UNIQUE CONTINUATION PRINCIPLES FOR A HIGHER ORDER FRACTIONAL LAPLACE EQUATION

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ABSTRACT. In this paper we prove the strong unique continuation principle and the unique continuation from sets of positive measure for solutions of a higher order fractional Laplace equation in an open domain. Our proofs are based on the Caffarelli-Silvestre [10] extension method combined with an Almgren type monotonicity formula. The corresponding extended problem is formulated as a system of two second order equations with singular or degenerate weights in a half-space, for which asymptotics estimates are derived by a blow-up analysis.

1. Introduction and main results

We study the following higher order fractional Laplace equation

$$(-\Delta)^s u = 0 \qquad \text{in } \Omega, \tag{1}$$

where $1 < s < 2$, $\Omega \subset \mathbb{R}^N$ is an open domain with $N > 2s$, and the fractional Laplacian $(-\Delta)^s$ of a function u defined over the whole $\mathbb{R}^N$ is defined by means of the Fourier transform:

$$\widehat{(-\Delta)^s u}(\xi) = |\xi|^{2s}\widehat{u}(\xi)\,.$$

Here by Fourier transform in $\mathbb{R}^N$ we mean

$$\widehat{u}(\xi) = \mathcal{F}u(\xi) := \frac{1}{(2\pi)^{N/2}}\int_{\mathbb{R}^N} e^{-ix\cdot\xi}u(x)\,dx\,.$$

In the sequel we will explain in more details what we mean by a weak solution of (1). Our main purpose is to prove the validity of unique continuation principles for solutions to (1).

Unique continuation properties and qualitative local behavior of solutions to fractional elliptic problems are a subject which was widely studied in the last years. In [15], the authors study a semilinear fractional elliptic problem containing a singular potential of Hardy type, a perturbation potential with a lower order singularity and a nonlinearity that is at most critical with respect to a suitable Sobolev exponent. In that paper the fractional differential operator is $(-\Delta)^s$ with power $0 < s < 1$; see also [16] for analogous results for relativistic Schrödinger operators. Unique continuation for fractional Laplacians with power $s \in (0, 1)$ was also investigated in [32] in presence of rough potentials and in [47] for fractional operators with variable coefficients.

Other results concerning qualitative properties of solutions of equations with the fractional Laplace operator $(-\Delta)^s$ can be found in [8, 23, 24, 41]. For more details on basic results on the fractional Laplace operator see [1, 6, 10, 12, 13]. Operators given by fractional powers of the Laplacian arise in the description of phenomena where long-term interactions and anomalous diffusion occur, see [27]. This happens in several fields of application, such as continuum mechanics, fluid mechanics, phase transition phenomena, population dynamics, financial mathematics, control theory, and game theory, see [9, 44]. Furthermore, fractional Laplace operators appear in Probability as infinitesimal generators of stable Lévy processes, see [50].

Up to our knowledge, unique continuation properties for higher order fractional elliptic equations were first studied in the paper [46]. Here the author states a strong unique continuation property for the Laplace equation (1) for any noninteger $s > 0$.

More precisely, in [46, Corollary 5.5] it is stated that the solutions to (1) vanishing of infinite order at a point are necessarily null in Ω. In [46] the proof of this result is not written in details;

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it is just observed that, following the classical argument by Garofalo and Lin [21], the boundedness of the Almgren frequency function for solutions of some extended problem, together with the Caffarelli-Silvestre type extension result given in [46], suffices to provide the strong unique continuation property. However, we think that the boundedness of the frequency function proved in [46] only shows the validity of a unique continuation principle for the extended function U (see (4)) and not for the solution u of equation (1); indeed, it is nontrivial to exclude that u vanishes of infinite order at a point when U does not.

It is easy to show the existence of functions defined in the half space $\overline{\mathbb{R}^{N+1}_+}$ that do not vanish of infinite order at a point $(x_0, 0) \in \mathbb{R}^N \times \{0\}$ but whose restrictions to $\mathbb{R}^N$ vanish of infinite order at the point $x_0 \in \mathbb{R}^N$. A similar situation can be observed in Figure 1.

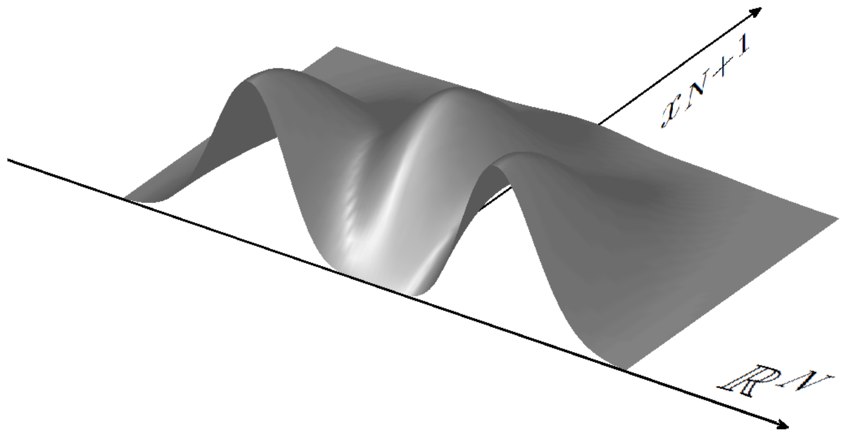

FIGURE 1.

It is our purpose to show that such a pathological situation cannot occur when dealing with solutions of (1); this seems far from being straightforward.

A first goal of the present paper is to give a complete proof of [46, Corollary 5.5] excluding such an occurrence by means of a blow-up analysis and a complete classification of local asymptotics of solutions for the extended problem. Nevertheless, we acknowledge the fundamental role of paper [46] since part of our approach to the unique continuation principle takes inspiration from the Caffarelli-Silvestre procedure [10] and the Almgren monotonicity formula performed by [46] in the higher order setting.

We point out that, among the all possible noninteger higher order powers of $-\Delta$, in the present paper we only consider the case $1 < s < 2$ just for technical reasons and in order to avoid excessive complications in the proofs. Indeed, as observed in [46], the case of non-integer $s > 2$ leads to a degenerate elliptic equation of order $2(m+1)$ with $m < s < m+1$ and consequently to an equivalent system of $m+1$ second order equations. An extension to all noninteger higher powers is then possible but requires the further technical effort to handle systems of more components, which are expected to create difficulties in the classification of blow-up profiles (see Theorem 5.7) as well as in exposition. Such extension is a matter of future studies.

The problem of unique continuation for higher order fractional Laplacians was also studied by I. Seo in [36, 37, 38] in presence of potentials in Morrey spaces; more precisely, in [36, 37, 38] Seo uses Carleman inequalities to prove a *weak* unique continuation result, i.e. vanishing of solutions which are zero on an open set; we recall that the *strong* unique continuation property instead requires the weaker assumption of infinite vanishing order at a point.

The major contribution of the present paper goes beyond bridging monotonicity formula for the extended problem and unique continuation for the original nonlocal equation, since our local analysis provides sharp results on the asymptotic behavior of solutions for the above mentioned extended problem, see (4), (6) below. Moreover our analysis allows us to prove a second version of the unique continuation principle which has, as an assumption, vanishing of solutions of (1) on sets of positive measure.

As already mentioned above, our approach is based on the Caffarelli-Silvestre procedure [10] and on an Almgren type monotonicity formula. But differently from [46], we combine the Almgren formula with a blow-up procedure with the purpose of proving asymptotic formulas for solutions of the extended problem. And it is by mean of this asymptotic formula that we are able to prove the validity of the two versions of the unique continuation principle.

As pointed out quoting the papers by I. Seo, other approaches in the proofs of unique continuation results are possible; in the present paper we chose a procedure which combines the biharmonic extension method with the Almgren type monotonicity formula, which allows proving a strong quantitative result, i.e. an asymptotic local analysis of solutions, which has the unique continuation principles as its consequence. Since the Almgren frequency function, defined as the ratio of local energy over mass near a point, has intrinsically a local nature, the possibility of realizing our nonlocal operator as a local one through the extension procedure plays a crucial role in the monotonicity approach.

Up to now, we succeeded in applying our method only to the fractional Laplace equation but we believe that similar results can be obtained in a more general setting by adding to equation (1) linear terms with singular potentials and subcritical nonlinearities, see Open Problem 1.3 for a more detailed explanation. A first step towards this goal is achieved in [19], where we prove the validity of an asymptotic formula and of unique continuation principles for problem

$$(-\Delta)^{3/2}u = h(x)u$$

in open domains of $\mathbb{R}^N$. The special case $s = \frac{3}{2}$ represents the "middle case" between the classical Laplace operator $-\Delta$ and the bilaplacian $(-\Delta)^2$ and produces a significant simplification when dealing with the Caffarelli-Silvestre extension, see (4) for more details.

Before stating the main results of the paper we introduce a suitable notion of weak solutions to (1). We define $\mathcal{D}^{s,2}(\mathbb{R}^N)$ as the completion of the space $C_c^\infty(\mathbb{R}^N)$ of C^∞ real compact supported functions, with respect to the scalar product

$$(u,v)_{\mathcal{D}^{s,2}(\mathbb{R}^N)} := \int_{\mathbb{R}^N} |\xi|^{2s}\, \widehat{u}(\xi)\overline{\widehat{v}(\xi)}\, d\xi\,.$$

We define a solution of (1) as a function $u \in \mathcal{D}^{s,2}(\mathbb{R}^N)$ satisfying

$$(u,\varphi)_{\mathcal{D}^{s,2}(\mathbb{R}^N)} = 0 \qquad \text{for any } \varphi \in C_c^\infty(\Omega)\,. \tag{2}$$

For a motivation of this definition see [12], where a detailed treatise on fractional Sobolev spaces and on $(-\Delta)^s$ in the case $0 < s < 1$ is provided. See also [15, (7)] for the definition of solution of a nonlinear problem with $(-\Delta)^s$ in the case $0 < s < 1$.

The first main result of the paper is the following strong unique continuation principle.

Theorem 1.1. *Assume that $1 < s < 2$ and $N > 2s$. Let $u \in \mathcal{D}^{s,2}(\mathbb{R}^N)$ be a solution of (1). Let us also assume that $(-\Delta)^s u \in (\mathcal{D}^{s-1,2}(\mathbb{R}^N))^\star$, where $(\mathcal{D}^{s-1,2}(\mathbb{R}^N))^\star$ denotes the dual space of $\mathcal{D}^{s-1,2}(\mathbb{R}^N)$, in the sense that the linear functional $\varphi \mapsto \int_{\mathbb{R}^N} |\xi|^{2s}\widehat{u}(\xi)\overline{\widehat{\varphi}(\xi)}\, d\xi$, $\varphi \in C_c^\infty(\mathbb{R}^N)$, is continuous with respect to the norm induced by $\mathcal{D}^{s-1,2}(\mathbb{R}^N)$. If there exists $x_0 \in \Omega$ such that $u(x) = O(|x-x_0|^k)$ as $x \to x_0$ for any $k \in \mathbb{N}$, then $u \equiv 0$ in Ω.*

We observe that the assumption $(-\Delta)^s u \in (\mathcal{D}^{s-1,2}(\mathbb{R}^N))^*$ is needed to prove that the trace of the weighted Laplacian of the extended function coincides, up to a multiplicative constant, with the Laplacian of u; the key of the proof of the unique continuation result then relies in showing that it can not occur that u vanishes of infinite order and its Laplacian does not, exploiting the blow-up analysis and the asymptotic estimates for the extended problem obtained in Theorem 1.6.

Now we state a second version of unique continuation principle where the condition on vanishing of infinite order around a point assumed in Theorem 1.1 is replaced by vanishing on a set of positive measure.

Theorem 1.2. *Assume that $1 < s < 2$ and $N > 2s$. Let $u \in \mathcal{D}^{s,2}(\mathbb{R}^N)$ be a solution of (1). Let us also assume that $(-\Delta)^s u \in (\mathcal{D}^{s-1,2}(\mathbb{R}^N))^\star$ in the sense explained in the statement of Theorem 1.1. If there exists a measurable set $E \subset \Omega$ of positive measure such that $u \equiv 0$ on E, then $u \equiv 0$ in Ω.*

As we mentioned before the statement of the main results, we believe that it should be interesting to extend the monotonicity approach to unique continuation to solutions of more general elliptic fractional equations. We leave this question as an open problem.

Open Problem 1.3. Let $u \in \mathcal{D}^{s,2}(\mathbb{R}^N)$ be a weak solution of

$$(-\Delta)^s u = h(x)u + f(x,u) \quad \text{in } \Omega, \tag{3}$$

with h and f satisfying

$$h \in W^{1,\infty}_{\rm loc}(\Omega \setminus \{0\}), \qquad |h(x)| + |x \cdot \nabla h(x)| \leqslant C_h |x|^{-2(s-1)+\varepsilon} \qquad \text{for a.e. } x \in \Omega,$$

and

$$f \in C^1(\Omega \times \mathbb{R}), \quad |f(x,\sigma)| \leqslant C_f |\sigma|^{p-1} \qquad \text{for a.e. } x \in \Omega \text{ and } \sigma \in \mathbb{R},$$

where $2 < p < 2^*(N, s-1) := \frac{2N}{N-2(s-1)}$. Develop a monotonicity formula for solutions to problem 3 and derive from that the validity of the two versions of unique continuation principle contained in Theorems 1.1-1.2 for solutions of (3).

The presence of a inhomogeneous right hand side in equation (3) would produce a coupling Neumann term in the system-type formulation of the extended problem (6), which makes the proof of a monotonicity formula more delicate because of the presence in the derivative of the frequency function $\mathcal{N}$ (87) of a term of the form

$$-r \int_{\partial B'_r} (hu + f(x,u)) v \, dS' + 2 \int_{B'_r} (hu + f(x,u))\, x \cdot \nabla_x v \, dx.$$

For $s = 3/2$, in [19] this term was estimated in terms of boundary integrals (see [19, Lemma 2.12]); however, such estimates seem to be quite more delicate to be derived for $s \in (1,2) \setminus \{3/2\}$, due to difficulties in handling the singular/degenerate weight appearing in the extension problem (see (4)). □

We mention that a progress in the study of unique continuation for higher order fractional equations with potentials was made, after the completion of the preprint version of the present paper, in the recent manuscript [33]; in particular in [33] strong unique continuation for equations of type (3) with $f \equiv 0$ and $\Omega = \mathbb{R}^N$ was established via Carleman inequalities.

Now, we explain in more details what we mean by the previously mentioned extended problem and we state which kind of asymptotic estimate we will prove on its solutions. Let $u \in \mathcal{D}^{s,2}(\mathbb{R}^N)$ be a solution of (1) in the sense given in (2) and let $U \in \mathcal{D}_b$ be the unique weak solution of the problem

$$\begin{cases} \Delta_b^2 U = 0 & \text{in } \mathbb{R}^{N+1}_+, \\ U(\cdot,0) = u(\cdot) & \text{in } \mathbb{R}^N, \\ \lim_{t\to 0^+} t^b U_t(\cdot,t) \equiv 0 & \text{in } \mathbb{R}^N, \end{cases} \tag{4}$$

where $b = 3 - 2s \in (-1,1)$, $\mathcal{D}_b$ is the functional space introduced in Section 3, and Δ_b is the operator defined at the beginning of Section 2.

For any function $u \in \mathcal{D}^{s,2}(\mathbb{R}^N)$, with u not necessarily a solution of (1), we say that $U \in \mathcal{D}_b$ is a weak solution of (4) if

$$\begin{cases} \displaystyle\int_{\mathbb{R}^{N+1}_+} t^b \Delta_b U \Delta_b \varphi \, dz = 0 & \text{for any } \varphi \in \mathcal{D}_b \text{ with } \mathrm{Tr}(\varphi) = 0, \\ \mathrm{Tr}(U) = u, \end{cases} \tag{5}$$

where $\mathrm{Tr} : \mathcal{D}_b \to \mathcal{D}^{s,2}(\mathbb{R}^N)$ is the trace map defined in Proposition 3.3. In Section 3 we prove the following existence and uniqueness result for solutions of (4):

Proposition 1.4. *For any $u \in \mathcal{D}^{s,2}(\mathbb{R}^N)$ problem* (4) *admits a unique weak solution $U \in \mathcal{D}_b$ in the sense of* (5).

Now, let $x_0 \in \Omega$ and let $R > 0$ be such that $B'_{2R}(x_0) \subset \Omega$ where, according with (8), $B'_{2R}(x_0)$ denotes the open ball in $\mathbb{R}^N$ of radius $2R$ centered at x_0. Then, if $u \in \mathcal{D}^{s,2}(\mathbb{R}^N)$ is a solution of

(1), putting $V := \Delta_b U$, the couple $(U,V) \in H^1(B_R^+(x_0);t^b) \times H^1(B_R^+(x_0);t^b)$ weakly solves the system

$$\begin{cases} \Delta_b U = V & \text{in } B_R^+(x_0)\,, \\ \Delta_b V = 0 & \text{in } B_R^+(x_0)\,, \\ \lim_{t\to 0^+} t^b U_t(\cdot,0) = 0 & \text{in } B'_R(x_0)\,, \\ \lim_{t\to 0^+} t^b V_t(\cdot,0) = 0 & \text{in } B'_R(x_0)\,, \end{cases} \tag{6}$$

see (8) and the successive part of Section 2 for the definition of the weighted Sobolev space $H^1(B_R^+(x_0);t^b)$. This means that the couple (U,V) satisfies

$$\int_{B_R^+(x_0)} t^b \nabla U \nabla \varphi \, dz = -\int_{B_R^+(x_0)} t^b V \varphi \, dz \quad \text{and} \quad \int_{B_R^+(x_0)} t^b \nabla V \nabla \varphi \, dz = 0$$

for any $\varphi \in H_0^1(\Sigma_R^+(x_0);t^b)$ with $H_0^1(\Sigma_R^+(x_0);t^b)$ as in Section 2.

In order to state our result on the local behavior of solutions of (6), we introduce the following eigenvalue problem:

$$\begin{cases} -\mathrm{div}_{\mathbb{S}^N_+}(\theta_{N+1}^b \nabla_{\mathbb{S}^N_+}\Psi) = \mu\, \theta_{N+1}^b \Psi & \text{in } \mathbb{S}^N_+, \\ \lim\limits_{\theta_{N+1}\to 0^+} \theta_{N+1}^b \nabla_{\mathbb{S}^N_+}\Psi \cdot \mathbf{e}_{N+1} = 0 & \text{on } \partial\mathbb{S}^N_+, \end{cases} \tag{7}$$

where $\mathbf{e}_{N+1} = (0,\dots,0,1) \in \mathbb{R}^{N+1}$, $\mathbb{S}^N_+ = \{(\theta_1,\dots,\theta_{N+1}) \in \mathbb{S}^N : \theta_{N+1} > 0\}$ and $\mathbb{S}^N$ is the N-dimensional unit sphere in $\mathbb{R}^{N+1}$.

By classical spectral theory the eigenvalue problem (7) admits a diverging sequence of real eigenvalues with finite multiplicity. We denote these distinct eigenvalues by μ_n and their multiplicity by M_n with $n \in \mathbb{N}\cup\{0\}$. Moreover, for any $n \geqslant 0$ let $\{Y_{n,m}\}_{m=1,\dots,M_n}$ be a $L^2(\mathbb{S}^N_+;\theta_{N+1}^b)$-orthonormal basis of the eigenspace of μ_n.

Combining the blow-up analysis performed in [15] with the regularity results proved in [40] for degenerate/singular problems arising from the Caffarelli-Silvestre extension, we can easily prove that the eigenvalues of problem (7) are in fact

$$\mu_n = n^2 + n(N+b-1), \quad n \in \mathbb{N}.$$

Remark 1.5. We observe that the eigenfunctions of problem (7) cannot vanish identically on $\partial\mathbb{S}^N_+$; indeed, if an eigenfunction Ψ vanishes on $\partial\mathbb{S}^N_+$, then the function $W(z) := |z|^{\sigma_\ell^+}\Psi(z/|z|)$ (with $\sigma_\ell^+ = -\frac{N+b-1}{2} + \sqrt{\mu_\ell + (N+b-1)^2/4} = \ell$) would be a weak solution to the equation $\mathrm{div}(t^b\nabla W) = 0$ in $\mathbb{R}^{N+1}_+$ satisfying both Dirichlet and weighted Neumann homogeneous boundary conditions; then its trivial extension to the entire space $\mathbb{R}^{N+1}$ would violate the unique continuation principle for elliptic equations with Muckenhoupt weights proved in [43] (see also [21], [39, Corollary 3.3], and [32, Proposition 2.2]).

We now state the main result on solutions to system (6).

Theorem 1.6. *Assume that $1 < s < 2$, $N > 2s$ and let $b = 3-2s \in (-1,1)$. For some $x_0 \in \mathbb{R}^N$ let $(U,V) \in H^1(B_R^+(x_0);t^b) \times H^1(B_R^+(x_0);t^b)$ be a nontrivial weak solution of* (6)*. Then there exists $\delta_1 \in \mathbb{N}$, a linear combination $\Psi_1 \not\equiv 0$ of eigenfunctions of* (7)*, possibly corresponding to different eigenvalues, and $\alpha \in (0,1)$ such that*

$$\lambda^{-\delta_1} U(z_0 + \lambda(z-z_0)) \to |z-z_0|^{\delta_1}\Psi_1\left(\tfrac{z-z_0}{|z-z_0|}\right)$$

in $H^1(B_1^+(x_0);t^b)$ and in $C^{1,\alpha}_{\mathrm{loc}}(B_1^+(x_0))$ as $\lambda \to 0^+$ where we put $z_0 = (x_0,0) \in \mathbb{R}^{N+1}$. Furthermore, if $V \not\equiv 0$, there exists $\delta_2 \in \mathbb{N}$, a linear combination $\Psi_2 \not\equiv 0$ of eigenfunctions of (7)*, possibly corresponding to different eigenvalues, and $\alpha \in (0,1)$ such that*

$$\lambda^{-\delta_2} V(z_0 + \lambda(z-z_0)) \to |z-z_0|^{\delta_2}\Psi_2\left(\tfrac{z-z_0}{|z-z_0|}\right)$$

in $H^1(B_1^+(x_0);t^b)$ and in $C^{1,\alpha}_{\mathrm{loc}}(B_1^+(x_0))$.

We observe that Theorem 1.6 implies a unique continuation principle from boundary points for solutions to (6); we refer to [2, 3, 18, 26, 42] for unique continuation from the boundary established via the Almgren monotonicity formula. Concerning unique continuation for systems of elliptic equations, we mention the recent papers [28] and [35].

Remark 1.7. We observe that Theorem 1.6 in general does not provide a sharp asymptotic formula around $x_0 \in \Omega$ for solutions to the original problem (1) when u and U are as in (4), even if u is the restriction to $B'_R(x_0)$ of U. This because we cannot exclude that the function Ψ_1 in Theorem 1.6 vanishes identically on $\partial\mathbb{S}^N$; what we can say is that this event cannot occur if Ψ_1 is an eigenfunction of (7) as explained in Remark 1.5. For this reason the unique continuation principles stated in Theorems 1.1–1.2 are not a direct consequence of Theorem 1.6 and additional arguments have to be employed in their proofs in order to exploit the asymptotic estimates of Theorem 1.6.

Remark 1.8. In the asymptotic profiles of Theorem 1.6, the appearence of eigenfunctions associated to possibly different eigenvalues basically originates from the fact that a homogeneous harmonic function of degree k multiplied by $|z|^2$ gives a homogeneous bi-harmonic function of degree $k+2$. An easy example can be constructed by taking, in the case $s = 3/2$, $U(z) = |z|^2 U_1(z) + U_2(z)$, where U_1, respectively U_2, is a harmonic function, homogeneous of degrees $k \in \mathbb{N}$, respectively $k+2$, even with respect to the hyperspace $\{t = 0\}$. The couple (U, V), with $V = 2(N + 2k + 1)U_1$, solves system (6) and $\lambda^{-k-2}U(\lambda\theta) = \psi_1(\theta) + \psi_2(\theta)$, where $\psi_1 = U_1\big|_{\mathbb{S}^N_+}$ is an eigenfunctions of (7) associated to μ_k and $\psi_2 = U_2|_{\mathbb{S}^N_+}$ is an eigenfunctions of (7) associated to μ_{k+2}.

Remark 1.9. The fact that a solution U to (6) asimptotically behaves as a homogeneous function of integer order leads to the natural conjecture that it is analytic with also its trace u solving (1). This is obviously true for $s = 3/2$ by standard regularity theory but it does not seem to be known in the degenerate/singular case $s \neq 3/2$. Of course an analyticity result for u would directly imply the unique continuation property proved in Theorem 1.1, so an alternative way to prove Theorem 1.1 could be given by the study of analyticity of solutions, e.g. by an iteration of our uniform asymptotic analysis. Our main reason for choosing the monotonicity approach to unique continuation relies in the possibility of obtaining a more detailed quantitative asymptotic statement and in our interest in developing a strategy of proof which could be applied to more general equations in future studies.

We observe that the proof of Theorem 1.6 presents substantial additional difficulties with respect to the lower order case $s \in (0, 1)$ treated in [15], since the corresponding Dirichlet-to-Neumann local problem is a fourth order equation (see (4)) which is equivalent to the second order system (6) with singular/degenerate weights and Neumann boundary conditions. In particular, several steps in our procedure, such as regularity and blow-up analysis, turn out to be more delicate for systems than for the single equation arising from the Caffarelli-Silvestre extension in the lower order case $s \in (0, 1)$.

We conclude this section by explaining how the rest of the paper is structured. Section 2 is devoted to some preliminary results and notations which will be used in the proofs of the main statements. In Section 3 we introduce a Caffarelli-Silvestre type extension for functions $u \in \mathcal{D}^{s,2}(\mathbb{R}^N)$ and we provide an alternative formulation for problem (1). In Section 4 we introduce an Almgren-type function and we prove a related monotonicity formula. In Section 5 we perform a blow-up procedure and we prove asymptotic estimates for the extended functions introduced in Section 3. Section 6 contains the proofs of the main results of the paper. Finally, Section 7 is an appendix devoted to weighted Sobolev spaces and related inequalities, Hölder regularity for solutions of a class of second order elliptic equations and systems with variable coefficients, and some properties of first kind Bessel functions.

2. Preliminaries and notations

Notations. We list below some notations used throughout the paper.

- $\mathbb{R}^{N+1}_+ = \{z = (z_1, \dots, z_{N+1}) \in \mathbb{R}^{N+1} : z_{N+1} > 0\}$.
- $\mathbb{S}^N = \{z \in \mathbb{R}^{N+1} : |z| = 1\}$ denotes the unit N-dimensional sphere in $\mathbb{R}^{N+1}$.
- $\mathbb{S}^N_+ = \{(\theta_1, \dots, \theta_{N+1}) \in \mathbb{S}^N : \theta_{N+1} > 0\} = \mathbb{S}^N \cap \mathbb{R}^{N+1}_+$.

- dS denotes the surface element in boundary integrals.
- $dz = dx\,dt$, $z = (x,t) \in \mathbb{R}^N \times \mathbb{R}$, denotes the $(N+1)$-dimensional volume element.
- $\Delta_b U = \Delta U + \frac{b}{t} U_t$ for any function $U = U(x,t)$ with $x \in \mathbb{R}^N$ and $t \in \mathbb{R}$, where ΔU denotes the classical Laplacian in $\mathbb{R}^{N+1}$ and U_t the partial derivative with respect to t.
- For any open set U and $k \in \mathbb{N}$, $C^k(U)$ denotes the space of k times continuously differentiable functions on U; $C^k(\overline{U})$ is the space of functions u in $C^k(U)$ such that $x \mapsto D^\alpha u$ admits a continuous extension to $\overline{U}$ for every multi-index α with length less or equal to k.

The main purpose of this section is to prove a regularity result for the boundary value problem (27). We observe that such a regularity result is needed to make the Almgren quotient (87) well-defined and seems to be taken for granted in [46] although not at all trivial. To prove the needed regularity we introduce two auxiliary problems, namely the eigenvalue problem (10) and the Poisson type equation (22).

For any $x_0 \in \mathbb{R}^N$, $t_0 \in \mathbb{R}$ and $R > 0$ we define

$$B_R(x_0,t_0) := \{(x,t) \in \mathbb{R}^{N+1} : |x-x_0|^2 + |t-t_0|^2 < R^2\}\,, \tag{8}$$

$$B_R^+(x_0) := \{(x,t) \in B_R(x_0,0) : t > 0\}\,, \quad B_R^-(x_0) := \{(x,t) \in B_R(x_0,0) : t < 0\}\,,$$

$$B'_R(x_0) := \{x \in \mathbb{R}^N : |x-x_0| < R\}\,,$$

$$S_R^+(x_0) := \{(x,t) \in \partial B_R(x_0,0) : t > 0\}\,, \quad S_R^-(x_0) := \{(x,t) \in \partial B_R(x_0,0) : t < 0\}\,,$$

$$\Sigma_R^+(x_0) := B_R^+(x_0) \cup (B'_R(x_0) \times \{0\})\,, \quad \Sigma_R^-(x_0) := B_R^-(x_0) \cup (B'_R(x_0) \times \{0\})\,,$$

$$Q_R(x_0) := B'_R(x_0) \times (-R,R)\,,$$

$$Q_R^+(x_0) := B'_R(x_0) \times (0,R)\,, \quad Q_R^-(x_0) := B'_R(x_0) \times (-R,0)\,,$$

$$\Gamma_R^+(x_0) := B'_R(x_0) \times [0,R)\,, \quad \Gamma_R^-(x_0) := B'_R(x_0) \times (-R,0]\,.$$

Given $b \in (-1,1)$, for any $x_0 \in \mathbb{R}^N$ and $R > 0$ we define the weighted Sobolev space $H^1(B_R^+(x_0);t^b)$ of functions $U \in L^2(B_R^+(x_0);t^b)$ such that $\nabla U \in L^2(B_R^+(x_0);t^b)$, endowed with the norm

$$\|U\|_{H^1(B_R^+(x_0);t^b)} := \left(\int_{B_R^+(x_0)} t^b |\nabla U(x,t)|^2 dx\,dt + \int_{B_R^+(x_0)} t^b (U(x,t))^2 dx\,dt\right)^{1/2}.$$

We also define the space $H_0^1(\Sigma_R^+(x_0);t^b)$ as the closure in $H^1(B_R^+(x_0);t^b)$ of $C_c^\infty(\Sigma_R^+(x_0))$.

In a completely similar way, we can introduce the Hilbert space $H^1(Q_R^+(x_0);t^b)$ and its subspace $H_0^1(\Gamma_R^+(x_0);t^b)$ defined as the closure in $H^1(Q_R^+(x_0);t^b)$ of $C_c^\infty(\Gamma_R^+(x_0))$.

