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Highlights

- We analyze Cournot oligopolies with heterogeneous firms of generic size.
- Rational and naive players are considered.
- Stability with respect to oligopoly composition is studied.
- In some settings, increasing the rational firms fraction introduces instability.

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Heterogeneity and the (de)stabilizing role of rationality

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ABSTRACT

In this paper we study oligopolies of generic size consisting of heterogeneous firms, which adopt best response adjustment mechanisms with either perfect foresight (rational firms) or static expectations (naive firms). Assuming an isoelastic demand function and larger total costs for the rational firms, we focus on the local stability of the Nash equilibrium. We show that, with respect to the oligopoly composition, described in terms of the fraction of rational firms, different scenarios are possible. We find that a high rationality degree may not always guarantee stability, in particular when rational firms have sufficiently larger marginal costs. In fact, in this situation, increasing the fraction of rational firms can even introduce instability. Besides the usual scenarios in which replacing some naive firms with rational ones leads to a stabilization of (or at least keeps unchanged) the dynamics, we provide a family of situations, characterized by costs ratio favorable to naive firms, in which equilibrium loses its stability when naive firms are replaced by rational ones. The results we present are both analytical and simulative.

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1. Introduction

In modern game theory, rationality concerns the capability of a player to collect information and to use it to compute his optimal strategy with respect to the other players' strategies. In an oligopolistic Cournotian competition, in which few firms controlling the market compete in the amount of output they produce, a firm is fully rational if it is able to forecast its competitors' strategies and to compute the production level that allows maximizing its profit. However, such a high degree of rationality is not always easy or realistic to achieve, since it implies a perfect knowledge of both market and competitors' strategies (*perfect foresight*). Even in the first formal theory of oligopoly developed by Cournot in 1838 (see [1]), the firms are supposed not to know their competitors' next period production levels, and so they use the so-called "static" expectations, namely they assume that the other firms will produce in the next period the same quantity of the previous one. If all the firms were supposed to be fully rational, they would be able to choose their optimal choice in one shot, simultaneously achieving the Nash equilibrium. Conversely, if some firms are not fully rational, they have to gradually adapt their production level, giving rise to a dynamical adjustment mechanism. Under suitable conditions, such dynamics can be convergent to the equilibrium, but in general they can also give rise to unstable, both periodic and chaotic, complex output level trajectories. Since in the framework in which all the firms are fully rational (which is actually static) the equilibrium is always stable, it is natural to assume a stabilizing role for the rational firms.

Considering oligopolies of variable size, several authors (see for instance the contributions by Palander in [2], Theocharis in [3], Bischi et al. in [4], Lampart in [5],

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Matsumoto and Szidarovski in [6], Naimzada and Tramontana in [7]) showed in various economic contexts that the equilibrium stability can severely change when the oligopoly size increases, having the size a general destabilizing role. However, the previous contributions concerned *homogeneous* oligopolies, i.e. oligopolies in which all firms adopt the same decision rule to choose their production levels. Conversely, the works concerning *heterogeneous* oligopolies, in which at least two firms adopt different decision mechanisms, mainly focused on studying different coupling of distinct decision mechanisms in oligopolies of fixed reduced sizes, usually duopolies or triopolies. We mention the papers by Agiza et al. [8–12], Angelini et al. [13], Bischi et al. [14–16], Matsumoto [17], Cavalli et al. [18–20] and Tramontana [21].

Our contribution aims to investigate the role of oligopoly size and composition for heterogeneous competitions. The heterogeneity we investigate concerns the degree of rationality of the firms, as we consider two different informational capabilities. Both the kind of firms we consider use *best reply* mechanisms to choose their strategy, but *rational* firms have full informational and computational capabilities to solve the resulting optimization problem and possess perfect foresight, while *naive* firms are not able to predict such strategies, and assume instead static expectations. We focus on an economy characterized by an isoelastic demand function and we consider linear total costs for all the firms. Moreover, we assume that naive firms, due to a limited rationality degree, may adopt a more cautious behavior, and so they do not immediately choose the production level they computed using the static expectations best response, but they more prudently adapt their strategy toward the expected profit maximizing production level.

