To appear in the *Journal of Difference Equations and Applications*
Vol. 00, No. 00, Month 20XX, 2–11

A multiscale time model with piecewise constant argument for a boundedly rational monopolist

F. Cavalli[a∗] and A. Naimzada[b†]

[a,b] *Dept. of Economics, Management and Statistics, University of Milano-Bicocca, U6 Building, Piazza dell'Ateneo Nuovo 1, 20126 Milano, Italy.*

()

1. Introduction

Observing that it is not realistic for a monopolist to have complete knowledge of the market and to be endowed with computational skills to solve the profits' optimization problem, a research strand investigated possible modelling approaches for "ignorant monopolists" (Clower [9]), focusing on simple learning mechanisms. In the studied decisional mechanisms, the agent, endowed with reduced rationality, tries to modify output levels toward the profit maximizing output through rule-of-thumb heuristics (see for example [4, 9]), so that the profit-maximizing equilibrium is not reached in one shot, but on the basis of repeated choices, following a dynamical adaptive process (see Colisnk [10]). These mechanisms, which are usually based on gradient rules, just require a local knowledge of the demand function, which can be for example obtained through periodical market experiments and surveys. More precisely, the learning activity aims at obtaining an estimate of the marginal profit. The monopolist modifies the production level depending on the realized marginal profit, so that increasing (respectively decreasing) profits lead to an increase (respectively decrease) of the production level, regulated by a reactivity parameter. Such mechanism, also known as myopic adjustment process, is widely investigated in the literature, see for example [12, 13, 29].

Starting from the work by Puu [27], the related literature of the last twenty years can be roughly divided into two main families, depending on the approach based on continuous differential rather than discrete difference equations.

In discrete difference models (see for instance [2, 3, 8, 14, 19–21, 25, 26, 28]) the agent is assumed to collect data about the economic environment and to choose the output level at each discrete time. In particular, in [14, 21], authors assume that the monopolistic firm, to decide the production level, uses an average of the current and of one or two past output values, suitably weighted. For a particular weight choice, for which the current output level is not taken into account, this means that the learning process has effect on the choice of the production level only after one or two lags. Conversely, in differential models, the time scale is continuous and both information about the profitability of the production choices and the decision

∗Corresponding author. Email: fausto.cavalli@unimib.it
†Email: ahmad.naimzada@unimib.it

about the output level are continuously updated. In [18–20, 22–24] Matsumoto and Szidarowski propose ways to introduce delays in the learning process and study the destabilizing effect on the resulting dynamics. However, we notice that, also when lags are considered, both the learning activity and the production decision occur at each discrete/continuous time t.

In our contribution, in which we generalize the technique and the example considered in [7], we propose and study a new modelling approach to take the best of both discrete and continuous monopoly modelling. The goal is to encompass into the model the more realistic assumption that the timing of the learning activity and of the output production activity are actually different. In fact, it is more reasonable to suppose that the costly data collection and organization process periodically occurs (for instance monthly or yearly), and it remains the basis of output decisions until the next learning activity, while indeed the production level is more frequently updated. To this end, we assume a continuous time scale for the production level and a discrete one for the learning activity, considering a multiscale time model. From the analytical viewpoint, this can be achieved using piecewise constant argument differential equations (DEPCA). Since the early 1980's, such kind of dynamical systems has been used to describe several real world phenomena, in engineering, physics, chemistry and biomedicine contexts (see for example [1, 6, 11, 30] and the references therein). Even if such modelling approach rests upon continuous differential equations, it encompasses the presence of an underlying discrete time level, and the resulting dynamical behavior can be studied by considering a nonlinear difference equation which describes the evolution of the output level at the discrete times of the learning activity. As noticed in [15], DEPCA provides a rich source for obtaining nonlinear difference equations of interesting types.