We observe that thanks to (145) the spaces $H_0^1(\Sigma_R^+(x_0);t^b)$ and $H_0^1(\Gamma_R^+(x_0);t^b)$ may be endowed with the equivalent norms

$$\|U\|_{H_0^1(\Sigma_R^+(x_0);t^b)} := \left(\int_{B_R^+(x_0)} t^b |\nabla U|^2 dx\,dt\right)^{\frac{1}{2}}, \quad \|U\|_{H_0^1(\Gamma_R^+(x_0);t^b)} := \left(\int_{Q_R^+(x_0)} t^b |\nabla U|^2 dx\,dt\right)^{\frac{1}{2}}.$$

For any $x_0 \in \Omega$ let

$$R = R(x_0) > 0 \quad \text{be such that} \quad B'_{2R}(x_0) \subset \Omega. \tag{9}$$

Here and in the sequel $\Omega \subset \mathbb{R}^N$ is an open domain.

Let us consider the eigenvalue problem

$$\begin{cases} -\Delta_b U = \lambda U & \text{in } Q_{2R}^+(x_0)\,, \\ U = 0 & \text{on } [\partial B'_{2R}(x_0) \times (0,2R)] \cup [B'_{2R}(x_0) \times \{2R\}]\,, \\ \lim\limits_{t \to 0^+} t^b U_t(\cdot,t) \equiv 0 & \text{on } B'_{2R}(x_0), \end{cases} \tag{10}$$

in a weak sense, i.e.

$$\int_{Q_{2R}^+(x_0)} t^b \nabla U \nabla \varphi\,dx\,dt = \lambda \int_{Q_{2R}^+(x_0)} t^b U \varphi\,dx\,dt, \quad \text{for all } \varphi \in H_0^1(\Gamma_R^+(x_0);t^b). \tag{11}$$

In the following proposition we construct a complete orthonormal system for $L^2(Q_{2R}^+(x_0);t^b)$ consisting of eigenfunctions of (10).

Proposition 2.1. *Let $b \in (-1,1)$, $x_0 \in \Omega$ and let $R > 0$ be as in (9). Define*

$$e_{n,m}(x,t) := \gamma_m\, t^\alpha J_{-\alpha}\left(\tfrac{j_{-\alpha,m}}{2R}\, t\right) e_n(x) \qquad \text{for any } n,m \in \mathbb{N}\setminus\{0\} \tag{12}$$

and

$$\lambda_{n,m} := \mu_n + \tfrac{j^2_{-\alpha,m}}{4R^2}\,, \qquad \text{for any } n,m \in \mathbb{N}\setminus\{0\} \tag{13}$$

where $\alpha := \frac{1-b}{2}$, $J_{-\alpha}$ is the first kind Bessel function with index $-\alpha$,

$$0 < j_{-\alpha,1} < j_{-\alpha,2} < \cdots < j_{-\alpha,m} < \cdots$$

are the zeros of $J_{-\alpha}$, $\gamma_m := \left\{ \int_0^{2R} t\big[J_{-\alpha}\big(\frac{j_{-\alpha,m}}{2R}\, t\big)\big]^2 dt\right\}^{-1/2}$, $\{e_n\}_{n\geqslant 1}$ denotes a complete system, orthonormal in $L^2(B'_{2R}(x_0))$, of eigenfunctions of $-\Delta$ in $B'_{2R}(x_0)$ with homogeneous Dirichlet boundary conditions and $\mu_1 < \mu_2 \leqslant \ldots \leqslant \mu_n \leqslant \ldots$ the corresponding eigenvalues.

Then for any $n,m \in \mathbb{N}\setminus\{0\}$, $e_{n,m}$ is an eigenfunction of (10) with corresponding eigenvalue $\lambda_{n,m}$. Moreover the set $\{e_{n,m} : n,m \in \mathbb{N}\setminus\{0\}\}$ is a complete orthonormal system for $L^2(Q_{2R}^+(x_0);t^b)$.

Proof. We look for nontrivial solutions of (10) in the form

$$U(x,t) = \sum_{n=1}^{+\infty} A_n(t) e_n(x)\,.$$

By (11) it follows that A_n must satisfy

$$t^2 A_n''(t) + btA_n'(t) + (\lambda-\mu_n)t^2 A_n(t) = 0, \quad \lim_{t\to 0^+} t^b A_n'(t) = 0, \quad A_n(2R) = 0. \tag{14}$$

Using well known properties of Bessel functions, see [4], it is easy to prove that nontrivial solutions of (14) exist if and only if $\lambda - \mu_n > 0$; in this case A_n is necessarily given by

$$A_n(t) = c_n t^\alpha J_{-\alpha}(\sqrt{\lambda-\mu_n}\,t) \tag{15}$$

with λ satisfying $J_{-\alpha}(2\sqrt{\lambda-\mu_n}\,R) = 0$ whenever $c_n \neq 0$. Then λ necessarily satisfies

$$\lambda = \mu_n + \tfrac{j^2_{-\alpha,m}}{4R^2}\,, \qquad \text{for some } n,m \in \mathbb{N},\ n,m \geqslant 1. \tag{16}$$

This proves that the eigenvalues of $-\Delta_b$ are the numbers which admit the representation (16). For any number $\lambda > 0$ we denote by $S(\lambda)$ the possibly empty set defined by

$$S(\lambda) := \{(n,m) \in (\mathbb{N}\setminus\{0\})^2 : (16) \text{ holds true}\}\,.$$

For any $\lambda > 0$, the set $S(\lambda)$ is finite since $\lim_{n\to+\infty}\mu_n = +\infty$ and $\lim_{m\to+\infty} j_{-\alpha,m} = +\infty$. Hence, if λ is an eigenvalue, then the corresponding eigenfunctions U are of the form

$$U(x,t) = \sum_{(n,m)\in S(\lambda)} c_{n,m}\, t^\alpha J_{-\alpha}\left(\tfrac{j_{-\alpha,m}}{2R}\, t\right) e_n(x)\,.$$

For any $(n,m) \in (\mathbb{N}\setminus\{0\})^2$, we define $e_{n,m}$ as in (12). We note that $\|e_{n,m}\|_{L^2(Q_{2R}^+(x_0);t^b)} = 1$. Moreover we have orthogonality in $L^2(Q_{2R}^+(x_0);t^b)$ of two distinct eigenfunctions e_{n_1,m_1}, e_{n_2,m_2}, as one can easily deduce from [4, Equation (4.14.2)].

Finally, completeness of the orthonormal system $\{e_{n,m} : n,m \in \mathbb{N}\setminus\{0\}\}$ in $L^2(Q_{2R}^+(x_0);t^b)$ follows from compactness of the embedding stated in Proposition 7.1) and the theory of compact self-adjoint operators. □

In the next proposition we prove some estimates on the eigenfunctions of (10).

Proposition 2.2. *Suppose that all the assumptions of Proposition 2.1 hold true. Then for any* $n, m \in \mathbb{N} \setminus \{0\}$ *and* $k \geqslant 0$, $e_{n,m} \in C^k\left(\overline{Q_{2R}^+(x_0)}\right)$ *and, letting* $\delta = [N/4] + [(k+1)/2] + 1$, *with* $[\cdot]$ *denoting the integer part of a number, we have*

$$\|e_{n,m}\|_{C^k\left(\overline{Q_{2R}^+(x_0)}\right)} = \begin{cases} O\left(\lambda_{n,m}^{\frac{k}{2}+\delta+\frac{1}{4}}\right) & \text{if } \alpha \in \left[\frac{1}{2}, 1\right)\,, \\ O\left(\lambda_{n,m}^{\frac{k-\alpha+1}{2}+\delta}\right) & \text{if } \alpha \in \left(0, \frac{1}{2}\right)\,, \end{cases} \quad \text{as } |(n,m)| = \sqrt{n^2+m^2} \to +\infty\,. \tag{17}$$

Moreover we also have that

$$\lim_{t\to 0^+} \partial_t e_{n,m}(\cdot, t) = 0 \tag{18}$$

uniformly in $B'_{2R}(x_0)$.

Proof. From classical elliptic estimates (see [5, Chapter V]) and Sobolev embeddings we have that, for any $k \in \mathbb{N}$, there exists a constant $C(N, R, k)$ depending only on N, R and k such that

$$\|e_n\|_{C^k(B'_{2R}(x_0))} \leqslant C(N, R, k)\mu_n^\delta \tag{19}$$

with δ as in the statement of the lemma.

In order to obtain a similar estimate for the function $\gamma_m\, t^\alpha J_{-\alpha}\left(\frac{j_{-\alpha,m}}{2R}\, t\right)$ we first observe that

$$\gamma_m = \left[\left(\frac{2R}{j_{-\alpha,m}}\right)^2 \int_0^{j_{-\alpha,m}} t(J_{-\alpha}(t))^2 dt\right]^{-1/2} \leqslant \left[4R^2 \int_0^{j_{-\alpha,1}} t(J_{-\alpha}(t))^2 dt\right]^{-1/2} j_{-\alpha,m}\,. \tag{20}$$

By (162) and (164) in Subsection 7.3 and direct computation one may check that

$$\left\|\frac{d^k}{dt^k}\left(t^\alpha J_{-\alpha}\left(\frac{j_{-\alpha,m}}{2R}\, t\right)\right)\right\|_{L^\infty(0,2R)} \leqslant \left(\frac{j_{-\alpha,m}}{2R}\right)^{-\alpha+k} C(\alpha, k)[1 + (j_{-\alpha,m})^{\alpha-1/2}] \tag{21}$$

for any $k \in \mathbb{N}$, where $C(\alpha, k)$ is a positive constant depending only on α and k. Using (20) and (21) we can then prove that, for any $k \in \mathbb{N}$,

$$\left\|\frac{d^k}{dt^k}\left(\gamma_m\, t^\alpha J_{-\alpha}\left(\tfrac{j_{-\alpha,m}}{2R}\, t\right)\right)\right\|_{L^\infty(0,2R)} = \begin{cases} O((j_{-\alpha,m})^{k+1/2}) & \text{if } \alpha \in \left(\frac{1}{2}, 1\right)\,, \\ O((j_{-\alpha,m})^{k-\alpha+1}) & \text{if } \alpha \in \left(0, \frac{1}{2}\right]\,, \end{cases} \quad \text{as } m \to +\infty,$$

which, together with (12), (13) and (19), implies (17), thus proving the first part.

Finally, from the series expansion of first kind Bessel functions, see [4, Section 4.5], we infer that $\lim_{t\to 0^+}(t^\alpha J_{-\alpha}(t))' = 0$ which, together with (12), implies $\lim_{t\to 0^+} \partial_t e_{n,m}(\cdot, t) = 0$ uniformly in $B'_{2R}(x_0)$. This completes the proof of the proposition. □

Given a function $\psi \in C_c^\infty\left(Q_{2R}^+(x_0)\right)$, consider the following Poisson equation

$$\begin{cases} -\Delta_b \varphi = \psi & \text{in } Q_{2R}^+(x_0)\,, \\ \varphi = 0 & \text{on } [\partial B'_{2R}(x_0) \times (0, 2R)] \cup [B'_{2R}(x_0) \times \{2R\}]\,, \\ \lim_{t\to 0^+} t^b \varphi_t(\cdot, t) = 0 & \text{on } B'_{2R}(x_0) \times \{0\}\,. \end{cases} \tag{22}$$

We prove below the existence of a smooth solution to (22).

Proposition 2.3. *Let* $b \in (-1, 1)$, $x_0 \in \Omega$ *and let* $R > 0$ *be as in* (9). *Then for any* $\psi \in C_c^\infty\left(Q_{2R}^+(x_0)\right)$, (22) *admits a unique solution* $\varphi \in C^\infty\left(\overline{Q_{2R}^+(x_0)}\right)$. *Moreover* φ *satisfies*

$$\lim_{t\to 0^+} \varphi_t(\cdot, t) = 0$$

uniformly in $B'_{2R}(x_0)$.

Proof. The datum ψ can be written in the form

$$\psi(x, t) = \sum_{n,m=1}^{+\infty} c_{n,m}\, e_{n,m}(x, t)\,.$$

Then the solution φ of (22) is formally given by

$$\varphi(x,t)=\sum_{n,m=1}^{+\infty}\frac{c_{n,m}}{\lambda_{n,m}}\,e_{n,m}(x,t)\,.$$

We observe that by integration by parts and the fact that $e_{n,m}$ is an eigenfunction of $-\Delta_b$ corresponding to the eigenvalue $\lambda_{n,m}$, we have

$$\begin{aligned}c_{n,m}&=\int_{Q^+_{2R}(x_0)}t^b\psi e_{n,m}\,dx\,dt=\frac{1}{\lambda_{n,m}}\int_{Q^+_{2R}(x_0)}-t^b\psi\Delta_b e_{n,m}\,dx\,dt\\&=\frac{1}{\lambda_{n,m}}\int_{Q^+_{2R}(x_0)}-t^b\Delta_b\psi\, e_{n,m}\,dx\,dt\,.\end{aligned}$$

Iterating this procedure, we deduce that, for any $\ell\in\mathbb{N}$,

$$c_{n,m}=\frac{1}{\lambda^{\ell}_{n,m}}\int_{Q^+_{2R}(x_0)}t^b(-\Delta_b)^{\ell}\psi\, e_{n,m}\,dx\,dt=:\frac{1}{\lambda^{\ell}_{n,m}}\,d_{n,m,\ell}\,. \tag{23}$$

Since $\psi\in C^\infty_c(Q^+_{2R}(x_0))$ then $(-\Delta_b)^\ell\psi\in C^\infty_c(Q^+_{2R}(x_0))$ and hence $(-\Delta_b)^\ell\psi\in L^2(Q^+_{2R}(x_0);t^b)$. This yields $\sum_{n,m=1}^{+\infty}d^2_{n,m,\ell}<+\infty$ and, in turn, $\lim_{|(n,m)|\to+\infty}d_{n,m,\ell}=0$. This, combined with (23), shows that for any $\ell\in\mathbb{N}$

$$c_{n,m}=o(\lambda^{-\ell}_{n,m})\qquad\text{as }|(n,m)|\to+\infty\,. \tag{24}$$

By (17) and (24), we obtain as $|(n,m)|\to+\infty$

$$\left\|\frac{c_{n,m}}{\lambda_{n,m}}\,e_{n,m}\right\|_{C^k\left(\overline{Q^+_{2R}(x_0)}\right)}=\begin{cases}O(c_{n,m}\,\lambda_{n,m}^{\frac k2+\delta-\frac34})=o(\lambda_{n,m}^{\frac k2+\delta-\frac34-\ell}) & \text{if }\alpha\in\left(\frac12,1\right)\,,\\ O(c_{n,m}\,\lambda_{n,m}^{\frac{k-\alpha-1}{2}+\delta})=o(\lambda_{n,m}^{\frac{k-\alpha-1}{2}+\delta-\ell}) & \text{if }\alpha\in\left(0,\frac12\right]\,.\end{cases} \tag{25}$$

We put $L:=\ell-\frac k2-\delta+\frac34$ if $\alpha\in\left(\frac12,1\right)$ and $L:=\ell-\frac{k-\alpha-1}{2}-\delta$ if $\alpha\in\left(0,\frac12\right]$. We may fix ℓ large enough such that $L>N$ in both cases.

By (13), (165) and Weyl's Law for the asymptotic behavior of eigenvalues of $-\Delta$ with Dirichlet boundary conditions (see [31, 34]), we infer that there exists a constant $C>0$ such that

$$\lambda_{n,m}\geqslant C(n^{\frac2N}+m^2)\geqslant C(n^{\frac2N}+m^{\frac2N})\geqslant C(n^2+m^2)^{\frac1N}\qquad\text{for any }n,m\geqslant1\,.$$

Combining this with (25) we obtain

$$\left\|\frac{c_{n,m}}{\lambda_{n,m}}\,e_{n,m}\right\|_{C^k\left(\overline{Q^+_{2R}(x_0)}\right)}=o\Big((n^2+m^2)^{-\frac LN}\Big)\qquad\text{as }|(n,m)|\to+\infty\,.$$

Since $L>N$, this proves that

$$\sum_{n,m=1}^{+\infty}\left\|\frac{c_{n,m}}{\lambda_{n,m}}\,e_{n,m}\right\|_{C^k\left(\overline{Q^+_{2R}(x_0)}\right)}<+\infty$$

for any $k\in\mathbb{N}$ thus showing that $\varphi\in C^\infty\left(\overline{Q^+_{2R}(x_0)}\right)$.

Finally, by (18) we also have

$$\lim_{t\to0^+}\varphi_t(\cdot,t)=0\qquad\text{uniformly in }B'_{2R}(x_0)\,. \tag{26}$$

This completes the proof of the proposition. □

We are ready to prove the main result of this section.

Proposition 2.4. *Let $\Omega\subset\mathbb{R}^N$ be open. Let $s\in(1,2)$ and $b=3-2s$. Let $g\in(\mathcal{D}^{s-1,2}(\mathbb{R}^N))^\star$, $f\in L^2_{\rm loc}(\mathbb{R}^{N+1}_+;t^b)$ and let $V\in L^2(\mathbb{R}^{N+1}_+;t^b)$ be a distributional solution of the problem*

$$\begin{cases}\operatorname{div}(t^b\nabla V)=t^bf & \text{in }\mathbb{R}^{N+1}_+,\\ \lim_{t\to0^+}t^bV_t(\cdot,t)=g & \text{in }\Omega,\end{cases} \tag{27}$$

namely

$$\int_{\mathbb{R}^{N+1}_+} V \operatorname{div}(t^b \nabla\varphi)\, dx\, dt = \int_{\mathbb{R}^{N+1}_+} t^b f\varphi\, dx\, dt \qquad \text{for any } \varphi \in C^\infty_c(\mathbb{R}^{N+1}_+)$$

and

$$\int_{\mathbb{R}^{N+1}_+} V \operatorname{div}(t^b \nabla\varphi)\, dx\, dt = \int_{\mathbb{R}^{N+1}_+} t^b f\varphi\, dx\, dt + {}_{(\mathcal{D}^{s-1,2}(\mathbb{R}^N))^\star}\Big\langle g, \varphi(x,0)\Big\rangle_{\mathcal{D}^{s-1,2}(\mathbb{R}^N)} \tag{28}$$

for any $\varphi \in C^\infty_c\big(\overline{\mathbb{R}^{N+1}_+}\big)$ *such that* $\operatorname{supp}(\varphi(\cdot,0)) \subset \Omega$ *and* $\lim_{t\to 0^+} \varphi_t(\cdot,t) \equiv 0$ *in* $\mathbb{R}^N$.

Then $V \in H^1(Q^+_R(x_0); t^b)$ *for any* $x_0 \in \Omega$ *and* $R > 0$ *satisfying* (9) *and moreover there exists a positive constant* C *depending only on* N, b, x_0, R *such that*

$$\|V\|_{H^1(Q^+_R(x_0);t^b)} \leqslant C\Big(\|f\|_{L^2(Q^+_{2R}(x_0);t^b)} + \|g\|_{(\mathcal{D}^{s-1,2}(\mathbb{R}^N))^\star} + \|V\|_{L^2(\mathbb{R}^{N+1}_+;t^b)}\Big). \tag{29}$$

Proof. Let $x_0 \in \Omega$ and let $R > 0$ be as in (9). Let $\eta_0 \in C^\infty([0,\infty))$ be such that $0 \leqslant \eta_0 \leqslant 1$ in $[0,\infty)$, $\eta_0 \equiv 1$ in $[0,R]$ and $\eta_0 \equiv 0$ in $[2R,\infty)$. We now define $\eta : \mathbb{R}^{N+1} \to \mathbb{R}$ as $\eta(x,t) := \eta_0(|x-x_0|)\eta_0(t)$ for any $(x,t) \in \mathbb{R}^{N+1}_+$ and $W(x,t) := \eta(x,t)V(x,t)$ for any $(x,t) \in \mathbb{R}^{N+1}_+$.

By (28) and the fact that $\lim_{t\to 0^+} (\eta\varphi)_t(\cdot,t) \equiv 0$ in $\mathbb{R}^N$ for any function $\varphi \in C^\infty_c\big(\overline{\mathbb{R}^{N+1}_+}\big)$ satisfying $\operatorname{supp}(\varphi(\cdot,0)) \subset \Omega$ and $\lim_{t\to 0^+} \varphi_t(\cdot,t) \equiv 0$ in Ω, it turns out that

$$\begin{aligned}\int_{\mathbb{R}^{N+1}_+} W \operatorname{div}(t^b \nabla\varphi)\, dx\, dt = \int_{\mathbb{R}^{N+1}_+} t^b f\eta\varphi\, dx\, dt + {}_{(\mathcal{D}^{s-1,2}(\mathbb{R}^N))^\star}\Big\langle g, \eta(x,0)\varphi(x,0)\Big\rangle_{\mathcal{D}^{s-1,2}(\mathbb{R}^N)} \\ - \int_{\mathbb{R}^{N+1}_+} V\left[\operatorname{div}(t^b\nabla\eta)\varphi + 2t^b\nabla\eta\nabla\varphi\right] dx\, dt,\end{aligned} \tag{30}$$

where we exploited the identity $\eta \operatorname{div}(t^b\nabla\varphi) = \operatorname{div}(t^b\nabla(\eta\varphi)) - 2t^b\nabla\eta\nabla\varphi - \operatorname{div}(t^b\nabla\eta)\varphi$.

From this we can deduce that W is a solution of the problem

$$\begin{cases} W \in L^2(Q^+_{2R}(x_0); t^b), \\ \displaystyle\int_{Q^+_{2R}(x_0)} W \operatorname{div}(t^b\nabla\varphi)\, dx\, dt = \int_{Q^+_{2R}(x_0)} t^b f\eta\varphi\, dx\, dt \\ \displaystyle + {}_{(\mathcal{D}^{s-1,2}(\mathbb{R}^N))^\star}\Big\langle g, \eta(x,0)\varphi(x,0)\Big\rangle_{\mathcal{D}^{s-1,2}(\mathbb{R}^N)} - \int_{Q^+_{2R}(x_0)} V\left[\operatorname{div}(t^b\nabla\eta)\varphi + 2t^b\nabla\eta\nabla\varphi\right] dx\, dt \\ \text{for any } \varphi \in C^\infty(\overline{Q^+_{2R}(x_0)}) \text{ such that } \varphi \equiv 0 \text{ on } \partial Q_{2R}(x_0) \cap \mathbb{R}^{N+1}_+ \\ \text{and } \lim_{t\to 0^+} \varphi_t(\cdot,0) \equiv 0 \text{ in } B'_{2R}(x_0), \end{cases} \tag{31}$$

where the duality product has to be interpreted as applied to a trivial extension of $\eta\varphi$.

We divide the remaining part of the proof into three steps.

Step 1. We prove that given V, g as in the statement and η as above, there exists a unique solution of (31).

Suppose that W_1, W_2 are two of these functions and denote by W their difference. Then we have that $W \in L^2(Q^+_{2R}(x_0); t^b)$ and it satisfies

$$\int_{Q^+_{2R}(x_0)} W \operatorname{div}(t^b\nabla\varphi)\, dx\, dt = 0 \tag{32}$$

for any $\varphi \in C^\infty(\overline{Q^+_{2R}(x_0)})$ with $\varphi \equiv 0$ on $\partial Q_{2R}(x_0) \cap \mathbb{R}^{N+1}_+$ and $\lim_{t\to 0^+} \varphi_t(\cdot,t) \equiv 0$ in $B'_{2R}(x_0)$.

Let $\psi \in C^\infty_c(Q^+_{2R}(x_0))$ and let φ be the unique solution of (22). We have shown that such a function φ belongs to $C^\infty(\overline{Q^+_{2R}(x_0)})$. This together with (26) implies that φ is an admissible test function in (32). This yields

$$\int_{Q^+_{2R}(x_0)} t^b W\psi\, dx\, dt = -\int_{Q^+_{2R}(x_0)} W \operatorname{div}(t^b\nabla\varphi)\, dx\, dt = 0$$

for any $\psi \in C^\infty_c(Q^+_{2R}(x_0))$. This shows that $W \equiv 0$ in $Q^+_{2R}(x_0)$ and completes the proof of Step 1.

Step 2. In this step we prove that, for V, g as in the statement of the proposition and η as above, there exists a unique function $Z \in H^1_0(\Gamma^+_{2R}(x_0); t^b)$ such that

$$\int_{Q^+_{2R}(x_0)} t^b \nabla Z \nabla \varphi \, dx \, dt = -\int_{Q^+_{2R}(x_0)} t^b f \eta \varphi \, dx \, dt \tag{33}$$
$$- \, {}_{(\mathcal{D}^{s-1,2}(\mathbb{R}^N))^\star}\Big\langle g, \eta(x,0)\varphi(x,0) \Big\rangle_{\mathcal{D}^{s-1,2}(\mathbb{R}^N)} + \int_{Q^+_{2R}(x_0)} V \left[\operatorname{div}(t^b \nabla \eta)\varphi + 2t^b \nabla \eta \nabla \varphi \right] dx \, dt$$

for any $\varphi \in H^1_0(\Gamma^+_{2R}(x_0); t^b)$. We recall that there exists a well-defined continuous trace embedding from $\mathcal{D}^{1,2}(\mathbb{R}^{N+1}_+; t^b)$ into $\mathcal{D}^{s-1,2}(\mathbb{R}^N)$, see (149). We observe that for any $\varphi \in H^1_0(\Gamma^+_{2R}(x_0); t^b)$ the function $\eta\varphi$, once it is trivially extended outside $Q^+_{2R}(x_0)$, belongs to $\mathcal{D}^{1,2}(\mathbb{R}^{N+1}_+; t^b)$. We denote the trace of $\eta\varphi$ simply by $\eta(\cdot, 0)\varphi(\cdot, 0) \in \mathcal{D}^{s-1,2}(\mathbb{R}^N)$. We have

$$\left| {}_{(\mathcal{D}^{s-1,2}(\mathbb{R}^N))^\star}\Big\langle g, \eta(x,0)\varphi(x,0) \Big\rangle_{\mathcal{D}^{s-1,2}(\mathbb{R}^N)} \right| \leqslant \|g\|_{(\mathcal{D}^{s-1,2}(\mathbb{R}^N))^\star} \|\eta(\cdot,0)\varphi(\cdot,0)\|_{\mathcal{D}^{s-1,2}(\mathbb{R}^N)} \tag{34}$$
$$\leqslant \text{const} \, \|g\|_{(\mathcal{D}^{s-1,2}(\mathbb{R}^N))^\star} \|\eta\varphi\|_{\mathcal{D}^{1,2}(\mathbb{R}^{N+1}_+; t^b)}$$
$$\leqslant \text{const} \, \|g\|_{(\mathcal{D}^{s-1,2}(\mathbb{R}^N))^\star} \|\varphi\|_{H^1_0(\Gamma^+_{2R}(x_0); t^b)}$$

for some const > 0 depending only on N, R, b and η.

On the other hand, from the fact that $\eta_t(\cdot, 0) \equiv 0$ in Ω and by (145), we deduce that

$$\left| \int_{Q^+_{2R}(x_0)} V \operatorname{div}(t^b \nabla \eta)\varphi \, dx \, dt \right| \leqslant \Big(b\|\eta_t/t\|_{L^\infty(\mathbb{R}^{N+1}_+)} + \|\Delta\eta\|_{L^\infty(\mathbb{R}^{N+1}_+)} \Big) \int_{Q^+_{2R}(x_0)} t^b |V| \, |\varphi| \, dx \, dt \tag{35}$$
$$\leqslant \Big(b\|\eta_t/t\|_{L^\infty(\mathbb{R}^{N+1}_+)} + \|\Delta\eta\|_{L^\infty(\mathbb{R}^{N+1}_+)} \Big) \|V\|_{L^2(Q^+_{2R}(x_0); t^b)} \tfrac{4\sqrt{2}R}{N+b-1} \, \|\varphi\|_{H^1_0(\Gamma^+_{2R}(x_0); t^b)}$$

and

$$\left| \int_{Q^+_{2R}(x_0)} t^b f \eta \varphi \, dx \, dt \right| \leqslant \tfrac{4\sqrt{2}R}{N+b-1} \, \|f\|_{L^2(Q^+_{2R}(x_0); t^b)} \, \|\varphi\|_{H^1_0(\Gamma^+_{2R}(x_0); t^b)} \tag{36}$$

for any $\varphi \in H^1_0(\Gamma^+_{2R}(x_0); t^b)$.

Finally we have

$$\left| \int_{Q^+_{2R}(x_0)} V t^b \nabla \eta \nabla \varphi \, dx \, dt \right| \leqslant \|\nabla \eta\|_{L^\infty(\mathbb{R}^{N+1}_+)} \|V\|_{L^2(Q^+_{2R}(x_0); t^b)} \, \|\varphi\|_{H^1_0(\Gamma^+_{2R}(x_0); t^b)} \,. \tag{37}$$

From (34)-(37) and the Lax-Milgram Theorem we deduce that (33) admits a unique solution $Z \in H^1_0(\Gamma^+_{2R}(x_0); t^b)$. An integration by parts yields

$$\int_{B'_{2R}(x_0)\times(\varepsilon,2R)} t^b \nabla Z \nabla \varphi \, dx \, dt = -\int_{B'_{2R}(x_0)} \varepsilon^b Z(x,\varepsilon)\varphi_t(x,\varepsilon) \, dx - \int_{B'_{2R}(x_0)\times(\varepsilon,2R)} Z \operatorname{div}(t^b \nabla \varphi) \, dx \, dt$$

for any $\varphi \in C^\infty(\overline{Q^+_{2R}(x_0)}) \cap H^1_0(\Gamma^+_{2R}(x_0); t^b)$ satisfying $\lim_{t\to 0^+} \varphi_t(\cdot, t) = 0$ uniformly in $B'_{2R}(x_0)$. Passing to the limit as $\varepsilon \to 0^+$ we obtain

$$\int_{Q^+_{2R}(x_0)} t^b \nabla Z \nabla \varphi \, dx \, dt = -\int_{Q^+_{2R}(x_0)} Z \operatorname{div}(t^b \nabla \varphi) \, dx \, dt \,. \tag{38}$$

Actually, one has to prove first (38) for smooth functions Z and then, by a density argument, for all functions in $H^1_0(\Gamma^+_{2R}(x_0); t^b)$. Combining (33) and (38) we obtain

$$\int_{Q^+_{2R}(x_0)} Z \operatorname{div}(t^b \nabla \varphi) \, dx \, dt = \int_{Q^+_{2R}(x_0)} t^b f \eta \varphi \, dx \, dt \tag{39}$$
$$+ \, {}_{(\mathcal{D}^{s-1,2}(\mathbb{R}^N))^\star}\Big\langle g, \eta(x,0)\varphi(x,0) \Big\rangle_{\mathcal{D}^{s-1,2}(\mathbb{R}^N)} - \int_{Q^+_{2R}(x_0)} V \left[\operatorname{div}(t^b \nabla \eta)\varphi + 2t^b \nabla \eta \nabla \varphi \right] dx \, dt$$

for any $\varphi \in C^\infty(\overline{Q^+_{2R}(x_0)}) \cap H^1_0(\Gamma^+_{2R}(x_0); t^b)$ with $\lim\limits_{t\to 0^+} \varphi_t(\cdot, t) \equiv 0$ in $B'_{2R}(x_0)$. From this we deduce that Z is a solution of (31).