The present work belongs to a research strand which includes [22], in which the same behavioral rules were studied in an economy characterized by a linear demand function. Moreover, in [22] also the possibility for the firms to switch the adjustment mechanisms was considered. We remark that, in the existing literature, the study of heterogeneous oligopolies of generic size can be found in the works by Anufriev et al. [23] (where the role of heterogeneity in learning is investigated in a Bertrand oligopoly), Banerjee and Weibull [24] (where an evolutive game with agents heterogeneous in the rationality degree is considered), Droste et al. [25] (where an infinite population of firms is studied with respect to the possibility to switch among different decisional mechanisms), Gale and Rosenthal [26] (in which experimentation and imitation behaviors are analyzed).

Conversely, in our contribution we study equilibrium stability in an oligopoly consisting of firms which can be different with respect to the degree of rationality, the technology and the adopted adjustment mechanism. The main question we address is the following: does an increase in the number of rational firms always improve the equilibrium stability? To try to give an answer, for any oligopoly size, we consider all the possible compositions of firms heterogeneous with respect to the rationality degree, parameterizing such compositions through the fraction ω of rational firms. This gives rise to a multidimensional discrete dynamical system, whose equilibrium stability we investigate on varying ω.

Our setting is similar to those studied by Hommes et al. in [27] and Bischi et al. in [28], which focus on oligopolies of generic sizes in which the firms can adopt different decisional mechanisms. Indeed, in [27] the authors study oligopolies of rational and naive firms which are heterogeneous just with respect to the rationality degree. Moreover, naive firms, at each time step, play the best response to the average global strategy, and not to the actual strategy of each other firm. Such assumptions only allow for stabilizing or unconditionally stable scenarios. Conversely, in [28] the authors study oligopolies in which all the firms are boundedly rational, as they can be either naive or use a local monopolistic approximation rule, extending to heterogeneous oligopolies the investigations about the effect on the equilibrium stability of such bounded rationality mechanisms, previously known for homogeneous settings only [4,5]. Moreover, both in [27] and [28] the evolutionary fraction setting, in which firms can switch among heuristics, is studied too.

Our main result concerns the existence of different possible behaviors with respect to the variation of the fraction of rational agents, as increasing their fraction ω in an oligopoly of given size may have, besides neutral or stabilizing, also a destabilizing role. To the best of our knowledge, the latter behavior has not been observed in the models studied in the existing literature, in which increasing the fraction of rational players can only have a stabilizing (or at least neutral) effect, as for example in [22,27]. This is mainly due to the presence in our model of two stability thresholds, with respect to the oligopoly composition, which occur when marginal costs of rational firms are different from those of naive firms. Conversely, in the examples investigated in the existing literature, stability is always regulated by a single threshold. In particular, also in [22], where a linear demand function is considered, stability is regulated by a unique threshold, even if the considered decision mechanisms are the same as in the present work. This suggests that the ambiguous effect on stability of the oligopoly composition is due to the presence of a nonlinear demand function. Conversely, both in the present work and in [22], when total costs are identical for all the firms, we recover the usual unambiguous stabilizing role for rational firms.

We show that the existence of two stability thresholds allows for the following possible scenarios, depending on the parameters configuration, and in particular on the value of the costs ratio.

- Unconditionally stable/unstable scenario: in this case, the equilibrium stability is unaffected by the fraction of rational firms, and the trajectories converge/do not converge to the equilibrium for any composition. If the dynamics do not converge, both periodic and chaotic dynamics are possible.
- Stabilizing scenario: in this case, when the fraction of rational firms is too small, the equilibrium is unstable, but its stability can be recovered replacing some naive firms with rational ones, namely increasing ω. It is a classical situation in which increasing the overall degree of rationality leads the dynamics to stabilization.
- Destabilizing scenario: in this case, increasing the fraction of rational players destabilizes the equilibrium. This is the most counterintuitive scenario, as it provides examples showing that increasing the overall rationality can impair stability. In particular, we show that this scenario

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can occur for any oligopoly size, but only if naive firms are technologically more efficient than rational firms.

