ON SIMPLE EIGENVALUES OF THE FRACTIONAL LAPLACIAN UNDER REMOVAL OF SMALL FRACTIONAL CAPACITY SETS

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Abstract. We consider the eigenvalue problem for the restricted fractional Laplacian in a bounded domain with homogeneous Dirichlet boundary conditions. We introduce the notion of fractional capacity for compact subsets, with the property that the eigenvalues are not affected by the removal of zero fractional capacity sets. Given a simple eigenvalue, we remove from the domain a family of compact sets which are concentrating to a set of zero fractional capacity and we detect the asymptotic expansion of the eigenvalue variation; this expansion depends on the eigenfunction associated to the limit eigenvalue. Finally, we study the case in which the family of compact sets is concentrating to a point.

Keywords. Fractional Laplacian; Asymptotics of eigenvalues; Fractional capacity.

MSC classification. 31C15, 35P20, 35R11

1. Introduction

In the present paper we consider the eigenvalue problem for the Dirichlet fractional Laplacian in a bounded domain of $\mathbb{R}^N$. Our aim is to provide asymptotic estimates of the eigenvalue variation when a small vanishing set is removed. In this context, the good notion of *smallness* ensuring stability of the eigenvalue variation is related to the *Gagliardo fractional capacity*, which generalizes to the fractional setting the condenser capacity appearing in the framework of the standard Laplace operator, see Definition 1.1 below.

In the classical setting of the Dirichlet Laplacian, Rauch and Taylor [26] observed that the spectrum does not change by imposing homogeneous Dirichlet conditions on a compact polar subset, i.e. on a subset of zero Newtonian capacity. Courtois [13] developed a perturbation theory for the Dirichlet spectrum of a domain with small holes, with the capacity of holes playing the role of a perturbation parameter. More precisely, in [13] it is proved that, if $K \subset \Omega$ is a compact set, the N-th Dirichlet eigenvalue of the Laplacian in $\Omega \setminus K$ is close to the N-th Dirichlet eigenvalue of the Laplacian in Ω if (and only if) the capacity of the removed set K in Ω is close to zero; furthermore, if the capacity of K is small, then the eigenvalue variation is even differentiable with respect to the capacity of K in Ω. In [1] asymptotic estimates for such eigenvalue variation were obtained, highlighting a sharp relation between the order of vanishing of an eigenfunction of the Dirichlet Laplacian at a point and the leading term of the asymptotic expansion of the eigenvalue, as a removed compact set concentrates at that point. We also mention [4, 5, 12, 16, 25] for related estimates of the eigenvalue variation for the Laplacian under removal of small sets.

In order to formulate our problem, let us first introduce a suitable functional setting. Let $\Omega \subset \mathbb{R}^N$, $N \geq 1$, be an open set (bounded or unbounded). For $s \in (0, \min\{1, N/2\})$, we define the homogeneous fractional Sobolev space $\mathcal{D}^{s,2}(\Omega)$ as the completion of $C_c^\infty(\Omega)$ with respect to the Gagliardo norm

$$[u]_{H^s(\mathbb{R}^N)} = \left(\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u(x) - u(y)|^2}{|x - y|^{N+2s}} \, dx \, dy \right)^{\frac{1}{2}}.$$

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We note that $\mathcal{D}^{s,2}(\Omega) \hookrightarrow \mathcal{D}^{s,2}(\mathbb{R}^N)$ continuously by trivial extension. $\mathcal{D}^{s,2}(\Omega)$ is a Hilbert space with the scalar product

$$(1.1)\quad (u,v)_{\mathcal{D}^{s,2}(\Omega)} = \frac{C(N,s)}{2}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{(u(x)-u(y))(v(x)-v(y))}{|x-y|^{N+2s}}\,dx\,dy = \int_{\mathbb{R}^N}|\xi|^{2s}\overline{\widehat{v}(\xi)}\widehat{u}(\xi)\,d\xi,$$

and the associated norm

$$\|u\|_{\mathcal{D}^{s,2}(\Omega)} = (u,u)^{1/2}_{\mathcal{D}^{s,2}(\Omega)} = \sqrt{\tfrac{C(N,s)}{2}}\,[u]_{H^s(\mathbb{R}^N)},$$

where

$$(1.2)\qquad C(N,s) = \pi^{-\frac{N}{2}}2^{2s}\frac{\Gamma\big(\frac{N+2s}{2}\big)}{\Gamma(2-s)}s(1-s),$$

Γ is the Gamma function, and $\widehat{u}$ denotes the unitary Fourier transform of u.

We observe that, if Ω is bounded, then an equivalent norm on $\mathcal{D}^{s,2}(\Omega)$ is

$$\|u\|_{L^2(\Omega)} + [u]_{H^s(\mathbb{R}^N)},$$

see [7, Corollary 5.2]. As observed in [8, 10], in general the space $\mathcal{D}^{s,2}(\Omega)$ is smaller than the space $H^s_0(\Omega)$ defined as the closure of $C^\infty_c(\Omega)$ with respect to the norm

$$\|u\|_{H^s(\Omega)} = \|u\|_{L^2(\Omega)} + [u]_{H^s(\Omega)}$$

where

$$[u]_{H^s(\Omega)} = \left(\int_\Omega\int_\Omega \frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}}\,dx\,dy\right)^{\frac12}.$$

The two spaces $\mathcal{D}^{s,2}(\Omega)$ and $H^s_0(\Omega)$ coincide when Ω is a bounded Lipschitz open set and $s \neq 1/2$, see [8, Proposition B.1]. Furthermore, defining $H^s(\Omega)$ as the space $\{u \in L^2(\Omega) : [u]_{H^s(\Omega)} < +\infty\}$ endowed with the norm $\|u\|_{H^s(\Omega)} = \|u\|_{L^2(\Omega)} + [u]_{H^s(\Omega)}$ and $\widetilde{H}^s(\Omega)$ as the space of $H^s(\mathbb{R}^N)$-functions that are zero in $\mathbb{R}^N \setminus \Omega$, it is known that, if Ω is bounded and Lipschitz, then

$$H^s_0(\Omega) = \widetilde{H}^s(\Omega) \quad \text{if } s \neq \frac12$$

and

$$H^s_0(\Omega) = \widetilde{H}^s(\Omega) = H^s(\Omega) \quad \text{if } s < \frac12,$$

see [19, Corollary 1.4.4.5].

A key role in the perturbation theory we are going to develop for singularly perturbed fractional eigenvalue problems is played by the *Gagliardo fractional capacity.*

Definition 1.1. Let $\Omega \subset \mathbb{R}^N$ be a bounded open set. Let $K \subset \Omega$ be a compact set and let $\zeta_K \in C^\infty_c(\Omega)$ be such that $\zeta_K = 1$ in a neighborhood of K. For every $s \in (0, \min\{1, N/2\})$, we define the *Gagliardo s-fractional capacity* of K in Ω as

$$\mathrm{Cap}^s_\Omega(K) = \inf\left\{\|u\|^2_{\mathcal{D}^{s,2}(\Omega)} : u \in \mathcal{D}^{s,2}(\Omega) \text{ and } u - \zeta_K \in \mathcal{D}^{s,2}(\Omega\setminus K)\right\}.$$

The Gagliardo s-capacity was introduced and studied in several recent papers. We refer e.g. to [27, Appendix A] for some basic properties of the s-capacity; we also mention [2, 3, 30, 32] for some related notions of fractional capacity.

From now on $\Omega \subset \mathbb{R}^N$ will denote a bounded open set. We consider the following eigenvalue problem with homogeneous Dirichlet boundary conditions for the *restricted fractional Laplacian*:

$$(1.3)\qquad \begin{cases}(-\Delta)^s u = \lambda u, & \text{in } \Omega,\\ u = 0, & \text{in } \mathbb{R}^N \setminus \Omega.\end{cases}$$

We refer to Section 2 for a quick review of the definition and main properties of the fractional Laplacian $(-\Delta)^s$. We say that $\lambda \in \mathbb{R}$ is an eigenvalue of problem (1.3) if there exists some $u \in \mathcal{D}^{s,2}(\Omega) \setminus \{0\}$ (called eigenfunction) such that

$$(u,v)_{\mathcal{D}^{s,2}(\Omega)} = \lambda\int_{\mathbb{R}^N} u(x)v(x)\,dx, \quad \text{for all } v \in \mathcal{D}^{s,2}(\Omega).$$

Since $(-\Delta)^s$ is a self-adjoint operator on $L^2(\Omega)$ with compact inverse, the Spectral Theorem implies that the eigenvalues have finite multiplicity and form a diverging sequence

$$0 < \lambda_1^s(\Omega) \le \lambda_2^s(\Omega) \le \lambda_3^s(\Omega) \le \ldots \to +\infty.$$

We notice that, in contrast with the local case, a connectedness assumption on the domain Ω would lead to some loss of generality. Indeed, in the classical case the spectrum of the Dirichlet Laplacian in a disconnected domain is the union of the spectra on the connected components, whereas in the fractional case the spectrum is influenced by the mutual position of the connected components due to the nonlocal effects, see [9, §2.3].

We shall consider the eigenfunctions normalized as follows

$$\int_\Omega |u_j(x)|^2\, dx = 1. \tag{1.4}$$

Our first result is the fractional counterpart of [13, Theorem 1.1] and establishes the continuity of the eigenvalue variation under the removal of small fractional capacity sets.

Theorem 1.2. *Let $\Omega \subset \mathbb{R}^N$ be a bounded open set. For $s \in (0, \min\{1, N/2\})$, $K \subset \Omega$ compact and $k \in \mathbb{N}_*$, let $\lambda_k^s(\Omega)$, respectively $\lambda_k^s(\Omega \setminus K)$, be the k-th eigenvalue of problem (1.3) in Ω, respectively $\Omega \setminus K$. There exist $C > 0$ and $\delta > 0$ (independent of K) such that, if $\mathrm{Cap}_\Omega^s(K) \le \delta$, then*

$$0 \le \lambda_k^s(\Omega \setminus K) - \lambda_k^s(\Omega) \le C \left(\mathrm{Cap}_\Omega^s(K)\right)^{1/2}.$$

In particular we have that $\lambda_k^s(\Omega \setminus K) \to \lambda_k^s(\Omega)$ as $\mathrm{Cap}_\Omega^s(K) \to 0$.

Let us now consider a family of compact sets concentrating to a set of zero capacity with the goal of detecting the leading term of the asymptotic expansion of the eigenvalue variation.

Definition 1.3. Let $\Omega \subset \mathbb{R}^N$ be a bounded open set. Let $\{K_\varepsilon\}_{\varepsilon>0}$ be a family of compact sets contained in Ω. We say that K_ε is concentrating to a compact set $K \subset \Omega$ if for every open set ω such that $K \subset \omega \subseteq \Omega$ there exists $\varepsilon_\omega > 0$ such that $K_\varepsilon \subset \omega$ for every $0 < \varepsilon < \varepsilon_\omega$.

We note that the limit set K appearing in the previous definition could be not unique. We comment on this definition in Appendix B, where in particular we discuss the relation between Definition 1.3 and the classical notion of convergence of sets in the sense of Mosco.

To state our main results in this direction, we need to introduce the notion of fractional u-capacity for a function $u \in \mathcal{D}^{s,2}(\Omega)$.

Definition 1.4. Let $\Omega \subset \mathbb{R}^N$ be a bounded open set, $K \subset \Omega$ a compact set and $s \in (0, \min\{1, N/2\})$. For every $u \in \mathcal{D}^{s,2}(\Omega)$, we define the *$s$-fractional u-capacity* of K in Ω as

$$\mathrm{Cap}_\Omega^s(K, u) = \inf\left\{ \|w\|^2_{\mathcal{D}^{s,2}(\Omega)} : w \in \mathcal{D}^{s,2}(\Omega) \text{ and } w - u \in \mathcal{D}^{s,2}(\Omega \setminus K) \right\}. \tag{1.5}$$

More generally, we can define the fractional relative u-capacity for every function $u \in H^s_{\mathrm{loc}}(\Omega)$. Indeed, letting $\zeta_K \in C_c^\infty(\Omega)$ be as in Definition 1.1, we have that $\zeta_K u \in \mathcal{D}^{s,2}(\Omega)$, so that we can define

$$\mathrm{Cap}_\Omega^s(K, u) = \inf\left\{ \|w\|^2_{\mathcal{D}^{s,2}(\Omega)} : w \in \mathcal{D}^{s,2}(\Omega) \text{ and } w - \zeta_K u \in \mathcal{D}^{s,2}(\Omega \setminus K) \right\}.$$

The following theorem provides a sharp asymptotic expansion of the eigenvalue variation under removing of a family of compact sets concentrating to a zero fractional capacity set. In the classical setting of the Dirichlet Laplacian an analogous result can be found in [1, Theorem 1.4], see also the proof of [13, Theorem 1.2].

Theorem 1.5. *Let $\Omega \subset \mathbb{R}^N$ be a bounded open set. For $s \in (0, \min\{1, N/2\})$ and $j \in \mathbb{N}_*$, let $\lambda_j^s(\Omega)$ be the j-th eigenvalue of (1.3). Let $\{K_\varepsilon\}_{\varepsilon>0}$ be a family of compact sets contained in Ω concentrating to a compact set $K \subset \Omega$ in the sense of Definition 1.3. If*

$$\lambda_j^s(\Omega) \text{ is simple} \qquad \text{and} \qquad \mathrm{Cap}_\Omega^s(K) = 0$$

then

$$\lambda_j^s(\Omega \setminus K_\varepsilon) - \lambda_j^s(\Omega) = \mathrm{Cap}_\Omega^s(K_\varepsilon, u_j) + o(\mathrm{Cap}_\Omega^s(K_\varepsilon, u_j)), \tag{1.6}$$

as $\varepsilon \to 0^+$, where $u_j \in \mathcal{D}^{s,2}(\Omega)$ is an eigenfunction associated to $\lambda_j^s(\Omega)$ normalized as in (1.4).

We can estimate the asymptotic behavior of the s-fractional u_j-capacity as the family of compact sets K_ε concentrates to a point, by exploiting some of the results in [15]. Without loss of generality, we can assume that the limit point is the origin, hence in the following we suppose that $0 \in \Omega$, with Ω being a bounded open set in $\mathbb{R}^N$. We study the asymptotic behaviour of the quantity $\mathrm{Cap}^s_\Omega(K_\varepsilon, u_j)$ when $K_\varepsilon = \varepsilon K$ for a given compact set $K \subset \mathbb{R}^N$ and $\varepsilon \to 0^+$. We observe that the family of compact sets $\{\varepsilon K\}_{\varepsilon>0}$ concentrates (in the sense of Definition 1.3) to the singleton $\{0\}$, which has zero s-capacity in Ω (see Example 2.5 ahead).

