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REGULAR ARTICLE

An evolutive financial market model with animal spirits: imitation and endogenous beliefs

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Abstract We propose a financial market model with optimistic and pessimistic fundamentalists who, respectively, overestimate and underestimate the true fundamental value due to ambiguity in the stock market. We assume that agents form their beliefs about the fundamental value through an imitative process, considering the relative ability shown by optimists and pessimists in guessing the realized stock price. We also introduce an endogenous switching mechanism, allowing agents to switch to the other group of speculators if they performed better in terms of relative profits. Moreover, the stock price is determined by a nonlinear mechanism. We study, via analytical and numerical tools, the stability of the unique steady state, its bifurcations and the emergence of complex behaviors, with possible multistability phenomena. To quantify the global propensity to optimism/pessimism of the market, we introduce an index, depending on pessimists' and optimists' beliefs and shares, thanks to which we are able to show that the occurrence of the waves of optimism and pessimism are due to the joint effect of imitation and switching mechanism. Finally, we perform

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a statistical analysis of a stochastically perturbed version of the model, which highlights fat tails and excess volatility in the returns distributions, as well as bubbles and crashes for stock prices, in agreement with the empirical literature.

Keywords Animal spirits · Imitative process · Evolutionary selection · Financial markets · Bifurcations · Complex dynamics

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1 Introduction

Stock price dynamics can exhibit very complex and erratic behaviors. The random-like dynamics can give rise to alternating periods with basically either inflating or falling prices (bubbles and crashes), with resulting distributions that significantly differ from normal. In trying to understand and explain such phenomena, psychological aspects of agents' behavior can not be neglected. In particular, a significant role is played by agents' beliefs about the stock price fundamental value. In the traditional descriptions of financial markets, it is assumed that such fundamental value is perfectly known.

However, complete knowledge and, more generally, full rationality are to a large extent unrealistic hypotheses, being costly in terms of both informational and computational skills. There is a wide literature concerning psychological aspects of taking decisions about uncertain events in which it is shown that agents more likely adopt intuitive rather than rational processes in their actions, relying on a small number of simple principles. Those principles allow them to reduce the huge task of prediction to simpler heuristic operations, which permit them to save time and resources (see e.g. Epstein 2003; Gilbert 2002; Tversky and Kahneman 1974; Wilson 2002). Such investigations about human decisional processes have been taken into account in several economically oriented frameworks, leading to modeling approaches based on the assumption of boundedly rational agents. Since the related economic literature is very vast, we limit ourselves to referring the reader to Hommes (2013) for some evidence of the boundedly rational behavior of human agents, also due to the complexity of the financial market structure,to the book edited by Dieci et al. in 2014 for a discussion on the modeling of financial markets, in particular as concerns nonlinearities and heterogeneous agents; to Kindleberger and Aliber (2005), and Barberis and Thaler (2003) for behavioral and empirical aspects. The thrust of these behavioral and economical investigations is that boundedly rational agents, not exclusively relying on complex, fully rational decisional mechanisms founded on the complete knowledge of the economic setting, drive the economy acting as "animal spirits" (Keynes 1936; Akerlof and Shiller 2009). Then, decisions are mostly a consequence of the agents' interactions among themselves and with the economic framework in which they are embedded.

Concerning the representation of animal spirits in behavioral finance, a relevant contribution is represented by the paper by De Grauwe and Rovira Kaltwasser (2012), in which it is illustrated how optimistic and pessimistic belief biases on the fundamental value can generate waves of optimism and pessimism that qualitatively resemble erratic price movements (see also De Grauwe and Macchiarelli 2015). Such beliefs are exogenously established, but agents, being either optimists or pessimists, can switch their behavioral rule, which is selected under an evolutionary competition based on the observation of a simple economic variable, namely, the relative profits realized by the two kinds of agents. In particular, the origin of this approach to the representation of animal spirits can be found in De Grauwe (2012), page 12, where the evolution of the probability that agents extrapolate a positive/negative output gap is interpreted in terms of the updating of the fractions of optimistic/pessimistic agents (but it could be read in terms of the dynamics of optimistic/pessimistic beliefs, as well). The emergence of waves of optimism/pessimism, which portrays the animal spirits' behavior, is driven by the endogenous fluctuations of the share of optimists/pessimists under evolutionary pressure. In this framework, optimism (respectively, pessimism) is then realized when there is a large fraction of agents adopting the optimistic (respectively, pessimistic) heuristic.

In Naimzada and Pireddu (2015b) the authors adopt a different approach, more related to Akerlof and Shiller (2009) and the concept of confidence therein. According to Akerlof and Shiller, all economic activities, such as investment, spending, employment, production, etc., require a certain confidence level, which can then be interpreted as an optimistic belief. A higher or lower confidence degree in their economic actions makes agents' beliefs become more optimistic or more pessimistic. On this basis, in Naimzada and Pireddu (2015b) a financial market model is studied with heterogeneous speculators, i.e., optimistic and pessimistic fundamentalists, who, respectively, overestimate and underestimate the true fundamental value due to ambiguity[1] in the stock market. This prevents them from relying on the true fundamental value in their speculations, even if they know it. In such paper it is assumed that agents use in its place values determined, as a consequence of the interaction among them, through an imitative mechanism. In this case, the behavioral rule of each agent is exogenously assumed (no switching mechanism is considered therein), and agents adapt their beliefs about the fundamental value within the assigned heuristic. The imitative process can generate endogenous fluctuations in the beliefs, which can be read as waves of optimism/pessimism. In this framework, optimism (respectively, pessimism) is realized when beliefs are suitably large (respectively, small).

The contributions by De Grauwe and Rovira Kaltwasser (2012) and by Naimzada and Pireddu (2015b) then provide two different ways (switching and imitation) to portray the emergence of animal spirits in financial markets, through alternating waves of optimism/pessimism. At this point, several questions arise, among which we will focus on the following three issues: which mechanism, between imitation and switching, is the most significant for the emergence of animal spirits? Is one of the two mechanisms negligible with respect to the other? How can we measure the

[1] We stress that such a notion of ambiguity differs from the one employed in general equilibrium theory, where agents, in making their choices in stochastic frameworks, are assumed to take into account different probability laws describing the distribution of relevant random variables. See e.g. Dow and Werlang (1992) and Ellsberg (1961).

general sentiment when both beliefs and shares of optimists/pessimists contribute to the emergence of optimism/pessimism waves?

The main goal of the present research consists in providing elements to answer the previous questions, trying to unify the two partial portraits depicted in the above mentioned papers. To the best of our knowledge, this is the first contribution in such unifying direction. On the basis of both the works by De Grauwe and Rovira Kaltwasser (2012) and by Naimzada and Pireddu (2015b), we aim indeed to go beyond the partial descriptions of optimism/pessimism proposed in those papers, which focus on single aspects of the complex phenomenon of animal spirits, in order to furnish a joined characterization of what optimism/pessimism are, as well as to introduce some tools to analyze the resulting framework.

More precisely, we take into account both endogenous belief biases and an endogenous switching rule between heuristics. This means that, due to the former mechanism, agents interact in order to form their beliefs about the fundamental value, using an imitative process the strength of which is regulated by a parameter μ, which we will call the *imitation degree*. On the other hand, as a consequence of the latter rule, the group sizes of optimists and pessimists are not fixed, being endogenously determined through an evolutionary selection mechanism, regulated by a parameter β which describes its *intensity of choice*. The model is completed by the stock price adjustment rule, consisting in a nonlinear, bounded mechanism that prevents negativity and divergence issues.

In such a framework, agents both update their beliefs on the fundamental value according to the relative ability shown by optimists and pessimists, albeit remaining optimists and pessimists, and can switch to the other group of speculators if they performed better in terms of relative profits. Indeed, in the present model, differently than in De Grauwe and Rovira Kaltwasser (2012) and Naimzada and Pireddu (2015b), the optimism/pessimism degree is regulated by the joint effect of the population shares and of the value of beliefs. In order to describe its evolution, we introduce a synthetic index $I_T(t)$, representing the mean of the agents' beliefs, weighted with the corresponding shares and averaged on a suitable period T. In this way, we take into account the resulting effect of both (beliefs and shares) aspects, so that we can say that optimism (respectively, pessimism) corresponds to those periods for which $I_T(t)$ exceeds (respectively, lies below) the true fundamental value F. As we shall see, in our setting, waves of optimism and pessimism are better understood in terms of $I_T(t)$ than of either beliefs or shares alone.

As concerns our results, we study, both analytically and through numerical simulations the stability of the unique steady state, its bifurcations, as well as the emergence of complex behaviors, with possible multistability phenomena, characterized by the presence of coexisting attractors. Several scenarios we find already arose in either De Grauwe and Rovira Kaltwasser (2012) or Naimzada and Pireddu (2015b) settings. For example, in the former work, two stability thresholds for the intensity of choice parameter were detected, corresponding, respectively, to a flip and a Neimark-Sacker bifurcation. On the other hand, before the flip bifurcation in that paper the system diverges and thus such bifurcation does not lead to complex behaviors, while in our framework, thanks to the presence of the bounded, nonlinear stock price adjustment mechanism, we do not face price divergence and the flip bifurcation is preceded by periodic and possibly chaotic motions. In regard to the latter work, it is there proven that the role of the imitation degree is ambiguous, meaning that the equilibrium can be stable only for intermediate values of μ, being instead unstable when μ lies below or above suitable thresholds. However, the setting we analyze allows us to show that such scenarios can be completely altered when the imitation and switching mechanisms are simultaneously taken into account. We prove that the erratic price behavior arising without the imitation mechanism is dampened and even stabilized by the introduction of the imitative process. Similarly, dynamics that are unstable when fixed fractions of optimists/pessimists are considered can be stabilized under a switching mechanism regulated by a suitable evolutionary pressure. We stress that the opposite situations can occur as well, with dynamics stable under just one between the imitation and the switching mechanisms that become unstable when both are taken into account.

Moreover, trying to deepen further the analysis of the emergence of waves of optimism/pessimism, we show that we can have either large or small beliefs together with either small or large fractions of pessimists, and that the occurrence of such combinations is in general uniformly distributed. Once more this means that we can not neglect any of the two mechanisms, since they jointly drive the behavior of the stock price market. These results allow us to give an answer to our first two research questions: to provide a whole description of animal spirits, both mechanisms should be considered.

Finally, by means of the sentiment index $I_T(t)$, we are able to answer the third question, showing that the emergence of optimism/pessimism waves can only be read considering and weighting the combined effects of imitation and switching. We have situations in which optimism is due to large beliefs, even if the share of pessimists is large, and situations in which it is mostly determined by the fraction of agents adopting an optimistic behavioral rule, even in the presence of small values of beliefs. This is both true for the deterministic and the stochastically perturbed versions of our model. In particular, we notice that the statistical analysis we perform highlights the fact that the behavior of our model is in agreement with the stylized facts reported in the empirical literature on financial markets (see e.g. Schmitt and Westerhoff 2014; Westerhoff 2009, and the references therein). Indeed, we observe fat tails and excess volatility in the returns distributions, as well as bubbles and crashes for stock prices.

The remainder of the paper is organized as follows. In Section 2 we introduce the model. In Section 3 we perform the corresponding stability analysis. In Section 4 we illustrate the role of the main parameters on the stability of the system and we present the bifurcation analysis. In Section 5 we interpret the observed dynamics from an economic viewpoint. In Section 6 we add stochastic shocks to the model and perform a statistical analysis. In Section 7 we discuss the results and propose some possible extensions. Proofs are collected in the Appendix.

2 The model

We introduce and analyze an evolutive financial market model with heterogeneous agents, governed by three crucial aspects: the behavioral rules of speculators, the mechanism of price formation (see Hommes 2013) and the switching mechanism between behavioral rules. More precisely, we will deal with two groups of fundamentalists (i.e., agents who, deeming that stock prices will return to their fundamental value F, buy stocks in undervalued markets and sell stocks in overvalued markets), whose behavior is captured by the dynamic motions of the corresponding beliefs about the fundamental value. In particular, we will consider optimists, who overestimate the true fundamental value, and pessimists, who instead underestimate it. We assume that the fundamentalists' decisional mechanism consists in forming time-varying beliefs about the fundamental value, as a result of an imitative process; due to the ambiguity in the financial market generated by the uncertainty about the future stock price, agents do not rely on the true fundamental value, although they know it. Considering F just as a reference value, they take into account the relative ability shown by optimists and pessimists in guessing the realized stock price $P(t)$ in the previous period and, still remaining optimists or pessimists, update their beliefs on the fundamental value proportionally to such criterion. The imitative updating mechanism of beliefs we assume is then described by

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$$\begin{aligned} X(t+1) &= \underline{f}\,\frac{e^{\mu(Y(t)-P(t))^2}}{e^{\mu(X(t)-P(t))^2}+e^{\mu(Y(t)-P(t))^2}} + F\,\frac{e^{\mu(X(t)-P(t))^2}}{e^{\mu(X(t)-P(t))^2}+e^{\mu(Y(t)-P(t))^2}} \\ Y(t+1) &= F\,\frac{e^{\mu(Y(t)-P(t))^2}}{e^{\mu(X(t)-P(t))^2}+e^{\mu(Y(t)-P(t))^2}} + \overline{f}\,\frac{e^{\mu(X(t)-P(t))^2}}{e^{\mu(X(t)-P(t))^2}+e^{\mu(Y(t)-P(t))^2}} \end{aligned} \tag{1}$$

where $X(t)$ (respectively, $Y(t)$) represents the belief on the fundamental value of pessimists (respectively, optimists), who always underestimate (overestimate) the true fundamental value F, i.e., $X(t) < F < Y(t)$. Moreover, $\underline{f} \in (0, F)$ (respectively, $\overline{f} \in (F, +\infty)$) is a lower (upper) bound for pessimists' (optimists') beliefs and $\mu \geq 0$ is a parameter measuring the intensity of the imitative process, i.e., how deeply agents are influenced by others' choices in updating their own beliefs. In particular, through (1) we express that both pessimistic and optimistic agents proportionally imitate those who have been more able in guessing the realized stock price.