A theoretically possible fourth framework is the mixed scenario, in which the fraction of rational firms has a further ambiguous role, since when it is too small, the equilibrium is stable, in an intermediate region of values for ω stability is recovered and then, if the number of rational firms is further increased, stability is lost again. In this work, we mainly focus our attention on dynamics involving strictly positive production levels. Under such hypothesis, we prove that the mixed scenario is actually impossible. If however we relax the positivity hypothesis and we consider also null production levels, we are able to show through simulations that the mixed scenario is possible and that the destabilizing scenario occurs for a wider parameters setting.

We remark that the previous scenarios are found in a context of general heterogeneity among the firms, which involves the informational endowment (different rationality degrees for the firms), the technology (different marginal costs) and the adjustment mechanisms (naive firms are assumed to cautiously adjust their production level). In particular, the destabilizing nature of the third scenario cannot be completely ascribed to the influence of rationality, as other heterogeneity aspects are involved. With this respect, we find that the destabilizing scenario is possible only if marginal costs are suitably unfavorable to the rational firms, while it cannot occur if a similar, but unfavorable to naive firms, level of heterogeneity in technology is considered. Moreover, in order to disentangle the effect of heterogeneity in the adjustment mechanisms, we investigate the consequences of assuming the same cautious adjustment rule for all the firms and we find that the destabilizing scenario still occurs.

The remainder of the paper is organized as follows. In Section 2 we describe the oligopolistic competition we want to study and we present the model. In Section 3 we perform the local stability analysis and we discuss the possible arising scenarios. In Section 4 we report some simulative results. In Section 5 we investigate the effects of assuming the same cautious adjustment mechanism for all the firms. In Section 6 we draw some conclusions and we outline future research developments. Finally, in Appendix A we collect some technical results used for the proofs in Section 3.

2. Oligopolistic Cournot game

In this section we present the oligopolistic market we want to study and the model through which it can be described. Since the firms we consider are not all endowed with full rationality, the resulting model is represented by a discrete dynamical system in which, at each time period, the firms repeatedly interact on the same market to maximize their profits. In this first part of the section, after describing the market, we present the oligopolies we focus on and the Nash equilibrium of the corresponding oligopolistic Cournotian competition. Moreover, we show in Proposition 2.1 how the components of the Nash equilibrium vary when modifying the total number of firms and the fraction of rational firms. Then we enter into detail about the decisional mechanisms and the dynamical model in Section 2.1.

We consider an economy characterized by a homogeneous market controlled by $N \geq 2$ firms producing quantities $q_i, i = 1, \dots, N$, of the same good, for which the price function is given by

$$p(Q) = \frac{1}{Q}, \tag{2.1}$$

where $Q = \sum_{i=1}^{N} q_i$ is aggregate demand. Function (2.1) is the same isoelastic one used for example by Puu [29], microfounded on Cobb–Douglas preferences. We suppose that firm $i = 1, \dots, N$ faces the linear total cost function

$$C(q_i) = c_i q_i + C_i,$$

where $c_i > 0 \in \{c_{\mathcal{R}}, c_{\mathcal{N}}\}, i = 1, \dots, N,$ are two possibly different (constant) marginal costs and $C_i > 0 \in \{C_{\mathcal{R}}, C_{\mathcal{N}}\}, i = 1, \dots, N,$ are two possibly different fixed costs. The choice to use subscripts $\mathcal{R}, \mathcal{N}$, which will refer to rational and naive agents, will become more clear in the next section. In particular, in agreement with the existing literature, we assume that $C_{\mathcal{R}} > C_{\mathcal{N}}$, as the higher informational costs incurred by rational firms are reflected by larger fixed costs. The possible difference in the marginal costs allows instead considering technological heterogeneity among the firms.

