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Introducing a price variation limiter mechanism into a behavioral financial market model

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In the present paper, we consider a nonlinear financial market model in which, in order to decrease the complexity of the dynamics and to achieve price stabilization, we introduce a price variation limiter mechanism, which in each period bounds the price variation so that the current price is forced to belong to a certain interval determined by the price realization in the previous period. More precisely, we introduce such mechanism into a financial market model in which the price dynamics are described by a sigmoidal price adjustment mechanism characterized by the presence of two asymptotes that bound the price variation and thus the dynamics. We show that the presence of our asymptotes prevents divergence and negativity issues. Moreover, we prove that the basins of attraction are complicated only under suitable conditions on the parameters and that chaos arises just when the price limiters are loose enough. On the other hand, for some suitable parameter configurations, we detect multistability phenomena characterized by the presence of up to three coexisting attractors. © *2015 AIP Publishing LLC*. [http://dx.doi.org/10.1063/1.4927831]

Since the introduction of the OGY method in Ref. 28, many papers have arisen in the literature on chaos control. Most of the employed methods are, however, designed for physics systems, and thus they are not well suited for economic models. Two exceptions are represented by the Delayed Feedback Control (DFC) method by Pyragas[29] and by the limiter method by Wieland and Westerhoff.[40] The limiter technique, simply consisting in setting constant upper and lower limiters to prices, is the technique that bears a stronger resemblance to the method we are going to employ in the present work. In fact, we here introduce a price variation limiter mechanism, which in each period bounds the price variation so that the current price is forced to belong to a certain interval determined by the price realization in the previous period. Our paper belongs indeed to the literature on the search of mechanisms of control so as to reduce the volatility of prices of commodities and of financial activities. Two are the main goals of our work. First, from a formal viewpoint, we aim to avoid divergence and negativity issues. Second, from a normative viewpoint, we aim to propose a method in view of reducing volatility and controlling chaos, that is, of diminishing turbulence, as well as of decreasing the number, the size, and the complexity of the attractor in the phase space, in order to achieve the convergence to a fixed point. To show the effectiveness and the functioning of our mechanism, we use as benchmark framework a special case of the model in Tramontana *et al.*[33]

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I. INTRODUCTION

In most of the papers that deal with speculative behavior in financial markets and with the dynamics of relevant variables, such as prices and traded quantities, authors introduce a mechanism that describes the disequilibrium price dynamics. Such mechanism links the price variation to excess demand and it is usually modeled as a linear relation (see, for instance, Refs. 16 and 17). This means that the ratio between the price variation and the excess demand is constant and in turns this implies that, for instance, if the latter assumes large values in absolute value, also the price variation will be large. Hence, such a mechanism easily leads to negativity issues and divergence of the dynamics because of an overreaction: indeed considering too large starting values for the stock price, the iterates may quickly limit towards minus infinity.

On the other hand, in those stock market models in which, even in the absence of stochastic shocks, the presence of nonlinearities in deterministic frameworks accounts for the dynamics of financial markets (see, e.g., Refs. 17, 37, and 39), instabilities and strong irregularities often arise, leading to the emergence of turbulence and chaotic phenomena.

Two are then the main goals of our work. First, from a formal viewpoint, we aim to avoid divergence and negativity issues. Second, from a normative viewpoint, we aim to propose a method in view of reducing volatility and controlling chaos, that is, of diminishing turbulence, as well as of decreasing the number, the size, and the complexity of the attractors in the phase space, in order to achieve the convergence to a fixed point.

Since the introduction of the OGY method in Ref. 28, many papers have arisen in the literature on chaos control.

Most of the employed methods are however designed for physics systems, and thus they are not well suited for economic models. In addition, for instance, to the contributions in Refs. 20, 27, and 31, two exceptions are represented by the DFC method by Pyragas,[29] used for example in Refs. 8, 13, and 21 and by the limiter method by Wieland and Westerhoff,[40] used, e.g., in Refs. 11, 30, and 12, where the interested reader may find an introduction to both methods, as well as a detailed survey of the literature on the topic. The limiter technique has been also employed in Refs. 32 and 36 and simply consists in setting constant upper and lower limiters to prices. This is the technique that bears a stronger resemblance to the method we are going to employ in the present work. In fact, we here introduce a price variation limiter mechanism, which in each period bounds the price variation so that the current price is forced to belong to a certain interval determined by the price realization in the previous period. Our paper belongs indeed to the literature on the search of mechanisms of control so as to reduce the volatility of prices of commodities and of financial activities. As regards the former strand of literature, we recall the papers by Athanasiou *et al.*,[2] Corron *et al.*,[7] He and Westerhoff,[15] and Mitra and Boussard.[22] As regards the latter strand of literature, we recall instead the papers by Athanasiou and Kotsios[3] and Westerhoff.[38] Making a comparison with such works, we stress that we deal with a financial market model, too, but that, differently from Refs. 3 and 38, we do not perform just a numerical analysis, but we also propose a formal treatment of the dynamical features of our model.

More precisely, in the present paper, in order to avoid overreaction phenomena and an excessive volatility in the stock market, we assume that the market maker is forced by a central authority to be more cautious in adjusting the stock price when excess demand is large, i.e., when the system is far from its equilibria, while he/she has more freedom when excess demand is small, that is, when the system is close to an equilibrium. This kind of diversified behavior may be represented by a nonlinear function only, which has also to be increasing and to pass through the origin. In particular, our price variation limiter mechanism is described by a sigmoidal price adjustment rule characterized by the presence of two asymptotes that bound the dynamics. In order to show the effectiveness and the functioning of our mechanism, we use as benchmark framework a special case of the model in Tramontana *et al.*[33]—see the 1D case in Section III therein. We make such a choice as we believe that context is interesting, both because of the presence of heterogeneous agents, i.e., chartists and fundamentalists, and due to the richness of the generated dynamics.

We find similar results to those obtained in Ref. 33, even if, as desired, the presence of our asymptotes prevents the divergence and negativity issues therein. Moreover, when dynamics are chaotic in Ref. 33, we can stabilize the system through a suitable choice of the asymptotes. Finally, when in Ref. 33 a unique attractor is present, we are able to limit its size and complexity, as well as to reach a complete stabilization of the system. On the other hand, for some intermediate values for the asymptotes and particular configurations of the other parameters, in addition to two coexisting attractors similar to those detected in Ref. 33, we find (at least) one more attractor, which may be periodic or chaotic, independently of the nature of the other two attractors. According to the considered parameter set, it may happen that the third attractor persists, while the other two disappear, or vice versa, that the third attractor disappears when the other two are still present. From a normative viewpoint, this means that just the imposition of our price variation limiter mechanism in the financial market is not sufficient in view of stabilizing the system. To such aim, we need the asymptotes bounding the price variation to be sufficiently close, because for intermediate distances between them, neither too large nor too small, we find multistability phenomena characterized by the coexistence of attractors with disconnected basins of attraction.

The specific results that we obtain can be summarized as follows.

In Proposition 3.1, we show that our system has the same steady states as the one studied in Ref. 33, i.e., a fundamental steady state, which is always unstable, and two steady states symmetric with respect to it, for which we derive the corresponding stability conditions. In Proposition 3.2, we prove that under suitable conditions on the asymptotes the map generating our dynamical system is increasing and we describe the basins of attraction in this simple framework, employing then similar arguments to exclude the presence of negative trajectories, as well as of divergent trajectories under any condition on the parameters. In Proposition 4.1, we show that the non-fundamental steady states lose stability via a flip bifurcation. By applying the method of the turbulent maps by Block and Coppel,[5] in Proposition 4.2, we prove the presence of chaotic dynamics and, in particular, the positivity of the topological entropy, when the price limits are loose enough. Globally, we find that the destabilization process implies multistability, i.e., the coexistence of different kinds of attractors, as discussed above.

As regards the remainder of the paper, in Section II, we introduce the model; in Section III, we derive the expression of the steady states and the stability conditions, comparing them with the corresponding ones in Ref. 33, and we show that our system admits no divergent trajectories; in Section IV, we investigate the occurrence of the first flip bifurcation for our steady states and we prove the existence of complex dynamics when the asymptotes are sufficiently distant; in Section V, we present some global scenarios with multistability phenomena, characterized by the presence of two or (at least) three coexisting attractors; finally, in Section VI, we draw some conclusions and discuss our results.

II. THE MODEL

We consider a stock market modeled in the sense of Day and Huang:[9] we deal in fact with the nonlinear interactions between heterogeneous agents, i.e., technical traders (or chartists) and fundamental traders (or fundamentalists). The fractions of technical and fundamental traders are fixed. Chartists may be either optimistic or pessimistic, depending on the stock price performance: in a bull market chartists buy stocks, while in a bear market they sell stocks. For the sake of simplicity, we assume that chartists rely on a linear trading rule. Fundamentalists have an opposite behavior: believing that stock prices will return to their fundamental value, they buy stocks in undervalued markets and sell stocks in overvalued markets. We suppose that fundamentalists rely on a nonlinear trading rule.

The market maker determines excess demand and adjusts the stock price for the next period: if aggregate excess demand is positive (negative), production increases (decreases). More precisely, price variation between price P_{t+1} in period $t+1$ and price P_t in period t is defined in the following way:

$$P_{t+1} - P_t = d\, g(D_t, P_t), \tag{2.1}$$

where d is the market maker reactivity, $D_t = D_t^C + D_t^F$ reflects the orders placed by chartists and fundamentalists, and g is a function increasing in D_t and vanishing for $D_t = 0$.

As stressed by one of the reviewers, it is often assumed that the adjustment coefficients belong to $(0, 1]$, since the framework with $d > 1$ may in general be interpreted as an overshoot or an overreaction. Nonetheless, along the paper, we shall consider $d > 0$, i.e., we will deal with the cases $d \in (0, 1]$ and $d > 1$, for both empirical and theoretical reasons.

Indeed, starting with the crucial paper by De Bondt and Thaler,[10] where overshooting of asset returns with respect to the fundamental value has been observed, a well-grounded empirical literature has arisen to show the presence of overreaction phenomena in financial markets. Subsequently, some authors have given further foundations to overreaction events through the formulation and analysis of mathematical models. To such strand of literature belong, for instance, the works by Barberis *et al.*,[4] by Hong and Stein,[18] and by Veronesi.[35] More precisely, in Ref. 4, the authors present a model of investor sentiment and find for stock prices underreaction to news such as earnings announcements and overreaction to a series of good or bad news. In Ref. 18, Hong and Stein consider a market populated by two groups of boundedly rational agents, i.e., "newswatchers" and "momentum traders," and observe for prices underreaction in the short run and overreaction in the long run. In Ref. 35, the author presents a dynamic, asset price equilibrium model with rational expectations in which, among other features, prices overreact to bad news in good times and underreact to good news in bad times.

From a theoretical viewpoint, we need to have the possibility of considering also $d > 1$ in order to strengthen the nonlinearity in the stock price adjustment mechanism and intensify the oscillations of the system, increasing (in modulus) the maximum and minimum values it may reach. Indeed, we are interested in investigating what happens if the price variation is limited, especially when the market maker reactivity is high. Namely, from a normative perspective, we aim to test the effectiveness of our price variation limiter mechanism in stabilizing the system and, in the cases in which we do not get a stabilization, to observe the arising dynamical phenomena, which turn out to be particularly interesting when sufficiently high values for d are taken into account. We remark that we could make the assumption $d \in (0, 1]$ also in the present context since, as it is clear from the stability condition in Proposition 3.1, instabilities and interesting dynamics could be obtained for $d \le 1$ as well, suitably increasing a_1, a_2, or e. On the other hand, we prefer to deal with d because, due to the multiplicative formulation of the price variation in (2.1), it plays a more global role and is not connected to a particular group of agents.