More precisely, by construction, $X(t)$ and $Y(t)$ are obtained as weighted averages in which the weights vary in $(0, 1)$, taking into account the relative distances between $P(t)$ and the realized values for $X(t) \in (\underline{f}, F)$ and $Y(t) \in (F, \overline{f})$. If $|X(t)-P(t)| = |Y(t)-P(t)|$ we have $X(t+1) = (\underline{f}+F)/2$ and $Y(t+1) = (F+\overline{f})/2$, i.e., $X(t+1)$ and $Y(t+1)$ lie at the middle point of the intervals in which they can, respectively, vary. If instead $|X(t) - P(t)| < |Y(t) - P(t)|$, i.e., pessimists performed better than optimists in guessing the realized stock price, then $X(t+1)$ will be closer to $\underline{f}$ than to F and $Y(t+1)$ will be closer to F than to $\overline{f}$, that is, both $X(t+1)$ and $Y(t+1)$ will be closer to the lowest possible value they can assume. The opposite conclusions hold in case $|X(t) - P(t)| > |Y(t) - P(t)|$.

We stress that, as in Naimzada and Pireddu (2015b), our updating mechanism bears resemblance to the so-called "Proportional Imitation Rules" in Schlag (1998).

However, differently from Naimzada and Pireddu (2015b), the updating of beliefs is no more grounded on the relative profits realized by the two kinds of agents, but rather on their relative ability in guessing the realized stock price, similarly to the rule used for the switching mechanism in Naimzada and Ricchiuti (2008, 2009), based on the squared errors between the perceived fundamental values and prices.

We notice that, when $\mu = 0$, then $X(t+1) \equiv \frac{1}{2}(\underline{f} + F)$ and $Y(t+1) \equiv \frac{1}{2}(F + \overline{f})$ and thus there is no imitation. When instead $\mu \to +\infty$, then if $(X(t) - P(t))^2 < (Y(t) - P(t))^2$, i.e., the squared error between the belief on the fundamental value and the stock price is lower for pessimists than for optimists, then $X(t+1) \to \underline{f}$ and $Y(t+1) \to F$, that is, both variables tend towards their lowest possible value, while if $(X(t) - P(t))^2 > (Y(t) - P(t))^2$, then $X(t+1) \to F$ and $Y(t+1) \to \overline{f}$.

Mechanism (1), which encompasses the effects of the imitative interaction among agents, does not establish a competition between optimistic and pessimistic behavioral rules, as Eq. 1 only affects the value of the beliefs of pessimists and optimists, but not the kind of decisional rule adopted by each speculator. The evolutionary competition between optimism and pessimism is instead described by an endogenous switching mechanism, allowing agents to switch to the other group of speculators, if they performed better in terms of relative profits. In particular, the population shares evolve according to the discrete choice model in Brock and Hommes (1997), used also in De Grauwe and Rovira Kaltwasser (2012). We assume a normalized population of size one, and we introduce $\omega(t) \in (0, 1)$, which represents the fraction of the population composed by pessimists at time t, so that total excess demand reads as

$$\begin{aligned} D(t) &= \omega(t)(X(t) - P(t)) + (1 - \omega(t))(Y(t) - P(t)) \\ &= \omega(t)X(t) + (1 - \omega(t))Y(t) - P(t). \end{aligned} \tag{2}$$

Indeed, fundamentalists' demand depends on the difference between the (beliefs on the) fundamental value and the stock price. We observe that orders placed by pessimistic and optimistic fundamentalists, which may be positive (meaning buy) or negative (meaning sell) according to the relative positions of the stock price with respect to the beliefs on the fundamental value (see also Fig. 5), are weighted with their corresponding shares.

Introducing the profits $\pi_X(t+1)$ and $\pi_Y(t+1)$ for the two kinds of speculators, the expression of which is given by

$$\pi_i(t+1) = (P(t+1) - P(t))(i(t) - P(t)), \ i \in \{X, Y\}, \tag{3}$$

we have the following formulation for the switching mechanism

$$\omega(t+1) = \frac{e^{\beta\pi_X(t+1)}}{e^{\beta\pi_X(t+1)} + e^{\beta\pi_Y(t+1)}} = \frac{1}{1 + e^{-\beta(\pi_X(t+1) - \pi_Y(t+1))}}. \tag{4}$$

Notice that the right-hand side of Eq. 4, which describes the relative profits of traders of type X (i.e., pessimists), coincides with the share in the next period of agents of type X in the discrete choice model (see Anderson et al. 1992; Brock and Hommes 1997). In particular, the positive parameter β represents the intensity of choice. In the limit $\beta \to 0$ there is no switching and both the population shares coincide with $1/2$; when instead $\beta \to +\infty$, the whole population moves towards optimism or pessimism, according to which option is more profitable.

The last aspect we need to specify in our model is the price adjustment mechanism. After gathering all the orders and computing excess demand, as usual the market maker sets the stock price for the next period. The majority of the existing literature on behavioral financial markets (for surveys, we refer the interested reader to Hommes 2006 and Chiarella et al. 2009) deals with a linear price adjustment mechanism. This means that the ratio between the price variation and the excess demand is constant and in turns this implies that, for instance, if the latter assumes large values in absolute value, the price variation will be large. Hence, such a mechanism easily leads to negativity issues and divergence of the dynamics because of an overreaction:[2] indeed, considering too large starting values for the stock price, the iterates may quickly limit towards minus infinity. In order to avoid overreaction phenomena and an excessive volatility in the stock market, central authorities often impose price limits. Recalling France et al. (1994), the aim of price limits is to "reduce the probability of an overreaction to news. By not allowing prices to move beyond a certain point, they discourage mob psychology and force prices to adjust slowly". For further discussions about price limits see Harris (1998) and Kyle (1988).

In agreement with such perspective, in the present work we assume the market maker is forced by a central authority to be more cautious in adjusting the stock price when excess demand is large, i.e., when the system is far from its equilibria, while he has more freedom when excess demand is small, that is, when the system is close to an equilibrium. This kind of diversified behavior may be represented by a nonlinear function only, which has also to be increasing and to pass through the origin.

In particular, our price variation limiter mechanism is described by a sigmoidal adjustment rule that determines a bounded price variation in every time period, thanks to the presence of two asymptotes that limit the dynamics, the formulation of which is given by

$$P(t+1) - P(t) = a_2 \left(\frac{a_1 + a_2}{a_1 e^{-\gamma D(t)} + a_2} - 1 \right), \tag{5}$$

where $D(t)$ is the excess demand defined in Eq. 2, γ is a positive parameter influencing the reactivity r of price variation with respect to changes in excess demand and a_1, a_2 are positive parameters limiting price variation and conditioning r. We notice that, in the literature, bounded price variation mechanisms are often obtained by considering piecewise linear maps (see for instance Weddepohl 1995). We rather chose to deal with the sigmoidal function in Eq. 5 because it is differentiable and thus simplifies the mathematical treatment of the model.

[2] Starting with the crucial paper by De Bondt and Thaler (1985), a well-grounded empirical literature has arisen to show the presence of overreaction phenomena in financial markets. We recall that, as said in France et al. (1994), page 19, overreaction is "defined as a movement in price that overshoots the equilibrium value and then subsequently returns to its true value". Subsequently, some authors have given further foundations to overreaction events through the formulation and analysis of mathematical models. See, for instance, the works by Barberis et al. (1998), by Hong and Stein (1999) and by Veronesi (1999).

The reactivity r of price variation with respect to changes in excess demand, i.e., the derivative of the right-hand side of Eq. 5 with respect to $D(t)$, computed in correspondence to a generic value of excess demand reads then as

$$r(D(t)) = \frac{\gamma a_1 a_2 (a_1 + a_2)}{(a_1 e^{-\gamma D(t)} + a_2)^2 \, e^{\gamma D(t)}} \, .$$

In particular, when $D(t) = 0$, i.e., in correspondence to a null excess demand, we simply obtain

$$r(0) = \widetilde{\gamma} = \frac{\gamma a_1 a_2}{a_1 + a_2}. \tag{6}$$

In this manner, $\widetilde{\gamma}$ may be interpreted as the market maker price adjustment reactivity when the excess demand vanishes, taking into account the combined effect of γ, a_1 and a_2. We stress that a_1 and a_2 play the role of horizontal asymptotes to the price variation. Indeed, as concerns price impact, that is, the impact of excess demand on price variation, we have that, with the choice in Eq. 5, $P(t+1) - P(t)$ is increasing in $D(t)$ and vanishes when $D(t) = 0$; moreover, price variation $P(t+1) - P(t)$ is bounded from below by $-a_2$ (obtained when $D(t) \to -\infty$) and from above by a_1 (obtained when $D(t) \to +\infty$). Hence, the price variations in Eq. 5 are gradual, and the presence of the two horizontal asymptotes prevents the dynamics of the stock price from diverging and helps avoide negativity issues. In particular, increasing (decreasing) such parameters we obtain an increase (decrease) in the possible price variations. Notice that γ influences the market maker price adjustment reactivity, without modifying the value of the asymptotes. We finally observe that, in the majority of the literature on the topic, it is assumed that the behavior of the market maker is symmetric with respect to variations in excess demand that have opposite signs but coincide in absolute value. Since in the present work we allow a_1 and a_2 to be possibly different, we can deal with more general settings in which the market maker can react in a different manner to a positive or to a negative excess demand.

We recall that nonlinear adjustment mechanisms determining a bounded price variation and similar to the one on the right-hand side of Eq. 5 have been already considered in Chiarella et al. (2009), Naimzada and Pireddu (2015b, c), Naimzada and Ricchiuti (2014), Tuinstra (2002) and Zhu et al. (2009).

The model we are going to study is obtained collecting (1), (4) and (5).

Actually, in order to simplify our analysis, since the most significant form of heterogeneity in the model is represented by the different attitudes of agents towards the reference value, from now on we shall assume that $\underline{f}$ and $\overline{f}$ lay at the same distance Δ from F, i.e., that $\underline{f} = F - \Delta$ and $\overline{f} = F + \Delta$. In this manner, $\Delta \geq 0$ may be used as bifurcation parameter in our analytical and numerical results about local stability. Indeed, in Sections 3 and 4 we will take μ, β and Δ as bifurcation parameters. We remark that we checked through simulations that the dynamics we will show in Sections 3 and 4 are not significantly affected by the presence of symmetric bounds for the beliefs with respect to F. In both the symmetric and asymmetric frameworks, the only relevant aspect seems to be the distance between optimistic and pessimistic beliefs, which is already represented by parameter Δ in a suitably general way in the symmetric setting.