For $s \in (0, \min\{1, N/2\})$ and $j \in \mathbb{N}_*$, let $\lambda^s_j(\Omega)$ be the j-th eigenvalue of problem (1.3) and let $u_j \in \mathcal{D}^{s,2}(\Omega)$ be an eigenfunction associated to $\lambda^s_j(\Omega)$ normalized as in (1.4). In view of [15], the asymptotic behavior of u_j at 0 can be described in terms of the eigenvalues and the eigenfunctions of the following eigenvalue problem

$$\begin{cases} -\operatorname{div}_{\mathbb{S}^N}(\theta_{N+1}^{1-2s}\nabla_{\mathbb{S}^N}\psi) = \mu\,\theta_{N+1}^{1-2s}\psi, & \text{in } \mathbb{S}^N_+, \\ -\lim_{\theta_{N+1}\to 0^+}\theta_{N+1}^{1-2s}\nabla_{\mathbb{S}^N}\psi\cdot\mathbf{e}_{N+1} = 0, & \text{on } \partial\mathbb{S}^N_+, \end{cases} \tag{1.7}$$

where $\mathbb{S}^N_+$ is the N-dimensional half-sphere

$$\mathbb{S}^N_+ = \{(\theta_1, \theta_2, \dots, \theta_{N+1}) \in \mathbb{S}^N : \theta_{N+1} > 0\} = \left\{ \tfrac{z}{|z|} : z \in \mathbb{R}^{N+1},\ z\cdot\mathbf{e}_{N+1} > 0 \right\},$$

with $\mathbf{e}_{N+1} = (0, \dots, 0, 1) \in \mathbb{R}^{N+1}$. From classical spectral theory, problem (1.7) admits a diverging sequence of real eigenvalues with finite multiplicity

$$\mu^s_1 \le \mu^s_2 \le \cdots \le \mu^s_k \le \cdots$$

Moreover $\mu^s_1 = 0$ and it is simple, i.e. $\mu^s_1 < \mu^s_2$. We note that, for $s = \frac{1}{2}$, by reflection eigenfunctions of (1.7) are spherical harmonics; then $\{\mu^{1/2}_k : k \ge 1\} = \{(N+k-2)(k-1) : k \ge 1\}$ and eigenfunctions associated to the eigenvalue $(N+k-2)(k-1)$ are spherical harmonics of degree $k-1$.

From [15, Theorem 4.1 and Lemma 4.2] there exist $k_0 \ge 1$ and $\psi \not\equiv 0$ eigenfunction of problem (1.7) associated to the eigenvalue $\mu^s_{k_0}$ such that, letting

$$\gamma_s = -\frac{N-2s}{2} + \sqrt{\left(\frac{N-2s}{2}\right)^2 + \mu^s_{k_0}}, \tag{1.8}$$

it holds

$$\tilde{u}_\varepsilon(x) := \varepsilon^{-\gamma_s} u_j(\varepsilon x) \to \hat{\psi}(x) \quad \text{in } H^s(B'_R) \quad \text{as } \varepsilon \to 0^+, \tag{1.9}$$

for every $R > 0$, where $B'_R = \{x \in \mathbb{R}^N : |x| < R\}$ and

$$\hat{\psi}(x) := |x|^{\gamma_s}\psi\left(\frac{x}{|x|}, 0\right). \tag{1.10}$$

We note that $\hat{\psi} \not\equiv 0$, see Section 2.1.

Theorem 1.6. *Let $\Omega \subset \mathbb{R}^N$ be a bounded open set with $0 \in \Omega$ and $K \subset \Omega$ compact. For every $\varepsilon > 0$ let $K_\varepsilon = \varepsilon K$. For $s \in (0, \min\{1, N/2\})$ and $j \in \mathbb{N}_*$, let $\lambda^s_j(\Omega)$ be the j-th eigenvalue of problem* (1.3) *and let $u_j \in \mathcal{D}^{s,2}(\Omega)$ be an eigenfunction associated to $\lambda^s_j(\Omega)$ normalized as in* (1.4)*. Then, as $\varepsilon \to 0^+$, it holds*

$$\mathrm{Cap}^s_\Omega(K_\varepsilon, u_j) = \varepsilon^{N+2(\gamma_s - s)}\left\{\mathrm{Cap}^s_{\mathbb{R}^N}(K, \hat{\psi}) + o(1)\right\}, \tag{1.11}$$

with γ_s and $\hat{\psi}$ as in (1.8) *and* (1.10) *respectively.*

As a consequence of Theorems 1.5 and 1.6, we deduce the following.

Theorem 1.7. *Let $\Omega \subset \mathbb{R}^N$ be a bounded open set with $0 \in \Omega$ and $K \subset \Omega$ compact. For every $\varepsilon > 0$ let $K_\varepsilon = \varepsilon K$. For $s \in (0, \min\{1, N/2\})$ and $j \in \mathbb{N}_*$, let $\lambda^s_j(\Omega)$ be the j-th eigenvalue of* *problem* (1.3) *and let* $u_j \in \mathcal{D}^{s,2}(\Omega)$ *be an associated eigenfunction satisfying* (1.4). *If* $\lambda_j^s(\Omega)$ *is simple, then, as* $\varepsilon \to 0^+$*, it holds*

$$\lambda_j^s(\Omega \setminus K_\varepsilon) - \lambda_j^s(\Omega) = \varepsilon^{N+2(\gamma_s - s)} \operatorname{Cap}_{\mathbb{R}^N}^s(K, \hat{\psi}) + o(\varepsilon^{N+2(\gamma_s-s)}), \tag{1.12}$$

with γ_s *and* $\hat{\psi}$ *as in* (1.8) *and* (1.10) *respectively.*

The asymptotic expansion (1.12) is sharp whenever $\operatorname{Cap}_{\mathbb{R}^N}^s(K, \hat{\psi}) \neq 0$, for example when K has nonzero Lebesgue measure in $\mathbb{R}^N$, as observed in Corollary 1.8 below. We mention that the fractional capacity $\operatorname{Cap}_{\mathbb{R}^N}^s(K, \hat{\psi})$ on the whole $\mathbb{R}^N$ appearing in the leading term of the expansion (1.12) is related to the weighted capacity of K in $\mathbb{R}^{N+1}$ with respect to the Muckenhoupt weight $|t|^{1-2s}$, see Remark 2.8; we refer to [20, Chapter 2] for a discussion on the properties of such capacity.

Corollary 1.8. *Under the same assumptions as in Theorem 1.7, suppose moreover that the N-dimensional Lebesgue measure of K is strictly positive. Then*

$$\lim_{\varepsilon \to 0^+} \frac{\lambda_j^s(\Omega \setminus K_\varepsilon) - \lambda_j^s(\Omega)}{\varepsilon^{N+2(\gamma_s - s)}} = \operatorname{Cap}_{\mathbb{R}^N}^s(K, \hat{\psi}) > 0. \tag{1.13}$$

Remark 1.9. It is worth mentioning that in the literature, besides the notion of *restricted fractional Laplacian* treated in the present paper, also the so called *spectral fractional Laplacian* (defined as the power of $-\Delta$ obtained by using its spectral decomposition) is often taken into consideration. The restricted and the spectral fractional Laplacians on bounded domains are different operators, as observed in [24] and [29]. The problem of spectral stability investigated in the present paper turns out to be much simpler for the spectral fractional Laplacian than for the restricted one, since the eigenvalues of the spectral fractional s-Laplacian are just the s-power of the eigenvalues of the classical Dirichlet Laplacian; hence the asymptotics of eigenvalues under removal of small sets can be easily deduced from the classical case treated in [1].

Denoting as $\{\lambda_j(\Omega)\}_{j=1}^\infty$ the eigenvalues the Laplacian in a bounded open set $\Omega \subset \mathbb{R}^N$ with homogeneous boundary conditions and by φ_j the eigenfunction associated to $\lambda_j(\Omega)$ normalized with respect to the $L^2(\Omega)$-norm, the spectral fractional Laplacian with homogeneous Dirichlet boundary conditions can be defined, for all $s \in (0,1)$, as

$$(-\Delta_{\text{spectral}})^s u(x) = \sum_{j=1}^{+\infty} (\lambda_j(\Omega))^s \left(\int_\Omega u \varphi_j \, dx \right) \varphi_j(x), \quad x \in \Omega.$$

The eigenvalues and the eigenfunctions of $(-\Delta_{\text{spectral}})^s$ are, respectively, $\nu_j^s(\Omega) := (\lambda_j(\Omega))^s$ and φ_j. Then, from [1, Theorem 1.4] it follows easily that, if $\lambda_j(\Omega)$ is simple and $\{K_\varepsilon\}_{\varepsilon>0}$ is a family of compact sets contained in Ω concentrating to a null capacity compact set, then

$$\nu_j^s(\Omega \setminus K_\varepsilon) - \nu_j^s(\Omega) = s(\lambda_j(\Omega))^{s-1} \operatorname{Cap}_\Omega(K_\varepsilon, \varphi_j) + o(\operatorname{Cap}_\Omega(K_\varepsilon, \varphi_j)),$$

as $\varepsilon \to 0^+$, where $\operatorname{Cap}_\Omega(K_\varepsilon, \varphi_j) = \inf\left\{ \int_\Omega |\nabla f|^2 : \ f \in H_0^1(\Omega) \text{ and } f - \varphi_j \in H_0^1(\Omega \setminus K_\varepsilon) \right\}$. Asymptotic expansions of $\operatorname{Cap}_\Omega(K_\varepsilon, \varphi_j)$ are obtained in [1] in several situations.

Comparing the above asymptotic expansion for the spectral fractional Laplacian with the expansion derived in Theorem 1.7, we note that only in the case of the restricted fractional Laplacian the vanishing order of the eigenvalue variation depends on the power s; hence the eigenvalues of the two operators exhibit quite different asymptotic behaviours under removal of small sets.

The paper is organized as follows. In Section 2 we collect some preliminary results. In Sections 3 and 4 we prove respectively Theorems 1.2 and 1.5. In Section 5 we present the proofs of Theorems 1.6, 1.7 and of Corollary 1.8. Finally, in Appendix A we prove an L^∞ bound for eigenfunctions which is needed in Section 3 and in Appendix B we discuss the Definition 1.3 of concentrating compact sets.

2. Preliminaries

In this section we recall some known facts and present some preliminary results.

2.1. **Restricted fractional Laplacian and Caffarelli-Silvestre extension.** The fractional Laplacian $(-\Delta)^s$ can be defined over the space $C_c^\infty(\mathbb{R}^N)$ by the principal value integral

$$(-\Delta)^s u(x) = C(N,s) \lim_{\varepsilon\to 0^+} \int_{\mathbb{R}^N\setminus B_\varepsilon(x)} \frac{u(x)-u(y)}{|x-y|^{N+2s}}\, dy,$$

where $C(N,s)$ is given in (1.2), or equivalently through the Fourier transform:

$$\mathcal{F}((-\Delta)^s u)(\xi) = |\xi|^{2s}\mathcal{F}u(\xi), \qquad \xi\in\mathbb{R}^N.$$

The scalar product of $\mathcal{D}^{s,2}(\mathbb{R}^N)$ defined in (1.1) is naturally associated to $(-\Delta)^s$, in the sense that $(-\Delta)^s$ can be extended to a bounded linear operator from $\mathcal{D}^{s,2}(\mathbb{R}^N)$ to its dual $(\mathcal{D}^{s,2}(\mathbb{R}^N))^*$, which actually coincides with the Riesz isomorphism of $\mathcal{D}^{s,2}(\mathbb{R}^N)$ with respect to the scalar product (1.1), i.e.

$${}_{(\mathcal{D}^{s,2}(\mathbb{R}^N))^*}\langle(-\Delta)^s u, v\rangle_{\mathcal{D}^{s,2}(\mathbb{R}^N)} = (u,v)_{\mathcal{D}^{s,2}(\mathbb{R}^N)}$$

for all $u,v\in\mathcal{D}^{s,2}(\mathbb{R}^N)$.

In [11] Caffarelli and Silvestre proved that $(-\Delta)^s$ can be realized as a Dirichlet-to-Neumann operator, i.e. as an operator mapping a Dirichlet boundary condition to a Neumann condition via an extension problem on the half space

$$\mathbb{R}^{N+1}_+ = \{(x,t)\in\mathbb{R}^{N+1} : x\in\mathbb{R}^N, t>0\}.$$

For every $U,V\in C_c^\infty(\overline{\mathbb{R}^{N+1}_+})$, let

$$q(U,V) = \int_{\mathbb{R}^{N+1}_+} t^{1-2s}\nabla U(x,t)\cdot\nabla V(x,t)\, dx\, dt.$$

We define $\mathcal{D}^{1,2}(\mathbb{R}^{N+1}_+;t^{1-2s})$ as the completion of $C_c^\infty(\overline{\mathbb{R}^{N+1}_+})$ with respect to the norm

$$\|U\|_{\mathcal{D}^{1,2}(\mathbb{R}^{N+1}_+;t^{1-2s})} = \sqrt{q(U,U)}.$$

There exists a well-defined continuous trace map

$$\mathrm{Tr}: \mathcal{D}^{1,2}(\mathbb{R}^{N+1}_+;t^{1-2s}) \to \mathcal{D}^{s,2}(\mathbb{R}^N) \tag{2.1}$$

which is onto (see for example [6]). By the Caffarelli-Silvestre extension theorem [11], given $u\in\mathcal{D}^{s,2}(\mathbb{R}^N)$, the minimization problem

$$\min\left\{q(W,W) : W\in\mathcal{D}^{1,2}(\mathbb{R}^{N+1}_+;t^{1-2s}),\ \mathrm{Tr}\,W = u\right\}$$

admits a unique minimizer $U=\mathcal{H}(u)\in\mathcal{D}^{1,2}(\mathbb{R}^{N+1}_+;t^{1-2s})$, which moreover satisfies

$$q(\mathcal{H}(u),W) = \kappa_s(u,\mathrm{Tr}\,W)_{\mathcal{D}^{s,2}(\mathbb{R}^N)}, \quad \text{for all } \varphi\in\mathcal{D}^{1,2}(\mathbb{R}^{N+1}_+;t^{1-2s}), \tag{2.2}$$

where

$$\kappa_s = \frac{\Gamma(1-s)}{2^{2s-1}\Gamma(s)},$$

i.e. $U=\mathcal{H}(u)$ weakly solves

$$\begin{cases} -\mathrm{div}(t^{1-2s}\nabla U) = 0, & \text{in } \mathbb{R}^{N+1}_+,\\ \lim_{t\to 0^+}\left(-t^{1-2s}\partial_t U\right) = \kappa_s(-\Delta)^s u, & \text{in } \mathbb{R}^N\times\{0\}.\end{cases}$$

From (2.2) it follows that

$$\|U\|^2_{\mathcal{D}^{1,2}(\mathbb{R}^{N+1}_+;t^{1-2s})} = \kappa_s\|u\|^2_{\mathcal{D}^{s,2}(\mathbb{R}^N)}. \tag{2.3}$$

As a consequence, if $\lambda_j^s(\Omega)$ is an eigenvalue of (1.3) for a certain $j\in\mathbb{N}_*=\mathbb{N}\setminus\{0\}$ and $u_j\in\mathcal{D}^{s,2}(\Omega)$ is an associated eigenfunction, the extension $U_j=\mathcal{H}(u_j)$ satisfies $\mathrm{Tr}\,U_j = u_j$ and

$$\begin{cases} -\mathrm{div}(t^{1-2s}\nabla U_j) = 0, & \text{in } \mathbb{R}^{N+1}_+,\\ \lim_{t\to 0^+}\left(-t^{1-2s}\partial_t U_j\right) = \lambda_j^s(\Omega)\kappa_s\,\mathrm{Tr}\,U_j, & \text{in } \Omega\times\{0\},\\ U_j = 0, & \text{in } (\mathbb{R}^N\setminus\Omega)\times\{0\},\end{cases} \tag{2.4}$$

in a weak sense, that is

$$
\begin{cases} U_j \in \mathcal{D}^{1,2}_{\Omega^c}(\mathbb{R}^{N+1}_+; t^{1-2s}), \\ q(U_j, \phi) = \lambda_j^s(\Omega)\kappa_s \displaystyle\int_{\Omega} \operatorname{Tr} U_j \operatorname{Tr} \phi \, dx \quad \text{for every } \phi \in \mathcal{D}^{1,2}_{\Omega^c}(\mathbb{R}^{N+1}_+; t^{1-2s}). \end{cases} \tag{2.5}
$$

