow for generic CS levels k_1 and k_2, but we will see that the vacuum equations admit solutions for non-trivial branches of the moduli space when $J_1 J_2$ is parabolic, *i.e.* $|\,\mathrm{tr}\, J_1 J_2| = 2$, or equivalently $k_1 k_2 = 0$ or 4.

Let us rewrite the quiver (5.5.1) in $\mathcal{N} = 2$ language:



(5.5.2)

with superpotential:

$$W = -\,\mathrm{tr}(A_1\varphi_1\widetilde{A}_1 + A_2\varphi_2\widetilde{A}_2) + \frac{1}{2}(k_1\varphi_1^2 + k_2\varphi_2^2) - 2\varphi_1\varphi_2 \,. \tag{5.5.3}$$

where we denoted in blue the contribution due to the two T-links, consisting of a mixed CS coupling. The vacuum equations are as follows:

$$A_1\varphi_1 = \widetilde{A}_1\varphi_1 = 0\,, \quad A_2\varphi_2 = \widetilde{A}_2\varphi_2 = 0\,, \tag{5.5.4}$$

and

$$\begin{aligned} k_1\varphi_1 - 2\varphi_2 &= (A_1)_a(\widetilde{A}_1)^a\,, \\ k_2\varphi_2 - 2\varphi_1 &= (A_2)_i(\widetilde{A}_2)^i. \end{aligned} \tag{5.5.5}$$

where $a, b, c = 1, \ldots, n_1$ and $i, j, k = 1, \ldots, n_2$.

The vacuum equations (5.5.4) and (5.5.5) admit the solutions in which $\varphi_1 = \varphi_2 = 0$, regardless of the CS levels. This branch of the moduli space is generated by the mesons $(M_1)_a^b = (A_1)_a(\widetilde{A}_1)^b$ and $(M_2)_i^j = (A_2)_i(\widetilde{A}_2)^j$ subject to the following relations:

$$\mathrm{rank}(M_{1,2}) \le 1\,, \qquad M_{1,2}^2 = 0\,, \tag{5.5.6}$$

where the first relations come from the fact that each of the matrices M_1 and M_2 is constructed as a product of two vectors, and the second matrix relations follow from (5.5.5). We refer to this branch of the moduli space as the Higgs branch, denoted by $\mathcal{H}_{(5.5.1)}$. Indeed, it is isomorphic to a product of the closures of the minimal nilpotent orbits:

$$\mathcal{H}_{(5.5.1)} = \overline{\mathcal{O}}^{SU(n_1)}_{\text{min}} \times \overline{\mathcal{O}}^{SU(n_2)}_{\text{min}} . \tag{5.5.7}$$

There are also other non-trivial branches of moduli spaces, which we are analysing in the following.

Let us consider the branch on which $\varphi_1 \neq 0$ and $\varphi_2 \neq 0$. From (5.5.4), we have $A_1 = \widetilde{A}_1 = A_2 = \widetilde{A}_2 = 0$. Equations (5.5.5) admit solutions only if:

$$k_1\varphi_1 = 2\varphi_2 , \qquad k_2\varphi_2 = 2\varphi_1 , \qquad k_1k_2 - 4 = 0 ; \tag{5.5.8}$$

the latter implies that J_1J_2 has to be parabolic such that either $(k_1, k_2) = (1, 4)$ or $(k_1, k_2) = (2, 2)$. (The case the $(k_1, k_2) = (4, 1)$ can be considered by simply exchanging n_1 and n_2.) We analyse these cases below.

- **The case of** $(k_1, k_2) = (2, 2)$**.** The first equation of (5.5.8) sets $\varphi_1 = \varphi_2 \equiv \varphi$. Since the real scalars in the vector multiplets belong to the same multiplets as $\varphi_{1,2}$, the magnetic fluxes of the monopole operators $V_{(m_1,m_2)}$ satisfy $m_1 = m_2 \equiv m$. The R-charge and the gauge charges with respect to the first and second nodes are respectively

$$\begin{aligned} R[V_{(m,m)}] &= \frac{1}{2}(n_1 + n_2)|m| , \\ q_1[V_{(m,m)}] &= -(k_1 m - 2m) = 0 , \quad q_2[V_{(m,m)}] = -(k_2 m - 2m) = 0 . \end{aligned} \tag{5.5.9}$$

  Observe that the $V_{(m,m)}$ are gauge neutral for all m. This branch is generated by the basic monopole operators $V_{\pm(1,1)}$ and φ (the latter has R-charge 1), satisfying the quantum relation.

$$V_{(1,1)} V_{-(1,1)} = \varphi^{n_1+n_2} . \tag{5.5.10}$$

  This branch is thus a Coulomb branch and it is isomorphic to

$$\mathcal{C}^{k_1=k_2=2}_{(5.5.1)} = \mathbb{C}^2/\mathbb{Z}_{n_1+n_2} . \tag{5.5.11}$$

  In the special case of one flavour, *i.e.* $(n_1, n_2) = (1, 0)$ or $(0, 1)$, we see that the Coulomb branch is isomorphic to $\mathbb{C}^2 \cong \mathbb{H}$. Indeed, the basic monopole operators decouple as a free hypermultiplet.

- **The case of** $(k_1, k_2) = (1, 4)$**.** In this case $\varphi_1 = 2\varphi_2 = 2\varphi$ and the allowed magnetic fluxes for the monopole operators $V_{(m_1,m_2)}$ are such that $m_1 = 2m_2 \equiv 2m$. The R-charge and the gauge charges with respect to the first and second nodes are respectively

$$\begin{aligned} R[V_{(2m,m)}] &= \frac{1}{2}(n_1|2m| + n_2|m|) = \left(n_1 + \frac{1}{2}n_2\right)|m| \\ q_1[V_{(2m,m)}] &= -[k_1(2m) - 2m] = 0 , \quad q_2[V_{(2m,m)}] = -[k_2(m) - 2(2m)] = 0 . \end{aligned} \tag{5.5.12}$$

Observe that $V_{(2m,m)}$ are gauge neutral for all m. This branch of the moduli space is generated by $V_{\pm(2,1)}$ and φ, satisfying the quantum relation:

$$V_{(2,1)}V_{-(2,1)} = \varphi^{2n_1+n_2} . \quad (5.5.13)$$

This branch is thus a Coulomb branch and it is isomorphic to

$$\mathcal{C}^{(k_1,k_2)=(1,4)}_{(5.5.1)} = \mathbb{C}^2/\mathbb{Z}_{2n_1+n_2} . \quad (5.5.14)$$

It is worth pointing out that for both $(k_1,k_2) = (2,2)$ and $(1,4)$, the vacuum equations admit the solutions such that there is a clear separation between the Higgs and Coulomb branches, in the same way as general 3d $\mathcal{N} = 4$ gauge theories. This is mainly due to the fact that the monopole operators are gauge neutral. Note also that both branches are hyperKähler cones.

Next, we analyse the case in which one of φ_1 and φ_2 is zero. For definiteness, let us take $\varphi_2 = 0$ and $0 \neq \varphi_1 \equiv \varphi$. From (5.5.4), we see that $A_1 = \widetilde{A}_1 = 0$, and so (5.5.5) admits a solution only if $k_1 = 0$. Let us suppose that

$$k_1 = 0 . \quad (5.5.15)$$

Observe that the CS levels $(0,k_2)$ satisfies the parabolic condition on $J_0J_{k_2}$, because $|\operatorname{Tr}(J_0J_{k_2})| = 2$ for any k_2. Then the second equation of (5.5.5) implies that

$$(A_2)_i(\widetilde{A}_2)^i = -2\varphi . \quad (5.5.16)$$

The fluxes (m_1,m_2) of the monopole operators $V_{(m_1,m_2)}$, satisfies $m_2 = 0$. For convenience, we write $m_1 = m$. The R-charge and the gauge charges of the monopole operators $V_{(m,0)}$ are

$$\begin{aligned} R[V_{(m,0)}] &= \frac{1}{2}(n_1|m| + n_2|0|) = \frac{1}{2}n_1|m| \\ q_1[V_{(m,0)}] &= -[k_1(m) - 2(0)] = 0 , \quad q_2[V_{(m,0)}] = -[k_2(0) - 2(m)] = 2m . \end{aligned} \quad (5.5.17)$$

In this case the monopole operator $V_{(m,0)}$ is no longer neutral under the gauge symmetry, but it carries charge $2m$ under the $U(1)_{k_2}$ gauge group. We can form the basic gauge invariant dressed monopole operators as follows:

$$(W^+)^{ij} = V_{(1,0)}(\widetilde{A}_2)^i(\widetilde{A}_2)^j , \qquad (W^-)_{ij} = V_{(-1,0)}(A_2)_i(A_2)_j . \quad (5.5.18)$$

These operators transform under the representation $[2,0,\ldots,0]$ and $[0,\ldots,0,2]$ of $SU(n_2)$ respectively. The carries R-charges

$$R[W^\pm] = \frac{1}{2}n_1 + 1 , \quad (5.5.19)$$

and satisfy the quantum relation

$$\operatorname{Tr}(W^+W^-) = (W^+)^{ij}(W^-)_{ji} = \varphi^{n_1+2} . \quad (5.5.20)$$

Since the dressed monopole operators $W^\pm$ are generators of this branch of the moduli space, we can regard this as a "mixed" Higgs and Coulomb branch.

Note that if we take instead $\varphi_1 = 0$ and $0 \neq \varphi_2 \equiv \varphi$, the situation is reversed. In order for the vacuum equations to admit a solution we must have $k_2 = 0$. This leads to the gauge invariant dressed monopole operators

$$(U^+)^{ij} = V_{(0,1)}(\widetilde{A}_1)^i(\widetilde{A}_1)^j\,, \qquad (U^-)_{ij} = V_{(0,-1)}(A_1)_i(A_1)_j\,, \tag{5.5.21}$$

which transform under the representation $[2, 0, \ldots, 0]$ and $[0, \ldots, 0, 2]$ of $SU(n_1)$ respectively. The carries R-charges $R[U^{\pm}] = \frac{1}{2}n_2 + 1$ and satisfy the quantum relation $\mathrm{Tr}(U^+U^-) = \varphi^{n_2+2}$.

Finally, we remark that if $(k_1, k_2) = (0, 0)$, which is another possibility for $J_{k_1}J_{k_2}$ to be parabolic, then both dressed monopole operators $W^{\pm}$ and $U^{\pm}$, as described above, are generators of the moduli space.

Chapter 6

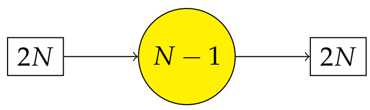
Duality walls in 4d $\mathcal{N} = 2$ theory

6.1 The 3d gauge theory with a monopole superpotential

The theory associated with the S-duality wall of the 4d $\mathcal{N} = 2$ $SU(N)$ gauge theory with $2N$ flavours is the 3d $\mathcal{N} = 2$ $U(N-1)$ gauge theory with $2N$ flavours and superpotential $W = V_+ + V_-$, where $V_\pm$ are the basic monopole operators of the latter theory [44]. For the sake of brevity, following [44], we refer to the aforementioned 3d theory as $\mathcal{T}_\mathfrak{M}$, where $\mathfrak{M}$ stands for the monopole superpotential. The identification of the theory on the S-duality wall of the 4d theory and the $\mathcal{T}_\mathfrak{M}$ theory[1] had been attempted by several authors, *e.g.* [41, 43, 42]. The main technique was to study a collection of the duality transformation coefficients of conformal blocks (also known as the kernel) of the Liouville or Toda theory, which are in the AGT correspondence [35, 36] with the 4d theory. The kernel was then interpreted as the partition function of the 3d theory associated with the duality wall [39]. Knowing the former allows one to identify the matter content of the 3d theory associated with the duality wall [41, 42]. In [42] it was observed that the R-charges of the chiral fields in the 3d theory were fixed to certain particular values. This was later interpreted in [44] as due to the monopole superpotential.

The $\mathcal{T}_\mathfrak{M}$ theory has a global symmetry $SU(2N) \times SU(2N)$. We represent this theory by the following quiver diagram:

$\mathcal{T}_\mathfrak{M}$:

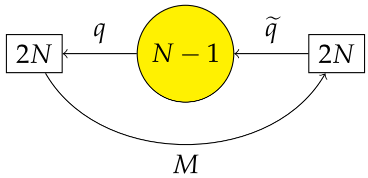


(6.1.1)

where we denoted the gauge node in yellow in order to indicate the monopole superpotential $W = V_+ + V_-$. Due to the monopole superpotential, the topological and axial symmetries are broken, and the R-charge r of the chiral fields is fixed to be $r = 1/2$ due to the relation $2N(1-r) - (N-1-1) = 2$.

[1] On the other hand, the 3d theory associated with the S-duality wall of the 4d $\mathcal{N} = 2^*$ $SU(N)$ gauge theory has been identified as the axial mass-deformed $T(SU(N))$ gauge theory by [40].

In fact, as pointed out in [44][2], theory (6.1.1) is dual to another theory with the same gauge group $U(N-1)$, also with $2N$ flavours and $4N^2$ singlets M



(6.1.2)

and superpotential $W = V_+ + V_- + Mq\tilde{q}$. In other words, we have duality

$$(6.1.1) \longleftrightarrow (6.1.2) \tag{6.1.3}$$

6.1.1 Indices of theories (6.1.1) and (6.1.2)

Our main tool is the supersymmetric index, which we shall refer to as *index* for the sake of brevity. It can be computed as the partition function on $\mathbb{S}^2 \times \mathbb{S}^1$. We have already summarised the necessary details in 5.1.