Exploiting the notation just introduced and observing that the weight coefficients in Eq. 1 sum up to 1, following Eq. 4 we may write our model as

$$\begin{cases} X(t+1) = F - \Delta\left(\frac{1}{1+e^{\mu((X(t)-P(t))^2-(Y(t)-P(t))^2)}}\right) \\ Y(t+1) = F + \Delta\left(\frac{1}{1+e^{-\mu((X(t)-P(t))^2-(Y(t)-P(t))^2)}}\right) \\ P(t+1) = P(t) + a_2\left(\frac{a_1+a_2}{a_1 e^{-\gamma(\omega(t)(X(t)-P(t))+(1-\omega(t))(Y(t)-P(t)))}+a_2} - 1\right) \\ \omega(t+1) = \frac{1}{1+e^{-\beta(\pi_X(t+1)-\pi_Y(t+1))}} \end{cases} \tag{7}$$

As concerns the meaning of Δ, it describes the degree of ambiguity in the financial market, which prevents agents from relying on the true fundamental value F in their speculations, even if they know it. We stress that Δ gives also a measure of the heterogeneity degree among agents and thus of the bias in their beliefs. Namely, it is defined as the maximum possible distance of optimists' and pessimists' beliefs from the true fundamental value F. In particular, for $\Delta = 0$ we have no ambiguity in the stock market and thus agents' heterogeneity disappears from the model. Indeed, when $\Delta = 0$ the first two equations in System (7) simply become $X(t+1) = Y(t+1) \equiv F$, and the dynamics are generated just by the stock price equation and by the switching mechanism. Moreover, we remark that, when setting $\mu = 0$, it holds that $X(t+1) = F - \frac{\Delta}{2}$ and $Y(t+1) = F + \frac{\Delta}{2}$. Hence, in this case there is no imitation in the updating of the beliefs on the fundamental value, which indeed are fixed. Thus, when $\mu = 0$, we enter the framework in De Grauwe and Rovira Kaltwasser (2012) with bias $a = \frac{\Delta}{2}$, except for the presence of our nonlinear price adjustment mechanism that replaces the linear price equation in De Grauwe and Rovira Kaltwasser (2012). When instead $\mu \neq 0$, the beliefs X and Y about the fundamental value are no more fixed and we generalize the constant belief setting in De Grauwe and Rovira Kaltwasser (2012), with the only exception being our nonlinear price equation. Finally, if $\beta = 0$ we fall within a particular case of Naimzada and Pireddu (2015b), in which ω is fixed and equal to 0.5.

As remarked by De Grauwe and Rovira Kaltwasser (2012), the presence of "animal spirits" affects reality, possibly leading to waves of optimism and pessimism. In this work, in which the beliefs of agents are exogenously determined and constant in time, waves of optimism and pessimism are identified with the alternation of small and large values of the share $\omega(t)$ of pessimists, while in the paper by Naimzada and Pireddu (2015b), in which ω is a fixed, exogenous parameter, waves of optimism and pessimism correspond to beliefs oscillations. On the other hand, in System (7) we can not disentangle the effects of the share $\omega(t)$ of pessimists from the role played by the value of beliefs. An optimistic behavior can be encompassed by values of $X(t)$ and $Y(t)$ sufficiently close to F and $F + \Delta$, respectively, even when $\omega(t)$ is large, as well as by a large share of optimists even if beliefs are small. To take into account such double nature of optimism/pessimism, it is appropriate to introduce a "sentiment index" that reflects the joint effects of $\omega(t)$ and $X(t)$. The simplest way to define it is to consider an average of optimists and pessimists beliefs weighted by their corresponding fractions, namely, $I_1(t) = \omega(t)X(t) + (1 - \omega(t))Y(t)$. We stress that $I_1(t)$ is completely consistent with both the De Grauwe and Rovira Kaltwasser (2012) and Naimzada and Pireddu (2015b) settings. In fact, in the former setting, since $X(t) = F - \Delta/2$ and $Y(t) = F + \Delta/2$, for all t, we have that $I_1(t) = F + \Delta(1-2\omega)/2$, which reflects the fact that optimism/pessimism is determined by the prevailing share of agents, i.e., by whether $\omega < 1/2$ or $\omega > 1/2$, respectively. Conversely, in the latter setting, we have that ω is constant in time, and hence $I_1(t)$ changes only with respect to variations of $X(t)$ and $Y(t)$.

Notice that $I_1(t)$ portrays the instantaneous sentiment at time t. On the other hand, in order to describe the temporal evolution of the waves of optimism and pessimism, it is sometimes crucial to consider several consecutive periods. Moreover, it may happen that the persistence of $I_1(t) > F$ over time is occasionally interrupted by some $\tilde{t}$ for which $I_1(\tilde{t}) < F$, or vice versa. This becomes particularly significant when stochastic shocks are introduced in System (7) (see Section 6), since they can temporarily affect the value of $I_1(t)$ in a determinant way.

To take into account the previous issues and to be able to neglect insignificant discording behaviors in isolated periods, we generalize $I_1(t)$ by introducing the following *optimism/pessimism persistence index*

$$I_T(t) = \sum_{j=t-T+1}^{t} \frac{\omega(j)X(j) + (1-\omega(j))Y(j)}{T}, \tag{8}$$

which is a moving average of $I_1(t)$ over the $T \geq 1$ time steps preceding t. We stress that T is an exogenous parameter reflecting at which scale we are looking at the optimistic/pessimistic behavior of agents. A small value of T is suitable to describe short-period sentiments, while large values of T allow us to neglect temporary effects and to describe better long-term persistence of optimism/pessimism. In such a perspective, $I_T(t)$ permits to consider the combined role played by beliefs and shares of pessimists/optimists, avoiding attaching too much relevance to small and isolated deviations in the trend behavior. To the best of our knowledge, no similar joint indexes have been introduced in the literature on financial markets. In fact, it bears a resemblance to the long-period optimism index considered in Naimzada and Pireddu (2015a), defined however, just as an average of the shares over several periods.

We remark that, from the ranges of variation of $\omega(t)$, $X(t)$ and $Y(t)$, it follows that $I_T(t) \in (F - \Delta, F + \Delta)$. We can then say that "optimism" is realized over periods of size T when $I_T(t) > F$, as well as that "pessimism" is prevalent when $I_T(t) < F$. The interpretation of the sentiment index in Eq. 8 and the influence of T on it depend on whether we are considering the completely deterministic model (7) or we are taking into account stochastic perturbations. As already mentioned, the index has particular relevance when non-deterministic shocks are encompassed, as it will become evident in Section 6. Nonetheless, the standard behavior is that, since I_T is a moving average of $I_1(t)$ over T periods, raising T has a smoothing effect on non constant time series, leading $I_T(t)$ to be in general closer and closer to F as T increases. Moreover, raising T produces a lagging effect on $I_T(t)$, which indeed displays a delay with respect to $I_1(t)$.

We start our analysis by studying the steady states of Eq. 7.

Proposition 1 *System* (7) *has a unique steady state in*

$$(X^*, Y^*, P^*, \omega^*) = \left(F - \frac{\Delta}{2}, F + \frac{\Delta}{2}, F, \frac{1}{2}\right).$$

Hence, the steady state values for X and Y are symmetric with respect to F and they lie at the middle points of the intervals in which they may respectively vary. In particular, when $\Delta = 0$ we find $X^* = Y^* = P^* = F$, as in the classical framework without belief biases and imitation, with identical agents which use F as fundamental value. As we shall see in Section 3, in this case the system inherits the stability/instability of the price adjustment mechanism. We remark that, at the equilibrium, index $I_T(t)$ coincides coincides with the fundamental value F and describes a neutral situation, where neither optimism nor pessimism prevails.

In the next result, we show that Eq. 7 is actually "equivalent", in a sense to be better specified, to a three-dimensional dynamical system we are going to analyze in Section 3.

Proposition 2 *The variables X and Y in* Eq. 7 *satisfy the following condition: $Y(t) = X(t) + \Delta$, for all $t \geq 1$.*

The above result means that, for all initial conditions $X(0) < F < Y(0)$, after just one time period the trajectory of $Y(t)$ converges on the trajectory of $X(t) + \Delta$ and since then they coincide. Hence, Proposition 1 may be rephrased by saying that, for $t \geq 1$, the dynamical system associated to Eq. 7 is equivalent to that associated to the three-dimensional map

$$G = (G_1, G_2, G_3) : (\underline{f}, F) \times (0, +\infty) \times (0, 1) \rightarrow \mathbb{R}^3,$$

$$(X(t), P(t), \omega(t)) \mapsto (G_1(X(t), P(t), \omega(t)), G_2(X(t), P(t), \omega(t)), G_3(X(t), P(t), \omega(t))),$$

defined as:

$$\begin{cases} X(t+1) = G_1(X(t), P(t), \omega(t)) \\ \qquad = F - \Delta\left(\frac{1}{1+e^{\mu((X(t)-P(t))^2-(X(t)+\Delta-P(t))^2)}}\right) \\ P(t+1) = G_2(X(t), P(t), \omega(t)) \\ \qquad = P(t) + a_2\left(\frac{a_1+a_2}{a_1 e^{-\gamma(\omega(t)(X(t)-P(t))+(1-\omega(t))(X(t)+\Delta-P(t)))}+a_2} - 1\right) \\ \omega(t+1) = G_3(X(t), P(t), \omega(t)) \\ \qquad = \frac{1}{1+e^{-\beta a_2 \Delta\left(\frac{a_1+a_2}{a_1 e^{-\gamma(\omega(t)(X(t)-P(t))+(1-\omega(t))(X(t)+\Delta-P(t)))}+a_2}-1\right)}} \end{cases} \tag{9}$$

in the sense that the two systems generate the same trajectories for $X(t)$, $P(t)$ and $\omega(t)$.

We will deal with G both to derive analytically in Section 3 the stability conditions for our model using the method in Farebrother (1973), and in the numerical simulations in Sections 4–6, where we will specify the initial conditions for $X(t)$, $P(t)$ and $\omega(t)$ only, implicitly taking $Y(0) = X(0) + \Delta$ for simplicity.

Before presenting the analytical results on local stability, we give some intuition about the effects, on the sentiment index in Eq. 8, of the potential dynamics of $\omega(t)$ and $X(t)$. Indeed, if trajectories converge toward the steady state, we have $I_T(t) \to \omega^* X^* + (1-\omega^*)Y^* = F$ for any value of T. When dynamics are periodic with period n, index $I_T(t)$ follows a periodic trajectory with period n, too, as long as $T \neq kn$, $k \in \mathbb{N}$, case in which $I_{kn}(t)$ converges to the average of $I_1(t)$ on a whole cycle of n time periods. If we have chaotic or quasi-periodic dynamics, it is not possible to provide analytical results about the dynamics of $I_T(t)$. However, we can report some qualitative facts for the quasi-periodic case; we shall numerically confirm these in Section 5. In general, index $I_1(t)$ inherits the quasi-periodic dynamics of $X(t)$ and $\omega(t)$, oscillating around F. If T moderately grows, such waves still occur and $I_T(t)$ gets increasingly smoothed around F. We notice that, when dynamics are quasi-periodic, time series of $\omega(t)$ and $X(t)$ can exhibit some significant positive autocorrelation coefficients for suitable lags $n > 1$. In such cases, the dynamics of $I_n(t)$ are qualitatively different from those of $\omega(t)$ and $X(t)$. This is essentially a consequence of computing the average over whole "quasi-periods", which allows enhancing deviations of quasi-periodic time series from a periodic trajectory. In such a case, we again observe waves of optimism/pessimism, which can last for larger time intervals, even if each wave may consist of a significantly different number of time periods. If T grows further, each time period has an increasingly smaller influence on $I_T(t)$, and thus the quasi-periodic underlying dynamics of $\omega(t)$ and $X(t)$ have less and less impact on the qualitative dynamics of $I_T(t)$, which again tend to exhibit a more irregular behavior than the quasi-periodic one.

3 Stability analysis

For our analytical results, we will deal with the three-dimensional framework in Eq. 9. We observe that, similarly to what was done in Proposition 1 it is possible to prove that Eq. 9 has a unique fixed point in $(X^*, P^*, \omega^*) = \left(F - \frac{\Delta}{2},\ F,\ \frac{1}{2}\right)$. In the next proposition, we derive the local stability conditions for System (9) at the steady state.

Proposition 3 *The steady state (X^*, P^*, ω^*) is locally asymptotically stable for System (9) provided that parameters fulfill the following conditions*

(i′) $\left(1 + \frac{\beta\widetilde{\gamma}\Delta^2}{4}\right)\left(1 - \frac{\mu\Delta^2}{2}\right) > \frac{\widetilde{\gamma}}{2}$;

(ii′) $\left(1 - \frac{\beta\widetilde{\gamma}\Delta^2}{4}\right)\left(1 + \frac{\beta\mu^2\widetilde{\gamma}\Delta^6}{16} + \frac{\mu\Delta^2}{2}\left(1 + \frac{\beta\widetilde{\gamma}\Delta^2}{4}\right)\right) + \frac{\beta\mu\widetilde{\gamma}^2\Delta^4}{8} > 0$;

(iii′) $6 + \mu\Delta^2 > \frac{\beta\widetilde{\gamma}\Delta^2}{4}(2 - \mu\Delta^2)$,

where $\widetilde{\gamma}$ is defined in Eq. 6.

