Profinite groups with a cyclotomic p-orientation

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To the memory of Vladimir Voevodsky

Abstract. Let p be a prime. A continuous representation

$$\theta\colon G \to \mathrm{GL}_1(\mathbb{Z}_p)$$

of a profinite group G is called a cyclotomic p-orientation if for all open subgroups $U \subseteq G$ and for all $k, n \geq 1$ the natural maps

$$H^k(U, \mathbb{Z}_p(k)/p^n) \longrightarrow H^k(U, \mathbb{Z}_p(k)/p)$$

are surjective. Here $\mathbb{Z}_p(k)$ denotes the $\mathbb{Z}_p$-module of rank 1 with U-action induced by $\theta|_U^k$. By the Rost-Voevodsky theorem, the cyclotomic character of the absolute Galois group $G_{\mathbb{K}}$ of a field $\mathbb{K}$ is, indeed, a cyclotomic p-orientation of $G_{\mathbb{K}}$. We study profinite groups with a cyclotomic p-orientation. In particular, we show that cyclotomicity is preserved by several operations on profinite groups, and that Bloch-Kato pro-p groups with a cyclotomic p-orientation satisfy a strong form of Tits' alternative and decompose as semi-direct product over a canonical abelian closed normal subgroup.

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1. Introduction

For a prime p let $\mathbb{Z}_p$ denote the ring of p-adic integers. For a profinite group G, we call a continuous representation $\theta\colon G \to \mathbb{Z}_p^\times = \mathrm{GL}_1(\mathbb{Z}_p)$ a *p-orientation* of G and call the couple (G, θ) a *p-oriented* profinite group. Given a p-oriented profinite group (G, θ), for $k \in \mathbb{Z}$ let $\mathbb{Z}_p(k)$ denote the left $\mathbb{Z}_p[\![G]\!]$-module

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induced by θ^k, namely, $\mathbb{Z}_p(k)$ is equal to the additive group $\mathbb{Z}_p$ and the left G-action is given by

$$g \cdot z = \theta(g)^k \cdot z, \qquad g \in G,\ z \in \mathbb{Z}_p(k). \tag{1.1}$$

Vice-versa, if M is a topological left $\mathbb{Z}_p[\![G]\!]$-module which as an abelian pro-p group is isomorphic to $\mathbb{Z}_p$, then there exists a unique p-orientation $\theta\colon G \to \mathbb{Z}_p^\times$ such that $M \simeq \mathbb{Z}_p(1)$.

The $\mathbb{Z}_p[\![G]\!]$-module $\mathbb{Z}_p(1)$ and the representation $\theta\colon G \to \mathbb{Z}_p^\times$ are said to be *k-cyclotomic*, for $k \geq 1$, if for every open subgroup U of G and every $n \geq 1$ the natural maps

$$H^k(U, \mathbb{Z}_p(k)/p^n) \longrightarrow H^k(U, \mathbb{Z}_p(k)/p)\ , \tag{1.2}$$

induced by the epimorphism of $\mathbb{Z}_p[\![U]\!]$-modules $\mathbb{Z}_p(k)/p^n \to \mathbb{Z}_p(k)/p$, are surjective. If $\mathbb{Z}_p(1)$ (respectively θ) is k-cyclotomic for every $k \geq 1$, then it is called simply a cyclotomic $\mathbb{Z}_p[\![G]\!]$-module (resp., cyclotomic p-orientation). Note that $\mathbb{Z}_p(1)$ is k-cyclotomic if, and only if, $H^{k+1}_{\mathrm{cts}}(U, \mathbb{Z}_p(k))$ is a torsion free $\mathbb{Z}_p$-module for every open subgroup $U \subseteq G$ — here H^*_{cts} denotes continuous cochain cohomology as introduced by J. Tate in [34] (see § 2.1).

Cyclotomic modules of profinite groups have been introduced and studied by C. De Clercq and M. Florence in [5]. Previously J.P. Labute, in [16], considered surjectivity of (1.2) in the case $k = 1$ and $U = G$ — note that demanding surjectivity for $U = G$ only is much weaker than demanding it for every open subgroup $U \subseteq G$, and this is what makes the definition of cyclotomic modules truly new.

Let $\mathbb{K}$ be a field, and let $\bar{\mathbb{K}}/\mathbb{K}$ be a separable closure of $\mathbb{K}$. If $\mathrm{char}(\mathbb{K}) \neq p$, the *absolute Galois group* $G_{\mathbb{K}} = \mathrm{Gal}(\bar{\mathbb{K}}/\mathbb{K})$ of $\mathbb{K}$ comes equipped with a canonical p-orientation

$$\theta_{\mathbb{K},p}\colon G_{\mathbb{K}} \longrightarrow \mathrm{Aut}(\mu_{p^\infty}(\bar{\mathbb{K}})) \simeq \mathbb{Z}_p^\times, \tag{1.3}$$

where $\mu_{p^\infty}(\bar{\mathbb{K}}) \subseteq \bar{\mathbb{K}}^\times$ denotes the subgroup of roots of unity of $\bar{\mathbb{K}}$ of p-power order. If $p = \mathrm{char}(\mathbb{K})$, we put $\theta_{\mathbb{K},p} = \mathbf{1}_{G_{\mathbb{K}}}$, the function which is constantly 1 on $G_{\mathbb{K}}$. The following result (cf. [5, Prop. 14.19]) is a consequence of the positive solution of the Bloch-Kato Conjecture given by M. Rost and V. Voevodsky with the "C. Weibel patch" (cf. [29, 36, 40]), which from now on we will refer to as the Rost-Voevodsky Theorem.

Theorem 1.1. *Let $\mathbb{K}$ be a field, and let p be prime number. The canonical p-orientation $\theta_{\mathbb{K},p}\colon G_{\mathbb{K}} \to \mathbb{Z}_p^\times$ is cyclotomic.*

Theorem 1.1 provides a fundamental class of examples of profinite groups endowed with a cyclotomic p-orientation. Bearing in mind the exotic character of absolute Galois groups, it also provides a strong motivation to the study of cyclotomically p-oriented profinite groups — which is the main purpose of this manuscript. In fact, one may recover several Galois-theoretic statements already for profinite groups with a 1-cyclotomic p-orientation — e.g., the only finite group endowed with a 1-cyclotomic p-orientation is the finite group C_2 of order 2, with non-constant 2-orientation $\theta\colon C_2 \to \{\pm 1\}$ (cf. [11, Ex. 3.5]), and this implies the Artin-Schreier obstruction for absolute Galois groups. In their paper, De Clercq and Florence formulated the "Smoothness Conjecture", which can be restated in this context as follows: for a p-oriented profinite group, 1-cyclotomicity implies k-cyclotomicity for all $k \geq 1$ (cf. [5, Conj. 14.25]).

A p-oriented profinite group (G, θ) is said to be *Bloch-Kato* if the $\mathbb{F}_p$-algebra

$$H^\bullet(U, \widehat{\theta}|_U) = \coprod_{k \geq 0} H^k(U, \mathbb{F}_p(k)), \tag{1.4}$$

where $\mathbb{F}_p(k) = \mathbb{Z}_p(k)/p$, with product given by cup-product, is quadratic for every open subgroup U of G. Note that if $\mathrm{im}(\theta) \subseteq 1 + p\mathbb{Z}_p$ and $p \neq 2$ then G acts trivially on $\mathbb{Z}_p(k)/p$. By the Rost-Voevodsky Theorem $(G_{\mathbb{K}}, \theta_{\mathbb{K},p})$ is, indeed, Bloch-Kato.

For a profinite group G, let $O_p(G)$ denote the maximal closed normal pro-p subgroup of G. A p-oriented profinite group (G, θ) has two particular closed normal subgroups: the kernel $\ker(\theta)$ of θ, and the *θ-center* of (G, θ), given by

$$\mathrm{Z}_\theta(G) = \left\{ x \in O_p(\ker(\theta)) \mid gxg^{-1} = x^{\theta(g)} \text{ for all } g \in G \right\}. \tag{1.5}$$

As $\mathrm{Z}_\theta(G)$ is contained in the center $\mathrm{Z}(\ker(\theta))$ of $\ker(\theta)$, it is abelian. The p-oriented profinite group (G, θ) will be said to be *θ-abelian*, if $\ker(\theta) = \mathrm{Z}_\theta(G)$ and if $\mathrm{Z}_\theta(G)$ is torsion free. In particular, for such a p-oriented profinite group (G, θ), G is a virtual pro-p group (i.e., G contains an open subgroup which is a pro-p group). Moreover, a θ-abelian pro-p group (G, θ) will be said to be *split* if $G \simeq \mathrm{Z}_\theta(G) \rtimes \mathrm{im}(\theta)$.

As $\mathrm{Z}_\theta(G)$ is contained in $\ker(\theta)$, by definition, the canonical quotient $\bar{G} = G/\mathrm{Z}_\theta(G)$ carries naturally a p-orientation $\bar{\theta}\colon \bar{G} \to \mathbb{Z}_p^\times$, and one has the following short exact sequence of p-oriented profinite groups.

$$\{1\} \longrightarrow \mathrm{Z}_\theta(G) \longrightarrow G \xrightarrow{\ \pi\ } \bar{G} \longrightarrow \{1\} \tag{1.6}$$

The following result can be seen as an analogue of the equal characteristic transition theorem (cf. [31, §II.4, Exercise 1(b), p. 86]) for cyclotomically p-oriented Bloch-Kato profinite groups.

Theorem 1.2. *Let (G, θ) be a cyclotomically p-oriented Bloch-Kato profinite group. Then* (1.6) *splits, provided that* $\mathrm{cd}_p(G) < \infty$*, and one of the following conditions hold:*

(i) *G is a pro-p group,*
(ii) *(G, θ) is an oriented virtual pro-p group (see §4),*
(iii) *$(\bar{G}, \bar{\theta})$ is cyclotomically p-oriented and Bloch-Kato.*

In the case that (G, θ) is the maximal pro-p Galois group of a field $\mathbb{K}$ containing a primitive p^{th}-root of unity endowed with the p-orientation induced by $\theta_{\mathbb{K},p}$, $\mathrm{Z}_\theta(G)$ is the inertia group of the maximal p-henselian valuation of $\mathbb{K}$ (cf. Remark 7.8).

Note that the 2-oriented pro-2 group $(C_2 \times \mathbb{Z}_2, \theta)$ may be θ-abelian, but θ is never 1-cyclotomic (cf. Proposition 6.5). As a consequence, in a cyclotomically 2-oriented pro-2 group every element of order 2 is self-centralizing.

For p odd it was shown in [25] that a Bloch-Kato pro-p group G satisfies a strong form of *Tits alternative*, i.e., either G contains a closed non-abelian free pro-p subgroup, or there exists a p-orientation $\theta\colon G \to \mathbb{Z}_p^\times$ such that G is θ-abelian. In Subsection 7.1 we extend this result to pro-2 groups with a cyclotomic orientation, i.e., one has the following analogue of R. Ware's theorem (cf. [38]) for cyclotomically oriented Bloch-Kato pro-p groups (cf. Fact 7.4).

Theorem 1.3. *Let (G, θ) be a cyclotomically p-oriented Bloch-Kato pro-p group. If $p = 2$ assume further that* $\mathrm{im}(\theta) \subseteq 1 + 4\mathbb{Z}_2$. *Then one — and only one — of the following cases hold:*

(i) *G contains a closed non-abelian free pro-p subgroup; or*
(ii) *G is θ-abelian.*

It should be mentioned that for $p = 2$ the additional hypothesis is indeed necessary (cf. Remark 5.8). The class of cyclotomically p-oriented Bloch-Kato profinite groups is closed with respect to several constructions.

Theorem 1.4. (a) *The inverse limit of an inverse system of cyclotomically p-oriented Bloch-Kato profinite groups with surjective structure maps is a cyclotomically p-oriented Bloch-Kato profinite group (cf. Corollary 3.3 and Corollary 3.6).*

(b) *The free profinite (resp. pro-p) product of two cyclotomically p-oriented Bloch-Kato profinite (resp. pro-p) groups is a cyclotomically p-oriented Bloch-Kato profinite (resp. pro-p) group (cf. Theorem 3.14).*

(c) *The fibre product of a cyclotomically p-oriented Bloch-Kato profinite group (G_1, θ_1) with a split θ_2-abelian profinite group (G_2, θ_2) is a cyclotomically p-oriented Bloch-Kato profinite group (cf. Theorem 3.11 and Theorem 3.13).*

(d) *The quotient of a cyclotomically p-oriented Bloch-Kato profinite group (G, θ) with respect to a closed normal subgroup $N \subseteq G$ satisfying $N \subseteq$* $\ker(\theta)$ *and N a p-perfect group is a cyclotomically p-oriented Bloch-Kato profinite group (cf. Proposition 4.6).*

Some time ago I. Efrat (cf. [7, 8, 9]) has formulated the so-called *elementary type conjecture* concerning the structure of finitely generated pro-p groups occurring as maximal pro-p quotients of an absolute Galois group. His conjecture can be reformulated in the class of cyclotomically p-oriented Bloch-Kato pro-p groups. Such a p-oriented pro-p group (G, θ) is said to be *indecomposable* if $\mathrm{Z}_\theta(G) = \{1\}$ and if G is not a proper free pro-p product. A positive answer to the following question would settle the elementary type conjecture affirmatively.

Question 1.5. Let (G, θ) be a finitely generated, torsion free, indecomposable, cyclotomically oriented Bloch-Kato pro-p group. Does this imply that G is a Poincaré duality pro-p group of dimension $\mathrm{cd}_p(G) \leq 2$?

The paper is organized as follows. In § 2 we give some equivalent definitions for cyclotomic p-orientations. In § 3 we study some operations of profinite groups (inverse limits, free products and fibre products) in relation with the properties of cyclotomicity and Bloch-Kato-ness, and we prove Theorem 1.4(a)-(b)-(c). In § 4 we study the quotients of cyclotomically p-oriented profinite groups over closed normal p-perfect subgroups — in particular, we introduce *oriented virtual pro-p groups* and we prove Theorem 1.4(d). In § 5 we study p-oriented profinite Poincaré duality groups. In § 6 we focus on the presence of torsion in cyclotomically 2-oriented pro-2 groups, and we prove that in a 1-cyclotomically 2-oriented pro-2 group every element of order 2 is self-centralizing (see Proposition 6.5). In § 7 we focus on the structure of cyclotomically p-oriented Bloch-Kato pro-p groups: we prove Theorems 1.2 and 1.3, and show that in many cases the θ-center is the maximal abelian closed normal subgroup (cf. Theorem 7.7).

2. Absolute Galois groups and cyclotomic p-orientations

Throughout the paper, we study profinite groups with a cyclotomic module $\mathbb{Z}_p(1)$. In contrast to [5, § 14], we refer to the associated representation $\theta\colon G \to \mathbb{Z}_p^\times$, rather than to the module itself. As we study several subgroups of G associated to this cyclotomic module $\mathbb{Z}_p(1)$, like $\ker(\theta)$ and $\mathrm{Z}_\theta(G)$, this choice of notation turns out to be convenient for our purposes. We follow the convention as established in [25, 26] and call such representations "p-orientations".[1] In the case that G is a pro-p group, the couple (G, θ) was called a *cyclotomic pro-p pair*, in [9, § 3].

2.1. The connecting homomorphism δ^k

Let G be a profinite group, and let $\theta\colon G \to \mathbb{Z}_p^\times$ be a p-orientation of G. For every $k \geq 0$ one has the short exact sequence of left $\mathbb{Z}_p[\![G]\!]$-modules

$$0 \longrightarrow \mathbb{Z}_p(k) \xrightarrow{\;p\cdot\;} \mathbb{Z}_p(k) \longrightarrow \mathbb{F}_p(k) \longrightarrow 0\ , \tag{2.1}$$

which induces the long exact sequence in cohomology

$$\cdots \xrightarrow{\;p\cdot\;} H^k_{\mathrm{cts}}(G, \mathbb{Z}_p(k)) \xrightarrow{\;\pi^k\;} H^k(G, \mathbb{F}_p(k)) \xrightarrow{\;\delta^k\;} H^{k+1}_{\mathrm{cts}}(G, \mathbb{Z}_p(k)) \xrightarrow{\;p\;} H^{k+1}_{\mathrm{cts}}(G, \mathbb{Z}_p(k)) \longrightarrow \cdots \tag{2.2}$$

with connecting homomorphism δ^k (cf. [34, §2]). In particular, δ^k is trivial if, and only if, multiplication by p on $H^{k+1}_{\mathrm{cts}}(G, \mathbb{Z}_p(k))$ is a monomorphism. This is equivalent to $H^{k+1}_{\mathrm{cts}}(G, \mathbb{Z}_p(k))$ being torsion free. Therefore, one concludes the following:

[1] For a Poincaré duality group G the representation associated to the *dualizing module* — which coincides with the cyclotomic module in the case of a Poincaré duality pro-p group of dimension 2 (cf. Theorem 5.7) — is sometimes also called the "orientation" of G.