Step 3. We conclude the proof of the proposition. We have shown that (31) admits a unique solution, hence Z coincides in $Q^+_{2R}(x_0)$ with the function $W = \eta V$ defined at the beginning of the proof. In particular $\eta V \in H^1(Q^+_{2R}(x_0); t^b)$ and, in turn, $V \in H^1(Q^+_R(x_0); t^b)$ being $\eta \equiv 1$ in $Q^+_R(x_0)$. The proof of (29) follows from the estimates of Step 2 and standard application of the continuous dependence from the data in Lax-Milgram Theorem. □

3. An alternative formulation of problem (1)

Inspired by [10] and [46], we introduce an alternative formulation for problem (1). For any $1 < s < 2$ as in (1) we define $b := 3 - 2s \in (-1, 1)$. Next we define $\mathcal{D}_b$ as the completion of

$$\mathcal{T} = \left\{U \in C^\infty_c(\overline{\mathbb{R}^{N+1}_+}) : U_t \equiv 0 \text{ on } \mathbb{R}^N \times \{0\}\right\} \tag{40}$$

with respect to the norm

$$\|U\|_{\mathcal{D}_b} = \left(\int_{\mathbb{R}^{N+1}_+} t^b |\Delta_b U(x,t)|^2 \, dx \, dt\right)^{1/2}.$$

Under the assumption $N > 2s$, the validity of the Hardy-type inequality provided by Proposition 7.2 makes the abstract completion defined above isomorphic to a concrete functions space.

Let now $u \in \mathcal{D}^{s,2}(\mathbb{R}^N)$ be a solution of (1) in the sense given in (2) and let $U \in \mathcal{D}_b$ be the unique solution of (4).

The existence of a solution for problem (4) is essentially contained in [46]. For completeness, we provide here a rigorous formulation for (4) and we prove the existence and uniqueness of its solutions, thus giving a proof of Proposition 1.4.

In order to do that, we need to show that the trace map $\mathrm{Tr} : \mathcal{D}_b \to \mathcal{D}^{s,2}(\mathbb{R}^N)$ is well defined and continuous, so that the first boundary condition in (4) can be interpreted in the sense of traces. The construction of this trace operator is one of the main goals of this section.

The second boundary condition in (4) is a forced condition coming from the functional space $\mathcal{D}_b$ and has the following meaning: any function $U \in \mathcal{D}_b$ is the limit with respect to the norm $\|\cdot\|_{\mathcal{D}_b}$ of a sequence $\{U_n\}$ of smooth functions satisfying $\lim_{t\to 0^+} t^b (U_n)_t(\cdot, t) \equiv 0$ in $\mathbb{R}^N$. In other words, the boundary condition $\lim_{t\to 0^+} t^b U_t(\cdot, 0) \equiv 0$ on $\mathbb{R}^N$ is equivalent to the validity of the following integration by parts formula

$$\int_{\mathbb{R}^{N+1}_+} t^b \psi \Delta_b U \, dx \, dt = -\int_{\mathbb{R}^{N+1}_+} t^b \nabla U \nabla \psi \, dx \, dt, \quad \text{for any } \psi \in C^\infty_c(\overline{\mathbb{R}^{N+1}_+}). \tag{41}$$

As mentioned above our main purpose now is to construct the trace map $\mathrm{Tr} : \mathcal{D}_b \to \mathcal{D}^{s,2}(\mathbb{R}^N)$. We define the weighted Sobolev space $V(0, \infty; t^b)$ as the completion of

$$\{\varphi \in C^\infty_c([0, \infty)) : \varphi'(0) = 0\} \tag{42}$$

with respect to the norm

$$\|\varphi\|_{V(0,\infty;t^b)} = \left(\int_0^\infty t^b \left[|\Delta_{b,t}\varphi|^2 + |\varphi'|^2 + |\varphi|^2\right] dt\right)^{1/2} \tag{43}$$

where $\Delta_{b,t}\varphi = \varphi'' + \frac{b}{t}\varphi'$.

Lemma 3.1. *Let $V(0, \infty; t^b)$ be the space defined in (42)–(43). Then the following facts hold true:*

(i) $V(0, \infty; t^b) \subset C^1([0, \infty))$;

(ii) φ'', $\frac{\varphi'}{t} \in L^2(0, \infty; t^b)$ *and* $\varphi'(0) = 0$ *for any* $\varphi \in V(0, \infty; t^b)$;

(iii) *for any* $\varphi \in V(0, \infty; t^b)$ *there exists a constant* $C > 0$ *independent of* t *but possibly dependent on* φ *such that*

$$|\varphi(t)| \leqslant C(1 + t^{\frac{3-b}{2}}) \qquad \text{for any } t \geqslant 0\,. \tag{44}$$

Proof. We divide the proof of the lemma into several steps.

Step 1. By integration by parts, combined with some easy computations, one can prove that, for any φ as in (42),

$$\int_0^\infty t^b \left[(\Delta_{b,t}\varphi)^2 + (\varphi')^2 + \varphi^2\right] dt = \int_0^\infty t^b \left[|\varphi''(t)|^2 + bt^{-2}|\varphi'(t)|^2 + |\varphi'(t)|^2 + |\varphi(t)|^2\right] dt. \tag{45}$$

Step 2. We prove that for any φ as in (42) we have

$$\frac{(b-1)^2}{4}\int_0^\infty t^{b-2}|\varphi'(t)|^2dt \leqslant \int_0^\infty t^b|\varphi''(t)|^2dt\,. \tag{46}$$

Indeed, integration by parts yields

$$\begin{aligned}
0 &\leqslant \int_0^\infty \left(t^{\frac b2}\varphi''(t)+\frac{b-1}{2}t^{\frac b2-1}\varphi'(t)\right)^2 dt\\
&= \int_0^\infty t^b|\varphi''(t)|^2dt+\frac{(b-1)^2}{4}\int_0^\infty t^{b-2}|\varphi'(t)|^2dt+(b-1)\int_0^\infty t^{b-1}\varphi'(t)\varphi''(t)\,dt\\
&= \int_0^\infty t^b|\varphi''(t)|^2dt-\frac{(b-1)^2}{4}\int_0^\infty t^{b-2}|\varphi'(t)|^2dt\,.
\end{aligned}$$

Step 3. We prove that the norm in (43) and the norm

$$\varphi\mapsto\left(\int_0^\infty t^b\left(|\varphi''(t)|^2+|\varphi'(t)|^2+|\varphi(t)|^2\right)dt\right)^{1/2}$$

are equivalent on the space defined in (42).

If $b\in[0,1)$ the equivalence of the two norms follows by (45) and (46).

If $b\in(-1,0)$ one of the two estimate is trivial and for the other we proceed in this way:

$$\int_0^\infty t^b\left(|\varphi''(t)|^2+bt^{-2}|\varphi'(t)|^2\right)dt\geqslant\left(1+\frac{4b}{(b-1)^2}\right)\int_0^\infty t^b|\varphi''(t)|^2dt=\left(\frac{b+1}{b-1}\right)^2\int_0^\infty t^b|\varphi''(t)|^2dt$$

where the above inequality follows from (46) and the fact that $b<0$.

Step 4. In this step we complete the proof of the lemma. From Step 2 and Step 3 and a density argument we deduce that

$$\int_0^\infty t^b|\varphi''(t)|^2dt\leqslant C\|\varphi\|^2_{V(0,\infty;t^b)}\quad\text{and}\quad\int_0^\infty t^{b-2}|\varphi'(t)|^2dt\leqslant C\|\varphi\|^2_{V(0,\infty;t^b)}$$

for any $\varphi\in V(0,\infty;t^b)$, where C is a positive constant independent of φ. This proves the first two assertions in (ii).

For any φ as in (42) and $t>s>0$ we have, for some positive constant C independent of s,t and φ,

$$\begin{aligned}
|\varphi(t)-\varphi(s)|&=\left|\int_s^t\tau^{\frac b2-1}\varphi'(\tau)\tau^{1-\frac b2}d\tau\right|\leqslant\left(\int_s^t\tau^{b-2}|\varphi'(\tau)|^2d\tau\right)^{1/2}\left(\int_s^t\tau^{2-b}d\tau\right)^{1/2}\\
&\leqslant C\|\varphi\|_{V(0,\infty;t^b)}\left|t^{3-b}-s^{3-b}\right|^{1/2}
\end{aligned}\tag{47}$$

where the last inequality follows from Step 2 and Step 3. By density we have that estimate (47) actually holds for any $\varphi\in V(0,\infty;t^b)$. This proves that any $\varphi\in V(0,\infty;t^b)$ is continuous in $[0,+\infty)$ being $3-b>0$. Moreover if we put $s=0$ in (47) we obtain

$$\left|\frac{\varphi(t)-\varphi(0)}{t}\right|\leqslant C\|\varphi\|_{V(0,\infty;t^b)}t^{\frac{1-b}{2}}\qquad\text{and}\qquad|\varphi(t)|\leqslant|\varphi(0)|+C\|\varphi\|_{V(0,\infty;t^b)}t^{\frac{3-b}{2}}.\tag{48}$$

Since $b<1$, from the first estimate in (48) we deduce that φ is differentiable at 0 and $\varphi'(0)=0$ so that the proof of (ii) is complete. The second estimate in (48) gives (44) and proves (iii).

It remains to complete the proof of (i). For any φ as in (42) and $t>s>0$ we have, for some positive constant C independent of s,t and φ,

$$\begin{aligned}
|\varphi'(t)-\varphi'(s)|&=\left|\int_s^t\tau^{\frac b2}\varphi''(\tau)\tau^{-\frac b2}d\tau\right|\leqslant\left(\int_s^t\tau^b|\varphi''(\tau)|^2d\tau\right)^{1/2}\left(\int_s^t\tau^{-b}d\tau\right)^{1/2}\\
&\leqslant C\|\varphi\|_{V(0,\infty;t^b)}\left|t^{1-b}-s^{1-b}\right|^{1/2}
\end{aligned}\tag{49}$$

where the last inequality follows from Step 2 and Step 3. By density we have that estimate (49) actually holds for any $\varphi\in V(0,\infty;t^b)$. Since $b<1$, we deduce that φ' is continuous in $[0,\infty)$ and this completes the proof of (i). □

Thanks to Lemma 3.1 we can now prove the existence of a classical solution of (4) when the datum u is sufficiently smooth.

Lemma 3.2. *Let $u \in C_c^\infty(\mathbb{R}^N)$. Then (4) admits a classical solution $U \in C^2(\overline{\mathbb{R}^{N+1}_+})$. Moreover $U \in \mathcal{D}_b$ and the following assertions hold true:*

(i) *there exists a constant $C_b > 0$ depending only on b such that*

$$\|U\|_{\mathcal{D}_b} = C_b \|u\|_{\mathcal{D}^{s,2}(\mathbb{R}^N)}\,; \tag{50}$$

(ii) *for any $V \in C_c^\infty(\overline{\mathbb{R}^{N+1}_+})$ such that $V(\cdot,0) \equiv u$ and $V_t(\cdot,0) \equiv 0$ in $\mathbb{R}^N$, we have*

$$\|U\|_{\mathcal{D}_b} \leqslant \|V\|_{\mathcal{D}_b}\,. \tag{51}$$

Proof. Given a function $u \in C_c^\infty(\mathbb{R}^N)$ we aim to solve problem (4) by using the Fourier transform. Writing the equation $\Delta_b^2 U = 0$ as $\Delta_x^2 U + 2\Delta_{b,t}\Delta_x U + \Delta_{b,t}^2 U = 0$ and applying the Fourier transform with respect to the x variable to both sides of the equation, we formally obtain

$$|\xi|^4 \widehat{U} - 2|\xi|^2 \Delta_{b,t}\widehat{U} + \Delta_{b,t}^2 \widehat{U} = 0\,. \tag{52}$$

Following [46], we look for a solution of (52) in the form $\widehat{U}(\xi,t) = \widehat{u}(\xi)\phi(|\xi|t)$ with $\phi(0) = 1$ and $\phi'(0) = 0$. From (52), ϕ has to be a solution of the equation

$$\Delta_{b,t}^2 \phi - 2\Delta_{b,t}\phi + \phi = 0\,. \tag{53}$$

We now divide the rest of the proof into several steps.

Step 1. In this step we prove the existence of a solution to equation (53) in $V(0,\infty;t^b)$. We introduce the functional $J : V(0,\infty;t^b) \to \mathbb{R}$ defined as

$$J(\varphi) = \int_0^\infty t^b \left[(\Delta_{b,t}\varphi)^2 + 2(\varphi')^2 + \varphi^2\right] dt = \int_0^\infty t^b (\Delta_{b,t}\varphi - \varphi)^2 dt\,.$$

We observe that the equality between the second and third term in the above formula follows from the fact that $\int_0^\infty t^b \varphi(\Delta_{b,t}\varphi)\,dt = \int_0^\infty \varphi(t^b\varphi')'\,dt = -\int_0^\infty t^b(\varphi')^2\,dt$.

Thanks to Lemma 3.1, it is possible to consider the minimization problem

$$\min\{J(\varphi) : \varphi \in V(0,\infty;t^b),\ \varphi(0) = 1\}\,.$$

Since the functional J is clearly coercive with respect to the norm of $V(0,\infty;t^b)$, the minimization problem admits a weak solution ϕ which solves equation (53) and satisfies the initial conditions $\phi(0) = 1$ and $\phi'(0) = 0$. In particular we have

$$\int_0^\infty t^b[\Delta_{b,t}\phi(t) - \phi(t)][\Delta_{b,t}\psi(t) - \psi(t)]\,dt = 0 \tag{54}$$

for any $\psi \in V(0,\infty;t^b)$ such that $\psi(0) = \psi'(0) = 0$.

Step 2. We prove that $\phi \in C^2([0,\infty))$. If we put $\zeta(t) := \Delta_{b,t}\phi(t) - \phi(t) \in L^2(0,\infty;t^b)$, by (54), we see that ζ is a distributional solution of the equation

$$\Delta_{b,t}\zeta - \zeta = 0 \qquad \text{in } (0,\infty)\,. \tag{55}$$

We claim that $\zeta \in C^\infty(0,\infty)$ and it solves (55) in a classical sense.

Indeed, if we put $F(t) := \int_1^t s^b \zeta(s)\,ds$ then $F \in H^1_{\rm loc}(0,\infty)$ being $\zeta \in L^2(0,\infty;t^b)$ and moreover $F'(t) = t^b\zeta(t)$ in the sense of distributions.

Hence, by (55), $(t^b\zeta'(t) - F(t))' = 0$ in the sense of distributions so that $t^b\zeta'(t) = F(t) + c$ in $(0,\infty)$. This implies $\zeta' \in H^1_{\rm loc}(0,\infty)$ and in particular $\zeta \in H^2_{\rm loc}(0,\infty)$. Now, with a bootstrap procedure which makes use of (55), we conclude that $\zeta \in C^\infty(0,\infty)$.

Now we claim that $\zeta \in C^0([0,\infty))$. For any $t > s > 0$, by (55), we have

$$\begin{aligned}
\left|t^b\zeta'(t) - s^b\zeta'(s)\right| &= \left|\int_s^t (\tau^b\zeta(\tau))'d\tau\right| = \left|\int_s^t \tau^b \Delta_{b,\tau}\zeta(\tau)\,d\tau\right| \\
&\leqslant \left(\int_s^t \tau^b|\Delta_{\tau,b}\zeta(\tau)|^2 d\tau\right)^{1/2}\left(\int_s^t \tau^b d\tau\right)^{1/2} = \tfrac{1}{\sqrt{b+1}}\left(\int_s^t \tau^b|\zeta(\tau)|^2 d\tau\right)^{1/2}\left|t^{b+1} - s^{b+1}\right|^{1/2} \\
&\leqslant \tfrac{1}{\sqrt{b+1}}\|\zeta\|_{L^2(0,\infty;t^b)}\left|t^{b+1} - s^{b+1}\right|^{1/2}\,.
\end{aligned} \tag{56}$$

Since $b > -1$, choosing $t = 1$ in (56) and letting $s \to 0^+$, we infer that $s^b\zeta'(s) = O(1)$ as $s \to 0^+$ and, in turn, $\zeta'(s) = O(s^{-b})$ as $s \to 0^+$. This proves that ζ' is integrable in a right neighborhood of 0 and hence ζ is continuous at 0, thus proving the claim.

Next, we can proceed by completing the proof of Step 2. By

$$(t^b\phi'(t))' = t^b[\phi(t)+\zeta(t)] \tag{57}$$

we deduce that $\phi \in C^\infty(0,\infty)$. Moreover, integrating (57), for any $0<s<t$, we obtain

$$t^b\phi'(t) - s^b\phi'(s) = \int_s^t \tau^b[\phi(\tau)+\zeta(\tau)]\,d\tau\,. \tag{58}$$

By Lemma 3.1 (i), the continuity of ζ and the fact that $b>-1$, it follows

$$\lim_{s\to 0^+} s^b\phi'(s) = t^b\phi'(t) - \int_0^t \tau^b[\phi(\tau)+\zeta(\tau)]\,d\tau \in \mathbb{R}\,.$$

This means that there exists $L\in\mathbb{R}$ such that $\lim_{t\to 0^+} t^b\phi'(t) = L$. We observe that $L=0$ since otherwise we would have

$$t^b\frac{(\phi'(t))^2}{t^2} \sim L^2 t^{-b-2} \qquad \text{as } t\to 0^+$$

and hence $t^b\frac{(\phi'(t))^2}{t^2}\notin L^1(0,R)$ for any $R>0$, in contradiction with Lemma 3.1 (ii).

Therefore, letting $s\to 0^+$ in (58), we infer that

$$\phi'(t) = t^{-b}\int_0^t \tau^b[\phi(\tau)+\zeta(\tau)]\,d\tau \tag{59}$$

and, in turn, by de L'Hôpital rule, we obtain

$$\lim_{t\to 0^+}\frac{\phi'(t)}{t} = \lim_{t\to 0^+}\frac{\int_0^t \tau^b[\phi(\tau)+\zeta(\tau)]\,d\tau}{t^{b+1}} = \frac{\phi(0)+\zeta(0)}{b+1}\,.$$

Finally, by (57), we have that

$$\lim_{t\to 0^+}\phi''(t) = \lim_{t\to 0^+}\left(-b\frac{\phi'(t)}{t}+\phi(t)+\zeta(t)\right) = \frac{1}{b+1}\,[\phi(0)+\zeta(0)]\,.$$

This completes the proof of Step 2.

Step 3. We show that the function U, defined in such a way that $\widehat{U}(\xi,t)=\widehat{u}(\xi)\phi(|\xi|t)$ with ϕ as in Step 1, satisfies $U\in C^2(\overline{\mathbb{R}^{N+1}_+})$, $U_t(\cdot,0)\equiv 0$ in $\mathbb{R}^N$ and it solves (4) in a classical sense.

First, we observe that, by Lemma 3.1 (iii) and (56), ϕ,ζ' and, in turn also ζ, have at most a polynomial growth at $+\infty$. Hence, by (59) also ϕ' has at most a polynomial growth at $+\infty$. Finally, from the equation $\Delta_{b,t}\phi=\phi+\zeta$, we also deduce that ϕ'' has at most a polynomial growth at $+\infty$.

Therefore, since $\phi\in C^2([0,\infty))$ and $\widehat{u}\in\mathcal{S}(\mathbb{R}^N)$, with $\mathcal{S}(\mathbb{R}^N)$ the space of rapidly decreasing $C^\infty(\mathbb{R}^N)$ functions, by the Dominated Convergence Theorem, one can deduce that the map $t\mapsto \widehat{u}(\xi)\phi(|\xi|t)$ belongs to the space of vector valued functions $C^2([0,\infty);L^2_{\mathbb{C}}(\mathbb{R}^N;(1+|\xi|^2)^\gamma))$ for any $\gamma\geqslant 0$. Here $L^2_{\mathbb{C}}(\mathbb{R}^N;(1+|\xi|^2)^\gamma)$ denotes the weighted complex L^2-space. This proves that the map $t\mapsto U(x,t)$ belongs to the space $C^2([0,\infty);H^\gamma(\mathbb{R}^N))$ for any $\gamma\geqslant 0$. From this we deduce that $U\in C^2(\overline{\mathbb{R}^{N+1}_+})$. Since

$$U_t(x,t) = \frac{1}{(2\pi)^{N/2}}\int_{\mathbb{R}^N} e^{i\xi\cdot x}\,\widehat{u}(\xi)\,|\xi|\,\phi'(|\xi|t)\,d\xi$$

and $\phi'(0)=0$ it follows that $U_t(x,0)=0$ for any $x\in\mathbb{R}^N$. By construction, we also have that U is a classical solution of (4).

Step 4. We prove that $\Delta_b U\in L^2(\mathbb{R}^{N+1}_+;t^b)$ and

$$\int_{\mathbb{R}^{N+1}_+} t^b|\Delta_b U(x,t)|^2 dx\,dt = J(\phi)\int_{\mathbb{R}^N}|\xi|^{2s}|\widehat{u}(\xi)|^2 d\xi\,. \tag{60}$$

By direct computation we see that

$$|\Delta_{b,t}\widehat{U}(\xi,t)-|\xi|^2\widehat{U}(\xi,t)|^2 = |\xi|^4|\widehat{u}(\xi)|^2[\Delta_{b,t}\phi(|\xi|t)-\phi(|\xi|t)]^2\,.$$

After integration, a change of variable with respect to t and Fubini-Tonelli Theorem, we obtain

$$\int_{\mathbb{R}^{N+1}_+} t^b|\Delta_{b,t}\widehat{U}(\xi,t)-|\xi|^2\widehat{U}(\xi,t)|^2 d\xi dt = \int_{\mathbb{R}^{N+1}_+} |\xi|^{3-b}|\widehat{u}(\xi)|^2 t^b[\Delta_{b,t}\phi(t)-\phi(t)]^2 d\xi dt = J(\phi)\int_{\mathbb{R}^N}|\xi|^{2s}|\widehat{u}(\xi)|^2 d\xi\,. \tag{61}$$

Since $\widehat{u}\in\mathcal{S}(\mathbb{R}^N)$, the last integral is finite and hence, by Fubini-Tonelli Theorem, for almost every $t\in(0,\infty)$ the map $\xi\mapsto\Delta_{b,t}\widehat{U}(\xi,t)-|\xi|^2\widehat{U}(\xi,t)=\widehat{\Delta_b U}(\xi,t)$ belongs to the complex space $L^2_{\mathbb{C}}(\mathbb{R}^N)$. Hence by Plancherel Theorem also the map $x\mapsto\Delta_b U(x,t)$ belongs to $L^2(\mathbb{R}^N)$ for almost every $t\in(0,\infty)$. Moreover

$$\int_{\mathbb{R}^N}|\Delta_b U(x,t)|^2 dx=\int_{\mathbb{R}^N}|\Delta_{b,t}\widehat{U}(\xi,t)-|\xi|^2\widehat{U}(\xi,t)|^2 d\xi \qquad \text{for almost every } t\in(0,\infty)\,.$$

Multiplying this identity by t^b, integrating in $(0,\infty)$ with respect to the variable t and applying Fubini-Tonelli Theorem we deduce that $\Delta_b U\in L^2(\mathbb{R}^{N+1}_+;t^b)$. Moreover (60) follows by exploiting (61).

Step 5. We prove that $U\in\mathcal{D}_b$.

We have to prove that U can be approximated with functions in $\mathcal{T}$ with respect to the norm $\|\cdot\|_{\mathcal{D}_b}$. Here $\mathcal{T}$ is the space defined in (40).

Combining Plancherel Theorem with the fact that $\widehat{u}\in\mathcal{S}(\mathbb{R}^N)$ and $\phi\in V(0,\infty;t^b)$ one can verify that $U\in L^2(\mathbb{R}^{N+1}_+;t^b)$ and $\nabla U\in L^2(\mathbb{R}^{N+1}_+;t^b)$. Therefore since $U\in C^2(\overline{\mathbb{R}^{N+1}_+})$ we also have that

$$\frac{U}{|x|^2+t^2}\in L^2(\mathbb{R}^{N+1}_+;t^b) \qquad\text{and}\qquad \frac{|\nabla U|}{\sqrt{|x|^2+t^2}}\in L^2(\mathbb{R}^{N+1}_+;t^b)\,. \tag{62}$$

Define $U_n(x,t)=\eta\left(\frac{|x|}{n}\right)\eta\left(\frac{t}{n}\right)U(x,t)$ where $\eta\in C^\infty([0,\infty))$, $\eta\equiv1$ in $[0,1]$ and $\eta\equiv0$ in $[2,\infty)$. We prove that

$$\int_{\mathbb{R}^{N+1}_+} t^b|\Delta_b(U_n-U)|^2 dx\,dt\to0 \qquad \text{as } n\to+\infty\,. \tag{63}$$

By direct computation one sees that

$$\begin{aligned}\Delta_b U_n(x,t)&=\eta\left(\tfrac{t}{n}\right)\Theta\left(\tfrac{x}{n}\right)\Delta_b U(x,t)+\eta\left(\tfrac{t}{n}\right)\left[\tfrac{1}{n^2}\Delta_x\Theta\left(\tfrac{x}{n}\right)U(x,t)+\tfrac{2}{n}\nabla_x\Theta\left(\tfrac{x}{n}\right)\nabla_x U(x,t)\right]\\&\quad+\Theta\left(\tfrac{x}{n}\right)\left[\tfrac{1}{n^2}\eta''\left(\tfrac{t}{n}\right)U(x,t)+\tfrac{2}{n}\eta'\left(\tfrac{t}{n}\right)U_t(x,t)+\tfrac{b}{t}\,\tfrac{1}{n}\eta'\left(\tfrac{t}{n}\right)U(x,t)\right]\end{aligned} \tag{64}$$

where we put $\Theta(x)=\eta(|x|)$. Then, we observe that there exists a positive constant C independent of x, t and n, such that

$$\begin{aligned}&t^b\left|\eta\left(\tfrac{t}{n}\right)\Theta\left(\tfrac{x}{n}\right)\Delta_b U(x,t)\right|^2\leqslant t^b|\Delta_b U(x,t)|^2\,, \quad \tfrac{t^b}{n^4}\left|\eta\left(\tfrac{t}{n}\right)\Delta_x\Theta\left(\tfrac{x}{n}\right)U(x,t)\right|^2\leqslant Ct^b\tfrac{U^2(z)}{|z|^4}\,,\\&\tfrac{4t^b}{n^2}\left|\eta\left(\tfrac{t}{n}\right)\nabla_x\Theta\left(\tfrac{x}{n}\right)\nabla_x U(x,t)\right|^2\leqslant Ct^b\tfrac{|\nabla U(z)|^2}{|z|^2}\,, \quad \tfrac{t^b}{n^4}\left|\Theta\left(\tfrac{x}{n}\right)\eta''\left(\tfrac{t}{n}\right)U(x,t)\right|^2\leqslant Ct^b\tfrac{U^2(z)}{|z|^4}\,,\\&\tfrac{4t^b}{n^2}\left|\Theta\left(\tfrac{x}{n}\right)\eta'\left(\tfrac{t}{n}\right)U_t(x,t)\right|^2\leqslant Ct^b\tfrac{|\nabla U(z)|^2}{|z|^2}\,, \quad \tfrac{b^2t^b}{n^4}\left|\Theta\left(\tfrac{x}{n}\right)\tfrac{\eta'(t/n)}{t/n}U(x,t)\right|^2\leqslant Ct^b\tfrac{U^2(z)}{|z|^4}\,,\end{aligned} \tag{65}$$

since $|z|\leqslant\sqrt{8}n$ for any $z\in\operatorname{supp}\left(\eta\left(\frac{t}{n}\right)\Theta\left(\frac{x}{n}\right)\right)$ where we put $z=(x,t)\in\mathbb{R}^{N+1}$.

By (62), (64), (65) and the Dominated Convergence Theorem, (63) follows.

This shows that for any $\varepsilon>0$ there exists a function $V\in C^2_c(\overline{\mathbb{R}^{N+1}_+})$ such that

$$\int_{\mathbb{R}^{N+1}_+} t^b|\Delta_b(U-V)|^2 dx\,dt<\varepsilon\,.$$

By Step 3 and the truncation argument introduced above, we deduce that we can choose V in such a way that $V_t(\cdot,0)\equiv0$ in $\mathbb{R}^N$.

A mollification argument allows us to approximate, with respect to the norm $\|\cdot\|_{\mathcal{D}_b}$, the function V found above, with a C^∞ compactly supported function W satisfying $W_t(\cdot,0)\equiv0$ in $\mathbb{R}^N$. Indeed, one can introduce a sequence of mollifiers $\{\rho_n\}$ and still denote by V the even extension with respect to the variable t to the whole $\mathbb{R}^{N+1}$. This extension satisfies $V\in C^2_c(\mathbb{R}^{N+1})$ since $V_t(x,0)=0$ for any $x\in\mathbb{R}^N$. We choose the functions ρ_n even with respect to the t variable. Then one can verify that the functions $W_n:=\rho_n*V\in C^\infty_c(\mathbb{R}^{N+1})$ are even with respect to t and the functions $\partial_t W_n$

are odd with respect to t; in particular $\partial_t W_n(\cdot,0)\equiv 0$ in $\mathbb{R}^N$. Exploiting the fact that for any $n\in\mathbb{N}$, $\partial_t W_n$ is odd with respect to t, one can show that $|\partial_t W_n(x,t)|\leqslant C|t|$ for any $(x,t)\in\mathbb{R}^{N+1}$ and $n\in\mathbb{N}$ where C is a constant independent of $(x,t)\in\mathbb{R}^{N+1}$ and $n\in\mathbb{N}$.

Combining this estimate with the fact that $V\in C^2_c(\mathbb{R}^{N+1})$, by the Dominated Convergence Theorem we obtain $\int_{\mathbb{R}^{N+1}}|t|^b\,|\Delta_b(W_n-V)|^2dx\,dt\to 0$ as $n\to+\infty$. We have just shown that $U\in\mathcal{D}_b$.

Step 6. In this step we complete the proof of the lemma. The proof of (i) follows from (60) once we put $C_b:=\sqrt{J(\phi)}$.