In the previous conditions, we may easily put in evidence β, μ and, for $\mu = 0$, also Δ. The possible resulting scenarios are summarized in the following corollary. We call a scenario destabilizing with respect to a parameter when the steady state is stable below a certain threshold of that parameter and unstable above it. We say that

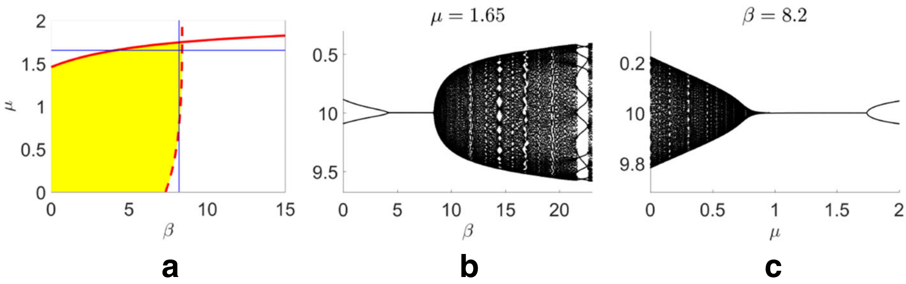

Fig. 1 **a**: Stability region (*in yellow*) for $F = 10$, $\Delta = 1$, $a_1 = 1.2$, $a_2 = 1$ and $\gamma = 1$. **b**: Bifurcation diagram on varying β for $\mu = 1.65$, corresponding to the *horizontal line* plotted in the stability diagram. **c**: Bifurcation diagram on varying μ for $\beta = 8.2$, corresponding to the *vertical line* plotted in the stability diagram

a scenario is mixed if the steady state is stable inside an interval of parameter values and unstable outside it. We say that a scenario is unconditionally unstable when the steady state is unstable for all the parameter values.

Corollary 1 *Let $\Delta \neq 0$, μ and $\widetilde{\gamma}$ be fixed. Then, on varying β, we can have destabilizing, mixed and unconditionally unstable scenarios.*

Let $\Delta \neq 0$, β and $\widetilde{\gamma}$ be fixed. Then, on varying μ, we can have destabilizing, mixed and unconditionally unstable scenarios.

Let $\beta \neq 0$, $\mu = 0$ and $\widetilde{\gamma}$ be fixed. Then, on varying Δ, we can have either a destabilizing or a mixed scenario.[3]

We notice that, usually in the literature, an increase in the intensity of choice has just a destabilizing effect (see for instance Hommes 2013), while for us it may also be stabilizing (cf. Figs. 1, 2, 3), as long as its value is not excessively large. In fact, also in the framework in De Grauwe and Rovira Kaltwasser (2012), we obtain for $\mu = 0$ except for the presence of the sigmoid function in our price adjustment mechanism, two stability thresholds for the intensity of choice parameter were found, corresponding, respectively, to a flip and a Neimark-Sacker bifurcation. On the other hand, some numerical simulations we performed suggest that, in that paper, before the flip bifurcation, the system diverges and thus such a bifurcation would not lead to complex behaviors: as we shall see in Section 4, in our framework, the flip bifurcation is instead preceded by periodic and possibly chaotic motions. It is immediately understood that the divergence issue encountered in De Grauwe and Rovira Kaltwasser (2012) is due to the linearity of the price adjustment mechanism, while our sigmoidal nonlinearity helps in avoiding negative or diverging dynamics. Also, in

[3] We only provide sufficient conditions for the occurrence of each scenario. We remark that we may indeed obtain the same scenario for other parameter configurations. Moreover, we notice that some of the stability intervals we shall derive below may in principle be empty for any parameter configuration. We will provide numerical examples in Section 4 to show that there exist parameter settings for which such intervals are nonempty and the corresponding scenarios do actually occur.

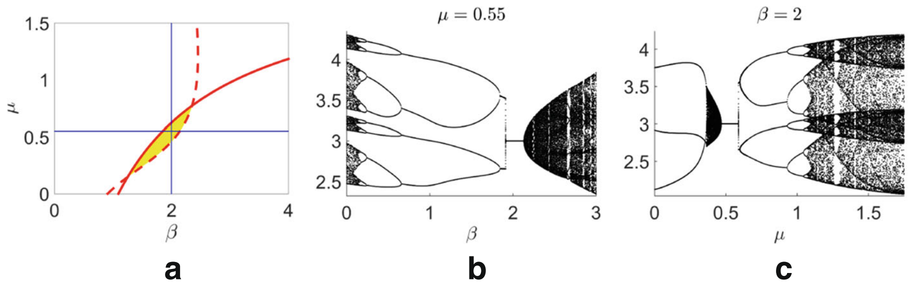

Fig. 2 **a**: Stability region (*in yellow*) for $F = 3$, $\Delta = 1$, $a_1 = 1.7$, $a_2 = 1$ and $\gamma = 7$. **b**: Bifurcation diagram on varying β for $\mu = 0.55$, corresponding to the *horizontal line* plotted in the stability diagram. **c**: Bifurcation diagram on varying μ for $\beta = 2$, corresponding to the *vertical line* plotted in the stability diagram

Chiarella et al. (2006), two stability thresholds for a different parameter (i.e, the population weighted reaction coefficient of the fundamentalists) were detected. However, as recalled in De Grauwe and Rovira Kaltwasser (2012), in Chiarella et al. (2006) the authors, in the numerical simulations, focused their attention on the case in which stability of the steady state is lost through a Neimark-Sacker bifurcation.

According to Corollary 1, increasing the imitation degree μ or the heterogeneity level Δ may have a stabilizing effect (see Figs. 1–4), so that the roles of μ and Δ are not univocally determined, depending on the value of the other parameters. We recall that we found a similar result for μ and Δ in Naimzada and Pireddu (2015b), where we detected a double stability threshold for the parameter governing the intensity of the imitative process, based in that paper on the past relative profits realized by optimists and pessimists, as well as for the heterogeneity level parameter. We, however, stress that, except for that contribution, to the best of our knowledge no other works in the related literature show the ambiguous effect of such parameters.

We notice that, as $\mu \to +\infty$, the resulting dynamics are unstable. In such case, it is easy to understand why beliefs (and consequently shares) do not settle on either

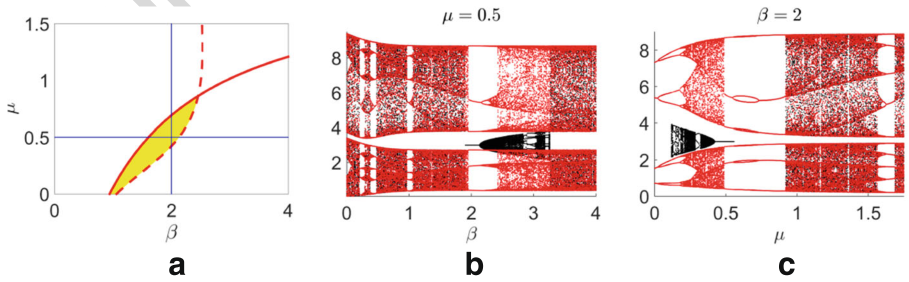

Fig. 3 **a**: Stability region (*in yellow*) for $F = 3$, $\Delta = 1$, $a_1 = 10.2$, $a_2 = 6$ and $\gamma = 1$. **b**: Bifurcation diagram on varying β for $\mu = 0.5$, corresponding to the *horizontal line* plotted in the stability diagram. **c**: Bifurcation diagram on varying μ for $\beta = 2$, corresponding to the *vertical line* plotted in the stability diagram. In both (**b**) and (**c**), *black diagrams* are obtained for initial conditions $X(0) = 2.6$, $P(0) = 3.0001$, $\omega(0) = 0.5$, while *red diagrams* for initial conditions $X(0) = 2.6$, $P(0) = 4$, $\omega(0) = 0.5$

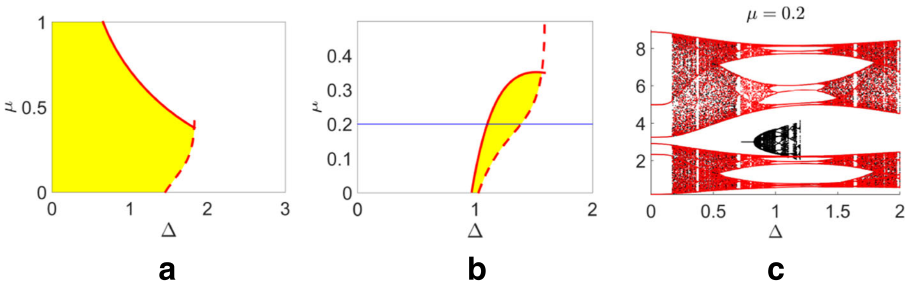

Fig. 4 Stability regions (*in yellow*) for $F = 3,\ \beta = 1,\ a_1 = 5.1,\ a_2 = 3$ in (**a**) and $F = 3,\ \beta = 1,\ a_1 = 10.2,\ a_2 = 6$ in (**b**). **c**: Bifurcation diagram on varying Δ for $\mu = 0.2$, corresponding to the *horizontal line* plotted in the stability diagram in (**b**). The *black diagram* is obtained for initial conditions $X(0) = 2.6,\ P(0) = 3.0001,\ \omega(0) = 0.5$, while the *red diagram* for initial conditions $X(0) = 2.6,\ P(0) = 4,\ \omega(0) = 0.5$

the upper or lower values of their ranges. In this respect, it is worth noticing that both imitation and switching mechanisms are affected by and affect price dynamics, and that the, possibly erratic, price trajectory must be taken into account. In fact, price dynamics have a qualitatively opposite effect on beliefs and shares dynamics; if the price is large, it is more likely that the best performance in guessing the price will be that of optimists, which means that beliefs will get close to their upper bound. On the other hand, if, at time t, the price is large, it is more likely, since prices are erratic when $\mu \to \infty$, that at time $t + 1$ the stock price will be small, inducing an increase in the share of pessimists. In the next time period, price changes again, so that a persistence of the previous optimistic beliefs/pessimistic shares configuration is not likely, and it is probable that they change to a pessimistic beliefs/optimistic shares configuration. The overall dynamics of (for example) pessimistic beliefs then result into an alternating behavior between values that are close to F and $F - \Delta/2$, with corresponding suitably (depending on β) small and large shares of pessimists.

As concerns the role of $\widetilde{\gamma}$, it is easy to prove that it has just a destabilizing effect, both when the dynamics are generated by the price adjustment mechanism only, i.e., for $\Delta = 0$, or in the limit case $\mu = \beta = 0$, as well as when all parameters are different from zero. This is confirmed by the increasing complexity of the dynamics when passing from $\widetilde{\gamma} = 1.88$ in Fig. 1 to $\widetilde{\gamma} = 3.77$ in Fig. 3, and finally to $\widetilde{\gamma} = 4.41$ in Fig. 2.

In particular, the isolated price adjustment mechanism is stable at $P^* = F$ if $\widetilde{\gamma} < 2$ and at $\widetilde{\gamma} = 2$ a flip bifurcation occurs. Thanks to its nonlinearity, the price dynamics stay bounded even when the isolated price adjustment mechanism is unstable (see, for instance, Fig. 4), differently from De Grauwe and Rovira Kaltwasser (2012), where the latter provides diverging trajectories.

Due to the destabilizing effect of $\widetilde{\gamma}$ even on the isolated price adjustment mechanism, we may explain the presence of two thresholds for stability with respect to μ and β as follows. When $\widetilde{\gamma}$ is large enough, the isolated price adjustment mechanism is unstable and small positive values for μ and β allow the transmission of such turbulence to the imitative process and thus to the dynamics of the beliefs on the fundamental value, as well as to the switching mechanism. When μ and β increase further, looking at the last equation in Eq. 7, we observe that intermediate values for β dampen large profits and this makes the population shares stabilize on the mean value they may assume, i.e., on their steady state values; similarly, looking at the first two equations in Eq. 7, we notice that intermediate values for μ dampen the role played by the difference between the squared errors between the beliefs on the fundamental value and prices, making the beliefs in the next period for both pessimists and optimists stabilize on the mean value they may assume, i.e., on their steady state values. On the other hand, when β and μ are too large, they become destabilizing. Indeed, large values for μ represent a high degree of nervousness in the imitation mechanism, leading to erratic fluctuations in the beliefs on the fundamental value; large values for β play a similar role in the switching mechanism.

We also notice that, when $\widetilde{\gamma}$ is sufficiently large, the left and right stability threshold values for μ and β get inverted and thus no values of such parameters can stabilize the system.

Since the analytical study of the role of Δ was possible only for $\mu = 0$, we postpone the description and explanation of the effects of Δ at the end of Section 4, after having presented the numerical investigations.

4 Bifurcation analysis

We now perform a numerical investigation of the local stability of the steady state, focusing in particular on the role of the imitation degree μ, the intensity of choice β and the level of ambiguity in the market Δ. Moreover, concerning global analysis, we give evidence of the occurrence of multistability phenomena characterized by the presence of different coexisting attractors. For simplicity, we just focus on the behavior of the stock price variable.

In the following two-dimensional stability diagrams we plot, for feasible parameter values and as long as they are real, with red color the stability threshold curves, obtained using for β, μ the upper and lower bounds derived along the proof of Corollary 1. In particular, when we numerically checked that the points belonging to a stability threshold curve correspond to parameter sets for which two eigenvalues of the Jacobian matrix in Eq. 13 are complex and lie on the unit circle, we used a dashed red curve. In this case, we then have that a Neimark-Sacker bifurcation occurs when a dashed red curve is crossed on varying a parameter. Conversely, when the points belonging to a stability threshold curve correspond to parameter sets for which exactly one eigenvalue of the Jacobian matrix (13) is equal to -1, we used a solid red line. In this case, a flip bifurcation occurs when the solid red curve is crossed on varying a parameter.