Here, the space $\mathcal{D}^{1,2}_{\Omega^c}(\mathbb{R}^{N+1}_+; t^{1-2s})$ is defined as the closure of $C^\infty_c(\mathbb{R}^{N+1}_+ \cup \Omega)$ in $\mathcal{D}^{1,2}(\mathbb{R}^{N+1}_+; t^{1-2s})$; we also have the equivalent characterization

$$
\mathcal{D}^{1,2}_{\Omega^c}(\mathbb{R}^{N+1}_+; t^{1-2s}) = \{U \in \mathcal{D}^{1,2}(\mathbb{R}^{N+1}_+; t^{1-2s}) : \operatorname{Tr} U \in \mathcal{D}^{s,2}(\Omega)\}. \tag{2.6}
$$

We can consider equivalently either (2.5) or (1.3) with $\lambda = \lambda_j^s(\Omega)$. In this extended setting, the eigenvalues admit the following Courant-Fisher minimax characterization

$$
\lambda_j^s(\Omega) = \min_{\mathcal{U} \in \mathcal{S}_j} \max_{\substack{U \in \mathcal{U} \\ \|\operatorname{Tr} U\|_{L^2(\Omega)} \neq 0}} \mathcal{R}(U) \tag{2.7}
$$

where $\mathcal{S}_j$ denotes the family of all j-dimensional subspaces of $\mathcal{D}^{1,2}_{\Omega^c}(\mathbb{R}^{N+1}_+; t^{1-2s})$ and $\mathcal{R}$ is the Rayleigh type quotient defined as

$$
\mathcal{R}(U) = \frac{q(U,U)}{\kappa_s \int_\Omega |\operatorname{Tr} U(x)|^2 \, dx}. \tag{2.8}
$$

Remark 2.1. If $\Omega \subset \mathbb{R}^N$ is bounded and open and $K \subset \Omega$ is a compact subset, in view of the Caffarelli-Silvestre extension result described above and, in particular, of (2.3), we can characterize the Gagliardo s-fractional capacity introduced in Definition 1.1 as

$$
\operatorname{Cap}^s_\Omega(K) = \frac{1}{\kappa_s} \inf \left\{ q(W,W) : W \in \mathcal{D}^{1,2}_{\Omega^c}(\mathbb{R}^{N+1}_+; t^{1-2s}) \text{ and } W - \eta_K \in \mathcal{D}^{1,2}_{\Omega^c \cup K}(\mathbb{R}^{N+1}_+; t^{1-2s}) \right\},
$$

where $\eta_K \in C^\infty_c(\mathbb{R}^{N+1}_+ \cup \Omega)$ is any fixed function such that $\eta_K = 1$ in a neighborhood of K.

Correspondingly, for any $u \in \mathcal{D}^{s,2}(\Omega)$, we can characterize the s-fractional u-capacity of K in Ω introduced in Definition 1.4 as

$$
\operatorname{Cap}^s_\Omega(K,u) = \tfrac{1}{\kappa_s} \inf \left\{ q(W,W) : W \in \mathcal{D}^{1,2}_{\Omega^c}(\mathbb{R}^{N+1}_+; t^{1-2s}),\ W - U \in \mathcal{D}^{1,2}_{\Omega^c \cup K}(\mathbb{R}^{N+1}_+; t^{1-2s}) \right\} \tag{2.9}
$$

where $U \in \mathcal{D}^{1,2}_{\Omega^c}(\mathbb{R}^{N+1}_+; t^{1-2s})$ is such that $\operatorname{Tr} U = u$.

2.2. **Local asymptotic behaviour of eigenfunctions and their extension.** For $j \in \mathbb{N}_*$ and $s \in (0, \min\{1, \frac{N}{2}\})$, let $\lambda_j^s(\Omega)$ be the j-th eigenvalue of problem (1.3) and let $U_j \in \mathcal{D}^{1,2}_{\Omega^c}(\mathbb{R}^{N+1}_+; t^{1-2s})$ be a solution to (2.4) such that its trace $u_j = \operatorname{Tr} U_j$ satisfies the normalization condition (1.4). In [15], the asymptotic behavior of U_j (and consequently of its trace u_j) at 0 has been described in terms of the eigenvalues and the eigenfunctions of problem (1.7). More precisely, in [15, Theorem 4.1 and Lemma 4.2] it has been proved that there exist $k_0 \geq 1$ and $\psi \not\equiv 0$ eigenfunction of problem (1.7) associated to the eigenvalue $\mu^s_{k_0}$ such that

$$
\tilde{U}_\varepsilon(z) := \varepsilon^{-\gamma_s} U_j(\varepsilon z) \to \tilde{\psi}(z) := |z|^{\gamma_s} \psi\left(\frac{z}{|z|}\right) \quad \text{in } H^1(B^+_R; t^{1-2s}) \quad \text{as } \varepsilon \to 0^+, \tag{2.10}
$$

for every $R > 0$, where $B^+_R = \{z = (x,t) \in \mathbb{R}^{N+1}_+ : |z| < R\}$, γ_s is given in (1.8), and the space $H^1(B^+_R; t^{1-2s})$ is defined in Section 2.3 below.

The convergence (1.9) stated in the introduction follows from (2.10) by passing to the traces.

Remark 2.2. We note that the limit profile $\hat{\psi} := \operatorname{Tr} \tilde{\psi}$ appearing in (1.9) is not identically null; indeed $\tilde{\psi}$ and $t^{1-2s}\partial_t \tilde{\psi}$ can not both vanish on $\partial\mathbb{R}^{N+1}_+$, because otherwise $\tilde{\psi}$ would be a weak solution to the equation $\operatorname{div}(t^{1-2s}\nabla\tilde{\psi}) = 0$ satisfying both Dirichlet and weighted Neumann homogeneous boundary conditions and its trivial extension in $\mathbb{R}^{N+1}$ would violate the unique continuation principle for elliptic equations with Muckenhoupt weights proved in [31] (see also [18], and [28, Proposition 2.2]).

2.3. **Sobolev and Hardy-type inequalities.** For every $s \in \big(0, \min\{1, \frac{N}{2}\}\big)$ (so that $N-2s>0$), let

$$2^*(s) = \frac{2N}{N-2s}. \tag{2.11}$$

The following Sobolev inequalities and compactness results can be found for example in [14].

Theorem 2.3 ([14, Theorems 6.5 and 6.7, Corollary 7.2])**.** *Let $\Omega \subset \mathbb{R}^N$, $N \geq 1$, be a bounded, open set of class $C^{0,1}$ and let $s \in (0, \min\{1, N/2\})$.*

(i) *There exists a positive constant $S_{N,s}$ such that*

$$S_{N,s}\|u\|_{L^{2^*(s)}(\mathbb{R}^N)} \leq \|u\|_{\mathcal{D}^{s,2}(\mathbb{R}^N)} \quad \text{for all } u \in \mathcal{D}^{s,2}(\mathbb{R}^N).$$

(ii) *There exists a positive constant $C = C(N,s,\Omega)$ such that for every $u \in H^s(\Omega)$ and for every $q \in [1, 2^*(s)]$ it holds*

$$\|u\|_{L^q(\Omega)} \leq C\|u\|_{H^s(\Omega)}.$$

(iii) *If $\mathcal{I}$ is a bounded subset of $H^s(\Omega)$, then $\mathcal{I}$ is pre-compact in $L^q(\Omega)$ for every $q \in [1, 2^*(s))$.*

Let us recall some fractional Hardy-type inequalities. For any $s \in (0,1)$, the following Hardy-type inequality for $\mathcal{D}^{s,2}(\mathbb{R}^N)$-functions was established in [22]:

$$\Lambda_{N,s}\int_{\mathbb{R}^N}\frac{u^2(x)}{|x|^{2s}}\,dx \leq \|u\|^2_{\mathcal{D}^{s,2}(\mathbb{R}^N)} \qquad \text{for all } u \in \mathcal{D}^{s,2}(\mathbb{R}^N), \tag{2.12}$$

where

$$\Lambda_{N,s} = 2^{2s}\frac{\Gamma^2\left(\frac{N+2s}{4}\right)}{\Gamma^2\left(\frac{N-2s}{4}\right)}.$$

By combining the (2.12) and (2.3), we obtain the following Hardy-trace inequality:

$$\Lambda_{N,s}\kappa_s\int_{\mathbb{R}^N}\frac{|\operatorname{Tr}U|^2}{|x|^{2s}}\,dx \leq \int_{\mathbb{R}^{N+1}_+} t^{1-2s}|\nabla U|^2\,dx\,dt, \quad \text{for all } U \in \mathcal{D}^{1,2}(\mathbb{R}^{N+1}_+; t^{1-2s}). \tag{2.13}$$

Relation (2.13) implies in particular that, if Ω is bounded,

$$\int_\Omega |\operatorname{Tr}U|^2\,dx \leq \frac{\operatorname{diam}(\Omega)^{2s}}{\Lambda_{N,s}\kappa_s}\|U\|^2_{\mathcal{D}^{1,2}(\mathbb{R}^{N+1}_+;t^{1-2s})}, \quad \text{for all } U \in \mathcal{D}^{1,2}_{\Omega^c}(\mathbb{R}^{N+1}_+; t^{1-2s}), \tag{2.14}$$

where $\operatorname{diam}(\Omega)$ is the diameter of Ω.

For $r>0$, let $B_r^+ = \{z = (x,t) \in \mathbb{R}^{N+1}_+ : |z| < r\}$. We define $H^1(B_r^+; t^{1-2s})$ as the completion of $C^\infty(\overline{B_r^+})$ with respect to

$$\|U\|_{H^1(B_r^+;t^{1-2s})} = \left(\int_{B_r^+} t^{1-2s}(|\nabla U|^2 + U^2)\,dx\,dt\right)^{1/2}.$$

The following Hardy type inequality with boundary terms was proved in [15].

Lemma 2.4 ([15, Lemma 2.4])**.** *Let $s \in (0, \min\{1, N/2\})$. For all $r>0$ and $U \in H^1(B_r^+; t^{1-2s})$, the following holds*

$$\left(\frac{N-2s}{2}\right)^2\int_{B_r^+} t^{1-2s}\frac{U^2(z)}{|z|^2}\,dz \leq \int_{B_r^+} t^{1-2s}\left(\nabla U(z)\cdot\frac{z}{|z|}\right)^2 dz + \left(\frac{N-2s}{2r}\right)\int_{S_r^+} t^{1-2s}U^2(z)dS,$$

where $S_r^+ = \{z = (x,t) \in \mathbb{R}^{N+1}_+ : |z| = r\}$ and dS denotes the volume element on S_r^+.

As a particular case of the inequality stated in Lemma 2.4, we obtain the following

$$\left(\frac{N-2s}{2}\right)^2\int_{\mathbb{R}^{N+1}_+} t^{1-2s}\frac{U^2(z)}{|z|^2}\,dz \leq \int_{\mathbb{R}^{N+1}_+} t^{1-2s}|\nabla U(z)|^2\,dz, \tag{2.15}$$

for all $U \in \mathcal{D}^{1,2}(\mathbb{R}^{N+1}_+; t^{1-2s})$ and $s \in (0, \min\{1, N/2\})$.

2.4. **Fractional capacities and capacitary potentials.** We observe that, by Stampacchia's Theorem, the infimum in Remark 2.1 is achieved by a unique function $V_{\Omega,K} \in \mathcal{D}^{1,2}_{\Omega^c}(\mathbb{R}^{N+1}_+; t^{1-2s})$, with $V_{\Omega,K} - \eta_K \in \mathcal{D}^{1,2}_{\Omega^c \cup K}(\mathbb{R}^{N+1}_+; t^{1-2s})$, so that

$$\mathrm{Cap}^s_\Omega(K) = \frac{1}{\kappa_s} q(V_{\Omega,K}, V_{\Omega,K}); \tag{2.16}$$

moreover $V_{\Omega,K}$ satisfies

$$q(V_{\Omega,K}, v - V_{\Omega,K}) \geq 0$$

for all $v \in \mathcal{D}^{1,2}_{\Omega^c}(\mathbb{R}^{N+1}_+; t^{1-2s})$ with $v - \eta_K \in \mathcal{D}^{1,2}_{\Omega^c \cup K}(\mathbb{R}^{N+1}_+; t^{1-2s})$. Equivalently, we have that $V_{\Omega,K} \in \mathcal{D}^{1,2}_{\Omega^c}(\mathbb{R}^{N+1}_+; t^{1-2s})$ is the unique function such that $V_{\Omega,K} - \eta_K \in \mathcal{D}^{1,2}_{\Omega^c \cup K}(\mathbb{R}^{N+1}_+; t^{1-2s})$ and

$$q(V_{\Omega,K}, \phi) = 0 \qquad \text{for all } \phi \in \mathcal{D}^{1,2}_{\Omega^c \cup K}(\mathbb{R}^{N+1}_+; t^{1-2s}), \tag{2.17}$$

that is to say, $V_{\Omega,K}$ is the unique weak solution of

$$\begin{cases} -\mathrm{div}(t^{1-2s} \nabla V_{\Omega,K}) = 0, & \text{in } \mathbb{R}^{N+1}_+, \\ \lim_{t \to 0^+} \left(-t^{1-2s} \partial_t V_{\Omega,K} \right) = 0, & \text{in } (\Omega \setminus K) \times \{0\}, \\ V_{\Omega,K} = 0, & \text{in } (\mathbb{R}^N \setminus \Omega) \times \{0\}, \\ V_{\Omega,K} = 1, & \text{in } K \times \{0\}. \end{cases} \tag{2.18}$$

We also observe that $\mathrm{Tr}\, V_{\Omega,K}$ attains the infimum in Definition 1.1.

Since $V^-_{\Omega,K}$ and $(V_{\Omega,K} - 1)^+$ belong to $\mathcal{D}^{1,2}_{\Omega^c \cup K}(\mathbb{R}^{N+1}_+; t^{1-2s})$, we can choose $\phi = V^-_{\Omega,K}$ and $\phi = (V_{\Omega,K} - 1)^+$ in (2.17); in this way we obtain that $V^-_{\Omega,K} = (V_{\Omega,K} - 1)^+ \equiv 0$, that is

$$0 \leq V_{\Omega,K} \leq 1 \quad \text{a.e. in } \mathbb{R}^{N+1}_+. \tag{2.19}$$

Example 2.5 (Capacity of a point). If $\Omega \subset \mathbb{R}^N$ is an open set, $s \in (0, \min\{1, N/2\})$, and $P \in \Omega$, then

$$\mathrm{Cap}^s_\Omega(\{P\}) = 0. \tag{2.20}$$

Indeed, for every $n \in \mathbb{N}_*$, let $W_n \in C^\infty(\mathbb{R}^{N+1})$ be such that $W_n(z) = 1$ for $|z - P| \leq \frac{1}{n}$, $W_n(z) = 0$ for $|z - P| \geq \frac{2}{n}$, and $|\nabla W_n(z)| \leq 2n$ for all $z \in \mathbb{R}^{N+1}$. Then, for n sufficiently large, the restriction $W_n\big|_{\mathbb{R}^{N+1}_+}$ belongs to $\mathcal{D}^{1,2}_{\Omega^c}(\mathbb{R}^{N+1}_+; t^{1-2s})$ and is equal to 1 in a neighborhood of $\{P\}$. Moreover

$$q(W_n, W_n) \leq \mathrm{const}\, n^2 \int_{1/n}^{2/n} r^{N+1-2s}\, dr = O(n^{2s-N}) = o(1) \quad \text{as } n \to +\infty,$$

thus proving (2.20).

