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Mann Iteration with Power Means

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We analyze the recurrence $x_{n+1} = f(z_n)$, where z_n is a weighted power mean of $x_0, \ldots, x_n$, which has been proposed to model a class of non-linear forward-looking economic models with bounded rationality. Under suitable hypotheses on weights, we prove the convergence of the sequence x_n. Then, to simulate a fading memory, we consider exponentially decreasing weights. Since in this case the resulting recurrence does not fulfill the hypotheses of the previous convergence theorem, it is studied by reducing it to an equivalent two-dimensional autonomous map, which shares the asymptotic behaviors with a particular one-dimensional map. This allows us to prove that a long memory with sufficiently large weights has a stabilizing effect. Finally, we numerically investigate what happens when the memory ratio is not sufficiently large to provide stability, showing that, depending on the power mean and the memory ratio, either delayed or early cascade of flip bifurcations occur.

Keywords: Forward-looking models, Learning, Mann Iterations, Non-autonomous difference equations.

1. Introduction

The canonical economic theory assumes that agents have perfect rationality: they have both the skills to exploit information achieved in the economic system and the ability to compute all actions needed to reach an optimal solution. Learning characterizes models in which agents with bounded rationality try to reconstruct key elements of the economic system using information available from the past experience.

For example, if the economic agents have to make forecasts about future, e.g. prices or taxes or incomes, in doing that they collect and analyze past data. In the early literature, for instance in the cobweb and Cournot oligopoly models, the *static expectation* hypothesis was widely used, with agents expecting that next period price will be on the same level of the current price. The *adaptive expectation* hypothesis became popular in the 1960s and 1970s. In this case, agents belief about next period price corresponds to a linear convex combination of currently observed price and predicted price. Based on the observation that static and adaptive expec-

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tations imply poor set of information and limited computational skills, in the 1980s the *rational expectation* hypothesis played a prominent role in economic theory. Under such assumption, agents are not supposed to make systematic forecasting errors, as they are assumed to have full knowledge of the economic system and all relevant information in order to make the optimal choice. The rational expectation hypothesis has been criticized for the unrealistic informational and computational requirements and for the contrast with the observed human behavior in laboratory experiments, for example by Sargent [25] and Conlisk [12]. Recently, a more realistic view about the forecasting activity has been proposed, in which agents act like statisticians or econometricians, collecting a large set of data from the past observations and using sophisticated algorithms, like regressions, sample means, recursive least squares, in order to form their expectations. Our contribution belongs to this research strand and tries to study the evolution of the system when it is supposed that economic agents have enough data and computational capabilities for forecasting future prices using weighted power means. Our way of modelling forecasting processes is intensely interweaved with the mathematical notion of *Mann iteration*, namely an iterative scheme of the form

$$x_{n+1} = f(z_n), \tag{1}$$

where $f : I \to I, I = [a, b] \subset \mathcal{R}^+$, and z_n is the arithmetic mean of all the previous values x_i, $0 \leq i \leq n$

$$z_n = \sum_{k=0}^{n} a_{nk} x_k \tag{2}$$

with

$$\sum_{k=0}^{n} a_{nk} = 1. \tag{3}$$

Such an iteration scheme has been used to model economic and social systems with agents who have not perfect foresight, so they learn from the past experiences using all the available information (that is present and past data), in order to calculate the expected values of future states. If n represents discrete time periods and x_n the value of the *state variable* in period n, z_n can be interpreted as the *expected value* (see e.g. the contributions of Bray [11], Lucas [20], Balasko and Royer [3], Bischi and Naimzada [9], Barucci [4], Foroni et al. [14]). Starting from the seminal paper of Mann [21], iterations (1) have been studied by many authors, among others, Borwein and Borwein [10], Rhoades [23], Aicardi and Invernizzi [1], Bischi and Gardini [7], Bischi et al. [8]. For example, Bray in [11] proposed a recurrence of the form (1), with z_n given by a uniform arithmetic mean

$$z_n = \frac{1}{n+1} \sum_{k=0}^{n} x_k,$$

as a learning mechanism. In this case, the Mann iteration coincides with the Cesáro iteration, whose dynamics are very simple since for each $x_0 \in I$ the resulting sequence $\{x_n\}$ converges to a fixed point of f, as shown by Franks and Marzek in

[15]. This suggests a strong stabilizing effect of a distributed uniform memory, since any kind of dynamics more complex than convergence toward a fixed point of f is excluded, being the existence of more than one fixed point of f in I the only possibility of non trivial dynamics, as in such situation different basins of attraction must be considered.

In this work, we propose a generalization of (2) expressed by the *power mean*

$$z_n = \left(\sum_{k=0}^{n} a_{nk} x_k^s\right)^{\frac{1}{s}}, \; s \neq 0. \tag{4}$$

The arithmetic mean (2) is a special case of (4) when $s = 1$, but other commonly used algebraic means can be obtained from (4), such as the weighted quadratic mean for $s = 2$ and the weighted harmonic mean for $s = -1$. Furthermore, the weighted geometric mean is obtained as a limiting case for $s \to 0$, since

$$\lim_{s \to 0} \left(\sum_{k=0}^{n} a_{nk} x_k^s\right)^{\frac{1}{s}} = \prod_{k=0}^{n} x_k^{a_{nk}}.$$

Of course, if $s < 0$, the further condition $x_i > 0$ for each i should be verified. For a detailed description of the properties of such means, as well as their applications, we refer to the book by Vajani [26], ch.6.

This study is motivated by the possibility that some learning mechanism can be expressed by the iteration scheme (1) with algebraic means of the form (4) with $s \neq 1$ (an example is given in Section 4).

The paper is organized as follows. In Section 2 the iteration scheme (1) with (4) is reduced to a first order non autonomous recurrence, and some convergence results are given, which generalize the results of Mann [21] and Borwein and Borwein [10], where only the arithmetic mean (2) is considered. In Section 3, the power mean (4) is considered with weights decreasing as terms of a geometric progression. In Section 4, we study an example coming from the literature on overlapping generations' models with a learning mechanism based on power iteration means.

2. Convergence of recurrences with power means

In what follows, we assume that the weights are obtained as

$$a_{nk} = \frac{\omega_k^{(n)}}{W_n}, \omega_k^{(n)} \geq 0,$$

where, for each $n \geq 0$, the $(n+1)$-dimensional vector of nonnegative weights

$$\omega^{(n)} = \left\{\omega_0^{(n)}, \omega_1^{(n)}, ..., \omega_n^{(n)}\right\}$$

defines the relative influence of each state $x_k, k = 0, ..., n$, in the computation of the average z_n, and

$$W_n = \sum_{k=0}^{n} \omega_k^{(n)},$$

so that (3) is satisfied.

In this section we assume, as in [10, 23], that at each n the vector of relative weights is obtained by adding the last component without any change of the previous ones, that is, from $\omega^{(n)} = (\omega_0, \omega_1, ..., \omega_n)$ we obtain $\omega^{(n+1)} = (\omega_0, \omega_1, ..., \omega_n, \omega_{n+1})$. In this case we have

$$W_{n+1} = W_n + \omega_{n+1}. \tag{5}$$

The iterative scheme (1) with a power mean (4) becomes

$$\begin{cases} x_{n+1} = f(z_n) \text{ with } z_n = \left(\displaystyle\sum_{k=0}^{n} \frac{\omega_k}{W_n} x_k^s \right)^{\frac{1}{s}}, \\ W_n = \displaystyle\sum_{k=0}^{n} \omega_k, \; s \neq 0, \end{cases} \tag{6}$$

where the continuous function f, mapping the compact set I into itself, has at least one fixed point in I.

Recurrence (6) with $s = 1$ is a *Mann iteration*, for which the following classical result holds.

THEOREM 2.1. Mann, 1953. *Let $s = 1$ and $W_n \to \infty$. If either of the sequences $\{x_n\}$ and $\{z_n\}$ converges then the other also converges to the same point and their common limit is a fixed point of f.*

In [10, 23] a Mann iteration (1) is reduced to the following non autonomous iteration, called *segmenting Mann iteration*

$$z_{n+1} = (1 - t_n) z_n + t_n f(z_n), \tag{7}$$

where $z_0 = x_0 \in I$, and

$$t_n = \frac{\omega_{n+1}}{W_{n+1}}. \tag{8}$$

From $\{z_n\}$, the sequence of states $\{x_n\}$ can be easily obtained as the images of z_n under f

$$x_{n+1} = f(z_n). \tag{9}$$

The following result is proved by Borwein and Borwein in [10].