It remains to prove (ii). Let $\Phi\in C^\infty_c(\overline{\mathbb{R}^{N+1}_+})$ such that $\Phi(x,0)=\Phi_t(x,0)=0$ for any $x\in\mathbb{R}^N$. Recalling that $\widehat{\Delta_b U}(\xi,t)=|\xi|^2\widehat{u}(\xi)[\Delta_{b,t}\phi(|\xi|t)-\phi(|\xi|t)]$, by Plancherel Theorem, Fubini-Tonelli Theorem and a change of variable, we have

$$\begin{aligned}(66)\quad &\int_{\mathbb{R}^{N+1}_+} t^b\Delta_b U(x,t)\Delta_b\Phi(x,t)\,dx\,dt\\ &=\int_{\mathbb{R}^N}\left(\int_0^\infty t^b|\xi|^2\,\widehat{u}(\xi)\,[\Delta_{b,t}\phi(|\xi|t)-\phi(|\xi|t)]\,\overline{\left[\Delta_{b,t}\widehat{\Phi}(\xi,t)-|\xi|^2\widehat{\Phi}(\xi,t)\right]}\,dt\right)d\xi\\ &=\int_{\mathbb{R}^N}\left(\int_0^\infty t^b|\xi|^{1-b}\,\widehat{u}(\xi)\,[\Delta_{b,t}\phi(t)-\phi(t)]\,\overline{\left[\Delta_{b,t}\widehat{\Phi}\left(\xi,\tfrac{t}{|\xi|}\right)-|\xi|^2\widehat{\Phi}\left(\xi,\tfrac{t}{|\xi|}\right)\right]}\,dt\right)d\xi\\ &=\int_{\mathbb{R}^N}|\xi|^{3-b}\,\widehat{u}(\xi)\left(\int_0^\infty t^b[\Delta_{b,t}\phi(t)-\phi(t)]\,\overline{\left[\Delta_{b,t}\left(\widehat{\Phi}\left(\xi,\tfrac{t}{|\xi|}\right)\right)-\widehat{\Phi}\left(\xi,\tfrac{t}{|\xi|}\right)\right]}\,dt\right)d\xi=0\end{aligned}$$

where the last identity follows from the fact that, for any $\xi\neq 0$, the real part and the imaginary part of the map $t\mapsto\widehat{\Phi}\left(\xi,\frac{t}{|\xi|}\right)$ are admissible test functions in (54) since they belong to $C^\infty_c([0,\infty))$ and they vanish at $t=0$ together with their first derivatives. By a density argument combined with the regularization procedure shown in Step 5, one can show that (66) actually holds for any $\Phi\in C^2(\overline{\mathbb{R}^{N+1}_+})$ such that

$$(67)\qquad \Delta_b\Phi\in L^2(\mathbb{R}^{N+1}_+;t^b)\,,\qquad \frac{|\nabla\Phi|}{\sqrt{|x|^2+t^2}}\in L^2(\mathbb{R}^{N+1}_+;t^b)\,,$$
$$\frac{\Phi}{|x|^2+t^2}\in L^2(\mathbb{R}^{N+1}_+;t^b)\,,\qquad \Phi(\cdot,0)\equiv\Phi_t(\cdot,0)\equiv 0\text{ in }\mathbb{R}^N\,.$$

Let V be as in the statement of the lemma and put $\Phi:=V-U$ is such a way that $\Phi\in C^2(\overline{\mathbb{R}^{N+1}_+})$ and it satisfies (67). By (66) we then have

$$\|V\|^2_{\mathcal{D}_b}=\|\Phi\|^2_{\mathcal{D}_b}+2\int_{\mathbb{R}^{N+1}_+}t^b\Delta_b U\Delta_b\Phi\,dx\,dt+\|U\|^2_{\mathcal{D}_b}=\|\Phi\|^2_{\mathcal{D}_b}+\|U\|^2_{\mathcal{D}_b}\geqslant\|U\|^2_{\mathcal{D}_b}\,.$$

This completes the proof of the lemma. □

Thanks to Lemma 3.2, in the next proposition we construct a trace map $\mathrm{Tr}:\mathcal{D}_b\to\mathcal{D}^{s,2}(\mathbb{R}^N)$.

Proposition 3.3. *Let $s\in(1,2)$ and let $b=3-2s\in(-1,1)$. Then there exists a linear continuous map $\mathrm{Tr}:\mathcal{D}_b\to\mathcal{D}^{s,2}(\mathbb{R}^N)$ such that $\mathrm{Tr}(V)=V_{|\mathbb{R}^N\times\{0\}}$ for any $V\in C^\infty_c(\overline{\mathbb{R}^{N+1}_+})$.*

Proof. Let $V\in C^\infty_c(\overline{\mathbb{R}^{N+1}_+})$ be such that $V_t(\cdot,0)\equiv 0$ in $\mathbb{R}^N$ and put $u=V_{|\mathbb{R}^N\times\{0\}}\in C^\infty_c(\mathbb{R}^N)$. By Lemma 3.2, we deduce that there exists $U\in C^2(\overline{\mathbb{R}^{N+1}_+})\cap\mathcal{D}_b$ such that

$$(68)\qquad U_{|\mathbb{R}^N\times\{0\}}=u\,,\quad \|U\|_{\mathcal{D}_b}=C_b\|u\|_{\mathcal{D}^{s,2}(\mathbb{R}^N)}\,,\quad \|U\|_{\mathcal{D}_b}\leqslant\|V\|_{\mathcal{D}_b}\,.$$

Therefore, if we put $\mathrm{Tr}(V):=u$ we have $\|\mathrm{Tr}(V)\|_{\mathcal{D}^{s,2}(\mathbb{R}^N)}\leqslant C_b^{-1}\|V\|_{\mathcal{D}_b}$. The conclusion follows by completion. □

We can now proceed with the proof of Proposition 1.4.

Proof of Proposition 1.4. Let $u\in\mathcal{D}^{s,2}(\mathbb{R}^N)$. Let $\{u_n\}\subset C^\infty_c(\mathbb{R}^N)$ be such that $u_n\to u$ in $\mathcal{D}^{s,2}(\mathbb{R}^N)$ and let $\{U_n\}\subset\mathcal{D}_b$ be the corresponding sequence of solutions of (4) whose existence is shown in Lemma 3.2. By (50) we deduce that $\{U_n\}$ is a Cauchy sequence in $\mathcal{D}_b$ and hence there exists a function $U\in\mathcal{D}_b$ such that $U_n\to U$ in $\mathcal{D}_b$. In particular, by Proposition 3.3 we have that

$\mathrm{Tr}(U) = u$ and moreover U solves (5) being U_n weak solutions of (4) corresponding to u_n for any n. This proves the existence of a solution of (4).

In order to prove uniqueness of solutions to (4) it is sufficient to consider (5) with $u \equiv 0$ and to prove that it admits only the trivial solution. Being $\mathrm{Tr}(U) = 0$, U becomes an admissible test function so that, choosing $\varphi = U$ in (5), it follows immediately that $U \equiv 0$. □

Let u be a solution of (1) and let $U \in \mathcal{D}_b$ be the corresponding solution to (4). From Lemma 3.2 it follows that $C_b^2 \|u\|^2_{\mathcal{D}^{s,2}(\mathbb{R}^N)} = \|U\|^2_{\mathcal{D}_b}$. Moreover by the proof of Proposition 3.3, for all $\varphi \in \mathcal{D}_b$ satisfying $\mathrm{Tr}\,(\varphi) = u$, we have that

$$C_b^2 \|\mathrm{Tr}\,(\varphi)\|^2_{\mathcal{D}^{s,2}(\mathbb{R}^N)} = \|U\|^2_{\mathcal{D}_b} \leqslant \|\varphi\|^2_{\mathcal{D}_b}, \tag{69}$$

which is equivalent to say that $U \in \mathcal{D}_b$ is a solution to the minimum problem

$$\min_{\varphi \in \mathcal{D}_b, \mathrm{Tr}\,(\varphi) = u} \left\{ \|\varphi\|^2_{\mathcal{D}_b} - C_b^2 \|\mathrm{Tr}\,(\varphi)\|^2_{\mathcal{D}^{s,2}(\mathbb{R}^N)} \right\}.$$

Therefore we have

$$(U, \psi)_{\mathcal{D}_b} = 0 \qquad \text{for any } \psi \in \mathcal{D}_b \text{ such that } \mathrm{Tr}\,(\psi) = 0\,. \tag{70}$$

Now, for any $\varphi \in \mathcal{D}_b$ we denote by $\Phi \in \mathcal{D}_b$ the solution of (4) corresponding to $\mathrm{Tr}\,(\varphi)$. By (68) we have that

$$\|U + \Phi\|^2_{\mathcal{D}_b} = C_b^2 \|u + \mathrm{Tr}\,(\varphi)\|^2_{\mathcal{D}^{s,2}(\mathbb{R}^N)} \quad \text{and} \quad \|U - \Phi\|^2_{\mathcal{D}_b} = C_b^2 \|u - \mathrm{Tr}\,(\varphi)\|^2_{\mathcal{D}^{s,2}(\mathbb{R}^N)}$$

and taking the difference we obtain

$$(U, \Phi)_{\mathcal{D}_b} = C_b^2 (u, \mathrm{Tr}\,(\varphi))_{\mathcal{D}^{s,2}(\mathbb{R}^N)}\,. \tag{71}$$

Since $\mathrm{Tr}\,(\varphi - \Phi) = 0$, combining (70) and (71) we obtain

$$(U, \varphi)_{\mathcal{D}_b} = (U, \Phi)_{\mathcal{D}_b} = C_b^2 (u, \mathrm{Tr}\,(\varphi))_{\mathcal{D}^{s,2}(\mathbb{R}^N)} \qquad \text{for any } \varphi \in \mathcal{D}_b. \tag{72}$$

Hence $u \in \mathcal{D}^{s,2}(\mathbb{R}^N)$ solves (2) if and only if the corresponding function $U \in \mathcal{D}_b$ solving (4) is a solution to

$$(U, \varphi)_{\mathcal{D}_b} = 0 \qquad \text{for all } \varphi \in \mathcal{D}_b \text{ s.t. } \mathrm{supp}\,(\mathrm{Tr}\,(\varphi)) \subset \Omega\,. \tag{73}$$

4. An Almgren type monotonicity formula

Let us assume that $U \in \mathcal{D}_b$ is a solution to (73). Let us set

$$V := \Delta_b U \in L^2(\mathbb{R}^{N+1}_+; t^b), \tag{74}$$

i.e., in view of (41) and Proposition 7.2,

$$\int_{\mathbb{R}^{N+1}_+} t^b V \varphi \, dz = - \int_{\mathbb{R}^{N+1}_+} t^b \nabla U \nabla \varphi \, dz\,, \qquad \text{for any } \varphi \in C_c^\infty(\overline{\mathbb{R}^{N+1}_+}). \tag{75}$$

Furthermore (73) yields

$$\int_{\mathbb{R}^{N+1}_+} V \,\mathrm{div}(t^b \nabla \varphi)\, dz = 0 \tag{76}$$

for any $\varphi \in C_c^\infty\left(\overline{\mathbb{R}^{N+1}_+}\right)$ such that $\mathrm{supp}(\varphi(\cdot, 0)) \subset \Omega$ and $\lim_{t \to 0^+} \varphi_t(\cdot, t) \equiv 0$ in $\mathbb{R}^N$. Proposition 2.4 then ensures that

$$V \in H^1(Q_R^+(x_0); t^b) \text{ for any } x_0 \in \Omega \text{ and } R > 0 \text{ satisfying } B'_{2R}(x_0) \subset \Omega. \tag{77}$$

Up to translation it is not restrictive to suppose that $x_0 = 0 \in \Omega$. Then we fix a radius $R > 0$ satisfying (77). For simplicity, the center x_0 of the sets introduced in (8) will be omitted whenever $x_0 = 0$.

By (76)-(77) we obtain

$$\int_{B_R^+} t^b \nabla V \nabla \varphi \, dz = 0 \tag{78}$$

for any $\varphi \in C_c^\infty(\Sigma_R^+(0))$ such that $\varphi_t(\cdot, 0) \equiv 0$ in B'_R.

Actually (78) still holds true for any $\varphi \in C_c^\infty(\Sigma_R^+(0))$ not necessarily satisfying $\varphi_t(\cdot,0)\equiv 0$ in B'_R. Indeed, for any $\varphi \in C_c^\infty(\Sigma_R^+(0))$, one can test (78) with $\varphi_k(x,t)=\varphi(x,t)-\varphi_t(x,0)\,t\,\eta(kt)$, $k\in\mathbb{N}$, where $\eta\in C_c^\infty(\mathbb{R})$, $0\leqslant\eta\leqslant 1$, $\eta(t)=1$ for any $t\in[-1,1]$ and $\eta(t)=0$ for any $t\in(-\infty,-2]\cup[2,+\infty)$, and pass to the limit as $k\to+\infty$.

By density we may conclude that

$$\int_{B_R^+} t^b \nabla V \nabla\varphi\, dz = 0 \qquad \text{for any } \varphi\in H_0^1(\Sigma_R^+;t^b)\,.$$

Hence, the couple $(U,V)\in\mathcal{D}_b\times L^2(\mathbb{R}_+^{N+1};t^b)$ is a weak solution to the system (6) in the sense that (75) and (76) hold together with the forced boundary condition (41). Thanks to Proposition 7.2 and (77), we may define the functions

$$D(r)=r^{-N-b+1}\left[\int_{B_r^+} t^b\left(|\nabla U|^2+|\nabla V|^2+UV\right)dz\right] \tag{79}$$

and

$$H(r)=r^{-N-b}\int_{S_r^+} t^b(U^2+V^2)\,dS\,. \tag{80}$$

We observe that the function $H=H(r)$ is well defined for every $r>0$ such that $B'_{2r}\subset\Omega$ since the trace operator

$$\mathrm{Tr}_{S_r}: H^1(B_r^+;t^b)\to L^2(S_r^+;t^b)$$

is well-defined and continuous being $b\in(-1,1)$, see [15, Subsection 2.2].

We now prove a Pohozaev-type identity for system (6).

Lemma 4.1. *Let U and V be as in* (73) *and* (74)*. Then for a.e. $r>0$ such that $B'_{2r}\subset\Omega$ we have*

$$\int_{B_r^+} t^b\left(|\nabla U|^2+|\nabla V|^2+UV\right)dz=\int_{S_r^+} t^b\left(\frac{\partial U}{\partial\nu}U+\frac{\partial V}{\partial\nu}V\right)dS \tag{81}$$

and

$$-\frac{N+b-1}{2}\int_{B_r^+} t^b\left(|\nabla U|^2+|\nabla V|^2\right)dz+\int_{B_r^+} t^b V(z\cdot\nabla U)\,dz + \frac{r}{2}\int_{S_r^+} t^b\left(|\nabla U|^2+|\nabla V|^2\right)dS = r\int_{S_r^+} t^b\left(\left|\frac{\partial U}{\partial\nu}\right|^2+\left|\frac{\partial V}{\partial\nu}\right|^2\right)dS\,. \tag{82}$$

Proof. The proof of this lemma can be obtained proceeding exactly as in the proof of Theorem 3.7 in [15]. Hence here we omit the details and we show only the main steps. Let us consider first identity (82). Let r be as in the statement of the lemma. Similarly to [15], for any $\delta>0$ we define the set

$$O_\delta := B_r^+\cap\{(x,t): t>\delta\}\,.$$

By (6) and exploiting [15, (51)] by replacing their $1-2s$ with our $b=3-2s$, we obtain

$$\begin{aligned}\frac{N+b-1}{2}\int_{O_\delta} t^b|\nabla U|^2 dz-\int_{O_\delta} t^b V(z\cdot\nabla U)\,dz &= -\frac12\,\delta^{b+1}\int_{B'_{\sqrt{r^2-\delta^2}}}|\nabla U(x,\delta)|^2dx\\ &\quad+\delta^{b+1}\int_{B'_{\sqrt{r^2-\delta^2}}}|U_t(x,\delta)|^2dx+\delta^b\int_{B'_{\sqrt{r^2-\delta^2}}}(x\cdot\nabla_x U(x,\delta))U_t(x,\delta)\,dx\\ &\quad+\frac{r}{2}\int_{S_r^+\cap\{t>\delta\}} t^b|\nabla U|^2dS-r\int_{S_r^+\cap\{t>\delta\}} t^b\left|\frac{\partial U}{\partial\nu}\right|^2dS.\end{aligned} \tag{83}$$

Now, arguing as in [15], on can show that there exists a sequence $\delta_n\downarrow 0$ such that

$$\delta_n^{b+1}\int_{B'_{\sqrt{r^2-\delta_n^2}}}|\nabla U(x,\delta_n)|^2dx\to 0\,,\qquad \delta_n^{b+1}\int_{B'_{\sqrt{r^2-\delta_n^2}}}|U_t(x,\delta_n)|^2dx\to 0\,, \tag{84}$$

as $n \to +\infty$. The local regularity estimates of Propositions 7.8 and 7.9 imply that $U, V \in C^{0,\alpha}(\overline{B_r^+})$, $\nabla_x U, \nabla_x V \in C^{0,\alpha}(\overline{B_r^+})$ and $t^b U_t, t^b V_t \in C^{0,\alpha}(\overline{B_r^+})$ for some $\alpha \in (0,1)$. These regularity estimates combined with the Dominated Convergence Theorem imply that

$$\lim_{\delta \to 0^+} \delta^b \int_{B'_{\sqrt{r^2-\delta^2}}} (x \cdot \nabla_x U(x,\delta)) U_t(x,\delta)\, dx = 0\,. \tag{85}$$

Next, by (84) and (85), one can pass to the limit in (83) with $\delta = \delta_n$ as $n \to +\infty$; summing such limit equality with its analogue for the function V (which can be derived with a similar argument), we obtain (82).

In order to prove (81) it is sufficient to test the equations in (6) with U and V respectively. □

Lemma 4.2. *Let U and V be as in* (73) *and* (74) *and let $D = D(r)$ and $H = H(r)$ be the functions defined in* (79) *and* (80)*. Suppose that $(U,V) \not\equiv (0,0)$. Then there exists $r_0 > 0$ such that $H(r) > 0$ for any $r \in (0, r_0)$.*

Proof. Suppose by contradiction that for any $r_0 > 0$ there exists $r \in (0, r_0)$ such that $H(r) = 0$. This means that U and V vanish on S_r^+. In particular, by (145) and (81), we have

$$0 = \int_{B_r^+} t^b(|\nabla U|^2 + |\nabla V|^2 + UV)\, dz \geqslant \left(1 - \frac{2r^2}{(N+b-1)^2}\right) \int_{B_r^+} t^b(|\nabla U|^2 + |\nabla V|^2)\, dz\,. \tag{86}$$

If r_0 is sufficiently small and $r \in (0, r_0)$, the parenthesis appearing in the right hand side of (86) becomes positive. This, in turn, implies $\int_{B_r^+} t^b(|\nabla U|^2 + |\nabla V|^2)\, dz = 0$ which, combined with (145), implies $U \equiv 0$ and $V \equiv 0$ in B_r^+. Since U and V are weak solutions of the equations $\Delta_b U = V$ and $\Delta_b V = 0$ in $\mathbb{R}^{N+1}_+$, by classical unique continuation principles for elliptic operators with smooth coefficients (see [49]), we deduce that U and V vanish in $\mathbb{R}^{N+1}$ thus contradicting the assumption $(U,V) \not\equiv 0$. □

The statement of Lemma 4.2 allows us to define the Almgren type function $\mathcal{N} : (0, r_0) \to \mathbb{R}$ as

$$\mathcal{N}(r) = \frac{D(r)}{H(r)} \qquad \text{for any } r \in (0, r_0)\,. \tag{87}$$

Lemma 4.3. *Let U and V be as in* (73) *and* (74) *and let R be as in* (77)*. Let D, H, $\mathcal{N}$ be the functions defined in* (79), (80) *and* (87) *respectively. Then there exists $\tilde r \in (0, r_0)$ such that*

$$D(r) \geqslant \frac{r^{-N-b+1}}{2} \int_{B_r^+} t^b(|\nabla U|^2 + |\nabla V|^2)\, dz - \frac{r^2}{N+b-1} H(r) \tag{88}$$

for any $r \in (0, \tilde r)$. In particular we have that

$$\mathcal{N}(r) \geqslant -\frac{r^2}{N+b-1}\,. \tag{89}$$

Moreover, there exist two positive constants C_1, C_2 independent of r such that $D(r) + C_2 H(r) \geqslant 0$ for any $r \in (0, \tilde r)$ and

$$\int_{B_r^+} t^b(U^2 + V^2)\, dz \leqslant C_1 r^{N+b+1}[D(r) + C_2 H(r)] \qquad \text{for any } r \in (0, \tilde r)\,. \tag{90}$$

Proof. By Young inequality and (145), we have

$$\begin{aligned} \left| \int_{B_r^+} t^b UV\, dz \right| &\leqslant \frac{1}{2} \int_{B_r^+} t^b(U^2 + V^2)\, dz \\ &\leqslant \frac{2r^2}{(N+b-1)^2} \left[\int_{B_r^+} t^b(|\nabla U|^2 + |\nabla V|^2)\, dz + \frac{N+b-1}{2r} \int_{S_r^+} t^b(U^2 + V^2)\, dS \right] \end{aligned} \tag{91}$$

from which we obtain

$$\begin{aligned} &\int_{B_r^+} t^b(|\nabla U|^2 + |\nabla V|^2 + UV)\, dz \\ &\quad \geqslant \left(1 - \frac{2r^2}{(N+b-1)^2}\right) \int_{B_r^+} t^b(|\nabla U|^2 + |\nabla V|^2)\, dz - \frac{r}{N+b-1} \int_{S_r^+} t^b(U^2 + V^2)\, dS \end{aligned}$$

for any $r \in (0, r_0)$. The proof of (88) and (89) then follows from the definitions of D, H and $\mathcal{N}$, choosing $\tilde r \in (0, r_0)$ sufficiently small. Combining (91) and (88) we also obtain (90). □

In order to prove the validity of an Almgren type monotonicity formula we need to compute the derivative of $\mathcal{N}$. In order to do that we first compute the derivatives of the functions D and H.

Lemma 4.4. *Let U and V be as in* (73) *and* (74) *and let R be as in* (77)*. Let $H = H(r)$ be the function defined in* (80)*. Then $H \in W^{1,1}_{\rm loc}(0, R)$ and moreover we have*

$$H'(r) = 2r^{-N-b} \int_{S_r^+} t^b \left(U \frac{\partial U}{\partial \nu} + V \frac{\partial V}{\partial \nu} \right) dS \quad \text{in a distributional sense and a.e. } r \in (0, R)\,, \tag{92}$$

and

$$H'(r) = \frac{2}{r}\, D(r) \quad \text{in a distributional sense and a.e. } r \in (0, R)\,. \tag{93}$$

Proof. See the proof of [15, Lemma 3.8]. □

Lemma 4.5. *Let U and V be as in* (73) *and* (74) *and let R be as in* (77)*. Let $D = D(r)$ be the function defined in* (79)*. Then $D \in W^{1,1}_{\rm loc}(0, R)$ and moreover we have*

$$\begin{aligned} D'(r) = & \frac{2}{r^{N+b-1}} \int_{S_r^+} t^b \left(\left| \frac{\partial U}{\partial \nu} \right|^2 + \left| \frac{\partial V}{\partial \nu} \right|^2 \right) dS + \frac{1}{r^{N+b-1}} \int_{S_r^+} t^b UV\, dS \\ & - \frac{2}{r^{N+b}} \int_{B_r^+} t^b V (z \cdot \nabla U)\, dz - \frac{N+b-1}{r^{N+b}} \int_{B_r^+} t^b UV\, dz \end{aligned} \tag{94}$$

in a distributional sense and a.e. $r \in (0, R)$.

Proof. The proof can be easily obtained by replacing (82) into

$$D'(r) = r^{-N-b} [(1 - N - b) I(r) + r I'(r)]\,,$$

where $I(r) = \int_{B_r^+} t^b \left(|\nabla U|^2 + |\nabla V|^2 + UV \right) dz$. □

Lemma 4.6. *Let U and V be as in* (73) *and* (74) *and let R be as in* (77)*. Let $\mathcal{N} = \mathcal{N}(r)$ and r_0 be as in* (87)*. Then $\mathcal{N} \in W^{1,1}_{\rm loc}(0, r_0)$ and moreover we have*

$$\mathcal{N}'(r) = \nu_1(r) + \nu_2(r) \tag{95}$$

in a distributional sense and for a.e. $r \in (0, r_0)$, where

$$\nu_1(r) = \frac{2r \Big[\Big(\int_{S_r^+} t^b \left(\left| \frac{\partial U}{\partial \nu} \right|^2 + \left| \frac{\partial V}{\partial \nu} \right|^2 \right) dS \Big) \Big(\int_{S_r^+} t^b (U^2 + V^2)\, dS \Big) - \Big(\int_{S_r^+} t^b \left(U \frac{\partial U}{\partial \nu} + V \frac{\partial V}{\partial \nu} \right) dS \Big)^2 \Big]}{\Big(\int_{S_r^+} t^b (U^2 + V^2)\, dS \Big)^2}$$

and

$$\nu_2(r) = \frac{r \int_{S_r^+} t^b UV\, dS - 2 \int_{B_r^+} t^b V (z \cdot \nabla U)\, dz - (N + b - 1) \int_{B_r^+} t^b UV\, dz}{\int_{S_r^+} t^b (U^2 + V^2)\, dS}\,. \tag{96}$$

Proof. The proof follows immediately from (92), (93) and (94). □

In the next result we obtain an estimate on the ν_2 component of the function $\mathcal{N}'$.

Lemma 4.7. *Under the same assumptions of Lemma 4.6 we have that*

$$\gamma := \lim_{r \to 0^+} \mathcal{N}(r) \tag{97}$$

exists, it is finite and moreover $\gamma \geqslant 0$.

Proof. Let ν_1 and ν_2 be the functions introduced in Lemma 4.6. By (88), (90) and (91), for any $r \in (0, \tilde r)$, with $\tilde r$ as in Lemma 4.3, we have

$$|\nu_2(r)| \leqslant \frac{r}{2} + \frac{r\int_{B_r^+} t^b V^2 dz + r\int_{B_r^+} t^b |\nabla U|^2 dz + (N+b-1)\left|\int_{B_r^+} t^b UV\, dz\right|}{\int_{S_r^+} t^b (U^2+V^2)\, dS} \tag{98}$$

$$\leqslant \frac{r}{2} + \frac{\widetilde C_1 r^{N+b} D(r) + \widetilde C_2 r^{N+b+1} H(r)}{r^{N+b} H(r)} = \widetilde C_1 \mathcal{N}(r) + \widetilde C_3 r$$

for some suitable constants $\widetilde C_1, \widetilde C_2, \widetilde C_2 > 0$ independent of r.

Therefore, since by Cauchy-Schwarz inequality we have that $\nu_1 \geqslant 0$, we obtain that

$$\mathcal{N}'(r) \geqslant -\widetilde C_1 \mathcal{N}(r) - \widetilde C_3 r \tag{99}$$

which yields

$$\mathcal{N}(r) \leqslant e^{-\widetilde C_1 r}\left[e^{\widetilde C_1 \tilde r}\mathcal{N}(\tilde r) + \widetilde C_3 \int_r^{\tilde r} \rho e^{\widetilde C_1 \rho} d\rho\right] \leqslant \widetilde C_4 \qquad \text{for any } r \in (0, \tilde r)\,. \tag{100}$$

This, combined with (98), yields boundedness of ν_2 in $(0, \tilde r)$.

This means that $\mathcal{N}'(r) = \nu_1(r) + \nu_2(r)$ is the sum of a nonnegative function and of a bounded function so that

$$\gamma := \lim_{r\to 0^+} \mathcal{N}(r) = \mathcal{N}(\tilde r) - \int_0^{\tilde r} \nu_2(\rho)\, d\rho - \lim_{r\to 0^+} \int_r^{\tilde r} \nu_1(\rho)\, d\rho$$

exists. Finally, by (89) and (100) we conclude that γ is finite and nonnegative. □

A first consequence of the previous monotonicity argument is the following estimate of the function H.

Lemma 4.8. *Letting γ be as in Lemma 4.7, we have that*

$$H(r) = O(r^{2\gamma}) \quad \text{as } r \to 0^+. \tag{101}$$

Furthermore, for any $\sigma > 0$ there exist $K(\sigma) > 0$ and $r_\sigma \in (0, r_0)$ depending on σ such that

$$H(r) \geqslant K(\sigma)\, r^{2\gamma+\sigma} \quad \text{for all } r \in (0, r_\sigma)\,. \tag{102}$$

Proof. The proof is quite standard once we have proved (97), see the proof of [15, Lemma 3.16] for the details. □

5. A blow-up procedure

In order to exploit the monotonicity formula obtained in Section 4 and to obtain asymptotic estimates on solutions to (6), we proceed with a blow-up argument. The blow-up analysis for a system with our type of coupling presents several new difficulties compared to the case of one single equation treated in [15]. In the first instance we are able to guarantee only that at least one of the components of the limit profile is nontrivial. Furthermore, the type of coupling in the system leads to a representation of the Fourier coefficients in the expansion (117) characterized by the coexistence in the same term of different homogeneity orders, see (119); this phenomenon is responsible for the possible appearance in the limit profiles of eigenfunctions associated with different eigenvalues, as in the example exhibited in Remark 1.8. In particular, it can occur that the first component, rescaled with the homogeneity identified by the limit of the Almgren function, tends to zero, so that it is necessary to identify its asymptotic rate by analyzing in details its expansion togheter with that of its weighted Laplacian, distinguishing several cases according to vanishing or non-vanishing of some sets of coefficients, as done in detail in the proof of Theorem 5.7.

Lemma 5.1. *Let $(U, V) \in H^1(B_R^+; t^b) \times H^1(B_R^+; t^b)$ be a nontrivial solution to* (6) *in the sense of* (75)–(76) *and* (41)*. Let $\mathcal{N}$ be the function defined in* (87) *and let γ be as in Lemma 4.7. Then the following statements hold true:*

(*i*) *there exists $\ell \in \mathbb{N}$ such that $\gamma = \ell$;*

(ii) *for any sequence* $\lambda_n \downarrow 0$ *there exists a subsequence* $\{\lambda_{n_k}\}_{k\in\mathbb{N}}$ *and* $2M_\ell$ *real constants* $\beta_{\ell,m}, \beta'_{\ell,m}$, $m = 1, \dots, M_\ell$, *such that* $\sum_{m=1}^{M_\ell} \left[(\beta_{\ell,m})^2 + (\beta'_{\ell,m})^2\right] = 1$ *and*

$$\frac{U(\lambda_{n_k} z)}{\sqrt{H(\lambda_{n_k})}} \to |z|^\gamma \sum_{m=1}^{M_\ell} \beta_{\ell,m} Y_{\ell,m}\left(\frac{z}{|z|}\right), \qquad \frac{V(\lambda_{n_k} z)}{\sqrt{H(\lambda_{n_k})}} \to |z|^\gamma \sum_{m=1}^{M_\ell} \beta'_{\ell,m} Y_{\ell,m}\left(\frac{z}{|z|}\right)$$

weakly in $H^1(B_1^+; t^b)$ *and strongly in* $H^1(B_r^+; t^b)$ *for any* $r \in (0,1)$, *with* $Y_{\ell,m}$ *as in Section 1 (see the definition of* $Y_{\ell,m}$ *below (7)).*

Proof. Let us define the following scaled functions

$$U_\lambda(z) := \frac{U(\lambda z)}{\sqrt{H(\lambda)}}, \qquad V_\lambda(z) := \frac{V(\lambda z)}{\sqrt{H(\lambda)}}, \tag{103}$$

which satisfy

$$\Delta_b U_\lambda = \lambda^2 V_\lambda \quad \text{and} \quad \int_{S_1^+} t^b (U_\lambda^2 + V_\lambda^2)\, dS = 1\,.$$

Using a change of variable, (88) and Lemma 4.7, one sees that

$$\int_{B_1^+} t^b (|\nabla U_\lambda|^2 + |\nabla V_\lambda|^2)\, dz \leqslant 2\mathcal{N}(\lambda) + \frac{2\lambda^2}{N+b-1} = O(1) \qquad \text{as } \lambda \to 0^+,$$

which combined with (145) yields that

$$\{U_\lambda\}_{\lambda\in(0,\tilde\lambda)} \quad \text{and} \quad \{V_\lambda\}_{\lambda\in(0,\tilde\lambda)} \qquad \text{are bounded in } H^1(B_1^+; t^b)$$

for some $\tilde\lambda$ small enough. Hence, for any sequence $\lambda_n \downarrow 0$, there exists a subsequence $\lambda_{n_k} \downarrow 0$ and two functions $\widetilde U, \widetilde V \in H^1(B_1^+; t^b)$ such that $U_{\lambda_{n_k}} \rightharpoonup \widetilde U$, $V_{\lambda_{n_k}} \rightharpoonup \widetilde V$ weakly in $H^1(B_1^+; t^b)$.