In order to prove that the spectrum of restricted fractional s-Laplacian in Ω does not change by removing a subset of zero fractional s-capacity, the following result is needed.

Proposition 2.6. *Let $\Omega \subset \mathbb{R}^N$ be an open set, $K \subset \Omega$ compact and $s \in (0, \min\{1, N/2\})$. The following three assertions are equivalent:*

(i) $\mathrm{Cap}^s_\Omega(K) = 0$;
(ii) $\mathcal{D}^{1,2}_{\Omega^c}(\mathbb{R}^{N+1}_+; t^{1-2s}) = \mathcal{D}^{1,2}_{\Omega^c \cup K}(\mathbb{R}^{N+1}_+; t^{1-2s})$;
(iii) $\mathcal{D}^{s,2}(\Omega) = \mathcal{D}^{s,2}(\Omega \setminus K)$.

Proof. It will be sufficient to prove that (i) is equivalent to (ii), since then the equivalence of (iii) follows from the fact that the restriction to Ω of the trace map Tr defined in (2.1) is onto and the characterization of spaces given in (2.6).

Suppose first that $\mathcal{D}^{1,2}_{\Omega^c}(\mathbb{R}^{N+1}_+; t^{1-2s}) = \mathcal{D}^{1,2}_{\Omega^c \cup K}(\mathbb{R}^{N+1}_+; t^{1-2s})$. Then we can take $\phi = V_{\Omega,K}$ as a test function in (2.17), so that

$$\mathrm{Cap}^s_\Omega(K) = q(V_{\Omega,K}, V_{\Omega,K}) = 0.$$

Now suppose that $\mathrm{Cap}^s_\Omega(K) = 0$. We have to show $\mathcal{D}^{1,2}_{\Omega^c}(\mathbb{R}^{N+1}_+; t^{1-2s}) \subset \mathcal{D}^{1,2}_{\Omega^c \cup K}(\mathbb{R}^{N+1}_+; t^{1-2s})$, the other inclusion being evident. To this aim, let $u \in \mathcal{D}^{1,2}_{\Omega^c}(\mathbb{R}^{N+1}_+; t^{1-2s})$. By the assumption that

$\mathrm{Cap}^s_\Omega(K) = 0$, for any $n \in \mathbb{N}$ there exists $\eta_n \in C^\infty_c(\mathbb{R}^{N+1}_+ \cup \Omega)$ such that $\eta_n \equiv 1$ in a neighborhood of K and

$$\int_{\mathbb{R}^{N+1}_+} t^{1-2s} |\nabla \eta_n|^2 \, dx\, dt < \frac{1}{n}.$$

On the other hand, by density of $C^\infty_c(\mathbb{R}^{N+1}_+ \cup \Omega)$ in $\mathcal{D}^{1,2}_{\Omega^c}(\mathbb{R}^{N+1}_+; t^{1-2s})$, for any $\varepsilon > 0$ there exists $u_\varepsilon \in C^\infty_c(\mathbb{R}^{N+1}_+ \cup \Omega)$ such that

$$\|u_\varepsilon - u\|^2_{\mathcal{D}^{1,2}(\mathbb{R}^{N+1}_+; t^{1-2s})} < \varepsilon.$$

In this way, the function $u_\varepsilon(1 - \eta_n) \in C^\infty_c(\mathbb{R}^{N+1}_+ \cup (\Omega \setminus K))$; we estimate

$$\begin{aligned}
\int_{\mathbb{R}^{N+1}_+} t^{1-2s} |\nabla(u_\varepsilon(1-\eta_n) - u)|^2 \, dx\, dt &= \int_{\mathbb{R}^{N+1}_+} t^{1-2s} |\nabla u_\varepsilon - \nabla u - \nabla(\eta_n u_\varepsilon)|^2 \, dx\, dt \\
&\le 2 \int_{\mathbb{R}^{N+1}_+} t^{1-2s} |\nabla u_\varepsilon - \nabla u|^2 \, dx\, dt + 2 \int_{\mathbb{R}^{N+1}_+} t^{1-2s} |\nabla(\eta_n u_\varepsilon)|^2 \, dx\, dt \\
&\le 2\varepsilon + 4 \int_{\mathbb{R}^{N+1}_+} t^{1-2s} |u_\varepsilon|^2 |\nabla \eta_n|^2 \, dx\, dt + 4 \int_{\mathbb{R}^{N+1}_+} t^{1-2s} |\eta_n|^2 |\nabla u_\varepsilon|^2 \, dx\, dt \\
&\le 2\varepsilon + \frac{4}{n} \sup |u_\varepsilon|^2 + 4 \left(\sup |\nabla u_\varepsilon|^2\right) \int_{\mathrm{supp}\, u_\varepsilon} t^{1-2s} |\eta_n|^2 \, dx\, dt \\
&\le 2\varepsilon + \frac{4}{n} \sup |u_\varepsilon|^2 + \frac{4}{n} \left(\frac{2}{N-2s}\right)^2 \sup |\nabla u_\varepsilon|^2 \sup_{z \in \mathrm{supp}\, u_\varepsilon} |z|^2,
\end{aligned}$$

where the last relation relies on (2.15).

This proves that u can be approximated in $\mathcal{D}^{1,2}(\mathbb{R}^{N+1}_+; t^{1-2s})$ with $C^\infty_c(\mathbb{R}^{N+1}_+ \cup (\Omega \setminus K))$-functions, so that $u \in \mathcal{D}^{1,2}_{\Omega^c \cup K}(\mathbb{R}^{N+1}_+; t^{1-2s})$. □

As a direct consequence of Proposition 2.6, we obtain that the removal of a zero fractional s-capacity set leaves the family of eigenvalues of $(-\Delta)^s$ unchanged.

Corollary 2.7. *Let $\Omega \subset \mathbb{R}^N$ be a bounded open set, $K \subset \Omega$ compact and $s \in (0, \min\{1, N/2\})$. It holds $\lambda^s_k(\Omega) = \lambda^s_k(\Omega \setminus K)$ for every $k \in \mathbb{N}_*$ if and only if $\mathrm{Cap}^s_\Omega(K) = 0$.*

Proof. The result follows from Proposition 2.6 combined with (2.7) and the Spectral Theorem. □

Remark 2.8. In the case $\Omega = \mathbb{R}^N$ and $K \subset \mathbb{R}^N$ compact, it holds

$$2\,\mathrm{Cap}^s_{\mathbb{R}^N}(K) = \mathrm{Cap}_{2,|t|^{1-2s}}(K, \mathbb{R}^{N+1}),$$

where the right hand side of the above expression is the $(2, |t|^{1-2s})$-capacity of the condenser $(K, \mathbb{R}^{N+1})$, as introduced in [20, Chapter 2]. To see this, it suffices to consider the function $V_K := V_{\mathbb{R}^N, K}$ that achieves $\mathrm{Cap}^s_{\mathbb{R}^N}(K)$ and its even extension

$$\tilde{V}_K(x, t) = \begin{cases} V_K(x, t), & \text{if } t \ge 0, \\ V_K(x, -t), & \text{if } t < 0, \end{cases}$$

and to notice that

$$\mathrm{Cap}^s_{\mathbb{R}^N}(K) = \frac{1}{2} \int_{\mathbb{R}^{N+1}} |t|^{1-2s} |\nabla \tilde{V}_K|^2 \, dx\, dt = \frac{1}{2} \mathrm{Cap}_{2,|t|^{1-2s}}(K, \mathbb{R}^{N+1}).$$

We remark that $|t|^{1-2s}$ is a 2-admissible weight (according to the definition given in [20, Chapter 2]), since $|t|^{1-2s}$ belongs to the Muckenhoupt class A_2.

Concerning the s-fractional u-capacity of K in Ω introduced in Definition 1.4 and characterized equivalently in (2.9), we have that, as it happens for $\mathrm{Cap}^s_\Omega(K)$, the infimum in (2.9) is achieved by a function $V_{\Omega,K,u} \in \mathcal{D}^{1,2}_{\Omega^c}(\mathbb{R}^{N+1}_+; t^{1-2s})$ and the infimum in (1.5) by $\mathrm{Tr}\, V_{\Omega,K,u}$, so that

$$\mathrm{Cap}^s_\Omega(K, u) = \frac{1}{\kappa_s} q(V_{\Omega,K,u}, V_{\Omega,K,u}) = \|\mathrm{Tr}\, V_{\Omega,K,u}\|^2_{\mathcal{D}^{s,2}(\Omega)}, \tag{2.21}$$

and $V_{\Omega,K,u}$ is the unique weak solution of

$$
\begin{cases}
-\mathrm{div}(t^{1-2s}\nabla V_{\Omega,K,u}) = 0, & \text{in } \mathbb{R}^{N+1}_+,\\
\lim_{t\to 0^+}\left(-t^{1-2s}\partial_t V_{\Omega,K,u}\right) = 0, & \text{in } (\Omega\setminus K)\times\{0\},\\
V_{\Omega,K,u} = 0, & \text{in } (\mathbb{R}^N\setminus\Omega)\times\{0\},\\
V_{\Omega,K,u} = u, & \text{in } K\times\{0\},
\end{cases}
\tag{2.22}
$$

in the sense that $V_{\Omega,K,u}\in\mathcal{D}^{1,2}_{\Omega^c}(\mathbb{R}^{N+1}_+;t^{1-2s})$, $V_{\Omega,K,u}-U\in\mathcal{D}^{1,2}_{\Omega^c\cup K}(\mathbb{R}^{N+1}_+;t^{1-2s})$ for some function $U\in\mathcal{D}^{1,2}_{\Omega^c}(\mathbb{R}^{N+1}_+;t^{1-2s})$ such that $\mathrm{Tr}\,U=u$, and

$$
q(V_{\Omega,K,u},\phi)=0 \quad \text{for all } \phi\in\mathcal{D}^{1,2}_{\Omega^c\cup K}(\mathbb{R}^{N+1}_+;t^{1-2s}).
\tag{2.23}
$$

3. Continuity of the eigenvalue variation

Proof of Theorem 1.2. For every $j\in\{1,2,\dots,k\}$, let $\lambda^s_j(\Omega)$ and $U_j\in\mathcal{D}^{1,2}_{\Omega^c}(\mathbb{R}^{N+1}_+;t^{1-2s})$ solve (2.4) and (1.4). Moreover we can choose the eigenfunctions U_j in such a way that

$$
\int_\Omega \mathrm{Tr}\,U_j(x)\,\mathrm{Tr}\,U_\ell(x)\,dx = 0 \quad \text{for } j\neq\ell.
\tag{3.1}
$$

Let us denote $u_j=\mathrm{Tr}\,U_j$ for all j. Let

$$
E=\mathrm{span}\{\Phi_j : j=1,2,\dots,k\}\ \subset\mathcal{D}^{1,2}_{\Omega^c\cup K}(\mathbb{R}^{N+1}_+;t^{1-2s})
$$

where $\Phi_j=U_j(1-V_{\Omega,K})$ and $V_{\Omega,K}$ is the capacitary potential of K satisfying (2.17)–(2.18). We denote $\varphi_j=\mathrm{Tr}\,\Phi_j$ for all j and $v_{\Omega,K}=\mathrm{Tr}\,V_{\Omega,K}$. We observe that, in view of (1.4), (3.1), (2.14) and Lemma A.1, we have, for all $j,\ell\in\{1,\dots,k\}$,

$$
\begin{aligned}
\left|\int_\Omega\varphi_j(x)\varphi_\ell(x)\,dx-\delta_{j\ell}\right| &= \left|-2\int_\Omega u_j(x)u_\ell(x)v_{\Omega,K}(x)\,dx+\int_\Omega u_j(x)u_\ell(x)v^2_{\Omega,K}(x)\,dx\right|\\
&\le\left(\max_{1\le j\le k}\|u_j\|_{L^\infty(\Omega)}\right)^2\left(2\int_\Omega|v_{\Omega,K}(x)|\,dx+\int_\Omega v^2_{\Omega,K}(x)\,dx\right)\\
&\le C\left((\mathrm{Cap}^s_\Omega(K))^{1/2}+\mathrm{Cap}^s_\Omega(K)\right)
\end{aligned}
\tag{3.2}
$$

for some constant $C>0$ independent of K. On the other hand

$$
\begin{aligned}
q(\Phi_j,\Phi_\ell) &= \int_{\mathbb{R}^{N+1}_+}t^{1-2s}(1-V_{\Omega,K})^2\nabla U_j\cdot\nabla U_\ell\,dx\,dt+\int_{\mathbb{R}^{N+1}_+}t^{1-2s}U_jU_\ell|\nabla V_{\Omega,K}|^2\,dx\,dt\\
&-\int_{\mathbb{R}^{N+1}_+}t^{1-2s}U_j(1-V_{\Omega,K})\nabla U_\ell\cdot\nabla V_{\Omega,K}\,dx\,dt-\int_{\mathbb{R}^{N+1}_+}t^{1-2s}U_\ell(1-V_{\Omega,K})\nabla U_j\cdot\nabla V_{\Omega,K}\,dx\,dt.
\end{aligned}
$$

Choosing $\phi=U_\ell(1-V_{\Omega,K})^2$ in (2.5) we obtain that

$$
\begin{aligned}
\int_{\mathbb{R}^{N+1}_+}t^{1-2s}(1-V_{\Omega,K})^2\nabla U_j\cdot\nabla U_\ell\,dx\,dt &= 2\int_{\mathbb{R}^{N+1}_+}t^{1-2s}(1-V_{\Omega,K})U_\ell\nabla U_j\cdot\nabla V_{\Omega,K}\,dx\,dt\\
&\quad+\kappa_s\lambda^s_j(\Omega)\int_\Omega\varphi_j(x)\varphi_\ell(x)\,dx,
\end{aligned}
$$

hence, thanks to Lemma A.1 and (3.2), for every $j,\ell\in\{1,2,\dots,k\}$,

$$
\begin{aligned}
&\left|\int_{\mathbb{R}^{N+1}_+}t^{1-2s}\nabla\Phi_j\cdot\nabla\Phi_\ell\,dx\,dt-\kappa_s\lambda^s_j(\Omega)\delta_{j\ell}\right|\\
&\quad=\left|\int_{\mathbb{R}^{N+1}_+}t^{1-2s}(1-V_{\Omega,K})U_\ell\nabla U_j\cdot\nabla V_{\Omega,K}\,dx\,dt+\kappa_s\lambda^s_j(\Omega)\left(\int_\Omega\varphi_j(x)\varphi_\ell(x)\,dx-\delta_{j\ell}\right)\right.\\
&\qquad\left.+\int_{\mathbb{R}^{N+1}_+}t^{1-2s}U_jU_\ell|\nabla V_{\Omega,K}|^2\,dx\,dt-\int_{\mathbb{R}^{N+1}_+}t^{1-2s}U_j(1-V_{\Omega,K})\nabla U_\ell\cdot\nabla V_{\Omega,K}\,dx\,dt\right|\\
&\quad\le C\left((\mathrm{Cap}^s_\Omega(K))^{1/2}+\mathrm{Cap}^s_\Omega(K)\right)
\end{aligned}
\tag{3.3}
$$

for some constant $C > 0$ independent of K. The above estimate implies there exists $\delta > 0$ independent of K such that $\Phi_1, \Phi_2, \dots, \Phi_k$ are linearly independent provided $\mathrm{Cap}^s_\Omega(K) < \delta$, so that E is a k-dimensional subspace of $\mathcal{D}^{1,2}_{\Omega^c\cup K}(\mathbb{R}^{N+1}_+; t^{1-2s})$ for $\mathrm{Cap}^s_\Omega(K) < \delta$.