THEOREM 2.2. Borwein and Borwein, 1991. *Suppose that $\{t_n\}$ tends to zero. Then the sequence $\{z_n\}$ converges.*

In this section we generalize these theorems to the case of power means with $s \neq 1$. This can be easily done once the iterative scheme (6) is put into a recursive form, for the expected variables z_n, similar to (7). In fact, even provided that $s \neq 1$, from (6) we get

$$z_{n+1} = \left(\sum_{k=0}^{n} \frac{\omega_k}{W_{n+1}} x_k^s + \frac{\omega_{n+1}}{W_{n+1}} x_{n+1}^s \right)^{\frac{1}{s}} = \left(\frac{W_n}{W_{n+1}} \sum_{k=0}^{n} \frac{\omega_k}{W_n} x_k^s + \frac{\omega_{n+1}}{W_{n+1}} x_{n+1}^s \right)^{\frac{1}{s}},$$

from which, by using the definition (8) of t_n and the identity (5), we obtain what we shall call *generalized segmenting Mann iteration*,

$$z_{n+1} = F(n, z_n) = \left((1 - t_n) z_n^s + t_n [f(z_n)]^s\right)^{\frac{1}{s}}. \tag{10}$$

Also in this case, the iterative process described by the non-autonomous first order difference equation is equivalent to the iterative process (6), in the sense that given an initial condition $z_0 = x_0$ the sequence of expected values obtained from (10) is the same as that obtained from (6) (and the sequence of states is given by (9)).

We recall that a fixed point (or stationary state) of the iteration (6) is defined as a value $x^* \in \mathcal{R}$ such that if $x_0 = x^*$ then (6) generates the sequence $x_n = x^*$ for each $n \geq 0$. The following results are straightforward.

Proposition 2.3. *Regarding iterations* (6) *and* (10)*, we have that*

(i) x^* *is a fixed point of the iteration* (6) *if and only if it is a fixed point of the function* f.
(ii) z^* *is a fixed point of* (10) *if and only if it is a fixed point of* f.

We recall that a fixed point (or stationary state) of the non-autonomous difference equation (10) is defined as a value z^* such that $F(n, z^*) = z^*$ for each n.

The following proposition generalizes the above quoted theorems.

Proposition 2.4. *Considering* (6) *and* (10)*, we have that*

(i) If $W_n \to \infty$ *then the sequence* $\{x_n\}$ *defined in* (6) *converges if and only if the sequence* $\{z_n\}$ *in* (10) *converges and the two sequences converge to a common limit which is a fixed point of f.*

(ii) If in (10) $\{t_n\}$ *is a positive sequence which tends to zero, then the sequence* $\{z_n\}$ *is convergent.*

Proof. (i) First we prove that, under the assumption $W_n \to \infty$, if x_n is convergent then also z_n converges to the same limit.

Let $x_n \to q > 0$ (the case $q = 0$ will be treated separately). Then, for each s, $x_n^s \to q^s$, i.e. for each $\varepsilon > 0$ an $N > 0$ exists such that

$$q^s - \varepsilon < x_n^s < q^s + \varepsilon \text{ for } n > N. \tag{11}$$

Now we prove that $(z_n^s - q^s) \to 0$, that is $z_n^s \to q^s$ which implies $z_n \to q$. For $n > N$

we have

$$(z_n^s - q^s) = \sum_{k=0}^{n} \frac{\omega_k}{W_n} x_k^s - q^s = \sum_{k=0}^{N} \frac{\omega_k}{W_n} x_k^s + \sum_{k=N+1}^{n} \frac{\omega_k}{W_n} x_k^s - q^s$$
$$= \frac{1}{W_n} \sum_{k=0}^{N} \omega_k x_k^s + \frac{W_n - W_N}{W_n} \sum_{k=N+1}^{n} \frac{\omega_k}{W_n - W_N} x_k^s - q^s.$$

From the right inequality in (11) we have

$$(z_n^s - q^s) \le \frac{1}{W_n} \sum_{k=0}^{N} \omega_k x_k^s + \frac{W_n - W_N}{W_n} (q^s + \varepsilon) - q^s, \tag{12}$$

because

$$\sum_{k=N+1}^{n} \frac{\omega_k}{W_n - W_N} = 1.$$

Similarly, from the left inequality in (11) we have

$$(z_n^s - q^s) \ge \frac{1}{W_n} \sum_{k=0}^{N} \omega_k x_k^s + \frac{W_n - W_N}{W_n} (q^s - \varepsilon) - q^s. \tag{13}$$

Since $W_n \to \infty$ and the ω_k are bounded, from (12) follows that

$$\overline{\lim_{n\to\infty}} (z_n^s - q^s) \le \varepsilon, \tag{14}$$

and, from (13),

$$\underline{\lim_{n\to\infty}} (z_n^s - q^s) \ge -\varepsilon. \tag{15}$$

Since ε is arbitrarily small, (14) and (15) prove that $\lim_{n\to\infty} (z_n^s - q^s) = 0$.

Consider now the case $x_n \to 0$. If $s > 0$ the previous arguments can be applied with no substantial modifications. If $s < 0$, since the x_n are supposed to be positive, we have that $x_n^s \to +\infty$, i.e. for each $M > 0$ an $N > 0$ exists such that $x_n^s > M$ for $n > N$. For $n > N$ we have

$$z_n^s = \sum_{k=0}^{N} \frac{\omega_k}{W_n} x_k^s + \sum_{k=N+1}^{n} \frac{\omega_k}{W_n} x_k^s > M \sum_{k=N+1}^{n} \frac{\omega_k}{W_n},$$

and, since M can be arbitrarily large, this implies $z_n^s \to +\infty$, from which, since $s < 0$, we have $z_n \to 0$.

To complete this part of the proof it remains to show that the common limit is a fixed point of f. Indeed, since f is continuous, from $z_n \to q$ follows that $f(z_n) \to f(q)$. But $x_{n+1} = f(z_n)$ so that $q = f(q)$.

We assume now that z_n converges and we prove that also x_n converges to the same limit. If $z_n \to r$, then $x_n \to f(r)$ because f is continuous. From the previous argument, it must also be $z_n \to f(r)$ which implies $r = f(r)$.

(ii) Since, for $z_0 \in I = [a, b] \subset \mathcal{R}^+$, the whole sequence $\{z_n\}$ is contained in I, it has at least one limit point. We show that it is unique. From (10), rewritten as

$$z_{n+1}^s - z_n^s = t_n\left([f(z_n)]^s - z_n^s\right), \tag{16}$$

we deduce that, since $t_n \to 0$, z_n and $f(z_n)$ are bounded, for each $\varepsilon > 0$ a $m > 0$ exists such that

$$\left|z_{n+1}^s - z_n^s\right| < \varepsilon \text{ for } n > m. \tag{17}$$

Following the argument used by Borwein and Borwein [10], let us assume, for sake of contradiction, that ξ and η, with $a \le \xi < \eta \le b$, are two distinct limit points. A consequence of this assumption is that $f(z) = z$ for each $z \in (\xi, \eta)$. In fact let c be a point such that $\xi < c < \eta$. If $f(c) > c$ then, by the continuity of f, a $\delta \in (0, c)$ exists such that

$$f(z) > z \text{ whenever } |z - c| < \delta. \tag{18}$$

Since η is a limit point for $\{z_n\}$ a $N > m$ exists such that $|z_N - \eta| < (\eta - c)$ which implies $z_N > c$. It follows that $z_n > c$ for each $n > N$. To prove this we separately analyze the cases of positive and of negative s. Consider first $s > 0$. If $c < z_N < c+\delta$, from (18) follows $f(z_N) > z_N$ which gives, since $s > 0$, $[f(z_N)]^s > z_N^s$. From (16) follows $z_{N+1}^s > z_N^s$ (remember that $t_n > 0$) and this implies $z_{N+1} > z_N$ because $s > 0$. If $z_N \ge c + \delta$ we have $z_N^s \ge (c + \delta)^s$ so that