By compactness of the trace map $H^1(B_1^+; t^b) \hookrightarrow L^2(S_1^+; t^b)$, see [15, Section 2.2], we obtain

$$\int_{S_1^+} t^b (\widetilde U^2 + \widetilde V^2)\, dS = 1\,, \tag{104}$$

which implies that $(\widetilde U, \widetilde V) \not\equiv (0,0)$. We observe that the couple (U_λ, V_λ) weakly solves

$$\begin{cases} \Delta_b U_\lambda = \lambda^2 V_\lambda & \text{in } B_1^+\,, \\ \Delta_b V_\lambda = 0 & \text{in } B_1^+\,, \\ \lim_{t\to 0^+} t^b \partial_t U_\lambda = \lim_{t\to 0^+} t^b \partial_t V_\lambda = 0 & \text{on } B'_1\,. \end{cases}$$

This means that

$$\int_{B_1^+} t^b \nabla U_\lambda \nabla \varphi\, dz = -\lambda^2 \int_{B_1^+} t^b V_\lambda \varphi\, dz \quad \text{and} \quad \int_{B_1^+} t^b \nabla V_\lambda \nabla \varphi\, dz = 0\,,$$

for any $\varphi \in H_0^1(\Sigma_1^+; t^b)$ with $H_0^1(\Sigma_1^+; t^b) = H_0^1(\Sigma_1^+(0); t^b)$ as in Section 2.

From the weak convergences $U_{\lambda_{n_k}} \rightharpoonup \widetilde U$, $V_{\lambda_{n_k}} \rightharpoonup \widetilde V$ in $H^1(B_1^+; t^b)$, we deduce that

$$\int_{B_1^+} t^b \nabla \widetilde U \nabla \varphi\, dz = 0\,, \quad \text{and} \quad \int_{B_1^+} t^b \nabla \widetilde V \nabla \varphi\, dz = 0\,, \quad \text{for any } \varphi \in H_0^1(\Sigma_1^+; t^b)\,,$$

which means that the couple $(\widetilde U, \widetilde V)$ weakly solves

$$\begin{cases} \Delta_b \widetilde U = 0 & \text{in } B_1^+\,, \\ \Delta_b \widetilde V = 0 & \text{in } B_1^+\,, \\ \lim_{t\to 0^+} t^b \partial_t \widetilde U = \lim_{t\to 0^+} t^b \partial_t \widetilde V = 0 & \text{on } B'_1\,. \end{cases} \tag{105}$$

By Propositions 7.8-7.9 we have that, for any $r \in (0,1)$,

$$\{\nabla_x U_\lambda\}_{\lambda\in(0,\tilde\lambda)}\,, \quad \{\nabla_x V_\lambda\}_{\lambda\in(0,\tilde\lambda)}\,, \quad \{t^b \partial_t U_\lambda\}_{\lambda\in(0,\tilde\lambda)}\,, \quad \{t^b \partial_t V_\lambda\}_{\lambda\in(0,\tilde\lambda)}$$

are bounded in $C^{0,\beta}(\overline{B_r^+})$ for some $\beta \in (0,1)$; hence by the Ascoli-Arzelà Theorem we deduce that these families of functions are uniformly convergent in $\overline{B_r^+}$ up to subsequences. In particular, we have that $U_{\lambda_{n_k}} \to \widetilde U$ and $V_{\lambda_{n_k}} \to \widetilde V$ strongly in $H^1(B_r^+; t^b)$ for any $r \in (0,1)$.

Now, for any $k \in \mathbb{N}$ and $r \in (0,1)$ we define the functions

$$D_k(r) := r^{-N-b+1} \int_{B_r^+} t^b \Big(|\nabla U_{\lambda_{n_k}}|^2 + |\nabla V_{\lambda_{n_k}}|^2 + \lambda_{n_k}^2 U_{\lambda_{n_k}} V_{\lambda_{n_k}} \Big)\, dz\,,$$

$$H_k(r) := r^{-N-b} \int_{S_r^+} t^b \Big(U_{\lambda_{n_k}}^2 + V_{\lambda_{n_k}}^2 \Big)\, dS\,.$$

We observe that

$$\mathcal{N}_k(r) := \frac{D_k(r)}{H_k(r)} = \frac{D(\lambda_{n_k} r)}{H(\lambda_{n_k} r)} = \mathcal{N}(\lambda_{n_k} r) \qquad \text{for any } r \in (0,1)\,. \tag{106}$$

Next, if we define

$$\widetilde{D}(r) := r^{-N-b+1} \int_{B_r^+} t^b \Big(|\nabla \widetilde{U}|^2 + |\nabla \widetilde{V}|^2 \Big)\, dz\,,$$

$$\widetilde{H}(r) := r^{-N-b} \int_{S_r^+} t^b \Big(\widetilde{U}^2 + \widetilde{V}^2 \Big)\, dS\,,$$

the strong convergences $U_{\lambda_{n_k}} \to \widetilde{U}$ and $V_{\lambda_{n_k}} \to \widetilde{V}$ in $H^1(B_r^+; t^b)$ yield

$$D_k(r) \to \widetilde{D}(r) \quad \text{and} \quad H_k(r) \to \widetilde{H}(r) \qquad \text{for any } r \in (0,1)\,. \tag{107}$$

We claim that $\widetilde{H}(r) > 0$ for any $r \in (0,1)$. Indeed, if there exists $\bar r \in (0,1)$ such that $\widetilde{H}(\bar r) = 0$ then by (105) and integration by parts we would have

$$0 = \int_{B_{\bar r}^+} \mathrm{div}(t^b \nabla \widetilde{U}) \widetilde{U}\, dz = - \int_{B_{\bar r}^+} t^b |\nabla \widetilde{U}|^2 dz\,. \tag{108}$$

Since $\widetilde{U} \in H_0^1(\Sigma_{\bar r}^+; t^b)$, combining (108) with (145), we conclude that $\widetilde{U} \equiv 0$ in $B_{\bar r}^+$ and, by the classical unique continuation principle for uniformly elliptic operators with regular coefficients, we conclude that $\widetilde{U} \equiv 0$ in B_1^+. With the same argument we also deduce that $\widetilde{V} \equiv 0$ in B_1^+. We have shown that $(\widetilde{U}, \widetilde{V}) \equiv (0,0)$ in B_1^+ thus contradicting (104).

The validity of the preceding claim allows to define the function $\widetilde{\mathcal{N}}(r) := \frac{\widetilde{D}(r)}{\widetilde{H}(r)}$ for any $r \in (0,1)$.

By (106), (107) and Lemma 4.7, we infer

$$\widetilde{\mathcal{N}}(r) = \lim_{k \to \infty} \mathcal{N}_k(r) = \lim_{k \to +\infty} \mathcal{N}(\lambda_{n_k} r) = \gamma\,. \tag{109}$$

This shows that $\widetilde{\mathcal{N}}$ is constant in $(0,1)$ so that $\widetilde{\mathcal{N}}'(r) = 0$ for any $r \in (0,1)$. Therefore, adapting Lemma 4.6 to the couple $(\widetilde{U}, \widetilde{V})$, we infer that

$$\int_{S_r^+} t^b \left(\left| \frac{\partial \widetilde{U}}{\partial \nu} \right|^2 + \left| \frac{\partial \widetilde{V}}{\partial \nu} \right|^2 \right) dS \cdot \int_{S_r^+} t^b (\widetilde{U}^2 + \widetilde{V}^2)\, dS - \left[\int_{S_r^+} t^b \left(\widetilde{U} \frac{\partial \widetilde{U}}{\partial \nu} + \widetilde{V} \frac{\partial \widetilde{V}}{\partial \nu} \right) dS \right]^2 = 0$$

for any $r \in (0,1)$. This represents an equality in the Cauchy-Schwarz inequality in the Hilbert space $L^2(S_r^+; t^b) \times L^2(S_r^+; t^b)$ thus showing that $(\widetilde{U}, \widetilde{V})$ and $\left(\frac{\partial \widetilde{U}}{\partial \nu}, \frac{\partial \widetilde{V}}{\partial \nu} \right)$ are parallel vectors in $L^2(S_r^+; t^b) \times L^2(S_r^+; t^b)$. Hence, there exists a function $\eta = \eta(r)$ defined for any $r \in (0,1)$ such that $\left(\frac{\partial \widetilde{U}}{\partial \nu}(r\theta), \frac{\partial \widetilde{V}}{\partial \nu}(r\theta) \right) = \eta(r)(\widetilde{U}(r\theta), \widetilde{U}(r\theta))$ for any $r \in (0,1)$ and $\theta \in \mathbb{S}_+^N$. By integration we obtain

$$\widetilde{U}(r\theta) = e^{\int_1^r \eta(s) ds} \widetilde{U}(\theta) = \varphi(r) \Psi_1(\theta), \quad r \in (0,1),\ \theta \in \mathbb{S}_+^N, \tag{110}$$

$$\widetilde{V}(r\theta) = e^{\int_1^r \eta(s) ds} \widetilde{V}(\theta) = \varphi(r) \Psi_2(\theta), \quad r \in (0,1),\ \theta \in \mathbb{S}_+^N, \tag{111}$$

where $\varphi(r) = e^{\int_1^r \eta(s) ds}$ and $\Psi_1 = \widetilde{U}\big|_{\mathbb{S}_+^N}$, $\Psi_2(\theta) = \widetilde{V}\big|_{\mathbb{S}_+^N}$. From (105), (110) and (111), it follows that

$$\begin{cases} r^{-N} \big(r^{N+b} \varphi'(r) \big)' \theta_{N+1}^b \Psi_1(\theta) + r^{b-2} \varphi(r) \mathrm{div}_{\mathbb{S}_+^N} (\theta_{N+1}^b \nabla_{\mathbb{S}_+^N} \Psi_1(\theta)) = 0 & \text{in } \mathbb{S}_+^N, \\ \displaystyle\lim_{\theta_{N+1} \to 0^+} \theta_{N+1}^b \nabla_{\mathbb{S}_+^N} \Psi_1(\theta) \cdot \mathbf{e}_{N+1} = 0\,, \end{cases} \tag{112}$$

and

$$\begin{cases} r^{-N}\big(r^{N+b}\varphi'(r)\big)'\theta_{N+1}^b\Psi_2(\theta)+r^{b-2}\varphi(r)\mathrm{div}_{\mathbb{S}^N_+}(\theta_{N+1}^b\nabla_{\mathbb{S}^N_+}\Psi_2(\theta))=0 & \text{in } \mathbb{S}^N_+,\\ \lim\limits_{\theta_{N+1}\to 0^+}\theta_{N+1}^b\nabla_{\mathbb{S}^N_+}\Psi_2(\theta)\cdot\mathbf{e}_{N+1}=0\,. \end{cases} \tag{113}$$

Taking r fixed, we deduce that Ψ_1,Ψ_2 are either zero or eigenfunctions of (7) associated to the same eigenvalue. Therefore there exist $\ell\in\mathbb{N}$, $\{\beta_{\ell,m},\beta'_{\ell,m}\}_{m=1}^{M_\ell}\subset\mathbb{R}$ such that

$$\begin{cases} -\mathrm{div}_{\mathbb{S}^N_+}(\theta_{N+1}^b\nabla_{\mathbb{S}^N_+}\Psi_1)=\mu_\ell\theta_{N+1}^b\Psi_1 & \text{in } \mathbb{S}^N_+,\\ \lim\limits_{\theta_{N+1}\to 0^+}\theta_{N+1}^b\nabla_{\mathbb{S}^N_+}\Psi_1(\theta)\cdot\mathbf{e}_{N+1}=0\,, \end{cases}$$

$$\begin{cases} -\mathrm{div}_{\mathbb{S}^N_+}(\theta_{N+1}^b\nabla_{\mathbb{S}^N_+}\Psi_2)=\mu_\ell\theta_{N+1}^b\Psi_2 & \text{in } \mathbb{S}^N_+,\\ \lim\limits_{\theta_{N+1}\to 0^+}\theta_{N+1}^b\nabla_{\mathbb{S}^N_+}\Psi_2(\theta)\cdot\mathbf{e}_{N+1}=0\,, \end{cases}$$

and

$$\Psi_1=\sum_{m=1}^{M_\ell}\beta_{\ell,m}Y_{\ell,m},\quad \Psi_2=\sum_{m=1}^{M_\ell}\beta'_{\ell,m}Y_{\ell,m}.$$

In view of (104) we have that $\int_{\mathbb{S}^N_+}\theta_{N+1}^b(\Psi_1^2+\Psi_2^2)\,dS=1$ and hence

$$\sum_{m=1}^{M_\ell}[(\beta_{\ell,m})^2+(\beta'_{\ell,m})^2]=1.$$

Since Ψ_1 and Ψ_2 are not both identically zero, from (112) and (113) it follows that $\varphi(r)$ solves the equation

$$\varphi''(r)+\frac{N+b}{r}\varphi'(r)-\frac{\mu_\ell}{r^2}\varphi(r)=0$$

and hence $\varphi(r)=c_1r^{\sigma_\ell^+}+c_2r^{\sigma_\ell^-}$ for some $c_1,c_2\in\mathbb{R}$ where

$$\begin{aligned} \sigma_\ell^+&=-\tfrac{N+b-1}{2}+\sqrt{\left(\tfrac{N+b-1}{2}\right)^2+\mu_\ell}=\ell,\\ \sigma_\ell^-&=-\tfrac{N+b-1}{2}-\sqrt{\left(\tfrac{N+b-1}{2}\right)^2+\mu_\ell}=-\ell-(N+b-1). \end{aligned} \tag{114}$$

Since either $|z|^{\sigma_\ell^-}\Psi_1(\frac{z}{|z|})\notin H^1(B_1^+;t^b)$ or $|z|^{\sigma_\ell^-}\Psi_2(\frac{z}{|z|})\notin H^1(B_1^+;t^b)$ as one can deduce by (151), (we recall that $(\Psi_1,\Psi_2)\not\equiv(0,0)$), we have that $c_2=0$ and $\varphi(r)=c_1r^{\sigma_\ell^+}$. Moreover, from $\varphi(1)=1$ we deduce that $c_1=1$. Therefore

$$\widetilde{U}(r\theta)=r^{\sigma_\ell^+}\Psi_1(\theta),\quad \widetilde{V}(r\theta)=r^{\sigma_\ell^+}\Psi_2(\theta),\quad \text{for all } r\in(0,1) \text{ and } \theta\in\mathbb{S}^N_+. \tag{115}$$

From (115) and the fact that

$$\int_{\mathbb{S}^N_+}\theta_{N+1}^b(\Psi_1^2+\Psi_2^2)\,dS=1\quad\text{and}\quad\int_{\mathbb{S}^N_+}\theta_{N+1}^b(|\nabla_{\mathbb{S}^N_+}\Psi_1|^2+|\nabla_{\mathbb{S}^N_+}\Psi_2|^2)\,dS=\mu_\ell$$

we infer

$$\widetilde{D}(r)=\sigma_\ell^+\,r^{2\sigma_\ell^+}\quad\text{and}\quad\widetilde{H}(r)=r^{2\sigma_\ell^+}\,.$$

By (109) we then have $\gamma=\widetilde{\mathcal{N}}(r)=\frac{\widetilde{D}(r)}{\widetilde{H}(r)}=\sigma_\ell^+=\ell$. The proof of the lemma is thereby complete. □

Lemma 5.2. *Suppose that all the assumptions of Lemma 5.1 hold true. Then the limit*

$$\lim_{r\to 0^+}r^{-2\gamma}H(r)$$

exists and it is finite.

Proof. Thanks to Lemma 4.8, it is sufficient to show that the limit exists.

By (93) and Lemma 4.7 we have

$$\frac{d}{dr}\frac{H(r)}{r^{2\gamma}} = 2r^{-2\gamma-1}H(r)[\mathcal{N}(r)-\gamma] = 2r^{-2\gamma-1}H(r)\int_0^r \mathcal{N}'(\rho)\,d\rho\,. \tag{116}$$

Since $\mathcal{N}$ is bounded in a right neighborhood of 0, by (99) we deduce that $\mathcal{N}'$ is bounded from below in a right neighborhood of 0. Hence there exist a constant $C>0$ and a nonnegative function $\omega \in L^1_{\rm loc}(0,r_0)$ such that $\mathcal{N}'(r) = -C + \omega(r)$ for any $r \in (0,r_0)$.

Therefore, integrating (116) in (r,r_0), we obtain

$$\frac{H(r_0)}{r_0^{2\gamma}} - \frac{H(r)}{r^{2\gamma}} = \int_r^{r_0} 2\rho^{-2\gamma-1}H(\rho)\left(\int_0^\rho \omega(\tau)\,d\tau\right)d\rho - 2C\int_r^{r_0}\rho^{-2\gamma}H(\rho)\,d\rho\,.$$

Since $\omega \geqslant 0$ then $\lim_{r\to 0^+}\int_r^{r_0} 2\rho^{-2\gamma-1}H(\rho)\left(\int_0^\rho \omega(\tau)\,d\tau\right)d\rho$ exists. Moreover we also have that $\lim_{r\to 0^+}\int_r^{r_0}\rho^{-2\gamma}H(\rho)\,d\rho = \int_0^{r_0}\rho^{-2\gamma}H(\rho)\,d\rho$ exists and it is finite being $\rho^{-2\gamma}H(\rho) \in L^1(0,r_0)$ thanks to (101). This completes the proof of the lemma. □

Let us expand U and V as

$$U(z) = U(\lambda\theta) = \sum_{k=0}^{\infty}\sum_{m=1}^{M_k}\varphi_{k,m}(\lambda)Y_{k,m}(\theta), \quad V(z) = V(\lambda\theta) = \sum_{k=0}^{\infty}\sum_{m=1}^{M_k}\widetilde{\varphi}_{k,m}(\lambda)Y_{k,m}(\theta) \tag{117}$$

where $\lambda = |z| \in (0,r_0)$, $\theta = z/|z| \in \mathbb{S}^N_+$, and

$$\varphi_{k,m}(\lambda) = \int_{\mathbb{S}^N_+}\theta_{N+1}^b U(\lambda\,\theta)Y_{k,m}(\theta)\,dS(\theta), \quad \widetilde{\varphi}_{k,m}(\lambda) = \int_{\mathbb{S}^N_+}\theta_{N+1}^b V(\lambda\,\theta)Y_{k,m}(\theta)\,dS(\theta). \tag{118}$$

Lemma 5.3. *Suppose that all the assumptions of Lemma 5.1 hold true. Let $\varphi_{\ell,m}$ and $\widetilde{\varphi}_{\ell,m}$, $m=1,\dots,M_\ell$, be as in (118). Then for any $1\leqslant m\leqslant M_\ell$ we have*

$$\varphi_{\ell,m}(\lambda) = c_1^{\ell,m}\lambda^\ell + \tfrac{d_1^{\ell,m}}{K(N,b,\ell)}\,\lambda^{\ell+2} \quad \textit{and} \quad \widetilde{\varphi}_{\ell,m}(\lambda) = d_1^{\ell,m}\lambda^\ell, \tag{119}$$

where $K(N,b,\ell) := (\ell+2)(\ell+1) + (N+b)(\ell+2) - \mu_\ell$,

$$d_1^{\ell,m} = R^{-\ell}\int_{\mathbb{S}^N_+}\theta_{N+1}^b V(R\theta)Y_{\ell,m}(\theta)\,dS(\theta)\,,\ c_1^{\ell,m} = R^{-\ell}\int_{\mathbb{S}^N_+}\theta_{N+1}^b U(R\theta)Y_{\ell,m}(\theta)\,dS(\theta) - \tfrac{d_1^{\ell,m}}{K(N,b,\ell)}\,R^2.$$

Furthermore $\varphi_{k,m} \equiv \widetilde{\varphi}_{k,m} \equiv 0$ *for any* $1 \leqslant k < \ell$ *and* $1 \leqslant m \leqslant M_k$.

Proof. From the Parseval identity it follows that

$$H(\lambda) = \sum_{k=0}^{\infty}\sum_{m=1}^{M_k}\left(\varphi^2_{k,m}(\lambda) + \widetilde{\varphi}^2_{k,m}(\lambda)\right), \qquad \text{for any } 0<\lambda\leqslant R. \tag{120}$$

By (6) we have that for any $m = 1,\dots,M_\ell$

$$\begin{cases} \varphi''_{\ell,m}(\lambda) + \frac{N+b}{\lambda}\,\varphi'_{\ell,m}(\lambda) - \frac{\mu_\ell}{\lambda^2}\,\varphi_{\ell,m}(\lambda) = \widetilde{\varphi}_{\ell,m}(\lambda)\,, \\ \widetilde{\varphi}''_{\ell,m}(\lambda) + \frac{N+b}{\lambda}\,\widetilde{\varphi}'_{\ell,m}(\lambda) - \frac{\mu_\ell}{\lambda^2}\,\widetilde{\varphi}_{\ell,m}(\lambda) = 0\,. \end{cases} \tag{121}$$

By direct calculation we obtain

$$\widetilde{\varphi}_{\ell,m}(\lambda) = d_1^{\ell,m}\lambda^{\sigma_\ell^+} + d_2^{\ell,m}\lambda^{\sigma_\ell^-}$$

for some constants $d_1^{\ell,m}, d_2^{\ell,m}$ where σ_ℓ^+ and σ_ℓ^- are defined in (114).

Now, by (120), (101) and the fact that $\gamma = \sigma_\ell^+ = \ell$, we infer $d_2^{\ell,m} = 0$ so that $\widetilde{\varphi}_{\ell,m}(\lambda) = d_1^{\ell,m}\lambda^\ell$.

In particular, (121) and direct calculation yield

$$\varphi_{\ell,m}(\lambda) = c_1^{\ell,m}\lambda^{\sigma_\ell^+} + c_2^{\ell,m}\lambda^{\sigma_\ell^-} + \frac{d_1^{\ell,m}}{(\sigma_\ell^+ +2)(\sigma_\ell^+ +1)+(N+b)(\sigma_\ell^+ +2)-\mu_\ell}\,\lambda^{\sigma_\ell^+ +2}$$

for some constants $c_1^{\ell,m}, c_2^{\ell,m}$. Exploiting again (120), (101) and the fact that $\gamma = \sigma_\ell^+ = \ell$ we deduce that $c_2^{\ell,m} = 0$. The proof of the first part of the lemma now easily follows. In order to prove the second part of the lemma one can proceed exactly as above replacing ℓ with k in (121) and solving the corresponding equation. The conclusion now follows from (120) and (101). □

Remark 5.4. We observe that the representation formula (119) actually holds for $\varphi_{k,m}$ and $\widetilde{\varphi}_{k,m}$ also for $k \neq \ell$; in this case to prove that $d_2^{k,m} = c_2^{k,m} = 0$ we can use the fact that $U, V \in H^1(B_R^+; t^b)$.

Lemma 5.5. *Suppose that all the assumptions of Lemma 5.1 hold true. Then we have*

$$\lim_{r \to 0^+} r^{-2\gamma} H(r) > 0 \,. \tag{122}$$

Proof. By Lemma 5.2 we know that the limit in (122) exists and it is nonnegative and finite. Suppose by contradiction that $\lim_{\lambda \to 0^+} \lambda^{-2\gamma} H(\lambda) = 0$. Then by (120) we deduce that for any $1 \leqslant m \leqslant M_\ell$, with ℓ as in Lemma 5.1,

$$\lim_{\lambda \to 0^+} \lambda^{-\gamma} \varphi_{\ell,m}(\lambda) = 0 \qquad \text{and} \qquad \lim_{\lambda \to 0^+} \lambda^{-\gamma} \widetilde{\varphi}_{\ell,m}(\lambda) = 0 \,.$$

We recall that by Lemma 5.1 we have $\gamma = \sigma_\ell^+$ and hence by Lemma 5.3 we infer $c_1^{\ell,m} = d_1^{\ell,m} = 0$ so that

$$\varphi_{\ell,m}(\lambda) = \widetilde{\varphi}_{\ell,m}(\lambda) = 0 \qquad \text{for any } \lambda \in (0,R) \text{ and } 1 \leqslant m \leqslant M_\ell \,. \tag{123}$$

From Lemma 5.1, for every sequence $\lambda_n \to 0^+$, there exist a subsequence $\{\lambda_{n_k}\}_{k \in \mathbb{N}}$ and $2M_\ell$ real constants $\beta_{\ell,m}, \beta'_{\ell,m}$, $m = 1, 2, \dots, M_\ell$, such that

$$\sum_{m=1}^{M_\ell} ((\beta_{\ell,m})^2 + (\beta'_{\ell,m})^2) = 1 \tag{124}$$

and

$$U_{\lambda_{n_k}} \to |z|^\ell \sum_{m=1}^{M_\ell} \beta_{\ell,m} Y_{\ell,m}\Big(\frac{z}{|z|}\Big), \qquad V_{\lambda_{n_k}} \to |z|^\ell \sum_{m=1}^{M_\ell} \beta'_{\ell,m} Y_{\ell,m}\Big(\frac{z}{|z|}\Big), \quad \text{as } k \to +\infty,$$

weakly in $H^1(B_1^+; t^b)$ and hence strongly in $L^2(S_1^+; t^b)$, where U_λ, V_λ have been defined in (103). Combining this with (123), it follows that, for any $m = 1, 2, \dots, M_\ell$,

$$\beta_{\ell,m} = \lim_{k \to +\infty} (U_{\lambda_{n_k}}, Y_{\ell,m})_{L^2(\mathbb{S}^N_+; \theta^b_{N+1})} = \lim_{k \to +\infty} \frac{\varphi_{\ell,m}(\lambda_{n_k})}{\sqrt{H(\lambda_{n_k})}} = 0 \,,$$

$$\beta'_{\ell,m} = \lim_{k \to +\infty} (V_{\lambda_{n_k}}, Y_{\ell,m})_{L^2(\mathbb{S}^N_+; \theta^b_{N+1})} = \lim_{k \to +\infty} \frac{\widetilde{\varphi}_{\ell,m}(\lambda_{n_k})}{\sqrt{H(\lambda_{n_k})}} = 0 \,,$$

thus contradicting (124). □

Then we prove the following lemma.

Lemma 5.6. *Let $(U,V) \in H^1(B_R^+; t^b) \times H^1(B_R^+; t^b)$ be a weak solution to system (6) such that $(U,V) \neq (0,0)$. Then there exists $\ell \in \mathbb{N}$ such that*

$$\lambda^{-\ell} U(\lambda z) \to |z|^\ell \sum_{m=1}^{M_\ell} \alpha_{\ell,m} Y_{\ell,m}\Big(\frac{z}{|z|}\Big), \quad \lambda^{-\ell} V(\lambda z) \to |z|^\ell \sum_{m=1}^{M_\ell} \alpha'_{\ell,m} Y_{\ell,m}\Big(\frac{z}{|z|}\Big),$$

strongly in $H^1(B_1^+; t^b)$ as $\lambda \to 0^+$, where

$$\begin{aligned} \alpha_{\ell,m} &= R^{-\ell} \int_{\mathbb{S}^N_+} \theta^b_{N+1} U(R\,\theta) Y_{\ell,m}(\theta)\, dS(\theta) - \tfrac{R^{2-\ell}}{K(N,b,\ell)} \int_{\mathbb{S}^N_+} \theta^b_{N+1} V(R\theta) Y_{\ell,m}(\theta)\, dS(\theta) \,, \\ \alpha'_{\ell,m} &= R^{-\ell} \int_{\mathbb{S}^N_+} \theta^b_{N+1} V(R\,\theta) Y_{\ell,m}(\theta)\, dS(\theta) \end{aligned} \tag{125}$$

with $K(N,b,\ell)$ as in Lemma 5.3 and

$$\sum_{m=1}^{M_\ell} ((\alpha_{\ell,m})^2 + (\alpha'_{\ell,m})^2) \neq 0 \,. \tag{126}$$

Moreover for any $1 \leqslant m \leqslant M_\ell$ we have

$$\varphi_{\ell,m}(\lambda) = \alpha_{\ell,m} \lambda^\ell + \frac{\alpha'_{\ell,m}}{K(N,b,\ell)} \lambda^{\ell+2} \,, \qquad \widetilde{\varphi}_{\ell,m}(\lambda) = \alpha'_{\ell,m} \lambda^\ell \,. \tag{127}$$

Proof. From Lemma 5.1 and (122) there exist $\ell \in \mathbb{N}$ such that, for every sequence $\lambda_n \to 0^+$, there exist a subsequence $\{\lambda_{n_k}\}_{k\in\mathbb{N}}$ and $2M_\ell$ real constants $\alpha_{\ell,m}, \alpha'_{\ell,m}$, $m = 1, 2, \dots, M_\ell$, such that $\sum_{m=1}^{M_\ell}((\alpha_{\ell,m})^2 + (\alpha'_{\ell,m})^2) \neq 0$ and

$$\lambda_{n_k}^{-\ell} U(\lambda_{n_k} z) \to |z|^\ell \sum_{m=1}^{M_\ell} \alpha_{\ell,m} Y_{\ell,m}\Big(\frac{z}{|z|}\Big), \quad \lambda_{n_k}^{-\ell} V(\lambda_{n_k} z) \to |z|^\ell \sum_{m=1}^{M_\ell} \alpha'_{\ell,m} Y_{\ell,m}\Big(\frac{z}{|z|}\Big), \tag{128}$$

strongly in $H^1(B_r^+; t^b)$ for all $r \in (0,1)$, and then, by homogeneity, strongly in $H^1(B_1^+; t^b)$.

By (118), (128) and Lemma 5.3 we deduce that

$$\begin{aligned}\alpha_{\ell,m} &= \lim_{k\to\infty} \lambda_{n_k}^{-\ell} \int_{\mathbb{S}^N_+} \theta_{N+1}^b U(\lambda_{n_k}\theta) Y_{\ell,m}(\theta)\, dS(\theta) \\ &= \lim_{k\to\infty} \lambda_{n_k}^{-\ell} \varphi_{\ell,m}(\lambda_{n_k}) = c_1^{\ell,m} \\ &= R^{-\ell} \int_{\mathbb{S}^N_+} \theta_{N+1}^b U(R\,\theta) Y_{\ell,m}(\theta)\, dS(\theta) - \tfrac{R^{2-\ell}}{K(N,b,\ell)} \int_{\mathbb{S}^N_+} \theta_{N+1}^b V(R\theta) Y_{\ell,m}(\theta)\, dS(\theta)\end{aligned}$$

and

$$\begin{aligned}\alpha'_{\ell,m} &= \lim_{k\to\infty} \lambda_{n_k}^{-\ell} \int_{\mathbb{S}^N_+} \theta_{N+1}^b V(\lambda_{n_k}\theta) Y_{\ell,m}(\theta)\, dS(\theta) \\ &= \lim_{k\to\infty} \lambda_{n_k}^{-\ell} \widetilde{\varphi}_{\ell,m}(\lambda_{n_k}) = d_1^{\ell,m} = R^{-\ell} \int_{\mathbb{S}^N_+} \theta_{N+1}^b V(R\,\theta) Y_{\ell,m}(\theta)\, dS(\theta)\,.\end{aligned}$$

We observe that the coefficients $\alpha_{\ell,m}, \alpha'_{\ell,m}$ depend neither on the sequence $\{\lambda_n\}_{n\in\mathbb{N}}$ nor on its subsequence $\{\lambda_{n_k}\}_{k\in\mathbb{N}}$. Hence the convergences in (128) hold as $\lambda \to 0^+$ and the lemma is proved. □

We now state and prove the following theorem.