From (2.7), the fact that $\lambda^s_i(\Omega) \le \lambda^s_k(\Omega)$ for all $i \le k$, (3.2) and (3.3) we have that

$$\begin{aligned}
\lambda^s_k(\Omega \setminus K) &\le \max_{\substack{(\alpha_1,\alpha_2,\dots,\alpha_k)\in\mathbb{R}^k\\ \sum_{i=1}^k \alpha_i^2=1}} \mathcal{R}\left(\sum_{i=1}^k \alpha_i \Phi_i\right)\\
&= \max_{\substack{(\alpha_1,\alpha_2,\dots,\alpha_k)\in\mathbb{R}^k\\ \sum_{i=1}^k \alpha_i^2=1}} \frac{\sum_{i,j=1}^k \alpha_i\alpha_j q(\Phi_i,\Phi_j)}{\kappa_s \sum_{i,j=1}^k \alpha_i\alpha_j \int_\Omega \varphi_i\varphi_j\,dx}\\
&= \max_{\substack{(\alpha_1,\alpha_2,\dots,\alpha_k)\in\mathbb{R}^k\\ \sum_{i=1}^k \alpha_i^2=1}} \frac{(\sum_{i=1}^k \alpha_i^2 \kappa_s \lambda^s_i(\Omega)) + O\left((\mathrm{Cap}^s_\Omega(K))^{1/2}\right)}{\kappa_s\left(1 + O\left((\mathrm{Cap}^s_\Omega(K))^{1/2}\right)\right)}\\
&\le \frac{\kappa_s\lambda^s_k(\Omega) + O\left((\mathrm{Cap}^s_\Omega(K))^{1/2}\right)}{\kappa_s\left(1 + O\left((\mathrm{Cap}^s_\Omega(K))^{1/2}\right)\right)} = \lambda^s_k(\Omega) + O\left((\mathrm{Cap}^s_\Omega(K))^{1/2}\right)
\end{aligned}$$

as $\mathrm{Cap}^s_\Omega(K) \to 0$. The proof is thereby complete. □

4. Asymptotic expansion of the eigenvalues under removal of small capacity sets

The aim of this section is to prove Theorem 1.5. The proof is inspired from that of [1, Theorem 1.4]. Let us start with some preliminary lemmas concerning the capacitary potential $V_{\Omega,K,u}$ defined in (2.21)–(2.22).

Lemma 4.1. *Let $\{K_\varepsilon\}_{\varepsilon>0}$ be a family of compact sets contained in the open set Ω concentrating, in the sense of Definition 1.3, to a compact set $K \subset \Omega$, with $\mathrm{Cap}^s_\Omega(K) = 0$. For every $u \in \mathcal{D}^{s,2}(\Omega)$ it holds*

$$\int_\Omega |\operatorname{Tr} V_{\Omega,K_\varepsilon,u}|^2\,dx = o(\mathrm{Cap}^s_\Omega(K_\varepsilon, u)) \qquad \text{as } \varepsilon \to 0. \tag{4.1}$$

Proof. Let $\mathcal{H}_\varepsilon = \mathcal{D}^{1,2}_{\Omega^c\cup K_\varepsilon}(\mathbb{R}^{N+1}_+; t^{1-2s})$. Suppose by contradiction that there exist a sequence $\varepsilon_n \to 0$, $\varepsilon_n > 0$, and a constant $C > 0$ such that

$$\int_\Omega |\operatorname{Tr} V_{\Omega,K_{\varepsilon_n},u}|^2\,dx \ge C\|V_{\Omega,K_{\varepsilon_n},u}\|^2_{\mathcal{D}^{1,2}_{\Omega^c}(\mathbb{R}^{N+1}_+;t^{1-2s})}$$

for every n. Letting

$$W_n = \frac{V_{\Omega,K_{\varepsilon_n},u}}{\|\operatorname{Tr} V_{K_{\Omega,K_{\varepsilon_n},u}}\|_{L^2(\Omega)}},$$

we have

$$\|\operatorname{Tr} W_n\|_{L^2(\Omega)} = 1 \qquad \text{and} \qquad \|W_n\|^2_{\mathcal{D}^{1,2}_{\Omega^c}(\mathbb{R}^{N+1}_+;t^{1-2s})} \le C^{-1}$$

for every n. By weak compactness of the unit ball of $\mathcal{D}^{1,2}_{\Omega^c}(\mathbb{R}^{N+1}_+; t^{1-2s})$ and by compactness of the trace operator $\operatorname{Tr} : \mathcal{D}^{1,2}_{\Omega^c}(\mathbb{R}^{N+1}_+; t^{1-2s}) \to L^2(\Omega)$ (which follows easily by combining the continuity of the trace map $\operatorname{Tr} : \mathcal{D}^{1,2}_{\Omega^c}(\mathbb{R}^{N+1}_+; t^{1-2s}) \to \mathcal{D}^{s,2}(\Omega)$ and part (iii) of Theorem 2.3), there exist a subsequence $(n_k)_{k\ge1}$ and $W \in \mathcal{D}^{1,2}_{\Omega^c}(\mathbb{R}^{N+1}_+; t^{1-2s})$ such that

$$W_{n_k} \rightharpoonup W \quad \text{in } \mathcal{D}^{1,2}_{\Omega^c}(\mathbb{R}^{N+1}_+; t^{1-2s}) \text{ as } k \to +\infty \qquad \text{and} \qquad \|\operatorname{Tr} W\|_{L^2(\Omega)} = 1. \tag{4.2}$$

Moreover, from (2.23) we deduce that

$$\int_{\mathbb{R}^{N+1}_+} t^{1-2s}\nabla W_{n_k}\cdot\nabla\phi\,dx\,dt = 0 \qquad \text{for every } \phi \in \mathcal{H}_{\varepsilon_{n_k}}.$$

For every $\phi \in C_c^\infty(\mathbb{R}^{N+1}_+ \cup (\Omega \setminus K))$, we have that $\phi \in \mathcal{H}_\varepsilon$ for ε sufficiently small. Therefore we can pass to the limit as $k \to +\infty$ above and obtain

$$\int_{\mathbb{R}^{N+1}_+} t^{1-2s} \nabla W \cdot \nabla \phi \, dx \, dt = 0 \qquad \text{for every } \phi \in C_c^\infty(\mathbb{R}^{N+1}_+ \cup (\Omega \setminus K)).$$

By density, the latter holds for every $\phi \in \mathcal{D}^{1,2}_{\Omega^c \cup K}(\mathbb{R}^{N+1}_+; t^{1-2s})$. Now, the assumption $\mathrm{Cap}^s_\Omega(K) = 0$ allows to deduce, through Proposition 2.6,

$$\int_{\mathbb{R}^{N+1}_+} t^{1-2s} \nabla W \cdot \nabla \phi \, dx \, dt = 0 \qquad \text{for every } \phi \in \mathcal{D}^{1,2}_{\Omega^c}(\mathbb{R}^{N+1}_+; t^{1-2s}).$$

Hence we can replace $\phi = W$ in the previous identity thus obtaining that $\|W\|^2_{\mathcal{D}^{1,2}(\mathbb{R}^{N+1}_+; t^{1-2s})} = 0$ and hence $W \equiv 0$ in $\overline{\mathbb{R}^{N+1}_+}$, thus contradicting (4.2). □

Lemma 4.2. *Let $\{K_\varepsilon\}_{\varepsilon>0}$ be a family of compact sets contained in the open set Ω concentrating, in the sense of Definition 1.3, to a compact set $K \subset \Omega$, with $\mathrm{Cap}^s_\Omega(K) = 0$. For every $u \in \mathcal{D}^{s,2}(\Omega)$ it holds*

$$\lim_{\varepsilon \to 0^+} \mathrm{Cap}^s_\Omega(K_\varepsilon, u) = \mathrm{Cap}^s_\Omega(K, u) = 0 \qquad \text{and} \qquad V_{\Omega, K_\varepsilon, u} \to V_{\Omega, K, u} \equiv 0$$

strongly in $\mathcal{D}^{1,2}(\mathbb{R}^{N+1}_+; t^{1-2s})$ as $\varepsilon \to 0^+$.

Proof. Let $U \in \mathcal{D}^{1,2}_{\Omega^c}(\mathbb{R}^{N+1}_+; t^{1-2s})$ be such that $\mathrm{Tr}\, U = u$ and let $V_{\Omega, K_\varepsilon, u} \in \mathcal{D}^{1,2}_{\Omega^c}(\mathbb{R}^{N+1}_+; t^{1-2s})$ achieve $\mathrm{Cap}^s_\Omega(K_\varepsilon, u)$. Then, by (2.23), $V_{\Omega, K_\varepsilon, u} - U \in \mathcal{D}^{1,2}_{\Omega^c \cup K_\varepsilon}(\mathbb{R}^{N+1}_+; t^{1-2s})$ and

$$q(V_{\Omega, K_\varepsilon, u}, \phi) = 0 \quad \text{for all } \phi \in \mathcal{D}^{1,2}_{\Omega^c \cup K_\varepsilon}(\mathbb{R}^{N+1}_+; t^{1-2s}). \tag{4.3}$$

As $V_{\Omega, K_\varepsilon, u}$ achieves (2.9), we have

$$\|V_{\Omega, K_\varepsilon, u}\|^2_{\mathcal{D}^{1,2}(\mathbb{R}^{N+1}_+; t^{1-2s})} \leq \|U\|^2_{\mathcal{D}^{1,2}(\mathbb{R}^{N+1}_+; t^{1-2s})},$$

so that $\{V_{\Omega, K_\varepsilon, u}\}_{\varepsilon>0}$ is bounded in $\mathcal{D}^{1,2}_{\Omega^c}(\mathbb{R}^{N+1}_+; t^{1-2s})$. Hence there exist a sequence $\varepsilon_n \to 0^+$ and $V \in \mathcal{D}^{1,2}_{\Omega^c}(\mathbb{R}^{N+1}_+; t^{1-2s})$ such that $V_{\Omega, K_{\varepsilon_n}, u} \rightharpoonup V$ weakly in $\mathcal{D}^{1,2}_{\Omega^c}(\mathbb{R}^{N+1}_+; t^{1-2s})$. Let us show that $V = V_{\Omega, K, u}$. On the one hand, $V - U \in \mathcal{D}^{1,2}_{\Omega^c \cup K}(\mathbb{R}^{N+1}_+; t^{1-2s})$ thanks to Proposition 2.6 and the assumption $\mathrm{Cap}^s_\Omega(K) = 0$. On the other hand, passing to the limit in (4.3) we obtain that $q(V, \phi) = 0$ for every $\phi \in C_c^\infty(\mathbb{R}^{N+1}_+ \cup (\Omega \setminus K))$ and so, by density, for every $\phi \in \mathcal{D}^{1,2}_{\Omega^c \cup K}(\mathbb{R}^{N+1}_+; t^{1-2s}) = \mathcal{D}^{1,2}_{\Omega^c}(\mathbb{R}^{N+1}_+; t^{1-2s})$. Therefore $V = V_{\Omega, K, u} \equiv 0$. In order to prove that the convergence is strong, take $\phi = V_{\Omega, K_{\varepsilon_n}, u} - U$ in (4.3) and pass to the limit to obtain

$$\lim_{n \to +\infty} \mathrm{Cap}^s_\Omega(K_{\varepsilon_n}, u) = \lim_{n \to +\infty} q(V_{\Omega, K_{\varepsilon_n}, u}, V_{\Omega, K_{\varepsilon_n}, u}) = \lim_{n \to +\infty} q(V_{\Omega, K_{\varepsilon_n}, u}, U) = q(V, U) = 0.$$

We conclude that $\mathrm{Cap}^s_\Omega(K_{\varepsilon_n}, u) \to 0$ and that $V_{\Omega, K_{\varepsilon_n}, u} \to 0$ strongly in $\mathcal{D}^{1,2}_{\Omega^c}(\mathbb{R}^{N+1}_+; t^{1-2s})$. Since these limits do not depend on the sequence $\varepsilon_n \to 0$, we reach the conclusion. □

Let us introduce the operator $A : \mathcal{D}^{1,2}_{\Omega^c}(\mathbb{R}^{N+1}_+; t^{1-2s}) \to \mathcal{D}^{1,2}_{\Omega^c}(\mathbb{R}^{N+1}_+; t^{1-2s})$ defined by

$$q(A(U), V) = \int_\Omega \mathrm{Tr}\, U \, \mathrm{Tr} V \, dx \tag{4.4}$$

for every $U, V \in \mathcal{D}^{1,2}_{\Omega^c}(\mathbb{R}^{N+1}_+; t^{1-2s})$. It is straightforward to see that A is symmetric, nonnegative, and compact. Letting, for $j \in \mathbb{N}_*$,

$$\mu_j = \frac{1}{\kappa_s \lambda^s_j(\Omega)}, \tag{4.5}$$

the spectrum of A is $\{0\} \cup \{\mu_j : j \in \mathbb{N}_*\}$; furthermore, since $\dim \ker A = +\infty$, 0 has infinite multiplicity as an eigenvalue of A, whereas the non-zero eigenvalues of A have finite multiplicity.

Proof of Theorem 1.5. Let $U_j = \mathcal{H}(u_j)$, so that U_j satisfies (2.4) and (1.4). To simplify the notation, in the rest of the proof we write $V_\varepsilon = V_{\Omega,K_\varepsilon,u_j}$ and $\mathcal{H}_\varepsilon = \mathcal{D}^{1,2}_{\Omega^c\cup K_\varepsilon}(\mathbb{R}^{N+1}_+;t^{1-2s})$. We divide the proof into three steps.