Without loss of generality, we order the firms so that the first ωN firms, indexed by $i = 1, 2, \dots, \omega N$, have marginal costs $c_{\mathcal{R}}$, while the remaining $N(1-\omega)$ firms, indexed by $i = \omega N + 1, \dots, N$ have marginal costs $c_{\mathcal{N}}$. In particular, ω represents the fraction of firms belonging to the first group. We want that each group consists of at least one firm, so we impose that

$$\frac{1}{N} \leq \omega \leq 1 - \frac{1}{N}. \tag{2.2}$$

In the previously described setting we have a game, where the players are represented by the N oligopolists, the feasible strategies are the nonnegative production levels $q_i \geq 0$ and the payoff functions are given by the profit functions

$$\pi_i = pq_i - c_i q_i - C_i, \quad i = 1, \dots, N.$$

We only consider the situation in which all firms have non null production levels, so that the resulting oligopoly will always be composed by N firms, and we focus on the internal Nash equilibrium, namely, the equilibrium consisting of strictly positive strategies. A simple but tedious computation shows that such internal equilibrium is characterized by

$$q_i^* = \frac{(c_{\mathcal{N}} N(1-\omega) - c_{\mathcal{R}}(N(1-\omega)-1))(N-1)}{N^2(c_{\mathcal{N}}(1-\omega) + c_{\mathcal{R}}\omega)^2} = q_{\mathcal{R}}^*, \quad i = 1, \dots, \omega N \tag{2.3}$$

and

$$q_i^* = \frac{(c_{\mathcal{R}} N\omega - c_{\mathcal{N}}(N\omega - 1))(N-1)}{N^2(c_{\mathcal{N}}(1-\omega) + c_{\mathcal{R}}\omega)^2} = q_{\mathcal{N}}^*, \quad i = \omega N + 1, \dots, N. \tag{2.4}$$

Introducing the cost ratio

$$r = \frac{c_{\mathcal{R}}}{c_{\mathcal{N}}}, \tag{2.5}$$

it is easy to see that quantities (2.3) and (2.4) are both strictly positive provided that

$$\frac{N\omega - 1}{N\omega} < r < \frac{N(1-\omega)}{N(1-\omega) - 1}, \tag{2.6}$$

which means that, for a fixed oligopoly size N and composition ω, only for sufficiently similar values of marginal costs for all firms both the production levels are nonnegative. We shall maintain (2.6) for the remainder of the paper. Notice that the interval given by (2.6) is always nonempty. In particular, we stress that the upper bound on r imposed by (2.6) increases with ω. For instance, when $\omega = 1 - \frac{1}{N}$ the above condition imposes no upper bound on r, while for $\omega \leq 1 - \frac{2}{N}$ the right inequality in (2.6) implies that $r < 2$. Similarly, the lower bound on r increases with ω. For instance, when $\omega = 1 - \frac{1}{N}$, we need $r > 1 - \frac{1}{N-1}$, while $\omega = 1/N$ does not impose any lower bound conditions on the cost ratio. We notice that the more unfavorable the cost ratio is to a particular group of firms, the more numerous that group shall be to preserve the positivity of the equilibrium. In what follows, we will focus on sets of oligopolies which, having the same size N, can differ for their composition ω. In particular, for a given oligopoly size N and cost ratio r, we will consider

$$\mathcal{F}_{N,r} = \{\omega : q_i^* > 0,\ i = 1, \dots, N\},$$

which represents the *family* of all the oligopoly compositions for which the equilibrium is strictly positive. The number of oligopoly compositions with positive equilibrium is then represented by the cardinality of the set $\mathcal{F}_{N,r}$, namely by $\#(\mathcal{F}_{N,r})$. From (2.6) and the previous considerations, for $r = 1$ we have that $\mathcal{F}_{N,r} = \{1/N, \dots, 1 - 1/N\}$, while for $r < 1$ (respectively $r > 1$) we have that $\mathcal{F}_{N,r} = \{1/N, \dots, n(r)/N\}$ (respectively $\mathcal{F}_{N,r} = \{1 - n(r)/N, \dots, 1 - 1/N\}$), where $n(r)$ is a suitable integer in $\{1, \dots, N-1\}$ which depends on r and represents the size of the family. For example, if we wanted to consider the complete family of heterogeneous oligopolies of a given size N, namely the family consisting of all the possible compositions $\omega = 1/N, \dots, 1 - 1/N$, we would need to restrict our attention to the most restrictive upper and lower constraints on the cost ratio. A straightforward computation shows that this implies $1 - 1/(N-1) < r < 1 + 1/(N-2)$.