As concerns the price adjustment mechanism, we stress that the majority of the existing literature on behavioral financial market models deals with a "linear" formulation, that is, with the case $g(D_t, P_t) = D_t$, like in Ref. 33. Even when a nonlinear price adjustment mechanism is considered, authors usually deal with a multiplicative formulation for g, i.e., $g(D_t, P_t) = D_t \cdot P_t$ (see, for instance, Refs. 26, 34, and 41), which admits $P = 0$ as steady state. However, both the linear and the multiplicative formulations do not impose any bound on the price variation and thus may allow overreaction phenomena and volatility, which in turn lead to instability and/or divergence issues.

Differently from such approaches, we consider a nonlinear price adjustment mechanism which determines a bounded price variation in every time period. In particular, we assume that the adjustment mechanism is S-shaped, and thus we specify the function g as

$$g(D_t, P_t) = a_2\left(\frac{a_1 + a_2}{a_1 \exp(-D_t) + a_2} - 1\right), \tag{2.2}$$

with a_1 and a_2 positive parameters. With this choice, g depends on D_t only, it is increasing in such variable, and it vanishes when $D_t = 0$. Moreover, g is bounded from below by $-a_2$ and from above by a_1 : hence, the price variations in (2.1) are gradual and the presence of the two horizontal asymptotes prevents the dynamics of the stock market from diverging and helps avoiding negativity issues. The above adjustment mechanism may be implemented assuming that the market maker is forced by a central authority to behave in a different manner according to the excess demand value. In particular, in order to avoid overreaction phenomena, he/she has to be more cautious in adjusting prices when excess demand is large, while he/she has more freedom when excess demand is small, i.e., when the system is close to an equilibrium. We recall that the adjustment mechanism in (2.2), determining a bounded variation of a given variable, has been already considered in Refs. 23–25. More precisely, in those works, we imposed a bound on the variation of the output variable in macroeconomic models, both without (see Ref. 23) and with (see Refs. 24 and 25) the stock market sector.

Adopting the nonlinear adjustment mechanism in (2.2), we can rewrite the dynamical equation in (2.1) as follows:

$$\Delta_{t+1} = P_{t+1} - P_t = da_2\left(\frac{a_1 + a_2}{a_1 \exp(-D_t) + a_2} - 1\right),$$

where Δ_{t+1} represents the price variation. We depict such relation in Figure 1.

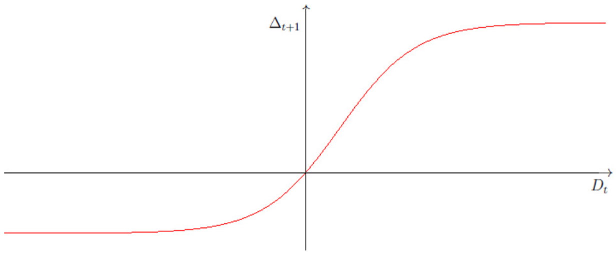

FIG. 1. The graph of Δ_{t+1} as a function of D_t.

Let us now specify the chartists' and fundamentalists' demand. Similarly to Ref. 33, we assume chartists' demand is linear with respect to the stock price and given by

$$D_t^C = e(P_t - F), \tag{2.3}$$

where $e > 0$ is the chartists' reactivity parameter and $F > 0$ is the fundamental value. As mentioned above, again like in Ref. 33, we suppose instead that the fundamentalists' demand is nonlinear and formalized by

$$D_t^F = f(F - P_t)^3, \tag{2.4}$$

where $f > 0$ is the fundamentalists' reactivity parameter. Hence, as long as the price is close to its fundamental value, fundamentalists are relatively cautious; however, the larger the difference between fundamental value and price, the more aggressive they become. As suggested by Day and Huang,[9] such a behavior is justified by increasing profit opportunities.

Like in Ref. 33, total excess demand then reads as

$$D_t = D_t^C + D_t^F = e(P_t - F) + f(F - P_t)^3.$$

Inserting its expression into (2.1) and recalling the definition of g in (2.2), we obtain the dynamic equation of the stock price in our nonlinear framework

$$P_{t+1} = P_t + da_2 \times \left(\frac{a_1 + a_2}{a_1 \exp(-(e(P_t - F) + f(F - P_t)^3)) + a_2} - 1 \right). \tag{2.5}$$

Notice that, introducing the variable $X_t = P_t - F$, it is possible to rewrite (2.5) in deviations from the fundamental value as

$$X_{t+1} = X_t + da_2 \left(\frac{a_1 + a_2}{a_1 \exp(-(eX_t - fX_t^3)) + a_2} - 1 \right). \tag{2.6}$$

The above equation generates the dynamical system we are going to analyze in what follows. We stress that, by construction, (2.6) represents the nonlinear counterpart of the system investigated in Ref. 33, obtained by implementing on the latter our price variation limiter mechanism in (2.2). For such reason, in order to test the effectiveness of our method, in Sections III–VI, we shall compare our findings with those in Ref. 33.

III. LOCAL ANALYSIS

In this section, we first discuss the existence and local stability of the steady states for our dynamical system and then we show the existence of an absorbing interval, which excludes the possibility of divergent dynamics.

In view of the subsequent analysis, it is expedient to introduce the map $\Phi : \mathbb{R} \to \mathbb{R}$ defined as

$$\begin{aligned} \Phi(X) &= X + da_2 \left(\frac{a_1 + a_2}{a_1 \exp(-(eX - fX^3)) + a_2} - 1 \right) \\ &= X + da_1 a_2 \left(\frac{1 - \exp(-(eX - fX^3))}{a_1 \exp(-(eX - fX^3)) + a_2} \right), \end{aligned} \tag{3.1}$$

associated to the dynamic equation in (2.6). Notice that in the case with $a_1 = a_2 = a$, the map Φ turns out to be odd and thus its graph is symmetric with respect to the origin. In fact, in such specific framework it holds that

$$\begin{aligned} \Phi(X) &= X + da \left(\frac{1 - \exp(-(eX - fX^3))}{\exp(-(eX - fX^3)) + 1} \right) \\ &= X + da \left(\frac{\exp(eX - fX^3) - 1}{1 + \exp(eX - fX^3)} \right) \\ &= -\left(-X + da \left(\frac{1 - \exp(eX - fX^3)}{1 + \exp(eX - fX^3)} \right) \right) = -\Phi(-X). \end{aligned}$$

In the next result, we show the existence of three steady states for our system and we investigate their local stability.

Proposition 3.1. *The dynamical system generated by the map Φ in (3.1) has the following steady states:*

$$X_1^* = -\sqrt{\frac{e}{f}}, \quad X_2^* = 0, \quad X_3^* = \sqrt{\frac{e}{f}}. \tag{3.2}$$

In particular, X_2^* is always unstable, while X_1^* and X_3^* are locally asymptotically stable if $de < \frac{1}{a_1} + \frac{1}{a_2}$.

Proof. The expression for X_i^*, $i \in \{1, 2, 3\}$, can be immediately found by solving the fixed point equation $\Phi(X) = X$.

As regards the local stability of the steady states, a straightforward computation shows that

$$\Phi'(X) = 1 - \frac{da_1 a_2 (a_1 + a_2)(3fX^2 - e)}{(a_1 \exp(-(eX - fX^3)) + a_2)^2 \exp(eX - fX^3)}. \tag{3.3}$$

Hence, $\Phi'(X_2^*) = 1 + \frac{da_1 a_2 e}{a_1 + a_2} > 1$, implying that the origin is always unstable.

In regard to X_1^* and X_3^*, it holds that $\Phi'(X_1^*) = \Phi'(X_3^*) = 1 - \frac{2dea_1 a_2}{a_1 + a_2} = 1 - \frac{2de}{\left(\frac{1}{a_1} + \frac{1}{a_2}\right)}$ and therefore, in order to get the local asymptotic stability of those steady states, we only need to impose $\Phi'(X_i^*) > -1$, $i \in \{1, 3\}$, which leads to the condition $de < \frac{1}{a_1} + \frac{1}{a_2}$. □

We observe that, according to Proposition 3.1, also in the case of an overshoot, i.e., when $d > 1$, it always possible to find a_1, a_2, and e large enough to guarantee that X_1^* and X_3^* are locally asymptotically stable.

Turning back for a while to the price variable P, we stress that the steady states for the original formulation in (2.5) are

$P_1^* = F - \sqrt{\frac{e}{f}}$, $P_2^* = F$ and $P_3^* = F + \sqrt{\frac{e}{f}}$, which are all positive when $F > \sqrt{\frac{e}{f}}$. Along the paper, we shall then maintain such assumption that, as it is easy to check, also guarantees a positive P_{t+1} in correspondence to $P_t = 0$. Actually, it is possible to prove that, when the fundamental value is large enough, then all P_{t+1}'s are positive, even the local minimum values, and thus our model makes economic sense. This can be shown by observing that for $\psi : [0, +\infty) \to \mathbb{R}$,

$$\psi(P) = P + da_2\left(\frac{a_1 + a_2}{a_1 \exp(-(e(P-F) + f(F-P)^3)) + a_2} - 1\right),$$

it holds that $\lim_{P\to+\infty} \psi(P) = +\infty$. Hence, ψ is positive for all the sufficiently large values of P, no matter the value of F. Let us then fix $\bar{P} > 0$ and show that $\psi(\bar{P}) > 0$ when F is large enough. This follows by noticing that $\lim_{F\to+\infty} \psi(\bar{P}) = \bar{P} + da_1 > 0$.

At the end of the present section, we will prove that in our model negativity issues can be avoided also by suitably tuning a_1 and a_2, via an argument based on Proposition 3.2.

In order to make a comparison with the results in Ref. 33, we recall that those authors found for their dynamical system, generated by the map

$$\phi(X) = X + d(eX - fX^3), \tag{3.4}$$

our same fixed points. Moreover, similarly to what we obtain, the origin was always unstable in that framework, too. On the other hand, in Ref. 33, only the parameters d and e play a role in determining the stability condition of the nonzero steady states, which reads as $de < 1$.

We stress instead that in our context also the parameters a_1 and a_2 enter the stability condition for X_1^* and X_3^*. In particular, according to Proposition 3.1, for any given value of d and e, either smaller or larger than 1, it is possible to find a_1 and a_2 sufficiently small, so that our stability condition is satisfied, even for those values of d and e that make the nonzero steady states in Ref. 33 unstable. This fact is illustrated in Figure 2(a), where we draw the bifurcation diagram for Φ with respect to $a_1 = a_2$ taking decreasing values in $[0.2, 2]$, and for $d = 0.35$, $e = 5$, $f = 0.7$, i.e., for the parameter values considered in the bifurcation diagram in Figure 6 of Ref. 33 (except for e there varying in $[0, 6]$). With such choices, since $de > 1$, ϕ is unstable at X_1^* and X_3^*, while Φ is locally asymptotically stable at X_1^* and X_3^* for $a_1 = a_2 < 1.142$. This shows that we may reach a complete stabilization of the system just by choosing a_1 and a_2 small enough, that is, by sufficiently reducing the range where the price variation may vary.

Of course, vice versa, fixed d and e, it is always possible to find a_1 and a_2 sufficiently large, so that our stability condition in Proposition 3.1 is not satisfied, even for those values of d and e that make the nonzero steady states in Ref. 33 locally asymptotically stable. We illustrate the latter phenomenon in Figure 2(b), where we draw the bifurcation diagram for Φ with $a_1 = a_2$ varying in $[3, 12]$, and for $d = 0.35$, $e = 1$, $f = 0.7$. With such choices, ϕ is locally asymptotically stable at X_1^* and X_3^*, while Φ is unstable at X_1^* and X_3^* for $a_1 = a_2 > 5.714$ and for $a_1 = a_2 > 7.5$, Φ turns out to be even chaotic.