$$z_{N+1}^s - c^s = z_{N+1}^s - z_N^s + z_N^S - c^s \ge z_{N+1}^s - z_N^s + (c+\delta)^s - c^s > -\varepsilon + (c+\delta)^s - c^s \tag{19}$$

where (17) has been used. Since $\delta < c$ from the binomial series we have

$$(c + \delta)^s = c^s + s\delta c^{s-1} + \frac{s(s-1)}{2}\delta^2 c^{s-2} + \frac{s(s-1)(s-2)}{3!}\delta^3 c^{s-3} + \dots$$

so that $(c + \delta)^s - c^s > s\delta c^{s-1}$ for $s \ge 1$, and $(c + \delta)^s - c^s > s\delta c^{s-2}(c - (1 - s)\delta/2)$ for $0 < s < 1$. Thus, if for $s \ge 1$ we take $0 < \varepsilon < s\delta c^{s-1}$ or, for $0 < s < 1$, $0 < \varepsilon < s\delta c^{s-2}(c - \frac{1-s}{2}\delta)$, (19) gives $z_{N+1}^s - c^s > 0$ which, for $s > 0$, implies $z_{N+1} > c$.

Consider now $s < 0$. If $c < z_N < c + \delta$ from (18) follows $f(z_N) > z_N$ which gives, since $s < 0$, $[f(z_N)]^s < z_N^s$. From (16) follows $z_{N+1}^s < z_N^s$ which implies $z_{N+1} > z_N$ because $s < 0$. If $z_N \ge c + \delta$ we have $z_N^s \le (c + \delta)^s$ so that

$$z_{N+1}^s - c^s = z_{N+1}^s - z_N^s + z_N^s - c^s \le z_{N+1}^s - z_N^s + (c+\delta)^s - c^s < \varepsilon + (c+\delta)^s - c^s, \tag{20}$$

where (17) has been used. From the binomial series with $s < 0$, we have $(c+\delta)^s - c^s < s\delta c^{s-1}$, so that if we take $0 < \varepsilon < -s\delta c^{s-1}$ (20) gives $z_{N+1}^s - c^s < 0$ which, for $s < 0$, implies again $z_{N+1} > c$.

Hence, by induction, $z_n > c$ for $n \ge N$ against the assumption that $\xi < c$ is a limit point of $\{z_n\}$.

If $f(c) < c$ a similar reasoning contradicts the assumption that η is a limit point. Thus $f(c) = c$ for each $\xi < c < \eta$.

Now, if for a given $\bar{n}$ we have $\xi < z_{\bar{n}} < \eta$ then $z_{\bar{n}+1} = z_{\bar{n}}$ and so $z_n = z_{\bar{n}}$ for each $n \geq \bar{n}$ which contradicts the fact that ξ and η are both limit points. If this is not the case, since $\{z_n\}$ cannot oscillate out of the interval (ξ, η) because of (17), taking $\varepsilon < (\eta - \xi)$ it remains $z_n > \eta$ or $z_n < \xi$ for each n, and again this excludes the possibility that $\xi < \eta$ be both limit points. Therefore $\{z_n\}$ converges to its unique limit point □

Of course, if f has a unique fixed point $x^* \in I$, then it is globally attracting in I, i.e. $x_n \to x^*$ for each $x_0 \in I$.

A typical example in which these propositions can be applied is that of a uniform power mean, that is with equal weights $\omega_k = \omega$ for any k. In fact in this case we have $t_n = 1/(n+1) \to 0$ and $W_n \to \infty$. This constitutes a generalization of the result by Franks and Marzek [15] on the Cesáro iteration, since it includes the uniform arithmetic mean for $s = 1$, the uniform harmonic mean for $s = -1$, the uniform geometric mean for $s \to 0$, and so on.

3. Asymptotic dynamics with exponentially decreasing weights

In this section we study the asymptotic dynamics arising when exponentially decreasing weights are assumed. These are often used in applications, since they describe, as suggested by Friedman in [16], agents which “form their expectations according to a weighted estimation procedure which exponentially discounts older observations”, that is, an *exponentially fading memory.* In this case some assumptions of previous section propositions are not satisfied, and more complex asymptotic dynamics can be obtained. The results of this section generalize, to the case of power means, the results given by Bischi and Gardini [6, 7] and by Bischi et al. [8] on Mann iterations which can be reduced to two-dimensional maps.

Exponentially decreasing weights can be defined by setting, at each n, in the vector of relative weights, a fixed value to the weight of the last state, say $\omega_n^{(n)} = \omega_0^{(0)} = 1$, while the values of the previous ones are obtained so that the ratio between two successive weights is fixed, say $\omega_k^{(n)}/\omega_{k+1}^{(n)} = \rho$. So, from $\omega^{(n)} = (\rho^n, \rho^{n-1}, ..., \rho, 1)$ we obtain $\omega^{(n+1)} = (\rho^{n+1}, \rho^n, ..., \rho, 1)$, or, more concisely,

$$\omega_k^{(n)} = \rho^{n-k}, 0 \leq k \leq n.$$

With these weights the following relation holds

$$W_{n+1} = 1 + \rho W_n, \tag{21}$$

and the recurrence with fading memory becomes

$$\begin{cases} x_{n+1} = f(z_n) \text{ with } z_n = \left(\displaystyle\sum_{k=0}^{n} \frac{\rho^{n-k}}{W_n} x_k^s \right)^{\frac{1}{s}}, \\ W_n = \sum_{k=0}^{n} \rho^{n-k} = \dfrac{1-\rho^{n+1}}{1-\rho}, s \neq 0. \end{cases} \tag{22}$$

As already stressed in Section 1 these weights are often used in economic modelling (see Gandolfo et al. [18], Aicardi and Invernizzi [1]) since, with a *memory ratio* $\rho \in (0,1)$, they represent the realistic assumption of an exponentially fading memory (see Friedman [16], Radner [22]). Let us first show that the relation (21) allows us to obtain, also in this case, a generalized segmenting Mann iteration. In fact we have

$$z_{n+1} = \left(\frac{1}{W_{n+1}} \left(\sum_{k=0}^{n} \rho \rho^{n-k} x_k^s + x_{n+1}^s \right) \right)^{\frac{1}{s}} = \left(\frac{\rho W_n}{W_{n+1}} z_n^s + \frac{1}{W_{n+1}} \left[f(z_n) \right]^s \right)^{\frac{1}{s}},$$

and defining

$$t_n = \frac{1}{W_{n+1}} = \frac{1-\rho}{1-\rho^{n+1}}, \tag{23}$$

and making use of the identity (21), we get the required non-autonomous difference equation (10). When $\rho \geq 1$ (non decreasing memory) the main results of Section 2 can be applied, without substantial changes, also to the case of geometric weights. In what follows, we consider the more realistic case of memory ratio $\rho \in (0,1)$, giving exponentially fading memory, for which the propositions of Section 2 do not apply, because the sequence of partial sums W_n converges to $W^* = 1/(1-\rho)$ and, consequently, sequence t_n, defined in (23), is not convergent to zero, being $t_n \to (1-\rho)$. For $\rho = 0$ (no memory of the past), the problem reduces to the study of the dynamics of an ordinary one-dimensional map $x_{n+1} = f(x_n)$. Since, as it is well known, the asymptotic dynamics of this iteration may be periodic of period $k \geq 1$, or even chaotic, depending on the shape of the function f, we can expect complex dynamics also for $\rho > 0$.

We saw in Section 2 that the only possible fixed points of the generalized Mann iteration are the fixed points of the function f. One may ask if also different asymptotic states, such as k-cycles, $k \geq 2$, are related to k-cycles of the map f. The answer is no.