In the next result we investigate how $q_{\mathcal{N}}^*$ and $q_{\mathcal{R}}^*$ vary when modifying ω and N.

Proposition 2.1. *It holds that:*

- *if* $r < 1$, *then* $\frac{\partial q_{\mathcal{N}}^*}{\partial \omega} < 0$ *and* $\frac{\partial q_{\mathcal{N}}^*}{\partial N} < 0$;
- *if* $1 < r < \frac{N\omega + N - 2}{N\omega}$, *then* $\frac{\partial q_{\mathcal{N}}^*}{\partial \omega} > 0$ *and* $\frac{\partial q_{\mathcal{N}}^*}{\partial N} < 0$;
- *if* $r > \frac{N\omega + N - 2}{N\omega}$, *then* $\frac{\partial q_{\mathcal{N}}^*}{\partial \omega} < 0$ *and* $\frac{\partial q_{\mathcal{N}}^*}{\partial N} > 0$;
- $\frac{\partial q_{\mathcal{N}}^*}{\partial \omega} = 0$ *for* $r = 1$ *or* $r = \frac{N\omega + N - 2}{N\omega}$, *and* $\frac{\partial q_{\mathcal{N}}^*}{\partial N} = 0$ *for* $r = \frac{N\omega + N - 2}{N\omega}$;
- *if* $r < \frac{N(1-\omega)}{2N - N\omega - 2}$, *then* $\frac{\partial q_{\mathcal{R}}^*}{\partial \omega} > 0$ *and* $\frac{\partial q_{\mathcal{R}}^*}{\partial N} > 0$;
- *if* $\frac{N(1-\omega)}{2N - N\omega - 2} < r < 1$, *then* $\frac{\partial q_{\mathcal{R}}^*}{\partial \omega} < 0$ *and* $\frac{\partial q_{\mathcal{R}}^*}{\partial N} < 0$;
- *if* $r > 1$, *then* $\frac{\partial q_{\mathcal{R}}^*}{\partial \omega} > 0$ *and* $\frac{\partial q_{\mathcal{R}}^*}{\partial N} < 0$;
- $\frac{\partial q_{\mathcal{R}}^*}{\partial \omega} = 0$ *for* $r = 1$ *or* $r = \frac{N(1-\omega)}{2N - N\omega - 2}$, *and* $\frac{\partial q_{\mathcal{R}}^*}{\partial N} = 0$ *for* $r = \frac{N(1-\omega)}{2N - N\omega - 2}$.

Proof. Direct computations show that

$$\frac{\partial q_{\mathcal{N}}^*}{\partial \omega} = \frac{(N-1)(c_{\mathcal{R}} - c_{\mathcal{N}})(c_{\mathcal{N}}N - 2c_{\mathcal{N}} + c_{\mathcal{N}}N\omega - c_{\mathcal{R}}N\omega)}{N^2(c_{\mathcal{R}}\omega + c_{\mathcal{N}} - c_{\mathcal{N}}\omega)^3}$$

and

$$\frac{\partial q_{\mathcal{N}}^*}{\partial N} = \frac{-(c_{\mathcal{N}}N - 2c_{\mathcal{N}} + c_{\mathcal{N}}N\omega - c_{\mathcal{R}}N\omega)}{N^3(c_{\mathcal{R}}\omega + c_{\mathcal{N}} - c_{\mathcal{N}}\omega)^2}.$$

Recalling the definition of r in (2.5), the desired conclusions on $q_{\mathcal{N}}^*$ easily follow. Similar computations allow to derive the desired conclusions on the behavior of $q_{\mathcal{R}}^*$, too. □