We now go on with the presentation of the analytical results on the map Φ showing that, under suitable conditions on a_1 and a_2, it can be strictly increasing and that in this case the basin of attraction of the locally asymptotically stable steady states X_1^* and X_3^* is connected. Notice indeed that if $\Phi'(X) > 0$, $\forall X \in \mathbb{R}$, then in particular, $\Phi'(X_i^*) > -1$, $i \in \{1, 3\}$, and, as we saw along the proof of Proposition 3.1, this is sufficient for the nonzero steady states to be locally asymptotically stable.

Proposition 3.2. *When* $\min\{a_1, a_2\} \to 0$, *then* Φ *in* (3.1) *is strictly increasing. In this case the basin of attraction of* X_1^* *is given by* $(-\infty, 0)$, *while the basin of attraction of* X_3^* *is given by* $(0, +\infty)$.

Proof. Let us prove at first that Φ is strictly increasing when $\min\{a_1, a_2\}$ is sufficiently small. When $a_1 \geq a_2$, we put in evidence a_1^2 and simplify it between numerator and denominator in the expression in (3.3), which can then be rewritten as

$$\Phi'(X) = 1 - \frac{da_2\left(1 + \frac{a_2}{a_1}\right)\left(3fX^2 - e\right)}{\left(\exp(-(eX - fX^3)) + \frac{a_2}{a_1}\right)^2 \exp(eX - fX^3)}. \tag{3.5}$$

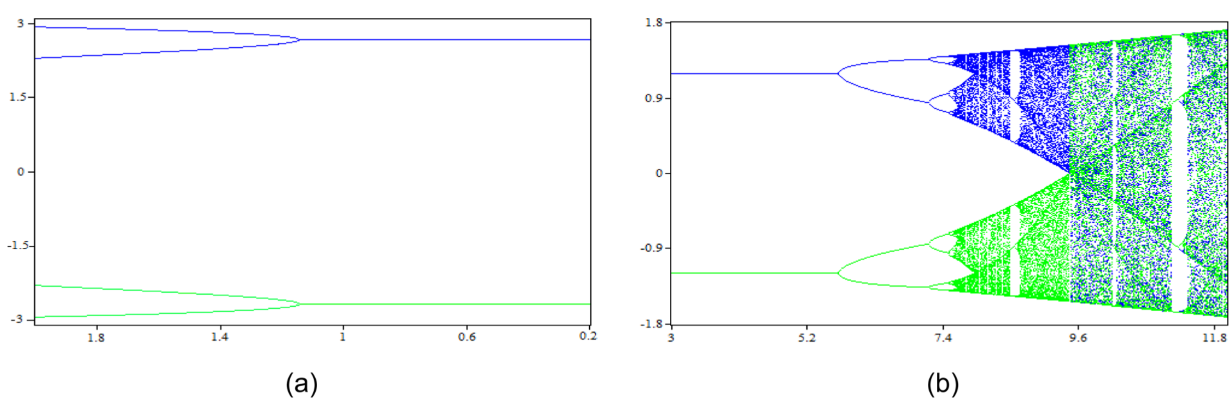

FIG. 2. In (a), the bifurcation diagram for Φ with respect to $a_1 = a_2$ taking decreasing values in $[0.2, 2]$. The green (light) dots refer to the initial condition $X(0) = -0.25$, while the blue (dark) dots refer to the initial condition $X(0) = 0.25$. In (b), the bifurcation diagram for Φ with respect to $a_1 = a_2 \in [3, 12]$. The green (light) dots refer to the initial condition $X(0) = -0.5$, while the blue (dark) dots refer to the initial condition $X(0) = 0.5$.

Since $0 < \frac{a_2}{a_1} \leq 1$, by (3.5) it follows that $\Phi'(X) \to 1$ when $a_2 = \min\{a_1, a_2\} \to 0$. Similarly, when $a_2 \geq a_1$, we put in evidence a_2^2 and simplify it between numerator and denominator in the expression in (3.3), which can then be rewritten as

$$\Phi'(X) = 1 - \frac{da_1\left(\frac{a_1}{a_2}+1\right)\left(3fX^2 - e\right)}{\left(\frac{a_1}{a_2}\exp\left(-\left(eX - fX^3\right)\right)+1\right)^2 \exp\left(eX - fX^3\right)}. \tag{3.6}$$

Since $0 < \frac{a_1}{a_2} \leq 1$, by (3.6) it follows that $\Phi'(X) \to 1$ when $a_1 = \min\{a_1, a_2\} \to 0$.

Hence, we have proved that when $\min\{a_1, a_2\}$ is small enough, then $\Phi'(X) > 0$ and thus Φ is strictly increasing, as desired.

Let us now show that, given a generic starting point $\bar{X} > 0$, its forward Φ-orbit will tend to X_3^*. By Proposition 3.1, X_2^* is always unstable. Moreover, when $X \to +\infty$ the graph of Φ admits an oblique asymptote, whose expression reads as $Y = X - da_2$. Then, since X_3^* is the only positive fixed point for Φ, by continuity, the graph of Φ lies above the 45° line in the interval $(0, X_3^*)$, while the graph lies below the 45° line in the interval $(X_3^*, +\infty)$. Moreover, from the proof of Proposition 3.1, it follows that X_3^* is locally asymptotically stable when $\min\{a_1, a_2\}$ is sufficiently small and Φ is strictly increasing. Hence, if $\bar{X} \in (0, X_3^*)$, its forward trajectory will tend to X_3^* in a strictly increasing manner, while if $\bar{X} \in (X_3^*, +\infty)$, its forward trajectory will tend to X_3^* in a strictly decreasing way. This shows that the basin of attraction of X_3^* contains $(0, +\infty)$. A completely symmetric argument, using the fact that when $X \to -\infty$ the graph of Φ admits an oblique asymptote, whose expression reads as $Y = X + da_1$, allows to show that the forward trajectory of any point in $(-\infty, 0)$ will tend to X_1^*. Hence, the basin of attraction of X_1^* contains $(-\infty, 0)$.

As the origin is a fixed point for Φ, we can infer that the basin of attraction of X_1^* is given by $(-\infty, 0)$, while the basin of attraction of X_3^* is given by $(0, +\infty)$.

This concludes the proof. □

Hence, Proposition 3.2 prevents the existence of interesting dynamics when a_1 or a_2 are sufficiently small, because in this case the map Φ is strictly increasing. On the other hand, for intermediate values of a_1 or a_2, neither too small nor too large, it is possible that, although X_1^* and X_3^* are locally asymptotically stable, they coexist with periodic or chaotic attractors (see, for instance, Figures 12 and 18).

An argument similar to the one employed in the proof of Proposition 3.2 allows to infer that the shape of the graph of the map Φ prevents the existence of diverging trajectories, even when Φ is no more monotone. In fact, since $\Phi(0) = 0$ and $\Phi'(0) > 1$ (as shown in the proof of Proposition 3.1), Φ is locally increasing at 0 and its graph lies below (above) the 45° line in a left (right) neighborhood of 0. If the map is not globally increasing, either $\Phi|_{(-\infty,0)}$ has a local maximum point, we call M_1, followed by a local minimum point, we call m_1, after which Φ monotonically grows towards the origin, or $\Phi|_{(0,+\infty)}$ has a local maximum point, we call M_2, followed by a local minimum point, we call m_2, and after that Φ monotonically grows towards infinity.[42] Of course, the map can have all the four critical points as well, like in the symmetric case in which $a_1 = a_2$ are large enough and Φ is odd, or for instance as in Figure 19(a).

For the sake of illustration, let us focus on the scenario with four critical points because, as we shall see in Section V, it is the most interesting from a dynamical viewpoint. In such setting, even when the values of both a_1 and a_2 are so large that[43] $\Phi(m_2) < 0 < \Phi(M_1)$, the position of the asymptotes ($Y = X + da_1$ for $X \to -\infty$ and $Y = X - da_2$ for $X \to +\infty$) with respect to the 45° line makes forward trajectories, starting from an arbitrary point, eventually bounce within the absorbing interval $J = (\min\{\Phi(m_1), \Phi(m_2)\}, \max\{\Phi(M_1), \Phi(M_2)\})$: in fact, when the starting point $\bar{X}$ belongs to the interval $(-\infty, X_1^*)$, either $\bar{X} \in J$ or its forward trajectory monotonically grows to reach J and, symmetrically, when the starting point $\bar{X}$ belongs to the interval $(X_3^*, +\infty)$, either $\bar{X} \in J$ or its forward trajectory monotonically decreases to reach J, and then, by construction, the orbit of $\bar{X}$ does not leave J anymore. On the other hand, it is easy to prove that $[X_1^*, X_3^*] \subset J$. Indeed, if $X \in [X_1^*, 0]$, then $\max\{\Phi(M_1), \Phi(M_2)\} \geq \Phi(M_2) > 0 \geq X > \Phi(X) \geq \Phi(m_1) \geq \min\{\Phi(m_1), \Phi(m_2)\}$ and thus $X \in J$. Similarly, if $X \in (0, X_3^*]$, then $\max\{\Phi(M_1), \Phi(M_2)\} \geq \Phi(M_2) \geq \Phi(X) > X > 0 > \Phi(m_1) \geq \min\{\Phi(m_1), \Phi(m_2)\}$ and thus $X \in J$, again.

When comparing our framework to the one in Ref. 33 in regard to the presence of divergence issues, we observe that, differently from the present context in which the existence of diverging orbits is prevented by a suitable choice of a_1 and a_2, in Ref. 33, it is possible to observe divergent dynamics. We illustrate such dissimilarity between the two papers in Figure 3. In particular, in Figure 3(a), we depict a scenario in which the map ϕ considered in Ref. 33 admits diverging trajectories; for the same parameter configuration, we show in Figure 3(b) how an intermediate choice for a_1 and a_2 allows instead our map Φ in (3.1) to display a periodic behavior around the steady states, while for smaller values of a_1 and a_2, which imply a further reduction of the range the price variation may belong to, we observe in Figure 3(c) a monotone convergence towards the steady states.

We finally observe that introducing our price variation limiter mechanism we are able to prevent not only divergent trajectories, but also negativity issues. In order to explore this point, we need to turn back for a while to the original formulation of our model in the price variable P (see (2.5)). Since it makes sense to consider only $P > 0$, in the following discussion, we shall restrict our attention to $X = P - F > -F$ and show that in our model, no matter the value of the other parameters, when a_1 and a_2 are sufficiently small it holds that $\Phi(X) > -F$, for any $X > -F$. In particular, we illustrate such fact in Figure 4, where we consider a framework in which for the map ϕ in Ref. 33 there exist instead trajectories along which prices are negative (that is, on which $X < -F$). Indeed, we stress that, except for the case in which $de = \frac{1}{2}$, it is always possible to find values for F so that $-F$ lies between the minimum of ϕ and X_1^*, i.e., so that in correspondence to its minimum point ϕ assumes a value lower than $-F$. Namely, straightforward computations show that

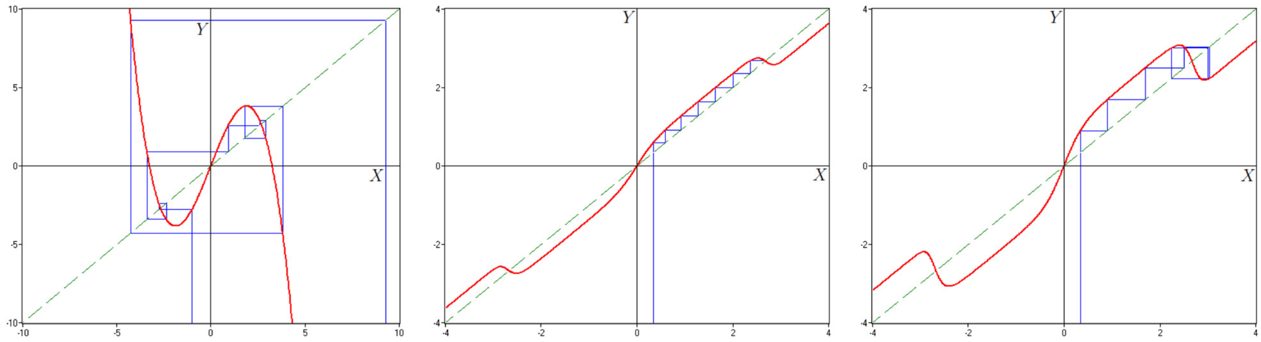

FIG. 3. For $d = 0.41$, $e = 5$, and $f = 0.7$, we illustrate: in (a), a diverging trajectory, starting from $X(0) = -1$, for the map ϕ in (3.4); in (b), a trajectory, starting from $X(0) = 0.15$, for the map Φ in (3.1) with $a_1 = a_2 = 2$, which tends to a period-two cycle surrounding X_3^*; in (c), a trajectory, starting from $X(0) = 0.15$, for the map Φ in (3.1) with $a_1 = a_2 = 0.9$, which converges to X_3^*.