To efficiently investigate through simulations the effects of the three parameters, we study stability in the (β, μ)- and (Δ, μ)-planes. In the following simulations, we consider different possible settings with respect to F, a_1, a_2 and γ. We stress that we chose not to present an analysis in the (β, Δ)-plane because the numerical investigations we performed highlighted the presence of dynamics and scenarios qualitatively similar to those in Sections 4.1 and 4.2.

4.1 Stability in (β, μ)-plane

For the first simulation, we set $F = 10$ and $\Delta = 1$, so that the steady state is $(X^*, P^*, \omega^*) = (9.5, 10, 0.5)$, and we investigate stability for $\beta \in (0, 15]$ and $\mu \in [0, 2]$. The two-dimensional stability region obtained for $a_1 = 1.2$, $a_2 = 1$ and $\gamma = 1$ is reported in Fig. 1a. The yellow region shows parameter values corresponding to locally stable steady states, while red curves represents stability thresholds.

Looking at Fig. 1a, we notice that, in the destabilizing scenario with respect to β, stability is lost through a Neimark-Sacker bifurcation, while in the mixed one, a period-halving bifurcation is then followed by a Neimark-Sacker one. Conversely, in the destabilizing scenario with respect to μ, stability is lost through a flip bifurcation, while in the mixed one, a Neimark-Sacker bifurcation is followed by a period-doubling one. We performed several simulations for different parameter settings that confirm that the previous behaviors are general. In Fig. 1b we focus on what happens on varying β, which corresponds to considering horizontal sections of the stability diagram of Fig. 1. For $\mu = 0$, we are in the framework considered in De Grauwe and Rovira Kaltwasser (2012), except for the presence of the sigmoidal adjustment mechanism for the stock price, and we find the same destabilizing role for the intensity of choice. A similar behavior with respect to β persists for sufficiently small values of the imitation degree μ. However, differently from that context, in which only the destabilizing role of β is illustrated, if we increase μ enough we have that intermediate values for β, neither too small nor too large, reduce the complexity of the system, until a complete stabilization of the dynamics, as shown by the bifurcation diagram for $\mu = 1.65$ with respect to β in Fig. 1b. In this case, we find that the left stability threshold, at which a flip bifurcation occurs, corresponds to $\beta_\ell = 4.0958$, while the right one, at which we have a Neimark-Sacker bifurcation, corresponds to $\beta_{r_1} = 8.3975$. Actually, also in De Grauwe and Rovira Kaltwasser (2012) a double stability threshold for β was found, but in that case just on the left of the flip bifurcation the system diverges and thus no interesting dynamics can be detected.

As argued in Section 3, an increasing value for the intensity of the imitative process or for the intensity of choice has usually just a destabilizing effect, while for us it may also be stabilizing. This happens because, when β is positive but close to 0, through the switching mechanism the instability of the price adjustment equation gets transmitted to the population dynamics, which inherit the periodic cycle of the isolated price mechanism. Increasing values for β intensify the oscillations due to a larger reactivity to higher profits, but, when β is sufficiently large, positive and negative excess demands for the two groups of agents balance out in the aggregate excess demand and such compensation causes smaller price oscillations, which in turn make the profit differential decrease and this leads to smaller variations in the population shares. When β increases further, agents become, however, very reactive in the switching mechanism, and quasi-periodic dynamics emerge. Finally, as μ increases, the stability interval with respect to β shrinks until it becomes empty. For such values of μ, the dynamics are unconditionally unstable for any β.

Similarly, if we look at the dynamical behavior on varying μ, we notice that, for small values of β, increasing the imitation degree is destabilizing, as we can see looking at vertical sections of the stability diagram in Fig. 1a. In this case, instability occurs through a flip bifurcation. As β increases, stability can only be guaranteed for intermediate values of μ and we have a mixed scenario, with a Neimark-Sacker (resp. flip) bifurcation for sufficiently small (resp. large) values of μ. For instance, for $\beta = 8.2$ we find $\mu_2 = 0.7660$ and $\mu_r = 1.7425$. Increasing further β leads to unconditionally unstable dynamics with respect to μ.

We notice that for the previous parameter setting the flip bifurcation only led to a period-two cycle. More complex evolutions of the period-doubling bifurcation leading to chaotic dynamics are possible considering different parameter configurations, as in the situation reported in Fig. 2. In this case, we set $F = 3$ and $\Delta = 1$, so that the steady state is $(X^*, P^*, \omega^*) = (2.5, 3, 0.5)$. The stability region in Fig. 2a is obtained for $a_1 = 1.7$, $a_2 = 1$ and $\gamma = 7$, for the same initial datum used for the previous simulation in Fig. 1. In this case, we notice that, if the beliefs about fundamentals are completely exogenous, i.e. for $\mu = 0$, the dynamics are unconditionally unstable with respect to β, but a moderate increase in the imitation degree allows for a stabilization of the dynamics for intermediate values of β, as, for example, shown in the bifurcation diagram reported in Fig. 2b for $\mu = 0.55$. A further increase in μ makes the equilibrium again unconditionally unstable. Similar considerations are valid also with respect to β, for which, with exogenously fixed fractions of optimists and pessimists ($\beta = 0$), the dynamics are unconditionally unstable. A mixed scenario is instead obtained for intermediate values of the intensity of choice, as reported in Fig. 2c. This means that intermediate values for both μ and β allow for the stabilization of frameworks in which equilibrium is initially unstable. In both Fig. 2b and c, the flip bifurcation gives rise to a complete cascade of period doublings leading to chaotic dynamics. Finally, we notice that the stability region reported in Fig. 2 is a subset of that reported in Fig. 1. This is a consequence of the increase in the reactivity at the equilibrium $\widetilde{\gamma}$, which passes from $\widetilde{\gamma} = 1.88$ in the former parameter setting to $\widetilde{\gamma} = 4.41$ in the latter. This confirms the destabilizing role of $\widetilde{\gamma}$.

All bifurcation diagrams reported in Figs. 1 and 2 are independent of the initial datum, as we computationally checked that any initial condition provides convergence toward the same attractors. On the other hand, this is no longer true if we increase the values of a_1 and a_2. In Fig. 3a we keep $F = 3$ and $\Delta = 1$, as for simulation in Fig. 2, and we report the stability region corresponding to the choice of $a_1 = 10.2, a_2 = 6$ and $\gamma = 1$. We notice that, for this parameter setting, the destabilizing scenario with respect to β no longer occurs. We also report two bifurcation diagrams, with respect to β (in Fig. 3b) and μ (in Fig. 3c), obtained for initial conditions $X(0) = 2.6$, $P(0) = 3.0001$, $\omega(0) = 0.5$ (black diagrams) and $X(0) = 2.6$, $P(0) = 4$, $\omega(0) = 0.5$ (red diagrams). We notice that, for the bifurcation diagram with respect to β, the stability thresholds of the steady state are $\beta_\ell = 1.6078$ and $\beta_r = 2.1294$, while for the bifurcation diagram with respect to μ, the stability thresholds of the steady state are $\mu_1 = 0.4219$ and $\mu_r = 0.6923$.

In the bifurcation diagrams in Fig. 3b and c we can observe the coexistence of the fixed point and of invariant curves with an external periodic or chaotic attractor in either two or more pieces, and the trajectories visit the internal or the external attractor according to the chosen initial condition. We remark that computational investigations we performed showed that the external attractor exists for all the parameter values considered in the stability diagram in Fig. 3a, while the internal attractor is present only for some parameter values. We stress that the diverging trajectories that would follow or precede the flip bifurcation in the case of a linear price adjustment mechanism (see De Grauwe and Rovira Kaltwasser 2012) in the present model are limited by the asymptotes a_1 and $-a_2$, setting the trajectories on the external attractor. We also notice the presence of "bubbles" (see Hommes 1991, 1994) for $\mu \approx 0.65$ in Fig. 3c.

4.2 Stability in (Δ, μ)-plane

We now investigate the dynamics arising for the parameter configurations used in the simulations in Figs. 1 and 3 when setting $\beta = 1$ and $F = 3$ and taking Δ and μ as bifurcation parameters. We omit reporting the analysis in (Δ, μ)-plane of the simulation in Fig. 2, as it provides very similar results to those obtained in Section 4.1. We notice that we presented Δ as the maximum possible degree of optimism and pessimism, but it may also be seen as the degree of ambiguity in the financial market, because, when it vanishes, agents use the true fundamental value in their speculations (see System (7)). We shall use this alternative point of view in the interpretation of the results at the end of the present subsection. We also recall that, in Corollary 1, we analytically studied stability with respect to Δ only for $\mu = 0$. Hence, when μ is different from zero we shall rely on numerical simulations.

In Fig. 4a and b we report the stability regions for $a_1 = 5.1$, $a_2 = 3$, and $a_1 = 10.2$, $a_2 = 6$, respectively, which correspond to the parameter settings used for the simulations in Figs. 1 and 3. We notice that, for $a_1 = 5.1$, $a_2 = 3$, we find for Δ as bifurcation parameter just the destabilizing scenario, while for μ we observe the destabilizing, mixed and unconditionally unstable scenarios, as predicted by Corollary 1. We stress that it can be checked that the solid curve, which represents in Fig. 4 the Neimark-Sacker stability threshold, is actually asymptotic to the vertical axis $\Delta = 0$, so that even for arbitrarily small positive values of Δ we do not have an unconditionally stable scenario with respect to μ (which is excluded by Corollary 1) but instead a destabilizing scenario. For $a_1 = 10.2$, $a_2 = 6$, we find again for μ as bifurcation parameter all possible scenarios, while for Δ we observe just the mixed and unconditionally unstable scenarios. We stress that, for both $a_1 = 5.1$, $a_2 = 3$ and $a_1 = 10.2$, $a_2 = 6$, increasing μ has an ambiguous effect on the stability interval with respect to Δ, as the latter initially grows and then starts shrinking, eventually becoming empty for $a_1 = 10.2$, $a_2 = 6$ and μ large enough. We also notice that increasing the value of Δ keeps fixed or raises the lower stability bound on μ, while for $a_1 = 5.1$, $a_2 = 3$, the upper bound on μ can either increase or decrease.

The bifurcation diagram in Fig. 4 is computed for the same pair of initial data used for the bifurcation diagrams reported in Fig. 3b and c. We again observe the coexistence of the fixed point first and of invariant curves with an external periodic or chaotic attractor. We notice that, when $a_1 = 10.2$, $a_2 = 6$, the external attractor exists for all the considered values of μ and Δ, being periodic or chaotic according to the value of such parameters, while the internal attractor is present only for intermediate values of Δ and for not too large values of μ.

Concluding, we briefly summarize the most significant dynamical results shown by the simulations of Sections 4.1 and 4.2. If we look at the stability regions reported in Figs. 1a, 3a, 4a and b and we focus on the role of μ, for the considered parameter configurations, we found that, when the system for $\mu = 0$, that is, in the context analyzed by De Grauwe and Rovira Kaltwasser (2012), is stable, a moderate increase in the imitation degree preserves the system stability, which is instead destroyed by a further, excessive increase in μ. Similarly, it is possible to show that, when the system for $\mu = 0$ displays periodic or quasi-periodic dynamics, a stabilization can be obtained for intermediate values of μ, neither too small, nor too large, even when dynamics are unconditionally unstable for $\mu = 0$ (see Figs. 1c, 2c, and 3c). On the other hand, when for $\mu = 0$ the system displays chaotic attractors, it seems that a complete stabilization cannot be achieved through an increase in the imitation degree. Figure 3b shows that, by increasing the value of β, it is instead possible to reach a complete stabilization of the dynamics, even when for $\beta = 0$ the system displays a chaotic behavior. Focusing now on the role of Δ, Fig. 4b shows that, for null or moderate values of the imitation degree, an intermediate level of ambiguity in the stock market may lead to a stabilization of the dynamics. This, at first sight, counterintuitive result may be explained as follows. If the level of ambiguity starts rising, agents no longer trust one another and thus they are discouraged from operating in the financial sector. The reduced amount of speculations causes in turn a reduction in the stock price volatility, stabilizing the dynamics. Such positive effect is, however, destroyed both by an excessive imitation degree, which makes agents too reactive to others' choices, and by a too high ambiguity level, which let orbits converge toward a periodic or chaotic attractor, rather than toward a fixed point. All the numerical simulations performed along the section confirm then the ambiguous role for our three main parameters, i.e., β, μ and Δ, captured also by the double stability thresholds analytically found in the proof of Corollary 1.