Proposition 2.1. *Let (G,θ) be a p-oriented profinite group. For $k \geq 1$ and $U \subseteq G$ an open subgroup the following are equivalent.*

(i) *The map* (1.2) *is surjective for every $n \geq 1$.*
(ii) *The map $\pi^k \colon H^k_{\mathrm{cts}}(U,\mathbb{Z}_p(k)) \to H^k(U,\mathbb{F}_p(k))$ is surjective.*
(iii) *The connecting homomorphism $\delta^k \colon H^k(U,\mathbb{F}_p(k)) \to H^{k+1}_{\mathrm{cts}}(U,\mathbb{Z}_p(k))$ is trivial.*
(iv) *The $\mathbb{Z}_p$-module $H^{k+1}_{\mathrm{cts}}(U,\mathbb{Z}_p(k))$ is torsion free.*

Proof. By the long exact sequence (2.2), the equivalences between (ii), (iii) and (iv) are straightforward. For $m \geq n \geq 1$ let $\pi^k_{m,n}$ denote the natural maps

$$\pi^k_{m,n} \colon H^k(U,\mathbb{Z}_p(k)/p^m) \longrightarrow H^k(U,\mathbb{Z}_p(k)/p^n)$$

(if $m=\infty$ we set $p^\infty = 0$). If condition (i) holds then the system

$$(H^k(U,\mathbb{Z}_p/p^n),\pi^k_{m,n})$$

satisfies the Mittag-Leffler property. In particular,

$$H^k(U,\mathbb{Z}_p(k)) \simeq \varprojlim_{n\geq 1} H^k(U,\mathbb{Z}_p(k)/p^n)$$

(cf. [28] and [23, Thm. 2.7.5]). Thus $\pi^k = \pi^k_{n,1} \circ \pi^k_{\infty,n}$ is surjective if, and only if, $\pi^k_{n,1}$ is surjective for every $n \geq 1$. Conversely, if π^k is surjective then $\pi^k = \pi^k_{n,1} \circ \pi^k_{\infty,n}$ yields the surjectivity of $\pi^k_{n,1}$ for every n. □

2.2. Profinite groups of cohomological p-dimension at most 1

Let G be a profinite group, and let $\theta \colon G \to \mathbb{Z}_p^\times$ be a p-orientation of G. Then

$$H^1_{\mathrm{cts}}(G,\mathbb{Z}_p(0)) = \mathrm{Hom}_{\mathrm{grp}}(G,\mathbb{Z}_p) \tag{2.3}$$

is a torsion free abelian group for every profinite group G, i.e., θ is 0-cyclotomic. If G is of cohomological p-dimension less or equal to 1, then $H^{m+1}_{\mathrm{cts}}(G,\mathbb{Z}_p(m)) = 0$ for all $m \geq 1$ showing that θ is cyclotomic. Moreover, $H^\bullet(G,\hat{\theta})$ is a quadratic $\mathbb{F}_p$-algebra for every profinite group with $\mathrm{cd}_p(G) \leq 1$ and for any p-orientation $\theta \colon G \to \mathbb{Z}_p^\times$. If G is of cohomological p-dimension less or equal to 1, one has $\mathrm{cd}_p(C) \leq 1$ for every closed subgroup C of G (cf. [31, §I.3.3, Proposition 14]). Thus one has the following.

Fact 2.2. *Let G be a profinite group with* $\mathrm{cd}_p(G) \leq 1$, *and let $\theta \colon G \to \mathbb{Z}_p^\times$ be a p-orientation for G. Then (G,θ) is Bloch-Kato and θ is cyclotomic.*

2.3. The m^{th}-norm residue symbol

Throughout this subsection we fix a field $\mathbb{K}$ and a separable closure $\bar{\mathbb{K}}$ of $\mathbb{K}$. For $p \neq \mathrm{char}(\mathbb{K})$, $\mu_{p^\infty}(\bar{\mathbb{K}})$ is a *divisible* abelian group. By construction, one has a canonical isomorphism

$$\varprojlim\nolimits_{k\geq 0}(\mu_{p^\infty}(\bar{\mathbb{K}}),p^k) \simeq \mathbb{Z}_p(1) \otimes_{\mathbb{Z}} \mathbb{Q}_p = \mathbb{Q}_p(1) \tag{2.4}$$

and a short exact sequence $0 \to \mathbb{Z}_p(1) \to \mathbb{Q}_p(1) \to \mu_{p^\infty}(\bar{\mathbb{K}}) \to 0$ of topological left $\mathbb{Z}_p[\![G_{\mathbb{K}}]\!]$-modules, where $G_{\mathbb{K}} = \mathrm{Gal}(\bar{\mathbb{K}}/\mathbb{K})$ is the absolute Galois group of $\mathbb{K}$.

Let $K_m^M(\mathbb{K})$, $m \geq 0$, denote the m^{th}-*Milnor K-group* of $\mathbb{K}$ (cf. [10, §24.3]). For $p \neq \mathrm{char}(\mathbb{K})$, J. Tate constructed in [34] a homomorphism of abelian groups

$$h_m(\mathbb{K})\colon K_m^M(\mathbb{K}) \longrightarrow H^m_{\mathrm{cts}}(G_{\mathbb{K}}, \mathbb{Z}_p(m)), \tag{2.5}$$

the so-called m^{th}-*norm residue symbol.* Let $K_m^M(\mathbb{K})_{/p} = K_m^M(\mathbb{K})/pK_m^M(\mathbb{K})$. Around ten years later S. Bloch and K. Kato conjectured in [1] that the induced map

$$h_m(\mathbb{K})_{/p}\colon K_m^M(\mathbb{K})_{/p} \longrightarrow H^m(G_{\mathbb{K}}, \mathbb{F}_p(m)) \tag{2.6}$$

is an isomorphism for all fields $\mathbb{K}$, $\mathrm{char}(\mathbb{K}) \neq p$, and for all $m \geq 0$. This conjecture has been proved by V. Voevodsky and M. Rost with a "patch" of C. Weibel (cf. [29, 36, 40]). In particular, since $K_\bullet^M(\mathbb{K})_{/p}$ is a quadratic $\mathbb{F}_p$-algebra and as $h_\bullet(\mathbb{K})_{/p}$ is a homomorphism of algebras, this implies that the absolute Galois group of a field $\mathbb{K}$ is Bloch-Kato (cf. [10, §23.4]). The Rost-Voevodsky Theorem has also the following consequence.

Proposition 2.3. *Let $\mathbb{K}$ be a field, let $G_{\mathbb{K}}$ denote its absolute Galois group, and let $\theta_{\mathbb{K},p}\colon G_{\mathbb{K}} \to \mathbb{Z}_p^\times$ denote its canonical p-orientation. Then $\theta_{\mathbb{K},p}$ is cyclotomic.*

Although Proposition 2.3 might be well known to specialists, we add a short proof of it. By Proposition 2.1, Proposition 2.3 in combination with Theorem 1.4-(d) is equivalent to [5, Prop. 14.19].

Proof of Proposition 2.3. If $\mathrm{char}(\mathbb{K}) = p$, then $\mathrm{cd}_p(G_{\mathbb{K}}) \leq 1$ (cf. [31, §II.2.2, Proposition 3]), and the p-orientation $\theta_{\mathbb{K},p}$ is cyclotomic by Fact 2.2. So we may assume that $\mathrm{char}(\mathbb{K}) \neq p$. In the commutative diagram

$$\begin{array}{ccccccc}
K_k^M(\mathbb{K}) & \xrightarrow{p} & K_k^M(\mathbb{K}) & \xrightarrow{\pi} & K_k^M(\mathbb{K})_{/p} & \longrightarrow & 0 \\
\downarrow{\scriptstyle h_k} & & \downarrow{\scriptstyle h_k} & & \downarrow{\scriptstyle (h_k)_{/p}} & & \\
H^k_{\mathrm{cts}}(G_{\mathbb{K}}, \mathbb{Z}_p(k)) & \xrightarrow{p} & H^k_{\mathrm{cts}}(G_{\mathbb{K}}, \mathbb{Z}_p(k)) & \xrightarrow{\alpha} & H^k(G_{\mathbb{K}}, \mathbb{F}_p(k)) & \xrightarrow{\beta} & H^{k+1}_{\mathrm{cts}}(G_{\mathbb{K}}, \mathbb{Z}_p(k))
\end{array} \tag{2.7}$$

the map π is surjective, and $(h_k)_{/p}$ is an isomorphism. Hence α must be surjective, and thus $\beta = 0$, i.e.,

$$p\colon H^{k+1}_{\mathrm{cts}}(G_{\mathbb{K}}, \mathbb{Z}_p(k)) \longrightarrow H^{k+1}_{\mathrm{cts}}(G_{\mathbb{K}}, \mathbb{Z}_p(k))$$

is an injective homomorphism of $\mathbb{Z}_p$-modules. Thus $H^{k+1}_{\mathrm{cts}}(G_{\mathbb{K}}, \mathbb{Z}_p(k))$ must be p-torsion free. Any open subgroup U of $G_{\mathbb{K}}$ is the absolute Galois group of $\bar{\mathbb{K}}^U$. Hence $\theta_{\mathbb{K},p}$ is cyclotomic, and this yields the claim. □

Remark 2.4. Let $\mathbb{K}$ be a number field, let S be a set of places containing all infinite places of $\mathbb{K}$ and all places lying above p, and let $G_{\mathbb{K}}^S$ be the Galois group of $\bar{\mathbb{K}}^S/\mathbb{K}$, where $\bar{\mathbb{K}}^S/\mathbb{K}$ is the maximal extension of $\bar{\mathbb{K}}/\mathbb{K}$ which is unramified outside S. Then $\theta_{\mathbb{K},p}\colon G_{\mathbb{K}} \to \mathbb{Z}_p^\times$ induces a p-orientation $\theta^S_{k,p}\colon G^S_{\mathbb{K}} \to \mathbb{Z}_p^\times$. However, it is well known (cf. [23, Prop. 8.3.11(ii)]) that,

$$H^1(G^S_{\mathbb{K}}, \mathbb{I}_p(1)) \simeq H^1(G^S_{\mathbb{K}}, \mathcal{O}^S_{\bar{\mathbb{K}}})_{(p)} \simeq \mathrm{cl}(\mathcal{O}^S_{\mathbb{K}})_{(p)} \tag{2.8}$$

(for the definition of $\mathbb{I}_p(1)$ see §3), where $\mathrm{cl}(\mathcal{O}_{\mathbb{K}}^S)$ denotes the *ideal class group* of the Dedekind domain $\mathcal{O}_{\mathbb{K}}^S$, and $\_{}_{(p)}$ denotes the p-primary component. Hence $(G_{\mathbb{K}}^S, \theta_{\mathbb{K},p}^S)$ is in general not cyclotomic (cf. Proposition 3.1).

3. Cohomology of p-oriented profinite groups

A homomorphism $\phi\colon (G_1, \theta_1) \to (G_2, \theta_2)$ of two p-oriented profinite groups (G_1, θ_1) and (G_2, θ_2) is a continuous group homomorphism $\phi\colon G_1 \to G_2$ satisfying $\theta_1 = \theta_2 \circ \phi$.

Let (G, θ) be a p-oriented profinite group. For $k \in \mathbb{Z}$, put $\mathbb{Q}_p(k) = \mathbb{Z}_p(k) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$, and also $\mathbb{I}_p(k) = \mathbb{Q}_p(k)/\mathbb{Z}_p(k)$, i.e., $\mathbb{I}_p(k)$ is a discrete left G-module and — as an abelian group — a divisible p-torsion module.

Let $\mathbb{I}_p = \mathbb{Q}_p/\mathbb{Z}_p$, and let $\_^* = \mathrm{Hom}_{\mathbb{Z}_p}(\_, \mathbb{I}_p)$ denote the Pontryagin duality functor. Then $\mathbb{I}_p(k)^*$ is a profinite left $\mathbb{Z}_p[\![G]\!]$-module which is isomorphic to $\mathbb{Z}_p(-k)$.

3.1. Criteria for cyclotomicity

The following proposition relates the continuous co-chain cohomology groups, Galois cohomology and the Galois homology groups as defined by A. Brumer in [3].

Proposition 3.1. *Let (G, θ) be a p-oriented profinite group, let k be an integer, and let m be a non-negative integer. Then the following are equivalent:*

(i) $H^{m+1}_{\mathrm{cts}}(G, \mathbb{Z}_p(k))$ *is torsion free;*
(ii) $H^m(G, \mathbb{I}_p(k))$ *is divisible;*
(iii) $H_m(G, \mathbb{Z}_p(-k))$ *is torsion free.*

Proof. The equivalence (i)⇔(ii) is a direct consequence of [34, Prop. 2.3], and (ii)⇔(iii) follows from [33, (3.4.5)]. □

The direct limit of divisible p-torsion modules is a divisible p-torsion module. From this fact — and Proposition 3.1 — one concludes the following.

Corollary 3.2. *Let (G, θ) be a cyclotomically p-oriented profinite group. Then $H^m(C, \mathbb{I}_p(m))$ is divisible for all $m \geq 0$ and all C closed in G.*

Proof. It suffices to show (ii)⇒(i). Let C be a closed subgroup of G. Then

$$H^m(C, \mathbb{I}_p(m)) \simeq \varinjlim_{U \in \mathfrak{B}_C} H^m(U, \mathbb{I}_p(m)),$$

where $\mathfrak{B}_C$ denotes the set of all open subgroups of G containing C (cf. [31, §I.2.2, Proposition 8]). Hence Proposition 3.1 yields the claim. □

In combination with [3, Corollary 4.3(ii)], Proposition 3.1 implies the following.

Corollary 3.3. *Let $(I, \preceq)$ be a directed set, let (G, θ) be a p-oriented profinite group, and let $(N_i)_{i\in I}$ be a family of closed normal subgroups of G satisfying $N_j \subseteq N_i \subseteq \ker(\theta)$ for $i \preceq j$ such that $\bigcap_{i\in I} N_i = \{1\}$ and the induced p-orientation $\theta_i\colon G/N_i \to \mathbb{Z}_p^\times$ is cyclotomic for all $i \in I$. Then $\theta\colon G \to \mathbb{Z}_p^\times$ is cyclotomic.*

Proof. Let $U \subseteq G$ be a open subgroup of G. Hypothesis (iii) implies that the group $H_m(UN_i/N_i, \mathbb{Z}_p(-m))$ is torsion free for all $i \in I$ (cf. Proposition 3.1). Thus, by [3, Corollary 4.3(ii)], $H_m(U, \mathbb{Z}_p(-m))$ is torsion free, and hence, by Proposition 3.1, $\theta\colon G \to \mathbb{Z}_p^\times$ is a cyclotomic p-orientation. □

3.2. The mod-p cohomology ring

An $\mathbb{N}_0$-graded $\mathbb{F}_p$-algebra $\mathbf{A} = \coprod_{k\geq 0} \mathbf{A}_k$ is said to be *anti-commutative* if for $x \in \mathbf{A}_s$ and $y \in \mathbf{A}_t$ one has $y \cdot x = (-1)^{st} \cdot x \cdot y$. E.g., if V is an $\mathbb{F}_p$-vector space, the *exterior algebra* $\Lambda_\bullet(V)$ (cf. [18, Chapter 4]) is an $\mathbb{N}_0$-graded anti-commutative $\mathbb{F}_p$-algebra. Moreover, if G is a profinite group, then its cohomology ring $H^\bullet(G, \mathbb{F}_p)$ is an $\mathbb{N}_0$-graded anti-commutative $\mathbb{F}_p$-algebra (cf. [23, Prop. 1.4.4]).