Indeed, if $0 < \rho < 1$, and a k-cycle of (10) exists, then, in general, it is not a k-cycle of map f. However, such cycles are related to those of another one-dimensional (autonomous) map. This can be intuitively justified on the basis of the observation that the sequences of the time-dependent coefficients in the right hand side of (23) are convergent, since $t_n \to (1-\rho)$, so that the right hand side of (10) possesses an autonomous *limiting form*

$$z_{n+1} = g_\rho(z_n), \text{ with } g_\rho(z) = \left(\rho z^s + (1-\rho)[f(z)]^s\right)^{\frac{1}{s}}. \tag{24}$$

It is natural to conjecture that the asymptotic behavior of (10) is related to that of the map $g_\rho(z)$, and this can be rigorously proved by making use of a two-dimensional map. Let us note, in fact, that the sequence of the partial sums W_n of the geometric weights can be defined recursively by (21), and this allows us to obtain a two dimensional map $(z_{n+1}, W_{n+1}) = T(z_n, W_n)$ defined as

$$T : \begin{cases} z_{n+1} = \left(\dfrac{\rho W_n}{1+\rho W_n} z_n^s + \dfrac{1}{1+\rho W_n} [f(z_n)]^s \right)^{\frac{1}{s}}, \\ W_{n+1} = 1 + \rho W_n. \end{cases} \tag{25}$$

This map is equivalent to (10) if the initial condition is taken with $W_0 = 1$, i.e.

$$(z_0, W_0) = (x_0, 1), x_0 \in I. \tag{26}$$

In fact, in such a case, the sequence $\{z_n\}$ given by (25) coincides with the sequence obtained from the generalized segmenting Mann iteration (10) related to the same initial condition z_0. In other words, the projection on the z-axis of an orbit of the map T (with initial condition as in (26)) is the orbit of the non-autonomous iterative process (10).

The map (25) is triangular, that is a map with the structure $T(z, W) = (T_1(z, W), T_2(W))$. We notice that the map T is not defined on the points of the line of equation $W = -1/\rho$, but, since the initial conditions have to be taken on the line $W = 1$, we shall consider the restriction of T to the half-plane $W > -1/\rho$. Moreover, this half-plane is mapped into itself by T, because the second difference equation in (25) gives an increasing sequence (the partial sums of the geometric series starting from $W = 1$) always converging to the limit

$$W^* = \frac{1}{1-\rho}.$$

This also implies that the line $W = W^*$ is mapped into itself by T (i.e. it is a trapping set), and it is globally attracting for T in the half-space $W > -1/\rho$ (which means that for any point in the domain $W > -1/\rho$, the limit set of its orbit belongs to the trapping line $W = W^*$). In particular, any initial condition (26) has an orbit which is bounded in the rectangle $S = I \times J$, with $J = [1, W^*]$, and the limit set of the orbit belongs to the segment of S on the line $W = W^*$, which is an invariant set of the restriction of T to the line $W = W^*$, namely of the one dimensional map $g_\rho(z)$ given in (24).

The considerations given above prove the following proposition.

Proposition 3.1. *Let $f : I \to I$, $0 < \rho < 1$, g_ρ defined in* (24) *and T defined in* (25)*. Then*

(i) The orbits of the non-autonomous equation (10) *are in one-to-one correspondence with the orbits of the autonomous two-dimensional map T associated with an initial condition on the line $W = 1$.*
(ii) The invariant sets of T belong to the line $W = W^*$.
(iii) The invariant sets of T and those of g_ρ are in one-to-one correspondence.
(iv) An invariant set of T is attracting (resp. repelling) if and only if the corresponding invariant set of g_ρ is attracting (resp. repelling).

Now we investigate if the knowledge of stability/instability of the cycles of the map g_ρ, may be useful in order to decide on the existence and on the stability of cycles for the non-autonomous recurrence (10).

An answer to this question can be obtained from an analysis of the global properties of T. In fact, from the properties of the limiting map g_ρ we know the local properties of T near the asymptotic line $W = W^*$ but, since the initial conditions for T must be taken on the line $W = 1$, we need a global study of the map T in order to obtain information on the properties of the non-autonomous equation (10). The following proposition gives an answer to this question.

Proposition 3.2. *Let A be a k-cycle, $k \geq 1$, of the map $g_\rho(z)$, $0 < \rho < 1$. Then*

(i) if A is attracting, or attracting from one side, for the limiting map g_ρ then it is an attracting cycle for the non-autonomous process (10), *and hence $f(A)$ is an attracting set of the iteration* (22)*;*

(ii) the basin of attraction D of the attractor $f(A)$ of (22) *is given by the intersection of the two-dimensional basin, say $\hat{D}$, of the cycle $\hat{A} = A \times \{W^*\}$ of the map T (located on the trapping line $W = W^*$) with the line of initial conditions $W = 1$, i.e. $\hat{D} \cap \{W = 1\} = D \times \{1\}$.*

In this proposition the term attracting $k-$cycle, for the process with memory, means that the process generated by (22) converges asymptotically to the cycle starting from a set of initial conditions of measure greater than zero. It can be noticed that the attracting sets are not, in general, invariant sets (as usual for the non-autonomous processes). This means that, starting from a point of an attracting $k-$cycle, the sequence $\{x_n\}$ generated by (22) may not converge to the $k-$cycle, that is, the basin of a given attractor may not contain the points of the cycle itself (see e.g. Bischi and Naimzada [9], Bischi and Gardini [7]).

If we consider (22) with $z_n = x_n$ (no memory case), then its asymptotic behavior is indeed given by the study of the map $f(z)$. Conversely, if we consider exponentially fading memory, limit sets of (22) must be searched among the invariant sets of another one-dimensional autonomous map, the limiting map g_ρ defined in (24). However, we remark that their basins of attraction can only be determined through a global study of the two-dimensional map T. As we have already observed, only the fixed points of the map g_ρ coincide with the fixed points of the map f, whereas the other invariant sets, $k-$cycles or chaotic sets, are in general different.

Of course, the shape of map g_ρ depends on that of f. From the definition (24), function $g_\rho(z)$ is a power mean of z and $f(z)$, so for each $z \in I$

$$\min(z, f(z)) \leq g_\rho(z) \leq \max(z, f(z)). \tag{27}$$

This means that the graph of g_ρ always belongs to the area between the bisector and the graph of f, and the graphs of f and g_ρ intersect at the common fixed points.

The derivative of function g_ρ is

$$g'_\rho(z) = \left(\rho z^s + (1-\rho)[f(z)]^s\right)^{\frac{1-s}{s}} \left(\rho z^{s-1} + (1-\rho)[f(z)]^{s-1} f'(z)\right),$$

and if z^* is a positive fixed point of f it becomes

$$g'_\rho(z^*) = \rho + (1-\rho) f'(z^*), \tag{28}$$

which implies

$$\min\left(1, f'(z^*)\right) \leq g'_\rho(z^*) \leq \max\left(1, f'(z^*)\right). \tag{29}$$

If $z^* = 0$, i.e. $f(0) = 0$, $g'_\rho(z^*)$ is not defined. However, in this case

$$\lim_{z \to 0^+} g'_\rho(z) = \left(\rho + (1-\rho)[f'(0)]^s\right)^{\frac{1}{s}}$$

so (29) holds even for $z^* = 0$.

If $-1 < f'(z^*) < 1$, so that z^* is an attracting fixed point of the map f, then (29) implies $-1 < g'_\rho(z^*) < 1$, thus z^* is attracting for the map g_ρ too. If $|f'(z^*)| > 1$, so that z^* is a repelling fixed point of f, then z^* may be attracting or repelling for g_ρ. In particular, if $f'(z^*) > 1$ then z^* is repelling also for g_ρ since from (29) we have $1 < g'_\rho(z^*) < f'(z^*)$, while $f'(z^*) < -1$ gives $f'(z^*) < g'_\rho(z^*) < 1$ and in this case z^* may be attracting for g_ρ.

More exactly if $f'(z^*) < -1$, let $\tilde{\rho} \in (0, 1)$ be defined as

$$\tilde{\rho} = \frac{f'(z^*) + 1}{f'(z^*) - 1}. \tag{30}$$

Then the sufficient condition for the stability of the fixed point of the map g_ρ, $\left|g'_\rho(z^*)\right| < 1$, which, from (28), can be written as $-(1 + \rho)/(1 - \rho) < f'(z^*) < 1$, is satisfied for $\tilde{\rho} < \rho < 1$, i.e. with a sufficiently strong memory. These arguments are summarized in the following proposition, which also states the stabilizing effect of a strong memory.