5 Economic interpretation of the results

In the present section, we will try to explain the rules governing the dynamics of the main variables of the model and to investigate the occurrence of waves of optimism and pessimism. More precisely, examining time series of beliefs, prices and shares of optimists/pessimists, we will point out the effects of simultaneously considering both the imitative process and the endogenous switching mechanism. Moreover, we will qualitatively compare the resulting dynamics obtained with model (7) with the significantly different dynamics that would have been obtained either with constant, exogenous beliefs (similarly to the De Grauwe and Rovira Kaltwasser 2012 setting) or with constant, exogenous shares of pessimists (similarly to the Naimzada and Pireddu 2015b setting).

First of all, we observe that the expression for the excess demand in Eq. 2 at time t becomes

$$D(t) = \omega(t)(X(t)-P(t))+(1-\omega(t))(Y(t)-P(t)) = -\omega(t)\Delta+Y(t)-P(t), \quad (10)$$

which means that $D(t)$ depends on term $-\omega(t)\Delta$, determined by the fraction of pessimists and not directly depending on the dynamics of beliefs, and on $Y(t) - P(t)$, which is affected by the endogenous nature of the beliefs on the fundamental value. It is easy to see that belief values increase from t to $t+1$ provided that $(X(t)-P(t))^2-(Y(t)-P(t))^2 > (X(t-1)-P(t-1))^2-(Y(t-1)-P(t-1))^2$, namely, if the relative performance of optimists improves from $t-1$ to t. We stress that an increase in $X(t)$ is more significant when the imitation degree μ is large. Moreover, from Eq. 5, the sign of $D(t)$ has direct influence on the value of $P(t+1)$, and the value of $P(t+1)-P(t)$ determines the fraction of pessimists (and hence of optimists) as, recalling (3), the profit differential between optimists and pessimists is given by

$$\pi_Y(t+1)-\pi_X(t+1)=(P(t+1)-P(t))(Y(t)-X(t)). \tag{11}$$

Hence, from the previous considerations, it is apparent the joint effect of the imitation degree (and in general of an endogenous belief formation mechanism) and of the intensity of choice, which both affect and determine dynamics. Making a comparison with the framework in De Grauwe and Rovira Kaltwasser (2012) (i.e., the present model with $\mu=0$), the sign of $Y(t)-P(t)$ depends there on the relative position of the stock price value with respect to the constant belief $F+\Delta/2$. If $\mu \neq 0$ and thus $Y(t)$ is no more constant, we can instead have situations in which prices are large but, being $Y(t)$ large as well, the excess demand is positive. Similarly, in the framework in Naimzada and Pireddu (2015b), only $Y(t)-P(t)$, which is constantly shifted by the exogenous value of $-\omega\Delta$, has a significant effect on $D(t)$.

In order to illustrate the previous aspects in more detail, we focus on a particular situation, obtained for the same parameters used for the bifurcation diagram in Fig. 1b and for $\beta=20.7$, for which the resulting dynamics are quasi-periodic. We set the initial conditions to $X(0)=1.3,\ P(0)=2.1,\ \omega(0)=0.2.$ and we consider 500 time periods after a transient of 1000 time periods. A portion of the corresponding time series of $X(t), Y(t), P(t), \omega(t)$ and $D(t)$ is reported in Fig. 5, where t represents the time period after the initial transient. We stress that the following considerations, even if concerning a particular parameter choice, are valid in general when unstable quasi-periodic dynamics arise, as we thoroughly checked by means of numerical experiments.

First we notice that all the possible combinations of either large or small values of beliefs together with either large or small shares of pessimists are possible. We report

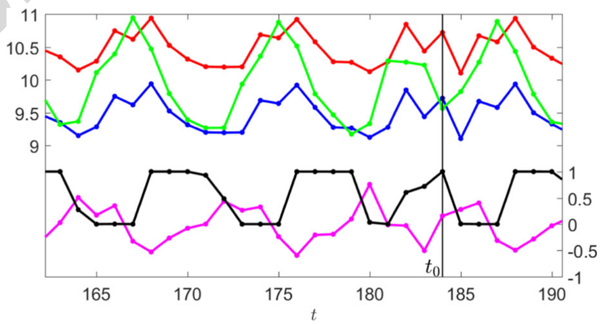

Fig. 5 Time series for $X(t)$ in *blue*, $Y(t)$ in *red*, $P(t)$ in *green*, with respect to the scale reported on the *left vertical axis*. Times series for $\omega(t)$ in *black* and $D(t)$ in *pink*, with respect to the scale reported on the *right vertical axis*. The parameter setting is the same used for the simulation reported in Fig. 1

some examples of the occurrence of each situation in Table 1, from which we also notice that each case occurs with a very similar frequency, so that they all represent significant scenarios for System (7). Indeed, within the framework analyzed by De Grauwe and Rovira Kaltwasser (2012), since $Y(t) = F + \Delta/2$, we can only distinguish between two cases ($\omega(t) \lessgtr 0.5$). Similarly, the model considered in Naimzada and Pireddu (2015b), depending on the exogenous share ω, can only provide two cases at a time (i.e., $X(t) \lessgtr F - \Delta/2$).

To explain the dynamics of stock prices, shares and beliefs, we focus our attention on the instant of time$t = t_0 = 184$ in Fig. 5. In this case, the share of pessimists $\omega(t_0) \approx 0.99$ is much larger than that of optimists and we have $P(t_0) \approx 9.57$, $Y(t_0) \approx 10.72$. In our situation, since $Y(t)$ is significantly larger than $P(t)$, the resulting excess demand $D(t_0) = 0.16$ is positive and hence we find that$P(t_0 + 1)$ is larger than $P(t_0)$. This also induces a relevant, due to the large value of β, decrease inthe pessimists' share ($\omega(t_0 + 1) < \omega(t_0)$). Moreover, since at $t = t_0$ the difference between squarederrors of pessimists and optimists is smaller than that at time $t_0 - 1$, both $X(t_0 + 1)$ and $Y(t_0 + 1)$ decrease with respect to $X(t_0)$ and $Y(t_0)$, in agreement with the considerations above about the independence of thevalues of beliefs and shares.

We notice that, in the De Grauwe and Rovira Kaltwasser (2012) setting, at the same instant of time $t = t_0$, we would observe a slightly negative excess demand $D(t_0) \approx -0.06$, since the beliefs $Y(t_0) = F + \Delta/2 = 10.5$ of optimists would not be sufficiently larger than price, with a consequent decrease in price, contrary to our findings.

We also observe that the dynamics obtained leaving the remaining parameters and the initial data unchanged, but setting, respectively, $\beta = 0$ and $\mu = 0$ (which correspond to the frameworks considered by De Grauwe and Rovira Kaltwasser 2012 and by Naimzada and Pireddu 2015b), are very different. As we can see from Fig. 6, setting $\beta = 0$ we obtain a dynamical behavior consisting in a period-2 cycle (in (a)), while the quasi-periodic trajectory (in (b)) arising when $\mu = 0$ is qualitatively different from that reported in Fig. 5.

With respect to both settings, the dynamics reported in Fig. 5 are much more interesting and complex. Westress that, from the considerations and the bifurcation

Table 1 Values of $X(t)$, $\omega(t)$ and $I_1(t)$ for some time periods t, for which all the possible combinations of large/small beliefs and large/small shares of pessimists realize

Case	t	$X(t)$	$\omega(t)$	$I_1(t)$	%
$X(t) < F - \Delta/2, \omega(t) < 0.5$	164	9.15	0.27	9.87	25.4%
	181	9.27	0.001	10.27	
$X(t) < F - \Delta/2, \omega(t) > 0.5$	171	9.20	0.93	9.27	26.6%
$X(t) > F - \Delta/2, \omega(t) < 0.5$	166	9.74	0.003	10.74	24.2%
$X(t) > F - \Delta/2, \omega(t) > 0.5$	169	9.52	0.99	9.52	23.8%
	182	9.84	0.60	10.24	

In the last column, we report the percentage of occurrence of each case during all the simulation. The parameter setting is the same used for the simulation reported in Fig. 1, and time periods belong to the interval reported in Fig. 5

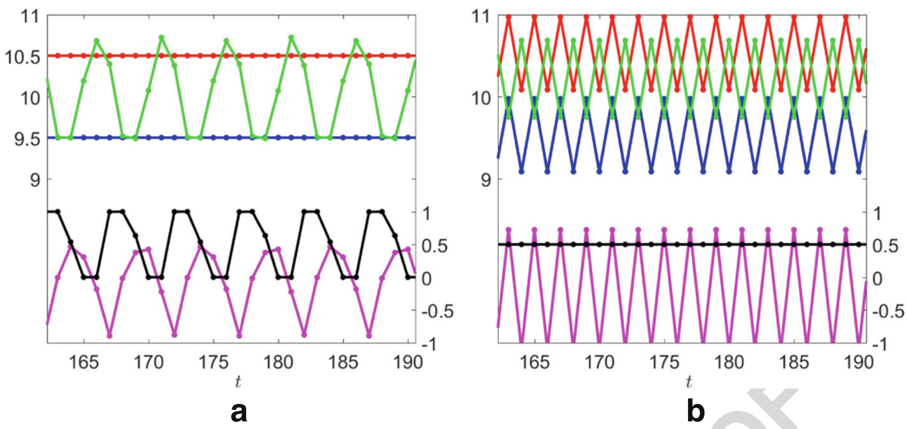

Fig. 6 Time series for $X(t)$ in *blue*, $Y(t)$ in *red*, $P(t)$ in *green*, with respect to the scale reported on the *left vertical axis*. Times series for $\omega(t)$ in *black* and $D(t)$ in *pink*, with respect to the scale reported on the *right vertical axis*. The considered parameter setting is similar to that used for the simulation reported in Fig. 1 with, respectively, the exception in (**a**) of $\mu = 0$ (no imitation) and in (**b**) of $\beta = 0$ (no switching mechanism)

diagrams reported in Section 4, we actually have that the converse is possible, too, with complex dynamics for either $\beta = 0$ or $\mu = 0$ which become stable when both endogenous mechanisms of switching and imitation are introduced. This once more emphasizes that considering just one of the two mechanisms only provides a partial description of the dynamics.

Now we turn our attention to the occurrence and alternation of optimistic and pessimistic scenarios. From the previous observations, also recalling the results reported in Table 1, the study of their occurrence must rely on the joint effect of both beliefs and shares of pessimists/optimists.

In Fig. 7 we show the time series of the instantaneous sentiment index $I_1(t)$ and of its moving average $I_4(t)$ over periods $[t-3, t]$. As arguable from Table 1, $X(t)$ and $\omega(t)$ provide a conflicting indication about the optimistic/pessimistic overall sentiment of the market in about the 50% of cases. However, for example, at the sign of the values of $I_1(t)$ corresponding to the pairs of times $t = 164, 180$ and $t = 169, 182$ reported in Table 1, we realize that such situations do not always result in a predominant behavior of either beliefs or shares. From a qualitative point of view, sentiment index $I_1(t)$ portrays the quasi-periodicity of the dynamics that gives rise to alternation of waves of optimism (e.g., $165 \leq t \leq 167$) and pessimism (e.g., $168 \leq t \leq 172$), each of which lasts for several periods.

If we consider $T > 1$, we can instead investigate the persistence of optimism/pessimism over time. In this case, especially if we deal with relatively small values of T, the deterministic quasi-periodic nature of dynamics is significant. If we take $1 < T < 8$, the results provided by $I_T(t)$ are qualitatively similar to those of $I_1(t)$, as we can notice looking at the behavior of $I_4(t)$ reported in Fig. 7. The remarkable difference is that $I_4(t)$, being a moving average of $I_1(t)$, provides lagged and smoothed values with respect to $I_1(t)$, but still shows alternating waves of optimism

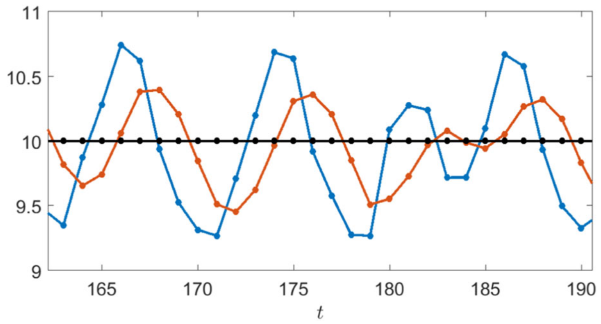

Fig. 7 Time series of sentiment indexes $I_1(t)$ (in *blue*) and $I_4(t)$ (in *orange*) related to the time series reported in Fig. 5

and pessimism. The situation changes for $T = 8$. From Fig. 5 it is evident the pattern of each variable, which qualitatively repeats itself after eight time periods. This is quantified by a significant autocorrelation coefficient $r_8 = 0.6602$. In Fig. 8 we report the time series of $I_8(t)$, from which it is evident that the quasi-periodic trend of values is canceled by the moving average, and former small deviations from periodicity become now significant. In this case, the oscillating nature of the time series of $I_1(t)$ is destroyed by the moving average, and the waves of optimism and pessimism last for longer periods. For instance, from $t = 171$ to $t = 185$ we have fifteen consecutive periods in which pessimism dominates (both $I_8(t)$ and $I_{15}(t)$ are smaller than F) followed by a lower number of consecutive periods in which optimism prevails. As T increases, the qualitative features of the scenario reported in Fig. 8 become more and more independent of the particular choice of T, and the short period quasi-periodic dynamical behavior disappears, as is noticeable observing the behavior of $I_{15}(t)$ in Fig. 8. The predominant smoothing effect of the averaging allows to portray the trend behavior of large periods $[t - T + 1, t]$, with the common feature of alternating waves of optimism and pessimism qualitatively similar to those reported in Fig. 8.