Step 1. We claim that

$$\lambda_j^s(\Omega\setminus K_\varepsilon)-\lambda_j^s(\Omega)=o\left(\sqrt{\mathrm{Cap}^s_\Omega(K_\varepsilon,u_j)}\right)\qquad\text{as }\varepsilon\to0^+. \tag{4.6}$$

Let

$$\psi_\varepsilon=U_j-V_\varepsilon\in\mathcal{H}_\varepsilon, \tag{4.7}$$

so that ψ_ε is the orthogonal projection of U_j on $\mathcal{H}_\varepsilon$ in the space $\mathcal{D}^{1,2}_{\Omega^c}(\mathbb{R}^{N+1}_+;t^{1-2s})$ endowed with the scalar product q, that is

$$q(\psi_\varepsilon-U_j,\phi)=0\qquad\text{for every }\phi\in\mathcal{H}_\varepsilon.$$

For every $\phi\in\mathcal{H}_\varepsilon$ we have, using (2.5),

$$\begin{aligned}q(\psi_\varepsilon,\phi)-\kappa_s\lambda_j^s(\Omega)\int_\Omega \mathrm{Tr}\,\psi_\varepsilon\,\mathrm{Tr}\,\phi\,dx&=q(U_j,\phi)-\kappa_s\lambda_j^s(\Omega)\int_\Omega \mathrm{Tr}\,\psi_\varepsilon\,\mathrm{Tr}\,\phi\,dx\\&=\kappa_s\lambda_j^s(\Omega)\int_\Omega \mathrm{Tr}\,V_\varepsilon\,\mathrm{Tr}\,\phi\,dx,\end{aligned}$$

so that

$$\int_\Omega \mathrm{Tr}\,\psi_\varepsilon\,\mathrm{Tr}\,\phi\,dx=\frac{1}{\kappa_s\lambda_j^s(\Omega)}q(\psi_\varepsilon,\phi)-\int_\Omega \mathrm{Tr}\,V_\varepsilon\,\mathrm{Tr}\,\phi\,dx\qquad\text{for ever }\phi\in\mathcal{H}_\varepsilon. \tag{4.8}$$

Let $A_\varepsilon:\mathcal{H}_\varepsilon\to\mathcal{H}_\varepsilon$ be defined by

$$q(A_\varepsilon(U),V)=\int_\Omega \mathrm{Tr}\,U\,\mathrm{TrV}\;dx\qquad\text{for every }U,V\in\mathcal{H}_\varepsilon. \tag{4.9}$$

Recalling the definition of μ_j in (4.5), the spectral theorem (see for instance [21, Proposition 8.20]) provides

$$\mathrm{dist}(\mu_j,\sigma(A_\varepsilon))\le\frac{\|A_\varepsilon\psi_\varepsilon-\mu_j\psi_\varepsilon\|_{\mathcal{H}_\varepsilon}}{\|\psi_\varepsilon\|_{\mathcal{H}_\varepsilon}}, \tag{4.10}$$

where $\sigma(A_\varepsilon)$ is the spectrum of A_ε.

Taking into account Lemma 4.2, we have that

$$|q(U_j,V_\varepsilon)|\le\sqrt{q(U_j,U_j)}\sqrt{q(V_\varepsilon,V_\varepsilon)}=\sqrt{\lambda_j^s(\Omega)\kappa_s}\sqrt{\mathrm{Cap}^s_\Omega(K_\varepsilon,u_j)}=o(1)$$

as $\varepsilon\to0$, then the denominator in the right hand side of (4.10) is easily estimated as follows

$$\begin{aligned}\|\psi_\varepsilon\|^2_{\mathcal{H}_\varepsilon}&=q(U_j-V_\varepsilon,U_j-V_\varepsilon)=q(U_j,U_j)+\mathrm{Cap}^s_\Omega(K_\varepsilon,u_j)-2q(U_j,V_\varepsilon)\\&=\lambda_j^s(\Omega)\kappa_s+o(1)\quad\text{as }\varepsilon\to0^+.\end{aligned} \tag{4.11}$$

In order to estimate the numerator in the right hand side of (4.10), let $Z_\varepsilon=A_\varepsilon\psi_\varepsilon-\mu_j\psi_\varepsilon\in\mathcal{H}_\varepsilon$. Using (4.9) and (4.8), we have

$$q(Z_\varepsilon,\phi)+\mu_jq(\psi_\varepsilon,\phi)=q(A_\varepsilon\psi_\varepsilon,\phi)=\int_\Omega \mathrm{Tr}\,\psi_\varepsilon\,\mathrm{Tr}\,\phi\,dx=\mu_jq(\psi_\varepsilon,\phi)-\int_\Omega \mathrm{Tr}\,V_\varepsilon\,\mathrm{Tr}\,\phi\,dx,$$

for every $\phi\in\mathcal{H}_\varepsilon$. Choosing $\phi=Z_\varepsilon\in\mathcal{H}_\varepsilon$ in the previous expression and using Theorem 2.3 (i) and (2.3), we obtain

$$\begin{aligned}\|Z_\varepsilon\|^2_{\mathcal{H}_\varepsilon}&=-\int_\Omega \mathrm{Tr}\,V_\varepsilon\,\mathrm{Tr}\,Z_\varepsilon\,dx\le\|\mathrm{Tr}\,V_\varepsilon\|_{L^2(\Omega)}|\Omega|^{\frac{s}{N}}\left(\int_\Omega|\mathrm{Tr}\,Z_\varepsilon|^{2^*(s)}\right)^{\frac{1}{2^*(s)}}\\&\le\|\mathrm{Tr}\,V_\varepsilon\|_{L^2(\Omega)}|\Omega|^{\frac{s}{N}}S_{N,s}^{-1}\|\mathrm{Tr}\,Z_\varepsilon\|_{\mathcal{D}^{s,2}(\mathbb{R}^N)}\le\|\mathrm{Tr}\,V_\varepsilon\|_{L^2(\Omega)}|\Omega|^{\frac{s}{N}}S_{N,s}^{-1}\kappa_s^{-1/2}\|Z_\varepsilon\|_{\mathcal{H}_\varepsilon}.\end{aligned} \tag{4.12}$$

Replacing (4.11) and (4.12) into (4.10), we find that there exists a constant C independent of ε such that

$$\mathrm{dist}(\mu_j,\sigma(A_\varepsilon))\le C\|\mathrm{Tr}\,V_\varepsilon\|_{L^2(\Omega)}. \tag{4.13}$$

Now, the assumption that $\lambda_j^s(\Omega)$ is simple and the continuity proved in Theorem 1.2 imply that

$$\lambda_{j,\varepsilon} := \lambda_j^s(\Omega \setminus K_\varepsilon) \qquad \text{is simple for } \varepsilon > 0 \text{ small enough.}$$

Denoting as

$$\mu_{j,\varepsilon} = 1/(\kappa_s \lambda_{j,\varepsilon}) \tag{4.14}$$

the j-th eigenvalue of A_ε, by the simplicity of μ_j as an eigenvalue of the operator A introduced in (4.4), and by Theorem 1.2 we have that

$$\operatorname{dist}(\mu_j, \sigma(A_\varepsilon)) = |\mu_j - \mu_{j,\varepsilon}| \quad \text{for } \varepsilon > 0 \text{ small enough.}$$

Then relation (4.13) provides, for ε small enough,

$$|\lambda_j^s(\Omega) - \lambda_{j,\varepsilon}| = \kappa_s \lambda_j^s(\Omega)\lambda_{j,\varepsilon}|\mu_j - \mu_{j,\varepsilon}| \le C\kappa_s \lambda_j^s(\Omega)\lambda_{j,\varepsilon}\| \operatorname{Tr} V_\varepsilon\|_{L^2(\Omega)}.$$

As C is independent of ε and $\lim_{\varepsilon\to 0^+}\lambda_{j,\varepsilon} = \lambda_j^s(\Omega)$, Lemma 4.1 provides the claim.

Step 2. We claim that

$$\|\psi_\varepsilon - \Pi_\varepsilon \psi_\varepsilon\|_{\mathcal{H}_\varepsilon} = o\left(\sqrt{\operatorname{Cap}^s_\Omega(K_\varepsilon, u_j)}\right) \qquad \text{as } \varepsilon \to 0^+, \tag{4.15}$$

where $\Pi_\varepsilon : \mathcal{D}^{1,2}_{\Omega^c}(\mathbb{R}^{N+1}_+; t^{1-2s}) \to \mathcal{H}_\varepsilon$ is defined as

$$\Pi_\varepsilon W = \left(\int_\Omega \operatorname{Tr} W \operatorname{Tr} U_{j,\varepsilon}\, dx\right) U_{j,\varepsilon} \quad \text{for any } W \in \mathcal{D}^{1,2}_{\Omega^c}(\mathbb{R}^{N+1}_+; t^{1-2s})$$

and $U_{j,\varepsilon}$ is a normalized eigenfunction associated to $\lambda_{j,\varepsilon}$, i.e.

$$\begin{cases} U_{j,\varepsilon} \in \mathcal{H}_\varepsilon, \\ q(U_{j,\varepsilon}, \phi) = \lambda_{j,\varepsilon}\kappa_s \displaystyle\int_\Omega \operatorname{Tr} U_{j,\varepsilon} \operatorname{Tr}\phi\, dx \quad \text{for every } \phi \in \mathcal{H}_\varepsilon, \\ \int_\Omega |\operatorname{Tr} U_{j,\varepsilon}(x)|^2\, dx = 1. \end{cases} \tag{4.16}$$

Let $\tilde{U}_\varepsilon = \psi_\varepsilon - \Pi_\varepsilon \psi_\varepsilon$ and notice that

$$\int_\Omega \operatorname{Tr}\tilde{U}_\varepsilon \operatorname{Tr} U_{j,\varepsilon}\, dx = 0. \tag{4.17}$$

Using the fact that $\Pi_\varepsilon\psi_\varepsilon$ is an eigenfunction associated to $\lambda_{j,\varepsilon}$ and relation (4.8), we see that the following holds for every $\phi \in \mathcal{H}_\varepsilon$

$$\begin{aligned} &q(\tilde{U}_\varepsilon, \phi) - \kappa_s\lambda_{j,\varepsilon}\int_\Omega \operatorname{Tr}\tilde{U}_\varepsilon \operatorname{Tr}\phi\, dx \\ &\quad= q(\psi_\varepsilon, \phi) - \kappa_s\lambda_{j,\varepsilon}\int_\Omega \operatorname{Tr}\psi_\varepsilon \operatorname{Tr}\phi\, dx - \left[q(\Pi_\varepsilon\psi_\varepsilon, \phi) - \kappa_s\lambda_{j,\varepsilon}\int_\Omega \operatorname{Tr}(\Pi_\varepsilon\psi_\varepsilon)\operatorname{Tr}\phi\, dx\right] \\ &\quad= q(\psi_\varepsilon, \phi) - \kappa_s\lambda_j^s(\Omega)\int_\Omega \operatorname{Tr}\psi_\varepsilon \operatorname{Tr}\phi\, dx + \kappa_s(\lambda_j^s(\Omega) - \lambda_{j,\varepsilon})\int_\Omega \operatorname{Tr}\psi_\varepsilon \operatorname{Tr}\phi\, dx \\ &\quad= \kappa_s\lambda_j^s(\Omega)\int_\Omega \operatorname{Tr} V_\varepsilon \operatorname{Tr}\phi\, dx + \kappa_s(\lambda_j^s(\Omega) - \lambda_{j,\varepsilon})\int_\Omega \operatorname{Tr}\psi_\varepsilon \operatorname{Tr}\phi\, dx. \end{aligned} \tag{4.18}$$

Let $\xi_\varepsilon = A_\varepsilon(\tilde{U}_\varepsilon) - \mu_{j,\varepsilon}\tilde{U}_\varepsilon \in \mathcal{H}_\varepsilon$. We use the definition of A_ε in (4.9), that of $\mu_{j,\varepsilon}$ in (4.14) and relation (4.18) evaluated at $\phi = \xi_\varepsilon$ to compute

$$\begin{aligned} \|\xi_\varepsilon\|^2_{\mathcal{H}_\varepsilon} &= q(A_\varepsilon(\tilde{U}_\varepsilon), \xi_\varepsilon) - \mu_{j,\varepsilon} q(\tilde{U}_\varepsilon, \xi_\varepsilon) = \int_\Omega \operatorname{Tr}\tilde{U}_\varepsilon \operatorname{Tr}\xi_\varepsilon\, dx \\ &\quad - \left[\int_\Omega \operatorname{Tr}\tilde{U}_\varepsilon \operatorname{Tr}\xi_\varepsilon\, dx + \frac{\lambda_j^s(\Omega)}{\lambda_{j,\varepsilon}}\int_\Omega \operatorname{Tr} V_\varepsilon \operatorname{Tr}\xi_\varepsilon\, dx + \frac{\lambda_j^s(\Omega) - \lambda_{j,\varepsilon}}{\lambda_{j,\varepsilon}}\int_\Omega \operatorname{Tr}\psi_\varepsilon \operatorname{Tr}\xi_\varepsilon\, dx\right] \\ &= -\frac{\lambda_j^s(\Omega)}{\lambda_{j,\varepsilon}}\int_\Omega \operatorname{Tr} V_\varepsilon \operatorname{Tr}\xi_\varepsilon\, dx - \frac{\lambda_j^s(\Omega) - \lambda_{j,\varepsilon}}{\lambda_{j,\varepsilon}}\int_\Omega \operatorname{Tr}\psi_\varepsilon \operatorname{Tr}\xi_\varepsilon\, dx, \end{aligned}$$

from which, taking into account (4.11) and (2.14), we deduce that

$$\|\xi_\varepsilon\|_{\mathcal{H}_\varepsilon} \le C\left(\|\operatorname{Tr} V_\varepsilon\|_{L^2(\Omega)} + |\lambda_j^s(\Omega) - \lambda_{j,\varepsilon}|\right),$$

for a constant C not depending on ε. Lemma 4.1 and relation (4.6) provide then

$$\|A_\varepsilon(\tilde U_\varepsilon) - \mu_{j,\varepsilon}\tilde U_\varepsilon\|_{\mathcal H_\varepsilon} = \|\xi_\varepsilon\|_{\mathcal H_\varepsilon} = o\left(\sqrt{\mathrm{Cap}^s_\Omega(K_\varepsilon,u_j)}\right) \qquad \text{as } \varepsilon\to 0^+. \tag{4.19}$$

Let

$$\mathcal K_\varepsilon = \mathrm{Ker}\,\Pi_\varepsilon|_{\mathcal H_\varepsilon} = \left\{W\in\mathcal H_\varepsilon : \int_\Omega \mathrm{Tr}\,W\,\mathrm{Tr}\,U_{j,\varepsilon}\,dx = 0\right\}$$

and note that $\tilde U_\varepsilon \in \mathcal K_\varepsilon$ thanks to (4.17). Moreover, in view of (4.9) and (4.16), $A_\varepsilon(U)\in\mathcal K_\varepsilon$ for all $U\in\mathcal K_\varepsilon$, hence, denoting as $\tilde A_\varepsilon$ the restriction of A_ε to $\mathcal K_\varepsilon$, we have $\tilde A_\varepsilon:\mathcal K_\varepsilon\to\mathcal K_\varepsilon$. As $\sigma(\tilde A_\varepsilon) = \sigma(A_\varepsilon)\setminus\{\mu_{j,\varepsilon}\}$, there exists $\delta>0$ independent of ε such that $\mathrm{dist}(\mu_{j,\varepsilon},\sigma(\tilde A_\varepsilon))\ge\delta$. We use this inequality, the spectral theorem, and relation (4.19) to obtain

$$\begin{aligned}\|\psi_\varepsilon - \Pi_\varepsilon\psi_\varepsilon\|_{\mathcal H_\varepsilon} = \|\tilde U_\varepsilon\|_{\mathcal H_\varepsilon} &\le \frac1\delta \mathrm{dist}(\mu_{j,\varepsilon},\sigma(\tilde A_\varepsilon))\|\tilde U_\varepsilon\|_{\mathcal H_\varepsilon}\\ &\le \frac1\delta\|\tilde A_\varepsilon(\tilde U_\varepsilon) - \mu_{j,\varepsilon}\tilde U_\varepsilon\|_{\mathcal H_\varepsilon} = o\left(\sqrt{\mathrm{Cap}^s_\Omega(K_\varepsilon,u_j)}\right)\end{aligned}$$

as $\varepsilon\to0^+$, thus proving (4.15).