Let $\mathbf{T}(V) = \coprod_{k\geq 0} V^{\otimes k}$ denote the *tensor algebra* generated by the $\mathbb{F}_p$-vector space V. A $\mathbb{N}_0$-graded associative $\mathbb{F}_p$-algebra $\mathbf{A}$ is said to be *quadratic* if the canonical homomorphism $\eta^{\mathbf{A}}\colon \mathbf{T}(\mathbf{A}_1) \to \mathbf{A}$ is surjective, and

$$\ker(\eta^{\mathbf{A}}) = \mathbf{T}(\mathbf{A}_1) \otimes \ker(\eta_2^{\mathbf{A}}) \otimes \mathbf{T}(\mathbf{A}_1) \tag{3.1}$$

(cf. [24, § 1.2]). E.g., $\mathbf{A} = \Lambda_\bullet(V)$ is quadratic.

If $\mathbf{A}$ and $\mathbf{B}$ are anti-commutative $\mathbb{N}_0$-graded $\mathbb{F}_p$-algebras, then $\mathbf{A} \otimes \mathbf{B}$ is again an anti-commutative $\mathbb{N}_0$-graded $\mathbb{F}_p$-algebra, where

$$(x_1 \otimes y_1) \cdot (x_2 \otimes y_2) = (-1)^{s_2 t_1} \cdot (x_1 \cdot x_2) \otimes (y_1 \cdot y_2), \tag{3.2}$$

for $x_1 \in \mathbf{A}_{s_1}$, $x_2 \in \mathbf{A}_{s_2}$ $y_1 \in \mathbf{B}_{t_1}$, $y_2 \in \mathbf{B}_{t_2}$. In particular, if $\mathbf{A}$ and $\mathbf{B}$ are quadratic, then $\mathbf{A} \otimes \mathbf{B}$ is quadratic as well.

A direct set $(I, \preceq)$ maybe considered as a small category with objects given by the set I and precisely one morphism $\iota_{i,j}$ for all $i \preceq j$, $i, j \in I$, i.e., $\iota_{i,i} = \mathrm{id}_i$. One has the following.

Fact 3.4. *Let $\mathbb{F}$ be a field, let $(I, \preceq)$ be a direct system, and let $\mathbf{A}\colon (I, \preceq) \to {}_{\mathbb{F}}\mathbf{qalg}$ be a covariant functor with values in the category of quadratic $\mathbb{F}$-algebras. Then $\mathbf{B} = \varinjlim_{i\in\mathbf{A}} \mathbf{A}(i)$ is a quadratic $\mathbb{F}$-algebra.*

Let (G, θ) be a p-oriented profinite group, and let $\widehat{\theta}\colon G \to \mathbb{F}_p^\times$ be the map induced by θ. If $\widehat{\theta} = \mathbf{1}_G$, then the *mod-p cohomology ring* of $H^\bullet(G, \widehat{\theta})$ coincides with $H^\bullet(G, \mathbb{F}_p)$ (see (1.4)), and hence it is anti-commutative. Furthermore, if $\widehat{\theta} \neq \mathbf{1}_G$ and $G^\circ = \ker(\widehat{\theta})$, restriction

$$\mathrm{res}^\bullet_{G,G^\circ}\colon H^\bullet(G, \hat{\theta}) \longrightarrow H^\bullet(G^\circ, \mathbb{F}_p) \tag{3.3}$$

is an injective homomorphism of $\mathbb{N}_0$-graded algebras. Hence the mod-p cohomology ring $H^\bullet(G, \theta)$ is anti-commutative. In particular, if $M_{(k)}$ denotes the homogeneous component of the left $\mathbb{F}_p[G/G^\circ]$-module M, on which G/G° acts by $\widehat{\theta}^k$, the Hochschild-Serre spectral sequence (cf. [23, § II.4, Exercise 4(ii)]) shows that

$$H^k(G, \widehat{\theta}) = H^k(G^\circ, \mathbb{F}_p)_{(-k)}. \tag{3.4}$$

From [31, §I.2.2, Prop. 8] and Fact 3.4 one concludes the following.

Corollary 3.5. *Let (G, θ) be a p-oriented profinite group which is Bloch-Kato. Then $H^\bullet(C, \widehat{\theta}|_C)$ is quadratic for all C closed in G.*

Corollary 3.6. *Let $(I, \preceq)$ be a directed set, let (G, θ) be a p-oriented profinite group, and let $(N_i)_{i\in I}$ be a family of closed normal subgroups of G, $N_j \subseteq N_i \subseteq \ker(\theta)$ for $i \preceq j$, such that $\bigcap_{i\in I} N_i = \{1\}$ and $(G/N_i, \widehat{\theta}_{N_i})$ is Bloch-Kato. Then (G, θ) is Bloch-Kato.*

Remark 3.7. Let G be a pro-p group with minimal presentation

$$G = \langle\, x_1, \ldots, x_d \mid [x_1, x_2][[x_3, x_4], x_5] = 1 \,\rangle,$$

with $d \geq 5$. In [22, Ex. 7.3] and [21, § 4.3] it is shown that G does not occur as maximal pro-p Galois group of a field containing a primitive p^{th}-root of unity, relying on the properties of *Massey products*. It would be interesting to know whether G admits a cyclotomic p-orientation $\theta\colon G \to \mathbb{Z}_p^\times$ such that (G, θ) is Bloch-Kato. By Theorem 1.1, a negative answer would provide a "Massey-free" proof of the aforementioned fact.

3.3. Fibre products

Let (G_1, θ_1), (G_2, θ_2) be p-oriented profinite groups. The *fibre product* $(G, \theta) = (G_1, \theta_1) \boxtimes (G_2, \theta_2)$ denotes the pull-back of the diagram

$$\begin{array}{ccc} G_1 & \xrightarrow{\theta_1} & \mathbb{Z}_p^\times \\ \uparrow & \overset{\theta}{\nearrow} & \uparrow{\scriptstyle \theta_2} \\ G & \dashrightarrow & G_2 \end{array} \tag{3.5}$$

Remark 3.8. By restricting to the suitable subgroups if necessary, for the analysis of a fibre product $(G, \theta) = (G_1, \theta_1) \boxtimes (G_2, \theta_2)$ one may assume that $\mathrm{im}(\theta_1) = \mathrm{im}(\theta_2)$. In particular, if (G_2, θ_2) is split θ_2-abelian and $G_2 \simeq A \rtimes \mathrm{im}(\theta_2)$ for some free abelian pro-p group A, then $G \simeq A \rtimes G_1$ with $gag^{-1} = a^{\theta_1(g)}$ for all $a \in A$ and $g \in G_1$.

Fact 3.9. *Let (G, θ) be a p-oriented profinite group, and let N be a finitely generated non-trivial torsion free closed subgroup of $\mathrm{Z}_\theta(G)$, i.e., $N \simeq \mathbb{Z}_p(1)^r$ as left $\mathbb{Z}_p[\![G]\!]$-modules for some $r \geq 1$. Then for $k \geq 0$ one has*

$$H^1(N, \mathbb{I}_p(k)) \simeq \mathbb{I}_p(k-1)^r \tag{3.6}$$

as left $\mathbb{Z}_p[\![G]\!]$-module.

The following property will be useful for the analysis of fibre products.

Lemma 3.10. *Let (G_1, θ) be a cyclotomically p-oriented profinite group, and set $(G, \theta) = (G_1, \theta_1) \boxtimes (G_2, \theta_2)$, where (G_2, θ_2) is split θ_2-abelian with $Z = \mathrm{Z}_{\theta_2}(G_2)$. Let $\pi \colon G \to G_1$ be the canonical projection, and let $U \subseteq G$ be an open subgroup. Then $U \simeq (Z \cap U) \rtimes \pi(U)$.*

Proof. Without loss of generality we may assume that $Z \simeq \mathbb{Z}_p$, so that $Z \cap U = Z^{p^k}$ for some $k \geq 0$. It suffices to show that there exists an open subgroup U_1 of U satisfying $Z \cap U_1 = \{1\}$ and $\pi(U_1) = \pi(U)$.

By choosing a section $\sigma \colon G_1 \to G$ (see Remark 3.8), one has a continuous homomorphism $\tau = \sigma \circ \pi \colon G \to G_1$ and a continuous function $\eta \colon G \to Z$ such that each $g \in G$ can be uniquely written as $g = \eta(g) \cdot \tau(g)$. In particular, for $h, h_1, h_2 \in U$ and $z \in Z \cap U = Z^{p^k}$ one has

$$\eta(z \cdot h) = z \cdot \eta(h) \qquad \text{and} \qquad \eta(h_1 \cdot h_2) = \eta(h_1) \cdot {}^{h_1}\eta(h_2). \tag{3.7}$$

Let $\eta_U = \chi \circ \eta|_U$, where $\chi \colon Z \to Z/Z^{p^k}$ is the canonical projection. By (3.7), η_U defines a crossed-homomorphism $\tilde{\eta}_U \colon \bar{U} \to Z/Z^{p^k}$, where $\bar{U} = U/Z^{p^k}$. As $\bar{U}$ is canonically isomorphic to an open subgroup of G_1, $(\bar{U}, \theta_1|_{\bar{U}})$ is cyclotomically p-oriented. (Note that $Z \simeq \mathbb{Z}_p(1)$ as $\mathbb{Z}_p[\![U]\!]$-modules.) Hence, $H^1_{\mathrm{cts}}(\bar{U}, \mathbb{Z}_p(1)) \to H^1(\bar{U}, \mathbb{Z}_p(1)/p^k)$ is surjective by Proposition 2.1, and the snake lemma applied to the commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{B}^1(\bar{U}, Z) & \longrightarrow & \mathcal{Z}^1(\bar{U}, Z) & \longrightarrow & H^1(\bar{U}, \mathbb{Z}_p(1)) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \mathcal{B}^1(\bar{U}, Z/Z^{p^k}) & \longrightarrow & \mathcal{Z}^1(\bar{U}, Z/Z^{p^k}) & \longrightarrow & H^1(\bar{U}, \mathbb{Z}_p(1)/p^k) & \longrightarrow & 0 \end{array} \tag{3.8}$$

where the left-side and right-side vertical arrows are surjective, shows that $\mathcal{Z}^1(\bar{U}, Z) \to \mathcal{Z}^1(\bar{U}, Z/Z^{p^k})$ is surjective. Thus there exists $\eta_\circ \in \mathcal{Z}^1(\bar{U}, Z)$ such that $\tilde{\eta}_U = \chi \circ \eta_\circ$. It is straightforward to verify that $U_1 = \{\eta_\circ(\bar{h}) \cdot \sigma(\bar{h}) \mid \bar{h} \in \bar{U}\}$ is an open subgroup of G_1 satisfying the requirements. □

Theorem 3.11. *Let (G_1, θ_1) be a cyclotomically p-oriented profinite group, and let (G_2, θ_2) be split θ_2-abelian. Then $(G_1, \theta_1) \boxtimes (G_2, \theta_2)$ is cyclotomically p-oriented.*

Remark 3.12. (a) If p is odd, then every θ-abelian profinite group (G, θ) is split. However, a 2-oriented θ-abelian profinite group (G, θ) is split if, and only if, it is cyclotomically 2-oriented (cf. Proposition 6.7).
(b) If (G, θ) is θ-abelian and $H \subseteq G$ is a closed subgroup, then $(H, \theta|_H)$ is also θ-abelian.

Proof of Theorem 3.11. Put $(G, \theta) = (G_1, \theta_1) \boxtimes (G_2, \theta_2)$ and $Z = \mathrm{Z}_{\theta_2}(G_2)$. We may also assume that $\mathrm{im}(\theta_1) = \mathrm{im}(\theta_2)$. As (G_2, θ_2) is split θ_2-abelian, one has $G = Z \rtimes G_1$.

We first show the claim for $Z \simeq \mathbb{Z}_p$. Let U be an open subgroup of G. By Lemma 3.10, $(U, \theta|_U) \simeq (U_1, \bar{\theta}_1) \boxtimes (U_2, \bar{\theta}_2)$ where U_1 is isomorphic to an open subgroup of G_1 and $(U_2, \bar{\theta}_2)$ is split $\bar{\theta}_2$-abelian with $N = \ker(\bar{\theta}_2)$ open in Z. As $\mathrm{cd}_p(N) = 1$, one has $H^m(N, \mathbb{I}_p(k)) = 0$ for $m \geq 2$ and $k \geq 0$.

Therefore, the E_2-term of the Hochschild-Serre spectral sequence associated to the short exact sequence of profinite groups

$$\{1\} \longrightarrow N \longrightarrow U \longrightarrow U_1 \longrightarrow \{1\} \tag{3.9}$$

and evaluated on the discrete $\mathbb{Z}_p[\![U]\!]$-module $\mathbb{I}_p(k)$, is concentrated on the first and the second row. In particular, $d_r^{s,t} = 0$ for $r \geq 3$. As (3.9) splits, and as $\mathbb{I}_p(k)$ is inflated from U_1, one has $E_2^{s,0}(\mathbb{I}_p(k)) = E_\infty^{s,0}(\mathbb{I}_p(k))$ for $s \geq 0$ (cf. [23, Prop. 2.4.5]). Hence $d_2^{s,t} = 0$ for all $s, t \geq 0$, i.e., $E_2^{s,t}(\mathbb{I}_p(k)) = E_\infty^{s,t}(\mathbb{I}_p(k))$, and the spectral sequence collapses. Thus, using the isomorphism (3.6), for every $k \geq 1$ one has a short exact sequence

$$0 \longrightarrow H^k(U_1, \mathbb{I}_p(k)) \xrightarrow{\text{inf}} H^k(U, \mathbb{I}_p(k)) \longrightarrow H^{k-1}(U_1, \mathbb{I}_p(k-1)) \longrightarrow 0, \tag{3.10}$$

where the right- and left-hand side are divisible p-torsion modules. As such $\mathbb{Z}_p$-modules are injective, (3.10) splits showing that $H^k(U, \mathbb{I}_p(k))$ is p-divisible. Therefore, by Proposition 3.1, (G, θ) is cyclotomic.

Thus, by induction the claim holds for all split θ_2-abelian groups (G_2, θ_2) satisfying $\mathrm{rk}(\mathrm{Z}_{\theta_2}(G_2)) < \infty$. In general, as Z is a torsion free abelian pro-p group, there exists an inverse system $(Z_i)_{i\in I}$ of closed subgroups of Z such that Z/Z_i is torsion free, of finite rank, and $Z = \varprojlim_{i\in I} Z/Z_i$. Since Z_i is normal in G and

$$(G/Z_i, \bar{\theta}) \simeq (G_1, \theta_1) \boxtimes (G_2/Z_i, \bar{\theta}_2)$$

is cyclotomically p-oriented, Corollary 3.3 yields the claim. □

The following theorem can be seen as a generalization of a result of A. Wadsworth [37, Thm. 3.6].

Theorem 3.13. *Let (G_i, θ_i), $i = 1, 2$, be p-oriented profinite groups satisfying $\mathrm{im}(\theta_1) = \mathrm{im}(\theta_2)$. Assume further that (G_2, θ_2) is split θ_2-abelian. Then for $(G, \theta) = (G_1, \theta_1) \boxtimes (G_2, \theta_2)$ one has that*

$$H^\bullet(G, \widehat{\theta}) \simeq H^\bullet(G_1, \widehat{\theta}_1) \otimes \Lambda_\bullet\left((\ker(\theta_2)/\ker(\theta_2)^p)^*\right). \tag{3.11}$$

Moreover, if (G_1, θ_1) is Bloch-Kato, then (G, θ) is Bloch-Kato.

Proof. Assume first that $\mathrm{d}(\mathrm{Z}_{\theta_2}(G_2))$ is finite. If $\mathrm{d}(\mathrm{Z}_{\theta_2}(G_2)) = 1$ then one obtains the isomorphism (3.11) from [37, Thm. 3.1], which uses the Hochschild-Serre spectral sequence associated to the short exact sequence of profinite groups

$$\{1\} \longrightarrow \mathrm{Z}_{\theta_2}(G_2) \longrightarrow G \longrightarrow G/\mathrm{Z}_{\theta_2}(G_2) \longrightarrow \{1\}$$

and evaluated on the discrete $\mathbb{Z}_p[\![G]\!]$-module $\mathbb{F}_p(k)$, to compute $H^\bullet(G, \widehat{\theta})$. If $\mathrm{d}(\mathrm{Z}_{\theta_2}(G_2)) > 1$, then applying induction on $\mathrm{d}(\mathrm{Z}_{\theta_2}(G_2))$ yields the isomorphism (3.11). Finally, if $\mathrm{Z}_{\theta_2}(G_2)$ is not finitely generated, then a limit argument similar to the one used in the proof Theorem 3.11 and Corollary 3.6 yield the claim. □

3.4. Coproducts

For two profinite groups G_1 and G_2 let $G = G_1 \amalg G_2$ denote the *coproduct* (or free product) in the category of profinite groups (cf. [27, § 9.1]). In particular, if (G_1, θ_1) and (G_2, θ_2) are two p-oriented profinite groups, the p-orientations θ_1 and θ_2 induce a p-orientation $\theta\colon G \to \mathbb{Z}_p^\times$ via the universal property of of the free product. Thus, we may interpret $\amalg$ as the coproduct in the category of p-oriented profinite groups (cf. [9, §3]). The same applies to $\amalg^p$ — the coproduct in the category of pro-p groups.