Theorem 5.7. *Let $(U,V) \in H^1(B_R^+; t^b) \times H^1(B_R^+; t^b)$ be a weak solution to system* (6) *such that $(U,V) \neq (0,0)$. Then there exists $\delta_1 \in \mathbb{N}$ and a linear combination $\Psi_1 \not\equiv 0$ of eigenfunctions of* (7)*, possibly corresponding to different eigenvalues, such that*

$$\lambda^{-\delta_1} U(\lambda z) \to |z|^{\delta_1} \Psi_1\left(\tfrac{z}{|z|}\right) \tag{129}$$

strongly in $H^1(B_1^+; t^b)$ as $\lambda \to 0^+$. Furthermore, if $V \not\equiv 0$, there exists $\delta_2 \in \mathbb{N}$ and a linear combination $\Psi_2 \not\equiv 0$ of eigenfunctions of (7)*, possibly corresponding to different eigenvalues, such that*

$$\lambda^{-\delta_2} V(\lambda z) \to |z|^{\delta_2} \Psi_2\left(\tfrac{z}{|z|}\right) \tag{130}$$

strongly in $H^1(B_1^+; t^b)$ as $\lambda \to 0^+$.

Proof. We treat separately the proofs of (129) and (130).

Proof of (129). Let ℓ be as in Lemma 5.6. If at least one of the numbers $\alpha_{\ell,1}, \dots, \alpha_{\ell,M_\ell}$ introduced in Lemma 5.6 is different from zero then the proof of (129) follows immediately with $\delta_1 = \ell$ and

$$\Psi_1(\theta) = \sum_{m=1}^{M_\ell} \alpha_{\ell,m} Y_{\ell,m}(\theta)\,.$$

Suppose now that $\alpha_{\ell,1} = \dots = \alpha_{\ell,M_\ell} = 0$. Let $k = \ell + 3$ and let

$$\Sigma := \{j \in \{\ell+1, \ell+2\} : \alpha_{j,m} \neq 0 \text{ for at least one } m \in \{1,\dots,M_j\}\}$$

with Σ being possibly empty. Here $\alpha_{j,m}$ is defined as in (125) replacing ℓ with j. When $\Sigma \neq \emptyset$ we put $J = \min \Sigma$.

We distinguish the two cases $\Sigma \neq \emptyset$ and $\Sigma = \emptyset$.

The case $\Sigma \neq \emptyset$. We put

$$\omega(z) := U(z) - \sum_{j=1}^{k-1}\sum_{m=1}^{M_j} \varphi_{j,m}(|z|)Y_{j,m}\left(\tfrac{z}{|z|}\right)$$
$$= U(z) - \sum_{j\in\Sigma}\sum_{m=1}^{M_j} \alpha_{j,m}|z|^j\, Y_{j,m}\left(\tfrac{z}{|z|}\right) - \sum_{j=\ell}^{k-1}\sum_{m=1}^{M_j} \frac{\alpha'_{j,m}}{K(N,b,j)}\,|z|^{j+2}\,Y_{j,m}\left(\tfrac{z}{|z|}\right)$$

for any $z\in B_R^+$, with $K(N,b,j) := (j+2)(j+1)+(N+b)(j+2)-\mu_j$. The last identity follows from the second part of Lemma 5.3 and Remark 5.4.

It is not restrictive to assume that $\omega \not\equiv 0$, otherwise the conclusion is trivial. We observe that ω is in the same position as the function U in Lemma 5.6 so that applying that result to ω we deduce that there exists $\widetilde{\ell}\geqslant 0$ such that

$$\lambda^{-\widetilde{\ell}}\omega(\lambda z) \to |z|^{\widetilde{\ell}} \sum_{m=1}^{M_{\widetilde{\ell}}} \widetilde{\alpha}_m\, Y_{\widetilde{\ell},m}\left(\frac{z}{|z|}\right), \qquad \lambda^{-\widetilde{\ell}}\Delta_b\omega(\lambda z) \to |z|^{\widetilde{\ell}} \sum_{m=1}^{M_{\widetilde{\ell}}} \widetilde{\alpha}'_m\, Y_{\widetilde{\ell},m}\left(\frac{z}{|z|}\right) \tag{131}$$

in $H^1(B_1^+;t^b)$ as $\lambda\to 0^+$, where $\widetilde{\alpha}_m$ and $\widetilde{\alpha}'_m$ satisfy (125) and (126) in which the roles of U and V in Lemma 5.6 are replaced by ω and $\Delta_b\omega$ respectively.

We claim that $\widetilde{\ell}\geqslant k$. We first observe that the Fourier coefficients $\varphi_{j,m}, \widetilde{\varphi}_{j,m}$ corresponding to ω are all zero for any $1\leqslant j\leqslant k-1$ and $1\leqslant m\leqslant M_j$. On the other hand, by (127) we deduce that at least one of the functions $\varphi_{\widetilde{\ell},m}, \widetilde{\varphi}_{\widetilde{\ell},m}$, $1\leqslant m\leqslant M_{\widetilde{\ell}}$, corresponding to ω is not the null function. This proves the validity of the claim.

Note that since $\widetilde{\ell}\geqslant k$, by (125) and the orthogonality of $\{Y_{j,m}\}_{j\geqslant 0, 1\leqslant m\leqslant M_j}$ in $L^2(\mathbb{S}^N_+;\theta^b_{N+1})$, we also deduce that $\widetilde{\alpha}_m = \alpha_{\widetilde{\ell},m}$ and $\widetilde{\alpha}'_m = \alpha'_{\widetilde{\ell},m}$ for any $1\leqslant m\leqslant M_{\widetilde{\ell}}$.

By (131) and the fact that $\widetilde{\ell}\geqslant k$, $\lambda^{-k}\omega(\lambda z)$ and $\lambda^{-k}\Delta_b\omega(\lambda z)$ remain uniformly bounded in $H^1(B_1^+;t^b)$ as $\lambda\to 0^+$.

We observe that from the definitions of ω, Σ and J we have $j+2\geqslant \ell+2 = k-1\geqslant i$ for any $\ell+1\leqslant i,j\leqslant k-1$.

Therefore, if $J<\ell+2$ the proof of (129) then follows with $\delta_1 = J$ and

$$\Psi_1(\theta) = \sum_{m=1}^{M_J} \alpha_{J,m}\, Y_{J,m}(\theta), \qquad \theta\in\mathbb{S}^N_+ .$$

Suppose now that $J=\ell+2=k-1$. In this case (129) follows with $\delta_1 = k-1 = \ell+2$ and

$$\Psi_1(\theta) = \sum_{m=1}^{M_{k-1}} \alpha_{k-1,m}\, Y_{k-1,m}(\theta) + \sum_{m=1}^{M_\ell} \frac{\alpha'_{\ell,m}}{K(N,b,\ell)}\, Y_{\ell,m}(\theta), \qquad \theta\in\mathbb{S}^N_+ .$$

The case $\Sigma = \emptyset$. As in the previous case we define

$$\omega(z) := U(z) - \sum_{j=1}^{k-1}\sum_{m=1}^{M_j} \varphi_{j,m}(|z|)Y_{j,m}\left(\tfrac{z}{|z|}\right) = U(z) - \sum_{j=\ell}^{k-1}\sum_{m=1}^{M_j} \frac{\alpha'_{j,m}}{K(N,b,j)}\,|z|^{j+2}Y_{j,m}\left(\tfrac{z}{|z|}\right),$$

for any $z\in B_R^+$, where the last identity follows from the second part of Lemma 5.3, Remark 5.4, and the fact that $\Sigma=\emptyset$. Proceeding as in this case $\Sigma\neq\emptyset$ we find $\widetilde{\ell}\geqslant k$ such that (131) holds with $\widetilde{\alpha}_m = \alpha_{\widetilde{\ell},m}$ and $\widetilde{\alpha}'_m = \alpha'_{\widetilde{\ell},m}$ for any $1\leqslant m\leqslant M_{\widetilde{\ell}}$. Again we have that $\lambda^{-k}\omega(\lambda z)$ and $\lambda^{-k}\Delta_b\omega(\lambda z)$ remain uniformly bounded in $H^1(B_1^+;t^b)$ as $\lambda\to 0^+$. Since $k>\ell+2$ and since $\alpha'_{\ell,m}\neq 0$ for at least one $1\leqslant m\leqslant M_\ell$, the (129) follows as well with $\delta_1 = \ell+2$ and

$$\Psi_1(\theta) = \sum_{m=1}^{M_\ell} \frac{\alpha'_{\ell,m}}{K(N,b,\ell)}\, Y_{\ell,m}(\theta), \qquad \theta\in\mathbb{S}^N_+ .$$

Proof of (130). If at least one of the numbers $\alpha'_{\ell,1},\dots,\alpha'_{\ell,M_\ell}$ introduced in Lemma 5.6 is different from zero then the proof of (130) follows immediately with $\delta_2=\ell$ and

$$\Psi_2(\theta)=\sum_{m=1}^{M_\ell}\alpha'_{\ell,m}Y_{\ell,m}(\theta)\,.$$

Suppose now that $\alpha'_{\ell,1}=\dots=\alpha'_{\ell,M_\ell}=0$. Let $k>\ell$ be the first integer for which at least one of the numbers $\alpha'_{k,1},\dots,\alpha'_{k,M_k}$ is different from zero (such k exists if $V\not\equiv 0$ in view of (117), Lemma 5.3, and Remark 5.4) and put

$$\begin{aligned}\omega(z)&:=U(z)-\sum_{j=1}^{k}\sum_{m=1}^{M_j}\varphi_{j,m}(|z|)Y_{j,m}\left(\tfrac{z}{|z|}\right)\\&=U(z)-\sum_{j=\ell}^{k}\sum_{m=1}^{M_j}\alpha_{j,m}|z|^j\,Y_{j,m}\left(\tfrac{z}{|z|}\right)-\sum_{m=1}^{M_k}\frac{\alpha'_{k,m}}{K(N,b,k)}\,|z|^{k+2}\,Y_{k,m}\left(\tfrac{z}{|z|}\right)\end{aligned}$$

for any $z\in B_R^+$. The last identity follows from the second part of Lemma 5.3 and Remark 5.4. Applying Lemma 5.6 to ω and proceeding as in the proof of (129), one can show that $\lambda^{-k}\omega(\lambda z)\to 0$ and $\lambda^{-k}\Delta_b\omega(\lambda z)\to 0$ in $H^1(B_1^+;t^B)$ as $\lambda\to 0^+$. The proof of (130) now follows with $\delta_2=k$ and

$$\Psi_2(\theta)=\sum_{m=1}^{M_k}\alpha'_{k,m}\,Y_{k,m}(\theta)$$

being $\Delta_b\omega(z)=V(z)-|z|^k\sum_{m=1}^{M_k}\alpha'_{k,m}\,Y_{k,m}\left(\frac{z}{|z|}\right)$. □

6. Proof of the main results

We start with the proof of Theorem 1.6 since the proofs of Theorems 1.1–1.2 are related to the asymptotic estimates stated in Theorem 1.6.

6.1. **Proof of Theorem 1.6.** Up to translation it is not restrictive to assume that $x_0=0$. The proof now follows from Theorem 5.7 and the regularity estimates of Proposition 7.9.

Once we have proved Theorem 1.6, we can proceed with the proofs of Theorems 1.1–1.2.

6.2. **Proof of Theorem 1.1.** Let u be as in the statement of the theorem and let $U\in\mathcal{D}_b$ be the corresponding solution of (4). According with Section 4 we also put $V=\Delta_bU$. Following the argument introduced at the beginning of Section 4, by assuming up to translation that $x_0=0$, we see that the couple $(U,V)\in H^1(B_R^+;t^b)\times H^1(B_R^+;t^b)$ is a solution of (6) with R as in (77). Since $(-\Delta)^su\in(\mathcal{D}^{s-1,2}(\mathbb{R}^N))^\star$, by (149) we deduce that the map

$$W\mapsto {}_{(\mathcal{D}^{s-1,2}(\mathbb{R}^N))^\star}\langle(-\Delta)^su,\mathrm{Tr}(W)\rangle_{\mathcal{D}^{s-1,2}(\mathbb{R}^N)}\,,\qquad W\in\mathcal{D}^{1,2}(\mathbb{R}^{N+1}_+;t^b)$$

belongs to $(\mathcal{D}^{1,2}(\mathbb{R}^{N+1}_+;t^b))^\star$.

Then, by classical minimization methods, we have that the minimum

$$\min_{W\in\mathcal{D}^{1,2}(\mathbb{R}^{N+1}_+;t^b)}\left[\frac12\int_{\mathbb{R}^{N+1}_+}t^b|\nabla W|^2\,dz+C_b^2\;{}_{(\mathcal{D}^{s-1,2}(\mathbb{R}^N))^\star}\langle(-\Delta)^su,\mathrm{Tr}(W)\rangle_{\mathcal{D}^{s-1,2}(\mathbb{R}^N)}\right]$$

is attained by some $\widetilde V\in\mathcal{D}^{1,2}(\mathbb{R}^{N+1}_+;t^b)$ weakly solving

$$-\int_{\mathbb{R}^{N+1}_+}t^b\nabla\widetilde V\nabla\Phi\,dz=C_b^2\;{}_{(\mathcal{D}^{s-1,2}(\mathbb{R}^N))^\star}\langle(-\Delta)^su,\mathrm{Tr}(\Phi)\rangle_{\mathcal{D}^{s-1,2}(\mathbb{R}^N)}\tag{132}$$

for any $\Phi\in\mathcal{D}^{1,2}(\mathbb{R}^{N+1};t^b)$. In particular we have

$$-\int_{\mathbb{R}^{N+1}_+}t^b\nabla\widetilde V\nabla\Phi\,dz=C_b^2\int_{\mathbb{R}^N}|\xi|^{2s}\widehat u\,\overline{\widehat{\mathrm{Tr}(\Phi)}}\,d\xi\qquad\text{for any }\Phi\in C_c^\infty(\overline{\mathbb{R}^{N+1}_+})\,.\tag{133}$$

Combining (133) and (72) we obtain

$$
\begin{aligned}
(134)\qquad -\int_{\mathbb{R}^{N+1}_+} t^b \nabla\widetilde{V}\nabla\Phi\,dz &= C_b^2 (u,\mathrm{Tr}(\Phi))_{\mathcal{D}^{s,2}(\mathbb{R}^N)}\\
&= (U,\Phi)_{\mathcal{D}_b} = \int_{\mathbb{R}^{N+1}_+} t^b \Delta_b U \Delta_b\Phi\,dz = \int_{\mathbb{R}^{N+1}_+} t^b V\Delta_b\Phi\,dz \quad \text{for any } \Phi\in\mathcal{T}
\end{aligned}
$$

with $\mathcal{T}$ as in (40).

Since $u\in\mathcal{D}^{s,2}(\mathbb{R}^N)$ and $(-\Delta)^s u\in(\mathcal{D}^{s-1,2}(\mathbb{R}^N))^\star$, with a mollification argument, it is possible to construct an approximating sequence of functions $\{u_n\}\subset\mathcal{D}^{s,2}(\mathbb{R}^N)$ such that $u_n\to u$ in $\mathcal{D}^{s,2}(\mathbb{R}^N)$, $(-\Delta)^s u_n\in C^\infty(\mathbb{R}^N)$, $(-\Delta)^s u_n\to(-\Delta)^s u$ weakly in $(\mathcal{D}^{s-1,2}(\mathbb{R}^N))^\star$.

Then we can construct the corresponding functions U_n, V_n and $\widetilde{V}_n$. First we observe that $U_n\to U$ in $\mathcal{D}_b$ and in particular $V_n\to V$ in $L^2(\mathbb{R}^{N+1}_+;t^b)$. Moreover $\widetilde{V}_n\rightharpoonup\widetilde{V}$ weakly in $\mathcal{D}^{1,2}(\mathbb{R}^{N+1};t^b)$.

Now we observe that for the functions V_n we have

$$
(135)\quad \int_{\mathbb{R}^{N+1}_+} t^b V_n\,\Delta_b\Phi\,dz = C_b^2(u_n,\mathrm{Tr}(\Phi))_{\mathcal{D}^{s,2}(\mathbb{R}^N)} = C_b^2\int_{\mathbb{R}^N}(-\Delta)^s u_n\,\mathrm{Tr}(\Phi)\,dx \quad \text{for any } \Phi\in\mathcal{T},
$$

and hence, since $(-\Delta)^s u_n\in(\mathcal{D}^{s-1,2}(\mathbb{R}^N))^\star$, by Proposition 2.4 one can show that, for any $r>0$, $V_n\in H^1(Q_r^+;t^b)$.

Combining (135) with (73) we obtain

$$
\int_{\mathbb{R}^{N+1}_+} t^b (V_n - V)\,\Delta_b\Phi\,dz = C_b^2\ {}_{(\mathcal{D}^{s-1,2}(\mathbb{R}^N))^\star}\langle(-\Delta)^s u_n - (-\Delta)^s u, \mathrm{Tr}(\Phi)\rangle_{\mathcal{D}^{s-1,2}(\mathbb{R}^N)}
$$

for any $\Phi\in\mathcal{T}$ such that $\mathrm{supp}(\Phi(\cdot,0))\subset\Omega$. Hence, by (29) we deduce that $V_n\rightharpoonup V$ weakly in $H^1(Q_R^+;t^b)$ and by Lemma 7.3 we also have

$$
(136)\qquad \mathrm{Tr}(V_n)\rightharpoonup\mathrm{Tr}(V)\qquad \text{weakly in } L^{2^*(N,s-1)}(B_R')\,.
$$

The fact that $V_n\in H^1(Q_r^+;t^b)$ implies

$$
\int_{\mathbb{R}^{N+1}_+} t^b V_n\,\Delta_b\Phi\,dz = -\int_{\mathbb{R}^{N+1}_+} t^b\nabla V_n\nabla\Phi\,dz \qquad \text{for any } \Phi\in\mathcal{T}
$$

and by (134) applied to V_n and $\widetilde{V}_n$ we obtain

$$
(137)\qquad \int_{\mathbb{R}^{N+1}_+} t^b\nabla(V_n-\widetilde{V}_n)\nabla\Phi\,dz = 0 \qquad \text{for any } \Phi\in\mathcal{T}\,.
$$

Actually we can prove that (137) still holds true for any $\Phi\in C_c^\infty(\overline{\mathbb{R}^{N+1}_+})$ not necessarily satisfying $\Phi_t(\cdot,0)\equiv 0$ in $\mathbb{R}^N\times\{0\}$, arguing as we did for (78). If we define

$$
\widetilde{W}_n(x,t) = \begin{cases} V_n(x,t)-\widetilde{V}_n(x,t) & \text{if } t\geqslant 0\,,\\ V_n(x,-t)-\widetilde{V}_n(x,-t) & \text{if } t<0\,,\end{cases}
$$

by (137) we obtain

$$
(138)\qquad \int_{\mathbb{R}^{N+1}} |t|^b\nabla\widetilde{W}_n\nabla\Phi\,dz = 0
$$

for any $\Phi\in C_c^\infty(\mathbb{R}^{N+1})$. Choosing a suitable sequence of test functions in (138) and passing to the limit, it is possible to prove that for any $x_0\in\mathbb{R}^N$ and $r>0$

$$
\int_{\partial B_r(x_0,0)} |t|^b\frac{\partial\widetilde{W}_n}{\partial\nu}\,dS = 0\,.
$$

From this identity, proceeding similarly to the proof of the mean value theorem for harmonic functions (see [22, Theorem 2.1]) and taking into account the Hölder regularity results stated in Proposition 7.4, one can prove that

$$
\widetilde{W}_n(x_0,0) = \frac{1}{\omega_{N,b}\,r^{N+b+1}}\int_{B_r(x_0,0)}|t|^b\,\widetilde{W}_n\,dz \qquad \text{for any } x_0\in\mathbb{R}^N \text{ and } r>0
$$

where $\omega_{N,b} = (N+b+1)^{-1}\int_{\partial B_1(0,0)} |t|^b\,dS$, see also [48, Lemma A.1] and [39, Lemma 2.6]. Hence we have

$$|\widetilde{W}_n(x_0,0)| \leqslant \frac{2}{\omega_{N,b}\, r^{N+b+1}}\left(\int_{B_r^+(x_0)} |t|^b|V_n|\,dz + \int_{B_r^+(x_0)} |t|^b|\widetilde{V}_n|\,dz\right)$$

$$\leqslant \frac{2}{\omega_{N,b}\, r^{N+b+1}}\left[r^{\frac{N+b+1}{2}}\Big(\frac{|B_1'|}{b+1}\Big)^{\frac12}\|V_n\|_{L^2(\mathbb{R}^{N+1}_+;t^b)} + r^{\frac{(N+b+1)(2^{**}(b)-1)}{2^{**}(b)}}\Big(\frac{|B_1'|}{b+1}\Big)^{\frac{2^{**}(b)-1}{2^{**}(b)}}\|\widetilde{V}_n\|_{L^{2^{**}(b)}(\mathbb{R}^{N+1}_+;t^b)}\right].$$

Letting $r \to +\infty$, we have that the right hand side of the previous inequality tends to zero, from which we deduce that $\widetilde{W}_n \equiv 0$ on $\mathbb{R}^N \times \{0\}$ and in particular that $V_n \equiv \widetilde{V}_n$ on $\mathbb{R}^N \times \{0\}$. But from the fact that $\widetilde{V}_n \rightharpoonup \widetilde{V}$ weakly in $\mathcal{D}^{1,2}(\mathbb{R}^{N+1}_+;t^b)$ and (150) we have that $\mathrm{Tr}(\widetilde{V}_n) \rightharpoonup \mathrm{Tr}(\widetilde{V})$ weakly in $L^{2^*(N,s-1)}(\mathbb{R}^N)$. Combining this with (136) we deduce that $\mathrm{Tr}(V) = \mathrm{Tr}(\widetilde{V})$ on B_R'.

Letting $\widetilde{v} := \mathrm{Tr}(\widetilde{V})$, by [6, 10] and (133) we deduce that there exists a positive constant $\kappa_{N,b}$ depending only on N and b such that

$$-(\widetilde{v},\varphi)_{\mathcal{D}^{s-1,2}(\mathbb{R}^N)} = \kappa_{N,b}(u,\varphi)_{\mathcal{D}^{s,2}(\mathbb{R}^N)} \qquad \text{for any } \varphi \in C_c^\infty(\mathbb{R}^N)$$

which means that $\widehat{\widetilde{v}}(\xi) = -\kappa_{N,b}|\xi|^2\,\widehat{u}(\xi)$ in $\mathbb{R}^N$ and hence $\widetilde{v} = \kappa_{N,b}\Delta u$ in $\mathbb{R}^N$.

Finally we have that $\mathrm{Tr}(V) = \widetilde{v} = \kappa_{N,b}\Delta u$ in B_R'. In the rest of the proof we denote by v the trace of V on B_R'.

Let us assume, by contradiction, that $u \not\equiv 0$. Then the couple $(U,V) \neq (0,0)$ is a weak solution to (6) in $H^1(B_R^+;t^b) \times H^1(B_R^+;t^b)$ for some $R > 0$.

From Lemma 5.6 and the fact that any eigenfunction of (7) cannot vanish on $\partial\mathbb{S}^N_+$, as observed in Remark 1.5, it follows that either u or v (which are the traces of U and V respectively) vanish of some order $\gamma \geqslant 0$ at 0. Since by assumption, u satisfies

$$u(x) = O(|x|^k) \qquad \text{as } x \to 0 \quad \text{for any } k \in \mathbb{N}, \tag{139}$$

we have that necessarily V vanishes of order γ, i.e. there exists $\Psi : \mathbb{S}^N_+ \to \mathbb{R}$, eigenfunction of (7), such that

$$\lambda^{-\gamma}V(\lambda z) \to |z|^\gamma \Psi\Big(\frac{z}{|z|}\Big) \text{ as } \lambda \to 0 \text{ strongly in } H^1(B_1^+;t^b).$$

In particular by (151) we also have

$$\lambda^{-\gamma}v(\lambda x) \to |x|^\gamma \Psi\Big(\frac{x}{|x|},0\Big) \text{ as } \lambda \to 0 \text{ strongly in } L^{2^*(N,s-1)}(B_1').$$

Let us denote

$$v_\lambda(x) = \lambda^{-\gamma}v(\lambda x) \quad \text{and} \quad \widetilde{u}_\lambda(x) = \lambda^{-2-\gamma}u(\lambda x),$$

so that

$$v_\lambda \to |x|^\gamma \Psi\Big(\frac{x}{|x|},0\Big) \text{ as } \lambda \to 0 \text{ strongly in } L^{2^*(N,s-1)}(B_1') \tag{140}$$

and

$$\kappa_{N,b}\Delta\widetilde{u}_\lambda = v_\lambda \quad \text{in } B'_{R/\lambda}.$$

For every $\varphi \in C_c^\infty(B_1')$ we have that, for λ small enough,

$$-\kappa_{N,b}\int_{\mathbb{R}^N} \widetilde{u}_\lambda(-\Delta\varphi)\,dx = -\kappa_{N,b}\int_{\mathbb{R}^N} \varphi(-\Delta\widetilde{u}_\lambda)\,dx = \int_{\mathbb{R}^N} \varphi v_\lambda\,dx. \tag{141}$$

From one hand, assumption (139) implies that

$$\lim_{\lambda\to0^+}\int_{\mathbb{R}^N} \widetilde{u}_\lambda(-\Delta\varphi)\,dx = 0$$

whereas convergence (140) yields

$$\lim_{\lambda\to0^+}\int_{\mathbb{R}^N} \varphi v_\lambda\,dx = \int_{\mathbb{R}^N} |x|^\gamma \Psi\Big(\frac{x}{|x|},0\Big)\varphi(x)\,dx.$$

Hence passing to the limit in (141) we obtain that

$$\int_{\mathbb{R}^N} |x|^\gamma \Psi\Big(\frac{x}{|x|},0\Big)\varphi(x)\,dx = 0 \quad \text{for every } \varphi \in C_c^\infty(B_1'),$$

thus contradicting the fact that $|x|^\gamma \Psi\Big(\frac{x}{|x|},0\Big) \not\equiv 0$.

6.3. **Proof of Theorem 1.2.** Let us assume by contradiction, that $u \not\equiv 0$ in Ω and $u(x) = 0$ a.e. in a measurable set $E \subset \Omega$ of positive measure.

Let U and V be defined as in the proof of Theorem 1.1. As we explained in the proof of Theorem 1.1, for any $x \in \Omega$ we have that $(U,V) \in H^1(B_R^+(x);t^b) \times H^1(B_R^+(x);t^B)$ for any $R > 0$ as in (77).

Hence, by Lebesgue's density Theorem (i.e. almost every point of E is a density point of E), there exists a point $y_0 \in E$ and $R > 0$ such that $B'_{2R}(y_0) \subset \Omega$, $|B'_R(y_0) \cap E|_N > 0$ and $(U,V) \in H^1(B_R^+(y_0);t^b) \times H^1(B_R^+(y_0);t^b)$ where $|\cdot|_N$ denotes the N-dimensional Lebesgue measure. With choice of y_0 and $R > 0$, proceeding as in the proof of Theorem 1.1, we deduce that $v = \kappa_{N,b}\Delta u$ in $B'_R(y_0)$ with $v = \mathrm{Tr}(V)$.

Since $\kappa_{N,b}\Delta u = v$ and by Lemma 7.3 $v \in L^{2^*(N,s-1)}(B'_R(y_0))$, by classical regularity theory we have that $u \in H^2_{\mathrm{loc}}(B'_R(y_0))$. Since $u(x) = 0$ for any $x \in E$, we have that $\nabla u(x) = 0$ for a.e. $x \in E \cap B'_R(y_0)$ and hence, since $\frac{\partial u}{\partial x_i} \in H^1_{\mathrm{loc}}(B'_R(y_0))$ for every i, $\Delta u = 0$ a.e. in $E \cap B'_R(y_0)$. In particular $u(x) = v(x) = 0$ for a.e. $x \in E' := E \cap B'_R(y_0)$.

Let x_0 be a density point of E'. Hence, for all $\varepsilon > 0$ there exists $r_0 = r_0(\varepsilon) \in (0,1)$ such that, for all $r \in (0, r_0)$,

$$\frac{|(\mathbb{R}^N \setminus E') \cap B'_r(x_0)|_N}{|B'_r(x_0)|_N} < \varepsilon \,. \tag{142}$$

Lemma 5.6 implies that there exist $\gamma \geqslant 0$, $\Psi_1, \Psi_2 : \mathbb{S}^N_+ \to \mathbb{R}$ solving (7) such that either $\Psi_1 \not\equiv 0$ or $\Psi_2 \not\equiv 0$ (and hence $\Psi_1 \not\equiv 0$ or $\Psi_2 \not\equiv 0$ on $\partial\mathbb{S}^N_+$ respectively as observed in Remark 1.5), and

$$\lambda^{-\gamma} u(x_0 + \lambda(x - x_0)) \to |x - x_0|^\gamma \Psi_1\Big(\frac{x - x_0}{|x - x_0|}, 0\Big) \tag{143}$$

and

$$\lambda^{-\gamma} v(x_0 + \lambda(x - x_0)) \to |x - x_0|^\gamma \Psi_2\Big(\frac{x - x_0}{|x - x_0|}, 0\Big) \tag{144}$$

as $\lambda \to 0$ strongly in $L^{2^*(N,s-1)}(B'_1(x_0))$.