Step 3. From the definition of ψ_ε (4.7), (1.4), Lemma 4.1, (4.15) and (2.14), we have

$$\begin{aligned}\|\mathrm{Tr}(\Pi_\varepsilon\psi_\varepsilon)\|_{L^2(\Omega)} &= \left(\int_\Omega |\mathrm{Tr}(\Pi_\varepsilon\psi_\varepsilon - \psi_\varepsilon) + u_j - \mathrm{Tr}\,V_\varepsilon|^2\,dx\right)^{1/2}\\ &= \left(1 + o\left(\sqrt{\mathrm{Cap}^s_\Omega(K_\varepsilon,u_j)}\right)\right)^{1/2}\\ &= 1 + o\left(\sqrt{\mathrm{Cap}^s_\Omega(K_\varepsilon,u_j)}\right) \qquad \text{as } \varepsilon\to0^+.\end{aligned} \tag{4.20}$$

Let

$$\Psi_\varepsilon = \frac{\Pi_\varepsilon\psi_\varepsilon}{\|\mathrm{Tr}(\Pi_\varepsilon\psi_\varepsilon)\|_{L^2(\Omega)}} \in \mathcal H_\varepsilon.$$

Noticing that

$$\Psi_\varepsilon - \psi_\varepsilon = \frac{\Pi_\varepsilon\psi_\varepsilon - \psi_\varepsilon + (1 - \|\mathrm{Tr}(\Pi_\varepsilon\psi_\varepsilon)\|_{L^2(\Omega)})\psi_\varepsilon}{\|\mathrm{Tr}(\Pi_\varepsilon\psi_\varepsilon)\|_{L^2(\Omega)}}$$

and using (4.20), (4.15) and (2.14), we deduce that

$$\|\mathrm{Tr}(\Psi_\varepsilon - \psi_\varepsilon)\|_{L^2(\Omega)} = o\left(\sqrt{\mathrm{Cap}^s_\Omega(K_\varepsilon,u_j)}\right) \qquad \text{as } \varepsilon\to0^+. \tag{4.21}$$

Similarly,

$$\|\mathrm{Tr}(\Psi_\varepsilon - U_j)\|_{L^2(\Omega)} = \|\mathrm{Tr}(\Psi_\varepsilon - \psi_\varepsilon - V_\varepsilon)\|_{L^2(\Omega)} = o\left(\sqrt{\mathrm{Cap}^s_\Omega(K_\varepsilon,u_j)}\right) \qquad \text{as } \varepsilon\to0^+. \tag{4.22}$$

We also remark, using the equation satisfied by V_ε (see (2.23)), the fact that $\psi_\varepsilon\in\mathcal H_\varepsilon$ and the equation satisfied by U_j, that

$$\mathrm{Cap}^s_\Omega(K_\varepsilon,u_j) = \frac1{\kappa_s}q(V_\varepsilon,V_\varepsilon) = \frac1{\kappa_s}q(V_\varepsilon,U_j-\psi_\varepsilon) = \frac1{\kappa_s}q(V_\varepsilon,U_j) = \lambda^s_j(\Omega)\int_\Omega u_j\,\mathrm{Tr}\,V_\varepsilon\,dx. \tag{4.23}$$

Noticing that Ψ_ε is an eigenfunction associated to $\lambda_{j,\varepsilon}$, relation (4.8) with $\phi=\Psi_\varepsilon$ provides

$$(\lambda_{j,\varepsilon} - \lambda^s_j(\Omega))\int_\Omega \mathrm{Tr}\,\Psi_\varepsilon\,\mathrm{Tr}\,\psi_\varepsilon\,dx = \lambda^s_j(\Omega)\int_\Omega \mathrm{Tr}\,\Psi_\varepsilon\,\mathrm{Tr}\,V_\varepsilon\,dx.$$

Therefore, by (4.22), (4.23) and Lemma 4.1, we have

$$\begin{aligned}(\lambda_{j,\varepsilon} - \lambda^s_j(\Omega))\int_\Omega \mathrm{Tr}\,\Psi_\varepsilon\,\mathrm{Tr}\,\psi_\varepsilon\,dx &= \lambda^s_j(\Omega)\int_\Omega u_j\,\mathrm{Tr}\,V_\varepsilon\,dx + \lambda^s_j(\Omega)\int_\Omega \mathrm{Tr}(\Psi_\varepsilon - U_j)\,\mathrm{Tr}\,V_\varepsilon\,dx\\ &= \mathrm{Cap}^s_\Omega(K_\varepsilon,u_j) + o(\mathrm{Cap}^s_\Omega(K_\varepsilon,u_j))\end{aligned}$$

as $\varepsilon \to 0^+$. As, by (4.21),

$$\int_\Omega \operatorname{Tr}\Psi_\varepsilon \operatorname{Tr}\psi_\varepsilon\,dx = \int_\Omega |\operatorname{Tr}\Psi_\varepsilon|^2\,dx + \int_\Omega \operatorname{Tr}\Psi_\varepsilon \operatorname{Tr}(\psi_\varepsilon - \Psi_\varepsilon)\,dx = 1 + o\left(\sqrt{\operatorname{Cap}^s_\Omega(K_\varepsilon, u_j)}\right),$$

we have concluded the proof. □

5. Asymptotics of capacities for scaling of a given set

In this section we will assume that $0 \in \Omega$. In order to prove Theorem 1.6, we first establish the following preliminary result.

Lemma 5.1. *Let $K \subset \Omega$ be compact and Ω' be an open set such that $K \subset \Omega' \Subset \Omega$. Let $f \in H^s_{\mathrm{loc}}(\Omega)$ and $(f_n)_{n\geq 1} \subset H^s_{\mathrm{loc}}(\Omega)$ be such that $f_n \to f$ as $n \to +\infty$ in $H^s(\Omega')$. Then*

$$V_{\Omega,K,f_n} \to V_{\Omega,K,f} \quad \text{in } \mathcal{D}^{1,2}(\mathbb{R}^{N+1}; t^{1-2s})$$

and

$$\lim_{n\to+\infty} \operatorname{Cap}^s_\Omega(K, f_n) = \operatorname{Cap}^s_\Omega(K, f).$$

Proof. Let $\tilde\eta_K \in C^\infty(\mathbb{R}^{N+1}_+ \cup \Omega')$ be such that $\tilde\eta_K \equiv 1$ in a neighborhood of K. Therefore $\tilde\eta_K f_n \to \tilde\eta_K f$ in $\mathcal{D}^{s,2}(\Omega')$ and, consequently, $\mathcal{H}(\tilde\eta_K f_n) \to \mathcal{H}(\tilde\eta_K f)$ in $\mathcal{D}^{1,2}(\mathbb{R}^{N+1}; t^{1-2s})$, where $\mathcal{H}$ is the extension operator introduced in (2.2).

Furthermore both $V_{\Omega,K,f_n} - \mathcal{H}(\tilde\eta_K f_n)$ and $V_{\Omega,K,f} - \mathcal{H}(\tilde\eta_K f)$ belong to $\mathcal{D}^{1,2}_{\Omega^c\cup K}(\mathbb{R}^{N+1}; t^{1-2s})$. Hence

$$q(V_{\Omega,K,f_n} - V_{\Omega,K,f}, V_{\Omega,K,f_n} - \mathcal{H}(\tilde\eta_K f_n)) = q(V_{\Omega,K,f_n} - V_{\Omega,K,f}, V_{\Omega,K,f} - \mathcal{H}(\tilde\eta_K f)) = 0,$$

so that, using the Hölder inequality,

$$\begin{aligned}\|V_{\Omega,K,f_n} - V_{\Omega,K,f}\|^2_{\mathcal{D}^{1,2}(\mathbb{R}^{N+1};t^{1-2s})} &= q(V_{\Omega,K,f_n} - V_{\Omega,K,f}, \mathcal{H}(\tilde\eta_K f_n) - \mathcal{H}(\tilde\eta_K f))\\ &\leq \|V_{\Omega,K,f_n} - V_{\Omega,K,f}\|_{\mathcal{D}^{1,2}(\mathbb{R}^{N+1};t^{1-2s})}\|\mathcal{H}(\tilde\eta_K f_n) - \mathcal{H}(\tilde\eta_K f)\|_{\mathcal{D}^{1,2}(\mathbb{R}^{N+1};t^{1-2s})}.\end{aligned}$$

Then

$$\lim_{n\to+\infty} \|V_{\Omega,K,f_n} - V_{\Omega,K,f}\|_{\mathcal{D}^{1,2}(\mathbb{R}^{N+1};t^{1-2s})} = 0,$$

concluding the proof. □

Proof of Theorem 1.6. For every $\varepsilon > 0$, let $V_{\Omega,K_\varepsilon,u_j}$ be the function that achieves $\operatorname{Cap}^s_\Omega(K_\varepsilon, u_j)$ as in (2.21) and let

$$\tilde V_\varepsilon(z) = \varepsilon^{-\gamma_s} V_{\Omega,K_\varepsilon,u_j}(\varepsilon z), \quad z \in \mathbb{R}^{N+1}_+.$$

Let $U_j = \mathcal{H}(u_j) \in \mathcal{D}^{1,2}_{\Omega^c}(\mathbb{R}^{N+1}_+; t^{1-2s})$ be the extension of u_j as in (2.2) and define $\tilde U_\varepsilon(z) := \varepsilon^{-\gamma_s} U_j(\varepsilon z)$ as in Section 2.2.

We notice that $\tilde V_\varepsilon \in \mathcal{D}^{1,2}_{(\Omega/\varepsilon)^c}(\mathbb{R}^{N+1}_+; t^{1-2s})$, $\tilde V_\varepsilon - \tilde U_\varepsilon \in \mathcal{D}^{1,2}_{(\Omega/\varepsilon)^c\cup K}(\mathbb{R}^{N+1}_+; t^{1-2s})$ and

$$q(\tilde V_\varepsilon, \phi) = 0 \quad \text{for all } \phi \in \mathcal{D}^{1,2}_{(\Omega/\varepsilon)^c\cup K}(\mathbb{R}^{N+1}_+; t^{1-2s}). \tag{5.1}$$

In particular,

$$\|\tilde V_\varepsilon\|^2_{\mathcal{D}^{1,2}(\mathbb{R}^{N+1};t^{1-2s})} = \kappa_s \operatorname{Cap}^s_{\Omega/\varepsilon}(K, \tilde u_\varepsilon), \tag{5.2}$$

where $\tilde u_\varepsilon = \operatorname{Tr}\tilde U_\varepsilon$.

Let $r_0 > 0$ be such that $K \subset B'_{r_0} = \{x \in \mathbb{R}^N : |x| < r_0\}$. For ε sufficiently small, we have that

$$B'_{r_0} \subset \frac{\Omega}{\varepsilon},$$

so that $\mathcal{D}^{1,2}_{(B'_{r_0})^c\cup K} \subseteq \mathcal{D}^{1,2}_{(\Omega/\varepsilon)^c\cup K}$ and, in turn,

$$\operatorname{Cap}^s_{\Omega/\varepsilon}(K, \tilde u_\varepsilon) \leq \operatorname{Cap}^s_{B'_{r_0}}(K, \tilde u_\varepsilon) \to \operatorname{Cap}^s_{B'_{r_0}}(K, \hat\psi) \tag{5.3}$$

as $\varepsilon \to 0^+$, where in the last step we used (1.9) and Lemma 5.1. Combining (5.2) and (5.3), we deduce that the family $\{\tilde{V}_\varepsilon\}_{\varepsilon>0}$ is bounded in the reflexive space $\mathcal{D}^{1,2}(\mathbb{R}^{N+1}_+; t^{1-2s})$. Then there exist a sequence $\varepsilon_n \to 0^+$ and $\tilde{V} \in \mathcal{D}^{1,2}(\mathbb{R}^{N+1}_+; t^{1-2s})$ such that

$$\tilde{V}_{\varepsilon_n} \rightharpoonup \tilde{V} \quad \text{weakly in } \mathcal{D}^{1,2}(\mathbb{R}^{N+1}_+; t^{1-2s}) \tag{5.4}$$

as $n \to +\infty$.

Let $\tilde{\eta}_K \in C^\infty_c(B^+_R)$ for some $R > 0$, be such that $\tilde{\eta}_k = 1$ on a neighborhood of K. Then $\tilde{V}_{\varepsilon_n} - \tilde{\eta}_K \tilde{U}_{\varepsilon_n} \in \mathcal{D}^{1,2}_K(\mathbb{R}^{N+1}_+; t^{1-2s}) = \{U \in \mathcal{D}^{1,2}(\mathbb{R}^{N+1}_+; t^{1-2s}) : \operatorname{Tr} U \in \mathcal{D}^{s,2}(\mathbb{R}^N \setminus K)\}$. Moreover, by (2.10) we have that

$$\tilde{\eta}_K \tilde{U}_\varepsilon \to \tilde{\eta}_K \tilde{\psi} \quad \text{in } \mathcal{D}^{1,2}(\mathbb{R}^{N+1}_+; t^{1-2s}). \tag{5.5}$$

Since $\mathcal{D}^{1,2}_K(\mathbb{R}^{N+1}_+; t^{1-2s})$ is closed in $\mathcal{D}^{1,2}(\mathbb{R}^{N+1}_+; t^{1-2s})$ (in the strong topology and then, being a subspace, in the weak topology), by (5.4) we conclude that $\tilde{V} - \tilde{\eta}_k \tilde{\psi} \in \mathcal{D}^{1,2}_K(\mathbb{R}^{N+1}_+; t^{1-2s})$. Moreover, relations (5.1) and (5.4) provide

$$q(\tilde{V}, \phi) = 0 \quad \text{for all } \phi \in C^\infty_c(\mathbb{R}^{N+1}_+ \cup (\mathbb{R}^N \setminus K)),$$

so that, by density,

$$q(\tilde{V}, \phi) = 0 \quad \text{for all } \phi \in \mathcal{D}^{1,2}_K(\mathbb{R}^{N+1}_+; t^{1-2s}).$$

In particular,

$$\|\tilde{V}\|^2_{\mathcal{D}^{1,2}(\mathbb{R}^{N+1}_+; t^{1-2s})} = \kappa_s \operatorname{Cap}^s_{\mathbb{R}^N}(K, \hat{\psi}) = q(\tilde{V}, \tilde{\eta}_K \tilde{\psi}). \tag{5.6}$$