Theorem 3.14. *Let (G_1, θ_1) and (G_2, θ_2) be two cyclotomically p-oriented profinite groups. Then their coproduct $(G, \theta) = (G_1, \theta_1) \amalg (G_2, \theta_2)$ is cyclotomically oriented. Moreover, if (G_1, θ_1) and (G_2, θ_2) are Bloch-Kato, then (G, θ) is Bloch-Kato.*

Proof. Let $(U, \theta|_U)$ be an open subgroup of (G, θ). Then, by the Kurosh subgroup theorem (cf. [27, Thm. 9.1.9]),

$$U \simeq \coprod_{s \in \mathcal{S}_1} ({}^sG_1 \cap U) \amalg \coprod_{t \in \mathcal{S}_2} ({}^tG_2 \cap U) \amalg F, \tag{3.12}$$

where ${}^yG_i = yG_iy^{-1}$ for $y \in G$. The sets $\mathcal{S}_1$ and $\mathcal{S}_2$ are sets of representatives of the double cosets $U \backslash G / G_1$ and $U \backslash G / G_2$, respectively. In particular, the sets $\mathcal{S}_1$ and $\mathcal{S}_2$ are finite, and F is a free profinite subgroup of finite rank.

Put $U_s = {}^sG_1 \cap U$ for all $s \in \mathcal{S}_1$, and $V_t = {}^tG_2 \cap U$ for all $t \in \mathcal{S}_2$. By [23, Thm. 4.1.4], one has an isomorphism

$$H^k(U, \mathbb{I}_p(k)) \simeq \bigoplus_{s \in \mathcal{S}_1} H^k(U_s, \mathbb{I}_p(k)) \oplus \bigoplus_{t \in \mathcal{S}_2} H^k(V_t, \mathbb{I}_p(k)), \tag{3.13}$$

for $k \geq 2$, and an exact sequence

$$M \xrightarrow{\alpha} H^1(U, \mathbb{I}_p(1)) \longrightarrow M' \longrightarrow 0. \tag{3.14}$$

If (G_1, θ_1) and (G_2, θ_2) are cyclotomically p-oriented, then, by hypothesis and (3.13), $H^k(U, \mathbb{I}_p(k))$ is a divisible p-torsion module for $k \geq 2$. In (3.14), the module M is a homomorphic image of a p-divisible p-torsion module, and the module M' is the direct sum of p-divisible p-torsion modules, showing that $H^1(U, \mathbb{I}_p(1))$ is divisible. Hence, by Proposition 3.1 and Corollary 3.3, (G, θ) is cyclotomically p-oriented.

Assume that (G_1, θ_1) and (G_2, θ_2) are Bloch-Kato. Then — for U as in (3.12) — one has by (3.13) and (3.14) that

$$H^\bullet(U, \widehat{\theta}|_U) \simeq \mathbf{A} \oplus \bigoplus_{s \in \mathcal{S}_1} H^\bullet(U_s, \widehat{\theta}|_{U_s}) \oplus \bigoplus_{t \in \mathcal{S}_2} H^\bullet(V_t, \widehat{\theta}|_{V_t}) \oplus H^\bullet(F, \widehat{\theta}|_F) \tag{3.15}$$

where $\mathbf{A}$ is a quadratic algebra, and $\oplus$ denotes the *direct sum* in the category of quadratic algebras (cf. [24, p. 55]). In particular, $H^\bullet(U, \widehat{\theta}|_U)$ is quadratic. □

For pro-p groups one has also the following.

Theorem 3.15. *Let (G_1, θ_1) and (G_2, θ_2) be two cyclotomically oriented pro-p groups. Then their coproduct $(G, \theta) = (G_1, \theta_1) \amalg^p (G_2, \theta_2)$ is cyclotomically oriented. Moreover, if (G_1, θ_1) and (G_2, θ_2) are Bloch-Kato, then (G, θ) is Bloch-Kato.*

Proof. The Kurosh subgroup theorem is also valid in the category of pro-p groups with $\amalg^p$ replacing $\amalg$ (cf. [27, Thm. 9.1.9]), and (3.13) and (3.14) hold also in this context (cf. [23, Thm. 4.1.4]). Hence the proof for cyclotomicity can be transferred verbatim. The Bloch-Kato property was already shown in [25, Thm. 5.2]. □

4. Oriented virtual pro-p groups

We say that a p-oriented profinite group (G, θ) is an *oriented virtual pro-p group* if $\ker(\theta)$ is a pro-p group. In particular, G is a virtual pro-p group. Since $\mathbb{Z}_2^\times$ is a pro-2 group, every oriented virtual pro-2 group is in fact a pro-2 group. For $p \neq 2$ let $\hat{\theta}\colon G \to \mathbb{F}_p{}^\times$ be the homomorphism induced by θ, and put $G^\circ = \ker(\hat{\theta})$. Then $G/G^\circ \simeq \operatorname{im}(\hat{\theta})$ is a finite cyclic group of order co-prime to p. The profinite version of the Schur-Zassenhaus theorem (cf. [14, Lemma 22.10.1]) implies that the short exact sequence of profinite groups

$$\{1\} \longrightarrow G^\circ \longrightarrow G \overset{\hat{\theta}}{\underset{\curvearrowleft\sigma}{\longrightarrow}} \operatorname{im}(\hat{\theta}) \longrightarrow \{1\} \tag{4.1}$$

splits. Indeed, if $C \subseteq G$ is a p'-Hall subgroup of G, then $\pi|_C\colon C \to \operatorname{im}(\hat{\theta})$ is an isomorphism, and $\sigma = (\pi|_C)^{-1}$ is a canonical section for $\hat{\theta}$.

Note that $\mathbb{Z}_p^\times = \mathbb{F}_p^\times \times \Xi_p$, where $\Xi_p = O_p(\mathbb{Z}_p^\times)$ is the pro-p Sylow subgroup of $\mathbb{Z}_p^\times$, and where we denoted by $\mathbb{F}_p^\times$ also the image of the Teichmüller section $\tau\colon \mathbb{F}_p^\times \to \mathbb{Z}_p^\times$. Hence a p-orientation $\theta\colon G \to \mathbb{Z}_p^\times$ on G defines a homomorphism $\hat{\theta}\colon G \to \mathbb{F}_p^\times$ and also a homomorphism $\theta^\vee : G \to \Xi_p$. On the contrary a pair of continuous homomorphisms $(\hat{\theta}, \theta^\vee)$, where $\hat{\theta}\colon G \to \mathbb{F}_p^\times$ and $\theta^\vee : G \to \Xi_p$, defines a p-orientation $\theta\colon G \to \mathbb{Z}_p^\times$ given by $\theta(g) = \hat{\theta}(g) \cdot \theta^\vee(g)$ for $g \in G$.

Fact 4.1. *Let $\hat{\theta}\colon G \to \mathbb{F}_p^\times$, $\sigma\colon \operatorname{im}(\hat{\theta}) \to G$ be homomorphisms of groups satisfying* (4.1)*. A homomorphism $\theta^\circ : G^\circ \to \Xi_p$ defines a p-orientation $\theta\colon G \to \mathbb{Z}_p^\times$, provided for all $c \in \operatorname{im}(\hat{\theta})$ and for all $g \in G^\circ$ one has*

$$\theta^\circ(\sigma(c) \cdot g \cdot \sigma(c)^{-1}) = \theta^\circ(g) \tag{4.2}$$

Proof. By (4.1), one has $G = G^\circ \rtimes_\beta \bar{\Sigma}$, where $\bar{\Sigma} = \operatorname{im}(\hat{\theta})$, $\beta\colon \bar{\Sigma} \to \operatorname{Aut}(G^\circ)$ and $\beta(c)$ is left conjugation by $\sigma(c)$ for $c \in \bar{\Sigma}$. Thus, by (4.2), the map $\theta^\vee : G \to \Xi_p$ given by $\theta^\vee(g, c) = \theta^\circ(g)$ is a continuous homomorphism of groups, and $(\iota, \theta^\vee)$, where $\iota\colon \bar{\Sigma} \to \mathbb{F}_p^\times$ is the canonical inclusion, defines a p-orientation of G. □

Let (G, θ) be an oriented virtual pro-p group satisfying (4.1). As $\theta\colon G \to \mathbb{Z}_p^\times$ is a homomorphism onto an abelian group one has

$$\theta(c \cdot g \cdot c^{-1}) = \theta(g) \tag{4.3}$$

for all $c \in C = \mathrm{im}(\sigma)$ and $g \in G$. Thus, if $i_c \in \mathrm{Aut}(G)$ denotes left conjugation by $c \in C$, one has

$$\theta = \theta \circ i_c \tag{4.4}$$

for all $c \in C$.

4.1. Oriented $\bar{\Sigma}$-virtual pro-p groups

From now on let p be odd, and fix a subgroup $\bar{\Sigma}$ of $\mathbb{F}_p^\times$. An oriented virtual pro-p group (G, θ) is said to be an oriented $\bar{\Sigma}$-virtual pro-p group, if $\mathrm{im}(\hat{\theta}) = \bar{\Sigma}$. Hence, by the previous subsection, for such a group one has a split short exact sequence

$$\{1\} \longrightarrow G^\circ \longrightarrow G \underset{\hat{\theta}}{\overset{\sigma}{\rightleftarrows}} \bar{\Sigma} \longrightarrow \{1\}\,. \tag{4.5}$$

By abuse of notation, we consider from now on (G, θ, σ) as an oriented $\bar{\Sigma}$-virtual pro-p group. As the following fact shows there is also *an alternative form* of a $\bar{\Sigma}$-virtual pro-p group.

Fact 4.2. *Let $\bar{\Sigma}$ be a subgroup of $\mathbb{F}_p^\times$. Let Q be a pro-p group, let $\theta^\circ\colon Q \to \Xi_p$ be a continuous homomorphism, and let $\gamma_Q\colon \bar{\Sigma} \to \mathrm{Aut}_c(Q)$ be a homomorphism of groups, where $\mathrm{Aut}_c(\_)$ is the group of continuous automorphisms, satisfying*

$$\theta^\circ(\gamma_Q(c)(q)) = \theta^\circ(q), \tag{4.6}$$

for all $q \in Q$ and $c \in \bar{\Sigma}$, then $(Q \rtimes_{\gamma_Q} \bar{\Sigma}, \theta, \iota)$ is an oriented $\bar{\Sigma}$-virtual pro-p group, where $\iota\colon \bar{\Sigma} \to Q \rtimes_{\gamma_Q} \bar{\Sigma}$ is the canonical map, and $\theta\colon Q \rtimes_{\gamma_Q} \bar{\Sigma} \to \mathbb{Z}_p^\times$ is the homomorphism induced by θ° (cf. Fact 4.1).

If $(G_1, \theta_1, \sigma_1)$ and $(G_2, \theta_2, \sigma_2)$ are oriented $\bar{\Sigma}$-virtual pro-p groups, a continuous group homomorphism $\phi\colon G_1 \to G_2$ is said to be a morphism of $\bar{\Sigma}$-virtual pro-p groups, if $\sigma_2 = \phi \circ \sigma_1$ and $\theta_1 = \theta_2 \circ \phi$. Similarly, if $(Q, \theta_Q^\circ, \gamma_Q)$ and $(R, \theta_R^\circ, \gamma_R)$ are $\bar{\Sigma}$-virtual pro-p groups in alternative form (cf. Fact 4.2), the continuous group homomorphism $\phi\colon Q \to R$ is a homomorphims of $\bar{\Sigma}$-virtual pro-p groups provided $\theta_R \circ \phi = \theta_Q$ and if for all $c \in \bar{\Sigma}$ and for all $q \in Q$ one has that

$$\gamma_R(c)(\phi(q)) = \phi(\gamma_Q(c)(q)). \tag{4.7}$$

With this slightly more sophisticated set-up the category of $\bar{\Sigma}$-virtual pro-p groups admits coproducts. In more detail, let $(Q, \theta_Q^\circ, \gamma_Q)$ and $(R, \theta_R^\circ, \gamma_R)$ be $\bar{\Sigma}$-virtual pro-p groups in alternative form. Put $X = Q \amalg^{\mathrm{p}} R$. Then for every element $c \in \bar{\Sigma}$ there exists an element $\delta(c) \in \mathrm{Aut}(X)$ making the diagram

$$\begin{array}{ccccc} Q & \xrightarrow{\iota_1} & X & \xleftarrow{\iota_2} & R \\ {\scriptstyle\gamma_Q(c)}\downarrow & & \downarrow{\scriptstyle\delta(c)} & & \downarrow{\scriptstyle\gamma_R(c)} \\ Q & \xrightarrow{\iota_1} & X & \xleftarrow{\iota_2} & R \end{array} \tag{4.8}$$

commute. Since Ξ_p is a pro-p group, there exists a continuous group homomorphism $\theta^\circ\colon X \to \Xi_p$ making the lower two rows of the diagram

$$\begin{array}{ccccc} Q & \xrightarrow{j_Q} & X & \xleftarrow{j_R} & R \\ {\scriptstyle\gamma_Q(c)}\downarrow & & \downarrow{\scriptstyle\delta(c)} & & \downarrow{\scriptstyle\gamma_R(c)} \\ Q & \xrightarrow{j_Q} & X & \xleftarrow{j_R} & R \\ & {\scriptstyle\theta^\circ_Q}\searrow & \downarrow{\scriptstyle\theta^\circ} & \swarrow{\scriptstyle\theta^\circ_R} & \\ & & \Xi_p & & \end{array} \tag{4.9}$$

commute. Since $\theta^\circ_{Q/R} = \theta^\circ_{Q/R} \circ \gamma_{Q/R}(c)$ for all $c \in \bar{\Sigma}$, one has $\theta^\circ = \theta^\circ \circ \delta(c)$ for all $c \in \bar{\Sigma}$. The commutativity of the diagram (4.9) yields that the group homomorphisms $j_Q\colon (Q, \theta^\circ_Q, \gamma_Q) \to (X, \theta^\circ, \delta)$ and $j_R\colon (R, \theta^\circ_R, \gamma_R) \to (X, \theta^\circ, \delta)$ are homomorphisms of oriented $\bar{\Sigma}$-virtual pro-p groups in alternative form. Moreover, one has the following.