In order to write the supersymmetric index of a theory with monopole superpotential one has to take into account suitable contributions of BF couplings with the global symmetries and the R-symmetry that make the monopole operators uncharged and exactly marginal. In the case of theory (6.1.1) we are considering the monopole superpotential breaks the topological as well as the axial symmetries. Hence, the index can be easily obtained from that of the $U(N-1)$ gauge theory with $2N$ flavours and zero superpotential, turning off the fugacities for the axial and the topological symmetries, as well as setting the R-charge of the chiral fields to $r = \frac{1}{2}$

$$\begin{aligned} &\mathcal{I}_{(6.1.1)}(x; \{\boldsymbol{\mu}, \boldsymbol{n}\}, \{\boldsymbol{\tau}, \boldsymbol{p}\}) \\ &= \sum_{\boldsymbol{m} \in \mathbb{Z}^{N-1}} \frac{1}{(N-1)!} \oint \prod_{a=1}^{N-1} \frac{du_a}{2\pi i\, u_a} Z_{vec}(x; \{\boldsymbol{u}, \boldsymbol{m}\}) Z_{chir}(x; \{\boldsymbol{u}, \boldsymbol{m}\}, \{\boldsymbol{\mu}, \boldsymbol{n}\}, \{\boldsymbol{\tau}, \boldsymbol{p}\}), \end{aligned} \tag{6.1.4}$$

where the contribution of the $\mathcal{N} = 2$ vector multiplet is

$$Z_{\text{vec}}(x; \{\boldsymbol{u}, \boldsymbol{m}\}) = \prod_{a,b=1}^{N-1} x^{-|m_a - m_b|} \left(1 - (-1)^{m_a - m_b} x^{|m_a - m_b|} \left(\frac{z_a}{z_b}\right)^{\pm 1}\right), \tag{6.1.5}$$

while that of the chiral multiplets is

$$\begin{aligned} &Z_{\text{chir}}(x; \{\boldsymbol{u}, \boldsymbol{m}\}, \{\boldsymbol{\mu}, \boldsymbol{n}\}, \{\boldsymbol{\tau}, \boldsymbol{p}\}) \\ &= \prod_{a=1}^{N-1} \prod_{i=1}^{2N} \left(u_a \mu_i^{-1} x^{1/2}\right)^{\frac{|n_i - m_a|}{2}} \frac{\left((-1)^{n_i - m_a} u_a \mu_i^{-1} x^{3/2 + |n_i - m_a|}; x^2\right)_\infty}{\left((-1)^{n_i - m_a} u_a^{-1} \mu_i x^{1/2 + |n_i - m_a|}; x^2\right)_\infty} \times \\ &\times \left(u_a^{-1} \tau_i x^{1/2}\right)^{\frac{|m_a - p_i|}{2}} \frac{\left((-1)^{m_a - p_i} u_a^{-1} \mu_i x^{3/2 + |m_a - p_i|}; x^2\right)_\infty}{\left((-1)^{m_a - p_i} u_a \mu_i^{-1} x^{1/2 + |m_a - p_i|}; x^2\right)_\infty} \end{aligned}$$

[2]More precisely, in [44], a more general duality relating the $U(N_c)$ gauge theory with N_f flavors and $W = V_+ + V_-$ and the $U(N_c - N_f)$ gauge theory with N_f flavors, N_f^2 singlets M and $W = V_+ + V_- + Mq\tilde{q}$ was proposed.

In the above expressions we denoted by $\{\boldsymbol{u}, \boldsymbol{m}\}$ the fugacities and the magnetic fluxes respectively for the gauge symmetry and with $\{\boldsymbol{\mu}, \boldsymbol{n}\}, \{\boldsymbol{\tau}, \boldsymbol{p}\}$ those of the two $SU(2N)$ global symmetries, which have to satisfy the constraints $\prod_{i=1}^{2N} \mu_i = \prod_{i=1}^{2N} \tau_i = 1$ and $\sum_{i=1}^{2N} n_i = \sum_{i=1}^{2N} p_i = 0$.

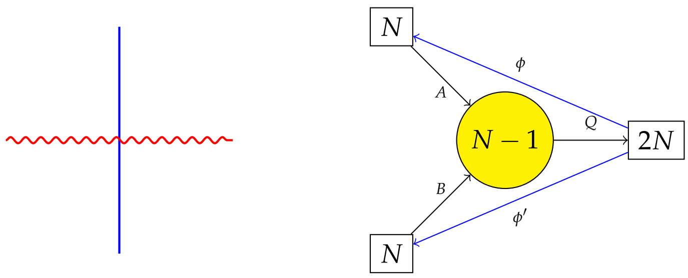
The index of the dual theory (6.1.2) is related to that of (6.1.1) by the following relation:

$$\begin{aligned} \mathcal{I}_{(6.1.2)}(x; \{\boldsymbol{\mu}, \boldsymbol{n}\}, \{\boldsymbol{\tau}, \boldsymbol{p}\}) &= \prod_{i,j=1}^{2N} \left(\mu_i \tau_j^{-1}\right)^{-\frac{|n_i - p_j|}{2}} \frac{\left((-1)^{n_i - p_j} \mu_i^{-1} \tau_j x^{1+|n_i - p_j|}; x^2\right)_\infty}{\left((-1)^{n_i - p_j} \mu_i \tau_j^{-1} x^{1+|n_i - p_j|}; x^2\right)_\infty} \times \\ &\times \mathcal{I}_{(6.1.1)}(x; \{\boldsymbol{\mu}^{-1}, -\boldsymbol{n}\}, \{\boldsymbol{\tau}^{-1}, -\boldsymbol{p}\}) \,, \end{aligned} \tag{6.1.6}$$

where the right hand side of the first line is the contribution of the $4N^2$ gauge singlets M. We point out that an analogous identity for the partition functions on $\mathbb{S}_b^3$ was actually derived in [44] as a limit of the identity for the 4d supersymmetric indices associated to Intriligator–Pouliot duality [131], where the latter was proven in [132]. Although we shall not provide an analytic proof[3] of the relation (6.1.6), it can be checked perturbatively by expanding both sides as power series in x and matching each order of the power expansion. Moreover, as a further support of (6.1.6), one may take an appropriate 2d limit of the index of each side in (6.1.6) to obtain certain complex integrals [133], which are related to CFT free field correlators; the equality of such integrals was proposed in [134, 135].

6.1.2 Inclusion of the 4d fields

As a theory realised on the wall, one of the $SU(2N)$ symmetries (say, the one associated with the left square node) can be decomposed into a subgroup $SU(N) \times SU(N) \times U(1)$, where we shall refer to the latter $U(1)$ as $U(1)_q$. Each of these $SU(N)$ can then be coupled to the $SU(N)$ gauge symmetry of the 4d theory on each side of the wall. Moreover, the 3d chiral fields of the theory on the wall also couple non-trivially to the chiral fields coming from the 4d theory. The appropriate quiver description for the 3d $\mathcal{N} = 2$ theory on the wall is



(6.1.7)

where ϕ is one of the chiral fields contained in the hypermultiplets of the 4d $\mathcal{N} = 2$ $SU(N)$ gauge theory with $2N$ flavours on one side of the wall restricted to the interface. The same is for ϕ' on the other side of the wall. The superpotential of (6.1.7) is

$$W_{(6.1.7)} = V_+ + V_- + Q\phi A + Q\phi' B \,. \tag{6.1.8}$$

[3] Relation (6.1.6) could, in principle, be derived in a similar way to the one for the $\mathbb{S}_b^3$ partition functions if a generalization of Rains' results for the lens space index [129], which is the partition function on $\mathbb{S}^3/\mathbb{Z}_p \times \mathbb{S}^1$, were known.

We shall, from now on, denote as blue arrows the chiral fields coming from the 4d theory.

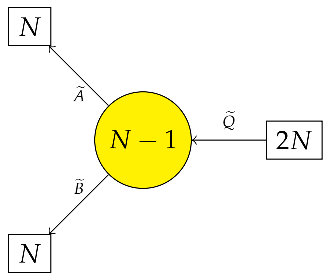
The arrows in the right diagram are consistent with the decomposition rule of the fundamental representation of $SU(2N)$ to $SU(N) \times SU(N) \times U(1)_q$:

$$[1, 0^{2N-2}] \longrightarrow q[1, 0^{N-2}; 0^{N-1}] + q^{-1}[0^{N-1}; 1, 0^{N-2}] \,, \tag{6.1.9}$$

which correspond to chiral fields A and B respectively. Note that Q carries zero charge under $U(1)_q$, and so from the superpotential, ϕ and ϕ' carry $U(1)_q$ charges -1 and $+1$ respectively.

Let us now explain the "skeleton" diagram on the left of (6.1.7). Each blue external leg (or each end of the blue line) denotes an $SU(N)$ global symmetry, and the wiggly red line denotes a duality wall, which brings about an $SU(2N) \times U(1)_q$ global symmetry. Note that the latter is the symmetry of the 4d $\mathcal{N} = 2$ $SU(N)$ gauge theory with $2N$ flavours, where $U(1)_q$ plays a role as the baryonic symmetry.

One may, in fact, apply the duality (6.1.3) to the yellow node in (6.1.7). As a result, ϕ and ϕ' disappear, and the arrows of A, B and Q are reversed. We denote the chiral fields in the dual theory as $\widetilde{A}$, $\widetilde{B}$ and $\widetilde{Q}$; they carry opposite $U(1)_q$ charges with respect to A, B and Q respectively. The dual theory is therefore



(6.1.10)

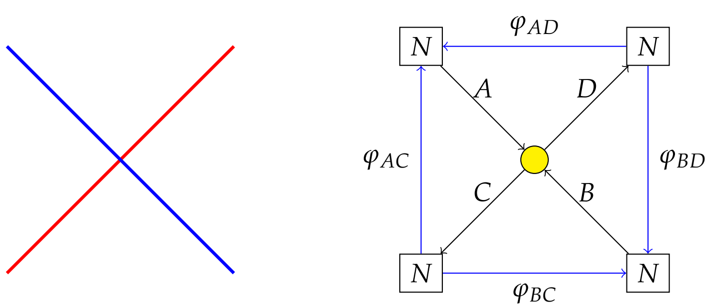
with the superpotential

$$W_{(6.1.10)} = V_+ + V_- \,. \tag{6.1.11}$$

Theory (6.1.7) will be used as as a basic building block to construct other theories. For the sake of readability, we shall suppress the number $N-1$ in the yellow node from now on.

6.1.3 Another representation of (6.1.7)

There is another *equivalent way* to represent theory (6.1.7). We further decompose the $SU(2N)$ flavour node in quiver (6.1.7) into $SU(N) \times SU(N) \times U(1)_p$. The resulting quiver is

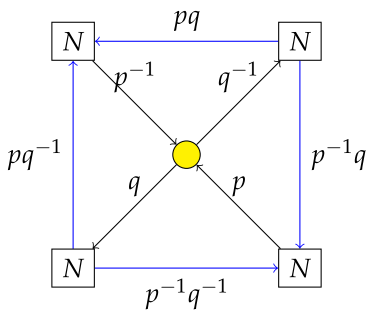


(6.1.12)

Here C and D are the chiral fields that come from the decomposition of Q in (6.1.7), and φ_{AD}, φ_{AC}, φ_{BC} and φ_{BD} are the fields that come from the 4d theory. The superpotential is

$$W_{(6.1.12)} = V_+ + V_- + A\varphi_{AD}D + A\varphi_{AC}C + B\varphi_{BC}C + B\varphi_{BD}D\,. \tag{6.1.13}$$

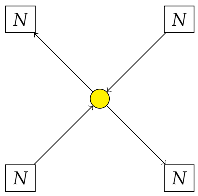
The $U(1)_p$ and $U(1)_q$ charge assignment is depicted as follows.



(6.1.14)

We use the "skeleton" diagram on the left of (6.1.12) to represent such a building block. Each red and blue external leg (or each end of the red and blue lines) corresponds to a flavour symmetry $SU(N)$. The red colour indicates that the two $SU(N)$ symmetries come from the group decomposition $SU(2N)$ due to the duality wall. The blue colour is the same as that used in (6.1.7). Observe the directions of the arrows of the chiral fields A, B, C, D that are transformed under each $SU(N)$ flavour symmetry associated with each external legs: it is ingoing for blue and outcoming for red.

Similar to the discussion around (6.1.10), we may get rid of the 4d chiral fields φ_{AD}, φ_{AC}, φ_{BC} and φ_{BD} using the duality (6.1.3). This results in



(6.1.15)

with the monopole superpotential $W = V_+ + V_-$.

6.2 Gluing basic building blocks

Having discussed the basic building block, we now consider construction involving multiple duality walls. The corresponding 3d theory can be obtained by gluing together the same number of basic building blocks in certain ways along the 4d fields (denoted by blue arrows in the quiver). In the following, we discuss the prescription for the gluing in detail. In fact, such a prescription is heavily motivated by that adopted in [1, 136, 137] in the context of compactifications of 6d theories on a Riemann surface with fluxes for the global symmetries. We discuss the motivation and the similarity of our set-up and that of [1] in the last subsection of this section.

6.2.1 Using basic building block (6.1.7)

We start by considering the cases in which we glue a number of copies of the basic block (6.1.7). This corresponds to set-up involving the same number of duality walls.

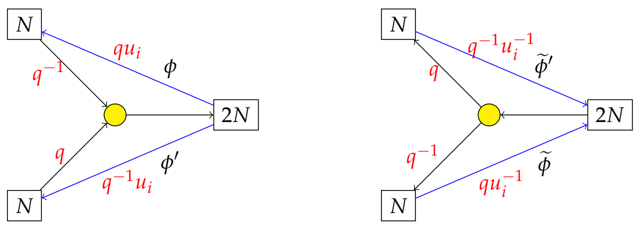
Prescription

Let us consider two copies of the basic building blocks (6.1.7). For the first copy, we assign the $U(1)_q \times SU(2N)$ fugacities $a_i = q\, u_i$ to ϕ (and hence $a'_i = q^{-1}\, u_i$ to ϕ'), where $i = 1, 2, \ldots, N$ and u_i are the parameters that have to satisfy $\prod_{i=1}^{2N} u_i = 1$ being $SU(2N)$ fugacities. For the second copy, let us call the 4d fields $\widetilde{\phi}$ and $\widetilde{\phi}'$ and assign the $U(1)_{\widetilde{q}} \times SU(2N)$ fugacities $\widetilde{a}_i = \widetilde{q}\, \widetilde{u}_i$ to $\widetilde{\phi}$ (and hence $\widetilde{a}'_i = \widetilde{q}^{-1}\, \widetilde{u}_i$ to $\widetilde{\phi}'$), again with the constraint $\prod_{i=1}^{2N} \widetilde{u}_i = 1$.