Since $u \equiv v \equiv 0$ a.e. in E', by (142) we have

$$\begin{aligned}
\int_{B'_r(x_0)} u^2(x)\,dx &= \int_{(\mathbb{R}^N\setminus E')\cap B'_r(x_0)} u^2(x)\,dx \\
&\leqslant \Big(\int_{(\mathbb{R}^N\setminus E')\cap B'_r(x_0)} |u(x)|^{2^*(N,s-1)}dx\Big)^{\frac{2}{2^*(N,s-1)}} |(\mathbb{R}^N \setminus E') \cap B'_r(x_0)|_N^{\frac{2^*(N,s-1)-2}{2^*(N,s-1)}} \\
&< \varepsilon^{\frac{2^*(N,s-1)-2}{2^*(N,s-1)}} |B'_r(x_0)|_N^{\frac{2^*(N,s-1)-2}{2^*(N,s-1)}} \Big(\int_{(\mathbb{R}^N\setminus E')\cap B'_r(x_0)} |u(x)|^{2^*(N,s-1)}dx\Big)^{\frac{2}{2^*(N,s-1)}}
\end{aligned}$$

and similarly

$$\int_{B'_r(x_0)} v^2(x)\,dx < \varepsilon^{\frac{2^*(N,s-1)-2}{2^*(N,s-1)}} |B'_r(x_0)|_N^{\frac{2^*(N,s-1)-2}{2^*(N,s-1)}} \Big(\int_{(\mathbb{R}^N\setminus E')\cap B'_r(x_0)} |v(x)|^{2^*(N,s-1)}dx\Big)^{\frac{2}{2^*(N,s-1)}}$$

for all $r \in (0, r_0)$. Then, letting $u^r(x) := r^{-\gamma}u(x_0 + r(x - x_0))$ and $v^r(x) := r^{-\gamma}v(x_0 + r(x - x_0))$,

$$\int_{B'_1(x_0)} |u^r(x)|^2 dx < \big(\tfrac{\omega_{N-1}}{N}\big)^{\frac{2^*(N,s-1)-2}{2^*(N,s-1)}} \varepsilon^{\frac{2^*(N,s-1)-2}{2^*(N,s-1)}} \Big(\int_{B'_1(x_0)} |u^r(x)|^{2^*(N,s-1)}dx\Big)^{\frac{2}{2^*(N,s-1)}},$$

$$\int_{B'_1(x_0)} |u^r(x)|^2 dx < \big(\tfrac{\omega_{N-1}}{N}\big)^{\frac{2^*(N,s-1)-2}{2^*(N,s-1)}} \varepsilon^{\frac{2^*(N,s-1)-2}{2^*(N,s-1)}} \Big(\int_{B'_1(x_0)} |u^r(x)|^{2^*(N,s-1)}dx\Big)^{\frac{2}{2^*(N,s-1)}},$$

for all $r \in (0, r_0)$, where $\omega_{N-1} = \int_{\mathbb{S}^{N-1}} 1\, dS'$. Letting $r \to 0^+$, from (143) and (144) we have that

$$\int_{B'_1(x_0)} |x - x_0|^{2\gamma} \Psi_i^2\Big(\tfrac{x-x_0}{|x-x_0|}, 0\Big)\, dx$$
$$\leqslant \big(\tfrac{\omega_{N-1}}{N}\big)^{\frac{2^*(N,s-1)-2}{2^*(N,s-1)}} \varepsilon^{\frac{2^*(N,s-1)-2}{2^*(N,s-1)}} \left(\int_{B'_1(x_0)} |x-x_0|^{\gamma \cdot 2^*(N,s-1)} \Big|\Psi_i\Big(\tfrac{x-x_0}{|x-x_0|}, 0\Big)\Big|^{2^*(N,s-1)} dx\right)^{\frac{2}{2^*(N,s-1)}}$$

for $i = 1, 2$ which yields a contradiction as $\varepsilon \to 0^+$, since either $\Psi_1 \not\equiv 0$ or $\Psi_2 \not\equiv 0$ on $\partial\mathbb{S}^N_+$.

7. Appendix

7.1. **Inequalities involving weighted Sobolev spaces.** Throughout this section, we will assume that $s \in (1, 2)$, $N > 2s$ and $b = 3 - 2s \in (-1, 1)$. For simplicity, the center x_0 of the sets introduced in (8) will be omitted whenever $x_0 = 0$.

Next we state the following Hardy-Sobolev inequality taken from [15, Lemma 2.4]. For any $R > 0$ and $U \in H^1(B^+_R; t^b)$ we have

$$\left(\frac{N+b-1}{2}\right)^2 \int_{B^+_R} t^b \frac{U^2}{|z|^2}\, dz \leqslant \int_{B^+_R} t^b |\nabla U|^2 dz + \frac{N+b-1}{2R} \int_{S^+_R} t^b U^2 dS.$$

In particular, for any $x_0 \in \mathbb{R}^N$ and $U \in H^1(B^+_R(x_0); t^b)$, we have

$$\left(\frac{N+b-1}{2R}\right)^2 \int_{B^+_R(x_0)} t^b U^2\, dz \leqslant \int_{B^+_R(x_0)} t^b |\nabla U|^2 dz + \frac{N+b-1}{2R} \int_{S^+_R(x_0)} t^b U^2 dS. \tag{145}$$

Now we state a Sobolev inequality involving a suitable critical Sobolev exponent. Let

$$2^{**}(b) = \begin{cases} \frac{2(N+b+1)}{N+b-1} & \text{if } 0 < b < 1\,, \\ \frac{2(N+1)}{N-1} & \text{if } -1 < b \leqslant 0\,. \end{cases}$$

By [29, Theorem 19.10] we have

$$\overline{S}(N, b) \left(\int_{B^+_1} t^b |U|^{2^{**}(b)} dz\right)^{\frac{2}{2^{**}(b)}} \leqslant \int_{B^+_1} t^b |\nabla U|^2 dz + \int_{B^+_1} t^b U^2 dz \quad \text{for any } U \in H^1(B^+_1; t^b), \tag{146}$$

for some constant $\overline{S}(N, b)$ depending only on N and b. The corresponding inequality in the half ball $B^+_R(x_0)$ can be obtained by (146) after scaling and translation.

Next we show that the embedding $H^1_0(\Gamma^+_R(x_0); t^b) \subset L^2(Q^+_R(x_0); t^b)$ is compact.

Proposition 7.1. *Let $x_0 \in \mathbb{R}^N$, $b \in (-1, 1)$ and $R > 0$. Then the embedding*

$$H^1_0(\Gamma^+_R(x_0); t^b) \subset L^2(Q^+_R(x_0); t^b)$$

is compact.

Proof. Let us define the function $d : Q^+_{3R}(x_0) \to [0, \infty)$ where

$$d(z) := \mathrm{dist}(z, \partial Q^+_{3R}(x_0)) \qquad \text{for any } z \in Q^+_{3R}(x_0)\,.$$

We immediately see that if $z = (x, t) \in Q^+_R(x_0)$ then $d(x, t) = t$. Let $\{U_n\} \subset H^1_0(\Gamma^+_R(x_0); t^b)$ be a sequence bounded in $H^1_0(\Gamma^+_R(x_0); t^b)$. For any n let us still denote by U_n the trivial extension to $Q^+_{3R}(x_0)$ so that $U_n \in H^1_0(\Gamma^+_{3R}(x_0); t^b)$. We observe that

$$\int_{Q^+_{3R}(x_0)} (d(z))^b\, |\nabla U_n|^2 dz = \int_{Q^+_R(x_0)} (d(z))^b\, |\nabla U_n|^2 dz = \int_{Q^+_R(x_0)} t^b\, |\nabla U_n|^2 dz,$$
$$\int_{Q^+_{3R}(x_0)} (d(z))^b\, U_n^2\, dz = \int_{Q^+_R(x_0)} (d(z))^b\, U_n^2\, dz = \int_{Q^+_R(x_0)} t^b U_n^2\, dz,$$

thus showing that $\{U_n\}$ is bounded in the weighted Sobolev space $W^{1,2}(Q^+_{3R}(x_0); d^b, d^b)$ where we used the notation of [29, Theorem 19.7]. By the same theorem in [29] we deduce that $\{U_n\}$ is, up to subsequences, strongly convergent in $L^2(Q^+_{3R}(x_0); d^b)$. But the functions U_n are supported in $Q^+_R(x_0)$ so that $\{U_n\}$ is strongly convergent in $L^2(Q^+_R(x_0); t^b)$. This completes the proof of the proposition. ∎

Now we state a Hardy-Rellich type inequality for functions in $\mathcal{D}_b$.

Proposition 7.2. *For every $U \in \mathcal{D}_b$, we have that $\frac{U}{|z|^2} \in L^2(\mathbb{R}^{N+1}_+; t^b)$ and $\frac{\nabla U}{|z|} \in L^2(\mathbb{R}^{N+1}_+; t^b)$. Furthermore*

$$(N-2s)^2 \int_{\mathbb{R}^{N+1}_+} t^b \frac{U^2}{|z|^4}\, dz + 2(N-2s) \int_{\mathbb{R}^{N+1}_+} t^b \frac{|\nabla U|^2}{|z|^2}\, dz \leqslant \int_{\mathbb{R}^{N+1}_+} t^b |\Delta_b U|^2\, dz \tag{147}$$

for every $U \in \mathcal{D}_b$.

Proof. By definition of $\mathcal{D}_b$, it is enough to prove inequality (147) for every $U \in C^\infty_c(\overline{\mathbb{R}^{N+1}_+})$ such that $U_t \equiv 0$ on $\mathbb{R}^N \times \{0\}$. Arguing as in [30], we have that, for every $\varepsilon > 0$ and $\lambda \in \mathbb{R}$,

$$\begin{aligned} 0 &\leqslant \left\| t^{b/2} \frac{z}{|z|} \Delta_b U + \lambda t^{b/2} U \frac{z}{|z|^3} \right\|^2_{L^2(\mathbb{R}^{N+1}_+ \setminus B_\varepsilon, \mathbb{R}^{N+1})} \\ &= \int_{\mathbb{R}^{N+1}_+ \setminus B_\varepsilon} t^b |\Delta_b U|^2\, dz + \lambda^2 \int_{\mathbb{R}^{N+1}_+ \setminus B_\varepsilon} t^b \frac{U^2(z)}{|z|^4}\, dz + 2\lambda \int_{\mathbb{R}^{N+1}_+ \setminus B_\varepsilon} t^b \frac{U \Delta_b U}{|z|^2}\, dz, \end{aligned}$$

where $z = (x,t)$ and $B_\varepsilon = \{z \in \mathbb{R}^{N+1} : |z| < \varepsilon\}$. Integration by parts yields

$$\begin{aligned} &\int_{\mathbb{R}^{N+1}_+ \setminus B_\varepsilon} t^b \frac{U \Delta_b U}{|z|^2}\, dz = \int_{\mathbb{R}^{N+1}_+ \setminus B_\varepsilon} \frac{U}{|z|^2} \operatorname{div}(t^b \nabla U)\, dz \\ &= -\int_{\{x \in \mathbb{R}^N : |x| > \varepsilon\}} \frac{U(x,0)}{|x|^2} \Big(\lim_{t \to 0^+} t^b U_t(x,t) \Big)\, dx - \int_{\mathbb{R}^{N+1}_+ \cap \partial B_\varepsilon} t^b \frac{U}{|z|^2} \nabla U(z) \cdot \frac{z}{|z|}\, dS \\ &\quad - \int_{\mathbb{R}^{N+1}_+ \setminus B_\varepsilon} t^b \nabla U \cdot \nabla \left(\frac{U}{|z|^2} \right) dz \\ &= 0 + O(\varepsilon^{b+N-2}) - \int_{\mathbb{R}^{N+1}_+ \setminus B_\varepsilon} t^b \frac{|\nabla U|^2}{|z|^2}\, dz + \int_{\mathbb{R}^{N+1}_+ \setminus B_\varepsilon} t^b \frac{\nabla(U^2) \cdot z}{|z|^4}\, dz \end{aligned}$$

and

$$\begin{aligned} \int_{\mathbb{R}^{N+1}_+ \setminus B_\varepsilon} t^b \frac{\nabla(U^2) \cdot z}{|z|^4}\, dz &= -\int_{\mathbb{R}^{N+1}_+ \cap \partial B_\varepsilon} t^b \frac{U^2}{|z|^3}\, dS - \int_{\mathbb{R}^{N+1}_+ \setminus B_\varepsilon} U^2 \operatorname{div}\left(t^b \frac{z}{|z|^4} \right) dz \\ &= O(\varepsilon^{b+N-3}) - (N+b-3) \int_{\mathbb{R}^{N+1}_+ \setminus B_\varepsilon} t^b \frac{U^2(z)}{|z|^4}\, dz. \end{aligned}$$

Combining the previous estimates we obtain that

$$\begin{aligned} 0 &\leqslant \int_{\mathbb{R}^{N+1}_+ \setminus B_\varepsilon} t^b |\Delta_b U|^2\, dz + \lambda^2 \int_{\mathbb{R}^{N+1}_+ \setminus B_\varepsilon} t^b \frac{U^2(z)}{|z|^4}\, dz \\ &\quad - 2\lambda \int_{\mathbb{R}^{N+1}_+ \setminus B_\varepsilon} t^b \frac{|\nabla U|^2}{|z|^2}\, dz - 2\lambda(N-2s) \int_{\mathbb{R}^{N+1}_+ \setminus B_\varepsilon} t^b \frac{U^2(z)}{|z|^4}\, dz + O(\varepsilon^{N-2s}). \end{aligned}$$

Choosing $\lambda = N - 2s$ and letting $\varepsilon \to 0^+$ we obtain that

$$(N-2s)^2 \int_{\mathbb{R}^{N+1}_+} t^b \frac{U^2(z)}{|z|^4}\, dz + 2(N-2s) \int_{\mathbb{R}^{N+1}_+} t^b \frac{|\nabla U|^2}{|z|^2}\, dz \leqslant \int_{\mathbb{R}^{N+1}_+} t^b |\Delta_b U|^2\, dz$$

thus completing the proof. □

If $N > 2\gamma$, the Sobolev embedding implies that there exists a positive constant $S(N,\gamma)$ depending only on N and γ, such that

$$S(N,\gamma) \|u\|^2_{L^{2^*(N,\gamma)}(\mathbb{R}^N)} \leqslant \|u\|^2_{\mathcal{D}^{\gamma,2}(\mathbb{R}^N)} \qquad \text{for any } u \in \mathcal{D}^{\gamma,2}(\mathbb{R}^N) \tag{148}$$

where $2^*(N,\gamma) = 2N/(N-2\gamma)$, see e.g. [11].

According with [6], we define $\mathcal{D}^{1,2}(\mathbb{R}^{N+1}_+; t^b)$ as the completion of the space $C^\infty_c(\overline{\mathbb{R}^{N+1}_+})$ with respect to the norm

$$\|U\|_{\mathcal{D}^{1,2}(\mathbb{R}^{N+1}_+; t^b)} := \left(\int_{\mathbb{R}^{N+1}_+} t^b |\nabla U|^2 dz \right)^{1/2}.$$

Arguing as in [6], we have that there exists a constant K_b depending only on $b \in (-1,1)$ such that

$$K_b \|\mathrm{Tr}\,(U)\|_{\mathcal{D}^{s-1,2}(\mathbb{R}^N)} \leqslant \|U\|_{\mathcal{D}^{1,2}(\mathbb{R}^{N+1}_+;t^b)} \qquad \text{for any } U \in \mathcal{D}^{1,2}(\mathbb{R}^{N+1}_+;t^b)\,. \tag{149}$$

Combining this with (148), we infer

$$S(N,s-1)K_b^2 \,\|\mathrm{Tr}\,(U)\|^2_{L^{2^*(N,s-1)}(\mathbb{R}^N)} \leqslant \|U\|^2_{\mathcal{D}^{1,2}(\mathbb{R}^{N+1}_+;t^b)} \qquad \text{for any } U \in \mathcal{D}^{1,2}(\mathbb{R}^{N+1}_+;t^b)\,. \tag{150}$$

Lemma 7.3. *For any $r > 0$ and any $U \in H^1(B_r^+;t^b)$ we have*

$$\widetilde{S}(N,b)\left(\int_{B_r'} |u|^{2^*(N,s-1)}dx\right)^{\frac{2}{2^*(N,s-1)}} \leqslant \int_{B_r^+} t^b|\nabla U|^2 dz + \frac{N+b-1}{2r}\int_{S_r^+} t^b U^2 dS \tag{151}$$

where $u = \mathrm{Tr}\,(U)$ and $\widetilde{S}(N,b)$ is a positive constant depending only on N and b.

Proof. See the proof of [15, Lemma 2.6]. □

7.2. **Hölder regularity of solutions.** This subsection is devoted to some results about Hölder regularity of solutions to systems of weighted elliptic equations in divergence form. Throughout this subsection, we will assume that $s \in (1,2)$, $N > 2s$ and $b = 3-2s \in (-1,1)$. As in Subsection 7.1 the center $x_0 \in \mathbb{R}^N$ of the sets introduced in (8) will be omitted whenever $x_0 = 0$.

We start with the following proposition which is a restatement, adapted to our setting, of some regularity results contained in [16], see also [23].

Proposition 7.4. *(Propositions 3-4 in [16]) Let $A, B \in L^{q_1}(B_1')$ for some $q_1 > \frac{N}{1-b}$ and let $D \in L^{q_2}(B_1^+;t^b)$ for some $q_2 > \frac{N+b+1}{2}$. Let $W \in H^1(B_1^+;t^b)$ be a weak solution of*

$$\begin{cases} -\mathrm{div}(t^b\nabla W) = t^b D(z) & \text{in } B_1^+\,,\\ -\lim_{t\to 0^+} t^b W_t = A(x)W + B(x) & \text{on } B_1'\,. \end{cases} \tag{152}$$

Then the following statements hold true:

(i) *$W \in C^{0,\alpha}(\overline{B_{1/2}^+})$ and in addition*

$$\|W\|_{C^{0,\alpha}(\overline{B_{1/2}^+})} \leqslant C\left(\|W\|_{L^2(B_1^+;t^b)} + \|B\|_{L^{q_1}(B_1')} + \|D\|_{L^{q_2}(B_1^+;t^b)}\right)$$

for some $C > 0$ and $\alpha \in (0,1)$ depending only on N, b and $\|A\|_{L^{q_1}(B_1')}$;

(ii) *if in addition to the previous assumptions we also suppose that $A, B \in W^{1,\infty}(B_1')$ and $D, \nabla_x D \in L^\infty(B_1^+)$ then we also have $\nabla_x W \in C^{0,\alpha}(\overline{B_{1/2}^+})$ and*

$$\begin{aligned}&\|W\|_{C^{0,\alpha}(\overline{B_{1/2}^+})} + \|\nabla_x W\|_{C^{0,\alpha}(\overline{B_{1/2}^+})}\\ &\quad\leqslant C\left(\|W\|_{L^2(B_1^+;t^b)} + \|A\|_{W^{1,\infty}(B_1')} + \|B\|_{W^{1,\infty}(B_1')} + \|D\|_{L^\infty(B_1^+)} + \|\nabla_x D\|_{L^\infty(B_1^+)}\right)\end{aligned}$$

for some $C > 0$ and $\alpha \in (0,1)$ depending only on N, b and $\|A\|_{L^\infty(B_1')}$.

In order to obtain a Hölder estimate for the t-derivative of a solution of (152) we need to adapt to our context some results from [8, 16, 15].

Proposition 7.5. *Let $t^b D_t \in L^\infty(B_1^+)$ and let $W \in H^1(B_1^+;t^b)$ be a weak solution of (152) with $A \equiv 0$ and $B \equiv 0$. Then $t^b W_t \in C^{0,\alpha}(\overline{B_{1/4}^+})$ and*

$$\|t^b W_t\|_{C^{0,\alpha}(\overline{B_{1/4}^+})} \leqslant C\left(\|W\|_{H^1(B_1^+;t^b)} + \|t^b D_t\|_{L^\infty(B_1^+)}\right)$$

for some $C > 0$ and $\alpha \in (0,1)$ depending only on N and b.

Proof. Since W is a weak solution of the problem

$$\begin{cases} -\mathrm{div}(t^b\nabla W) = t^b D(z) & \text{in } B_1^+\,,\\ \lim_{t\to 0^+} t^b W_t = 0 & \text{on } B_1'\,, \end{cases}$$

it is clear that the even reflection of W with respect to t, which we denote by $\widetilde{W}$, belongs to $H^1(B_1;|t|^b)$ and it is a weak solution of

$$-\mathrm{div}(|t|^b\nabla\widetilde{W}) = |t|^b\widetilde{D}(z) \qquad \text{in } B_1\,,$$

where we denote by $\widetilde{D}$ the even reflection of D. In other words

$$\int_{B_1} |t|^b\nabla\widetilde{W}\nabla\varphi\,dz = \int_{B_1} |t|^b\widetilde{D}(z)\varphi\,dz \qquad \text{for any } \varphi\in H^1_0(B_1;|t|^b)\,. \tag{153}$$

Let now ψ be a function in $C^\infty_c(B_1)$ such that $\psi_t(x,0)=0$ for any $x\in B'_1$. Then the function $\varphi(x,t)=|t|^{-b}\psi_t(x,t)$ belongs to $H^1(B_1;|t|^b)$. Since $\mathrm{supp}(\varphi)\subset B_1$ then $\varphi\in H^1_0(B_1;|t|^b)$ as one can deduce from [25, Theorem 2.5] and a standard truncation argument.

With this particular choice of φ in (153) we obtain

$$\begin{aligned}\int_{B_1}\widetilde{D}(z)\psi_t(z)\,dz &= \int_{B_1}\nabla\widetilde{W}\nabla(\psi_t)\,dz - \int_{B_1}\frac{b}{t}\,\widetilde{W}_t\psi_t\,dz = -\int_{B_1}\widetilde{W}(\Delta\psi)_t\,dz - \int_{B_1}\frac{b}{t}\,\widetilde{W}_t\psi_t\,dz\\ &= \int_{B_1}\widetilde{W}_t\left(\Delta\psi - \frac{b}{t}\,\psi_t\right)dz = \int_{B_1}|t|^b\widetilde{W}_t\,\mathrm{div}(|t|^{-b}\nabla\psi)\,dz\,.\end{aligned}$$

This proves that the function $\Psi(x,t):=|t|^b\widetilde{W}_t(x,t)\in L^2(B_1;|t|^{-b})$ satisfies

$$-\int_{B_1}\Psi\,\mathrm{div}(|t|^{-b}\nabla\psi)\,dz = \int_{B_1}\widetilde{D}_t(z)\psi\,dz \tag{154}$$

for any $\psi\in C^\infty_c(B_1)$ such that $\psi_t(x,0)=0$ for all $x\in B'_1$.

By Proposition 2.4 we deduce that $\Psi\in H^1(B_{1/2};|t|^{-b})$ being $|t|^b\widetilde{D}_t\in L^2(B_1;|t|^{-b})$.

In particular, by (154) we have that

$$\int_{B_{1/2}}|t|^{-b}\nabla\Psi\nabla\psi\,dz = \int_{B_{1/2}}\widetilde{D}_t(z)\psi\,dz \tag{155}$$

for any $\psi\in C^\infty_c(B_{1/2})$ such that $\psi_t(x,0)=0$ for all $x\in B'_{1/2}$.

In order to remove the condition $\psi_t(\cdot,0)\equiv 0$ on $B'_{1/2}$, it is enough to test (155) with

$$\psi_k(x,t)=\psi(x,t)-\psi_t(x,0)\,t\,\eta(kt)\,,\quad k\in\mathbb{N}\,,\quad \text{for any }\psi\in C^\infty_c(B_{1/2})\,,$$

where $\eta\in C^\infty_c(\mathbb{R})$, $0\leqslant\eta\leqslant 1$, $\eta(t)=0$ for any $t\in(-\infty,-2]\cup[2,+\infty)$ and $\eta(t)=1$ for any $t\in[-1,1]$, and to pass to the limit as $k\to+\infty$.

In other words, we have shown that $\Psi\in H^1(B_{1/2};|t|^{-b})$ is a weak solution in the usual sense of the equation

$$-\mathrm{div}(|t|^{-b}\nabla\Psi)=\widetilde{D}_t(z) \qquad \text{in } B_{1/2}\,.$$

Since by assumption $t^bD_t\in L^\infty(B_1^+)$ then $|t|^b\widetilde{D}_t\in L^\infty(B_1)$ and hence $\widetilde{D}_t/|t|^{-b}\in L^p(B_{1/2};|t|^{-b})$ for any $1\leqslant p<\infty$. In particular $\widetilde{D}_t/|t|^{-b}\in M_\sigma(B_{1/2},|t|^{-b})$ for some $\sigma>0$ (see Definition 2.4 and Remark 2.6 in [51]). Recalling that the weight $|t|^{-b}$ belongs to the Muckenhoupt class A_2, by Theorem 5.2 in [51] we deduce that $\Psi\in C^{0,\alpha}(\overline{B_{1/4}})$ for some $\alpha\in(0,1)$ and there exists a constant $C>0$ such that

$$\|\Psi\|_{C^{0,\alpha}(\overline{B_{1/4}})}\leqslant C\left(\|\Psi\|_{L^2(B_{1/2};|t|^{-b})}+\|\,|t|^b\widetilde{D}_t\|_{L^\infty(B_{1/2})}\right)\leqslant 2C\left(\|W\|_{H^1(B_1^+;t^b)}+\|t^bD_t\|_{L^\infty(B_1^+)}\right).$$

The proof of the theorem now follows from the definition of Ψ. □

In order to apply the last two propositions to system (6), we prove the following Brezis-Kato type result for a system of two equations with a potential in the boundary conditions and forcing terms both in the equation and in the boundary conditions.

Proposition 7.6. *Let $A,B\in L^{\frac{N}{2(s-1)}}(B'_1)$. Suppose that $U,V\in H^1(B_1^+;t^b)$ weakly solve the system*

$$\begin{cases}\mathrm{div}(t^b\nabla U)=t^bV & \text{in } B_1^+\,,\\ \mathrm{div}(t^b\nabla V)=0 & \text{in } B_1^+\,,\\ \lim_{t\to0^+}t^bU_t=0 & \text{on } B'_1\,,\\ -\lim_{t\to0^+}t^bV_t=A(x)U+B(x) & \text{on } B'_1\,.\end{cases} \tag{156}$$

Then $U, V \in L^q(B^+_{1/2}; t^b)$, $U(\cdot,0), V(\cdot,0) \in L^q(B'_{1/2})$ for any $1 \leqslant q < \infty$ and moreover there exists a constant K_1 depending only on $N, b, q, \|A\|_{L^{\frac{N}{2(s-1)}}(B'_1)}$ and $\|B\|_{L^{\frac{N}{2(s-1)}}(B'_1)}$ such that

$$\|U\|_{L^q(B^+_{1/2};t^b)} \leqslant K_1\Big(1 + \|U\|_{L^{2^{**}(b)}(B^+_1;t^b)} + \|V\|_{L^{2^{**}(b)}(B^+_1;t^b)}\Big),$$
$$\|U(\cdot,0)\|_{L^q(B'_{1/2})} \leqslant K_1\Big(1 + \|U\|_{L^{2^{**}(b)}(B^+_1;t^b)} + \|V\|_{L^{2^{**}(b)}(B^+_1;t^b)}\Big),$$
$$\|V\|_{L^q(B^+_{1/2};t^b)} \leqslant K_1\Big(1 + \|U\|_{L^{2^{**}(b)}(B^+_1;t^b)} + \|V\|_{L^{2^{**}(b)}(B^+_1;t^b)}\Big),$$
$$\|V(\cdot,0)\|_{L^q(B'_{1/2})} \leqslant K_1\Big(1 + \|U\|_{L^{2^{**}(b)}(B^+_1;t^b)} + \|V\|_{L^{2^{**}(b)}(B^+_1;t^b)}\Big).$$

Proof. The proof is quite standard and it is based on a Moser-Trudinger iteration scheme inspired by the paper of Brezis-Kato [7].

If we combine (146) with (145) we obtain

$$\overline{C}(N,b)\left(\int_{B^+_1} t^b |W|^{2^{**}(b)} dz\right)^{2/2^{**}(b)} \leqslant \int_{B^+_1} t^b |\nabla W|^2 dz \qquad \text{for any } W \in H^1_0(\Sigma^+_1; t^b), \tag{157}$$

where $\overline{C}(N,b) = \overline{S}(N,b)\cdot\left[1 + \left(\frac{2}{N+b-1}\right)^2\right]^{-1}$.

Let $\frac12 < r_U < 1$ and let $\eta_U \in C^\infty_c(\overline{\mathbb{R}^{N+1}_+})$ be a cut-off function such that $\mathrm{supp}(\eta_U) \subset \Sigma^+_1$ and $\eta_U \equiv 1$ in $\Sigma^+_{r_U}$. For any $n \in \mathbb{N}$, set $U^n := \min\{|U|, n\}$, $V^n := \min\{|V|, n\}$. Put $\alpha_0 = 2^{**}(b)$. Testing the first equation in (156) with $\eta_U^2 (U^n)^{\alpha_0-2} U$ and exploiting the respective boundary condition, we obtain

$$\int_{B^+_1} t^b \nabla U \nabla\left(\eta_U^2 (U^n)^{\alpha_0-2} U\right) dz = -\int_{B^+_1} t^b V \eta_U^2 (U^n)^{\alpha_0-2} U \, dz. \tag{158}$$

By direct computation (see the proof of Lemma 9.1 in [20] for more details), one can verify that if we put $C(q) = \min\{\frac14, \frac{4}{q+4}\}$ for any $q \geqslant 1$, we have

$$\begin{aligned} &C(\alpha_0)\int_{B^+_1} t^b \left|\nabla\Big(\eta_U (U^n)^{\frac{\alpha_0-2}{2}} U\Big)\right|^2 dz \\ &\leqslant \int_{B^+_1} t^b \nabla U \nabla\Big(\eta_U^2 (U^n)^{\alpha_0-2} U\Big) dz + \left(2 + C(\alpha_0)\frac{\alpha_0+2}{2}\right)\int_{B^+_1} t^b (U^n)^{\alpha_0-2} |U|^2 |\nabla \eta_U|^2 dz. \end{aligned}$$

Combing this with (158), using Young inequality and the fact that $U^n \leqslant |U|$, we obtain

$$\begin{aligned} &C(\alpha_0)\int_{B^+_1} t^b \left|\nabla\Big(\eta_U (U^n)^{\frac{\alpha_0-2}{2}} U\Big)\right|^2 dz \\ &\leqslant -\int_{B^+_1} t^b V \eta_U^2 (U^n)^{\alpha_0-2} U \, dz + \left(2 + C(\alpha_0)\frac{\alpha_0+2}{2}\right)\int_{B^+_1} t^b (U^n)^{\alpha_0-2} |U|^2 |\nabla \eta_U|^2 dz \\ &\leqslant \frac{1}{\alpha_0}\int_{B^+_1} t^b \eta_U^2 |V|^{\alpha_0} dz + \frac{\alpha_0-1}{\alpha_0}\int_{B^+_1} t^b \eta_U^2 (U^n)^{\frac{\alpha_0(\alpha_0-2)}{\alpha_0-1}} |U|^{\frac{\alpha_0}{\alpha_0-1}} dz \\ &\quad + \left(2 + C(\alpha_0)\frac{\alpha_0+2}{2}\right)\int_{B^+_1} t^b (U^n)^{\alpha_0-2} |U|^2 |\nabla \eta_U|^2 dz \\ &\leqslant \frac{1}{\alpha_0}\int_{B^+_1} t^b \eta_U^2 |V|^{\alpha_0} dz + \frac{\alpha_0-1}{\alpha_0}\int_{B^+_1} t^b \eta_U^2 (U^n)^{\alpha_0-2} |U|^2 dz \\ &\quad + \left(2 + C(\alpha_0)\frac{\alpha_0+2}{2}\right)\int_{B^+_1} t^b (U^n)^{\alpha_0-2} |U|^2 |\nabla \eta_U|^2 dz. \end{aligned}$$

Since $U, V \in H^1(B^+_1; t^b) \subset L^{\alpha_0}(B^+_1; t^b)$, letting $n \to +\infty$, by Fatou Lemma, we deduce that $\nabla\Big(\eta_U |U|^{\frac{\alpha_0-2}{2}} U\Big) \in L^2(B^+_1; t^b)$. Moreover, since $\eta_U |U|^{\frac{\alpha_0-2}{2}} U \in L^2(B^+_1; t^b)$, being $U \in L^{\alpha_0}(B^+_1; t^b)$, then $\eta_U |U|^{\frac{\alpha_0-2}{2}} U \in H^1(B^+_1; t^b)$.