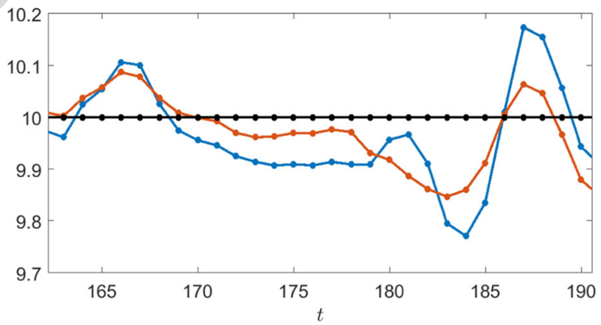

Fig. 8 Time series of sentiment indexes $I_8(t)$ (in *blue*) and $I_{15}(t)$ (in *orange*) related to the time series reported in Fig. 5

6 Adding stochastic shocks

Following the approach in De Grauwe and Rovira Kaltwasser (2012), we introduce a stochastic term into the dynamical system in (9) by assuming that the true fundamental value follows the random walk

$$F(t+1) = F(t) + \varepsilon(t+1), \tag{12}$$

where $\varepsilon(t)$ is a normally distributed random variable with standard deviation $\sigma > 0$, which can be interpreted as a news arrival process. The resulting stochastic model is then obtained replacing F with $F(t)$ in System (9), to which equation (12) has to be added.

Our goal consists in checking whether the resulting stochastically perturbed system is able to reproduce several features of financial markets, such as bubbles and crashes for stock prices and fat tails and excess volatility in the distributions of returns, which are defined by

$$R(t+1) = 100(\log(P(t+1)) - \log(P(t))).$$

For a survey on stylized facts of financial markets, we refer the interested reader to Westerhoff (2009) and the references therein.

Throughout this section, we consider the parameter setting used for the simulation reported in Section 5 with the addition of the stochastic perturbation in Eq. 12 with $\sigma = 0.1$ and we compare $I_T(t)$ with the average true fundamental value $F_T(t) = F(t)/T$ computed over the same amount of periods.

As we can see from Fig. 9, considering $I_{15}(t)$ in this more realistic situation with noise, we have that the waves of optimism (when the orange line is above the blue one) and pessimism (when the orange line is below the blue one) are definitely more evident and last for more periods than in Fig. 8. We remark that the behavior of $I_1(t)$, we do not report for the sake of brevity, is qualitatively similar to that of Fig. 7.

Now we investigate the statistical behavior of the model. We performed 5000 simulations, in each of which we used a different random sequence of values of the

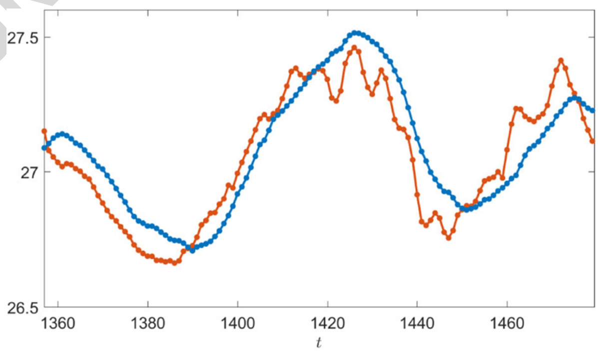

Fig. 9 Time series of $I_{15}(t)$ (in *orange*) for the same parameter configuration considered in Section 5, adding now a stochastic shock to F. The *blue line* represents the average true fundamental value computed over 15 periods

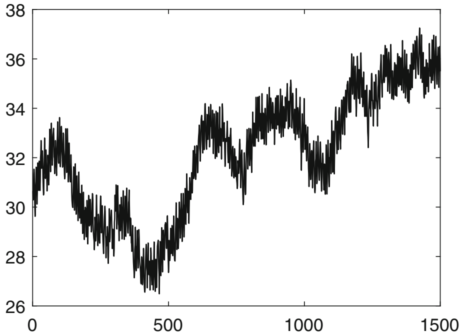

Fig. 10 Time series of stock price $P(t)$, displaying bubbles and crashes

stochastic variable σ. For every simulation, we collected 10000 values for $R(t)$, after a transient of 1500 time periods. We note that the plots in Figs. 9 and 10 refer to the last 1500 time periods, in which t is shifted leftwards by 10000 to improve readability. The returns distributions we found are characterized by strong non-normality, with a large mean kurtosis (≈ 5.32) and volatility (≈ 1.02), as we can also infer from the plots of the simulations reported in Figs. 10 and 11.

In more detail, in Fig. 10 we report a typical plot for time series of stock price $P(t)$, which shows the very erratic price movements with both bubbles and crashes. We remark that the qualitative behavior of beliefs is similar to that of $P(t)$ reported in Fig. 10, while time series of $\omega(t)$ look like those reported in Fig. 5 for the deterministic simulation.

In Fig. 11a and b, we report the histogram of returns distribution and its Q-Q test plot,while the corresponding time series of returns are reported in Fig. 11c. We recall that Q-Q plot (Quantile-Quantile) plots the quantiles of one distribution against those of the normal, contrasting the two cumulative distribution functions. If the variable under analysis is normally distributed, then its plot lies on the 45-degree line, which corresponds to the normal distribution. Moreover, if the considered variable is not normally distributed and its left (right) tail lies below (above) the 45-degree line, then its distribution is fat-tailed. The opposite situation, with left (right) tail laying above (below) the 45-degree line, corresponds to a thin-tailed distribution. The Q-Q plot in Fig. 11b confirms that the distribution of returns is leptokurtic.

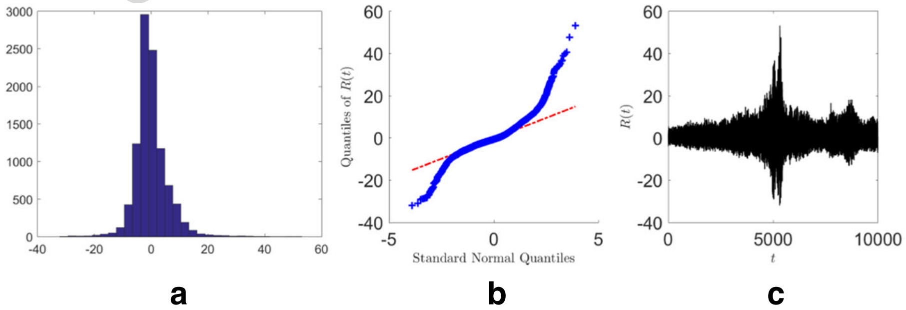

Fig. 11 **a**: Histogram of $R(t)$. **b**: Q-Q test plot of returns, in which it is evident that $R(t)$ has a non-normal distribution. **c**: Corresponding time series of returns $R(t)$

The presence of fat tails implies that, in our model, there are large returns, corresponding to strong movements in prices, and thus more volatility in the financial market than is compatible with a normal distribution.

We remark that, similarly to the returns, also variables $X(t)$, $P(t)$ and $\omega(t)$ display distributions that significantly deviate from normality.

We also stress that we checked through simulations that the previous results are qualitatively the same on varying the parameter setting, provided that we consider quasi-periodic dynamics, and for values of σ in the interval [0.07, 0.5].

All the findings above confirm that the stochastically perturbed version of our model is able to reproduce qualitatively several stylized facts about distributions of prices and returns in financial markets.

7 Conclusions and possible extensions

In the present paper, we proposed a financial market model with optimistic and pessimistic fundamentalists, which form their beliefs about the fundamental value on the basis of an updating mechanism grounded on the relative agents ability in guessing the realized stock price. We assumed a bounded nonlinear adjustment mechanism in the stock price formation in order to avoid divergence and negativity issues. Finally, we allowed agents to switch from being optimists to pessimists, and vice versa, on the basis of the realized profits. The framework we dealt with generalizes the model considered in De Grauwe and Rovira Kaltwasser (2012) and permitted us to study the effect on the dynamics of an endogenous belief formation mechanism (as wished for in the same De Grauwe and Rovira Kaltwasser 2012), as well as to analyze a more refined cognitive process of the agents, together with its consequences on the emergence of waves of optimism and pessimism.

The analytical and numerical investigation of the stability of the equilibrium highlighted that stability is affected by the three main model parameters (i.e., the imitation degree in the updating of the beliefs on the fundamental value, the maximum possible level of optimism and pessimism, and the intensity of choice in the switching mechanism) in an ambiguous way, so that each parameter has, despite the usual destabilizing effect, a stabilizing role, as well. To the best of our knowledge, no other works in the related literature show the ambiguous effect of the imitation degree or the heterogeneity level parameters, with the sole exception of Naimzada and Pireddu (2015b).

We performed a bifurcation analysis using the three main model parameters, which also highlighted the emergence of complex behaviors, with possible multistability phenomena, characterized by the presence of coexisting attractors.

We explained the rules governing the dynamics of the model variables, and we showed that, in the present model, the emergence of waves of optimism and pessimism can not be attributed to the behavior of a single variable, but it can only be understood studying the combined effect of beliefs and shares of behavioral rules. With regard to this, we introduced an index to estimate the degree of optimism/pessimism over a suitable interval of time steps, and we showed the alternation of sequences of consecutive time periods all characterized by either optimistic or pessimistic behavior.

Finally, we added a stochastic noise to the deterministic framework and showed through a statistical analysis that the behavior of our model is in agreement with the stylized facts observed in the empirical literature on financial markets.

Our next research activity aims to extend the model in various directions. A possible generalization may concern, similarly to Brock and Hommes (1998) and De Grauwe and Rovira Kaltwasser (2012), the introduction in our model of a group of unbiased fundamentalists (or contrarians) and a group of unbiased chartists (or trend followers), whose degrees of optimism and pessimism are null, and who use the true fundamental value in their speculations, in order to investigate the effects of such further groups of agents on the dynamics of the system. The goal is to check whether, as in De Grauwe and Rovira Kaltwasser (2012), the former group has a stabilizing role, i.e., its presence makes the stability region become larger, while the latter group is destabilizing. Another development of the present work might consist in transforming the optimism/pessimism persistence index into a variable on which agents base their decisions, in addition to considering price and profit dynamics, so that such index would not play anymore just a descriptive role, but it would rather be directly taken into account by speculators in making their choices.

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Compliance with Ethical Standards

Conflict of interests The authors declare that they have no conflict of interest.

Appendix

Proof of Proposition 1 The expression of the steady state for the population shares follows by noticing that, in equilibrium, $P(t+1) = P(t)$ and thus, by Eq. 3, we have $\pi_X = \pi_Y = 0$ at the steady state, so that $\omega^* = \frac{1}{2}$. Moreover, from the stock price equation we find $P^* = \omega^* X^* + (1-\omega^*)Y^* = \frac{X^*+Y^*}{2}$, so that $X^* = F - \frac{\Delta}{2}$ and $Y^* = F + \frac{\Delta}{2}$. Inserting such expressions in P^*, we get $P^* = F$, as desired. $\square$

Proof of Proposition 2 Since from Eq. 7 it follows that

$$\begin{aligned} F &= X(t+1) + \frac{\Delta}{1 + e^{\mu((X(t)-P(t))^2-(Y(t)-P(t))^2)}} \\ &= Y(t+1) - \frac{\Delta}{1 + e^{-\mu((X(t)-P(t))^2-(Y(t)-P(t))^2)}}, \end{aligned}$$

then we find

$$\begin{aligned} Y(t+1) &= X(t+1) + \Delta \left(\frac{1}{1 + e^{\mu((X(t)-P(t))^2 - (Y(t)-P(t))^2)}} + \frac{1}{1 + e^{-\mu((X(t)-P(t))^2 - (Y(t)-P(t))^2)}} \right) \\ &= X(t+1) + \Delta, \end{aligned}$$

as desired. □

Proof of Proposition 3 To prove $(i') - (iii')$, we use the conditions in Farebrother (1973).