The model presented in this paper represents, to the best of our knowledge, the first application of piecewise constant argument differential equations to economic modelling, together with [7]. The resulting modelling approach belongs to the wide family of the so-called *hybrid models*, in which both continuous and discrete time levels are present. The application of hybrid models to economic problems, which are widely used in several scientific contexts, is quite recent and to the best of our knowledge we can only mention the contribution by Lamantia and Radi [16], in which a continuous time resource growth coexists with impulsive, discrete time, changes of strategies. After deriving the model for both a general economic context and agent's reactivity, we prove conditions under which the steady state, which coincides with the profit maximizing output level, is stable. We show that differently from the classical continuous argument differential model without lags, equilibrium is only conditionally stable, for any parameters' choices. The equilibrium stability depends on the agent's reaction function, on the sensitivity of the marginal profits at the equilibrium q^* and on the size σ of the time interval between two consecutive learning activities. As usual, a reactive behavior is destabilizing, as well as an increase in the marginal profit sensitivity $\pi''(q^*)$. More noteworthy is that σ is destabilizing, namely stability interval is increasingly smaller as the learning activity is less frequently carried out. We remark that, with the introduction of the discrete time scale for the learning process, the equilibrium is never unconditionally stable, independently of σ, differently from the continuous argument differential models with lags, in which there exists a threshold for the considered lag under which dynamics are unconditionally stable. Finally we show that instability occurs through a flip bifurcation and, focusing on a linear monopoly model, that the subsequent cascade of period doublings can lead to chaotic dynamics.

The remainder of the paper is organized as follows: in Section 2 we present the canonical static monopoly model, in Section 3 we introduce and analyze the multiscale dynamical monopoly model, in Section 4 we consider the case of a linear monopoly model and in Section 5 we present some possible generalization of the proposed approach.

2. The Monopoly model

The setting we consider consists of a single seller, the monopolist, who faces price-taking consumers in a market over one or several periods. Having market power, he can either determine the price for the product or the supplied quantity (output level). In what follows, we shall focus on the case of a monopolist that, in order to maximize profits, sets the output level, represented by variable $q > 0$. We assume that the inverse demand curve is represented by a twice differentiable, strictly decreasing function $p : (0, \bar{q}) \to (0, +\infty)$, while costs are described by the twice differentiable, increasing function $c : (0, \bar{q}) \to [0, +\infty)$. We notice that we may also have $\bar{q} = +\infty$. As in the book by Bischi et al. [5, p. 52], we suppose that functions p and c fulfill conditions

$$qp''(q) + p'(q) \leq 0, \quad p'(q) - c''(q) < 0 \tag{1}$$

for all feasible values of q. The first condition of (1), which is usually called the decreasing marginal revenue condition, specifies that marginal revenue decreases as output increases, while the second condition of (1) imposes a lower bound on the convexity/concavity of the cost function with respect to the (negative) slope of the inverse demand function. Thanks to (1), the profit function is strictly convex and so optimization problem $\arg\max_{q>0} p(q)q - c(q)$ has at most one solution. We notice that previous problem has no solution when $p(q)q - c(q)$ is decreasing, which corresponds to an optimum production level $q \to 0$. Since we aim at studying economically interesting situations in which the optimal production level is not null, we limit to those (inverse) demand and cost functions for which the optimization problem has a solution, which equivalently corresponds to assume that $p(q)q - c(q)$ is increasing for $q \to 0^+$. The unique internal solution q^* is then obtained by imposing first and second order conditions

$$\begin{aligned} &p(q^*) + p'(q^*)q^* - c'(q^*) = 0, \\ &2p'(q^*) + p''(q^*)q^* - c''(q^*) \leq 0. \end{aligned}$$

After finding the monopolist's profit-maximizing output level, the correspondent market price level is simply provided by $p(q^*)$.

3. Piecewise constant argument monopoly model

The assumption of optimizing behavior requires that economic agents have high computational capabilities and complete information about their environment. Actually, it is more realistic to assume that, for instance, monopolist has a limited and local knowledge of the demand function ([4, 9]). The lack of information is due to the costly and time consuming nature of the collecting information activity. The

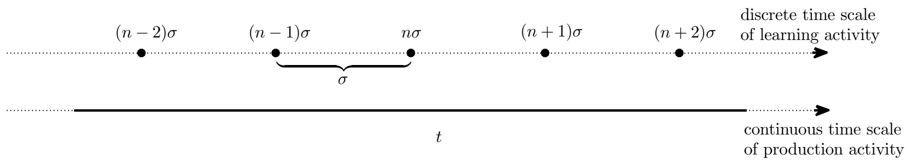

Figure 1. Time scales of learning and production activities.