The prescription is that two building blocks can be glued along ϕ and $\widetilde{\phi}'$ (or along ϕ' and $\widetilde{\phi}$) if and only if one of the following conditions is satisfied:

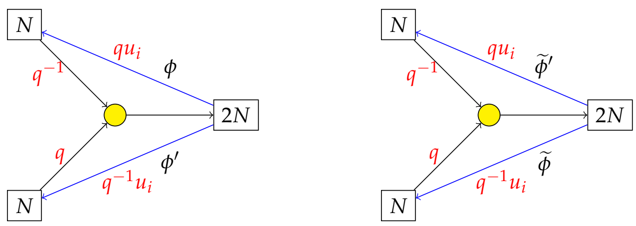
$$\begin{aligned} &\Phi\text{-gluing:} \qquad a_i = \widetilde{a}'_i\,, \\ &S\text{-gluing:} \qquad a_i = \frac{1}{\widetilde{a}'_i}\,, \qquad\qquad \forall i = 1, \cdots, 2N\,. \end{aligned} \tag{6.2.1}$$

Let us illustrate this using explicit examples. We can perform an S-gluing, but not a Φ-gluing, for these two models along ϕ and $\widetilde{\phi}'$ (or along ϕ' and $\widetilde{\phi}$):



(6.2.2)

On the other hand, it is possible to perform a Φ-gluing, but not an S-gluing, for these two model along ϕ and $\widetilde{\phi}'$ (or along ϕ' and $\widetilde{\phi}$):



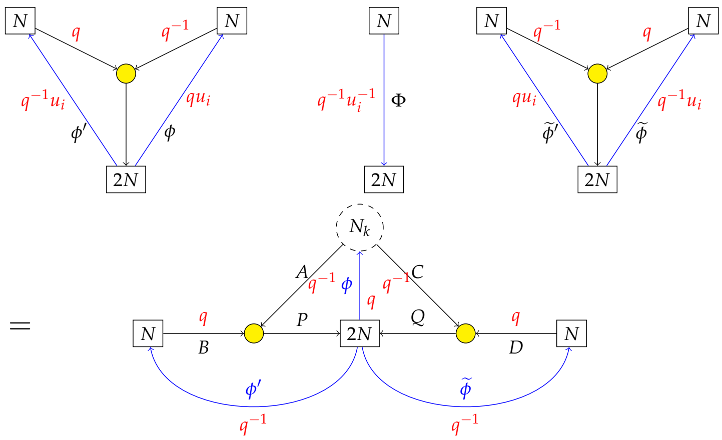
(6.2.3)

The next step is to turn on some superpotential terms to identify the 4d fields along which we glue.

The Φ-gluing. To identify ϕ with $\widetilde{\phi}'$, we introduce an additional set of chiral fields Φ that are coupled to the 4d fields via the superpotential term

$$\delta W = \Phi(\phi - \widetilde{\phi}')\,, \tag{6.2.4}$$

where the contraction of indices is understood. This is a mass term for the fields Φ, ϕ and $\widetilde{\phi}'$ and integrating them out we are left with only one combination of ϕ and $\widetilde{\phi}'$. In the process, the equations of motion of Φ precisely identify $\phi = \widetilde{\phi}'$ as desired. Moreover, this superpotential breaks the two $SU(N)$ symmetries from each copy of the building blocks to a diagonal combination, which we gauge with Chern–Simons (CS) level k. Similarly, the two copies of the $SU(2N)$ symmetry are also broken to a diagonal subgroup, which remains as a flavour symmetry in the resulting theory. In the quiver description, the Φ-gluing and the resulting model are

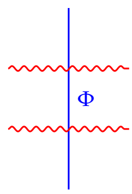


(6.2.5)

where the superpotential is

$$W = V_+^{(1)} + V_-^{(1)} + V_+^{(2)} + V_-^{(2)} + A\phi P + C\phi Q + B\phi' P + D\widetilde{\phi} Q\,, \tag{6.2.6}$$

and we drop the fugacity u_i in the lower diagram (the transformation rule of each chiral field under $SU(2N)$ is clear from the arrow). We denote by N_k in a dashed circle the $SU(N)$ gauge group with CS level k. Notice that the charges and the representations of all the chiral fields under the global symmetries implied by the Φ-gluing condition are compatible with the cubic superpotential terms corresponding to each loop in the quiver. We use the following skeleton diagram to denote the Φ-gluing (6.2.5):



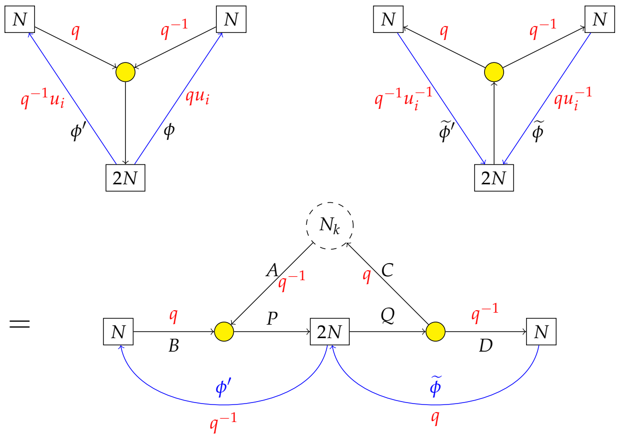
(6.2.7)

The two blue external legs correspond to the two $SU(N)$ flavour nodes in the lower diagram of (6.2.5). As discussed before, the two $SU(2N)$ symmetries coming from each duality wall (red wiggle line) are broken to a diagonal subgroup by the aforementioned superpotential, and this is denoted by the square node labelled by $2N$ in the bottom quiver in (6.2.5).

The S-gluing. The S-gluing can be implemented by introducing the superpotential term

$$\delta W = \phi \widetilde{\phi}'\,. \tag{6.2.8}$$

This implies that both ϕ and $\widetilde{\phi}'$ are integrated out and we are left with no field. Again the two $SU(N)$ symmetries are broken to a diagonal combination, which we gauge with a CS level k. The two $SU(2N)$ symmetries are also broken to a diagonal subgroup. In the quiver description, the S-gluing and the resulting model are

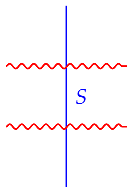


(6.2.9)

The superpotential of the resulting theory is

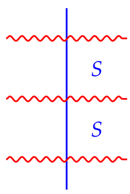
$$W = V_+^{(1)} + V_-^{(1)} + V_+^{(2)} + V_-^{(2)} + CAPQ + B\phi' P + D\widetilde{\phi} Q\,. \tag{6.2.10}$$

Notice again that the charges and the representations of all the chiral fields under the global symmetries implied by the S-gluing condition are compatible with the cubic and quartic superpotential terms corresponding to each loop in the quiver. We use the following skeleton diagram to denote the S-gluing (6.2.9):



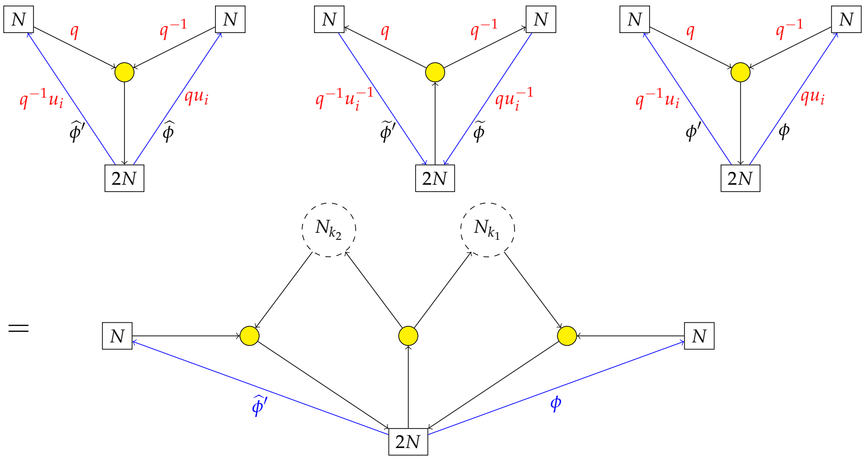
(6.2.11)

Note that we can also treat odd number of duality walls in a similar way as described above. For example, in the case of three duality walls, we can perform S-gluing in the following way:



(6.2.12)

The corresponding theory is

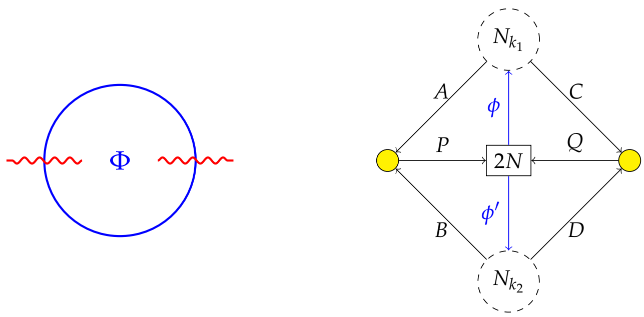


(6.2.13)

Self-gluing: closing external legs

With the prescription for the Φ- and S-gluing one can construct several other models, either adding more basic building blocks or gauging the remaining flavour symmetries. The latter corresponds to closing external legs of the skeleton diagram. For example, in (6.2.5) and (6.2.9) we can "self-glue" the theory along ϕ' and $\widetilde{\phi}$ which results in gauging together the two remaining $SU(N)$ flavour symmetries.

In the model (6.2.5), obtained from the Φ-gluing of two basic building blocks, we can only perform a further Φ-gluing along ϕ' and $\widetilde{\phi}$. The latter is because both ϕ' and $\widetilde{\phi}$ carry the same $U(1)_q$ charge and transform the same way under $SU(N) \times SU(2N)$. This leads to the model

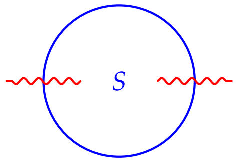


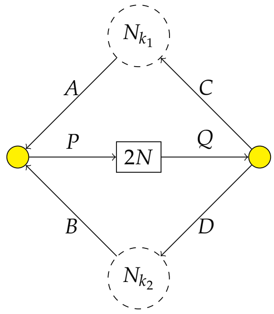
(6.2.14)

with superpotential

$$W = V_+^{(1)} + V_-^{(1)} + V_+^{(2)} + V_-^{(2)} + A\phi P + B\phi' P + C\phi Q + D\phi' Q\,. \tag{6.2.15}$$

On the other hand, in the model (6.2.9), obtained from the S-gluing, we can only perform a further S-gluing along ϕ' and $\widetilde{\phi}'$. This is because ϕ' and $\widetilde{\phi}$ carry opposite $U(1)_q$ charges and transform the opposite way under $SU(N) \times SU(2N)$. We thus arrive at the following model





(6.2.16)

with superpotential

$$W = V_+^{(1)} + V_-^{(1)} + V_+^{(2)} + V_-^{(2)} + PQCA + PQDB\,. \tag{6.2.17}$$

Notice that the two previous models have similar structures, apart from the fact that in (6.2.16) the 4d fields ϕ and ϕ' are absent, and the $U(1)$ charges as well as the arrows of the right half of the quiver are inverted with respect to (6.2.14). This is very similar to the difference between the models (6.1.7) and (6.1.10). Indeed, by applying duality (6.1.3) locally on the right yellow node of (6.2.16), one obtains (6.2.14). Models (6.2.14) and (6.2.16) are actually dual to each other for any $N \geq 2$:

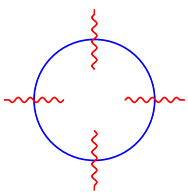
$$(6.2.14) \overset{(6.1.3)}{\longleftrightarrow} (6.2.16)\,. \tag{6.2.18}$$

As a result there is no need to specify Φ or S when we draw the skeleton diagram with all external legs being closed.

This result can be generalised for any *even* number of duality walls. We state a general result as follows.

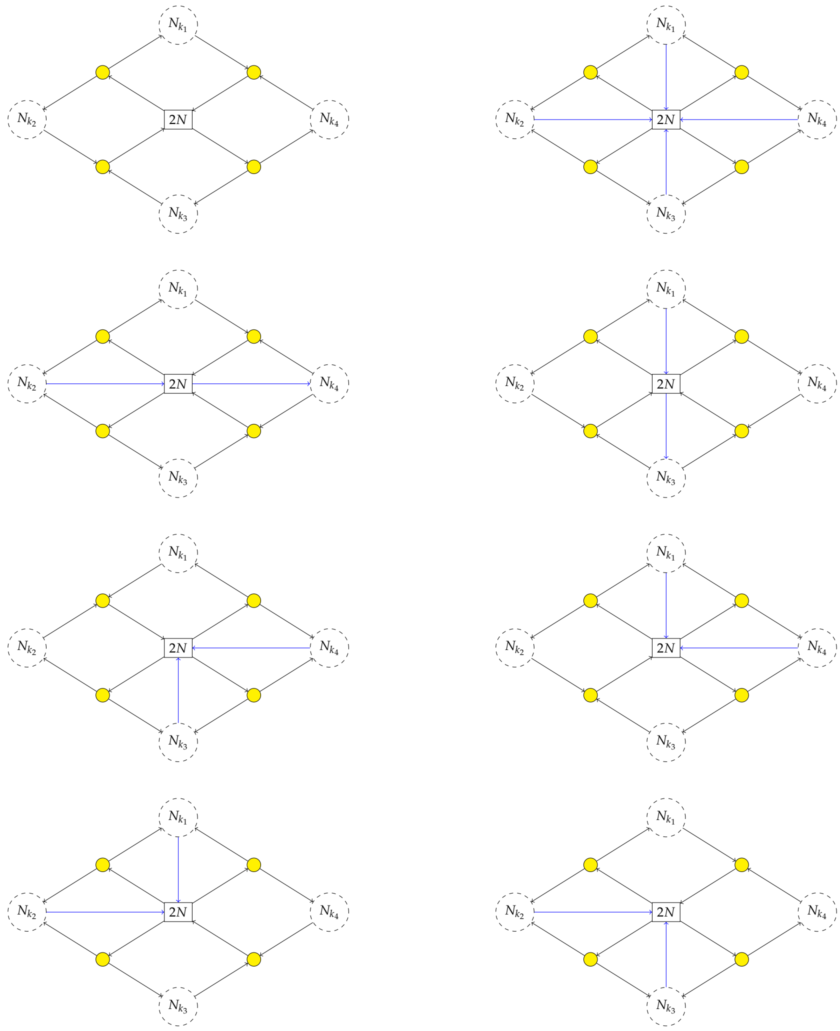
For given N and the Chern–Simons levels as well as a topology of the skeleton diagram, if all external legs of the latter are closed, the theories associated with the Φ-gluing and/or S-gluing of an even number of walls are dual to each other.

Let us provide an example for theories associated with four duality walls such that all external legs are closed.



(6.2.19)

We have eight duality frames with an "octality" that relates them to each other.



(6.2.20)

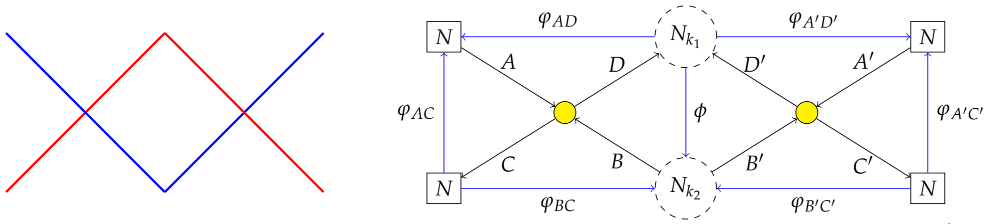
The superpotential of each theory contains the basic monopole operators from each yellow node; the cubic terms coming from every closed triangular loop that contains one blue line as an edge; and the quartic terms coming from every closed rectangular loop that does not contain a blue line.