Proposition 4.3. *The oriented $\bar{\Sigma}$-virtual pro-p group $(X, \theta^\circ, \delta)$ together with the homomorphisms $j_Q\colon Q \to X$, and $j_R\colon R \to X$ is a coproduct in the category of oriented $\bar{\Sigma}$-virtual pro-p groups.*

Proof. Let (H, θ_H, γ_H) be an oriented $\bar{\Sigma}$-virtual pro-p group in alternative form, and let $\phi_Q\colon Q \to H$ and $\phi_R\colon R \to H$ be homomorphisms of oriented $\bar{\Sigma}$-virtual pro-p groups in alternative form. Then there exists a unique homomorphism of pro-p groups $\phi\colon X \to H$ making the diagram concentrated on the second and third row of

$$\begin{array}{ccccc} Q & \xrightarrow{j_Q} & X & \xleftarrow{j_R} & R \\ {\scriptstyle\gamma_Q(c)}\downarrow & & \downarrow{\scriptstyle\delta(c)} & & \downarrow{\scriptstyle\gamma_R(c)} \\ Q & \xrightarrow{j_Q} & X & \xleftarrow{j_R} & R \\ & {\scriptstyle\phi_Q}\searrow & \downarrow{\scriptstyle\phi} & \swarrow{\scriptstyle\phi_R} & \\ {\scriptstyle\theta^\circ_Q} & & H & & {\scriptstyle\theta^\circ_R} \\ & & \downarrow{\scriptstyle\theta^\circ_H} & & \\ & & \Xi_p & & \end{array} \tag{4.10}$$

commute. Since $\phi_{Q/R} \circ \gamma_{Q/R}(c) = \gamma_H(c) \circ \phi_{Q/R}$ for all $c \in \bar{\Sigma}$, the uniqueness of ϕ implies that $\phi \circ \delta(c) = \gamma_H(c) \circ \phi$ for all $c \in \bar{\Sigma}$. As $\phi_Q\colon Q \to H$ and $\phi_R\colon R \to H$ are homomorphisms of $\bar{\Sigma}$-virtual pro-p groups, one has that

$\theta^\circ_{Q/R} = \theta^\circ_H \circ \phi_{Q/R}$. This implies that $(\theta^\circ_H \circ \phi) \circ j_{Q/R} = \theta^\circ_{Q/R}$, and from the construction of $\theta^\circ \colon X \to \Xi_p$ one concludes that $\theta^\circ = \theta^\circ_H \circ \phi$. This implies that ϕ is a homomorphism of oriented $\bar{\Sigma}$-virtual pro-p groups. □

Example 4.4. For $p = 3$ set $\bar{\Sigma} = \mathbb{F}_3^\times = \{1, s\}$. Then the free product $(\mathbb{Z}_3^\times, \mathrm{id}) \amalg^{\bar{\Sigma}} (\mathbb{Z}_3^\times, \mathrm{id})$ is isomorphic to $F \rtimes \bar{\Sigma}$, where $F = \langle x, y \rangle$ is a free pro-3 group of rank 2 and the induced isomorphism $s \colon F \to F$ satisfies $s(x) = x^{-1}$, $s(y) = y^{-1}$.

Proposition 4.5. *Let (Q, θ_Q, γ_Q) be an oriented $\bar{\Sigma}$-virtual pro-p group, and let Z be a normal $\bar{\Sigma}$-invariant subgroup of Q isomorphic to $\mathbb{Z}_p$, which is not contained in the Frattini subgroup $\Phi(Q) = \mathrm{cl}([Q,Q]Q^p)$ of Q. Then there exists a maximal closed subgroup M of Q which is $\bar{\Sigma}$-invariant, such that $M \cdot \mathrm{Z} = Q$ and $M \cap \mathrm{Z} = \mathrm{Z}^p$.*

Proof. Let $\bar{Q} = Q/\Phi(Q)$. Then γ_Q induces a homomorphism $\bar{\gamma}_{\bar{Q}} \colon \bar{\Sigma} \to \mathrm{Aut}_c(\bar{Q})$ making $\bar{Q}$ a compact $\mathbb{F}_p[\bar{\Sigma}]$-module. Let $\Omega = \mathrm{Hom}^c_{\bar{\Sigma}}(\bar{Q}, \mathbb{F}_p)$, where $\mathbb{F}_p$ denotes the finite field $\mathbb{F}_p$ with canonical left $\bar{\Sigma}$-action. By Pontryagin duality, one has $\bigcap_{\omega \in \Omega} \ker(\omega) = \{0\}$. Thus, by hypothesis, there exists $\psi \in \Omega$ such that $\psi|_{\mathrm{Z}} \neq 0$. Hence $M = \ker(\psi)$ has the desired properties. □

4.2. The maximal oriented virtual pro-p quotient

For a prime p and a profinite group G we denote by $O^p(G)$ the closed subgroup of G generated by all Sylow pro-ℓ subgroups of G, $\ell \neq p$. In particular, $O^p(G)$ is *p-perfect*, i.e., $H^1(O^p(G), \mathbb{F}_p) = 0$, and one has the short exact sequence

$$\{1\} \longrightarrow O^p(G) \longrightarrow G \longrightarrow G(p) \longrightarrow \{1\}\ ,$$

where $G(p)$ denotes the *maximal pro-p quotient* of G.

For a p-oriented profinite group (G, θ), we denote by

$$G(\theta) = G/O^p(G^\circ)$$

the *maximal p-oriented virtual pro-p quotient* of G (for the definition of G° see the beginning of § 4). By construction, it carries naturally a p-orientation $\theta \colon G(\theta) \to \mathbb{Z}_p^\times$ inherited by G.

Note that if $\mathrm{im}(\theta)$ is a pro-p group, then $G^\circ = G$, and $G(\theta) = G(p)$.

Proposition 4.6. *Let (G, θ) be a p-oriented Bloch-Kato profinite group, and let $O \subseteq G$ be a p-perfect subgroup such that $O \subseteq \ker(\theta)$. Then the inflation map*

$$\mathrm{inf}^k(M) \colon H^k_{\mathrm{cts}}(G/O, M) \longrightarrow H^k_{\mathrm{cts}}(G, M), \tag{4.11}$$

is an isomorphism for all $k \geq 0$ and all $M \in \mathrm{ob}({}_{\mathbb{Z}_p[\![G/O]\!]}\mathbf{prf})$, where ${}_{\mathbb{Z}_p[\![G/O]\!]}\mathbf{prf}$ denotes the abelian category of profinite left $\mathbb{Z}_p[\![G/O]\!]$-modules.

Proof. As $O \subseteq \ker(\theta)$, $\mathbb{Z}_p(k)$ is a trivial $\mathbb{Z}_p[\![O]\!]$-module for every $k \in \mathbb{Z}$. Since O is p-perfect, and as the $\mathbb{F}_p$-algebra $H^\bullet(O, \mathbb{F}_p)$ is quadratic, $H^\bullet(O, \mathbb{F}_p)$ is 1-dimensional concentrated in degree 0. By Pontryagin duality, this is equivalent to $H_k(O, \mathbb{F}_p) = 0$ for all $k > 0$, where $H_k(O, \_)$ denotes Galois homology as defined by A. Brumer in [3]. Thus, the long exact sequence in Galois homology implies that $H_k(O,\mathbb{Z}_p)=0$ for all $k>0$.

Let $(P_\bullet,\partial_\bullet,\varepsilon)$ be a projective resolution of the trivial left $\mathbb{Z}_p[\![G]\!]$-module in the category ${}_{\mathbb{Z}_p[\![G]\!]}\mathbf{prf}$. For a projective left $\mathbb{Z}_p[\![G]\!]$-module $P\in\mathrm{ob}({}_{\mathbb{Z}_p[\![G]\!]}\mathbf{prf})$ define

$$\mathrm{def}(P)=\mathrm{def}^G_{G/O}(P)=\mathbb{Z}_p[\![G/O]\!]\,\widehat{\otimes}_G\,P, \tag{4.12}$$

where $\widehat{\otimes}$ denotes the completed tensor product as defined in [3]. Then, by the Eckmann-Shapiro lemma in homology, one has that

$$H_k(\mathrm{def}(P_\bullet),\mathrm{def}(\partial_\bullet))\simeq H_k(O,\mathbb{Z}_p). \tag{4.13}$$

Hence, by the previously mentioned remark, $(\mathrm{def}(P_\bullet),\mathrm{def}(\partial_\bullet))$ is a projective resolution of $\mathbb{Z}_p$ in the category ${}_{\mathbb{Z}_p[\![G/O]\!]}\mathbf{prf}$.

Let $M\in\mathrm{ob}({}_{\mathbb{Z}_p[\![G/O]\!]}\mathbf{prf})$. Then for every projective profinite left $\mathbb{Z}_p[\![G]\!]$-module P, one has a natural isomorphism

$$\mathrm{Hom}_{G/O}(\mathrm{def}(P),M)\simeq\mathrm{Hom}_G(P,M). \tag{4.14}$$

Hence $\mathrm{Hom}_{G/O}(\mathrm{def}(P_\bullet),M)$ and $\mathrm{Hom}_G(P_\bullet,M)$ are isomorphic co-chain complexes, and the induced maps in cohomology — which coincide with $\mathrm{inf}^\bullet(M)$ — are isomorphisms. □

Corollary 4.7. *Let (G,θ) be a p-oriented profinite group which is Bloch-Kato, respectively cyclotomically oriented. Then the maximal oriented virtual pro-p quotient $(G(\theta),\theta)$ is Bloch-Kato, respectively cyclotomically oriented.*

5. Profinite Poincaré duality groups and p-orientations

5.1. Profinite Poincaré duality groups

Let G be a profinite group, and let p be a prime number. Then G is called a *p-Poincaré duality group* of dimension d, if

- (PD_1) $\mathrm{cd}_p(G)=d$;
- (PD_2) $|H^k_{\mathrm{cts}}(G,A)|<\infty$ for every finite discrete left G-module A of p-power order;
- (PD_3) $H^k_{\mathrm{cts}}(G,\mathbb{Z}_p[\![G]\!])=0$ for $k\neq d$, and $H^d_{\mathrm{cts}}(G,\mathbb{Z}_p[\![G]\!])\simeq\mathbb{Z}_p$.

Although quite different at first glance, for a pro-p group our definition of p-Poincaré duality coincides with the definition given by J-P. Serre in [31, §I.4.5]. However, some authors prefer to omit the condition (PD_2) in the definition of a p-Poincaré duality group (cf. [23, Chap. III, §7, Definition 3.7.1]).

For a profinite p-Poincaré duality group G of dimension d the profinite right $\mathbb{Z}_p[\![G]\!]$-module $D_G=H^d_{\mathrm{cts}}(G,\mathbb{Z}_p[\![G]\!])$ is called the *dualizing module*. Since D_G is isomorphic to $\mathbb{Z}_p$ as a pro-p group, there exists a unique p-orientation $\eth_G\colon G\to\mathbb{Z}_p^\times$ such that for $g\in G$ and $z\in D_G$ one has

$$z\cdot g=z\cdot\eth_G(g)=\eth_G(g)\cdot z.$$

We call $\eth_G$ the *dualizing p-orientation*.

Let ${}^{\times}D_G$ denote the associated profinite left $\mathbb{Z}_p[\![G]\!]$-module, i.e., setwise ${}^{\times}D_G$ coincides with D_G and for $g \in G$ and $z \in {}^{\times}D_G$ one has

$$g \cdot z = z \cdot g^{-1} = \eth_G(g^{-1}) \cdot z.$$

For a profinite p-Poincaré duality group of dimension d the usual standard arguments (cf. [2, §VIII.10] for the discrete case) provide natural isomorphisms

$$\begin{aligned} \operatorname{Tor}_k^G(D_G, \_) &\simeq H^{d-k}_{\mathrm{cts}}(G, \_), \\ \operatorname{Ext}^k_G({}^{\times}D_G, \_) &\simeq H_{d-k}(G, \_), \end{aligned} \tag{5.1}$$

where $\operatorname{Tor}^G_{\bullet}(\_,\_)$ denotes the left derived functor of $\_\hat{\otimes}_G\_$, and $\operatorname{Ext}^{\bullet}_G(\_,\_)$ denotes the right derived functors of $\operatorname{Hom}_G(\_,\_)$ in the category ${}_{\mathbb{Z}_p[\![G]\!]}\mathbf{prf}$ (cf. [3]).

If A is a discrete left G-module which is also a p-torsion module, then A^* carries naturally the structure of a left (profinite) $\mathbb{Z}_p[\![G]\!]$-module (cf. [27, p. 171]). Then, by [31, § I.3.5, Proposition 17], Pontryagin duality and [33, (3.4.5)], one obtains for every finite discrete left $\mathbb{Z}_p[\![G]\!]$-module A of p-power order that

$$H^d_{\mathrm{cts}}(G, A) \simeq \operatorname{Hom}_G(A, I_G)^* \simeq \operatorname{Hom}_G(I_G^*, A^*)^* \simeq (I_G^*)^{\times} \hat{\otimes}_G A, \tag{5.2}$$

where I_G denotes the discrete left dualizing module of G (cf. [31, §I.3.5]). In particular, by (5.1), $D_G \simeq (I_G^*)^{\times}$.

Example 5.1. Let $G_{\mathbb{K}}$ be the absolute Galois group of an ℓ-adic field $\mathbb{K}$. Then $G_{\mathbb{K}}$ satisfies p-Poincaré duality of dimension 2 for all prime numbers p. One has $I_G \simeq \mu_{p^\infty}(\bar{\mathbb{K}})$ (cf. [31, §II.5.2, Theorem 1]). Hence ${}^{\times}D_{G_{\mathbb{K}}} \simeq \mathbb{Z}_p(-1)$ with respect to the cyclotomic p-orientation $\theta_{\mathbb{K},p}\colon G_{\mathbb{K}} \to \mathbb{Z}_p^{\times}$, i.e., $\eth_{G_{\mathbb{K}}} = \theta_{\mathbb{K},p}$.

As we will see in the next proposition, the final conclusion in Example 5.1 is a consequence of a general property of Poincaré duality groups.

Proposition 5.2. *Let G be a p-Poincaré duality group of dimension d, and let $\theta\colon G \to \mathbb{Z}_p^{\times}$ be a cyclotomic p-orientation of G. Then $\theta^{d-1} = \eth_G$ and ${}^{\times}D_G \simeq \mathbb{Z}_p(1-d)$.*

Proof. By (5.1) and the hypothesis, $H^d_{\mathrm{cts}}(G, \mathbb{Z}_p(d-1)) \simeq D_G \hat{\otimes} \mathbb{Z}_p(d-1)$ is torsion free, and hence isomorphic to $\mathbb{Z}_p$. This implies $\eth_G = \theta^{d-1}$. □

5.2. Finitely generated θ-abelian pro-p groups

Recall that (G,θ) is said to be θ-abelian if $\ker(\theta) = \mathrm{Z}_\theta(G)$ and $\mathrm{Z}_\theta(G)$ is p-torsion free — in particular $\ker(\theta)$ is an abelian pro-p group. If G is finitely generated then one has an isomorphism of left $\mathbb{Z}_p[\![G]\!]$-modules $N \simeq \mathbb{Z}_p(1)^r$ for some non-negative integer r, and either $\Gamma = \operatorname{im}(\theta)$ is a finite group of order coprime to p, or Γ is a p-Poincaré duality group of dimension 1 satisfying $\eth_\Gamma = \mathbf{1}_\Gamma$ (cf. [23, Prop. 3.7.6]). Moreover, one has isomorphisms of left $\mathbb{Z}_p[\![G]\!]$-modules

$$H_k(N, \mathbb{Z}_p) \simeq \Lambda_k(N) \simeq \mathbb{Z}_p(k)^{\binom{r}{k}}, \tag{5.3}$$

where $\Lambda_\bullet(\_)$ denotes the exterior algebra over the ring $\mathbb{Z}_p$. Since $\mathrm{cd}_p(\Gamma) \leq 1$, the Hochschild-Serre spectral sequence for homology (cf. [39, § 6.8])

$$E^2_{s,t} = H_s(\Gamma, H_t(N, \mathbb{Z}_p(-m))) \implies H_{s+t}(G, \mathbb{Z}_p(-m)) \tag{5.4}$$

is concentrated in the first two columns. Hence, the spectral sequence collapses at the E^2-term, i.e., $E^2_{s,t} = E^\infty_{s,t}$. Thus, for $n \geq 1$ one has a short exact sequence

$$0 \longrightarrow H_{n-1}(N, \mathbb{Z}_p(-m))^\Gamma \longrightarrow H_n(G, \mathbb{Z}_p(-m)) \longrightarrow H_n(N, \mathbb{Z}_p(-m))_\Gamma \longrightarrow 0 \tag{5.5}$$

if $\mathrm{cd}_p(\Gamma) = 1$, and isomorphisms

$$H^n(G, \mathbb{Z}_p(-m)) \simeq H_n(N, \mathbb{Z}_p(-m))_\Gamma \tag{5.6}$$

if Γ is a finite group of order coprime p. Here we used the fact that $H_0(\Gamma, \_) = \_{}_\Gamma$ coincides with the coinvariants of Γ, and that $H_1(\Gamma, \_) = \_{}^\Gamma$ coincides with the invariants of Γ if Γ is a p-Poincaré duality group of dimension 1 with $\eth_\Gamma = \mathbf{1}_\Gamma$. Since $H_{m-1}(N, \mathbb{Z}_p(-m))^\Gamma$ is a torsion free abelian pro-p group, and as

$$H_m(N, \mathbb{Z}_p(-m))_\Gamma = (H_m(N, \mathbb{Z}_p) \otimes \mathbb{Z}_p(-m))_\Gamma \simeq \Lambda_m(N) \tag{5.7}$$

by (5.3), one concludes from (5.5) and (5.6) that $H_m(G, \mathbb{Z}_p(-m))$ is torsion free.