Proposition 3.3. *Let z^* be a fixed point of f.*

(i) If $|f'(z^*)| < 1$ then also $\left|g'_\rho(z^*)\right| < 1$ for each $\rho \in (0, 1)$;
(ii) If $f'(z^*) < -1$ a value $\tilde{\rho} \in (0, 1)$ exists, given by (30), such that $\left|g'_\rho(z^*)\right| < 1$ for $\tilde{\rho} < \rho < 1$;
(iii) If $f'(z^*) > 1$ then also $g'_\rho(z^*) > 1$.

This proposition allows us to distinguish, among the fixed points of map f, those attracting for the process with a sufficiently strong memory (in particular with a uniform memory, obtained in the limiting case $\rho \to 1$).

4. Power mean learning mechanism in an OLG model

In this section we apply the learning mechanisms studied in the previous sections to an economic system modelled by a law expressed in the forward-looking form

$$x_n = f(x_{n+1}^{(e)}), \tag{31}$$

where $x_{n+1}^{(e)}$ represents the expected value of the state variable x for the next time period. The example we consider belongs to the family of Overlapping Generations' Models (OLG), which, among others, was studied by Samuelson [24], Diamond [13], Gale [17], Benhabib and Day [5], Azariadis [2]. In the setting we consider, population is constant in time and we assume that the economy is characterized by a single perishable consumption good and an asset called money. In each period time n, money is exchanged against good at a price p_n. Economic agents are consumers of a single type, hence we will only consider one agent as representative of the whole population. The representative agent lives two periods, in which she is referred to as *young* (period 1) and *old* (period 2) respectively, and she possesses time-invariant endowments w_1 and w_2 of the good for each period of the life. We suppose that the preferences about current (c_n) and future (c_{n+1}) consumptions are given by the following additive lifetime utility function

$$U(x_1, x_2) = u(c_n) + v(c_{n+1}),$$

in which both u and v are assumed to be twice differentiable, increasing and strictly concave real functions.

In the economy there is a nominal quantity of money, denoted by M, exogenously determined by the government, which is used to transfer wealth from one period to the next one in the following way: the young consumer saves part of her first period endowment using money and then she consumes her second period endowment and the saving when old. In this framework, it is supposed that all the money at the beginning of each period is held by the old agent.

The young consumer, on the base of the expected price p^e_{n+1}, must choose the level of consumption of the two periods of life and the nominal amount of money $m_d \geq 0$ to save for the second period of life. This can be achieved by solving the optimization problem

$$\max[u(c_n) + v(c_{n+1})] \text{ under constraints } \begin{cases} p_n c_n + m_d = p_n w_n, \\ p^e_{n+1} c_{n+1} = p^e_{n+1} w_2 + m_d, \end{cases}$$

which straightforwardly leads to

$$p^e_{n+1} u'\left(w_1 - \frac{m_d}{p_n}\right) = p_n v'\left(w_2 + \frac{m_d}{p^e_{n+1}}\right). \tag{32}$$

If we introduce the demand optimal excess of the good

$$z_i = c_i - w_i, \quad i = 1, 2,$$

we can write the money demand as

$$m_d(p_n, p^e_{n+1}) = -p_n z_1\left(\frac{p_n}{p^e_{n+1}}\right) = p^e_{n+1} z_2\left(\frac{p_n}{p^e_{n+1}}\right).$$

The excess demand of the good by the old consumer during period n is M/p_n, so, in any given period n, a competitive equilibrium is described by equating the demand and the supply in both the good and the money market, namely

$$z_1\left(\frac{p_n}{p^e_{n+1}}\right) + \frac{M}{p_n} = 0, \qquad m_d(p_n, p^e_{n+1}) = M.$$

Since by Walras' Law the two previous equilibrium conditions are equivalent, we can rewrite (32) as

$$p^e_{n+1} u'\left(w_1 - \frac{M}{p_n}\right) - p_n v'\left(w_2 + \frac{M}{p^e_{n+1}}\right) = 0$$

The previous relation can be expressed in compact form as $G(p_n, p^e_{n+1}) = 0$ from which, assuming that $\partial G/\partial p_n \neq 0$, we can obtain an equation of the form (31). In order to run the dynamics of the model, we still have to specify how the agents form expectations about next period price. A possible choice is given by a relation of the form

$$p^e_{n+1} = \psi(p_{n-1}, p_{n-2}, \ldots, p_0), \tag{33}$$

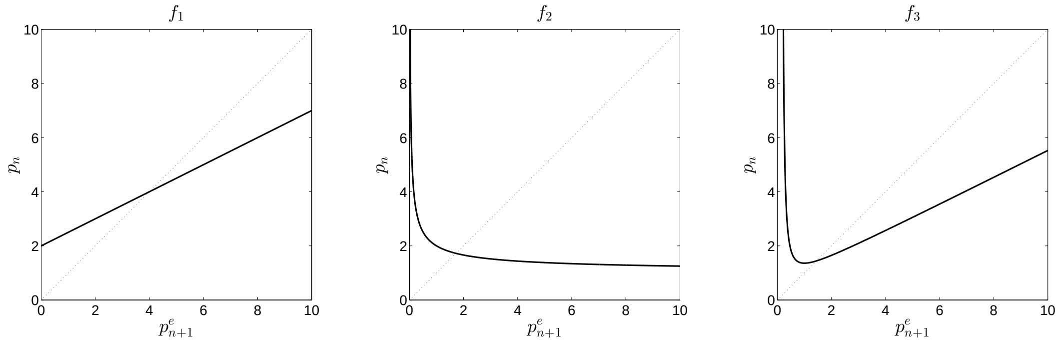

Figure 1. Maps of OLG models obtained with utility functions (35), (37) and (39).

in which ψ is a prevision function which must satisfy

$$x = \psi(x, x, \ldots, x). \tag{34}$$

In particular, if we assume that the agents have long memory and the capabilities to compute the power mean expressed in (4), we obtain the general iterative mechanism of the power means. We remark that the no-memory case $\rho = 0$ actually corresponds to the so-called myopic expectations.

In what follows, we will focus on three different examples of the previous OLG model, obtained considering different utility functions, and we study through simulations the effect of power means with fading memory. These three examples differ for the monotonicity of the map (31), which is either increasing, decreasing or unimodal (see Figure 1). The main goal is to investigate the case of exponentially fading memory.

In the first example, we consider Cobb-Douglas utility functions

$$u(c_n) = \alpha_1 \log c_n, \quad v(c_{n+1}) = \alpha_2 \log c_{n+1}, \tag{35}$$

where $\alpha_i > 0, i = 1, 2$ and $\alpha_1 + \alpha_2 = 1$, which gives the linear temporary equilibrium map

$$p_n = f_1(p^e_{n+1}) = \frac{w_2}{w_1} p^e_{n+1} + \frac{2M}{w_1}. \tag{36}$$

If we consider the Samuelson case (see [24]) and we assume $w_1 > w_2$, the r.h.s. of (36) is increasing with respect to p^e_{n+1} and, provided that p^e_{n+1} satisfies (33) with (34), has the unique positive equilibrium

$$p^* = \frac{2M}{w_1 - w_2}.$$

The plot of map (36) for $w_1 = 1, w_2 = 0.5$ and $M = 1$ is reported in Figure 1. Since we assume $w_2/w_1 < 1$, we have that equilibrium is unconditionally stable for both myopic and power mean expectations, i.e. for any ρ and s. As there is just one equilibrium, the dynamics are always convergent to p^* and they can only be different with respect to how quickly they approach, up to a desired precision, the equilibrium, namely their speed of convergence. Setting $w_1 = 1, w_2 = 0.5, M = 1$, in the left plot of Figure 2 we compare the time series of p_n obtained for $\rho = 0$