Hence, $q_{\mathcal{N}}^*$ is increasing with ω when $1 < r < \frac{N\omega + N - 2}{N\omega}$ and decreasing with ω for the remaining values of r, while $q_{\mathcal{N}}^*$ is increasing with N when $r > \frac{N\omega + N - 2}{N\omega}$ and decreasing with N otherwise; $q_{\mathcal{R}}^*$ is increasing with ω for $r < \frac{N(1-\omega)}{2N - N\omega - 2}$ or $r > 1$, and decreasing with ω for the remaining values of r, while $q_{\mathcal{R}}^*$ is increasing with N for $r < \frac{N(1-\omega)}{2N - N\omega - 2}$ and decreasing with N otherwise. When the marginal costs for all firms coincide, i.e., when $r = 1$, then $q_{\mathcal{N}}^*$ and $q_{\mathcal{R}}^*$ are not influenced by ω, but both of them are decreasing with the number of firms. In fact, in this particular case it holds that

$$q_{\mathcal{N}}^* = q_{\mathcal{R}}^* = \frac{N-1}{cN^2}.$$

2.1. The decision mechanisms

If all the firms were endowed with full rationality, namely they had complete informational and computational capabilities, the firms would simply choose at once the Nash equilibrium (2.3) and (2.4). Conversely, in this work we assume that only a subset of the firms are fully rational, so that they have complete knowledge of the market and they are able to optimally respond to the other firms strategies. In addition to this, they are endowed with perfect foresight, so that they exactly forecast the next time production levels of their opponents. We will refer to such firms as to *rational firms*. To encompass their high informational capability, we associate rational firms to the first group of firms, which have larger total costs. Conversely, the firms belonging to the second group lack in perfect foresight and we assume for them static expectations (*naive firms*). This setting gives rise to a dynamic adjustment of the production levels, in order to maximize the one-period profits, and the resulting oligopoly turns out to be heterogeneous in terms of the decision mechanisms. This means that, if we denote by $q_{j,t+1}^{e,i}$ the production for period $t+1$ of firm j expected by firm i, we have

$$q_{j,t+1}^{e,i} = q_{j,t+1}, \quad 1 \leq i \leq \omega N,\ 1 \leq j \leq N, \tag{2.7}$$

for the rational firms and

$$q_{j,t+1}^{e,i} = q_{j,t}, \quad \omega N + 1 \leq i \leq N,\ 1 \leq j \leq N, \tag{2.8}$$

for the naive ones. We remark that ω now also represents the fraction of rational firms. Condition (2.2) assures that the considered oligopoly is always heterogeneous. Since for $N = 2$ we have only one possible heterogeneous composition consisting of a rational and a naive firm, in what follows we will focus on the case $N > 2$, in which it is meaningful to investigate the effect of varying the oligopoly composition. Moreover, the case with $N = 2$ firms has already been studied in [20].

Let us now specify the strategies for the two groups of firms. At each time, all the players try to choose their production level in order to maximize their one-period profits. If $Q_{t+1}^{e,i}$ is the total output expected for the next period by firm i

$$Q_{t+1}^{e,i} = \sum_{j=1,\, j\neq i}^{N} q_{j,t+1}^{e,i} + q_{i,t+1},$$

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we have that its expected profit at $t+1$ is

$$\pi^e_{i,t+1} = \frac{q_{i,t+1}}{\sum_{j=1,\, j\neq i}^{N} q^{e,i}_{j,t+1} + q_{i,t+1}} - c_s q_{i,t+1} - C_s,$$

where $s=\mathcal{R}$ for $i=1,\ldots,\omega N$ and $s=\mathcal{N}$ for $i=\omega N+1,\ldots,N$, we can write that the expected marginal profit is

$$\frac{\partial \pi^e_{i,t+1}}{\partial q_{i,t+1}} = \frac{(Q^{e,i}_{t+1} - q_{i,t+1})}{(Q^{e,i}_{t+1})^2} - c_s, \quad s=\mathcal{R},\mathcal{N}. \tag{2.9}$$

First of all we notice that, thanks to perfect foresight assumption (2.7), at each time step, all the rational firms have the same best response to the strategies of the other agents, namely, if i is a rational firm, we have that $q_{i,t+1}=q_{\mathcal{R},t+1}$ for $1\le i\le \omega N$, from which we then find $Q^{e,i}_{t+1}=Q_{t+1}=Q_{\mathcal{R},t+1}+Q_{\mathcal{N},t+1}=N\omega q_{\mathcal{R},t+1}+\sum_{i=\omega N+1}^{N} q_{i,t+1}$, where we introduced the aggregate production quantities $Q_{\mathcal{R}}$ and $Q_{\mathcal{N}}$ of the rational and naive firms, respectively. Inserting the expression for $Q^{e,i}_{t+1}$ into the right hand side of (2.9) and imposing the first order condition, we find

$$(N\omega q_{\mathcal{R},t+1} + Q_{\mathcal{N},t+1} - q_{\mathcal{R},t+1}) = c_{\mathcal{R}}(N\omega q_{\mathcal{R},t+1} + Q_{\mathcal{N},t+1})^2.$$