Proposition 5.3. *Let (G, θ) be a θ-abelian p-oriented virtual pro-p group such that $N = \ker(\theta)$ is a finitely generated torsion free abelian pro-p group, and that $\Gamma = \mathrm{im}(\theta)$ is p-torsion free. Then G is a p-Poincaré duality group of dimension $d = \mathrm{cd}(G)$, and θ is cyclotomic.*

Proof. By hypothesis, G is a p-torsion free p-adic analytic group. Hence the former assertion is a direct consequence of M. Lazard's theorem (cf. [33, Thm. 5.1.5]). The latter follows from Proposition 3.1. □

From Proposition 5.2 one concludes the following:

Corollary 5.4. *Let (G, θ) be a θ-abelian pro-p group. If $p = 2$ assume further that* $\mathrm{im}(\theta)$ *is torsion free.*

(a) *The orientation θ is cyclotomic.*
(b) *Suppose that G is finitely generated with minimun number of generators $d = d(G) < \infty$. If $p = 2$ assume further that* $\mathrm{im}(\theta) \subseteq 1 + 4\mathbb{Z}_2$. *Then G is a Poincaré duality pro-p group of dimension d. Moreover, $\eth_G = \theta^{d-1}$.*
(c) *If G satisfies the hypothesis of (b) and $d(G) \geq 2$, then for p odd, any cyclotomic orientation $\theta' \colon G \to \mathbb{Z}_p^\times$ of G must coincide with θ, i.e., $\theta' = \theta$. For $p = 2$ any cyclotomic orientation $\theta' \colon G \to \mathbb{Z}_2^\times$ satisfying* $\mathrm{im}(\theta') \subseteq 1 + 4\mathbb{Z}_2$ *must coincide with θ.*

Proof. (a) follows from Proposition 5.3.
(b) By hypothesis, G is uniformly powerful (cf. [6, Ch. 4]), or equi-p-value, as it is called in [17]. Hence the claim follows from Proposition 5.3. By Proposition 5.2, $\eth_G = \theta^{d-1}$.

(c) An element $\phi \in \mathrm{Hom}_{\mathrm{grp}}(G, \mathbb{Z}_p^\times)$ has finite order if, and only if, $\mathrm{im}(\phi)$ is finite. Proposition 5.2 and part (b) imply that

$$\theta^{d-1} = \eth_G = (\theta')^{d-1}.$$

Hence $(\theta^{-1}\theta')^{d-1} = \mathbf{1}_G$. For p odd, $\mathrm{Hom}_{\mathrm{grp}}(G, \mathbb{Z}_p^\times)$ does not contain non-trivial elements of finite order. Hence $\theta' = \theta$. For $p = 2$ the hypothesis implies that $\mathrm{im}(\theta^{-1}\theta') \subseteq 1 + 4\mathbb{Z}_2$. Hence $(\theta^{-1}\theta')^{d-1} = \mathbf{1}_G$ implies that $\theta' = \theta$. □

Note that, by Fact 2.2, Corollary 5.4(c) cannot hold if $d(G) = 1$.

5.3. Profinite p-Poincaré duality groups of dimension 2

As the following theorem shows, for a profinite p-Poincaré duality group G of dimension 2, the dualizing p-orientation $\eth_G : G \to \mathbb{Z}_p^\times$ is always cyclotomic.

Theorem 5.5. *Let G be a profinite p-Poincaré duality group of dimension 2. Then $\eth_G : G \to \mathbb{Z}_p^\times$ is a cyclotomic p-orientation.*

Proof. As every p-oriented profinite group is 0-cyclotomic, it suffices to show that $H^2_{\mathrm{cts}}(U, \mathbb{Z}_p(1))$ is torsion free for every open subgroup $U \subseteq G$. By Proposition 5.2, $\mathbb{Z}_p(-1) \simeq {}^\times D_G$. Hence, from the Eckmann-Shapiro lemma in homology and (5.1), one concludes that

$$\begin{aligned} H_1(U, \mathbb{Z}_p(-1)) &= \mathrm{Tor}_1^U(\mathbb{Z}_p, \mathbb{Z}_p(-1)) \simeq \mathrm{Tor}_1^U(\mathbb{Z}_p(-1)^\times, \mathbb{Z}_p) \\ &\simeq \mathrm{Tor}_1^G(D_G, \mathbb{Z}_p[\![G/U]\!]) \simeq H^1_{\mathrm{cts}}(G, \mathbb{Z}_p[\![G/U]\!]) \\ &\simeq \mathrm{Hom}_{\mathrm{grp}}(U, \mathbb{Z}_p). \end{aligned} \tag{5.8}$$

Hence $H_1(U, \mathbb{Z}_p(-1))$ is a torsion free $\mathbb{Z}_p$-module, and, by Proposition 3.1, $H^2_{\mathrm{cts}}(U, \mathbb{Z}_p(1))$ is torsion free as well. □

Remark 5.6. Let G be a profinite p-Poincaré duality group of dimension 2, and let $\eth_G : G \to \mathbb{Z}_p^\times$ be the dualizing p-orientation. Then $(G, \eth_G)$ is not necessarily Bloch-Kato, as the following example shows.

Let $p = 2$ and let $A = \mathrm{PSL}_2(q)$ where $q \equiv 3 \mod 4$. Then there exists a p-Frattini extension $\pi : G \to A$ of A such that G is a 2-Poincaré duality group of dimension 2, i.e., $\ker(\pi)$ is a pro-2 group contained in the Frattini subgroup of G (cf. [41]). In particular, G is perfect, and thus $\eth_G = \mathbf{1}_G$. Hence $\mathbb{F}_2(1) = \mathbb{F}_2(0)$ is the trivial $\mathbb{F}_2[\![G]\!]$-module, and — as G is perfect — $H^1(G, \mathbb{F}_2(1)) = 0$. Moreover, $H^2(G, \mathbb{F}_2(2)) \simeq \mathbb{F}_2$, as G is a profinite 2-Poincaré duality group of dimension 2 with $\eth_G = \mathbf{1}_G$. Therefore, $H^\bullet(G, \mathbf{1}_G)$ is not quadratic.

A pro-p group G which satisfies p-Poincaré duality in dimension 2 is also called a *Demuškin group* (cf. [23, Def. 3.9.9]). For this class of groups one has the following.

Corollary 5.7. *Let G be a Demuškin pro-p group. Then G is a Bloch-Kato pro-p group, and $\eth_G : G \to \mathbb{Z}_p^\times$ is a cyclotomic p-orientation.*

Proof. By Theorem 5.5, it suffices to show that $(G, \eth_G)$ is Bloch-Kato. It is well known that $H^\bullet(G, \hat{\eth}_G)$ is quadratic (cf. [31, §I.4.5]). Moreover, every open subgroup U of G is again a Demuškin group, with $\eth_U = \eth_G|_U$ (cf. [23, Thm. 3.9.15]). Hence $(G, \eth_G)$ is Bloch-Kato. □

Remark 5.8. [The Klein bottle pro-2 group] Let G be the pro-2 group given by the presentation

$$G = \langle\, x, y \mid xyx^{-1}y = 1 \,\rangle \tag{5.9}$$

Then G is a Demuškin pro-2 group containing the free abelian pro-2 group $H = \langle\, x^2, y \,\rangle$ of rank 2. Thus, by Corollary 5.7 $(G, \eth_G)$ is cyclotomic. Since $H^1(G, \mathbb{I}_2(0)) \simeq \mathbb{I}_2 \oplus \mathbb{Z}/2\mathbb{Z}$, Proposition 3.1 implies that $\eth_G \neq \mathbf{1}_G$ is non-trivial. In particular, since $\eth_G|_H = \mathbf{1}_H$, this implies that $\mathrm{im}(\eth_G) = \{\, \pm 1 \,\}$. Note that $H = \ker(\eth_G)$ and that one has a canonical isomorphism

$$H = \langle\, x^2 \,\rangle \oplus \langle\, y \,\rangle \simeq \mathbb{Z}_2(0) \oplus \mathbb{Z}_2(1). \tag{5.10}$$

In particular, $(G, \eth_G)$ is not $\eth_G$-abelian.

Example 5.9. Let G be the pro-p group with presentation

$$G = \langle\, x, y, z \mid [x, y] = z^{-p} \,\rangle.$$

If $p = 2$ then G is a Demuškin group, and $\eth_G \colon G \to \mathbb{Z}_2^\times$ is given by $\eth_G(x) = \eth_G(y) = 1$, $\eth_G(z) = -1$. On the other hand, if $p \neq 2$ then G is not a Demuškin group, and any p-orientations $\theta \colon G \to \mathbb{Z}_p^\times$ is not 1-cyclotomic (cf. [11, Thm. 8.1]). However, $H^\bullet(G, \hat{\theta})$ is still quadratic.

6. Torsion

It is well known that a Bloch-Kato pro-p group may have non-trivial torsion only if, $p = 2$. More precisely, a Bloch-Kato pro-2 group G is torsion if, and only if, G is abelian and of exponent 2. Moreover, any such group is a Bloch-Kato pro-2 group (cf. [25, §2]). The following result — which appeared first in [26, Prop. 2.13] — holds for 1-cyclotomically oriented pro-p groups (see also [11, Ex. 3.5] and [5, Ex. 14.27]).

Proposition 6.1. *Let (G, θ) be a 1-cyclotomically oriented pro-p group.*

(a) *If $\mathrm{im}(\theta)$ is torsion free, then G is torsion free.*
(b) *If G is non-trivial and torsion, then $p = 2$, $G \simeq C_2$ and θ is injective.*

Remark 6.2. Let $\theta \colon C_2 \to \mathbb{Z}_2^\times$ be an injective homomorphism of groups. Then $\mathbb{Z}_2(1) \simeq \omega_{C_2}$ is isomorphic to the augmentation ideal

$$\omega_{C_2} = \ker(\mathbb{Z}_2[C_2] \to \mathbb{Z}_2).$$

Hence — by dimension shifting —

$$H^2(C_2, \mathbb{Z}_2(1)) = H^1(C_2, \mathbb{Z}_2(0)) = 0.$$

Thus — as C_2 has periodic cohomology of period 2 — one concludes that $H^s(C_2, \mathbb{Z}_2(t)) = 0$ for s odd and t even, and also for s even and t odd. Hence (C_2, θ) is cyclotomic.

From Proposition 6.1 and the profinite version of Sylow's theorem one concludes the following corollary, which can be seen as a version of the Artin-Schreier theorem for 1-cyclotomically p-oriented profinite groups.

Corollary 6.3. *Let p be a prime number, and let (G,θ) be a profinite group with a 1-cyclotomic p-orientation.*

(a) *If p is odd, then G has no p-torsion.*
(b) *If $p=2$, then every non-trivial 2-torsion subgroup is isomorphic to C_2. Moreover, if $\mathrm{im}(\theta)$ has no 2-torsion, then G has no 2-torsion.*

Remark 6.4. Let $\theta\colon \mathbb{Z}_2 \to \mathbb{Z}_2^\times$ be the homomorphism of groups given by $\theta(1+\lambda)=-1$ and $\theta(\lambda)=1$ for all $\lambda\in 2\mathbb{Z}_2$. Then θ is a 2-orientation of $G=\mathbb{Z}_2$ satisfying $\mathrm{im}(\theta)=\{\pm1\}$. As $\mathrm{cd}_2(\mathbb{Z}_2)=1$, Fact 2.2 implies that $(\mathbb{Z}_2,\theta)$ is Bloch-Kato and cyclotomically 2-oriented. However, $\mathrm{im}(\theta)$ is not torsion free.

6.1. Orientations on $C_2\times\mathbb{Z}_2$

As we have seen in Proposition 5.3, for p odd, every θ-abelian oriented pro-p group is cyclotomically p-oriented. For $p=2$, this is not true. Indeed, one has the following.

Proposition 6.5. *Any 2-orientation $\theta\colon G\to\mathbb{Z}_2^\times$ on $G\simeq C_2\times\mathbb{Z}_2$ is not 1-cyclotomic.*

Proof. Suppose that (G,θ) is 1-cyclotomically 2-oriented. Let x,y be elements of G such that $x^2=1$ and $\mathrm{ord}(y)=2^\infty$, and that x,y generate G. Proposition 6.1 applied to the cyclic pro-2 group generated by x yields $\theta(x)=-1$. Put $\theta(y)=1+2\lambda$ for some $\lambda\in\mathbb{Z}_2$. By [16, Prop. 6], if θ is 1-cyclotomic then for any pair of elements $c_x,c_y\in\mathbb{Z}_2(1)$ there exists a continuous crossed-homomorphism $c\colon G\to\mathbb{Z}_2(1)$ (i.e., a map satisfying $c(g_1g_2)=c(g_1)+\theta(g_1)c(g_2)$, cf. [23, p. 15]) such that $c(x)=c_x$, $c(y)=c_y$. Set $c_x=c_y=1$. Then one computes

$$c(xy)=c_x+\theta(x)c_y=1-1=0,\qquad\text{and}$$
$$c(yx)=c_y+\theta(y)c_x=1+1+2\lambda,$$

which yields $\lambda=-1$. The element xy has the same properties as y. Hence the previously mentioned argument applied to the element xy yields $\theta(xy)=1-2=-1$, whereas $\theta(xy)=\theta(x)\theta(y)=1$, a contradiction. □

Remark 6.6. From Proposition 6.1 and Proposition 6.5 one deduces that in a 1-cyclotomically 2-oriented pro-2 group, every element of order 2 is self-centralizing, which is a remarkable property of absolute Galois groups (cf. [4, Prop. 2.3] and [19, Cor. 2.3]).

Proposition 6.7. *Let (G,θ) be a θ-abelian oriented pro-2 group. Then θ is cyclotomic if, and only if, either*

(a) $\mathrm{im}(\theta)$ *is torsion free; or*
(b) $\mathrm{im}(\theta)$ *has order* 2.

In both these cases (G,θ) is split θ-abelian.

Proof. Assume first that $\mathrm{im}(\theta)$ is torsion free. Then the short exact sequence $\{1\} \to \ker(\theta) \to G \to \mathrm{im}(\theta) \to \{1\}$ splits, as $\mathrm{im}(\theta) \simeq \mathbb{Z}_2$ is a projective pro-2 group. Moreover, (G, θ) is cyclotomic by Proposition 5.3.

Second assume that θ is cyclotomic, $p = 2$ and that $\mathrm{im}(\theta) \supseteq \{\pm 1\}$. If $g \in G$ satisfies $\theta(g) = -1$, then $g^2 \in \ker(\theta) = \mathrm{Z}_\theta(G)$, and consequently

$$g^2 = g \cdot g^2 \cdot g^{-1} = (g^2)^{\theta(g)} = g^{-2},$$

i.e., $g^4 = 1$. Since $(\ker(\theta), \mathbf{1})$ is cyclotomically 2-oriented, $\ker(\theta)$ is torsion free, and one deduces that $g^2 = 1$. Therefore, the short exact sequence

$$\{1\} \longrightarrow H \longrightarrow G \longrightarrow C_2 \longrightarrow \{1\}$$

splits (here $H = \ker(\pi \circ \theta)$, where π is the canonical epimorphism $\mathbb{Z}_2^\times \twoheadrightarrow \{\pm 1\}$). Since $(H, \theta|_H)$ is again cyclotomically 2-oriented and as $\mathrm{im}(\theta|_H)$ is torsion free, $(H, \theta|_H)$ is split $\theta|_H$-abelian by the previously mentioned argument. We claim that $H = \ker(\theta)$. Indeed, suppose there exists $h \in H$ such that $\theta(h) \neq 1$. Put $\lambda = (1 + \theta(h))/2$ and let $z = ghgh^{-1} = [g, h^{-1}] \in \ker(\theta)$. Then — as $g = g^{-1}$ and $\theta(g) = -1$ — one has

$$\begin{aligned} g(z^\lambda h^2)g^{-1} &= (gzg)^\lambda \cdot gh^2 g \\ &= z^{-\lambda} \cdot (ghg)^2 = z^{-\lambda} \cdot (ghgh^{-1} \cdot h)^2 \\ &= z^{-\lambda} \cdot (zhzh^{-1} \cdot h^2) = z^{-\lambda+1+\theta(h)} h^2 \\ &= z^\lambda h^2, \end{aligned}$$

i.e., g and $z^\lambda h^2$ commute which implies that $\langle\, g, z^\lambda h^2 \,\rangle \simeq C_2 \times \mathbb{Z}_p$ contradicting Proposition 6.5. Therefore, $H = \ker(\theta)$ is a free abelian pro-2 group, and $G \simeq H \rtimes C_2$.