monopolistic firm is hence not able to reach the equilibrium in one shot and, consequently, can only try to adapt production choices toward the profit maximizing output level. This can be achieved by means of a local estimate of the marginal profit π', obtained, for example, through market experiments ([25]). The adaptive process amounts to a learning activity of revising decisions after considering past decisions and outcomes. We suppose that the monopolist employs a rule-of-thumb as a local (or myopic) profit maximizer. In particular, he looks at how a variation of quantity affects the variation of profits. A positive (negative) variation of profits will induce the monopolist to change the quantity in the same (opposite) direction from that of the preceding period. If the profits are stationary, the production is not changed. We suppose that the time evolution is continuous and that the firm takes output decisions continuously. As we said, the firm must carry out a learning activity of collection and organization of information and data about the environment in which it acts, in order to produce a formal representation of the profitability situation (marginal profit). Such demanding learning process is usually performed only periodically, so it is not realistic to suppose that it takes place continuously, with the same timing of the production activity. On the contrary, we can assume that it regularly takes place, for example at discrete times $n = 0, \sigma, 2\sigma, \ldots$, so that output decisions, during each time period $n\sigma \le t < (n+1)\sigma, n = 0, 1, 2\ldots$, are based on the representation given by the learning activity carried out at time $t = n\sigma$. We notice that $\sigma > 0$ is the (constant) time interval between two consecutive learning processes. For example, if $\sigma = 1$, the learning activity occurs at each integer value of t. A sketch of the two time scales is reported in Figure 1.

In other words, the decisional process has two different temporal scales, a first one, in which output decisions are taken and updated and which is suitably described in terms of a continuous time evolution, and a second one, in which the learning activity takes place and which is better modeled through a discrete time evolution. The whole decisional mechanism can be represented through the following nonlinear equation with piecewise constant argument

$$\frac{dq(t)}{dt} = k(q(t))\frac{d\pi}{dq}\left(\left[\frac{t}{\sigma}\right]\sigma\right),\ t \ge 0,\ q(0) = q_0 > 0, \tag{2}$$

where $[\cdot]$ denotes the integer part of its argument and $k : [0, +\infty) \to [0, +\infty)$ is a positive and increasing function with $k(0) = 0$, which represents the extent of production variation of the monopolist following a certain profit signal and depends on the current monopolistic firm dimension, given by the production volume. Problem (2) is studied in the following proposition.

Proposition 3.1. *If k is locally Lipschitz and sublinear for $q \to +\infty$, then differential equation with piecewise argument* (2) *has a unique strictly positive solution* *given, for* $t \in [n\sigma, (n+1)\sigma)$, $n \in \mathbb{N}$ *by*

$$q(t) = F^{-1}\Big(F(q(n\sigma)) + \pi'(q(n\sigma))(t - n\sigma)\Big), \tag{3}$$

with $q(0) = q_0 > 0$ *and* F *is an antiderivative of* $1/k(q)$.

Proof. When $\sigma = 1$, equation (2) belongs to the family of differential equations with piecewise argument $x'(t) = f(x(t), x[t])$, studied for example in [30, ch. 2]. We notice that (2) satisfies assumptions of Theorem 2.1 in [30, p. 84], since, for each μ, $x'(t) = f(x(t), \mu) = k(x(t))\pi'(\mu)$ has a unique solution on $[0, +\infty)$ thanks to the assumptions on k. With minor adjustments, such theorem still holds for $\sigma \neq 1$, too. Then we have that there exists a unique continuous function, differentiable for each $t \in [0, +\infty) \setminus n\sigma$ and with one-sided derivatives for $t = n\sigma$ which satisfies (2) on each interval $[n\sigma, (n+1)\sigma)$ with $n \in \mathbb{N}$. Such solution, thanks to the uniqueness and since for $q_0 = 0$ we would have the null solution, is strictly positive since $q_0 > 0$. We notice that, since $k(q)$ is strictly positive on $(0, +\infty)$, then F is strictly increasing and invertible. It is easy to see that (3) matches all the previous regularity requirements and satisfy (2) on $[n\sigma, (n+1)\sigma)$, so, thanks to uniqueness, it must be the solution of (2). □