Solving with respect to $q_{\mathcal{R},t+1}$, we get two solutions, one of which is negative. Hence the only admissible solution is given by

$$\begin{aligned} q_{\mathcal{R},t+1} &= f(Q_{\mathcal{N},t+1}) \\ &= \frac{Nw-1-2c_{\mathcal{R}}N\omega Q_{\mathcal{N},t+1}+\sqrt{(Nw-1)^2+4c_{\mathcal{R}}N\omega Q_{\mathcal{N},t+1}}}{2c_{\mathcal{R}}N^2\omega^2}, \end{aligned} \tag{2.10}$$

where $f:(0,+\infty)\to\mathbb{R}$ is a positive function provided that $Q_{\mathcal{N},t+1}<1/c_{\mathcal{R}}$. In what follows, we will mainly focus on dynamics in which the production levels stay strictly positive. If not, the best response to $Q_{N,t+1}$ would be $q_{\mathcal{R},t+1}=0$.

Conversely, if $\omega N+1\le i\le N$ is a naive firm, the assumption of static expectations (2.8) gives $Q^{e,i}_{t+1}=\sum_{j=1,\, j\neq i}^{N} q_{j,t}+q_{i,t+1}$, which, inserted in (2.9), provides

$$\left(\sum_{j=1,\, j\neq i}^{N} q_{j,t}\right) = c_{\mathcal{N}}\left(\sum_{j=1,\, j\neq i}^{N} q_{j,t} + q_{i,t+1}\right)^2.$$

Solving with respect to $q_{i,t+1}$, we find two solutions, one of which is negative.

Calling $Q_{-i,t}$ the aggregate quantity produced by all the firms but the ith naive one, the other solution is given by

$$q_{i,t+1} = h(Q_{-i,t}) = \sqrt{\frac{Q_{-i,t}}{c_{\mathcal{N}}}} - Q_{-i,t}, \tag{2.11}$$

where $h:(0,+\infty)\to\mathbb{R}$, $Q_{-i,t}\mapsto h(Q_{-i,t})$, is the best response of the ith naive player provided that $Q_{-i,t}<1/c_{\mathcal{N}}$, otherwise $q_{i,t+1}=0$. We remark that $Q_{-i,t}$ is actually a function of $q_{j,t}$ with $j\neq i$, that is,

$$\begin{aligned} Q_{-i,t} &= Q_t - q_{i,t} = \sum_{j=1}^{\omega N} q_{j,t} + \sum_{j=\omega N+1,\, j\neq i}^{N} q_{j,t} \\ &= \omega N q_{\mathcal{R},t} + Q_{\mathcal{N},t} - q_{i,t}. \end{aligned} \tag{2.12}$$

Inserting the expression for $q_{\mathcal{R},t}$ from (2.10) into (2.12) we obtain

$$Q_{-i,t} = \omega N f(Q_{\mathcal{N},t}) + Q_{\mathcal{N},t} - q_{i,t}, \tag{2.13}$$

and consequently (2.11) becomes

$$\begin{aligned} q_{i,t+1} = &\sqrt{\frac{\omega N f(Q_{\mathcal{N},t}) + Q_{\mathcal{N},t} - q_{i,t}}{c_{\mathcal{N}}}} \\ &-(\omega N f(Q_{\mathcal{N},t}) + Q_{\mathcal{N},t} - q_{i,t}). \end{aligned} \tag{2.14}$$