the minimum point for ϕ is given by $\bar{X} = -\sqrt{\frac{1+de}{3df}}$ and that $\phi(\bar{X}) < X_1^*$ is always satisfied, except when $de = \frac{1}{2}$, case in which the previous inequality becomes an equality. More precisely, in Figure 4 we show that, suitably choosing the fundamental value F, the map ϕ in (3.4) assumes negative P-values (look, for instance, at the minimum point)—and there may possibly exist periodic cycles on which all P-values are negative—while this does not happen for Φ, which for the considered choice of a_1 and a_2 is increasing. We claim that this is sufficient to avoid that the Φ-image of some values for X larger than $-F$ is lower than $-F$. Indeed, since we imposed $F > \sqrt{\frac{e}{f}}$ in order for the steady states to be positive in the original formulation of the model in terms of prices (see the discussion after Proposition 3.1), it is easy to show that $\Phi(-F) > -F$. Since Φ is increasing we can then conclude that $\Phi(X) > \Phi(-F) > -F$, for all $X > -F$, as desired.

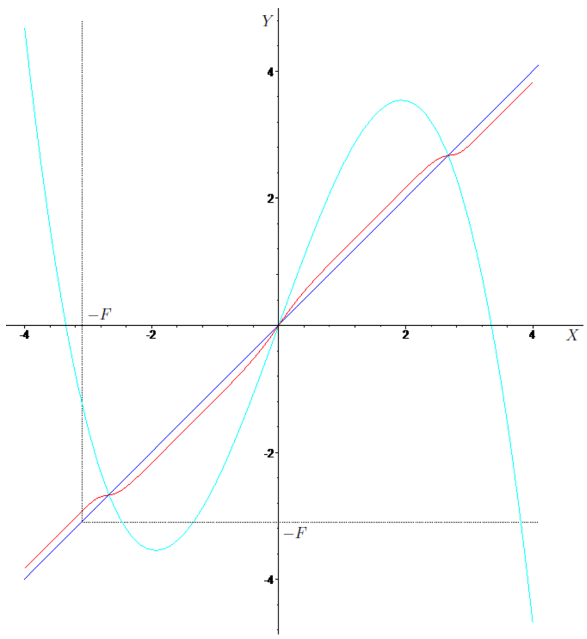

FIG. 4. The graphs of ϕ (in cyan) and Φ (in red) with $d = 0.35$, $e = 5$, $f = 0.7$, and for Φ we also fix $a_1 = a_2 = 0.5$. For $F = 3.1$, we show that the map ϕ assumes negative P-values (i.e., the ϕ-image of some values for X larger than $-F$ is below $-F$: look, for instance, at the minimum point), while this does not happen for Φ, which in particular, for the considered parameter configuration is increasing.

IV. BIFURCATIONS AND CHAOTIC DYNAMICS

We start the present section by investigating in the next result which kind of bifurcation occurs for the map Φ at the symmetric steady states X_1^* and X_3^* when they lose their stability.

Proposition 4.1. *For the map Φ in (3.1), a flip bifurcation occurs at X_1^* and X_3^* when $de = \frac{1}{a_1} + \frac{1}{a_2}$.*

Proof. According to the proof of Proposition 3.1, the steady states X_1^* and X_3^* are locally asymptotically stable when $\Phi'(X_i^*) > -1$, for $i \in \{1, 3\}$. Then, the map Φ satisfies the canonical conditions required for a (double) flip bifurcation (see Ref. 14) and the desired conclusion follows. Indeed, when $\Phi'(X_i^*) = -1$, $i \in \{1, 3\}$, i.e., for $de = \frac{1}{a_1} + \frac{1}{a_2}$, then X_1^* and X_3^* are non-hyperbolic fixed points; when $de < \frac{1}{a_1} + \frac{1}{a_2}$, they are attracting and, finally, when $de > \frac{1}{a_1} + \frac{1}{a_2}$, they are repelling. □

Notice that the above result is confirmed by Figure 2, where the first flip bifurcation for Φ occurs at X_1^* and X_3^* in (a) for $a_1 = a_2 = \frac{2}{de} \approx 1.142$ and in (b) for $a_1 = a_2 = \frac{2}{de} \approx 5.714$, respectively.

We now prove the existence of complex dynamics for our system via the method of the turbulent maps in Ref. 5. Such technique, like similar ones in the literature on topological dynamics (see, for instance, the more general covering interval approach in Ref. 6), requires to find for a given continuous map $f : J \to \mathbb{R}$, where $\emptyset \neq J \subset \mathbb{R}$ is a compact interval, two nonempty compact subintervals (if they exist) J_0 and J_1 of J with at most one common point such that

$$J_0 \cup J_1 \subseteq f(J_0) \cap f(J_1). \tag{4.1}$$

If the latter property is fulfilled, then the map f is called turbulent in Ref. 5 and it is therein shown to display some of the typical features associated to the concept of chaos like, e.g., existence of periodic points of each period, semi-conjugacy to the Bernoulli shift and thus positive topological entropy.

We will use the just described methodology along the proof of Proposition 4.2.

We notice that in the statement of the next result we focus on a particular parameter configuration for which there exists a unique chaotic attractor for our system. However, the presence of a further external attractor similar to those depicted in blue in Figures 12 and 13 would not affect the argument in the next result and thus the applicability of the method of the turbulent maps.

Proposition 4.2. *Let us consider the map* Φ *in* (3.1) *for the following parameter configuration:* $a_1 = a_2 = 1.1$, $d = 3$, $e = 1.5$, $f = 1.2$. *Setting* $J = [0, 1.336]$, $J_0 = [0, 0.439]$ *and* $J_1 = [1.056, 1.336]$, *it holds that*

$$\Phi(J_i) = J, \quad i = 0, 1. \tag{4.2}$$

Hence, the map Φ *is turbulent and, in particular, it follows that* $h_{\text{top}}(\Phi) \geq \log(2)$, *where we denoted by* $h_{\text{top}}(\Phi)$ *the topological entropy of* Φ.

Proof. The map Φ is turbulent because it is continuous on $\mathbb{R}$, and thus on J, and J_0 and J_1 are nonempty compact disjoint subintervals of J for which (4.2) is satisfied. In fact, the extreme points of J (coinciding with the left extreme point of J_0 and the right extreme point of J_1, respectively) are the smallest nonnegative solutions to the equation $\Phi(X) = 0 = X_2^*$, with X_2^* as in (3.2). Hence, one solution is the fixed point X_2^*, while we call $\bar{X}$ the smallest positive solution. Moreover, the right extreme point of J_0 and the left extreme point of J_1 are the solutions to the equation $\Phi(X)|_{[0,X_3^*]} = \bar{X}$. Thus, by construction, it holds that $\Phi(J_i) = J$, $i = 0, 1$, as desired. □

Notice that the construction in the proof of Proposition 4.2 is possible because, for the considered parameter configuration, the equation $\Phi(X) = 0$ admits the positive solution $\bar{X}$ and the maximum value of $\Phi|_{[0,X_3^*]}$ exceeds $\bar{X}$. If the latter condition were not satisfied, the equation $\Phi(X) = \bar{X}$ would not admit solutions on $[0, X_3^*]$. See Figure 5 for a graphical illustration of the result.

Moreover, we stress that, although in the statement of Proposition 4.2, we have fixed some particular parameter values, the result is robust, as the same conclusions hold for several different sets of parameter values, as well. For instance, we do not need to assume that $a_1 = a_2$, as another possible configuration for which the same conclusions hold is given by $a_1 = 3.7$, $a_2 = 3.2$, $d = 2$, $e = 0.7$, $f = 1.3$. In this case, the intervals J, J_0, and J_1 that satisfy (4.2) can be chosen as $J = [0, 0.875]$, $J_0 = [0, 0.287]$, and $J_1 = [0.692, 0.875]$, respectively. In addition we remark that, once that a result analogous to Proposition 4.2 is proven for a certain parameter configuration, by continuity, the same conclusions still hold, suitably modifying the intervals J, J_0, and J_1, also for small variations in those parameters. Hence, for instance, Proposition 4.2 actually allows to infer the existence of complex dynamics for the map Φ when a_1 and a_2 are possibly different and both lie in a neighborhood of 1.1, and for some suitable values of the other parameters.

More generally, a simple computation suggests that the map Φ is chaotic when a_1 and a_2 are large enough. Indeed, when a_1 and a_2 increase, the positive maximum point M_2 belongs to $[0, X_3^*]$ and the slope of Φ at $X = 0$ and $X = X_3^*$ increases in absolute value. This implies that when a_1 and a_2 are sufficiently large the map Φ on $[0, X_3^*]$ is steep enough to allow the construction described along the proof of Proposition 4.2 and depicted in Figure 5. Namely, $\Phi'(X_2^*) = 1 + \frac{da_1a_2e}{a_1+a_2} > 1$ and $\Phi'(X_3^*) = 1 - \frac{2de}{\left(\frac{1}{a_1}+\frac{1}{a_2}\right)} < 0$ for a_1 and a_2 sufficiently large, so that $M_2 \in [0, X_3^*]$. Moreover, it holds that

$$\frac{\partial^2\Phi}{\partial X \partial a_1}(0) = \frac{de}{\left(\frac{a_1}{a_2}+1\right)^2} > 0, \quad \frac{\partial^2\Phi}{\partial X \partial a_1}(X_3^*) = \frac{-2de}{\left(\frac{a_1}{a_2}+1\right)^2} < 0;$$

$$\frac{\partial^2\Phi}{\partial X \partial a_2}(0) = \frac{de}{\left(1+\frac{a_2}{a_1}\right)^2} > 0, \quad \frac{\partial^2\Phi}{\partial X \partial a_2}(X_3^*) = \frac{-2de}{\left(1+\frac{a_2}{a_1}\right)^2} < 0.$$

Hence, the steepness of Φ at $X = 0$ and $X = X_3^*$ increases when a_1 and a_2 increase. Notice however that when the difference (and thus one of the ratios) between a_1 and a_2 is too large in absolute value then some of the above derivatives limit to zero. Thus, in order to have complex dynamics for the map Φ we need both a_1 and a_2 to be large enough.

The latter conclusion may be interpreted from an economic viewpoint by saying that the system displays a complex behavior when the price limits are loose enough, that is, when the admissible price variations are sufficiently large.