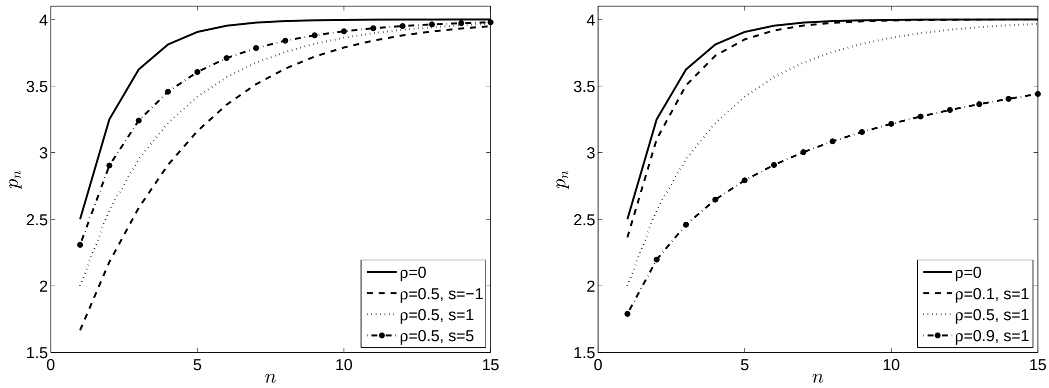

Figure 2. Time series of (36) without memory (solid line) and with memory (dashed, dotted and dash-dotted lines) for different exponents s (left plot) and memory ratios ρ (right plot). As s increases, convergence becomes more and more fast, while increasing ρ reduces convergence speed.

and for $\rho = 0.5$ and $s = -1, 1, 5$. The fastest convergence is achieved by the process without memory, and increasing the value of power s seems to improve the speed of convergence. Similarly, if we set $s = 1$ and we compare the time series obtained for different values of ρ, we can see that as the weight increases, convergence speed becomes more and more slow (left plot of Figure 2).

The remaining examples are inspired by utility functions studied by Benhabib and Day in [5]. Firstly, we consider

$$u(c_n) = \log c_n, \quad v(c_{n+1}) = \frac{c_{n+1}^{1-\alpha}}{1-\alpha}, \tag{37}$$

where $\alpha \neq 1$. We remark that in (37) functions u and v are swapped with respect to the example proposed in [5].

Assuming $\omega_1 = \omega, \omega_2 = 0$, the resulting temporary equilibrium function is given by

$$p_n = f_2(p^e_{n+1}) = \frac{M}{\omega_1} + \frac{M^\alpha}{\omega_1 (p^e_{n+1})^{\alpha-1}}. \tag{38}$$

We notice that, for $\alpha > 1$, the r.h.s. of (38) is decreasing with respect to p^e_{n+1} (see the middle plot of Figure 1, obtained setting $w_1 = 1, M = 1$ and $\alpha = 1.6$), so it has a unique equilibrium p^* the expression of which, however, can not be analytically obtained. Moreover, if no memory is considered ($\rho = 0$), we have that (38) can only converge to either the stable equilibrium or a period-2 cycle. If $\rho < 1$ is sufficiently large, from Proposition 3.3 we have that the power mean iteration scheme allows p_n to converge for any value of s to the equilibrium. In Figure 3 we compare the iterates obtained setting $M = 1, \omega = 1$ and $\alpha = 50$ for $\rho = 0.6$ and $s = -1, 1, 5$. We remark that with myopic expectations, we would have a period-2 cycle, in which p_n alternates between values respectively close to 1 and 2. Introducing a suitable weight allows for converging prices, with more and more slow convergence as the power s increases. Similarly, increasing ρ leads to a slow convergence, which, however, for small values of ρ exhibits an oscillating behavior.

Now we investigate the effect on p_n of varying ρ, setting again $M = 1, \omega = 1$ and $\alpha = 50$. We already noticed that if $\rho = 0$, the price dynamic can exhibit at most a period-2 cycle. However, introducing a positive memory ratio $\rho > 0$, we can have

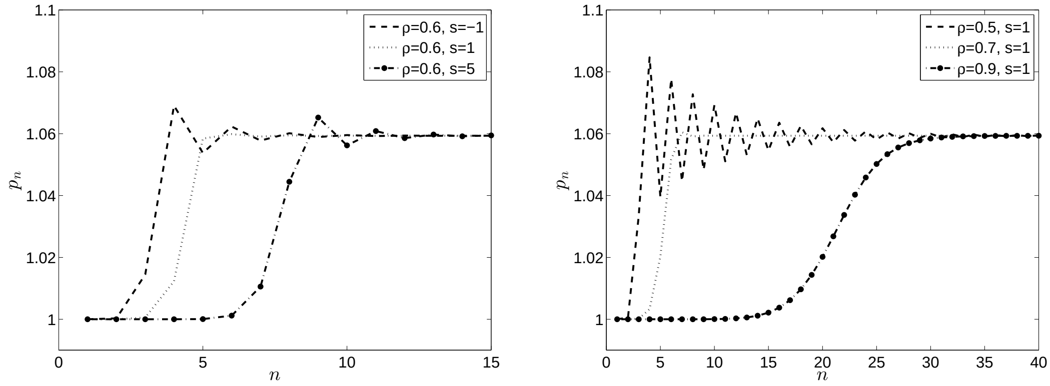

Figure 3. Time series of (38) for different exponents s (left plot) and memory ratios ρ (right plot). As s and ρ increases, convergence is more and more slow, but, if ρ is sufficiently small, it can be oscillating.

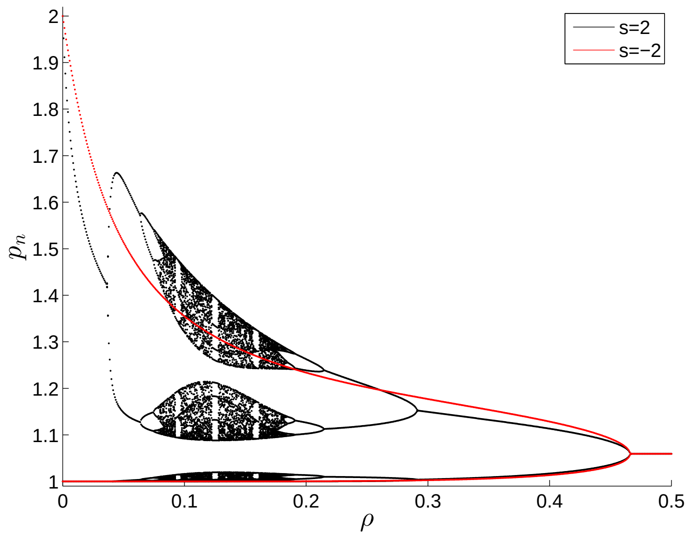

Figure 4. Bifurcation diagrams for (38) on varying ρ, with power means prediction function, for powers $s = 2$ (black) and $s = -2$ (red). Even if without memory at most a period-2 cycle is possible, when $\rho > 0$ both a qualitatively similar level of complexity ($s = -2$) or a complete sequence of period doubling/halving ($s = 2$) are possible.

an initial sequence of period doublings leading to chaos, which, when ρ is further increased, develops a sequence of period halvings leading to convergence toward the equilibrium for $\rho > 0.47$, as shown in Figure 4 for $s = 2$. Such phenomena are known as bubbling (see for example [19]). This means that for $\rho < \tilde{\rho}$, fading memory can also introduce an initial, with respect to ρ, complexity increasing. We also remark that such increasing of complexity does not occur for each power s, as the red bifurcation diagram reported in Figure 4 shows. The simulations we performed indicate that large s produces qualitatively more complex dynamic scenarios. If we keep ρ fixed and we compare the behavior of p_n on varying α, we can notice that, if $\rho < \tilde{\rho}$, even if, as predicted by Proposition 3.3, the loss of stability of the equilibrium occurs for the same value of α for any power s, the subsequent route to chaos can be different. For example, the second period doubling occurs for different values of α, as shown in Figure 5, which once more suggests that qualitatively simpler behaviors are induced by larger values of s (we point out that only one branch of the period-4 bifurcation is actually visible from Figure 5, as that arising from the value close to

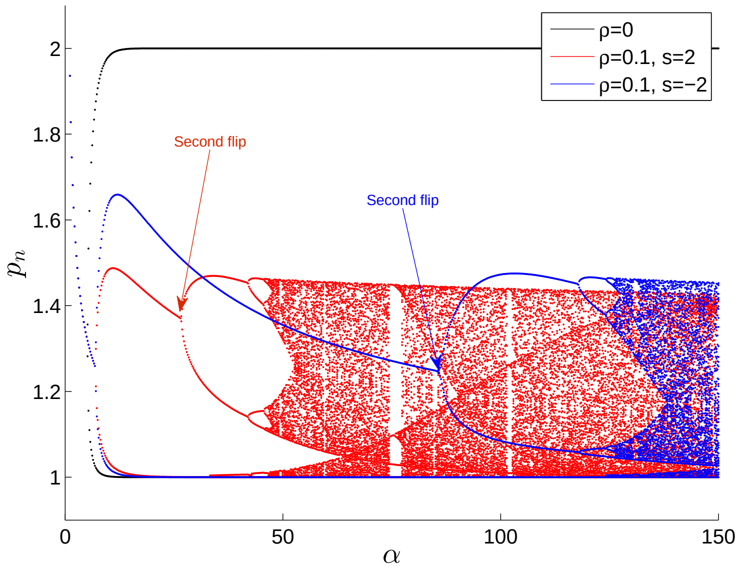

Figure 5. Bifurcation diagrams for (38) on varying α with power means prediction function, with myopic expectations ($\rho = 0$, black) and for $\rho = 0.1$ and $s = 2$ (red) and $s = -2$ (blue). After the first simultaneous period doubling, the subsequent ones occur for different values of α.