To such aim, we need to compute the Jacobian matrix for G in correspondence to (X^*, P^*, ω^*), which reads as

$$\begin{bmatrix} \frac{-\mu\Delta^2}{2} & \frac{\mu\Delta^2}{2} & 0 \\ \widetilde{\gamma} & 1 - \widetilde{\gamma} & -\widetilde{\gamma}\Delta \\ \frac{-\beta\widetilde{\gamma}\Delta}{4} & \frac{\beta\widetilde{\gamma}\Delta}{4} & \frac{\beta\widetilde{\gamma}\Delta^2}{4} \end{bmatrix}. \tag{13}$$

The Farebrother conditions are the following:

(i) $1 - C_1 + C_2 - C_3 > 0$;
(ii) $1 - C_2 + C_1C_3 - (C_3)^2 > 0$;
(iii) $3 - C_2 > 0$;
(iv) $1 + C_1 + C_2 + C_3 > 0$,

where C_i, $i \in \{1, 2, 3\}$, are the coefficients of the characteristic polynomial

$$\lambda^3 + C_1\lambda^2 + C_2\lambda + C_3 = 0.$$

In our framework, we have:

$$C_1 = \frac{\mu\Delta^2}{2} + \widetilde{\gamma} - 1 - \frac{\beta\widetilde{\gamma}\Delta^2}{4}, \quad C_2 = \frac{-\mu\Delta^2}{2}\left(1 + \frac{\beta\widetilde{\gamma}\Delta^2}{4}\right) + \frac{\beta\widetilde{\gamma}\Delta^2}{4}, \quad C_3 = \frac{\mu\beta\widetilde{\gamma}\Delta^4}{8},$$

and thus simple computations allow to notice that conditions $(i) - (iii)$ above read, respectively, as $(i') - (iii')$, while (iv) reduces to $\widetilde{\gamma} > 0$, which is indeed true. □

Proof of Corollary 1 Firstly, we keep $\Delta \neq 0$, μ and $\widetilde{\gamma}$ fixed and we solve conditions $(i') - (iii')$ with respect to β.

If $2 - \mu\Delta^2 \leq 0$, we have that condition (i') is not satisfied by any $\beta > 0$, which means that we are in the unconditionally unstable scenario.

Let us now consider $2 - \mu\Delta^2 > 0$. Condition (i') is then equivalent to

$$\beta > \beta_\ell = \frac{4}{\widetilde{\gamma}\Delta^2}\left(\frac{\widetilde{\gamma}}{2 - \mu\Delta^2} - 1\right). \tag{14}$$

From (iii') we have

$$\beta < \beta_{r_2} = \frac{4}{\widetilde{\gamma}\Delta^2} \cdot \frac{6 + \mu\Delta^2}{2 - \mu\Delta^2}. \tag{15}$$

Finally, we notice that we can rearrange condition (ii') as $k_1\beta^2 + k_2\beta + k_3 > 0$ with $k_1 = -\Delta^8\widetilde{\gamma}^2\mu^2 - 2\Delta^6\widetilde{\gamma}^2\mu$, $k_2 = 4\Delta^6\widetilde{\gamma}\mu^2 + 8\Delta^4\widetilde{\gamma}^2\mu - 16\Delta^2\widetilde{\gamma}$ and $k_3 = 32\mu\Delta^2 + 64$.

If $\mu > 0$, then $k_1 < 0 < k_3$, and thus condition (ii') is fulfilled by

$$\beta_{r_0} < \beta < \beta_{r_1}, \tag{16}$$

with $\beta_{r_0} < 0 < \beta_{r_1}$ given by

$$\beta_{r_0} = \frac{2}{\mu\widetilde{\gamma}\Delta^4} \cdot \frac{\mu^2\Delta^4 + 2\mu\widetilde{\gamma}\Delta^2 - 4 - \sqrt{\xi}}{2 + \mu\Delta^2}, \quad \beta_{r_1} = \frac{2}{\mu\widetilde{\gamma}\Delta^4} \cdot \frac{\mu^2\Delta^4 + 2\mu\widetilde{\gamma}\Delta^2 - 4 + \sqrt{\xi}}{2 + \mu\Delta^2},$$

where

$$\xi = \mu^4\Delta^8 + 4\mu^2\widetilde{\gamma}^2\Delta^4 + 16 + 4\mu^3\widetilde{\gamma}\Delta^6 - 16\mu\widetilde{\gamma}\Delta^2 + 8\mu^3\Delta^6 + 24\mu^2\Delta^4 + 32\mu\Delta^2.$$

Combining Eqs. 14–16 and recalling that $\beta > 0$, we find that system $(i') - (iii')$ is equivalent to

$$\max\{\beta_\ell, 0\} < \beta < \min\{\beta_{r_1}, \beta_{r_2}\}. \tag{17}$$

In particular, if $\beta_\ell \le 0$, i.e., if $\widetilde{\gamma} \le 2 - \mu\Delta^2$, and $\min\{\beta_{r_1}, \beta_{r_2}\} > 0$, we have the destabilizing scenario, while when $0 < \beta_\ell < \min\{\beta_{r_1}, \beta_{r_2}\}$ the mixed scenario occurs; in all other cases we have the unconditionally unstable scenario.

If, instead, $\mu = 0$, then $k_1 = 0$ and it is easy to see that conditions $(i') - (iii')$ are equivalent to $\max\{\beta_\ell, 0\} < \beta < \beta_{r_1}$, with $\beta_\ell = \frac{4}{\widetilde{\gamma}\Delta^2}\left(\frac{\widetilde{\gamma}}{2} - 1\right)$ and $\beta_{r_1} = \frac{4}{\widetilde{\gamma}\Delta^2}$. Then, in this case we have a destabilizing scenario ($\beta_\ell \le 0 < \beta_{r_1}$) if and only if $\widetilde{\gamma} \in (0, 2]$, while a mixed scenario ($0 < \beta_\ell < \beta_{r_1}$) occurs if and only if $\widetilde{\gamma} \in (2, 4)$.

Now we keep $\Delta \neq 0$, β and $\widetilde{\gamma}$ fixed and we solve conditions $(i') - (iii')$ with respect to μ. Solving (i') we find

$$\mu < \mu_r = \frac{2}{\Delta^2} \cdot \frac{\beta\widetilde{\gamma}\Delta^2 + 4 - 2\widetilde{\gamma}}{\beta\widetilde{\gamma}\Delta^2 + 4}, \tag{18}$$

while (iii') provides

$$\mu > \mu_\ell = \frac{2}{\Delta^2} \cdot \frac{\beta\widetilde{\gamma}\Delta^2 - 12}{\beta\widetilde{\gamma}\Delta^2 + 4}. \tag{19}$$

Condition (ii') can be rewritten as

$$q_1\mu^2 + q_2\mu + q_3 > 0, \tag{20}$$

where $q_1 = \Delta^6\beta\widetilde{\gamma}(4 - \beta\widetilde{\gamma}\Delta^2)$, $q_2 = -2\Delta^2(\Delta^4\beta^2\widetilde{\gamma}^2 - 4\Delta^2\beta\widetilde{\gamma}^2 - 16)$ and $q_3 = 16(4 - \beta\widetilde{\gamma}\Delta^2)$.

First we notice that, if $q_1 = 0$, i.e., if $\beta = 4/(\widetilde{\gamma}\Delta^2)$, then also $q_3 = 0$ and thus (20) is satisfied for all $\mu > 0$ if and only if $\Delta^4\beta^2\widetilde{\gamma}^2 = \Delta^4\left(4/(\widetilde{\gamma}\Delta^2)\right)^2\widetilde{\gamma}^2 = 16 < 4\Delta^2\beta\widetilde{\gamma}^2 + 16 = 32$, which is indeed true. We stress that if $\mu = 0 = q_1$, then condition (ii') is again never fulfilled and thus we do not have stability for $\mu = 0$ if $q_1 = 0$. Hence, when $q_1 = 0$ conditions $(i') - (iii')$ simply reduce to $\mu \in (\mu_\ell, \mu_r) \cap (0, +\infty)$, and this provides the destabilizing scenario for $\mu_\ell < 0 < \mu_r$, the mixed scenario for $0 < \mu_\ell < \mu_r$, and the unconditionally unstable scenario for $\mu_r \le \max\{\mu_\ell, 0\}$.

Let us now assume that $q_1 \neq 0$. If $\chi = q_2^2 - 4q_1q_3 > 0$, we can introduce the real numbers

$$\mu_1 = \frac{\beta^2\widetilde{\gamma}^2\Delta^4 - 16 - 4\beta\widetilde{\gamma}^2\Delta^2 - \sqrt{\chi}}{\beta\widetilde{\gamma}\Delta^4(4 - \beta\widetilde{\gamma}\Delta^2)},$$

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and

$$\mu_2 = \frac{\beta^2\widetilde{\gamma}^2\Delta^4 - 16 - 4\beta\widetilde{\gamma}^2\Delta^2 + \sqrt{\chi}}{\beta\widetilde{\gamma}\Delta^4(4 - \beta\widetilde{\gamma}\Delta^2)},$$

where

$$\begin{aligned}\chi &= 256 + 16\beta^2\widetilde{\gamma}^4\Delta^4 + \beta^4\widetilde{\gamma}^4\Delta^8 + 128\beta\widetilde{\gamma}^2\Delta^2 + 96\beta^2\widetilde{\gamma}^2\Delta^4 \\ &\quad -8\beta^3\widetilde{\gamma}^4\Delta^6 - 16\beta^3\widetilde{\gamma}^3\Delta^6 - 256\beta\widetilde{\gamma}\Delta^2.\end{aligned}$$

We then have that Eq. 20 is solved by $\mu < \mu_1 \cup \mu > \mu_2$ if $q_1 > 0$ and by $\mu_1 < \mu < \mu_2$ if $q_1 < 0$. Let us examine the former case. Since we have $q_1 > 0$ and $q_3 > 0$, the sign of $\mu_{1/2}$ is the same of $-q_2$. Since from $q_1 > 0$ we have $\beta < 4/(\widetilde{\gamma}\Delta^2)$, we obtain

$$\beta^2\widetilde{\gamma}^2\Delta^4 - 16 - 4\beta\widetilde{\gamma}^2\Delta^2 < \left(\frac{4}{\Delta^2\widetilde{\gamma}}\right)^2\widetilde{\gamma}^2\Delta^4 - 16 - 4\beta\widetilde{\gamma}^2\Delta^2 = -4\beta\widetilde{\gamma}^2\Delta^2 < 0,$$

so that, in such case, we have $q_2 > 0$ and thus $\mu_1 < \mu_2 < 0$ and Eq. 20 is fulfilled by any $\mu \geq 0$. Combining this with Eqs. 18 and 19 we obtain $\mu \in (\mu_\ell, \mu_r) \cap [0, +\infty)$, which provides the destabilizing scenario if $\mu_\ell < 0 < \mu_r$, the mixed scenario if $0 < \mu_\ell < \mu_r$ and the unconditionally unstable scenario if $\mu_r \leq \max\{\mu_\ell, 0\}$.

Conversely, if $q_1 < 0$, combining $\mu_1 < \mu < \mu_2$ with Eqs. 18 and 19 we have $\mu \in (\max\{\mu_1, \mu_\ell\}, \min\{\mu_2, \mu_r\}) \cap [0, +\infty)$, which can again give rise to either a destabilizing, a mixed or an unconditionally unstable scenario.

If $q_2^2 - 4q_1q_3 < 0$, we have that Eq. 20 is always fulfilled if $4 - \beta\widetilde{\gamma}\Delta^2 > 0$ and never fulfilled when $4 - \beta\widetilde{\gamma}\Delta^2 < 0$. In the former case, recalling Eqs. 18 and 19, we obtain $\mu \in (\mu_\ell, \mu_r) \cap [0, +\infty)$, and thus we can have destabilizing, mixed and unconditionally unstable scenarios. If instead $4 - \beta\widetilde{\gamma}\Delta^2 < 0$, conditions $(i') - (iii')$ can not be satisfied and we just find the unconditionally unstable scenario.

It can be easily seen that the remaining situation $q_2^2 - 4q_1q_3 = 0$ can only provide the previous scenarios.[4]

Finally, the stability conditions with respect to Δ when $\mu = 0$ and $\beta \neq 0$ read as

$$\sqrt{\frac{2\widetilde{\gamma} - 4}{\widetilde{\gamma}\beta}} < \Delta < \frac{2}{\sqrt{\widetilde{\gamma}\beta}},$$

when $\widetilde{\gamma} \geq 2$, and simply as $0 \leq \Delta < \frac{2}{\sqrt{\widetilde{\gamma}\beta}}$, when $\widetilde{\gamma} < 2$. For $\widetilde{\gamma} \geq 2$ we then find the mixed scenario and for $\widetilde{\gamma} < 2$ the destabilizing scenario. The proof is complete. □

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[4] Actually, if $\mu_1 = \mu_2 \in (\max\{\mu_\ell, 0\}, \mu_r)$ we find two disjoint adjacent stability intervals. Such limit case can however be encompassed in the mixed scenario.

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Q1. Equations are renumbered, Please check if correct.