Firstly, Proposition (3.1) guarantees existence and uniqueness of the solution of (2) provided that $k(q)$ is sufficiently regular. In literature, it is usually assumed either a constant or a linear expression for $k(q)$, both fulfilling the regularity assumptions of Theorem 3.1. Moreover, Proposition 3.1 also provides the explicit expression of the solution of (2). We stress that (3) provides a strictly positive trajectory for any $q(0) > 0$. Since however inverse demand function may be only defined on a bounded subset $(0, \bar{q}) \subset \mathbb{R}$, trajectories are actually feasible and economically meaningful only when the parameter configuration guarantees that $q(t) \in (0, \hat{q})$ for any t.

It is interesting to focus on the evolution of output quantities at $t = n\sigma$ for $n \in \mathbb{N}$, namely when the learning process is carried out and the new information about the marginal profit is collected. Thanks to the continuity of the solution of (2), we can write

$$q((n+1)\sigma) = \lim_{t \to (n+1)\sigma^-} = F^{-1}\Big(F(q(n\sigma)) + \pi'(q(n\sigma))\Big). \tag{4}$$

Setting $q(n\sigma) = q_n$ for $n \in \mathbb{N}$, from (4) we obtain the discrete difference equation

$$q_{n+1} = f(q_n) = F^{-1}\Big(F(q_n) + \sigma\pi'(q_n)\Big),\ n \geq 0, \tag{5}$$

which establishes a recurrence relation between on the discrete times $n\sigma$ at which the learning process occurs. Looking at the solution (3) of (2), we notice that since F^{-1} is strictly increasing, $q(t)$ is strictly increasing (resp. decreasing) on $[n\sigma, (n+1)\sigma)$ when $\pi'(q(n\sigma)) > 0$ (resp. $\pi'(q(n\sigma)) < 0$), while it is constantly equal to $q(n\sigma)$ if and only if $\pi'(q(n\sigma)) = 0$. This means that to study the steady state of (2) and its stability, we can equivalently focus on the continuous time differential equation (2) and on the discrete time difference equation (5), since no oscillating behaviors can occur inside each interval $(n\sigma, (n+1)\sigma)$.

In the following proposition we investigate discrete model (5).

Proposition 3.2. *The only steady state of* (5) *is* q^*, *which is stable provided that*

$$\sigma k(q^*)\pi''(q^*) > -2. \tag{6}$$

Proof. Steady states are obtained by setting $f(q) = q$, i.e. $q = F^{-1}(F(q) + \sigma\pi'(q))$ which, thanks to the monotony of F is equivalent to $F(q) = F(q) + \sigma\pi'(q)$ and hence to $\pi'(q) = 0$. Under the assumptions of the previous section, it has the unique solution $q = q^*$. Local asymptotic stability of a steady state x is obtained from $|f'(x)| < 1$. We have

$$f'(q^*) = \frac{F'(q^*) + \sigma\pi''(q^*)}{F'(F^{-1}(F(q^*)))} = \frac{F'(q^*) + \sigma\pi''(q^*)}{F'(q^*)} \tag{7}$$

in which we used first order condition and invertibility of F. Since $F'(q^*) = 1/k(q^*)$, the last right hand side of equations (7) becomes $1 + \sigma k(q^*)\pi''(q^*)$, which, since $k(q^*) > 0$ and thanks to the concavity of π, is smaller than 1. Imposing $f'(q^*) > -1$ allows concluding. □