As a final remark, we point out that in the case of odd number of duality walls, it is not possible to close all external legs in the skeleton diagram. For example, in (6.2.13), $\widehat{\varphi}'$ and φ carry the fugacities $q^{-1}u_i$ and qu_i respectively. These do not satisfy the gluing condition (6.2.1) and so we cannot glue the theory along $\widehat{\varphi}'$ and φ and hence the external legs cannot be closed. A way to evade this problem is to use (6.1.12) as a basic building block instead of (6.1.7). We discuss this in further detail in section 6.2.2.

6.2.2 Using basic building block (6.1.12)

Rectangular gluing

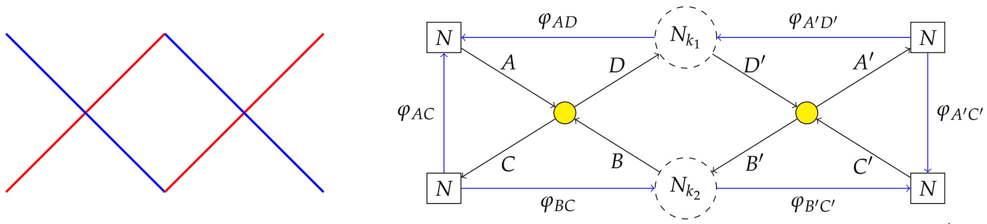
Instead of using (6.1.7), we can perform a Φ gluing or an S-gluing for multiple copies of the building block (6.1.12). For example, if we take two copies of (6.1.12) and perform a **Φ-gluing** along φ_{BD} in both copies, the resulting theory is



(6.2.21)

with superpotential containing the basic monopole operators from both yellow nodes and the cubic terms coming from every closed triangular loop in the quiver that contains one blue line. Upon gluing, we have gauged the upper and lower $SU(N)$ symmetries with CS levels k_1 and k_2 respectively. In the skeleton diagram, for the Φ-gluing, a blue (*resp.* red) line joins with another blue (*resp.* red) line. Topologically, the skeleton diagram has genus 1, as well as 2 red and 2 blue external legs.

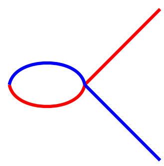
Let us now consider the S-**gluing**. We take two copies of (6.1.12) and glue them along φ_{BD} of one copy and φ_{AC} of the other copy. As a result we obtain



(6.2.22)

The superpotential of the resulting theory contains the basic monopole operators from each yellow node; the cubic terms coming from every closed triangular loop in the quiver that contains one blue line; and the quartic term $DD'B'B$ coming from the middle rectangular loop. In the skeleton diagram, for the S-gluing, a blue (*resp.* red) line joins with another red (*resp.* blue) line – this is opposite to the Φ-gluing.

Observe that as a result of such gluing, which involves **two pairs** of external legs at the same time, we end up with a rectangle in the skeleton diagram. We will refer to these types of gluing as **rectangular Φ-gluing** and **rectangular S-gluing** respectively. There is also another type of gluing which is not a rectangular gluing. For example, one may self-glue the left part of the skeleton diagram of (6.1.12) to obtain



(6.2.23)

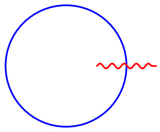
First of all, the "loop" on the left is not rectangular. Secondly, this type of gluing involves only one pair of external legs, not two pairs as for the rectangular gluing. We postpone the discussion of the non-rectangular gluing until later.

Theories (6.2.21) and (6.2.22) will be analysed in detail in section 6.5.

Gluing amusement. As a final remark, we can further perform a rectangular self-Φ-gluing on (6.2.21) such that the blue (*resp.* red) external leg on the left is joined with the blue (*resp.* red) external leg on the right. As a result, we obtain the skeleton diagram (as well as the quiver diagram) whose topology is an "strip", whose face containing two rectangles. Similarly, we can further perform a rectangular self-S-gluing on (6.2.22) such that the blue external legs are joined with the red ones. The topology of the diagram is also a strip, but with half of the face "flipped" with respect to the former.

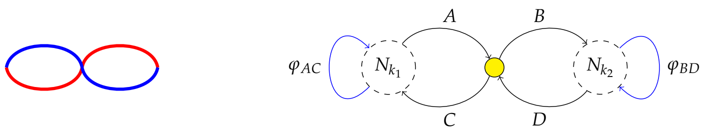
Odd number of basic building blocks

As we have discussed in the paragraph below (6.2.20), it is not possible to close all external legs for odd number of duality walls, provided that we use (6.1.7) as a basic building block. This can also be seen in the case of one duality wall. In particular, it is not possible to perform the following self-gluing:



(6.2.24)

This is because none of the conditions in (6.2.1) is satisfied, since ϕ carries a fugacity $q\,u_i$, whereas ϕ' carries a fugacity $q^{-1}\,u_i$). However, if we instead use (6.1.12) as a basic building block, we can perform a **rectangular (self-)S-gluing** along the opposite blue edges, namely along $(\varphi_{AD}, \varphi_{BC})$ or along $(\varphi_{AC}, \varphi_{BD})$. For definiteness, let us consider the former option. In terms of the skeleton diagram, we can identify the left (*resp.* right) blue external leg with the left (*resp.* right) red external leg. As a result, we obtain

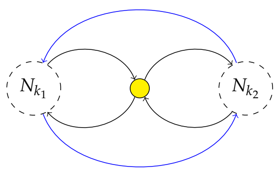


(6.2.25)

$$W = V_+ + V_- + C\varphi_{AC}A + B\varphi_{BD}D$$

Note that the skeleton diagram is rectangular in the sense that it has four sides. Also, since this gluing involves two pairs of external legs at the same time, it is qualified as a rectangular gluing. Moreover, the blue lines that connect $SU(N)_{k_1}$ and $SU(N)_{k_2}$ disappear because we have performed an S-gluing. We can further apply duality (6.1.3) to the yellow node of the quiver (6.2.25) and obtain the following dual theory:

$$(6.2.25) \quad \overset{(6.1.3)}{\longleftrightarrow}$$

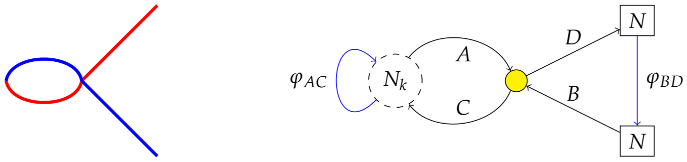


(6.2.26)

with the monopole superpotential and the two cubic terms coming from the upper and lower triangular loops. We further explore these theories in section 6.3.2, where we find two more dual theories. These four theories are then related to each other by a quadrality as shown in (6.3.20).

Non-rectangular gluing

Let us now consider a closure of one pair of external legs. We propose the following prescription:



(6.2.27)

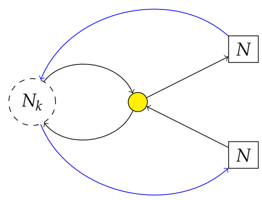
$$W = V_+ + V_- + C\varphi_{AC}A + D\varphi_{BD}B$$

When a pair of external legs is glued together, the corresponding $SU(N)$ flavour symmetries associated to those legs are commonly gauged with a certain CS level k. The 4d fields that was connecting the two $SU(N)$ flavour symmetries becomes an adjoint field and a singlet under the gauge group $SU(N)_k$ (this is φ_{AC} in the above example). We also remove the 4d fields connecting the $SU(N)_k$ gauge groups to other $SU(N)$ flavour symmetries (hence φ_{AD} and φ_{BC} are absent in the above example).

The reason we proposed such a prescription for the non-rectangular gluing is the consistency with (6.2.25). Observe that when we also close the right pair of external legs in (6.2.27), we obtain precisely (6.2.25).

Notice also that the above prescription for closing a pair of external legs *commutes* with duality (6.1.3). In (6.2.26), we first closed all external legs and then applied duality (6.1.3) to the yellow node to obtain the right quiver diagram. Now suppose that we first apply duality (6.1.3) to the yellow node in (6.2.27) to obtain[4]

(6.2.27) $\longleftrightarrow$

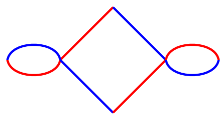


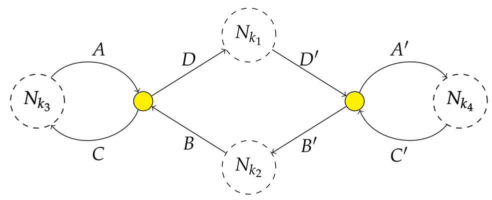
(6.2.28)

with the monopole superpotential and the two cubic terms coming from the upper and lower triangular loops. Upon closing the right pair of external legs using the above prescription, one obtain precisely the quiver in (6.2.26). In section 6.3.1, we analyse (6.2.27) and (6.2.28) in more detail.

[4]We emphasise that, upon applying duality (6.1.3), all black arrows in (6.2.28) have to be reversed with respect to those in (6.2.27). (The directions of the blue arrows are then fixed.) However, since the quiver has a horizontal symmetry, we draw the quiver as it is in (6.2.28). One should keep in mind that the roles of the upper and lower nodes in (6.2.28) are reversed with respect to those of (6.2.27).

This prescription can, of course, be applied to a more complicated theory. For example, we have





$$W = V_+^{(1)} + V_-^{(1)} + V_+^{(2)} + V_-^{(2)} + BDD'B' \tag{6.2.29}$$

6.2.3 Comparison with the gluing prescription in [1]

As mentioned earlier, the gluing prescription adopted is heavily motivated by that used in [1]. The reason why we adopted the latter is due to the similarity of our construction and [1].

Let us first briefly summarise the construction of [1]. In that reference, the basic building block arises from the 6d E-string theory compactified on a sphere with two punctures (a tube) with a particular choice of flux that breaks the E_8 symmetry of the E-string theory to $E_7 \times U(1)_F$. The latter was then realised from the 5d E-string theory with a duality domain wall [138], which gives rise to a subgroup $SU(8) \times U(1)_F$ of the former symmetry. The $U(1)_F$ charge on one side of the domain wall flips its sign as we cross to the other side. Let us mention that the compactification of higher-rank E-string theory has been recently discussed in [137] and further analysed in [139]. Some of the results obtained in [137] have been applied, in the spirit of duality walls in five-dimensional gauge theories [138], to study various global and supersymmetry enhancement of 4d $\mathcal{N} = 1$ theories [140].

We now turn to our construction. We consider duality domain walls in 4d $\mathcal{N} = 2$ $SU(N)$ gauge theory with $2N$ flavours. In this case, the duality wall gives rise to a symmetry $SU(2N) \times U(1)_q$, which is also the flavour symmetry of the 4d theory. The analog of $U(1)_F$ in [1] is indeed $U(1)_q$. As we explained around (6.1.7), each of the two $SU(N)$ flavour symmetries are coupled to the $SU(N)$ gauge symmetry of the 4d theory on each side of the wall. Since A and B as well as ϕ and ϕ' carry opposite charges under $U(1)_q$, we see that, indeed, the $U(1)_q$ charge on the left flips its sign on the right of the duality wall. The cubic superpotential terms also appear in the same way as described in [1].

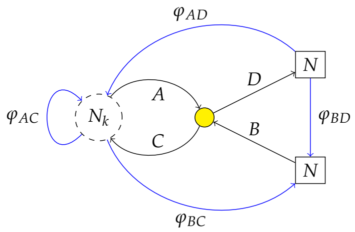
Although we do not have a realisation of our theory as coming from a 5d theory on a Riemann surface (analog of 6d E-string theory on a Riemann surface in [1]), we have a very similar geometric analog of the Riemann surface, namely the skeleton diagram. The genus and the external legs of the latter play the same roles as the genus and the puncture of the Riemann surface in [1]. It would be nice to understand the theory studied here as coming from compactification of a higher dimensional theory.

6.3 A single duality wall

In this section we consider the case of a single duality wall, whose skeleton diagram has genus one. We first discuss theories with two external legs and then move on to those with zero external legs and genus two. We study indices of such theories and discuss various dualities among them.

6.3.1 Two external legs

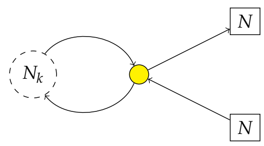
We have already introduced two dual theories associated with the skeleton diagram with genus one and two external legs, namely (6.2.27) and (6.2.28). In this subsection, we introduce two more theories that are closely related to the former. The first one is



(6.3.1)

$$W = V_+ + V_- + C\varphi_{AC}A + D\varphi_{BD}B + D\varphi_{AD}A + C\varphi_{BC}B$$

To obtain the second theory, we apply duality (6.1.3) to the yellow node. We get rid of φ_{AD}, φ_{BC}, φ_{BD} and φ_{AC}, and reverse all the black arrows. However, since the quiver has a horizontal symmetry, we can draw the quiver for the dual theory as follows:



(6.3.2)

$$W = V_+ + V_-$$

where we emphasise that the roles of the upper and lower flavour nodes are reversed with respect to that of (6.3.1).

Let us summarise the four closely related theories:

$$\begin{aligned} (6.3.1) &\overset{(6.1.3)}{\longleftrightarrow} (6.3.2) \\ (6.2.27) &\overset{(6.1.3)}{\longleftrightarrow} (6.2.28) \end{aligned} \qquad (6.3.3)$$

where each pair is related by the duality (6.1.3). We shall discuss in the next subsection that, for $N = 2$, the four theories are, in fact, dual to each other. However, for $N > 2$, the theories in the first line are not dual to those in the second line.

Quadrality for the case of $N = 2$

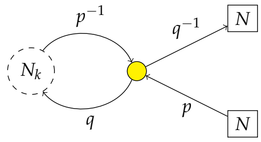
In the special case of $N = 2$, as we shall discuss below, the indices of the four models in (6.3.3) are equal. We thus conjecture that the four models are related by a quadrality:

$$(6.3.1) \overset{(6.1.3)}{\longleftrightarrow} (6.3.2) \overset{\text{for } N=2}{\longleftrightarrow} (6.2.28) \overset{(6.1.3)}{\longleftrightarrow} (6.2.27) \qquad (6.3.4)$$

The indices for the theories in (6.3.4) with $N = 2$

Let us first fix the convention in drawing the quivers in (6.3.4). All black lines in every model in (6.3.4) are drawn in the following way and carry the following

$U(1)_p \times U(1)_q$ fugacities:



(6.3.5)

The $U(1)_p \times U(1)_q$ charges of the chiral fields corresponding to the blue line then follow from the superpotential. For example, φ_{BD} in (6.3.1) carries the $U(1)_p \times U(1)_q$ fugacity $p^{-1}q$.