Finally, let $p = 2$ and assume that $\mathrm{im}(\theta) = \{\pm 1\}$. By Remark 6.2, we may also assume that $\ker(\theta)$ is non-trivial. Then, either
Case I: $\theta^{-1}(\{-1\})$ contains an element of order 2 and (G, θ) is split θ-abelian, i.e., $G \simeq \ker(\theta) \rtimes C_2$ with $\ker(\theta)$ a free abelian pro-2 group, or
Case II: all elements in $x \in \theta^{-1}(\{-1\})$ are of infinite order. Then for $y \in \ker(\theta)$, the group $K = \langle\, x, y \,\rangle$ must be isomorphic to the Klein bottle pro-2 group which is impossible as G is θ-abelian and thus contains only θ-abelian closed subgroups (cf. Remark 3.12(b)). Hence Case II is impossible.

By Lemma 3.10, if $U \subseteq G$ is an open subgroup, then either $U \subseteq \ker(\theta)$, or $U \simeq V \rtimes C_2$ for some open subgroup V of $\ker(\theta)$. In the first case, $(U, \mathbf{1})$ is cyclotomically 2-oriented by Proposition 5.3. For the second case, we claim that $H^k(U, \mathbb{I}_2(k))$ is 2-divisibe for all $k \geq 1$.

Recall that $\mathbb{Z}_2[C_2]$ has periodic cohomology (of period 2), and that one has the equalities of $\mathbb{Z}_2[\![U]\!]$-modules $\mathbb{I}_2(k) = \mathbb{I}_2(0)$ for k even and $\mathbb{I}_2(k) = \mathbb{I}_2(-1)$ for k odd. Moreover,

$$\begin{aligned} \hat{H}^0(C_2, \mathbb{I}_2(0)) &= \mathbb{I}_2(0)^{C_2}/N_{C_2}\mathbb{I}_2(0) = \mathbb{I}_2(0)/2 \cdot \mathbb{I}_2(0) = 0, \\ \hat{H}^{-1}(C_2, \mathbb{I}_2(-1)) &= \ker(N_{C_2})/\omega_{C_2}\mathbb{I}_2(-1) = \mathbb{I}_2(-1)/2 \cdot \mathbb{I}_2(-1) = 0, \end{aligned} \tag{6.1}$$

where $\hat{H}^k$ denotes Tate cohomology, $N_{C_2} = \sum_{x \in C_2} x \in \mathbb{Z}_2[C_2]$ is the norm element, and ω_{C_2} is the augmentation ideal of the group algebra $\mathbb{Z}_2[C_2]$ (cf. [23, § I.2]). Thus, by (6.1), one has

$$H^m(C_2, \mathbb{I}_2(m)) = \hat{H}^m(C_2, \mathbb{I}_2(m)) \simeq \hat{H}^k(C_2, \mathbb{I}_2(k)) = 0, \tag{6.2}$$

for all positive integers $m > 0$ and $m \equiv k(\mod 2)$.

Suppose first that $V \simeq \mathbb{Z}_2$. As in the proof of Theorem 3.11, the E_2-term of the Hochschild-Serre spectral sequence associated to the short exact sequence $\{1\} \to V \to U \to C_2 \to \{1\}$ evaluated on $\mathbb{I}_2(k)$ is concentrated in the first and the second row. In particular, $d_2^{\bullet,\bullet} = 0$ and thus $E_2^{s,t}(\mathbb{I}_2(k)) = E_\infty^{s,t}(\mathbb{I}_2(k))$. Thus, by Fact 3.9, for every $k \geq 1$ one has a short exact sequence

$$0 \longrightarrow H^k(C_2, \mathbb{I}_2(k)) \longrightarrow H^k(U, \mathbb{I}_2(k)) \longrightarrow H^{k-1}(C_2, \mathbb{I}_2(k-1)) \longrightarrow 0\ ,$$

and $H^k(C_2, \mathbb{I}_2(k)) = 0$ by (2.6). Hence, $(U, \theta|_U)$ is cyclotomically 2-oriented by Proposition 3.1. If $V \simeq \mathbb{Z}_2^n$ with $n > 1$, then $H^k(U, \mathbb{I}_2(k)) = 0$ by induction on n and the previously mentioned argument. Finally, Corollary 3.3 yields the claim in case V not finitely generated. □

7. Cyclotomically oriented pro-p groups

For a cyclotomically oriented pro-2 group (G, θ) satisfying $\mathrm{im}(\theta) \subseteq 1 + 4\mathbb{Z}_2$ one has the following.

Fact 7.1. *Let (G, θ) be a pro-2 group with a cyclotomic orientation satisfying* $\mathrm{im}(\theta) \subseteq 1+4\mathbb{Z}_2$. *Then $\chi \cup \chi = 0$ for all $\chi \in H^1(G, \mathbb{F}_2)$, i.e., the first Bockstein morphism $\beta^1 \colon H^1(G, \mathbb{F}_2) \to H^2(G, \mathbb{F}_2)$ vanishes.*

Proof. Since $\mathrm{im}(\theta) \subseteq 1 + 4\mathbb{Z}_2$, the action of G on $\mathbb{F}_2(1)$ is trivial. The epimorphism of $\mathbb{Z}_2[\![G]\!]$-modules $\mathbb{Z}_2(1)/4 \to \mathbb{F}_2$ induces a long exact sequence

$$\cdots \xrightarrow{2\cdot} H^1(G, \mathbb{Z}_2(1)/4) \xrightarrow{\pi^1_{2,1}} H^k(G, \mathbb{F}_2) \xrightarrow{\beta^1} H^2(G, \mathbb{F}_2) \xrightarrow{2\cdot} H^2(G, \mathbb{Z}_2(1)/4) \longrightarrow \cdots \tag{7.1}$$

where the connecting homomorphism is the first Bockstein morphism. Since θ is cyclotomic, the map $\pi^1_{2,1}$ is surjective, and thus β^1 is the 0-map. □

Remark 7.2. As before for a finitely generated pro-p group G let $d(G)$ denote its minimum number of generators. If p is odd and G is a finitely generated Bloch-Kato pro-p group, the cohomology ring $(H^\bullet(G, \mathbb{F}_p), \cup)$ is a quotient of the exterior $\mathbb{F}_p$-algebra $\Lambda_\bullet = \Lambda_\bullet(H^1(G, \mathbb{F}_p))$. In particular, $\mathrm{cd}_p(G) \leq d(G)$. Moreover, $\Lambda_{d(G)}$ is the unique minimal ideal of $\Lambda_\bullet$. Hence equality of $\mathrm{cd}_p(G)$ and $d(G)$ is equivalent to $H^\bullet(G, \mathbb{F}_p)$ being isomorphic to $\Lambda_\bullet$. It is well known that this implies that G is uniformly powerful (cf. [33, Thm. 5.1.6]), and that there exists a p-orientation $\theta \colon G \to \mathbb{Z}_p^\times$ such that G is θ-abelian (cf. [25, Thm. 4.6]).

Let $p = 2$, and let (G, θ) be a cyclotomically oriented Bloch-Kato pro-2 group satisfying $\mathrm{im}(\theta) \subseteq 1 + 4\mathbb{Z}_2$. Then Proposition 7.1 implies that the cohomology ring $(H^\bullet(G, \mathbb{F}_2), \cup)$ is a quotient of the exterior $\mathbb{F}_2$-algebra $\Lambda_\bullet = \Lambda_\bullet(H^1(G, \mathbb{F}_2))$, and hence $\mathrm{cd}_2(G) \leq d(G)$. If $\mathrm{cd}_2(G) = d(G)$, the previously mentioned argument, Proposition 7.1 and [42] imply that G is uniformly powerful. Finally, [25, Thm. 4.11] yields that G is θ'-abelian for some orientation $\theta' \colon G \to \mathbb{Z}_2^\times$. Thus, if $d(G) \geq 2$, one has $\theta = \theta'$ by Corollary 5.4(c).

From the above remark and J-P. Serre's theorem (cf. [30]) one concludes the following fact.

Fact 7.3. *Let (G, θ) be a finitely generated cyclotomically oriented torsion free Bloch-Kato pro-2 group. Then $\mathrm{cd}_2(G) < \infty$.*

7.1. Tits' alternative

From Remark 7.2 one concludes the following.

Fact 7.4. (a) *Let p be odd, and let G be a Bloch-Kato pro-p group satisfying $d(G) \leq 2$. Then G is either isomorphic to a free pro-p group, or G is θ-abelian for some orientation $\theta \colon G \to \mathbb{Z}_p^\times$.*
(b) *Let $p = 2$, and let (G, θ) be a cyclotomically oriented Bloch-Kato pro-2 group satisfying $\mathrm{im}(\theta) \subseteq 1 + 4\mathbb{Z}_2$ and $d(G) \leq 2$. Then G is either isomorphic to a free pro-2 group, or G is θ-abelian.*

In [25, Thm. 4.6] it was shown, that for p odd any Bloch-Kato pro-p group satisfies a strong form of Tits' alternative (cf. [35]), i.e., either G contains a closed non-abelian free pro-p subgroup, or there exists a p-orientation $\theta \colon G \to \mathbb{Z}_p^\times$ such that G is θ-abelian. Using the results from the previous subsection and [25, Thm. 4.11], one obtains the following version of Tits' alternative if p is equal to 2.

Proposition 7.5. *Let (G, θ) be a cyclotomically oriented virtual pro-2 group which is also Bloch-Kato, such that $\mathrm{im}(\theta) \subseteq 1 + 4\mathbb{Z}_2$. Then either G contains a closed non-abelian free pro-2 subgroup; or G is θ-abelian.*

Proof. As $\mathrm{im}(\theta) \subseteq 1 + 4\mathbb{Z}_2$, Proposition 6.1-(a) implies that G is torsion free. From Proposition 7.1 one concludes that the first Bockstein morphism β^1 vanishes. Thus, the hypothesis of [25, Thm. 4.11] are satisfied (cf. Remark 7.2), and this yields the claim. □

Remark 7.6. Note that Proposition 7.5 without the hypothesis $\mathrm{im}(\theta) \subseteq 1 + 4\mathbb{Z}_2$ does not remain true (cf. Remark 5.8).

7.2. The θ-center

One has the following characterization of the θ-center for a cyclotomically oriented Bloch-Kato pro-p group (G, θ).

Theorem 7.7. *Let (G, θ) be a cyclotomically oriented torsion free Bloch-Kato pro-p group. If $p = 2$ assume further that $\mathrm{im}(\theta) \subseteq 1 + 4\mathbb{Z}_2$. Then $\mathrm{Z}_\theta(G)$ is the unique maximal closed abelian normal subgroup of G contained in $\ker(\theta)$.*

Proof. Let $A \subseteq \ker(\theta)$ be a closed abelian normal subgroup of G, let $z \in A$, $z \neq 1$, and let $x \in G$ be an arbitrary element. Put $C = \mathrm{cl}(\langle x, z \rangle) \subseteq G$. Then either $C \simeq \mathbb{Z}_p$ or C is a 2-generated pro-p group. Thus, by Fact 7.4, one has to distinguish three cases:

(i) $d(C) = 1$;
(ii) $d(C) = 2$ and C is isomorphic to a free pro-p group; or
(iii) $d(C) = 2$ and C is θ'-abelian for some p-orientation $\theta' \colon C \to \mathbb{Z}_p^\times$.

In case (i), x and z commute. If C is generated by z, then $C \subseteq \ker(\theta)$ and $\theta(x) = 1$. If C is generated by x, then $z = x^\lambda$ for some $\lambda \in \mathbb{Z}_p$, and $1 = \theta(z) = \theta(x)^\lambda$. Hence $\theta(x) = 1$, as $\mathrm{im}(\theta)$ is torsion free. In both cases

$$xzx^{-1} = z = z^{\theta(x)}.$$

Case (ii) cannot hold: by hypothesis, $A \cap C \neq \{1\}$, but free pro-p groups of rank 2 do not contain non-trivial closed abelian normal subgroups.

Suppose that case (iii) holds. Then $\theta' = \theta|_C$ by Corollary 5.4(c), and $z \in \ker(\theta|_C) = \mathrm{Z}_{\theta|_C}(C)$. Therefore,

$$xzx^{-1} = z^{\theta|_C(x)} = z^{\theta(x)}.$$

Hence we have shown that for all $z \in A$ and all $x \in G$ one has that $xzx^{-1} = z^{\theta(x)}$. This yields the claim. □

The above result can be seen as the group theoretic generalization of [12, Corollary 3.3] and [13, Thm. 4.6]. Note that in the case $p = 2$ the additional hypothesis in Theorem 7.7 is necessary (cf. Remark 5.8). Indeed, if G is the Klein bottle pro-2 group then $\langle x^2 \rangle$ is another maximal closed abelian normal subgroup of G contained in $\ker(\eth_G)$.

Remark 7.8. Let $\mathbb{K}$ be a field containing a primitive p^{th}-root of unity. Theorem 7.7, together with [12, Thm. 3.1] and [13, Thm. 4.6], implies that the $\theta_{\mathbb{K},p}$-center of the maximal pro-p Galois group $G_{\mathbb{K}}(p)$ is the inertia group of the maximal *p-henselian valuation* admitted by $\mathbb{K}$.

7.3. Isolated subgroups

Let G be a pro-p group, and let $S \subseteq G$ be a closed subgroup of G. Then S is called *isolated*, if for all $g \in G$ for which there exists $k \geq 1$ such that $g^{p^k} \in S$ follows that $g \in S$. Hence a closed normal subgroup N of G is isolated if, and only if, G/N is torsion free.

Proposition 7.9. *Let (G, θ) be an oriented Bloch-Kato pro-p group. In the case $p = 2$ assume further that $\mathrm{im}(\theta) \subseteq 1 + 4\mathbb{Z}_2$ and that θ is 1-cyclotomic. Then $\mathrm{Z}_\theta(G)$ is an isolated subgroup of G.*

Proof. Suppose there exists $x \in G \smallsetminus \mathrm{Z}_\theta(G)$ and $k \geq 1$ such that $x^{p^k} \in \mathrm{Z}_\theta(G)$. By changing the element x if necessary, we may assume that $k = 1$, i.e., $x^p \in \mathrm{Z}_\theta(G)$. As G is torsion free (cf. Corollary 6.3), one has that $x^p \neq 1$.

For an arbitrary $g \in G$, the subgroup $C(g) = \mathrm{cl}(\langle g, x \rangle) \subseteq G$ is not free, as $gx^pg^{-1} = x^{p\theta(g)}$. Thus, from Fact 7.4 one concludes that $C(g)$ is

$\theta|_{C(g)}$-abelian. Moreover, as $\mathrm{im}(\theta)$ is torsion-free, $\theta(x^p) = \theta(x)^p = 1$ implies that

$$x \in \ker(\theta|_{C(g)}) = \mathrm{Z}_{\theta|_{C(g)}}(C(g)).$$

Thus, $x \in \bigcap_{g\in G} \mathrm{Z}_{\theta_{C(g)}}(C(g)) \subseteq \mathrm{Z}_\theta(G)$. □

Proposition 7.9 generalises to profinite groups as follows.