Stability condition (6) clarifies the role of the agent's reactivity, of the sensitivity of the marginal profit at the equilibrium (represented by $\pi''(q^*)$) and of the learning activity. Since π is concave, the stability constraint actually imposes an upper bound on the size of $\sigma, k(q^*)$ and $\pi''(q^*)$. Firstly we notice that the steady state is only conditionally stable for any parameter configurations, in particular for any $\sigma > 0$. This is a different behavior with respect to continuous argument models with lags, in which, as shown for example in [20], there exists a lower bound on time lags under which the dynamics are unconditionally stable. We notice that, letting $\sigma > 0$, namely letting the discrete time scale of the learning activity become continuous, we formally recover the unconditional stability of the continuous argument model without lags. This confirms that the conditional stability of the steady state is induced by the presence of the two time scales. Actually, the presence of the discrete time scale also affects the size of the stability interval, as it is easy to see that the steady state is stable provided that

$$\sigma < \sigma^* = -\frac{2}{k(q^*)\pi''(q^*)}.$$

Finally, as well established in the literature, the dynamics becomes more unstable as agent's reactivity increases, as well as the sensitivity of the marginal profit becomes larger. We remark that when (6) is violated, instability generally occurs through a period doubling bifurcation, provided that supplementary second order conditions are fulfilled (see for example [17, p. 92]).

Typically, agent's reactivity is represented by an exogenous positive constant or through the endogenous linear mechanism $k(q(t)) = vq(t)$, so that the the output change rate is proportional to the size of the production, regulated by $v > 0$ which gives the relative speed of adjustment. In this latter case, we can provide a more explicit expression for the solution of (2).

Corollary 3.3. *Assuming* $k(q(t)) = vq(t)$, *differential equation with piecewise ar-*

gument (2) is solved by

$$q(t) = q(n\sigma)\exp\Big(v\pi'(q(n\sigma))(t-n\sigma)\Big),\ t\in[n\sigma,(n+1)\sigma),\ n\in\mathbb{N}, \tag{8}$$

where $q(0) = q_0$*. Setting* $q(n\sigma) = q_n$ *for* $n \in \mathbb{N}$*, from (8) we have the discrete difference equation*

$$q_{n+1} = q_n \exp\Big(v\pi'(q_n)\Big),\ n\geq 0,$$

which has the only steady state q^**, which is stable provided that* $v\sigma q^*\pi''(q^*) > -2$.

4. The case of linear monopoly

In this section we focus on the particular example of a monopoly model characterized by linear demand, cost and agent's reactivity functions, i.e. $k(q) = vq$ and

$$p(q) = a - bq,\ a > 0, b > 0,$$

and

$$c(q) = dq,\ d > 0.$$

Profit function is $\pi(q) = (a-d)q - bq^2$, so that the first order condition is $a - d - 2bq = 0$ and second order condition $2b > 0$ is always satisfied. Assuming that $a > d$, the maximizing output level and the corresponding optimal price are given by the positive values $q^* = (a-d)/(2b)$, and $p^* = a - (a-d)/2$. Equation (2) becomes

$$\frac{dq(t)}{dt} = vq(t)\left(a - d - 2bq\left[\frac{q}{\sigma}\right]\sigma\right), t\geq 0, q(0) > 0, \tag{9}$$

to which correspond the nonlinear difference equation

$$q_{n+1} = q_n \exp\left[v\sigma(a - d - 2(b+e)q_n)\right]. \tag{10}$$

We want to compare the classical smooth argument differential model for the boundedly rational monopolist with the present one. In the classical framework, the differential equation which governs the dynamics is

$$\frac{dq(t)}{dt} = vq(t)\left[a - d - 2bq(t)\right],\ t\geq 0,\ q(0) > 0, \tag{11}$$

which has the unique solution

$$q(t) = \frac{q(0)\,(a-b)\exp\left(v\,(a-b)\,t\right)}{a - b - 2q(0)b + 2q(0)b\exp\left(v\,(a-b)\,t\right)},\ t\geq 0. \tag{12}$$

In order to compare the classical and the piecewise constant argument models, from (12) we derive a recurrence relation defined for q at discrete times $n\sigma, n\in\mathbb{N}$, which,