In the rest of this proof, in order to simplify the notation, we will denote the critical exponent $2^*(N,s-1)=\frac{2N}{N-2s+2}$ by 2^*. By Lemma 7.3 and (157), we have that $\eta_U|U|^{\frac{\alpha_0-2}{2}}U\in L^{2^{**}(b)}(B_1^+;t^b)$ and $\eta_U(\cdot,0)|U(\cdot,0)|^{\frac{\alpha_0-2}{2}}U(\cdot,0)\in L^{2^*}(B_1')$. This implies that

$$U(\cdot,0)\in L^{\frac{\alpha_0\cdot 2^*}{2}}(B'_{r_U})\qquad\text{and}\qquad U\in L^{\frac{\alpha_0\cdot 2^{**}(b)}{2}}(B^+_{r_U};t^b)\,. \tag{159}$$

Now, let $\frac12<r_V<r_U$ and let $\eta_V\in C_c^\infty(\overline{\mathbb{R}^{N+1}_+})$ be such that $\mathrm{supp}(\eta_V)\subset\Sigma^+_{r_U}$ and $\eta_V\equiv1$ in $\Sigma^+_{r_V}$.

Testing the second equation in (156) with $\eta_V^2(V^n)^{\beta_0-2}V$, being $\beta_0=\frac{2^*\cdot\alpha_0}{2(2^*-1)}\in(2,2^{**}(b))$, and exploiting the corresponding boundary condition, we obtain

$$\int_{B_1^+}t^b\nabla V\nabla\Big(\eta_V^2(V^n)^{\beta_0-2}V\Big)dz
=\int_{B_1'}[A(x)U(x,0)+B(x)]\,\eta_V^2(x,0)(V^n(x,0))^{\beta_0-2}V(x,0)\,dx\,.$$

Proceeding as above we infer

$$\begin{aligned}
&C(\beta_0)\int_{B_1^+}t^b\left|\nabla\Big(\eta_V(V^n)^{\frac{\beta_0-2}{2}}V\Big)\right|^2dz\\
&\leqslant\int_{B_1'}[A(x)U(x,0)+B(x)]\,\eta_V^2(x,0)(V^n(x,0))^{\beta_0-2}V(x,0)\,dx\\
&\quad+\Big(2+C(\beta_0)\tfrac{\beta_0+2}{2}\Big)\int_{B_1^+}t^b(V^n)^{\beta_0-2}|V|^2|\nabla\eta_V|^2dz\,,
\end{aligned}$$

$$\begin{aligned}
&\text{(by Young inequality)}\leqslant\tfrac{\lambda^{1-\beta_0}}{\beta_0}\int_{B_1'}\eta_V^2(x,0)|A(x)|\,|U(x,0)|^{\beta_0}dx\\
&+\tfrac{\lambda(\beta_0-1)}{\beta_0}\int_{B_1'}|A(x)|\eta_V^2(x,0)(V^n(x,0))^{\frac{\beta_0(\beta_0-2)}{\beta_0-1}}|V(x,0)|^{\frac{\beta_0}{\beta_0-1}}dx\\
&+\tfrac{\lambda^{1-\beta_0}}{\beta_0}\int_{B_1'}\eta_V^2(x,0)|B(x)|\,dx+\tfrac{\lambda(\beta_0-1)}{\beta_0}\int_{B_1'}|B(x)|\eta_V^2(x,0)(V^n(x,0))^{\frac{\beta_0(\beta_0-2)}{\beta_0-1}}|V(x,0)|^{\frac{\beta_0}{\beta_0-1}}dx\\
&+\Big(2+C(\beta_0)\tfrac{\beta_0+2}{2}\Big)\int_{B_1^+}t^b(V^n)^{\beta_0-2}|V|^2|\nabla\eta_V|^2dz\,,
\end{aligned}$$

(by Hölder inequality and the fact that $V^n\leqslant|V|$)

$$\begin{aligned}
&\leqslant\tfrac{\lambda^{1-\beta_0}}{\beta_0}\|A\|_{L^{\frac{N}{2(s-1)}}(B_1')}|B_1'|^{\frac{2(s-1)[N-2(s-1)]}{N[N+2(s-1)]}}\left(\int_{B_1'}\eta_V^2(x,0)|U(x,0)|^{\beta_0(2^*-1)}dx\right)^{\frac{1}{2^*-1}}\\
&+\tfrac{\lambda(\beta_0-1)}{\beta_0}\int_{B_1'}|A(x)|\Big(\eta_V(x,0)(V^n(x,0))^{\frac{\beta_0-2}{2}}|V(x,0)|\Big)^2dx\\
&+\tfrac{\lambda^{1-\beta_0}}{\beta_0}\|B\|_{L^{\frac{N}{2(s-1)}}(B_1')}|B_1'|^{\frac{N-2(s-1)}{N}}+\tfrac{\lambda(\beta_0-1)}{\beta_0}\int_{B_1'}|B(x)|\Big(\eta_V(x,0)(V^n(x,0))^{\frac{\beta_0-2}{2}}|V(x,0)|\Big)^2dx\\
&+\Big(2+C(\beta_0)\tfrac{\beta_0+2}{2}\Big)\int_{B_1^+}t^b(V^n)^{\beta_0-2}|V|^2|\nabla\eta_V|^2dz
\end{aligned}$$

$$\begin{aligned}
&\leqslant\tfrac{\lambda^{1-\beta_0}}{\beta_0}\|A\|_{L^{\frac{N}{2(s-1)}}(B_1')}|B_1'|^{\frac{2(s-1)[N-2(s-1)]}{N[N+2(s-1)]}}\left(\int_{B'_{r_U}}|U(x,0)|^{\beta_0(2^*-1)}dx\right)^{\frac{1}{2^*-1}}\\
&+\tfrac{\lambda(\beta_0-1)}{\beta_0}\Big(\|A\|_{L^{\frac{N}{2(s-1)}}(B_1')}+\|B\|_{L^{\frac{N}{2(s-1)}}(B_1')}\Big)\left(\int_{B_1'}\left|\eta_V(x,0)(V^n(x,0))^{\frac{\beta_0-2}{2}}V(x,0)\right|^{2^*}dx\right)^{\frac{2}{2^*}}\\
&+\tfrac{\lambda^{1-\beta_0}}{\beta_0}\|B\|_{L^{\frac{N}{2(s-1)}}(B_1')}|B_1'|^{\frac{N-2(s-1)}{N}}+\Big(2+C(\beta_0)\tfrac{\beta_0+2}{2}\Big)\int_{B_1^+}t^b(V^n)^{\beta_0-2}|V|^2|\nabla\eta_V|^2dz\,,
\end{aligned}$$

and finally by (151)

$$\leqslant \tfrac{\lambda^{1-\beta_0}}{\beta_0}\|A\|_{L^{\frac{N}{2(s-1)}}(B_1')}|B_1'|^{\frac{2(s-1)[N-2(s-1)]}{N[N+2(s-1)]}}\left(\int_{B'_{r_U}}|U(x,0)|^{\beta_0(2^*-1)}dx\right)^{\frac{1}{2^*-1}}$$
$$+\tfrac{\lambda(\beta_0-1)}{\beta_0}\Big(\|A\|_{L^{\frac{N}{2(s-1)}}(B_1')}+\|B\|_{L^{\frac{N}{2(s-1)}}(B_1')}\Big)\widetilde{S}(N,b)^{-1}\int_{B_1^+}t^b\left|\nabla\Big(\eta_V(V^n)^{\frac{\beta_0-2}{2}}V\Big)\right|^2dz$$
$$+\tfrac{\lambda^{1-\beta_0}}{\beta_0}\|B\|_{L^{\frac{N}{2(s-1)}}(B_1')}|B_1'|^{\frac{N-2(s-1)}{N}}+\left(2+C(\beta_0)\frac{\beta_0+2}{2}\right)\int_{B_1^+}t^b(V^n)^{\beta_0-2}|V|^2|\nabla\eta_V|^2dz\,.$$

Choosing $\lambda>0$ small enough, in such a way that the constant

$$K:=C(\beta_0)-\tfrac{\lambda(\beta_0-1)}{\beta_0}\Big(\|A\|_{L^{\frac{N}{2(s-1)}}(B_1')}+\|B\|_{L^{\frac{N}{2(s-1)}}(B_1')}\Big)\widetilde{S}(N,b)^{-1}$$

becomes positive, we obtain

$$\text{(160)}\quad K\int_{B_1^+}t^b\left|\nabla\Big(\eta_V(V^n)^{\frac{\beta_0-2}{2}}V\Big)\right|^2dz$$
$$\leqslant \tfrac{\lambda^{1-\beta_0}}{\beta_0}\|A\|_{L^{\frac{N}{2(s-1)}}(B_1')}|B_1'|^{\frac{2(s-1)[N-2(s-1)]}{N[N+2(s-1)]}}\left(\int_{B'_{r_U}}|U(x,0)|^{\beta_0(2^*-1)}dx\right)^{\frac{1}{2^*-1}}$$
$$+\tfrac{\lambda^{1-\beta_0}}{\beta_0}\|B\|_{L^{\frac{N}{2(s-1)}}(B_1')}|B_1'|^{\frac{N-2(s-1)}{N}}+\Big(2+C(\beta_0)\tfrac{\beta_0+2}{2}\Big)\int_{B_1^+}t^b(V^n)^{\beta_0-2}|V|^2|\nabla\eta_V|^2dz\,.$$

We observe that by (159) and the definition of β_0 we have that the integral in the right hand side of (160) involving the function U is finite and so it is the one involving the function V since $V\in L^{\beta_0}(B_1^+;t^b)$ being $\beta_0\in(2,2^{**}(b))$.

Passing to the limit as $n\to+\infty$, by Fatou Lemma, we have that $\nabla\Big(\eta_V|V|^{\frac{\beta_0-2}{2}}V\Big)\in L^2(B_1^+;t^b)$ and hence $\eta_V|V|^{\frac{\beta_0-2}{2}}V\in H^1(B_1^+;t^b)$. By (157) we then have $V\in L^{\frac{2^{**}(b)\cdot\beta_0}{2}}(B_{r_V}^+;t^b)$.

Now we want to iterate the procedures previously applied to the functions U and V to improve their summability. To this purpose we define two sequences of radii in the following way:

$$\rho_0=\frac{3}{4}\,,\quad r_0=\frac{7}{8}\,,\quad \rho_{k+1}:=\frac{1}{2}\left(\rho_k+\frac{1}{2}\right)\,,\quad r_{k+1}:=\frac{1}{2}\left(\rho_k+\rho_{k+1}\right)\qquad\text{for any }k\geqslant 0\,.$$

Then we define two sequences of exponents in the following way:

$$\alpha_{k+1}:=\beta_k\cdot\frac{2^{**}(b)}{2}\,,\quad \beta_{k+1}:=\frac{2^*\cdot\alpha_{k+1}}{2(2^*-1)}\qquad\text{for any }k\geqslant 0\,.$$

We observe that

$$\text{(161)}\qquad \alpha_{k+1}<\alpha_k\cdot\frac{2^{**}(b)}{2}\qquad\text{and}\qquad \beta_{k+1}<\beta_k\cdot\frac{2^{**}(b)}{2}\,.$$

We apply inductively the two procedures to U and V respectively, replacing every time r_U with r_k, r_V with ρ_k, α_0 with α_k and β_0 with β_k.

If after a certain step we obtained that $U(\cdot,0)\in L^{\frac{\alpha_k\cdot2^*}{2}}(B'_{r_k})$, $U\in L^{\frac{\alpha_k\cdot2^{**}(b)}{2}}(B_{r_k}^+;t^b)$ and $V\in L^{\frac{\beta_k\cdot2^{**}(b)}{2}}(B_{\rho_k}^+;t^b)$, then at the beginning of the subsequent step, by (161), we have in particular $U\in L^{\alpha_{k+1}}(B_{r_k}^+;t^b)$ and $V\in L^{\beta_{k+1}}(B_{\rho_k}^+;t^b)$. Applying the two procedures first to U and then to V, we obtain $U(\cdot,0)\in L^{\frac{\alpha_{k+1}\cdot2^*}{2}}(B'_{r_{k+1}})$, $U\in L^{\frac{\alpha_{k+1}\cdot2^{**}(b)}{2}}(B_{r_{k+1}}^+;t^b)$ and $V\in L^{\frac{\beta_{k+1}\cdot2^{**}(b)}{2}}(B_{\rho_{k+1}}^+;t^b)$.

It is easy to check that $\beta_{k+1}/\beta_k=\frac{2^*\cdot2^{**}(b)}{4(2^*-1)}>1$ so that $\lim_{k\to+\infty}\alpha_k=\lim_{k\to+\infty}\beta_k=+\infty$. Since $r_k>\rho_k>\frac12$ for any k the proof of the lemma then follows. □

Remark 7.7. We observe that in Propositions 7.4, 7.5, 7.6 the equations are set in the half ball in $\mathbb{R}^{N+1}$ of radius 1 and that the regularity or summability result is obtained in the half ball of radius 1/2 or 1/4. The special choice of those radii was made only for simplicity of notation but it easy to understand that completely similar results still hold true with the equations set in a half ball of arbitrary radius R_1 and with the conclusion on regularity or summability obtained on a half ball of arbitrary radius $R_2<R_1$. □

We now state a Hölder regularity result for solutions of system (156).

Proposition 7.8. *Let $s \in (1,2)$, $b = 3-2s \in (-1,1)$, $A \in L^{\bar q}(B_1')$, $B \in L^{\bar q}(B_1')$ for some $\bar q > \frac{N}{2(s-1)}$. If $U, V \in H^1(B_1^+;t^b)$ weakly solve (156) then $U, V \in C^{0,\alpha}(\overline{B}_{1/2}^+)$ for some $\alpha \in (0,1)$ and moreover there exists a constant K_2 depending only on $N, b, \|A\|_{L^{\bar q}(B_1')}, \|B\|_{L^{\bar q}(B_1')}, \|U\|_{L^{2^{**}(b)}(B_1^+;t^b)}$ and $\|V\|_{L^{2^{**}(b)}(B_1^+;t^b)}$ such that*

$$\|U\|_{C^{0,\alpha}(\overline{B}_{1/2}^+)} \leqslant K_2\,, \qquad \|V\|_{C^{0,\alpha}(\overline{B}_{1/2}^+)} \leqslant K_2\,.$$

Proof. We first apply Proposition 7.6 to U and V and, taking into account Remark 7.7, we obtain $U, V \in L^q(B_r^+;t^b)$ and $U, V \in L^q(B_r')$ for any $1 \leqslant q < \infty$ and $r \in (1/2, 1)$. Then, by (156), by the assumptions on A and B, by Proposition 7.4 (i) applied to U and V respectively and by Remark 7.7, we obtain $U, V \in C^{0,\alpha}(\overline{B}_{1/2}^+)$ for some $\alpha \in (0,1)$. □

We are now ready to prove a Hölder regularity estimate for derivatives of solutions (U,V) of (156).

Proposition 7.9. *Let $s \in (1,2)$, $b = 3-2s \in (-1,1)$, $A, B \in W^{1,\bar q}(B_1')$ for some $\bar q > \frac{N}{2(s-1)}$. Then the following statements hold true:*

(i) *if $U, V \in H^1(B_1^+;t^b) \cap C^{0,\alpha}(\overline{B}_1^+)$, for some $\alpha \in (0,1)$, weakly solve (156) then $\nabla_x U, \nabla_x V \in C^{0,\beta}(\overline{B}_{1/2}^+)$ for some $\beta \in (0,\alpha)$ and moreover there exists a constant K_3 depending only on $N, b, \|A\|_{W^{1,\bar q}(B_1')}, \|B\|_{W^{1,\bar q}(B_1')}, \|U\|_{C^{0,\alpha}(\overline{B}_1^+)}$ and $\|V\|_{C^{0,\alpha}(\overline{B}_1^+)}$ such that*

$$\|\nabla_x U\|_{C^{0,\beta}(\overline{B}_{1/2}^+)} \leqslant K_3\,, \qquad \|\nabla_x V\|_{C^{0,\beta}(\overline{B}_{1/2}^+)} \leqslant K_3\,.$$

(ii) *if we also assume $A, B \in C^{0,\alpha}(B_1')$ for some $\alpha \in (0,1)$ and if $U, V \in H^1(B_1^+;t^b) \cap C^{0,\alpha}(\overline{B}_1^+)$ weakly solve (156) in B_1^+ then $t^b U_t, t^b V_t \in C^{0,\beta}(\overline{B}_{1/2}^+)$ for some $\beta \in (0,\alpha)$ and moreover there exists a constant K_4 depending only on $N, b, \|A\|_{C^{0,\alpha}(B_1')}, \|B\|_{C^{0,\alpha}(B_1')}$, $\|U\|_{C^{0,\alpha}(\overline{B}_1^+)}$ and $\|V\|_{C^{0,\alpha}(\overline{B}_1^+)}$ such that*

$$\|t^b U_t\|_{C^{0,\beta}(\overline{B}_{1/2}^+)} \leqslant K_4\,, \qquad \|t^b V_t\|_{C^{0,\beta}(\overline{B}_{1/2}^+)} \leqslant K_4\,.$$

Proof. In order to prove (i) we proceed as in the proof of Lemma 3.3 in [15]. We define for any $\xi \in \mathbb{R}^N$ with $|\xi|$ small enough the functions

$$U^\xi(x,t) := \frac{U(x+\xi,t) - U(x,t)}{|\xi|} \qquad \text{and} \qquad V^\xi(x,t) := \frac{V(x+\xi,t) - V(x,t)}{|\xi|}$$

for any $(x,t) \in B_{3/4}^+$. Then we have

$$\begin{cases} \operatorname{div}(t^b \nabla U^\xi) = t^b V^\xi & \text{in } B_{3/4}^+\,, \\ \operatorname{div}(t^b \nabla V^\xi) = 0 & \text{in } B_{3/4}^+\,, \\ \lim_{t\to 0^+} t^b U_t^\xi = 0 & \text{on } B_{3/4}'\,, \\ -\lim_{t\to 0^+} t^b V_t^\xi = A(x) U^\xi + B_\xi & \text{on } B_{3/4}'\,, \end{cases}$$

where

$$B_\xi(x) := \frac{A(x+\xi) - A(x)}{|\xi|}\, U(x+\xi,0) + \frac{B(x+\xi) - B(x)}{|\xi|}\,.$$

We observe that

$$\|B_\xi\|_{L^{\bar q}(B_{3/4}')} \leqslant \|A\|_{W^{1,\bar q}(B_1')} \|U\|_{C^{0,\alpha}(\overline{B}_1^+)} + \|B\|_{W^{1,\bar q}(B_1')}\,.$$

Applying Proposition 7.8 to U^ξ and V^ξ and taking into account Remark 7.7, we infer that $\|U^\xi\|_{C^{0,\beta}(\overline{B}_{1/2}^+)}, \|V^\xi\|_{C^{0,\beta}(\overline{B}_{1/2}^+)}$ are uniformly bounded with respect to ξ small for some $\beta \in (0,\alpha)$. Passing to the limit as $\xi \to 0$, by the Ascoli-Arzelà Theorem we deduce that $\nabla_x U, \nabla_x V \in C^0(\overline{B}_{1/2}^+)$. Finally, exploiting the uniform Hölder estimates for U^ξ and V^ξ, passing to the limit as $\xi \to 0$, we obtain the validity of the Hölder estimates for $\nabla_x U$ and $\nabla_x V$ on $\overline{B}_{1/2}^+$. This completes the proof of (i).

It remains to prove (ii). We first observe that $A(x)U(x,0)+B(x)\in C^{0,\alpha}(B_1')$ and hence by [8, Lemma 4.5], applied to the function V, we obtain $t^bV_t\in C^{0,\beta}(\overline{B}^+_{1/2})$ for some $\beta\in(0,\alpha)$. In turn, applying Proposition 7.5 to the function U, we also obtain the Hölder continuity of the function t^bU_t over $\overline{B}^+_{1/2}$. This completes the proof of (ii). □

7.3. **Properties of Bessel functions.** We start by recalling an asymptotic estimate for first kind Bessel functions as $t\to+\infty$:

$$J_\nu(t)=O(t^{-1/2})\qquad \text{as } t\to+\infty\,. \tag{162}$$

This property can be deduced from the asymptotic expansion [4, (4.8.5)]. In order to obtain a similar estimate for derivatives of J_ν we start from the following identity

$$J_\nu'(t)=-J_{\nu+1}(t)+\nu t^{-1}J_\nu(t)\,, \tag{163}$$

see for example [4, Section 4.6]. From this identity we immediately see that $J_\nu'(t)=O(t^{-1/2})$ ad $t\to+\infty$.

Using iteratively (163), we deduce that

$$\frac{d^nJ_\nu}{dt^n}(t)=O(t^{-1/2})\qquad \text{as } t\to+\infty\,. \tag{164}$$

We conclude this subsection with an asymptotic estimate for the zeros of J_ν as $m\to+\infty$:

$$j_{\nu,m}\sim\pi\,m\qquad \text{as } m\to+\infty\,. \tag{165}$$

For more details on (165), see [45, Page 506] and also [14, Eq. (1.5)].

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References

[1] N. Abatangelo, S. Jarohs, A. Saldaña, *Green function and Martin kernel for higher-order fractional Laplacians in balls*, Nonlinear Anal. 175 (2018), 173–190.

[2] V. Adolfsson, L. Escauriaza, *$C^{1,\alpha}$ domains and unique continuation at the boundary*, Comm. Pure Appl. Math. 50 (1997), 935–969.

[3] V. Adolfsson, L. Escauriaza, C. Kenig, *Convex domains and unique continuation at the boundary*, Rev. Mat. Iberoamericana 11 (1995), no. 3, 513–525.

[4] G. E. Andrews, R. Askey, R. Roy, *Special Functions*, Cambridge University Press (1999).

[5] S. Agmon, A. Douglis, L. Nirenberg, *Estimates near the boundary for solutions of elliptic partial differential equations satisfying general boundary conditions. I* Comm. Pure App. Math. 12 (1959), 623–727.

[6] C. Brändle, E. Colorado, A. de Pablo, U. Sánchez, *A concave-convex elliptic problem involving the fractional Laplacian*, Proc. Roy. Soc. Edinburgh Sect. A 143 (2013), no. 1, 39–71.

[7] H. Brézis, T. Kato, *Remarks on the Schrödinger operator with singular complex potentials*, J. Math. Pures Appl. (9), 58 (1979), no. 2, 137–151.

[8] X. Cabré, Y. Sire, *Nonlinear equations for fractional Laplacians, I: Regularity, maximum principles, and Hamiltonian estimates*, Ann. Inst. H. Poincaré Anal. Non Linéaire 31 (2014), no. 1, 23–53.

[9] L. Caffarelli, *Non-local diffusions, drifts and games*, Nonlinear partial differential equations, 37–52, Abel Symp., 7, Springer, Heidelberg, 2012.

[10] L. Caffarelli, L. Silvestre, *An extension problem related to the fractional Laplacian*, Comm. Partial Differential Equations 32 (2007), no. 7-9, 1245–1260.

[11] A. Cotsiolis, N. Tavoularis, *Best constants for Sobolev inequalities for higher order fractional derivatives*, J. Math. Anal. Appl. 295 (2004), no. 1, 225–236.

[12] E. Di Nezza, G. Palatucci, E. Valdinoci, *Hitchhiker's guide to the fractional Sobolev spaces*, Bull. Sci. Math. 136 (2012), no. 5, 521–573.

[13] S. Dipierro, H.-Ch. Grunau, *Boggio's formula for fractional polyharmonic Dirichlet problems*, Ann. Mat. Pura Appl. (4) 196 (2017), no. 4, 1327–1344.

[14] Á. Elbert, *Some recent results on the zeros of Bessel functions and orthogonal polynomials*, Comput. Appl. Math. 133 (2001), no. 1–2, 65–83.

[15] M. M. Fall, V. Felli, *Unique continuation property and local asymptotics of solutions to fractional elliptic equations*, Comm. Partial Differential Equations 39 (2014), no. 2, 354–397.

[16] M. M. Fall, V. Felli, *Unique continuation properties for relativistic Schrödinger operators with a singular potential*, Discrete Contin. Dyn. Syst. 35 (2015), no. 12, 5827–5867.

[17] M. M. Fall, T. Weth, *Nonexistence results for a class of fractional elliptic boundary value problems*, J. Funct. Anal. 263 (2012), no. 8, 2205–2227.

[18] V. Felli, A. Ferrero, *Almgren-type monotonicity methods for the classification of behaviour at corners of solutions to semilinear elliptic equations*, Proc. Roy. Soc. Edinburgh Sect. A 143 (2013), no. 5, 957–1019.

[19] V. Felli, A. Ferrero, *Unique continuation and classification of blow-up profiles for elliptic systems with Neumann boundary coupling and applications to higher order fractional equations*, Nonlinear Analysis 196 (2020), Article no. 111826.
[20] V. Felli, A. Ferrero, S. Terracini, *Asymptotic behavior of solutions to Schrödinger equations near an isolated singularity of the electromagnetic potential*, J. Eur. Math. Soc. (JEMS) 13 (2011), no. 1, 119–174.
[21] N. Garofalo, F.-H. Lin, *Monotonicity properties of variational integrals, A_p weights and unique continuation*, Indiana Univ. Math. J. 35 (1986), no. 2, 245–268.
[22] D. Gilbarg, N. Trudinger, *Elliptic Partial Differential Equations of Second Order*, Springer-Verlag, Berlin, Heidelberg, New York, Tokyo, (1983).
[23] T. Jin, Y. Y. Li, J. Xiong, *On a fractional Nirenberg problem, part I: blow up analysis and compactness of solutions*, J. Eur. Math. Soc. (JEMS) 16 (2014), no. 6, 1111–1171.
[24] T. Jin, Y. Y. Li, J. Xiong, *On a fractional Nirenberg problem, Part II: Existence of solutions*, Int. Math. Res. Not. IMRN 2015, no. 6, 1555–1589.
[25] T. Kilpeläinen, *Weighted Sobolev spaces and capacity*, Ann. Acad. Sci. Fenn. Ser. A I Math. 19 (1994), no. 1, 95–113.
[26] I. Kukavica, K. Nyström, *Unique continuation on the boundary for Dini domains*, Proc. Amer. Math. Soc. 126 (1998), no. 2, 441–446.
[27] R. Metzler, J. Klafter, *The random walks guide to anomalous diffusion: a fractional dynamics approach*, Phys. Rep. 339 (2000) 1–77.
[28] E. Moreira dos Santos, G. Nornberg, N. Soave, *On unique continuation principles for some elliptic systems*, Preprint 2019, `arXiv:1909.07948`.
[29] B. Opic, A. Kufner, *Hardy-type inequalities*, Pitman Research Notes in Mathematics Series, 219, Longman Scientific & Technical, Harlow, (1990).
[30] M. Pérez-Llanos, A. Primo, *Semilinear biharmonic problems with a singular term*, J. Differential Equations 257 (2014), no. 9, 3200–3225.
[31] M. Reed, B. Simon, *Methods of Modern Mathematical Physics. IV. Analysis of Operators*, Academic Press, New York-London, 1978.
[32] A. Rüland, *Unique continuation for fractional Schrödinger equations with rough potentials*, Comm. Partial Differential Equations 40 (2015), no. 1, 77–114.
[33] A. Rüland, M.-Á. García-Ferrero, *Strong unique continuation for the higher order fractional Laplacian*, Preprint 2019, `arXiv:1902.09851`.
[34] Yu. Safarov, D. Vassiliev, *The Asymptotic Distribution of Eigenvalues of Partial Differential Operators*, Translations of Mathematical Monographs, 155, American Mathematical Society, Providence, RI, 1997.
[35] A. Saldaña, H. Tavares, *Least energy nodal solutions of Hamiltonian elliptic systems with Neumann boundary conditions*, J. Differential Equations 265 (2018), no. 12, 6127–6165.
[36] I. Seo, *On unique continuation for Schrödinger operators of fractional and higher orders*, Math. Nachr. 287 (2014), no. 5-6, 699–703.
[37] I. Seo, *Unique continuation for fractional Schrödinger operators in three and higher dimensions*, Proc. Amer. Math. Soc. 143 (2015), no. 4, 1661–1664.
[38] I. Seo, *Carleman inequalities for fractional Laplacians and unique continuation*, Taiwanese J. Math. 19 (2015), no. 5, 1533–1540.
[39] Y. Sire, S. Terracini, G. Tortone, *On the nodal set of solutions to degenerate or singular elliptic equations with an application to s-harmonic functions*, Preprint 2018, `arXiv:1808.01851`.
[40] Y. Sire, S. Terracini, S. Vita, *Liouville type theorems and regularity of solutions to degenerate or singular problems part I: even solutions*, Preprint 2019, `arXiv:1904.02143`.
[41] J. Tan, J. Xiong, *A Harnack inequality for fractional Laplace equations with lower order terms*, Discrete Contin. Dyn. Syst. 31 (2011), no. 3, 975–983.
[42] X. Tao, S. Zhang, *Boundary unique continuation theorems under zero Neumann boundary conditions*, Bull. Austral. Math. Soc. 72 (2005), no. 1, 67–85.
[43] X. Tao, S. Zhang, *Weighted doubling properties and unique continuation theorems for the degenerate Schrödinger equations with singular potentials*, J. Math. Anal. Appl. 339 (2008), no. 1, 70–84.
[44] J. L. Vázquez, *Nonlinear diffusion with fractional Laplacian operators*, Nonlinear partial differential equations, 271298, Abel Symp., 7, Springer, Heidelberg, 2012.
[45] G. N. Watson, *A Treatise on the Theory of Bessel Functions*, 2nd Edition, Cambridge University Press, Cambridge, (1944).
[46] R. Yang, *On higher order extensions for the fractional Laplacian*, arXiv:1302.4413, (2013).
[47] H. Yu, *Unique continuation for fractional orders of elliptic equations*, Ann. PDE 3 (2017), no. 2, Art. 16, 21 pp.
[48] K. Wang, J. Wei, *On the uniqueness of solutions of a nonlocal elliptic system*, Math. Ann. 365 (2016), no. 1–2, 105–153.
[49] T. H. Wolff, *A property of measures in $\mathbb{R}^N$ and an application to unique continuation*, Geom. Funct. Anal. 2 (1992), no. 2, 225–284.
[50] W. Woyczynski, *Nonlinear PDE's driven by Lévy diffusions and related statistical issues*, Probability and partial differential equations in modern applied mathematics, 247–258, IMA Vol. Math. Appl., 140, Springer, New York, 2005.

[51] P. Zamboni, *Hölder continuity for solutions of linear degenerate elliptic equations under minimal assumptions*, J. Differential Equations 182 (2002), no. 1, 121–140.

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