Corollary 7.10. *Let (G, θ) be a torsion free p-oriented Bloch-Kato profinite group. For $p = 2$ assume also that $\mathrm{im}(\theta) \subseteq 1+4\mathbb{Z}_2$ and that θ is 1-cyclotomic. Then $\mathrm{Z}_\theta(G)$ is an isolated subgroup of G.*

Proof. Let $x \in \mathrm{Z}_\theta(G)$, $y \in G$ and $n \in \mathbb{N}$ such that $x = y^n$. Then $Y = \mathrm{cl}(\langle\, y\,\rangle)$ is pro-cyclic and virtually pro-p. Thus, as G is torsion free by hypothesis, Y is a cyclic pro-p group, and n is a p-power. Let $P \in \mathrm{Syl}_p(G)$ be a pro-p Sylow subgroup of G containing Y. Then $(P, \theta|_P)$ satisfies the hypothesis of Proposition 7.9, which yields the claim. □

7.4. Split extensions

Proposition 7.11. *Let (G, θ) be a p-oriented Bloch-Kato pro-p group of finite cohomological dimension satisfying $\mathrm{im}(\theta) \subseteq 1 + p\mathbb{Z}_p$ (resp. $\mathrm{im}(\theta) \subseteq 1 + 4\mathbb{Z}_2$ if $p = 2$), and let Z be a closed normal subgroup of G isomorphic to $\mathbb{Z}_p$ such that G/Z is torsion free. Then $Z \not\subseteq G^p[G, G]$.*

Proof. Let $d = \mathrm{cd}_p(G)$. As $\mathrm{cd}(Z) = 1$, and as $H^1(Z, \mathbb{F}_p) \simeq \mathbb{F}_p$, one has $\mathrm{vcd}_p(G/Z) = d-1$ (cf. [43]). Thus, as G/Z is torsion free, J-P. Serre's theorem (cf. [30]) implies that $\mathrm{cd}_p(G/Z) = d - 1$.

Suppose that $Z \subseteq G^p[G, G]$. Then $\mathrm{inf}^1_{G,Z} \colon H^1(G/Z, \mathbb{F}_p) \to H^1(G, \mathbb{F}_p)$ is an isomorphism. For $\chi \in H^1(G, \mathbb{F}_p)$, set $\bar\chi \in H^1(G/Z, \mathbb{F}_p)$ such that $\chi = \mathrm{inf}^1_{G,Z}(\bar\chi)$. Then, by [23, Prop. 1.5.3] one has

$$\chi_1 \cup \ldots \cup \chi_k = \mathrm{inf}^1_{G,Z}(\bar\chi_1) \cup \ldots \cup \mathrm{inf}^1_{G,Z}(\bar\chi_k) = \mathrm{inf}^k_{G,Z}(\bar\chi_1 \cup \ldots \cup \bar\chi_k)$$

for any $\chi_1, \ldots, \chi_k \in H^1(G, \mathbb{F}_p)$, i.e.,

$$\mathrm{inf}^k_{G,Z} \colon H^k(G/Z, \mathbb{F}_p) \longrightarrow H^k(G, \mathbb{F}_p) \tag{7.2}$$

is surjective for all $k \geq 0$. Let

$$(E^{st}_r, d_r) \Rightarrow H^{s+t}(G, \mathbb{F}_p), \qquad E^{st}_2 = H^s\left(G/Z, H^t(Z, \mathbb{F}_p)\right) \tag{7.3}$$

denote the Hochschild-Serre spectral sequence associated to the extension of pro-p groups $Z \to G \to G/Z$ with coefficients in the discrete G-module $\mathbb{F}_p$. We claim that E^{st}_∞ is concentrated on the buttom row, i.e., $E^{st}_\infty = 0$ for all $t \geq 1$. Since $\mathrm{cd}_p(Z) = 1$ and $\mathrm{cd}_p(G/Z) = d - 1$, one has $E^{st}_2 = 0$ for $t \geq 2$ or $s \geq d$. Hence, d^{st}_r is the 0-map for every $s, t \geq 0$ and $r \geq 3$, i.e., $E^{st}_\infty \simeq E^{st}_3$. The total complex $\mathbf{tot}_\bullet(E^{\bullet\bullet}_\infty)$ of the graded $\mathbb{F}_p$-bialgebra $E^{\bullet\bullet}_\infty$ coincides with $H^\bullet(G, \mathbb{F}_p)$, which is quadratic by hypothesis. Thus $E^{\bullet\bullet}_\infty$ is generated by

$$\mathbf{tot}_1(E^{\bullet\bullet}_\infty) = E^{1,0}_\infty = E^{1,0}_2.$$

Hence, $E^{st}_3 = 0$ for $t \geq 1$.

On the other hand, $H^1(Z,\mathbb{F}_p)$ is a trivial G/Z-module isomorphic to $\mathbb{F}_p$, and thus, as $\mathrm{cd}_p(G/Z)=d-1$, one has

$$E_2^{d-1,1}=H^{d-1}\left(G/Z,H^1(Z,\mathbb{F}_p)\right)\neq 0. \tag{7.4}$$

Moreover, $d_2^{d-1,1}$ is the 0-map, thus $E_3^{d-1,1}=\ker(d_2^{d-1,1})=E_\infty^{d-1,1}\neq 0$, a contradiction, and this yields the claim. □

Proposition 7.11 has the following consequence.

Proposition 7.12. *Let (G,θ) be a p-oriented Bloch-Kato pro-p group (resp. virtual pro-p group) of finite cohomological p-dimension, and let Z be a closed normal subgroup of G isomorphic to $\mathbb{Z}_p$ such that G/Z is torsion free. Then there exists a Z-complement C in G, i.e., the extension of profinite groups*

$$\{1\}\longrightarrow Z\longrightarrow G\longrightarrow G/Z\longrightarrow\{1\} \tag{7.5}$$

splits.

Proof. Assume first that G is a pro-p group. By Proposition 7.11, one has that $Z\not\subseteq\Phi(G)=G^p[G,G]$. Hence there exists a maximal closed subgroup C_1 of G such that

$$C_1Z=G \qquad \text{and} \qquad Z_1=C_1\cap Z=Z^p.$$

Moreover, Z_1 is a closed normal subgroup in C_1 such that C_1/Z_1 is torsion free and $Z_1\simeq\mathbb{Z}_p$. From Proposition 7.11 again, one concludes that $Z_1\not\subseteq\Phi(C_1)$. Thus repeating this process one finds open subgroup C_k of G of index p^k such that

$$C_k\,Z=G \qquad \text{and} \qquad Z_k=C_k\cap Z=Z^{p^k}.$$

Hence $C=\bigcap_{k\geq 1}C_k$ is a Z-complement in G.

If G is a p-oriented virtual pro-p group, then G is a $\bar\Sigma$-virtual pro-p group for $\bar\Sigma=\mathrm{im}(\hat\theta)$ (cf. 4.1), and thus corresponds to $(O_p(G),\theta^\circ,\gamma)$ in alternative form. In particular, the maximal subgroup C_1 and hence all closed subgroups C_k can be chosen to be $\bar\Sigma$-invariant (cf. Proposition 4.5). Hence $C=\bigcap_{k\in\mathbb{N}}C_k$ carries canonically a left $\bar\Sigma$-action, and thus defines a Z complement $H=C\rtimes\bar\Sigma$ in G. □

The proof of Theorem 1.2 can be deduced from Proposition 7.12 as follows.

Proof of Theorem 1.2. Assume first that G is either pro-p, or virtually pro-p. To prove statement (i) (and (ii)), we proceed by induction on $d=\mathrm{cd}_p(G)=\mathrm{cd}(G)$. For $d=1$, G is free (resp. virtually free) (cf. [23, Prop. 3.5.17]), and thus $\mathrm{Z}_\theta(G)=\{1\}$. So assume that $d\geq 1$, and that the claim holds for $d-1$. Note that $\mathrm{Z}_\theta(G)$ is a finitely generated abelian pro-p group satisfying

$$d_\circ=d(\mathrm{Z}_\theta(G))=\mathrm{cd}_p(\mathrm{Z}_\theta(G))\leq d.$$

If $d_\circ=0$, there is nothing to prove. If $d_\circ\geq 1$, $\mathrm{Z}_\theta(G)$ contains an isolated closed subgroup Z satisfying $d(Z)=1$. By definition, Z is normal in G. Hence Proposition 7.12 implies that there exists a subgroup $C\subseteq G$ satisfying

$C \cap Z = \{1\}$ and $C\,Z = G$. As $C \simeq G/Z$, the main result of [43] implies that $\mathrm{cd}(C) = \mathrm{vcd}(C) = d-1$. Since $\mathrm{Z}_{\theta|_C}(C)\,Z = \mathrm{Z}_\theta(G)$, the claim then follows by induction.

To prove statement (iii), let $G^\circ = \ker(\hat{\theta}\colon G \to \mathbb{F}_p^\times)$ and $\bar{G}^\circ = \ker(\hat{\bar{\theta}}\colon \bar{G} \to \mathbb{F}_p^\times)$, and put $\bar{O} = O^p(\bar{G}^\circ)$ and

$$O = \{\, g \in G^\circ \mid g\mathrm{Z}_\theta(G) \in \bar{O}^p(\bar{G})\,\}. \tag{7.6}$$

Then, by construction, $\mathrm{im}(\hat{\bar{\theta}}|_{\bar{O}})$ is a pro-p group and hence trivial. In particular, the left $\mathbb{F}_p[\![\bar{O}]\!]$-module $\mathbb{F}_p(1)$ is the trivial module. Thus, as $\bar{O}$ is p-perfect, one concludes that

$$H^1(\bar{O}, \mathbb{F}_p(1)) = 0. \tag{7.7}$$

By hypothesis, $(\bar{G}, \bar{\theta})$ is Bloch-Kato, and therefore $(\bar{O}, \mathbf{1})$ is Bloch-Kato. Hence (7.7) yields that

$$H^k(\bar{O}, \mathbb{F}_p(j)) = H^k(\bar{O}, \mathbb{F}_p(0)) = 0 \tag{7.8}$$

for all positive integers k, j. Note that $\mathbb{Z}_p(1)$ is the trivial $\mathbb{Z}_p[\![\bar{O}]\!]$-module isomorphic to $\mathbb{Z}_p$ as abelian pro-p group. The cyclotomicity of $(\bar{O}, \mathbf{1})$ implies that $H^2(\bar{O}, \mathbb{Z}_p(1))$ is p-torsion free, and from the exact sequence

$$0 \longrightarrow H^2(\bar{O}, \mathbb{Z}_p(1)) \xrightarrow{\cdot p} H^2(\bar{O}, \mathbb{Z}_p(1)) \longrightarrow H^2(\bar{O}, \mathbb{F}_p(1)) \longrightarrow 0 \tag{7.9}$$

one concludes that

$$H^2(\bar{O}, \mathbb{Z}_p(1)) = 0. \tag{7.10}$$

By hypothesis, $\mathrm{cd}_p(\mathrm{Z}_\theta(G)) \leq \mathrm{cd}_p(G) < \infty$, and thus $\mathrm{Z}_\theta(G) \simeq \mathbb{Z}_p(1)^r$ is a trivial left $\mathbb{Z}_p[\![\bar{O}]\!]$-module and a finitely generated free (abelian pro-p group). Hence

$$H^2(\bar{O}, \mathrm{Z}_\theta(G)) = 0, \tag{7.11}$$

which implies that

$$\{1\} \longrightarrow \mathrm{Z}_\theta(G) \longrightarrow O \xrightarrow{\pi} \bar{O} \longrightarrow \{1\} \tag{7.12}$$

is a split short exact sequence of profinite groups. From this fact one concludes that

$$O = \mathrm{Z}_\theta(G) \cdot O^p(G^\circ) \qquad \text{and} \qquad \mathrm{Z}_\theta(G) \cap O^p(G^\circ) = \{1\}. \tag{7.13}$$

Let $\tilde{G} = G/O^p(G^\circ)$. Then for all abelian pro-p groups M with a continuous left $\mathbb{Z}_p[\![\tilde{G}]\!]$-action inflation induces an isomorphism in cohomology

$$\mathrm{inf}_{\tilde{G}}^{G}(-)\colon H^k_{\mathrm{cts}}(\tilde{G}, M) \longrightarrow H^k_{\mathrm{cts}}(G, M) \tag{7.14}$$

(cf. Proposition 4.6). Moreover, as $\theta|_O = \mathbf{1}$ is the constant 1 function, θ induces a p-orientation $\tilde{\theta}\colon \tilde{G} \to \mathbb{Z}_p^\times$ on $\tilde{G}$. In particular, from (7.14) one concludes that $\mathrm{cd}_p(\tilde{G}) < \infty$, and that $(\tilde{G}, \tilde{\theta})$ is cyclotomic and Bloch-Kato. Thus, by part (i), the exact sequence of virtual pro-p groups

$$\{1\} \longrightarrow \frac{\mathrm{Z}_\theta(G)O^p(G^\circ)}{O^p(G^\circ)} \longrightarrow \tilde{G} \xrightarrow{\tilde{\pi}} \bar{G}/\bar{O} \longrightarrow \{1\} \tag{7.15}$$

splits. Let $\tilde{H} \subset \tilde{G}$ be a complement for $\mathrm{Z}_\theta(G)O^p(G^\circ)/O^p(G^\circ)$ in $\tilde{G}$, and let

$$H = \{\, g \in G^\circ \mid gO^p(G^\circ) \in \tilde{H} \,\}. \tag{7.16}$$

Then, by construction, $H \cap \mathrm{Z}_\theta(G)O^p(G^\circ) \subseteq O^p(G^\circ)$. Thus $HO^p(G^\circ)$ is a complement of $\mathrm{Z}_\theta(G)$ in G. □

Finally, we ask whether the converse of Theorem 3.13 holds true.

Question 7.13. Let (G, θ) be a cyclotomically p-oriented Bloch-Kato pro-p group, and suppose that

$$H^\bullet(G, \mathbb{F}_p) \simeq H^\bullet(C, \mathbb{F}_p) \otimes \Lambda_\bullet(V),$$

for some subgroup $C \subseteq G$ and some nontrivial subspace $V \subseteq H^1(G, \mathbb{F}_p)$. Does there exist an isolated closed subgroup $\mathrm{Z} \subseteq \mathrm{Z}_\theta(G)$ such that $G = C\mathrm{Z}$ and $\mathrm{Z}/\mathrm{Z}^p \simeq V^* = \mathrm{Hom}(V, \mathbb{F}_p)$?

7.5. The elementary type conjecture

In order to formulate a conjecture concerning the maximal pro-p Galois groups of fields, I. Efrat introduced in [9] the class $\mathcal{C}_{\mathrm{FG}}$ of p-oriented pro-p groups (resp. cyclotomic pro-p pairs) of *elementary type*.

This class consists of all finitely generated p-oriented pro-p groups which can be constructed from $\mathbb{Z}_p$ and Demuškin groups using coproducts and fibre products (cf. [9, § 3]).

Efrat's *elementary type conjecture* asks whether every pair $(G_{\mathbb{K}}(p), \theta_{\mathbb{K},p})$ for which $\mathbb{K}$ contains a primitive p^{th}-root of unity and $G_{\mathbb{K}}(p)$ is finitely generated, belongs to $\mathcal{C}_{\mathrm{FG}}$ (see [7], and also [15] for the case $p = 2$). This conjecture originates from the theory of quadratic forms (cf. [20], [10, p. 268]).

One may extend slightly Efrat's class by defining the class $\mathcal{E}_{\mathrm{CO}}$ of *cyclotomically p-oriented Bloch-Kato pro-p groups of elementary type* to be the smallest class of cyclotomically p-oriented pro-p groups containing

(a) (F, θ), with F a finitely generated free pro-p group and $\theta \colon F \to \mathbb{Z}_p^\times$ any p-orientation;
(b) $(G, \eth_G)$, with G a Demuškin pro-p group;
(c) $(\mathbb{Z}/2\mathbb{Z}, \theta)$, with $\mathrm{im}(\theta) = \{\pm 1\}$ in case that $p = 2$;

and which is closed under coproducts and under fibre products with respect to finitely generated split θ-abelian pro-p groups, i.e., if (G_1, θ_1) and (G_2, θ_2) are contained in $\mathcal{E}_{\mathrm{CO}}$, then

(d) $(G, \theta) = (G_1, \theta_1) \amalg (G_2, \theta_2) \in \mathcal{E}_{\mathrm{CO}}$; and
(e) $(G, \theta) = \mathbb{Z}_p \rtimes_{\theta_1} (G_1, \theta_1) \in \mathcal{E}_{\mathrm{CO}}$.

Question 1.5 asks whether every finitely generated cyclotomically p-oriented Bloch-Kato pro-p group belongs to the class $\mathcal{E}_{\mathrm{CO}}$. By Theorem 1.1, Question 1.5 is stronger than Efrat's elementary type conjecture. Nevertheless, it is stated in purely group theoretic terms.

Remark 7.14. Recently, Question 1.5 has received a positive solution in the class of *trivially p-oriented right-angled Artin pro-p groups*: I. Snopce and P.A. Zalesskiĭ proved that the only indecomposable right-angled Artin pro-p group which is Bloch-Kato and cyclotomically p-oriented is $(\mathbb{Z}_p, \mathbf{1})$ (cf. [32]).

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