We first examine theory (6.3.1). For $N = 2$ and the CS level $k \geq 2$[5], the index reads

$$\begin{aligned}&\mathcal{I}^{N=2}_{(6.3.1)}(x; \boldsymbol{y}, \boldsymbol{z}, p, q)\\&= 1 + C_1(\boldsymbol{y}, \boldsymbol{z}, p, q)x + C_2(\boldsymbol{y}, \boldsymbol{z}, p, q)x^2 + C_3(\boldsymbol{y}, \boldsymbol{z}, p, q)x^3 + \ldots\ .\end{aligned} \tag{6.3.6}$$

where the coefficients C_1, C_2 and C_3 are as follows:

$$\begin{aligned}C_1(\boldsymbol{y}, \boldsymbol{z}, p, q) &= p^{-1}q[1;1] + pq^{-1}\\C_2(\boldsymbol{y}, \boldsymbol{z}, p, q) &= p^{-2}q^2[2;2] + [1;1] + p^2q^{-2} - ([2;0] + [0;2] + 2[0;0])\\C_3(\boldsymbol{y}, \boldsymbol{z}, p, q) &= p^{-3}q^3[3;3] + p^{-1}q[2;2] + 2pq^{-1}[1;1] + p^{-1}q + p^3q^{-3}\\&\quad - p^{-1}q([1;3] + [3;1]) - pq^{-1}([2;0] + [0;2]) - 2p^{-1}q[1;1]\\&\quad - p^{-3}q^{-1}[2;0] - 2pq^{-1}\ .\end{aligned} \tag{6.3.7}$$

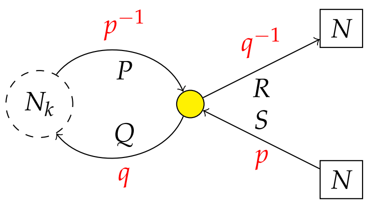
Here we use the shorthand notation $[a;b]$ to denote the characters $\chi^{SU(2)}_{[a]}(\boldsymbol{y})\chi^{SU(2)}_{[b]}(\boldsymbol{z})$ of the representation $[a;b]$ of the global symmetry $SU(2) \times SU(2)$, with the first slot a corresponding to the upper node and second slot to the lower node.

We find that the indices of the other theories in (6.3.4) are related to that of (6.3.6) by the following relation:

$$\begin{aligned}\mathcal{I}^{N=2}_{(6.3.1)}(x; \boldsymbol{y}, \boldsymbol{z}, p, q) &= \mathcal{I}^{N=2}_{(6.3.2)}(x; \boldsymbol{y}, \boldsymbol{z}, p^{-1}, q^{-1})\\= \mathcal{I}^{N=2}_{(6.2.27)}(x; \boldsymbol{z}, \boldsymbol{y}, p, q) &= \mathcal{I}^{N=2}_{(6.2.28)}(x; \boldsymbol{z}, \boldsymbol{y}, p^{-1}, q^{-1})\ .\end{aligned} \tag{6.3.8}$$

This serves as a non-trivial test for the quadrality proposed in (6.3.4).

Let us label the chiral fields in (6.3.2) and their $U(1)$ charges as follows.



(6.3.9)

Recall that for this theory we need to invert p and q in (6.3.7). The terms in the coefficient coefficient C_1 correspond to the following gauge invariant quantities:

$$\begin{aligned}pq^{-1}[1;1]: &\qquad X_i^{i'} := R_i S^{i'}\ ,\\p^{-1}q: &\qquad Y := P^\alpha Q_\alpha\ .\end{aligned} \tag{6.3.10}$$

[5]We take the CS level to be generic; unless specified otherwise, we take its absolute value to be larger than or equal to 2. When the CS level is taken to be 0 or 1, for example, the index may diverge depending on the cases we are considering.

where $i, j = 1, 2$ and $i', j' = 1, 2$ are the flavour indices for the upper and lower square nodes respectively, and $\alpha, \beta = 1, 2$ are the $SU(2)_k$ gauge indices. These are relevant operators. The positive terms in C_2 correspond to

$$\begin{aligned} p^2q^{-2}[2;2] :& \quad X_i^{i'} X_j^{j'} \,, \\ [1;1] :& \quad X_i^{i'} Y \,, \\ p^2q^{-2} :& \quad Y^2 \,. \end{aligned} \tag{6.3.11}$$

These are marginal operators. The negative terms in C_2 indicate that the global symmetry is $SU(2)^2 \times U(1)^2$, as is manifest in the quiver diagram.

The indices for the case of $N = 3$

Let us take $N = 3$ and $k \geq 2$. The indices of (6.3.1) and (6.3.2) are given by

$$\begin{aligned} \mathcal{I}_{(6.3.2)}(\boldsymbol{y}, \boldsymbol{z}, p, q) &= \mathcal{I}_{(6.3.1)}(\boldsymbol{y}, \boldsymbol{z}, p^{-1}, q^{-1}) \\ &= 1 + C_1(\boldsymbol{y}, \boldsymbol{z}, p, q)x + C_2(\boldsymbol{y}, \boldsymbol{z}, p, q)x^2 + \dots \,, \end{aligned} \tag{6.3.12}$$

where

$$\begin{aligned} C_1(\boldsymbol{y}, \boldsymbol{z}, p, q) &= pq^{-1}[1,0;0,1] + p^{-1}q \\ C_2(\boldsymbol{y}, \boldsymbol{z}, p, q) &= p^2q^{-2}[2,0;0,2] + 2p^{-2}q^2 + p^2q^{-2}[0,1;1,0] + 2[1,0;0,1] \\ &\quad - [1,1;0,0] - [0,0;1,1] - 2 - p^2q^{-2} \,. \end{aligned} \tag{6.3.13}$$

Here the notation $[\mathbf{R}_1; \mathbf{R}_2]$ denote a representation of the $SU(3) \times SU(3)$ flavour symmetry, where the first slot corresponds to the lower node and the second slot corresponds to the upper node of (6.3.2) (which becomes the upper and lower nodes of the dual theory (6.3.1)). Let us use the notation as in (6.3.9) and take $N = 3$. Now the yellow node is $U(2)$, whose indices will be denoted by $a, b = 1, 2$. The terms in the coefficient C_1 correspond to

$$\begin{aligned} pq^{-1}[1,0;0,1] :& \quad X_i^{i'} := R_i^a S_a^{i'} \,, \\ p^{-1}q :& \quad Y := P_a^\alpha Q_\alpha^a \,. \end{aligned} \tag{6.3.14}$$

These are the relevant operators. The positive terms in C_2 correspond to

$$\begin{aligned} p^2q^{-2}[2,0;0,2] :& \quad X_i^{i'} X_j^{j'} \,, \\ 2p^{-2}q^2 :& \quad Y^2 \,, \quad P_a^\alpha Q_\beta^a P_b^\beta Q_\alpha^b \,, \\ p^2q^{-2}[0,1;1,0] :& \quad \epsilon_{ab}\epsilon^{cd} R_i^a R_j^b S_c^{i'} S_d^{j'} \,, \\ 2[1,0;0,1] :& \quad X_i^{i'} Y \,, \quad S_a^{i'} Q_\alpha^a P_b^\alpha R_i^b \,. \end{aligned} \tag{6.3.15}$$

These are the marginal operators.

On the other hand, the indices of (6.2.27) and (6.2.28) are given by

$$\begin{aligned} \mathcal{I}_{(6.2.28)}(\boldsymbol{y}, \boldsymbol{z}, p, q) &= \mathcal{I}_{(6.2.27)}(\boldsymbol{y}, \boldsymbol{z}, p^{-1}, q^{-1}) \\ &= 1 + c_1(\boldsymbol{y}, \boldsymbol{z}, p, q)x + c_2(\boldsymbol{y}, \boldsymbol{z}, p, q)x^2 + \dots \,, \end{aligned} \tag{6.3.16}$$

where

$$\begin{aligned} c_1(\boldsymbol{y}, \boldsymbol{z}, p, q) &= pq^{-1}[1,0;0,1] + p^{-1}q \\ c_2(\boldsymbol{y}, \boldsymbol{z}, p, q) &= p^2q^{-2}[2,0;0,2] + 2p^{-2}q^2 + [1,0;0,1] + (1+p^2q^{-2})[0,1;1,0] \\ &\quad - [1,1;0,0] - [0,0;1,1] - 2 - p^2q^{-2} \,. \end{aligned} \tag{6.3.17}$$

Let us analyse theory (6.2.28). The relevant operators, corresponding to the terms in c_1, are

$$\begin{aligned} pq^{-1}[1,0;0,1]: &\qquad X_i^{i'} := B_a^{i'} D_i^a \,, \\ p^{-1}q: &\qquad Y := A_a^\alpha C_\alpha^a \,. \end{aligned} \tag{6.3.18}$$

The marginal operators, corresponding to the terms in c_2, are

$$\begin{aligned} p^2q^{-2}[2,0;0,2]: &\qquad X_i^{i'} X_j^{j'} \\ 2p^{-2}q^2: &\qquad Y^2 \,, \quad A_a^\alpha C_\beta^a A_b^\beta C_\alpha^b \\ [1,0;0,1]: &\qquad X_i^{i'} Y \\ [0,1;1,0]: &\qquad (\varphi_{AD})^i_a (\varphi_{BC})^a_{i'} \\ p^2q^{-2}[0,1;1,0]: &\qquad \epsilon_{ab}\epsilon^{cd} D_i^a D_j^b B_c^{i'} B_d^{j'} \end{aligned} \tag{6.3.19}$$

Let us now compare the two sets of results. Observe that the operators in (6.3.14) are in correspondence with (6.3.18), and so as the first two lines of (6.3.15) and (6.3.19). However, the last two lines of (6.3.15) do not agree with (6.3.19). In particular, in the former the representation $[1,0;0,1]$ appears with multiplicity 2, whereas it appears with multiplicity 1 in the latter. For this reason we conclude that for $N > 2$, the two sets of theories stated in (6.3.3) are not dual to each other.

Superconformal fixed points

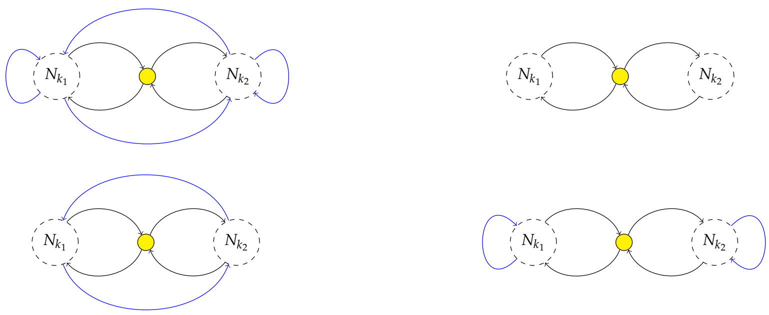
Let us focus on $N = 3$, and *assume* that theories (6.3.1), (6.3.2), (6.2.27) and (6.2.28) flow to superconformal fixed points. Due to the dualities (6.3.3), theory (6.3.1) flows to the same fixed point as theory (6.3.2), and theory (6.2.27) flows to the same fixed point as theory (6.2.28). Due to the previous discussion, we expect that the two fixed points are different for $N = 3$.

Under the assumption of the existence of the superconformal fixed point, the negative terms in C_2 of (6.3.13) and those in c_2 of (6.3.17) correspond to the conserved current of each set of theories. Both contain a term $-p^2q^{-2}$, which should correspond to a $U(1)$ conserved current and should appear in the index as 1 (since its character is 1). Therefore our assumption on the conformality forces us to set $p = q$. The terms $2p^{-2}q^2 - 2 - p^2q^{-2}$ thus combine into -1, and we are left with the negative terms $-[1,1;0,0] - [0,0;1,1] - 1$ in both C_2 and c_2. These indicate that the global symmetries of both superconformal fixed points are indeed $SU(3) \times SU(3) \times U(1)$, where the fugacity of such a $U(1)$ symmetry is identified with $p = q$. Another possible interpretation of this phenomenon is as follows: if we deform theory (6.3.13) or theory (6.3.17) by two real mass deformations, one associated with $U(1)_p$ and the other with $U(1)_q$, then we reach the aftermentioned fixed point only when the two real masses are set to be equal.

6.3.2 Zero external leg and quadrality

In this subsection, the two $SU(N)$ global symmetry in each of the theories in (6.3.3) are commonly gauged with CS level k_2, and let us denote the CS level k for the former $SU(N)$ gauge group by k_1. We have introduced actually two of the resulting theories in (6.2.25) and (6.2.26), whose skeleton diagram has genus two and zero external leg. In this subsection we discuss the relation between the four theories after such gauging.

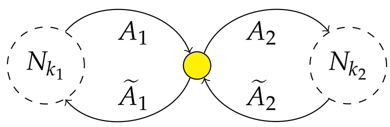
We find that the indices of the following four theories are equal for **any** $N \geq 2$ and for $k_1, k_2 \geq 2$:



(6.3.20)

where, for each quiver, there is a monopole superpotential due to the yellow node and the cubic superpotential terms coming from every closed triangular loop that contains one blue line as an edge. We thus claim that these four theories are related to each other by a *quadrality*. Note that for the special case of $N = 2$, such a quadrality is an immediate consequence of that discussed in (6.3.4).

Let us analyse such theories in more detail. For definiteness, we choose one of the theories from the above list, say



(6.3.21)

with the superpotential

$$W = V_+ + V_- \,. \tag{6.3.22}$$

The case of $N = 2$

For $k_1 \geq 1$ and $k_2 \geq 2$ (or $k_2 \geq 1$ and $k_1 \geq 2$), the first few orders of the power expansion of the index are[6]

$$\mathcal{I}_{(6.3.21)}(x; u) = 1 + C_1(p, q)x + C_2(p, q)x^2 + C_3(p, q)x^3 + C_4(p, q)x^4 + \ldots \,, \tag{6.3.23}$$

[6] We find that for $k_1 = k_2 = 1$, the index is equal to unity, and if either k_1 or k_2 is zero, the index diverges.

where

$$\begin{aligned} C_1(p,q) &= pq^{-1} + p^{-1}q \\ C_2(p,q) &= (p^2q^{-2} + 1 + p^{-2}q^2) - 2 \\ C_3(p,q) &= p^3q^{-3} + p^{-3}q^3 \\ C_4(p,q) &= p^4q^{-4} + p^{-4}q^4 + p^2q^2 + p^{-2}q^{-2} + c_{k_1,k_2} , \end{aligned} \tag{6.3.24}$$

with c_{k_1,k_2} a positive interger that depends on the values of k_1 and k_2. For example, $c_{2,2} = 1$, $c_{2,k} = 2$ for $k \geq 3$, and $c_{k_1,k_2} = 3$ for $k_1, k_2 \geq 3$.