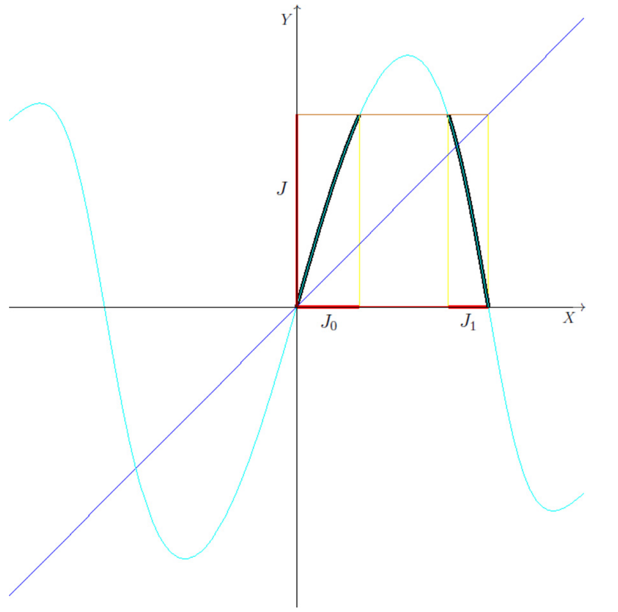

FIG. 5. A pictorial illustration of Proposition 4.2. For the map Φ with $a_1 = 1.1$, $a_2 = 1.1$, $d = 3$, $e = 1.5$, $f = 1.2$, we draw with a ticker line the interval J and its subintervals J_0 and J_1.

V. GLOBAL ANALYSIS

In Sections III and IV, we presented some theoretical results about the dynamics of our system. We saw that different types of behaviors can occur, such as the existence of absorbing intervals, locally asymptotically stable steady states, periodic cycles and chaotic regimes. In the present section, we investigate, using bifurcation diagrams, the global behavior of the system. In particular, we split our analysis according to the number of coexisting attractors we found, i.e., two or three.[44] In fact, we chose not to deal with the scenario with a unique attractor separately, as we find it when there are two coexisting attractors that merge through a homoclinic bifurcation or when there are three coexisting attractors and two of them disappear and just one survives.

We recall that the coexistence of different kinds of attractors is also known as multistability. This feature may be considered as a source of richness for the framework under analysis because, other parameters being equal, i.e., under the same institutional and economic conditions, it allows explaining different trajectories and evolutionary paths. The initial conditions, leading to the various attractors, represent indeed a summary of the past history, which in the presence of multistability phenomena does matter in determining the evolution of the system. Such property, in the literature on complex systems, is also called "path dependence" (see Ref. 1).

A. Two coexisting attractors

In this first subsection about multistability, we analyze a framework in which we have the coexistence of two attractors in the symmetric scenario with $a_1 = a_2 = a$. In particular, such setting allows us to describe some important steps determined by the increase of a. In our explanation, we will follow, as far as possible, the steps in Ref. 33. Indeed, under the symmetry condition, the two systems behave in a similar manner, except for the disappearance in Ref. 33 of the chaotic attractor due to divergence issues (see Figure 6 therein).

In performing our analysis, we start by drawing in Figure 6 the bifurcation diagram with respect to $a_1 = a_2 = a \in [0.2, 2]$ for the map Φ with $d = 3$, $e = 1.5$, and $f = 1.2$.

We now better investigate the various transformations undergone by the graph of the map Φ when a increases, while keeping the other parameters fixed like in Figure 6. In particular, recalling the name given to the critical points in Section III, i.e., $M_1 < m_1 < 0 < M_2 < m_2$, where m_1 and m_2 are the minimum points and M_1 and M_2 are instead the maximum points (see also Figure 7 for a graphical illustration), we will first focus on the position of $\Phi(M_1)$ and $\Phi(m_2)$ with respect to the X-axis and then on the position of $\Phi(m_1)$ and $\Phi(M_2)$ with respect to the "break-through intervals," i.e., the sets of points around M_1 and m_2 whose Φ-iterate has already crossed the X-axis (see Figures 8(a) and 9).

At first, we draw in Figure 7 the framework with $a_1 = a_2 = 0.5$. In this case, $\Phi(M_1)$ and $\Phi(m_2)$ have not crossed the X-axis yet, the attractors around X_1^* and X_3^* are still distinct and the basins of attraction are connected, coinciding respectively with $(-\infty, 0)$ and $(0, +\infty)$. In particular, for such parameter configuration, the attractor around X_3^* is a period-two cycle assuming values close to 0.905 and 1.249.

In Figure 8, we represent instead the framework with $a_1 = a_2 = 0.69$. In this case, $\Phi(M_1)$ and $\Phi(m_2)$ have crossed the X-axis (the crossing occurs indeed for $a_1 = a_2 \approx 0.608$) and we show in Figure 8(a) that the forward iterates of m_2, as well as (some of) its preimages, limit towards the chaotic attractor around X_1^*. More generally, in Figure 8(b), we try to give an idea of how complex and disconnected are the basins of attraction of the chaotic attractors around X_1^* and X_3^* for the same parameter values. Notice in particular, that the open intervals I_1 and J_1 coincide with the "break-through intervals" highlighted on the X-axis in Figure 8(a). With this respect we observe that, since $\Phi(m_1)$ and $\Phi(M_2)$ for $a_1 = a_2 = 0.69$ do not belong to such intervals, all forward iterates of the points in I_1 are negative and symmetrically all forward iterates of the points in J_1 are positive: hence, the attractors around X_1^* and X_3^* are still distinct.

When increasing $a_1 = a_2$ further, we obtain the framework in Figure 9, where we set $a_1 = a_2 = 0.8$. In such case, the chaotic attractors around X_1^* and X_3^* have disappeared and in their place a new unique attractor has emerged surrounding both those two fixed points. As argued above, this change is caused by the fact that $\Phi(m_1)$ and $\Phi(M_2)$ now belong to the "break-through intervals," highlighted on the X-axis in Figure 9, or, equivalently, by the fact that the homoclinic bifurcation of X_2^* has already occurred. Indeed, it takes place

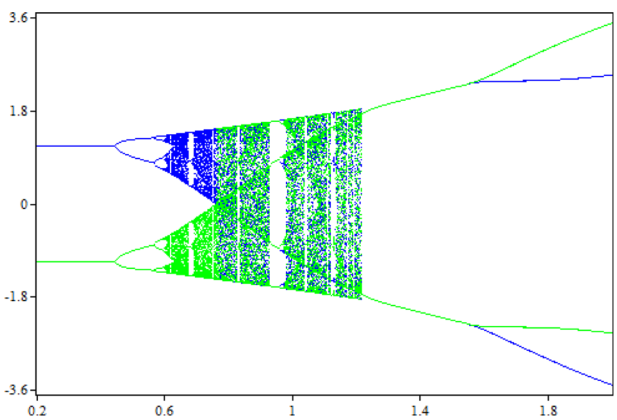

FIG. 6. The bifurcation diagram with respect to $a_1 = a_2 \in [0.2, 2]$ for the map Φ with $d = 3$, $e = 1.5$, and $f = 1.2$. The green dots refer to the initial condition $X(0) = -0.83$, while the blue dots refer to the initial condition $X(0) = 0.83$.

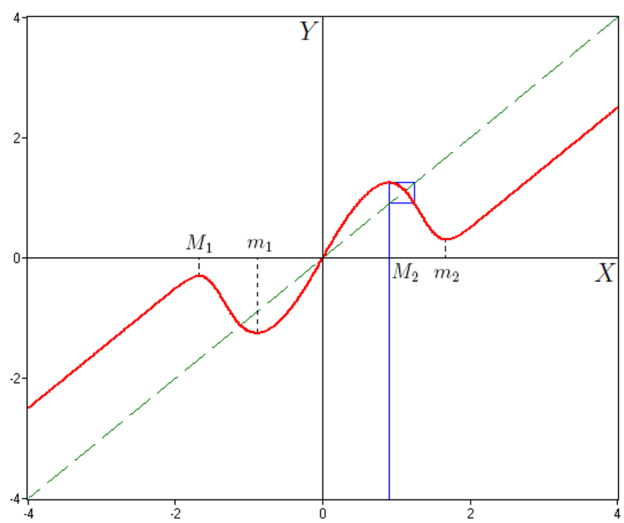

FIG. 7. The graph of the first iterate of the map Φ with $a_1 = a_2 = 0.5$, $d = 3$, $e = 1.5$, and $f = 1.2$, together with the forward iterates of $X(0) = 0.905$.

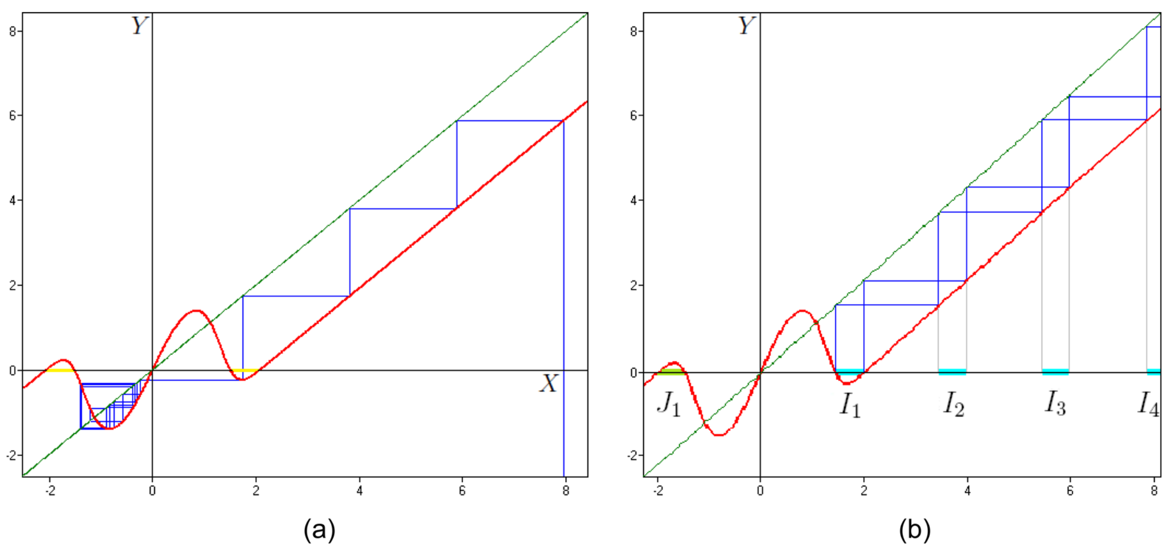

FIG. 8. The graph of the first iterate of the map Φ with $a_1 = a_2 = 0.69$, $d = 3$, $e = 1.5$, and $f = 1.2$. In (a), we also show the forward iterates of $X(0) = 7.95$ and the sets highlighted on the X-axis are what we called the "break-through intervals." Notice that, for the considered parameter values, $\Phi(m_1)$ and $\Phi(M_2)$ do not belong to those intervals and thus the chaotic attractors around X_1^* and X_3^* are still separated. In fact, in (b), we highlight in cyan the open interval I_1, as well as some of its preimages, we called I_2, I_3, and I_4, whose points limit towards the chaotic attractor around X_1^*. Symmetrically, the points belonging to the green open interval J_1 limit towards the chaotic attractor around X_3^*.

when $a_1 = a_2$ are such that $\Phi^2(M_2) = \Phi^2(m_1) = 0$, that is, for $a_1 = a_2 \approx 0.765$, value for which the attractors around X_1^* and X_3^* merge, with the consequent emergence of a unique attractor, as in Figure 9.