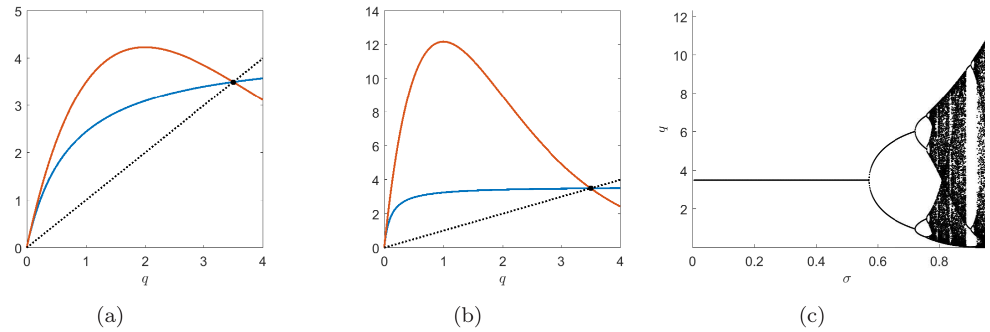

Figure 2. (a,b) Plots of maps (10) (red color) and (13) (blue color) for different values of σ. (c) Bifurcation diagram for (10) on varying σ

setting $q(n\sigma) = q_n$, results

$$q_{n+1} = \frac{q_n (a-b) \exp(v\sigma(a-b))}{a - b - 2q_n b + 2q_n b \exp(v\sigma(a-b))}. \tag{13}$$

In Figures 2 (a) and (b) we report maps (10) and (13) for $a = v = 1, b = d = 0.5$ and for two different time interval sizes $\sigma = 0.5$ and $\sigma = 1$. As we can notice, in the continuous argument model the steady state q^* is stable for both parameter configurations, while in the piecewise argument model it is stable for $\sigma = 0.5$ and unstable for $\sigma = 1$. Finally in Figure 2 (c) we report a bifurcation diagram on varying σ, in which it is evident how increasing the time interval between two consecutive learning processes can induce instability though a cascade of period doubling bifurcations which lead to chaos.

5. Conclusions

We presented a dynamical model of a boundedly rational monopolistic firm in which the production decisions and the learning activity take place at different time scales. We showed that if production levels are updated continuously, while the collection of data about the economic environment is only periodically carried out, instabilities can arise in the resulting dynamics. The proposed approach is based on a piecewise continuous differential equation which leads to a nonlinear difference equation, describing the evolution of output levels at the discrete time level of the learning activity. The resulting model exhibits a multiscale time structure in which the discrete time scale, at which the learning process takes place, is the cause of the possible destabilization of the equilibrium. We showed, considering generic demand, cost and agent's reactivity functions, that increasing the size of the time interval between two consecutive learning processes can lead the dynamics to become unstable, and, considering a linear monopoly, we showed that this can occur through a cascade of period doublings leading to chaos. Future generalizations of the proposed approach include the study of the effect of endogenizing the times at which the learning activity occurs. In this way, the agent is allowed to decide, on the basis of some fitness measure, when the learning activity has to be carried out. Moreover, we aim to investigate the possible applications of the proposed approach and of piecewise continuous argument differential equations to oligopolistic competitions and, more generally, to game theory.

References

[1] M. Akhmet, *Nonlinear Hybrid Continuous/Discrete-Time Models*, Atlantis Studies in Mathematics for Engineering and Science, Vol. 8, Atlantis press, 2011.

[2] B. Al-Hdaibat, W. Govaerts, and N. Neirynck, *On periodic and chaotic behavior in a two-dimensional monopoly model*, Chaos Solitons Fractals 70 (2015), pp. 27–37.

[3] S. Askar, *On complex dynamics of monopoly market*, Econom. Model. 31 (2013), pp. 586–589.

[4] W.J. Baumol and R.E. Quandt, *Rules of thumb and optimally imperfect decisions*, Amer. Econ. Rev. 54 (1964), pp. 23–46.

[5] G. Bischi, C. Chiarella, M. Kopel, and F. Szidarowski, *Nonlinear Oligopolies - Stability and Bifurcations*, Springer, 2010.

[6] K. Busenberg and L. Cooke, *Models of Vertically Transmitted Diseases with Sequential–Continuous Dynamics*, in *Nonlinear Phenomena in Mathematical Sciences*, Academic Press, New York, 1982, pp. 179–187.