Using the assignment as in (6.1.14), we see that the relevant operators, corresponding to the terms pq^{-1} and $p^{-1}q$ in $C_1(p,q)$, are

$$X_1 := (A_1)^\alpha (\widetilde{A}_1)_\alpha , \qquad X_2 := (A_2)_{\alpha'} (\widetilde{A}_2)^{\alpha'} , \tag{6.3.25}$$

where $\alpha = 1, 2, \ldots, N$ and $\alpha' = 1, 2, \ldots, N$ are the gauge indices for $SU(N)_{k_1}$ and $SU(N)_{k_2}$ respectively. The marginal operators, corresponding to the terms p^2q^{-2}, 1 and $p^{-2}q^2$ in $C_2(p,q)$, are X_1^2, X_1X_2, X_2^2. The term -2 in $C_2(p,q)$, corresponding to the conserved current, confirms that the global symmetry of the theory is $U(1)_p \times U(1)_q$. Note that due to terms in $C_4(p,q)$, it is not possible to rewrite the fugacities p, q in terms of characters of $SU(2)$ representations.

The case of $N = 3$

For $N = 3$ with $k_1 \geq 2$ and $k_2 \geq 2$, the first few coefficients of the index (6.3.23) are

$$\begin{aligned} C_1(p,q) &= pq^{-1} + p^{-1}q \\ C_2(p,q) &= (x_{k_1}p^2q^{-2} + 2 + x_{k_2}p^{-2}q^2) - 2 , \end{aligned} \tag{6.3.26}$$

where

$$x_k = \begin{cases} 1 & \text{if } k = 2 \\ 2 & \text{if } k \geq 3 \end{cases} . \tag{6.3.27}$$

The term -2 in $C_2(p,q)$ in (6.3.26) indicates that the global symmetry of the theory is $U(1) \times U(1)$, whose fugacities are denoted by p and q. Let us now explain the other terms in $C_2(p,q)$, as well as those in $C_1(p,q)$.

Let us first consider the case of $k_1 = k_2 = 2$, so that $x_{k_1} = x_{k_2} = 1$. A crucial difference between the coefficient $C_2(p,q)$ for $N = 3$ and that for $N = 2$ in (6.3.24) is that there is an extra marginal operator that carry zero charges under both $U(1)_p$ and $U(1)_q$ in the former case. For $N \geq 3$, the relevant operators are similar to (6.3.25):

$$X_1 = (A_1)^\alpha_a (\widetilde{A}_1)^a_\alpha , \qquad X_2 := (A_2)^a_{\alpha'} (\widetilde{A}_2)^{\alpha'}_a , \tag{6.3.28}$$

where $a, b = 1, 2, \ldots, N-1$ are the indices for the $U(N-1)$ gauge group denoted by the yellow node. These correspond to the terms in the coefficient $C_1(p,q)$. In addition to X_1^2, X_1X_2, X_2^2, there is an extra marginal operator given by

$$Q = (A_1)^\alpha_a (A_2)^a_{\alpha'} (\widetilde{A}_2)^{\alpha'}_b (\widetilde{A}_1)^b_\alpha , \tag{6.3.29}$$

which is different from X_1X_2, for $N \geq 3$, and is neutral under both $U(1)_p$ and $U(1)_q$. These four marginal operators correspond to the terms in the bracket in the coefficient $C_2(p,q)$.

Now let us assume that one of k_1 and k_2 or both are strictly greater than 2. The left or right gauge nodes can be regarded as $SU(3)_k$ with 2 flavours, where k is either k_1 or k_2. To analyse this, we find that it is convenient to apply the duality (3.23) of [119]. The dual theory is $U(k-1)_{-k,-1}$ with 2 flavours q, $\widetilde{q}$, the chiral field M in the adjoint representation of the yellow $U(2)$ node in (6.3.21), and the superpotential $W = Mq\widetilde{q}$. Let us denote by M_1 and M_2 the adjoint fields of the yellow $U(2)$ node that arise from dualising the $SU(3)_{k_1}$ and $SU(3)_{k_2}$ respectively. The gauge invariant quantities $\mathrm{tr}(M_1)$ and $\mathrm{tr}(M_2)$ in this dual theory can be mapped to X_1 and X_2 in the original theory (6.3.28).

Let us consider the case of $k=2$. The dual theory has the $U(1)_{-1}$ gauge group. The F-terms with respect to q and $\widetilde{q}$ are $M^a_b q^b = 0$ and $M^a_b \widetilde{q}_a = 0$. Then, M can be regarded as a two by two matrix of rank 1, since M maps a vector to zero and so the dimension of its kernel is one. Since M has rank 1, it can be written as a product of two vectors and it follows that $\mathrm{tr}(M^2) = (\mathrm{tr}\, M)^2$. Therefore, in the case of $k_1 = k_2 = 1$, the operators X_1^2, X_1X_2, X_2^2, Q can be mapped to $\mathrm{tr}(M_1^2)$, $\mathrm{tr}(M_1)\,\mathrm{tr}(M_2)$, $\mathrm{tr}(M_2^2)$, $\mathrm{tr}(M_1M_2)$, where $M_{1,2}$ satisfy $\mathrm{tr}(M_1^2) = (\mathrm{tr}\, M_1)^2$ and $\mathrm{tr}(M_2^2) = (\mathrm{tr}\, M_2)^2$.

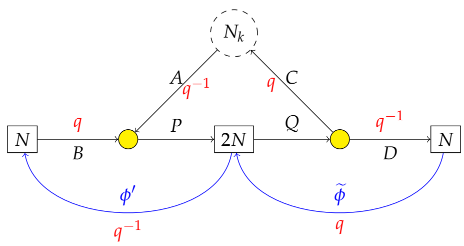
However, when $k > 2$, the dual gauge group $U(k-1)_{-k,1}$ has a higher rank. On the contrary to the case of $k=2$, M has rank greater than 1. As a consequence, $\mathrm{tr}(M^2)$ and $(\mathrm{tr}\, M)^2$ are not identical and they correspond to two different operators. This explains the presence of x_{k_1} and x_{k_2} in $C_2(p,q)$ in (6.3.26). In particular, if $k_1, k_2 > 2$, the marginal operators corresponding to the terms in the brackets in $C_2(p,q)$ are $\mathrm{tr}(M_1^2)$, $(\mathrm{tr}\, M_1)^2$, $\mathrm{tr}(M_2^2)$, $(\mathrm{tr}\, M_2)^2$, corresponding to $2p^2q^{-2} + 2q^2p^{-2}$, and $\mathrm{tr}(M_1)\,\mathrm{tr}(M_2), \mathrm{tr}(M_1M_2)$, corresponding to 2.

6.4 Two duality walls: using (6.1.7) as a building block

Let us now consider the theories associated with two duality walls. As discussed in section 6.2, if we use (6.1.7) as a basic building block, we obtain two theories (6.2.5) and (6.2.9) from Φ-gluing and S-gluing respectively. One can perform further gauging to close the external legs and obtain (6.2.14) and (6.2.16). We discuss in detail the four models in this section. One the other hand, we discuss the case of two duality walls if we use (6.1.12) as a basic building block in section 6.5.

6.4.1 Indices of models (6.2.5) and (6.2.9) for $N=2$

We first analyse model (6.2.9).



(6.4.1)

For $k \geq 2$, the index for $N=2$ reads

$$\mathcal{I}^{N=2}_{(6.2.9)}(x; y_L, \boldsymbol{y}, y_R) = 1 + C_1 x + C_2 x^2 + \dots \tag{6.4.2}$$

where

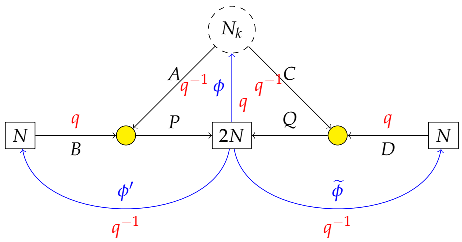
$$\begin{aligned} C_1 &= q^{-1}[1;\, 1,0,0;\, 0] + q[0;\, 0,0,1;\, 1]\,, \\ C_2 &= q^{-2}[2;\, 2,0,0;\, 0] + q^2[0;\, 0,0,2;\, 2] + \{[1;\, 1,0,1;\, 1] + [1;\, 0,0,0;\, 1]\} \\ &\quad - [2;\, 0,0,0;\, 0] - [0;\, 1,0,1;\, 0] - [0;\, 0,0,0;\, 2] - 1\,. \end{aligned} \tag{6.4.3}$$

The unrefined index for this model is

$$\mathcal{I}^{N=2}_{(6.2.9)}(x; y_L = 1, \boldsymbol{y} = 1, y_R = 1) = 1 + 16x + 102x^2 + 288x^3 + 396x^4 + \ldots\,. \tag{6.4.4}$$

The terms in C_1 in (6.4.3) correspond to φ' and $\widetilde{\varphi}$ respectively. The terms in the curly brackets in C_2 come from the tensor product of the two terms in C_1. The second symmetric power of the representation $q^{-1}[1;\, 1,0,0;\, 0]$ in C_1 is $q^{-2}[2;\, 2,0,0;\, 0] + q^{-2}[0;\, 0,1,0;\, 0]$. However, the gauge invariant combinations $\epsilon^{\alpha\beta}M^{ij}_{\alpha\beta}$[7], with $M^{ij}_{\alpha\beta} = (\varphi')^i_\alpha(\varphi')^j_\beta$ associated with the latter representation vanish in the chiral ring, due to the quantum rank condition (in the same way as in the Seiberg duality). This can be seen by applying duality (6.1.3) to the left yellow node; we see that $(\varphi')^i_\alpha$ is identified with $\widetilde{B}_\alpha\widetilde{P}^i$, where $\widetilde{B}$ and $\widetilde{P}$ are the chiral fields (whose arrows are in the opposite direction to B and P) in the dual theory, and so $\epsilon^{\alpha\beta}M^{ij}_{\alpha\beta} = \epsilon^{\alpha\beta}\widetilde{B}_\alpha\widetilde{B}_\beta\widetilde{P}^i\widetilde{P}^j = 0$. This is the reason why only $q^{-2}[2;\, 2,0,0;\, 0]$ survives in the index. The similar argument can be applied to the second symmetric power of $q[0;\, 0,0,1;\, 1]$. The negative terms in C_2 tell us that the global symmetry of the theory is $SU(2)^2 \times SU(4) \times U(1)$.

Now let us analyse model (6.2.5).



(6.4.5)

The index for $N = 2$ and for $k \geq 2$ reads

$$\mathcal{I}^{N=2}_{(6.2.5)}(x; y_L, \boldsymbol{y}, y_R) = 1 + c_1 x + c_2 x^2 + \ldots \tag{6.4.6}$$

where

$$\begin{aligned} c_1 &= q^{-1}[1;\, 1,0,0;\, 0] + q^{-1}[0;\, 1,0,0;\, 1]\,, \\ c_2 &= q^{-2}[2;\, 2,0,0;\, 0] + q^{-2}[0;\, 2,0,0;\, 2] + \{q^{-2}[1;\, 0,1,0;\, 1] + q^{-2}[1;\, 2,0,0;\, 1]\} \\ &\quad + q^{-2}[0;\, 0,1,0;\, 0] - q^2[0;\, 0,1,0;\, 0] - [2;\, 0,0,0;\, 0] - [0;\, 1,0,1;\, 0] \\ &\quad - [0;\, 0,0,0;\, 2] - 1\,. \end{aligned} \tag{6.4.7}$$

[7]Recall that we take $N = 2$. Here $i, j = 1, \ldots, 4$ are the $SU(4)$ flavour indices, and $\alpha, \beta = 1, 2$ are the indices for the left $SU(2)$ flavour node. Note also that, due to the definition of M, $M^{[ij]}_{\alpha\beta}$, with an antisymmetrisation on i and j, can be written as $\epsilon^{\alpha\beta}M^{ij}_{\alpha\beta}$.

The unrefined index of this theory turns out to be equal to that of (6.2.5), which is given by (6.4.4):

$$\mathcal{I}^{N=2}_{(6.2.5)}(x; y_L = 1, \boldsymbol{y} = 1, y_R = 1) = \mathcal{I}^{N=2}_{(6.2.9)}(x; y_L = 1, \boldsymbol{y} = 1, y_R = 1) \, . \qquad (6.4.8)$$

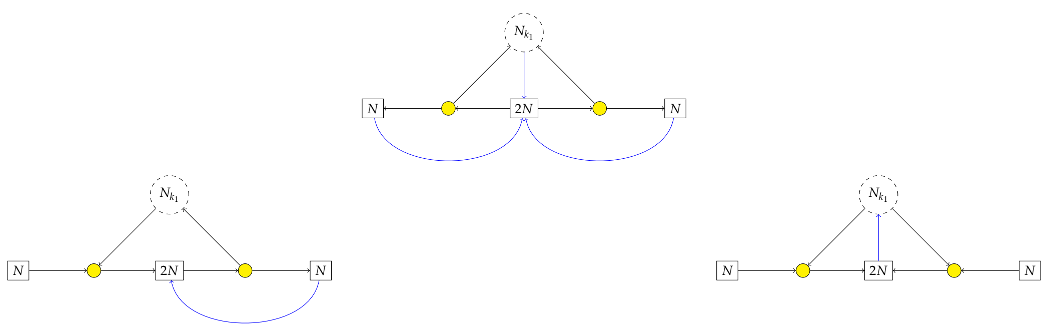
The interpretation for (6.4.7) is very similar to the above. The terms in c_1 correspond to ϕ' and $\widetilde{\phi}$. The terms in the curly brackets in c_2 come from the tensor product of the two terms in c_1. The term $+q^{-2}[0;\, 0,1,0;\ 0]$ can be conveniently explained using another duality frame. If we dualise both left and right yellow nodes using duality (6.1.3), the chiral fields ϕ' and $\widetilde{\phi}$ disappear and we replace ϕ by a chiral field χ, whose arrow is in the opposite direction of ϕ and carrying the $U(1)_q$ fugacity q^{-1}. (The arrows for A, B, P, C, Q, D also reverse their directions.) We can construct the gauge invariant quantity $\epsilon_{ab}\chi^a_i \chi^b_j$, where $a, b = 1, 2$ is the $SU(2)_k$ gauge indices and $i, j = 1, \dots, 4$ are the $SU(4)$ flavour indices, in the representation $q^{-2}[0;\, 0,1,0;\ 0]$, as required.

It is interesting to point out that even though the unrefined indices of the two models are equal, their refined indices are different. In particular, the representation $[1;\, 1,0,1;\ 1] + [1;\, 0,0,0;\ 1]$ that appears in the former but not the latter, and the representation [1; 0,1,0; 1] +[1; 0,0,2; 1] that appears in the latter but not in the former. Although their dimensions are equal and they both come from the tensor products of the two terms of the coefficient of x, their characters are different.

Another important point is the negative term $-q^{-2}[0;\, 0,1,0;\ 0]$ that appears in c_2 in (6.4.7). Since this is not the adjoint representation, it cannot correspond to a conserved current. If we *assume* that theory (6.2.5) flows to a fixed point, this negative term cannot be there by itself. Indeed, if we set $q = 1$, such a term cancels with another positive term (both are indicated in blue). After the cancellation, the negative terms indicate that the global symmetry of the theory is $SU(2)^2 \times SU(4) \times U(1)$. Since the fugacity q has already been set to 1, the index no longer has a manifest $U(1)$ fugacity, and we interpret such a $U(1)$ global symmetry as emergent in the infrared.