We stress that, before the homoclinic bifurcation of X_2^*, there have actually been other such bifurcations, i.e., the homoclinic bifurcations of X_1^* and X_3^*, which take place when $\Phi^3(X_i^*) = X_i^*$, for $i = 1, 3$, that is, for $a_1 = a_2 \approx 0.627$. Before their occurrence, the chaotic attractors around such steady states consist of two chaotic intervals each, like shown in Figure 10(a) for X_3^* for $a_1 = a_2 = 0.605$, while from this moment on the chaotic intervals around X_1^* and X_3^* merge into two chaotic intervals, as shown in Figure 10(b)

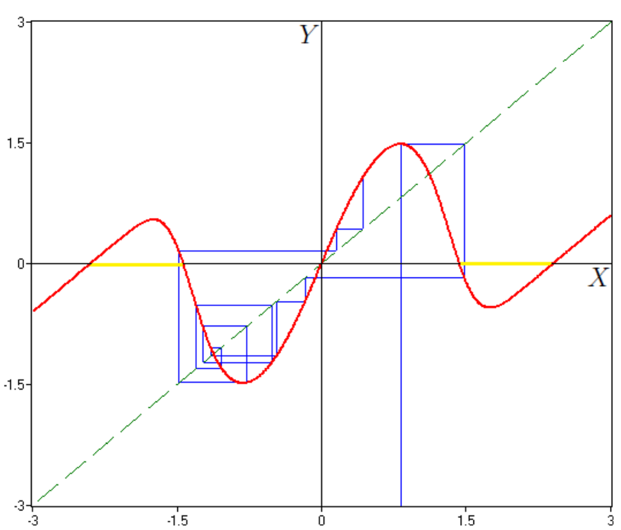

FIG. 9. The graph of the first iterate of the map Φ with $a_1 = a_2 = 0.8$, $d = 3$, $e = 1.5$, and $f = 1.2$, together with the forward iterates of $X(0) = 0.83$. For such parameter values, $\Phi(m_1)$ and $\Phi(M_2)$ belong to the "break-through intervals," highlighted on the X-axis, causing the emergence of a unique chaotic attractor surrounding both X_1^* and X_3^*.

for X_3^* for $a_1 = a_2 = 0.628$. We remark that, by Li-Yorke Theorem (see Ref. 19 (Theorem 1)), when the map Φ has a period-three orbit, then it admits periodic orbits of all periods, while before that moment only even-period orbits are allowed.

Finally, in Figure 11, we show that for $a_1 = a_2 \approx 1.216$, through a double fold bifurcation of Φ^2 in correspondence to the black squares in Figure 11(a), together with an unstable period-two orbit, also a stable period-two cycle arises, whose basin of attraction is internal to the pre-existing chaotic attractor and which captures all the orbits previously attracted by the chaotic set. In particular, in Figure 11(b), we show that the forward iterates of M_2 are attracted by the new period-two cycle, thanks to the presence of M_1 and m_2, as well as of our oblique asymptotes, which with their folding help avoiding the divergence issues in Ref. 33. This makes the old chaotic attractor disappear and just the new stable period-two cycle persists, which becomes unstable for $a_1 = a_2 \approx 1.560$, when two stable period-two cycles emerge through a double pitchfork bifurcation of Φ^2 (see Figure 6).

When comparing again our findings with the ones in Ref. 33 (Proposition 3.1), we observe that both maps ϕ and Φ display a similar cascade of period-doubling bifurcations at the non-fundamental, symmetric steady states, until there homoclinic bifurcations occur, followed by a homoclinic bifurcation of the fundamental steady state, i.e., the origin, which unifies the previously separated chaotic attractors into a unique attractor surrounding all the steady states. On the other hand, after the chaotic regime, our system reaches a stable period-two cycle, while in Ref. 33, after the final bifurcation, no attractors exist anymore and the generic trajectory is divergent. Another difference between the two works concerns the features of the corresponding basins of attraction, which in Ref. 33 are complex and disconnected also for small values of the parameters, while in the present context the complexity arises for intermediate values of the

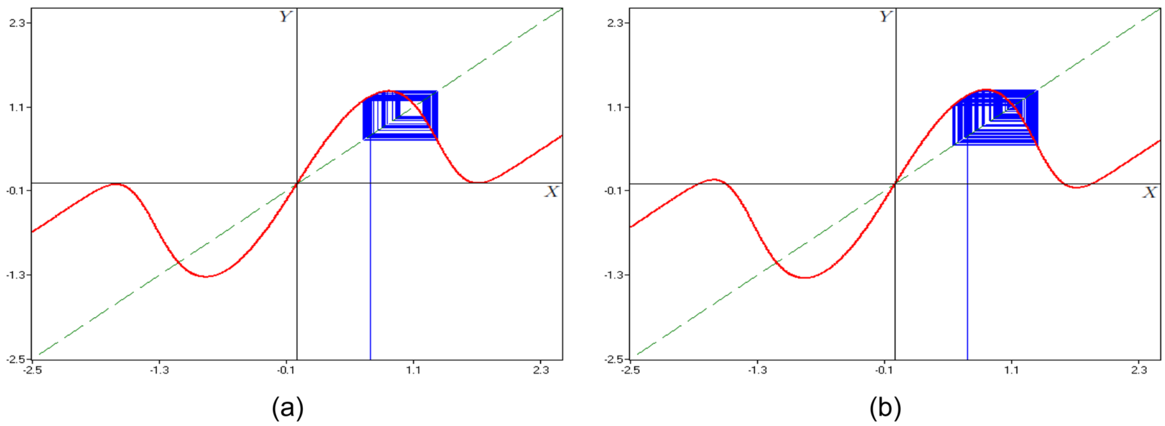

FIG. 10. The graph of the first iterate of the map Φ with $d = 3$, $e = 1.5$, and $f = 1.2$, in (a) for $a_1 = a_2 = 0.605$ together with the forward iterates of $X(0) = 0.69$, in (b) for $a_1 = a_2 = 0.628$ together with the forward iterates of $X(0) = 0.68$.

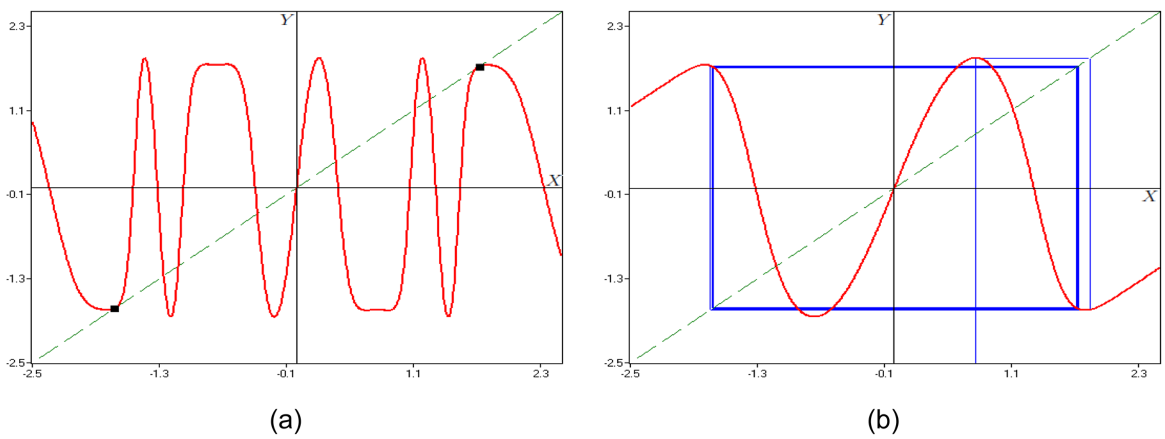

FIG. 11. For the map Φ with $d = 3$, $e = 1.5$, $f = 1.2$, and $a_1 = a_2 = 1.216$, we draw in (a) the graph of the second iterate and in (b) the graph of the first iterate together with the forward iterates of $X(0) = 0.76$.

parameters a_1 and a_2, i.e., when $\Phi(M_1)$ and $\Phi(m_2)$ have already crossed the X-axis, but $\Phi(m_1)$ and $\Phi(M_2)$ have not entered the "break-through intervals," yet. We stress however that for larger parameter values both in the present framework and in Ref. 33 a unique attractor emerges and thus the above mentioned complexity in the basins of attraction disappears.

From an analytical viewpoint, the dissimilarities between our findings and the ones in Ref. 33 are motivated by the different behavior at infinity of our map Φ in (3.1)

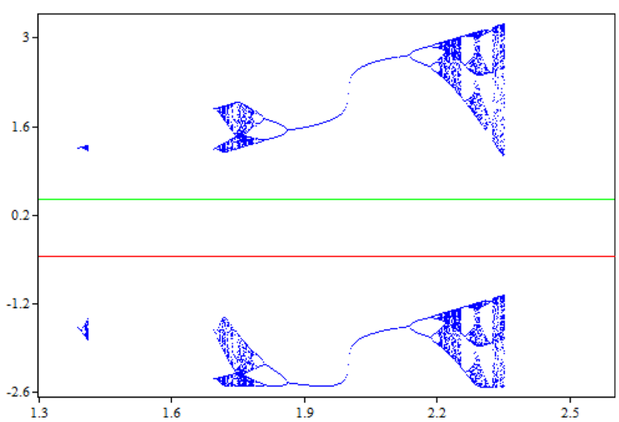

FIG. 12. The bifurcation diagram with respect to $a_1 \in [1.3, 2.8]$ for the map Φ with $a_2 = 2$, $d = 2.14$, $e = 0.2$, $f = 1$, with initial conditions $X(0) = 0.1$ for the green dots, $X(0) = -0.1$ for the red dots, and $X(0) = 1.26$ for the blue dots, respectively, which highlights a multistability scenario characterized by the coexistence of an external chaotic or periodic attractor with two internal stable fixed points.

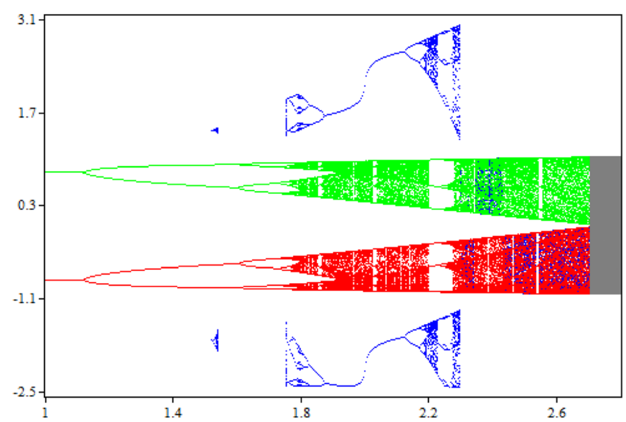

FIG. 13. The bifurcation diagram with respect to $a_1 \in [1, 2.8]$ for the map Φ with $a_2 = 2$, $d = 2.14$, $e = 0.65$, $f = 1$, with initial conditions $X(0) = 0.1$ for the green dots, $X(0) = -0.1$ for the red dots, and $X(0) = 1.45$ for the blue dots, respectively, which highlights a multistability framework characterized by the coexistence of an external chaotic or periodic attractor with two internal chaotic or periodic attractors.

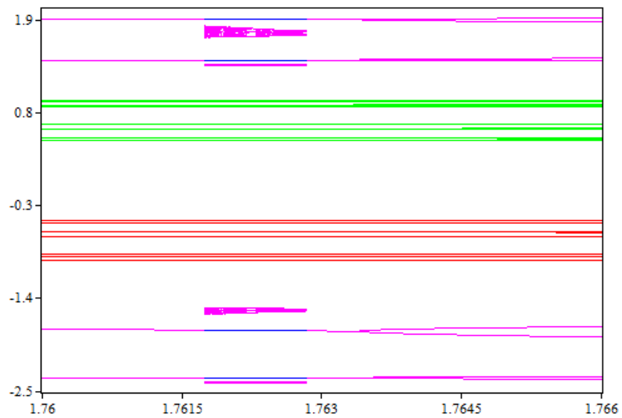

FIG. 14. A magnification of the bifurcation diagram in Figure 13, zooming on $a_1 \in [1.7600, 1.7660]$, which highlights the presence for $a_1 \in [1.7615, 1.7630]$ of a fourth attractor (in violet), which is chaotic and composed by four pieces, coexisting with the external period-four cycle and the two internal period-eight cycles. The initial conditions are $X(0) = 0.1$ for the green dots, $X(0) = -0.1$ for the red dots, $X(0) = 1.9$ for the blue dots, and $X(0) = 2.7$ for the violet dots.

and the map ϕ in (3.4) (see Figure 4). From an interpretative point of view, the dissimilarities between the two works originate from the absence in Ref. 33 of our price-limiters, which prevent sudden strong price variations and reduce the complexity of the system.