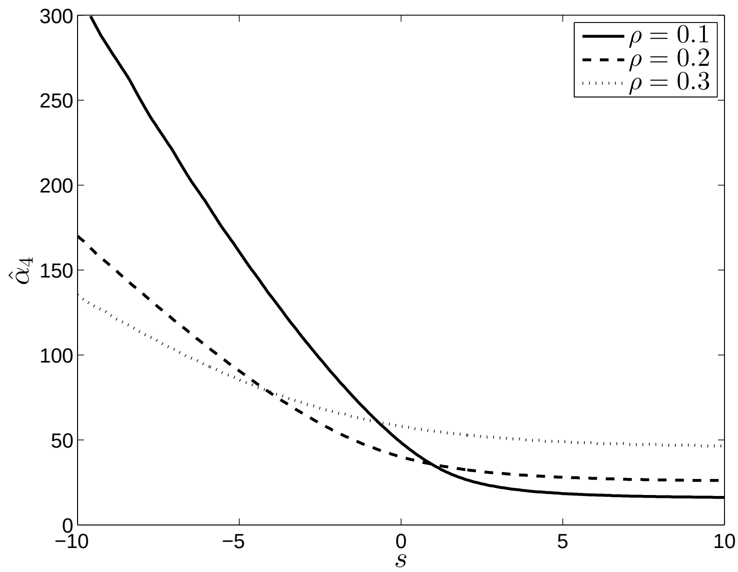

Figure 6. Function $\hat{\alpha}_4(s)$, obtained through simulations. For (38), as s increases, period-4 cycle earlier occurs.

1 is very flat).

To investigate this aspect, we numerically computed the value of α at which the second period doubling occurs, giving rise to the period-4 cycle. To this end, we define function $\hat{\alpha}_4 : \mathrm{R} \to (1, +\infty]$ that, for any given s, provides the infimum of the values of bifurcation parameter α for which a period-4 cycle occurs. If, for a given s, a period-4 cycle never occurred, we would have $\hat{\alpha}_4(s) = +\infty$. We computationally estimate $\hat{\alpha}_4(s)$ and we report the results in Figure 6, in which we can see that $\hat{\alpha}_4$ is a decreasing function, which implies that we have an increasingly retarded route toward chaos for small values of s (we remark that the larger is $\hat{\alpha}_4$, the later the period-4 cycle and subsequent cascade of flip bifurcations occurs).

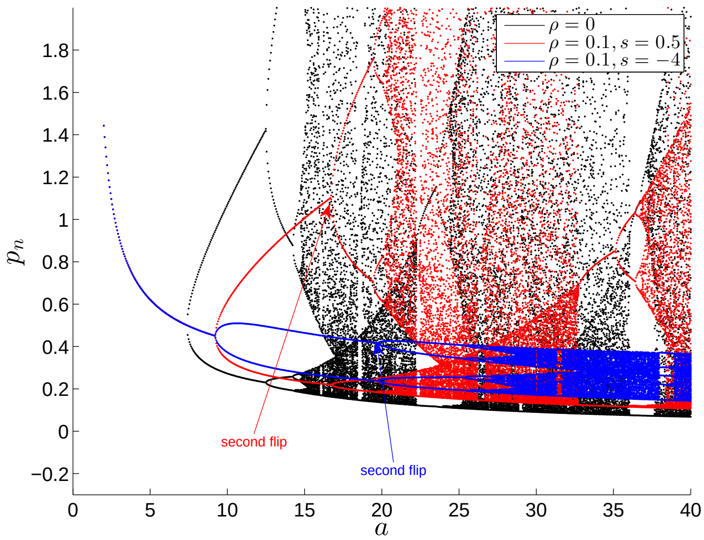

Figure 7. Bifurcation diagrams of (40), in which both myopic expectations (black) and memory ratio with two different values of s (red,blue) are considered.

In the last example we set

$$u(c_n) = bc_n, \quad v(c_{n+1}) = -a\mathrm{e}^{-c_{n+1}}, \tag{39}$$

where a, b are positive constants. For any initial endowments w_1, w_2, the resulting temporary equilibrium function is given by

$$p_n = f_3(p^e_{n+1}) = \frac{(p^e_{n+1})^2}{a\mathrm{e}^{-c_{n+1}}}, \tag{40}$$

which, for $a > 1$, has the unique equilibrium

$$p^* = \frac{M}{\log(a)}.$$

Function f_3 is unimodal, as shown in the right plot of Figure 1, obtained for $a = 2$ and $M = 1$. As predicted by Proposition 3.3, the dynamic with power means is convergent for $\rho > \tilde{\rho}$ defined by (30), which, since $f'(p^*) = 1 - \log(a)$, is $\tilde{\rho} = (\log(a) - 2)/(\log(a))$. As the bifurcation parameter a varies, differently from (38), the dynamic generated by (40) with myopic expectations ($\rho = 0$) consists in a complete cascade of flip bifurcations toward chaos. Introducing a non null memory ratio, the power mean iteration scheme is able to stabilize the equilibrium for $a \in (1, \exp(2/(1-\rho))$, and so setting $a = \exp(2/(1-\rho))$, for any value of s a flip bifurcation occurs. However, for (40) too, the subsequent behavior depends on s, as qualitatively described by the bifurcation diagrams in Figure 7. As for (38), we define function $\hat{a}_4 : \mathrm{R} \to (1, +\infty]$ so that $\hat{a}_4(s)$ represents the infimum of the set of values of a for which a period-4 cycle occurs. We compute $\hat{a}_4$ numerically for $\rho = 0.1$ and we report the results in Figures 8 and 9, distinguishing between $s < \hat{s}$ and $s > \hat{s}$ with $\hat{s} \approx -0.648$. When s is sufficiently small, increasing s corresponds to an increase of $\hat{a}_4$, as reported in Figure 8. Conversely, when $s > -0.648$, the behavior

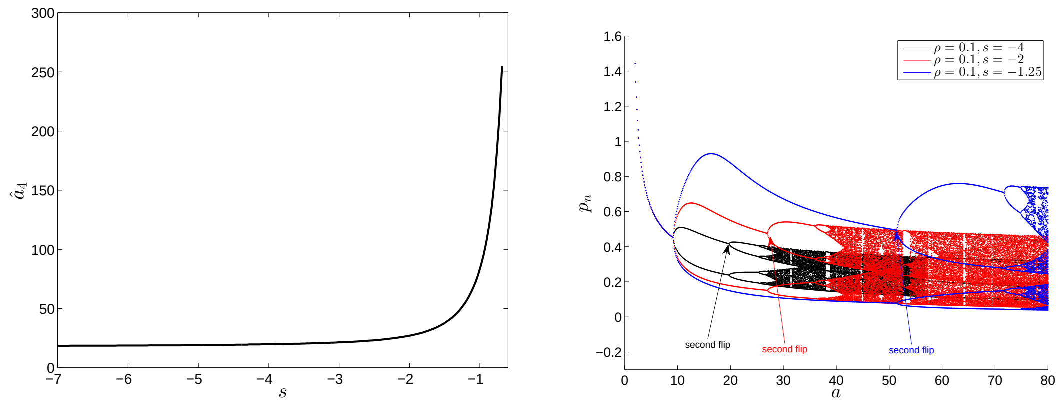

Figure 8. (Left plot) Function $\hat{a}_4(s)$ for $s < -0.648$, obtained through simulations. In this subdomain, the function is increasing. (Right plot) Bifurcation diagrams for three different values of s.