6.4.2 Various dualities for any $N \geq 2$

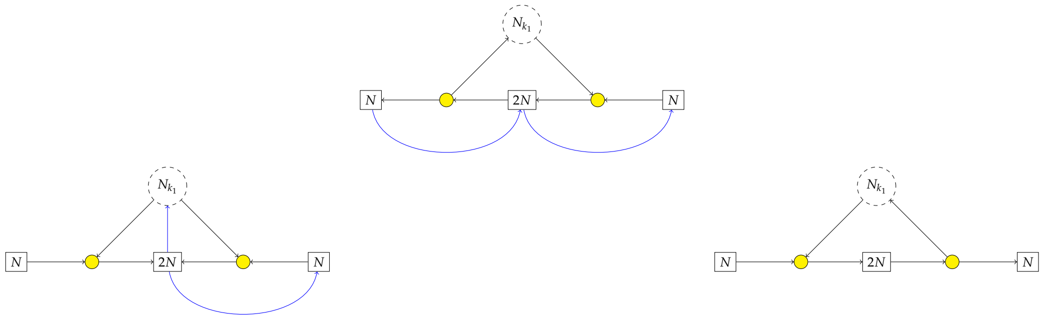
Given models (6.2.5) and (6.2.9), we can generate a number of dualities that hold for any $N \geq 2$ by applying duality (6.1.3) to each yellow node. For (6.2.5), we have a *triality* between the following theories:



(6.4.9)

where the top and bottom left theories are related by dualising the left yellow node, and the bottom left and bottom right theories are related by dualising the right yellow node. For each quiver, there is a monopole superpotential due to the yellow node, the cubic superpotential terms coming from every closed triangular loop that contains one blue line as an edge, and there is also a quartic term for the bottom left quiver coming from the middle triangle.

For (6.2.9), we have a *triality* between the following theories:



(6.4.10)

The superpotential for each quiver is in the same way as described above.

If we commonly gauge the two $SU(N)$ flavour symmetry corresponding to the left and right square nodes, we obtain models (6.2.14) and (6.2.16) and their *duality*. We discuss this in detail below.

Duality between models (6.2.14) and (6.2.16)

Applying duality (6.1.3) to either of the yellow nodes, we find that models (6.2.14) and (6.2.16) are dual to each other for **any** $N \geq 2$. Indeed, we find that the indices for (6.2.14) and (6.2.16) are equal.

In particular, for $N = 2$ and $k_1, k_2 \geq 2$, their indices are

$$\mathcal{I}^{N=2}_{(6.2.14)}(x; q, \boldsymbol{y}) = \mathcal{I}^{N=2}_{(6.2.16)}(x; q, \boldsymbol{y}) = 1 + 0x + 0x^2 + 0x^3 + C_4(q, \boldsymbol{y})x^4 + \ldots\,, \qquad (6.4.11)$$

where the coefficients of x, x^2, x^3 vanish, and

$$C_4(q) = \chi^{SU(4)}_{[1,0,1]}(\boldsymbol{y}) + \chi^{SU(4)}_{[0,2,0]}(\boldsymbol{y}) + 2(q^2 + q^{-2})\chi^{SU(4)}_{[0,1,0]}(\boldsymbol{y}) + q^4 + 1 + q^{-4}\,, \qquad (6.4.12)$$

where $\boldsymbol{y} = (y_1, y_2, y_3)$ are fugacities of the $SU(4)$ flavour symmetry and q is a fugacity of the $U(1)$ global symmetry.

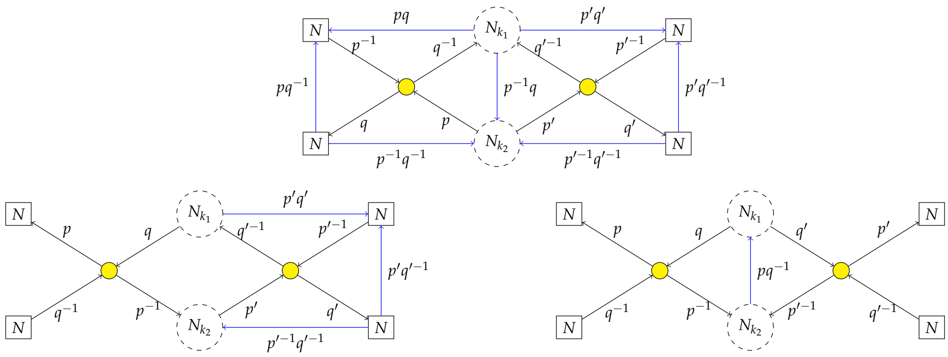
The vanishing coefficient of x^2 in (6.4.11) deserves some explanations. Models (6.2.14) and (6.2.16) in fact have the global symmetry $SU(4) \times U(1)$. The contribution $-\chi^{SU(4)}_{[1,0,1]}(\boldsymbol{y}) - 1$ at order x^2 of the conserved current is cancelled by the contribution $\chi^{SU(4)}_{[1,0,1]}(\boldsymbol{y}) + 1$ of the marginal operators. For model (6.2.16), such marginal operators are $A^\alpha C_\alpha Q^i P_j$, corresponding to the close path in the upper triangle. Note that these are equal to $-B^{\alpha'} D_{\alpha'} Q^i P_j$, corresponding to the close path in the lower triangle, due to the F-terms that are the derivatives with respect to P_j of the superpotential (6.2.17).

6.5 Two duality walls: using (6.1.12) as a building block

In this section, we consider the theories associated with two duality walls, using (6.1.12) as a basic building block. We consider the theories arising from Φ-gluing and S-gluing and their dual theories. We finally compute their indices and discuss the duality for the case of $N = 2$.

Φ-gluing

The theory associated with the Φ-gluing of two building blocks has already been introduced in (6.2.21). We present such a theory, with the fugacities for $U(1)_p \times U(1)_q \times U(1)_{p'} \times U(1)_{q'}$ for each chiral fields, along with its duals below.



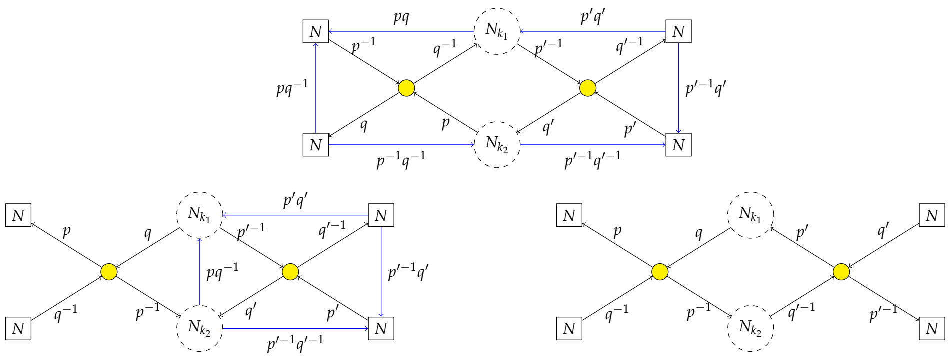
(6.5.1)

where the bottom left and right quivers come from applying (6.1.3) to the left yellow node and to both yellow nodes of the top diagram, respectively. There are monopole superpotential terms, the cubic superpotential terms coming from each triangular loop in the quiver that contains one blue line as an edge, and the quartic superpotential term for the bottom left quiver coming from rectangular loop in the middle. Such a superpotential imposes the following condition on the $U(1)$ fugacities:

$$p^{-1}qp'q'^{-1} = 1 \,. \tag{6.5.2}$$

S-gluing

The theory associated with the S-gluing of two building blocks has already been introduced in (6.2.22). We present such a theory, with the fugacities for $U(1)_p \times U(1)_q \times U(1)_{p'} \times U(1)_{q'}$ for each chiral fields, along with its duals below.



(6.5.3)

where the bottom left and right quivers come from applying (6.1.3) to the left yellow node and to both yellow nodes of the top diagram, respectively. There are monopole superpotential terms, the cubic superpotential terms coming from each triangular loop in the quiver that contains one blue line as an edge, and the quartic superpotential term for the top and bottom right quivers coming from the rectangular loop in the middle. Such a superpotential imposes the following condition on the $U(1)$ fugacities:

$$pq^{-1}p'^{-1}q' = 1 \,. \tag{6.5.4}$$

6.5.1 The indices of (6.5.1) and (6.5.3) for $N = 2$

We focus only on the case of $N = 2$ and $k_1, k_2 \geq 2$.

Theory (6.5.3)

The index of this theory is

$$\mathcal{I}^{N=2}_{(6.5.3)}(x; p, q, p', q', y_1, \ldots, y_4) = 1 + C_1 x + C_2 x^2 + \ldots \,, \qquad (6.5.5)$$

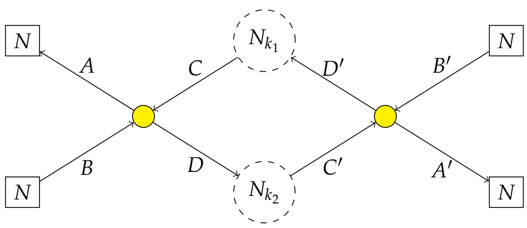
where the coefficients C_i are functions of $p, q, p', q', y_1, \ldots, y_4$. Here we report only the two coefficients C_1 and C_2 in full:

$$\begin{aligned} C_1 &= \frac{p}{q}\begin{bmatrix}1 & 0\\ 1 & 0\end{bmatrix} + \frac{q'}{p'}\begin{bmatrix}0 & 1\\ 0 & 1\end{bmatrix} \overset{(6.5.4)}{=} \frac{p}{q}\begin{bmatrix}1 & 0\\ 1 & 0\end{bmatrix} + \frac{q}{p}\begin{bmatrix}0 & 1\\ 0 & 1\end{bmatrix}\,, \\ C_2 &= \frac{p^2}{q^2}\begin{bmatrix}2 & 0\\ 2 & 0\end{bmatrix} + \frac{q'^2}{p'^2}\begin{bmatrix}0 & 2\\ 0 & 2\end{bmatrix} + \frac{pq'}{qp'}\begin{bmatrix}1 & 1\\ 1 & 1\end{bmatrix} + pqp'q'\begin{bmatrix}1 & 1\\ 0 & 0\end{bmatrix} + \frac{1}{pqp'q'}\begin{bmatrix}0 & 0\\ 1 & 1\end{bmatrix} \\ &\quad - \begin{bmatrix}2 & 0\\ 0 & 0\end{bmatrix} - \begin{bmatrix}0 & 2\\ 0 & 0\end{bmatrix} - \begin{bmatrix}0 & 0\\ 2 & 0\end{bmatrix} - \begin{bmatrix}0 & 0\\ 0 & 2\end{bmatrix} - 4 + \underbrace{\frac{qp'}{pq'}}_{\overset{(6.5.4)}{=}1}\,. \end{aligned} \qquad (6.5.6)$$

We have used the notation $\begin{bmatrix}a & b\\ c & d\end{bmatrix}$ to denote the representation $[a; b; c; d]$ of the flavour symmetry $SU(2)^4$ associated with each corner of the quiver. Upon setting $p, q, p', q', y_1, \ldots, y_4$ to 1, the unrefined index for $(k_1, k_2) = (2, 2)$ is

$$1 + 8x + 27x^2 + 24x^3 - 14x^4 + \ldots \,. \qquad (6.5.7)$$

We now focus on gauge invariant combinations of chiral fields corresponding to various terms in the index. For convenience, we consider the bottom right quiver in (6.5.3) and label the chiral fields as follows:



(6.5.8)

Explicitly, the superpotential of the above quiver is $W = V_+ + V_- + V'_+ + V'_- + CDC'D'$. Let us use the indices $\begin{bmatrix}i, j & m, n\\ i', j' & m', n'\end{bmatrix}$, each of which takes values 1, 2, for the flavour symmetry $SU(2)^4$ associated with each corner of the quiver. We use $a, b = 1, 2$ and $a', b' = 1, 2$ to denote the $SU(2)_{k_1}$ and $SU(2)_{k_2}$ gauge indices respectively.

The terms in the coefficient C_1 corresponds to the following gauge invariant combinations:

$$X_i^{i'} = A_i B^{i'}\,, \qquad (X')^m_{m'} = A'_{m'} B'^m\,. \qquad (6.5.9)$$

Indeed, X and X' are the relevant operators. The positive terms of the coefficient C_2 correspond to the following gauge invariant combinations:

$$\begin{aligned} &X_i^{i'} X_j^{j'} , \quad (X')^m_{m'} (X')^n_{n'} , \quad X_i^{i'} (X')^m_{m'} , \\ &Y_i^m := A_i C^a D'_a B'^m , \; Y'^{i'}_{m'} := B^{i'} D_a C'^a B'^{i'} \end{aligned} \tag{6.5.10}$$

These are the marginal operators. From the negative terms in the coefficient C_2, we see that the global symmetry of the theory is $SU(2)^4 \times U(1)^3$. Indeed, the $SU(2)^4$ symmetry is manifest as the four square nodes in the quiver, and the three copies of $U(1)$ correspond to the fugacities p, q, p', q' subject to (6.5.4).

Theory (6.5.1)

The index of this theory is

$$\mathcal{I}^{N=2}_{(6.5.1)}(x; p, q, p', q', y_1, \ldots, y_4) = 1 + c_1 x + c_2 x^2 + \ldots , \tag{6.5.11}$$

where the coefficients c_i are functions of $p, q, p', q', y_1, \ldots, y_4$. We report only c_1 and c_2 in full:

$$\begin{aligned} c_1 &= \frac{p}{q}\begin{bmatrix}1 & 0\\ 1 & 0\end{bmatrix} + \frac{p'}{q'}\begin{bmatrix}0 & 1\\ 0 & 1\end{bmatrix} \overset{(6.5.2)}{=} \frac{p}{q}\begin{bmatrix}1 & 0\\ 1 & 0\end{bmatrix} + \frac{p}{q}\begin{bmatrix}0 & 1\\ 0 & 1\end{bmatrix} , \\ c_2 &= \frac{p^2}{q^2}\begin{bmatrix}2 & 0\\ 2 & 0\end{bmatrix} + \frac{p'^2}{q'^2}\begin{bmatrix}0 & 2\\ 0 & 2\end{bmatrix} + \frac{pp'}{qq'}\begin{bmatrix}1 & 1\\ 1 & 1\end{bmatrix} + pqp'q'\begin{bmatrix}1 & 1\\ 0 & 0\end{bmatrix} + \frac{1}{pqp'q'}\begin{bmatrix}0 & 0\\ 1 & 1\end{bmatrix} \\ &\quad - \begin{bmatrix}2 & 0\\ 0 & 0\end{bmatrix} - \begin{bmatrix}0 & 2\\ 0 & 0\end{bmatrix} - \begin{bmatrix}0 & 0\\ 2 & 0\end{bmatrix} - \begin{bmatrix}0 & 0\\ 0 & 2\end{bmatrix} + \frac{p^2}{q^2} - \frac{q^2}{p^2} - 4 + \underbrace{\frac{pq'}{qp'}}_{\overset{(6.5.2)}{=} 1} . \end{aligned} \tag{6.5.12}$$

Upon setting $p, q, p', q', y_1, \ldots, y_4$ to 1, the unrefined index of this theory for $(k_1, k_2) = (2, 2)$ is

$$1 + 8x + 27x^2 + 24x^3 - 14x^4 + \ldots . \tag{6.5.13}$$

From (6.5.7) and (6.5.13), we see the unrefined indices of theory (6.5.1) and theory (6.5.3) are equal to each other.