B. Three coexisting attractors

In this second subsection about multistability, we analyze some frameworks with three coexisting attractors and we look at their evolution when a_1 or e increase. In fact, unlike in Subsection V A, we here generally do not assume that $a_1 = a_2$ and thus the map Φ is possibly no more symmetric.

In Figure 12, we show that there exists a range of values for a_1 for which two stable fixed points coexist with a third attractor, which may be chaotic (in two or more pieces) or periodic. In such framework, dynamics tend towards the locally stable fixed points only for suitable past histories, i.e., if the considered initial conditions are sufficiently close to those steady states. Indeed, for intermediate values of a_1 and a_2, neither too small nor too large, X_1^* and X_3^* are locally asymptotically stable, but they may coexist with other attractors. According to Proposition 3.2, this cannot happen when a_1 or a_2 are small enough, because in that case the map Φ is strictly increasing and thus X_1^* and X_3^* are globally stable and no other attractors can exist.

In Figure 13, we illustrate instead a numerical example where, for a value of parameter e larger than in the previous picture, there exists a range of values for a_1 for which we have the coexistence of three chaotic or periodic attractors of period greater than one. In Figure 14, zooming on a particular range of values for a_1 while keeping the other parameters fixed as in Figure 13, we actually show the presence of a fourth chaotic attractor in four pieces. However, it exists only for a small set of values for a_1, and thus it is not visible when considering the larger range in Figure 13. This is the reason why we chose to classify such scenario under the framework with three coexisting attractors, rather than dealing with it in a separate subsection.

Focusing our attention on Figure 12, we try to explain the crucial steps that lead to the emergence of the external attractor, to its main transformations, and finally to its disappearance.

For $a_1 \approx 1.386$, a fold bifurcation of Φ^2 occurs, through which a stable and an unstable period-two cycles for Φ emerge. Around the former we observe for increasing values of a_1 a cascade of period-doubling bifurcations, after which, for $a_1 \approx 1.412$, the iterates of $M_1 \approx -1.57$ (corresponding to the most negative minimum point of Φ^2 in Figure 15(a)) go beyond the unstable period-two cycle, which acts as a "separatrix" from the basin of attraction of X_1^* (see Figure 19(b), where, for different parameter values, we represent such unstable period-two cycle through black squares) and limit towards such steady state. This makes the external attractor disappear, since the orbits previously attracted to it, tend now towards X_1^* or X_3^*.

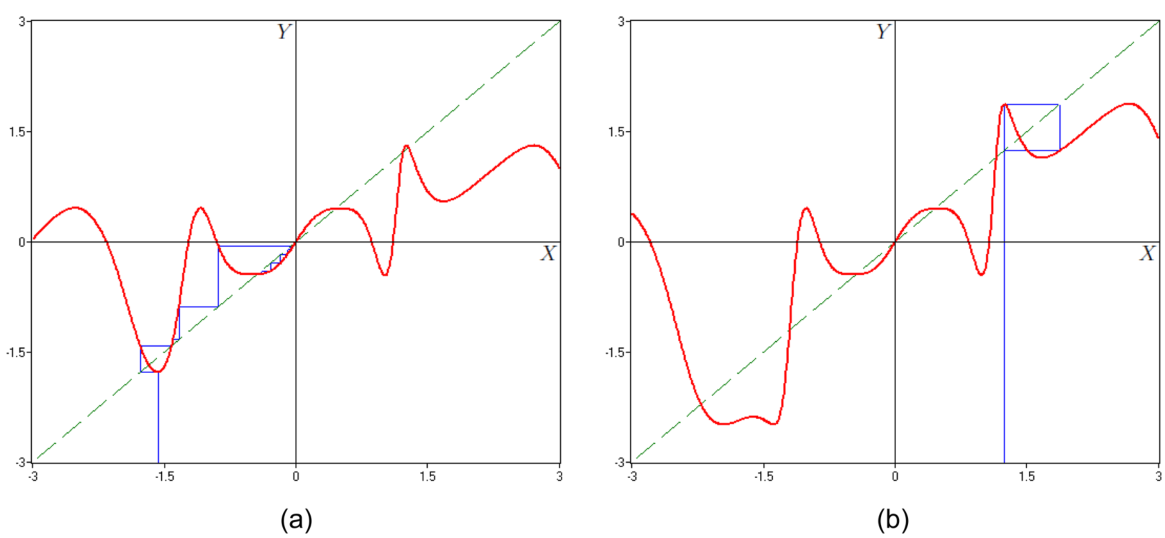

FIG. 15. In regard to the scenario depicted in Figure 12, through the graph of Φ^2, in (a), we show for $a_1 = 1.412$ how the iterates of $X(0) = -1.57$ limit towards X_1^*, while in (b) we show for $a_1 = 1.695$ how the iterates of $X(0) = 1.24$ generate a period-two cycle for Φ^2.

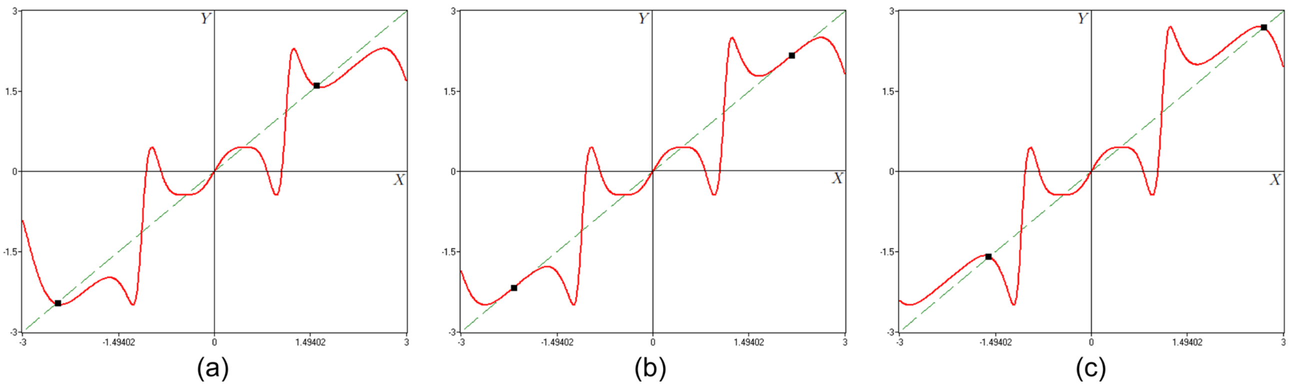

FIG. 16. For the same parameter values considered in Figure 12, we show how the shape of Φ^2 varies in correspondence to the stable external period-two cycle, denoted by black squares, when increasing a_1 from 1.9 in (a), to 2 in (b), and further to 2.1 in (c), while keeping $a_2 = 2$.

Increasing the value of a_1 further, the rightmost maximum points of Φ^2 raise, as well as its rightmost minimum point, until for $a_1 \approx 1.695$ a period-two cycle for Φ^2, i.e., a period-four cycle for Φ, emerges on the right on the separating unstable period-two cycle for Φ (see Figure 15(b)). Notice that this is the global bifurcation, not preceded by period-doubling bifurcations, through which the external attractor in Figure 12 reappears.

For still larger values of a_1, through a cascade of period doubling bifurcations, we reach a chaotic regime, after which, via period-halving bifurcations, the external attractor becomes a period-two cycle, because the minimum point at the right of the separating unstable period-two cycle for Φ raises and the inclination of the increasing branch changes. In particular, for $a_1 = 2$, we observe in Figure 12 two points which look like "inflexion points" for the map Φ, at which the values of the period-two cycle drastically change, as illustrated in Figure 16, where we investigate what happens to the period-two cycle when moving a_1 from 1.9 to 2.1, while keeping $a_2 = 2$. In particular, in Figure 16, we focus on the graph of Φ^2 and show how it intersects the 45° line in correspondence to the external stable period-two cycle, which we denote by black squares. We observe that the graph of Φ^2 raises in a neighborhood of such intersection points with the increase of a_1 and that Φ^2 is tangent to the 45° line exactly when $a_1 = a_2 = 2$, i.e., when Φ is odd. Such sensibility of the map for a_1 and a_2 in a neighborhood of the symmetry condition determines the occurrence of the "inflexion points."

Further increasing a_1, we observe again the emergence of complex regimes for the external attractor in Figure 12, which disappears for $a_1 \approx 2.355$. The reason of such sudden change is once more easy to find when looking at the behavior of Φ^2. Indeed, in Figure 17, we show that for $a_1 = 2.36$ the Φ^2-iterates starting from the region previously attracted by the external chaotic attractor now go beyond the separating unstable period-two cycle for Φ and tend towards X_1^* (in (a)) or towards X_3^* (in (b)). Again, this leads to the disappearance of the external attractor in Figure 12.

As concerns Figure 13, the external attractor therein is similar to that in Figure 12 and evolves in an analogous manner. Namely, it starts for $a_1 \approx 1.522$ with a stable period-two cycle born via a fold bifurcation of Φ^2, at which also an unstable period-two cycle arises. The former undergoes a cascade of period-doubling bifurcations and disappears for $a_1 \approx 1.540$, when the iterates previously attracted by it go

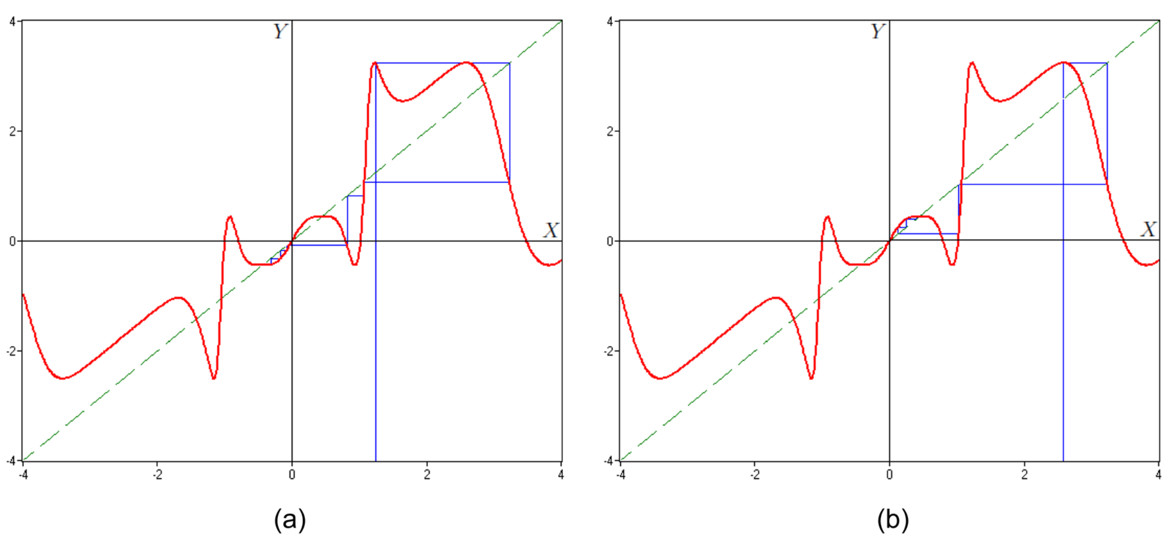

FIG. 17. In regard to the scenario depicted in Figure 12 for $a_1 = 2.36$, we show in (a) how the forward iterates of Φ^2 starting from $X(0) = 1.239$ tend towards X_1^*, and in (b) how the forward iterates of Φ^2 starting from $X(0) = 2.56$ tend towards X_3^*.