Similarly, since $\tilde{V}_\varepsilon - \tilde{\eta}_K \tilde{U}_\varepsilon \in \mathcal{D}^{1,2}_{(\Omega/\varepsilon)^c \cup K}(\mathbb{R}^{N+1}_+; t^{1-2s})$ for $\varepsilon > 0$ sufficiently small, using also relations (5.1), (5.4), (5.5) and (5.6), we obtain

$$\|\tilde{V}_{\varepsilon_n}\|^2_{\mathcal{D}^{1,2}(\mathbb{R}^{N+1}_+; t^{1-2s})} = q(\tilde{V}_{\varepsilon_n}, \tilde{\eta}_K \tilde{U}_{\varepsilon_n}) \to q(\tilde{V}, \tilde{\eta}_K \tilde{\psi}) = \kappa_s \operatorname{Cap}^s_{\mathbb{R}^N}(K, \hat{\psi}), \tag{5.7}$$

as $n \to +\infty$. By the Urysohn's subsequence principle we conclude that the above convergence holds as $\varepsilon \to 0^+$ and not only along the sequence ε_n. To conclude the proof it suffices to notice that, by a change of variables,

$$\operatorname{Cap}^s_\Omega(K_\varepsilon, u_j) = \frac{1}{\kappa_s} \|V_{\Omega, K_\varepsilon, u_j}\|^2_{\mathcal{D}^{1,2}(\mathbb{R}^{N+1}_+; t^{1-2s})} = \frac{1}{\kappa_s} \varepsilon^{N+2(\gamma_s - s)} \|\tilde{V}_\varepsilon\|^2_{\mathcal{D}^{1,2}(\mathbb{R}^{N+1}_+; t^{1-2s})}$$

and to replace (5.7) into the previous expression. □

Proof of Theorem 1.7. The family of sets $\{\varepsilon K\}_{\varepsilon>0}$ concentrates to the compact set $\{0\}$, which satisfies $\operatorname{Cap}^s_\Omega(\{0\}) = 0$ by Example 2.5, so that Theorem 1.5 applies in our situation. By combining it with Theorem 1.6, we obtain the stated result. □

Proof of Corollary 1.8. Let V_K be the function that achieves the infimum in (2.9) with $u = \hat{\psi}$ and $\Omega = \mathbb{R}^N$, so that $\operatorname{Cap}^s_{\mathbb{R}^N}(K, \hat{\psi}) = \frac{1}{\kappa_s} q(V_K, V_K)$. The Hardy-trace inequality (2.13) provides

$$\begin{aligned}\operatorname{Cap}^s_{\mathbb{R}^N}(K, \hat{\psi}) = \frac{1}{\kappa_s} q(V_K, V_K) &\geq \Lambda_{N,s} \int_{\mathbb{R}^N} \frac{|\operatorname{Tr} V_K|^2}{|x|^{2s}} \, dx \\ &\geq \Lambda_{N,s} \int_K \frac{|\operatorname{Tr} V_K|^2}{|x|^{2s}} \, dx = \Lambda_{N,s} \int_K |x|^{-2s} |\hat{\psi}(x)|^2 \, dx.\end{aligned}$$

If, by contradiction, $\operatorname{Cap}^s_{\mathbb{R}^N}(K, \hat{\psi}) = 0$ the above inequality would imply $\hat{\psi} = 0$ a.e. in K. Since the N-dimensional Lebesgue measure of K is strictly positive and $\hat{\psi}$ weakly solves $(-\Delta)^s \hat{\psi} = 0$ in $\mathbb{R}^N$, the Unique Continuation Principle from sets of positive measure proved in [15, Theorem 1.4] would imply that $\hat{\psi} \equiv 0$ in $\mathbb{R}^N$, giving rise to a contradiction in view of Remark 2.2. □

Appendix A. Boundedness of eigenfunctions

To prove boundedness of eigenfunctions we need the following Sobolev-trace inequality which follows from combination of Theorem 2.3 (i) and continuity of the trace map (2.1) (see also [6, Theorem 2.1]): there exists a positive constant $\tau_{N,s} > 0$ such that

$$\tau_{N,s}\| \operatorname{Tr} W\|^2_{L^{2^*(s)}(\mathbb{R}^N)} \le \int_{\mathbb{R}^{N+1}_+} t^{1-2s}|\nabla W|^2\,dt\,dx, \quad \text{for all } W \in \mathcal{D}^{1,2}(\mathbb{R}^{N+1}_+; t^{1-2s}), \tag{A.1}$$

where $2^*(s)$ is defined in (2.11). In the following lemma we prove that the extensions of eigenfunctions of (1.3) are bounded in $\mathbb{R}^{N+1}_+$.

Lemma A.1. *Let $\Omega \subset \mathbb{R}^N$, $N \ge 1$, be a bounded open set and $s \in (0, \min\{1, N/2\})$. Let $\alpha \in \mathbb{R}$ and $W \in \mathcal{D}^{1,2}_{\Omega^c}(\mathbb{R}^{N+1}_+; t^{1-2s})$ be a weak solution to*

$$\begin{cases} -\operatorname{div}(t^{1-2s}\nabla W) = 0, & \text{in } \mathbb{R}^{N+1}_+,\\ \lim_{t\to 0^+}\left(-t^{1-2s}\partial_t W\right) = \alpha W, & \text{in } \Omega \times \{0\},\\ W = 0, & \text{in } (\mathbb{R}^N \setminus \Omega) \times \{0\}, \end{cases} \tag{A.2}$$

in the sense that

$$\int_{\mathbb{R}^{N+1}_+} t^{1-2s}\nabla W \cdot \nabla \phi\,dx\,dt = \alpha \int_\Omega \operatorname{Tr} W \operatorname{Tr}\phi\,dx \tag{A.3}$$

for every $\phi \in \mathcal{D}^{1,2}_{\Omega^c}(\mathbb{R}^{N+1}_+; t^{1-2s})$. Then $W \in L^\infty(\mathbb{R}^{N+1}_+)$ and $\operatorname{Tr} W \in L^\infty(\Omega)$.

Proof. The fact that $\operatorname{Tr} W \in L^\infty(\Omega)$ can be found in [9, Theorem 3.1, Remark 3.2], see also [17]. Let us prove the statement about its extension. From the Poisson formula for problem (A.2) given in [11] we have that, for some constant $C_{N,s}$,

$$W(x,t) = C_{N,s}\int_{\mathbb{R}^N} \frac{t^{2s}}{(|x-\xi|^2+t^2)^{\frac{N+2s}{2}}}\operatorname{Tr}(W)(\xi)\,d\xi \quad \text{for all } (x,t) \in \mathbb{R}^{N+1}_+,$$

hence

$$\begin{aligned} |W(x,t)| &\le \|\operatorname{Tr} W\|_{L^\infty(\Omega)}|C_{N,s}|\int_{\mathbb{R}^N} \frac{t^{2s}}{(|x-\xi|^2+t^2)^{\frac{N+2s}{2}}}\,d\xi\\ &= \|\operatorname{Tr} W\|_{L^\infty(\Omega)}|C_{N,s}|\int_{\mathbb{R}^N} \frac{t^{2s}}{(|\xi|^2+t^2)^{\frac{N+2s}{2}}}\,d\xi\\ &= \|\operatorname{Tr} W\|_{L^\infty(\Omega)}|C_{N,s}|\int_{\mathbb{R}^N} \frac{d\xi'}{(|\xi'|^2+1)^{\frac{N+2s}{2}}}\,d\xi' \end{aligned}$$

for all $(x,t) \in \mathbb{R}^{N+1}_+$, thus implying that $W \in L^\infty(\mathbb{R}^{N+1}_+)$ and completing the proof. □

Appendix B. Fractional convergence of sets in the sense of Mosco

We give the following definition which is the analogue of the standard sets convergence in the sense of Mosco ([23]).

Definition B.1. Let $\Omega \subset \mathbb{R}^N$ be a bounded open set. Let $\{K_\varepsilon\}_{\varepsilon>0}$ be a family of compact sets contained in Ω. We say that $\{\Omega \setminus K_\varepsilon\}_{\varepsilon>0}$ *converges to* $\Omega \setminus K$ *in the fractional sense of Mosco* if the following two properties hold:

(i) the weak limit points in $\mathcal{D}^{s,2}(\mathbb{R}^N)$ of every family of functions $u_\varepsilon \in \mathcal{D}^{s,2}(\Omega \setminus K_\varepsilon)$ belong to $\mathcal{D}^{s,2}(\Omega \setminus K)$;

(ii) for every $u \in \mathcal{D}^{s,2}(\Omega \setminus K)$, there exists a family of functions $u_\varepsilon \in \mathcal{D}^{s,2}(\Omega \setminus K_\varepsilon)$ such that $u_\varepsilon \to u$ in $\mathcal{D}^{s,2}(\mathbb{R}^N)$.

In this appendix we prove that the notion of concentration introduced in Definition 1.3 implies the convergence of $\Omega \setminus K_\varepsilon$ to $\Omega \setminus K$ in the fractional sense of Mosco if $\operatorname{Cap}^s_\Omega(K) = 0$.

Lemma B.2. *Let $\Omega \subset \mathbb{R}^N$ be a bounded open set and $K \subset \Omega$ be a compact set with $\mathrm{Cap}^s_\Omega(K) = 0$. Let $\{K_\varepsilon\}_{\varepsilon>0}$ be a family of compact sets contained in Ω concentrating to K in the sense of Definition 1.3. Then $\Omega \setminus K_\varepsilon$ converges to $\Omega \setminus K$ in the fractional sense of Mosco as $\varepsilon \to 0^+$.*

Proof. We first prove that condition (i) in Definition B.1 is satisfied. Let us consider a family $\{u_\varepsilon\}_{\varepsilon>0} \subset \mathcal{D}^{s,2}(\Omega \setminus K_\varepsilon)$ such that $u_\varepsilon \rightharpoonup u$ in $\mathcal{D}^{s,2}(\mathbb{R}^N)$. We need to show that $u \in \mathcal{D}^{s,2}(\Omega \setminus K)$. Obviously $\{u_\varepsilon\}_{\varepsilon>0} \subset \mathcal{D}^{s,2}(\Omega)$ and $\mathcal{D}^{s,2}(\Omega)$ is a closed subspace of $\mathcal{D}^{s,2}(\mathbb{R}^N)$. Then $u \in \mathcal{D}^{s,2}(\Omega)$ since this space is closed in the weak topology. Furthermore, being $\mathrm{Cap}^s_\Omega(K) = 0$, Proposition 2.6 provides $\mathcal{D}^{s,2}(\Omega) = \mathcal{D}^{s,2}(\Omega \setminus K)$.

We now address item (ii) in Definition B.1. Let $u \in \mathcal{D}^{s,2}(\Omega \setminus K)$ and $U = \mathcal{H}(u)$ be its Caffarelli-Silvestre extension as in (2.2). We need to exhibit a sequence u_ε in $\mathcal{D}^{s,2}(\Omega \setminus K_\varepsilon)$ which converges to u in $\mathcal{D}^{s,2}(\mathbb{R}^N)$. We note that for every $\delta > 0$ there exists $\varphi_\delta \in C^\infty_c(\mathbb{R}^{N+1}_+ \cup \Omega)$ such that

$$\|\varphi_\delta - U\|_{\mathcal{D}^{1,2}(\mathbb{R}^{N+1}_+;t^{1-2s})} < \delta.$$

Since by assumption $\mathrm{Cap}^s_\Omega(K) = 0$, then for every $n \in \mathbb{N}$ there exists $\varepsilon_n > 0$ and $\eta_n \in C^\infty_c(\overline{\mathbb{R}^{N+1}_+})$ such that $\{\varepsilon_n\}$ is strictly decreasing to zero, $\eta_n \equiv 0$ in $\mathbb{R}^N \setminus \Omega$, $\eta_n \equiv 1$ in a neighborhood of K_ε for all $\varepsilon \in (0, \varepsilon_n)$ and

$$\int_{\mathbb{R}^{N+1}_+} t^{1-2s} |\nabla \eta_n|^2 \, dx \, dt < \frac{1}{n}.$$

Let us define $W_n := \varphi_\delta(1 - \eta_n)$. We note that $W_n \in \mathcal{D}^{1,2}_{\Omega^c \cup K_\varepsilon}(\mathbb{R}^{N+1}_+; t^{1-2s})$ for all $\varepsilon \in (0, \varepsilon_n)$. Then, using (2.15) we obtain

$$\begin{aligned}
\int_{\mathbb{R}^{N+1}_+} t^{1-2s} |\nabla W_n - \nabla \varphi_\delta|^2 \, dx \, dt &= \int_{\mathbb{R}^{N+1}_+} t^{1-2s} |\nabla(\varphi_\delta \eta_n)|^2 \, dx \, dt \\
&\le 2 \int_{\mathbb{R}^{N+1}_+} t^{1-2s} |\varphi_\delta|^2 |\nabla \eta_n|^2 \, dx \, dt + 2 \int_{\mathbb{R}^{N+1}_+} t^{1-2s} |\eta_n|^2 |\nabla \varphi_\delta|^2 \, dx \, dt \\
&\le \frac{2 \sup |\varphi_\delta|^2}{n} + 2 \left(\sup |\nabla \varphi_\delta|^2\right) \left(\sup_{z \in \mathrm{supp} \varphi_\delta} |z|^2\right) \int_{\mathbb{R}^{N+1}_+} t^{1-2s} \frac{|\eta_n|^2}{|x|^2 + t^2} \, dx \, dt \\
&\le \frac{2 \sup |\varphi_\delta|^2}{n} + \frac{8}{n(N-2s)^2} \left(\sup |\nabla \varphi_\delta|^2\right) \left(\sup_{z \in \mathrm{supp} \varphi_\delta} |z|^2\right).
\end{aligned}$$

Hence there exists n_δ such that

$$\|W_n - \varphi_\delta\|_{\mathcal{D}^{1,2}(\mathbb{R}^{N+1}_+;t^{1-2s})} < \delta \quad \text{for all } n \ge n_\delta.$$

For all $\varepsilon \in (0, \varepsilon_1)$ we let $U_\varepsilon := W_n$ where n is such that $\varepsilon_{n+1} \le \varepsilon < \varepsilon_n$. The above argument then yields that $U_\varepsilon \in \mathcal{D}^{1,2}_{\Omega^c \cup K_\varepsilon}(\mathbb{R}^{N+1}_+; t^{1-2s})$ and

$$\|U_\varepsilon - \varphi_\delta\|_{\mathcal{D}^{1,2}(\mathbb{R}^{N+1}_+;t^{1-2s})} < \delta \quad \text{for all } \varepsilon \in (0, \varepsilon_{n_\delta}).$$

Hence $\|U_\varepsilon - U\|_{\mathcal{D}^{1,2}(\mathbb{R}^{N+1}_+;t^{1-2s})} \le \|U_\varepsilon - \varphi_\delta\|_{\mathcal{D}^{1,2}(\mathbb{R}^{N+1}_+;t^{1-2s})} + \|\varphi_\delta - U\|_{\mathcal{D}^{1,2}(\mathbb{R}^{N+1}_+;t^{1-2s})} < 2\delta$ for all $\varepsilon \in (0, \varepsilon_{n_\delta})$.

We conclude that $U_\varepsilon \to U$ in $\mathcal{D}^{1,2}(\mathbb{R}^{N+1}_+; t^{1-2s})$ and therefore $u_\varepsilon = \mathrm{Tr}\, U_\varepsilon \in \mathcal{D}^{s,2}(\Omega \setminus K_\varepsilon)$ converges to $u = \mathrm{Tr}\, U$ in $\mathcal{D}^{s,2}(\mathbb{R}^N)$ by continuity of the trace map (2.1). □

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