[7] F. Cavalli, *A model of monopoly with lags in the planning and production activity*, Tech. Rep., DEMS Working Paper Series N. 326 SSRN, 2016, http://dx.doi.org/10.2139/ssrn.2728987.

[8] F. Cavalli and A. Naimzada, *Effect of price elasticity of demand in monopolies with gradient adjustment*, Chaos Solitons Fractals 76 (2015), pp. 47–55.

[9] R.W. Clower, *Some theory of an ignorant monopolist*, Econom. J. 69 (1959), pp. pp. 705–716.

[10] J. Colinsk, *Why bounded rationality*, J. Econom. Lit. 34 (1996), pp. 669–700.

[11] K. Cooke and J. Wiener, *A survey of differential equations with piecewise continuous arguments*, in *Delay Differential Equations and Dynamical Systems*, Springer, 1991.

[12] L. Corchon and A. Mas-Colell, *On the stability of best reply and gradient systems with applications to imperfectly competitive models*, Econom. Lett. 51 (1996), pp. 59–65.

[13] A. Dixit, *Comparative statics for oligopoly*, Internat. Econom. Rev. 27 (1986), pp. 107–122.

[14] A.A. Elsadany and A.M. Awad, *Dynamical analysis of a delayed monopoly game with a log-concave demand function*, Oper. Res. Lett. 44 (2016), pp. 33–38.

[15] V.L. Kocic and G. Ladas, *Global Behavior of Nonlinear Diffeence Equations of Higher Order with Applications*, Kluwer Academic Publishers, 1993.

[16] F. Lamantia and D. Radi, *Exploitation of renewable resources with differentiated technologies: An evolutionary analysis*, Math. Comp. Simulation 108 (2015), pp. 155–174.

[17] M. Martelli, *Introduction to Discrete Dynamical Systems and Chaos*, Wiley Series in Discrete Mathematics and Optimization (Book 53), Wiley-Interscience, 1999.

[18] A. Matsumoto and F. Szidarovszky, *Nonlinear delay monopoly with bounded rationality*, Chaos Solitons Fractals 45 (2012), pp. 507 – 519.

[19] A. Matsumoto and F. Szidarovszky, *Complex dynamics of monopolies with gradient adjustment*, Econom. Model. 42 (2014), pp. 220–229.

[20] A. Matsumoto and F. Szidarovszky, *Discrete and continuous dynamics in nonlinear monopolies*, Appl. Math. Comput. 232 (2014), pp. 632–642.

[21] A. Matsumoto and F. Szidarovszky, *Discrete-time delay dynamics of boundedly rational monopoly*, Decis. Econ. Finance 37 (2014), pp. 53–79, Available at www.scopus.com.

[22] A. Matsumoto and F. Szidarovszky, *Nonlinear Economic Dynamics and Financial Modelling*, chap. Boundedly rational monopoly with single continuously distributed time delay, Nonlinear Economic Dynamics and Financial Modelling: Essays in Honour of Carl Chiarella, Springer, 2014, pp. 83–107.

[23] A. Matsumoto and F. Szidarovszky, *Dynamic monopoly with multiple continuously distributed time delays*, Math. Comp. Simulation 108 (2015), pp. 99–118.

[24] A. Matsumoto and F. Szidarovszky, *Learning monopolies with delayed feedback on price expectations*, Commun. Nonlinear Sci. Numer. Simul. 28 (2015), pp. 151–165.
[25] A. Naimzada and G. Ricchiuti, *Complex dynamics in a monopoly with a rule of thumb*, Appl. Math. Comput. 203 (2008), pp. 921–925.
[26] A. Naimzada and G. Ricchiuti, *Monopoly with local knowledge of demand function*, Econom. Model. 28 (2011), pp. 299–307.
[27] T. Puu, *The chaotic monopolist*, Chaos Solitons Fractals 5 (1995), pp. 35–44.
[28] G. Sarafopoulos, *Complexity in a monopoly market with a general demand and quadratic cost function*, Procedia Econ. Finance 19 (2015), pp. 122–128.
[29] H. Varian, *Microeconomic Analysis*, W. W. Norton & Company, 1992.
[30] J. Wiener, *Generalized Solutions of Functional Differential Equations*, World Scientific, 1993.