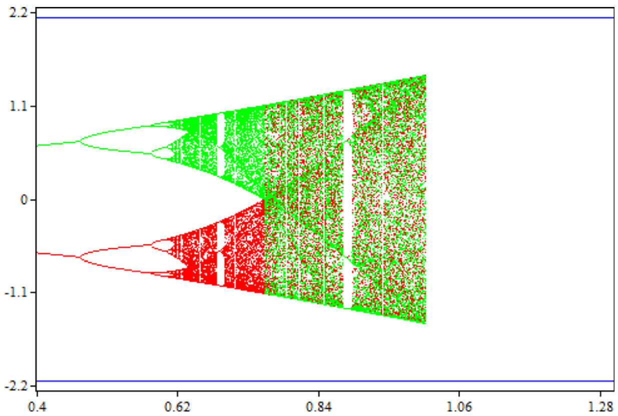

FIG. 18. The bifurcation diagram with respect to $e \in [0.4, 1.3]$ for the map Φ with $a_1 = a_2 = 2$, $d = 2.14$, $f = 1$, with initial conditions $X(0) = 0.1$ for the green dots, $X(0) = -0.1$ for the red dots, and $X(0) = 2.25$ for the blue dots, respectively, which highlights a multistability framework in which at $e \approx 1.006$ the internal attractor disappears and only the external one persists.

beyond the unstable period-two cycle, which acts as a "separatrix" from the basins of attraction of the internal attractors. The external attractor reappears again for $a_1 \approx 1.751$ with a chaotic band followed by a period-four cycle, which undergoes some period-doubling bifurcations leading to a period-sixteen orbit, followed by some period-halving bifurcations leading to a period-two cycle. As in Figure 12, we observe two "inflexion points" when $a_1 = a_2 = 2$ and then the period-two cycle undergoes a cascade of period-doubling bifurcations leading to the emergence of the chaotic regime, interrupted by some periodicity windows, and finally that attractor disappears for $a_1 \approx 2.295$, when the iterates previously attracted by it go beyond the separating unstable period-two cycle.

The two internal attractors emerge instead in the form of stable fixed points, losing their stability through period-doubling bifurcations, after which, through a classical cascade of period-doubling bifurcations, they become two chaotic attractors, which then merge into a unique chaotic attractor through a homoclinic bifurcation of X_2^*.[45] Notice that, as long as the external attractor does not exist yet, the internal attractors are not chaotic, and the maximum degree of complexity they both reach is given by a period-four orbit.

Finally, in Figure 18, we depict a framework in which, letting e vary and keeping the other parameters fixed, we find symmetric conclusions with respect to Figures 12 and 13. Indeed, in Figure 18, we still have up to three coexisting attractors, but now the two central ones, which become chaotic after a cascade of period-doubling bifurcations and merge for $e \approx 0.754$ via a homoclinic bifurcation of X_2^*, disappear for $e \approx 1.006$ and only the external attractor (i.e., the period-two cycle) survives. This happens because the iterates of m_1 and M_2 fall within the basin of attraction of the external period-two cycle. We illustrate such phenomenon in Figure 19 for $e = 1.009$, where in (a) we show that the forward iterates of M_2 converge towards the external stable period-two cycle, as they enter the basin of attraction of such cycle, we paint in light blue in (b).

We stress that the external (periodic or chaotic) attractor was not present in Ref. 33, where in fact the only possible attractors were the two central ones which, like in the present paper, could be given by fixed points, periodic orbits, or chaotic sets. The emergence of those attractors, their nature, and the way they merged bear a remarkable resemblance to our corresponding phenomena: on the other hand, as just mentioned, we have in addition the external attractor for suitable parameter configurations. Moreover, as argued in the Subsection V A, in our framework, the presence of the price-limiters reduces the complexity of the basins of attraction and prevents negativity and divergence issues.

VI. CONCLUSIONS

In our paper, we presented a stock market modeled in the sense of Day and Huang,[9] describing the nonlinear interactions between heterogeneous agents, i.e., chartists and fundamentalists.

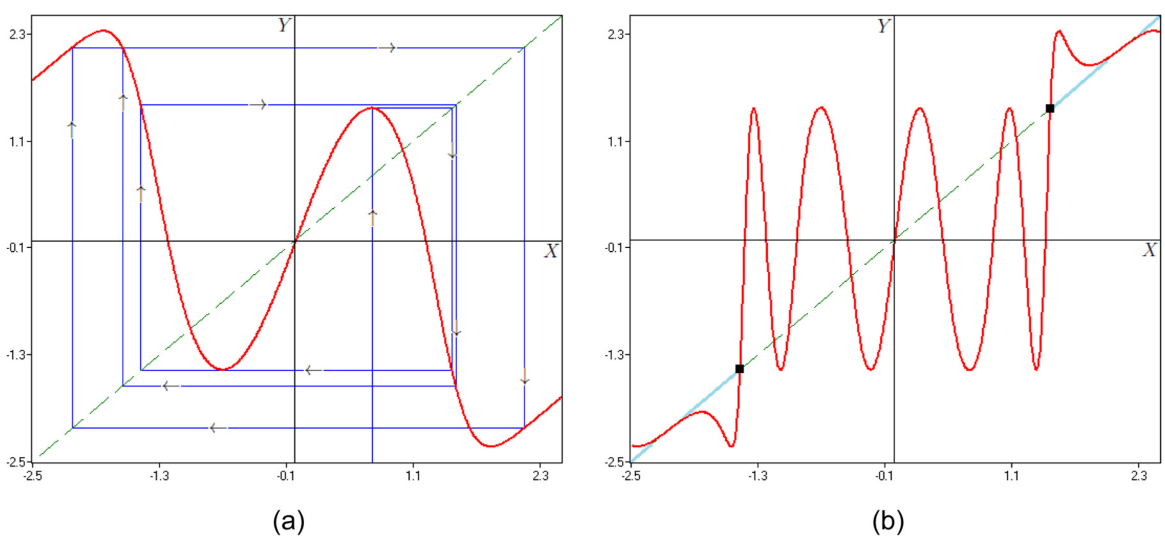

FIG. 19. In regard to the scenario depicted in Figure 18 for $e = 1.009$, we draw in (a) the graph of Φ, together with the forward iterates of M_2, that is, of $X(0) = 0.71$, which converge towards the external stable period-two cycle, and in (b) the graph of Φ^2, in which we put in evidence through black squares the unstable period-two cycle, that bounds the basin of attraction of the stable period-two cycle, we paint in light blue.

We chose a simple context in order to show how to fix some issues typical of those works dealing with speculative behavior in financial markets and with the dynamics of relevant variables, such as prices and traded quantities. First, from a formal viewpoint, we aimed to avoid the divergence and negativity issues caused by an overreaction when modeling the mechanism that describes disequilibrium price dynamics as a linear relation. Second, from a normative viewpoint, we aimed to propose a method in view of controlling the instabilities and strong irregularities that often arise when introducing nonlinearities into deterministic frameworks, so as to reduce turbulence, as well as to decrease the size of the attractors in the phase space and their complexity, in order to achieve the convergence to a fixed point.

In order to reach our goals, we exploited a technique for the chaos control, which bears resemblance to the limiter method by Wieland and Westerhoff.[40] In fact, to avoid overreaction phenomena and an excessive volatility in the stock market, we assumed a market maker forced by a central authority to be more cautious in adjusting the stock price when excess demand is large. This kind of diversified behavior can be represented by a nonlinear function only, which has also to be increasing and to pass through the origin. In particular, our price variation limiter mechanism was described by a sigmoidal adjustment rule characterized by the presence of two asymptotes that bound the price variation and thus the dynamics.

In view of showing the functioning of our mechanism and its effectiveness in achieving price stabilization, we chose as benchmark framework a special case of the model in Ref. 33, in which the presence of a cubic demand function for fundamentalists leads to multistability phenomena characterized by the coexistence of different attractors, which merge through a homoclinic bifurcation.

We found similar results to those obtained in Ref. 33, even if, as desired, the presence of our asymptotes prevents the divergence and negativity issues therein. Moreover, when dynamics are chaotic in Ref. 33, we can stabilize the system through a suitable choice of the asymptotes. Finally, when in Ref. 33 a unique attractor is present, we are able to limit its size and complexity, as well as to reach again a complete stabilization.

On the other hand, for some intermediate values of the asymptotes and particular configurations of the remaining parameters, in addition to two coexisting attractors similar to those in Ref. 33, we found a third attractor, which may be periodic or chaotic, independently of the nature of the other two attractors. Moreover, according to the considered parameter set, it may happen that the third attractor persists, while the other two disappear, or vice versa, that the third attractor disappears when the other two are still present. From a normative viewpoint, this means that just the imposition of our price variation limiter mechanism in the financial market is not sufficient in view of stabilizing the system. To such aim we need the asymptotes bounding the dynamics to be sufficiently close, because for intermediate distances between them, neither too large nor too small, we find multistability phenomena characterized by the coexistence of attractors with disconnected basins of attraction.

In the present paper, we proposed a simple financial model, in order to show the efficacy of the price variation limiter mechanism in decreasing the complexity of the dynamics and in achieving price stabilization. Of course, the same technique may be applied to commodity market models such as the cobweb and the oligopoly models.

ACKNOWLEDGMENTS

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[42]For instance, when $a_1 = 0.5$, $a_2 = 0.01$, $d = 2$, $e = 5$ and $f = 4$, there exist M_1 and m_1 with $M_1 < m_1 < X_1^*$, while M_2 and m_2 have not emerged yet.
[43]In order to show that $\Phi(m_2) < 0 < \Phi(M_1)$ when a_1 and a_2 are large enough, we can proceed as follows. First of all, let us observe that when a_1 and a_2 are sufficiently large, then $\Phi'(X_1^*) = \Phi'(X_3^*) = 1 - \frac{2ed}{\left(\frac{1}{a_1}+\frac{1}{a_2}\right)} < 0$ and thus $M_1 < X_1^* < m_1 < 0 < M_2 < X_3^* < m_2$. Let us now consider separately the cases $a_1 \le a_2$ and $a_1 \ge a_2$. If $a_1 \le a_2$, we rewrite Φ as follows: $\Phi(X) = X + da_1\left(\frac{1-\exp(-(eX-fX^3))}{\frac{a_1}{a_2}\exp(-(eX-fX^3))+1}\right)$, observing that $0 < \frac{a_1}{a_2} \le 1$ and that $1-\exp(-(eX-fX^3)) > 0$ for $X < -\sqrt{\frac{e}{f}} = X_1^*$, while $1-\exp(-(eX-fX^3)) < 0$ for $X > \sqrt{\frac{e}{f}} = X_3^*$. Hence, since $M_1 < X_1^*$ and $X_3^* < m_2$, when $a_1 \le a_2$ and a_1 is large enough we have that $\Phi(m_2) < 0 < \Phi(M_1)$, as desired. A very similar argument can be employed for the case in which $a_1 \ge a_2$, suitably rewriting Φ.
[44]We stress that, when drawing a bifurcation diagram, we chose the starting points by looking closely at the modifications in the shape of the graph of the first iterates of the map Φ and also at the time series of the state variable, on varying the value of the parameter under consideration. Hence, we do believe our choices being well pondered and thus preventing loss of information, such as the existence of further branches not considered in the paper, because their presence is excluded by the shape of Φ and of its iterates. Moreover, because of the way we selected the initial conditions for our bifurcation diagrams, our choices about the number of decimals cannot have a deep influence on the obtained results. Nonetheless, in order to make the discussion more consistent, we tried to homogenize the number of utilized decimals along the paper, setting it equal to three, especially in the descriptions of the bifurcation diagrams. We remark that when the number of decimals is less than three is because those are exact numbers and thus we do not need to round or truncate them.
[45]We stress that there are two cases in which we end up with a unique attractor: the first one is when, like in the present framework, there are no (more) external attractors and the internal ones merge through a homoclinic bifurcation; the second possibility, which we observe in Figure 18, is instead that the internal attractor disappears and all orbits are attracted by the external one.