Let us consider (6.5.12) in more detail. Notice that the coefficient c_2 contains a negative term $-\frac{q^2}{p^2}$. If we *assume* that theory (6.5.1) flows to a superconformal fixed point, the negative terms in c_2 must correspond to a conserved current. Let us proceed with this assumption. The $-\frac{q^2}{p^2}$ term should correspond to a $U(1)$ conserved current and should appear in the index as 1 (since its character is 1). Therefore our assumption on the conformality forces us to set $p = q$. It follows from (6.5.2) that $p' = q'$. Therefore (6.5.12) can be rewritten as

$$\begin{aligned} c_1 &= \begin{bmatrix}1 & 0\\ 1 & 0\end{bmatrix} + \begin{bmatrix}0 & 1\\ 0 & 1\end{bmatrix} , \\ c_2 &= \begin{bmatrix}2 & 0\\ 2 & 0\end{bmatrix} + \begin{bmatrix}0 & 2\\ 0 & 2\end{bmatrix} + \begin{bmatrix}1 & 1\\ 1 & 1\end{bmatrix} + p^2 p'^2 \begin{bmatrix}1 & 1\\ 0 & 0\end{bmatrix} + \frac{1}{p^2 p'^2}\begin{bmatrix}0 & 0\\ 1 & 1\end{bmatrix} \\ &\quad - \begin{bmatrix}2 & 0\\ 0 & 0\end{bmatrix} - \begin{bmatrix}0 & 2\\ 0 & 0\end{bmatrix} - \begin{bmatrix}0 & 0\\ 2 & 0\end{bmatrix} - \begin{bmatrix}0 & 0\\ 0 & 2\end{bmatrix} - 3 . \end{aligned} \tag{6.5.14}$$

For the coefficient c_3, we report the result only for $y_1 = y_2 = y_3 = y_4 = 1$:

$$c_3 = 8 + 16\left(p^2 p'^2 + \frac{1}{p^2 p'^2}\right) - 8\left(\frac{p^2}{p'^2} + \frac{p'^2}{p^2}\right) . \tag{6.5.15}$$

It can be see from the negative terms in c_2 that the theory has a global symmetry $SU(2)^4 \times U(1)^3$. Although $SU(2)^4$ is manifest in the quiver, not all three $U(1)$ symmetries is manifest. Since we have two fugacities p and p' appearing in the index, only two $U(1)$ symmetries is manifest. We conjecture that the other $U(1)$ global symmetry emerges at the superconformal fixed point in the infrared.

In fact, it is important to emphasise that the indices of (6.5.3) and (6.5.1) are equal if we set $p = q$ and $p' = q'$:

$$\mathcal{I}^{N=2}_{(6.5.1)}(x; p = q, p' = q', y_1, \ldots, y_4) = \mathcal{I}^{N=2}_{(6.5.3)}(x; p = q, p' = q', y_1, \ldots, y_4) . \tag{6.5.16}$$

We have checked this relation up to order x^6 for various (k_1, k_2). We conjecture that theories (6.5.1) and (6.5.3) are dual to each other, in the sense that they flow to the same fixed point in the infrared. For (6.5.3), the global symmetry $SU(2)^4 \times U(1)^3$ is manifest in the quiver description, and it is therefore possible to refine all of the corresponding fugacities in the index. For (6.5.1), the global symmetry is also $SU(2)^4 \times U(1)^3$, but among all global fugacities, it is possible to refine only two $U(1)$ fugacities in the index, since the other $U(1)$ is emergent in the infrared. This interpretation is consistent with the relation (6.5.16). An immediate consequence of this conjecture is that we have six dual descriptions, namely

$$(6.5.1) \overset{\text{for } N=2}{\longleftrightarrow} (6.5.3) . \tag{6.5.17}$$

Let us end this subsection by briefly discussing the case of $N = 3$. We find that the indices of models (6.5.3) and (6.5.1) are not equal, and so the two theories are not dual. In particular, for $N = 3$ and $(k_1, k_2) = (2, 2)$, their unrefined indices are

$$\begin{aligned} (6.5.1):& \qquad 1 + 18x + 136x^2 + 562x^3 + \ldots \\ (6.5.3):& \qquad 1 + 18x + 154x^2 + 832x^3 + \ldots . \end{aligned} \tag{6.5.18}$$

Chapter 7

Summary and perspectives

In this thesis we have studied various properties of three-dimensional supersymmetric gauge theories whose origin comes from the existence of S-duality domain walls in four dimensions. The main character four dimensional theories have been the $\mathcal{N} = 4$ SYM and the $\mathcal{N} = 2$ $SU(N)$ theory with $2N$ flavours, that give rise respectively to the class of $T(G)$ theories and to the 3d $\mathcal{N} = 2$ $U(N-1)$ theory with $2N$ flavours and a linear monopole superpotential. The general idea has been to take the aforementioned theories as *basic building blocks* to construct a variety of quiver theories, and analyse various aspects, such as their moduli spaces and their symmetries in the infrared, either global or supersymmetry itself.

In chapter 3 we studied the moduli space of quiver theories arising from the Hanany–Witten brane system, with an insertion of S-folds. We find that such theories have the Higgs and the Coulomb branches, the former given by the hyperKähler quotient description, and the latter computed in a similar way to the usual 3d $\mathcal{N} = 4$ gauge theories, with the remark that the vector multiplets of the gauge nodes linked by $T(U(N))$ are frozen and do not contribute to the Coulomb branch. Such a rule has been dubbed "freezing rule". We check that this proposal is consistent with mirror symmetry. Moreover, in the case of J-folds, we examine the moduli space of the abelian theories with $T(U(1))$ links and non-zero Chern–Simons levels systematically. With the inclusion of bifundamental and fundamental hypermultiplets into the quiver, the moduli space can be non-trivial, and in many cases the vacuum equations admit many branches of solutions. Finally, for the case of non-abelian theories with $T(U(N))$ links and non-zero Chern–Simons levels, we do not have a general prescription to compute the moduli space of such theories. Nevertheless, we demonstrate the computation of the Hilbert series for an example that belongs to a special class of models arising from multiple M2-branes probing Calabi–Yau 4-fold singularities.

In chapter 4 we propose new classes of three dimensional S-fold CFTs and study their moduli spaces. We explored the possibility of replacing $T(U(N))$ by a more general $T(G)$ theory, where G is self-dual under the S-duality, restricting our attention to the cases where G is taken be $SO(2N)$, $USp'(2N)$ and G_2. For $G = SO(2N)$ and $USp'(2N)$, we propose that the quiver can be realised from an insertion of an S-fold into a brane configuration that involves D3 branes on top of orientifold three-planes, possibly with NS5 and D5 branes [21]. In which case, the S-fold needs to be inserted in an interval of the D3 brane where the orientifold is of the $O3^-$ type or the $\widetilde{O3}^+$ type for $G = SO(2N)$ or $USp'(2N)$, respectively. The resulting quiver contains alternating orthogonal and symplectic gauge groups, along with a $T(G)$ theory connecting two gauge groups G in the quiver. Moreover, we also obtain the mirror theory by performing S-duality on the brane system. Under the action of latter, the $O3^-$ and $\widetilde{O3}^+$ planes, as well as the S-fold remain invariant. Hence the resulting mirror theory can be obtained from the S-dual configuration discussed in [21], with an S-fold inserted in the position corresponding to the original set-up. The freezing rule conjectured in the first chapter has been successfully tested by study the moduli spaces of various quiver theories and their mirrors. The results turned out be nicely consistent with mirror symmetry, namely the Higgs and Coulomb branches of the original theory get exchanged with the Coulomb and Higgs branches of the mirror theory. We perform similar consistency checks. We find that the freezing rule still holds for the quiver with $T(SO(2N))$ and $T(USp'(2N))$ and the results are consistent with mirror symmetry. Such consistency supports the statement that the S-fold can be present in the background of $\mathrm{O3}^-$ and $\widetilde{\mathrm{O3}}^+$ planes. Following the same logic, we also investigate the presence of the S-fold in the background of orientifold fiveplanes. In particular, we examine the insertion of the S-fold into the brane configurations involving orientifold fiveplanes, studied in [25]. The corresponding quiver contains several interesting features such as the presence of the antisymmetric hypermultiplet, along with the $T(U(N))$ link connecting two unitary gauge groups. The mirror configuration consists of an ON plane that gives rise to a bifurcation in the mirror quiver [24], with the S-fold inserted in the position corresponding to the original set-up. An important result that we discover for this class of theories is that, in order for the freezing rule to hold and for the moduli spaces of the mirror pair to be consistent with mirror symmetry, the S-fold must not be inserted "too close" to the orientifold plane; there must be a sufficient number of NS5 branes that separate the S-fold from the orientifold plane. This suggests that the NS5 branes provide a certain "screening effect" or "shielding effect" in the combination of the orientifold plane and the S-fold. We hope to understand this phenomenon better in the future. Finally, we propose a novel class of circular quivers that contains the exceptional G_2 gauge groups. In particular, the quiver contains alternating G_2 and $USp'(4)$ gauge groups, possibly with flavours under $USp'(4)$. Although the Type IIB brane configuration for this class of theories is not known and the S-fold supergravity solution for the exceptional group is not available, we propose that it is possible to "insert an S-fold" into the G_2 and/or $USp'(4)$ gauge groups in the aforementioned quivers. This results in the presence of the $T(G_2)$ link connecting two G_2 gauge groups, and/or the $T(USp'(4))$ link connecting two $USp'(4)$ gauge groups. Furthermore, we propose the mirror theory which is also a circular quiver, consisting of the G_2, $USp'(4)$ and possibly $SO(5)$ gauge groups if the original theory has the flavours under $USp'(4)$. To the best of our knowledge, such mirror pairs are new and have never been studied in the literature before. We check, using the Hilbert series, that moduli spaces of such pairs satisfy the freezing rule and are consistent with mirror symmetry. This, again, serves as strong evidence for the existence of an S-fold of the G_2 type.

In chapter 5 several properties of 3d S-fold SCFTs have been investigated using supersymmetric indices. We have found several theories whose indices exhibit supersymmetry enhancement, due to the presence of extra-supersymmetry current multiplets. Dualities between different S-fold quiver theories have also been explored and the indices allow us to establish the operator map between such theories. Moreover, we studied S-fold theories whose gauge symmetries have different global structures, namely $SU(2)$ and $SU(2)/\mathbb{Z}_2$. We found that the indices of the latter reveal interesting properties regarding the discrete topological symmetry as well as supersymmetry enhancement in a certain case.

Finally, in chapter 6, we study 3d $\mathcal{N}=2$ gauge theories associated with S-duality walls in the 4d $\mathcal{N}=2$ $SU(N)$ gauge theory with $2N$ flavours. Motivated by [1], we propose a prescription in gluing theories associated with multiple duality walls as well as self-gluing for arbitrary number of walls. The analog of the geometric view point of [1], involving Riemann surfaces, is presented using the skeleton diagram. Using supersymmetric indices, we find a number of dualities between different theories, some of them hold only for $N = 2$ and many of them are true for all $N \geq 2$. In particular, we find that for an even number of walls, if all external legs of the skeleton diagrams are closed, the theories associated with the same topology of the skeleton diagram (for given rank and CS levels of the gauge groups) are dual to each other, independent of the way we glue the basic building block (6.1.7). The gluing that has been used can also be viewed as a generalisation of the S-fold theory [80, 81, 124, 126, 14] associated with duality walls in the 4d $\mathcal{N} = 4$ super-Yang-Mills to a theory with lower amounts of supersymmetry, which is the 4d $\mathcal{N} = 2$ gauge theory in our case.

Let us end with some future perspectives and open questions. First of all, it would be nice to find a general prescription to compute the moduli space of non-abelian theories with $T(U(N))$ links, non-zero Chern–Simons levels and possibly with bifundamental and fundamental hypermultiplets. One could ask if it is possible to replace the $T(U(N))$ link between two $U(N)$ gauge groups by the $T^{\sigma}_{\sigma}(U(N))$ link, with an appropriate σ, between two G_σ gauge groups (where G_σ is a subgroup of $U(N)$ that is left unbroken by σ). Since $T^{\sigma}_{\sigma}(U(N))$ is invariant under mirror symmetry, we expect this to be a good candidate to replace $T(U(N))$ in the quiver diagram. Observe that we restricts ourself to models with equal-rank gauge nodes; this avoids problem arising from non-complete Higgsing of the gauge symmetries. It would be interesting to generalise all the result to the unequal-rank cases. This amounts to consider S-fold configurations with fractional branes. Moreover, regarding the analysis of S-fold theories with $T(G)$ with $G \neq U(N)$, we have taken the Chern–Simons levels of all gauge groups connected by the T-link to be zero. It would be interesting to study the moduli spaces as well as the duality between theories with non-zero Chern–Simons levels. Finally, it should be possible to generalise our result on the G_2 group to other exceptional groups, including F_4 and $E_{6,7,8}$, which are also invariant under the S-duality. It is natural to expect that the S-fold associated with such groups should exist and, in that case, it should be possible to find quivers as well as their mirror theories that describe such S-fold CFTs. It would be nice to find a string or an F-theoretic construction for such theories. The investigation of theories involving the $\mathcal{T}_{\mathfrak{M}}$ as a component have also led to a number of open problems that deserve a further investigation in the future. First of all, it would be interesting to understand the geometric origin, such as compactification of a higher dimensional theory, for our theories and, in particular, the skeleton diagrams. Another related line of future research would be to understand the holographic dual of this class of theories along the line of [14, 141]. Finally, it would interesting to understand properties of the moduli space of vacua of the 3d $\mathcal{N} = 2$ theories along the line of [50], as well as to generalise our result to 4d $\mathcal{N} = 2$ gauge theory with orthogonal, symplectic and exceptional gauge groups in analogy to those studied in [51].

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