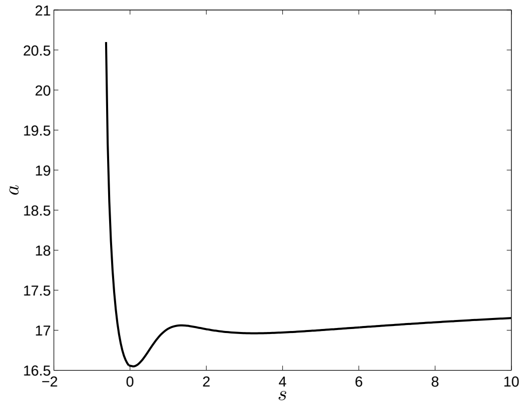

Figure 9. Function $\hat{a}_4(s)$ for $s > -0.648$, obtained through simulations. For these values of s, $\hat{a}_4(s)$ is non monotonic.

seems more complicated, as reported in Figure 9, from which we can notice that, when s lies in a right neighborhood of $\hat{s}$, the threshold $\hat{a}_4(s)$ is much smaller than that for $s \to \hat{s}^-$.

To better understand the discrepancy between the behaviors with $s < \hat{s}$ and $s > \hat{s}$, we refer to the bifurcation diagrams reported in Figure 10. Looking at the behavior of $\hat{a}_4$ reported in Figure 8, we expect that $\hat{a}_4(s)$ be increasingly larger than 250 for $s > \hat{s}$. Indeed, if we look at bifurcation diagrams reported in the right column of Figure 10, we can see that a period-4 indeed occurs for increasingly larger values of a for $s > \hat{s}$ too. However, if we compute the same bifurcation diagrams for small values of a (left column plots of Figure 10), we can see that a new, transient, increase of complexity occurs, giving rise to a bubble. For values of s sufficiently close to $\hat{s}$, we have only a couple of period doubling/halving appears, while slightly increasing s, an increasingly complex sequence of period doublings and halvings occur. In any case, it seems that function $\hat{a}_4$ attains its minimum value when (approximate) geometric mean is considered ($s \approx 0$).

We found that $\hat{a}_4$ shows a behavior similar to those reported in Figures 8 and 9 also when different parameters setting or values of ρ are considered. However, we remark that the discriminating value $\hat{s}$ seems to depend on ρ.

Even if the results we reported considering (36), (38), and (40) are only numerical, we can indeed say that, when the more realistic case of fading memory $\rho < 1$ is considered, the choice of a particular power mean is relevant. The value of s affects

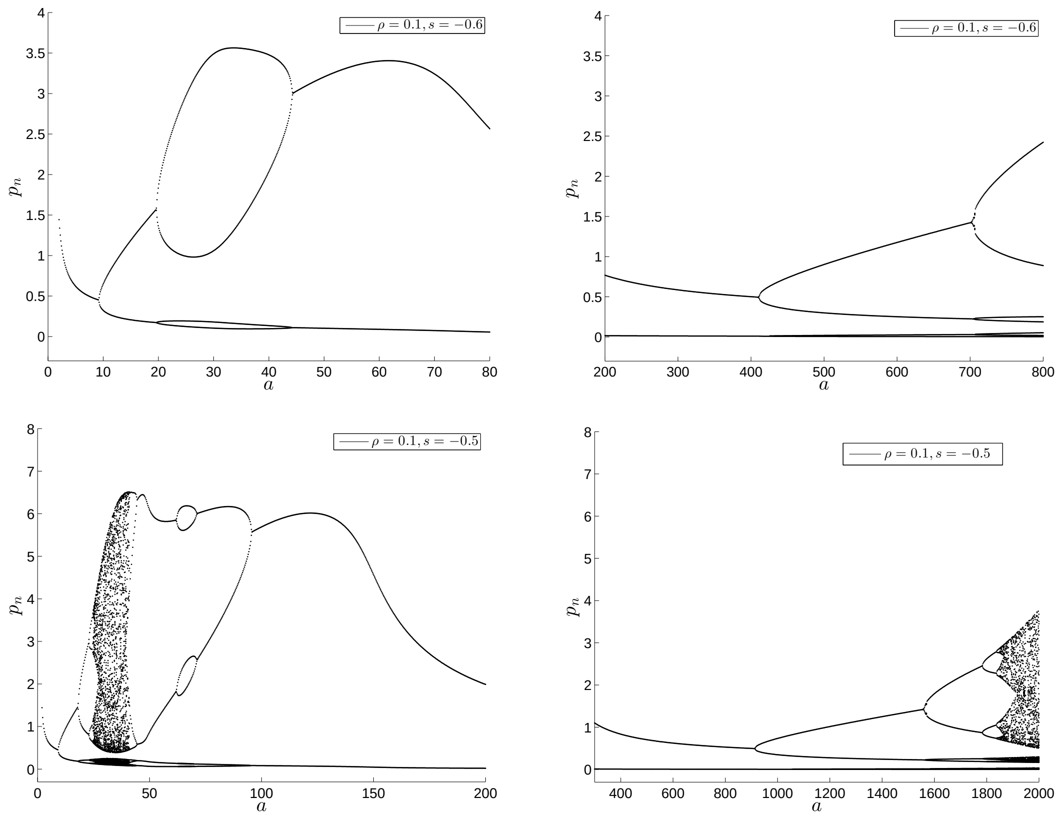

Figure 10. Bifurcation diagrams for model (40) for $\rho = 0.1, s = -0.6$ (first row) and $s = -0.5$ (second row). Left column: for small values of a, after the first period doubling, a bubble develops, consisting of either a period-4 cycle (top plot) or a complex sequence of period doubling/halving, leading to a period-2 cycle again. Right column: for large values of a, the period-2 cycle evolves through a sequence of period doublings.

not only the number of iterations required to approach the equilibrium (and this would indeed be true also for $\rho \geq 1$ and for the averaging processes considered in Section 2), but can also give rise to very different behaviors when stability is lost. In particular, the increasing in complexity can occur for very different values of the parameters. Moreover, the most natural choice of arithmetic mean could not be the most efficient one, as shown for instance by the last example, where the values close to $\hat{s}^-$ allow for the most retarded route toward chaos.

An analytical investigation of the previous phenomena is beyond the purposes of the present work, but it is indeed one of the research aim we want to focus on, to better understand the behavior of the period-4 appearance and of the further complexity increasing with respect to the shape of the recurrence function (monotone, unimodal, multimodal) and to investigate the conditions under which bubbles appear.

5. Conclusions

In this paper we studied an iterative scheme of the form $x_{n+1} = f(z_n)$, where z_n is a weighted power mean of all the previous state variables $x_0, ..., x_n$. Our results extend, to a general class of commonly used algebraic means (including arithmetic, quadratic, harmonic, and geometric means) some existing results about arithmetic mean only.

These iterative schemes can be used to model learning mechanisms in economic and social systems, where the agents use all available past data to compute expected values by some averaging method.

A particular distribution of weights, exponentially decreasing like the terms of a geometric series of ratio ρ has been used to investigate the effects of a fading memory on the asymptotic properties of the discrete process. This has been obtained through the reduction of the problem to the study of an equivalent two-dimensional triangular map whose asymptotic behavior is governed by a one-dimensional map.

This allows us to state that the presence of a strong memory, that is, with a memory ratio suitably close to 1, has a stabilizing effect. Conversely, if memory ratio is too small, we may not have convergence. In this case, we numerically investigated the route toward chaos, focusing on three particular examples arising from OLG models. The computational analysis suggests that the choice of the particular exponent s and memory ratio ρ conditions the (retarded or early) appearance of period-4 cycles and subsequent cascades of flip bifurcations. We aim to analyze such aspects in future researches.

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