Naive firms, due to their bounded rationality, act in an uncertainty context, as they are aware that adopting static expectations is not able to guarantee that the production level they choose for the next period actually provides their next period optimal profits. Then, a more cautious behavior can be expected for naive firms, due to the reduced degree of confidence in their own expectations. For a discussion about firms behavior we refer to the book of Sterman [30]. Examples of cautious adjustment mechanisms for naive firms can be found in [6,29]. Hence, we assume that naive firms do not immediately choose the production level they computed using the static expectations best response, but they more prudently adapt their strategy toward the expected profit maximizing production level. To model such effects, we suppose that the naive firms adopt the following adjustment mechanism

$$q_{i,t+1} = q_{i,t} + \sigma(h(Q_{-i,t}) - q_{i,t}), \quad i=\omega N+1,\ldots,N, \tag{2.15}$$

where $\sigma:\mathbb{R}\to\mathbb{R}$ is a function describing the possible production variation. This function is differentiable, strictly increasing and bounded, with $\sigma(0)=0$, so that it preserves the steady states of the best response functions for the naive players in (2.11).

The resulting model is then obtained considering the iterates of the $N(1-\omega)$-dimensional system generated by (2.15). We will call g the associated map.

We underline that, since ω can assume the values $1/N, 2/N, \ldots, (N-1)/N$, the dimension of (2.15) can vary between $N-1$ and 1.

We notice that the oligopolistic competition we study is described by the evolution of the strategies of both rational and naive players, namely by the ωN identical equations in (2.10) and the $(1-\omega)N$ equations in (2.15). On the other hand, the dynamical system that has to be analytically investigated is made up by the $(1-\omega)N$-dimensional system (2.15) only. We remark that it is easy to prove that when the strategies of the naive players converge, they necessarily converge to $q^*_{\mathcal{N}}$ (and, consequently, the strategies of the rational players converge to $q^*_{\mathcal{R}}$).

3. Stability analysis

In order to perform the local stability analysis for model (2.15), at first we need to distinguish between the case $\omega<1-1/N$, in which the model consists of a system of $N(1-\omega)$ equations, and the case $\omega=1-1/N$, in which the model consists of a single equation.

In the framework with $\omega<1-1/N$, we need to evaluate the $N(1-\omega)\times N(1-\omega)$ Jacobian matrix $J=(J_{ij})=\partial_{q_j} g_i$ at the Nash equilibrium, where g_i represents the r.h.s. of each equation in (2.15). Recalling (2.13)–(2.15), we have that

$$\begin{aligned} \partial_{q_i} g_i &= 1 + \sigma'(h(Q_{-i}) - q_i)(h'(Q_{-i})\partial_{q_i} Q_{-i} - 1) \\ &= 1 + \sigma'(h(Q_{-i}) - q_i)(\omega N h'(Q_{-i}) f'(Q_{\mathcal{N}}) - 1) \end{aligned}$$

and, for $i \neq j$,

$$\partial_{q_j} g_i = \sigma'(h(Q_{-i}) - q_i)h'(Q_{-i})\partial_{q_j} Q_{-i} \\ = \sigma'(h(Q_{-i}) - q_i)h'(Q_{-i})(\omega N f'(Q_{\mathcal{N}}) + 1),$$

where

$$f'(Q_{\mathcal{N}}) = \frac{1}{N\omega\sqrt{(1-N\omega)^2 + 4NQ_{\mathcal{N}}c_{\mathcal{R}}\omega}} - \frac{1}{N\omega}.$$

We denote by $J^* = (J^*_{ij})$ the Jacobian matrix J evaluated at the Nash equilibrium, and, noticing that $h(Q^*_{-i}) = q^*_{\mathcal{N}}$, we have

$$J^*_{ii} = 1 + \sigma'(0)(\omega N h'(Q^*_{-i})f'(Q^*_{\mathcal{N}}) - 1) = a \tag{3.1a}$$

and, for $i \neq j$,

$$J^*_{ij} = \sigma'(0)h'(Q^*_{-i})(\omega N f'(Q^*_{\mathcal{N}}) + 1) = b. \tag{3.1b}$$

This means that J^* matrix evaluated at the equilibrium has the structure

$$J(q^*) = \begin{pmatrix} a & b & b & \dots & b \\ b & a & b & \dots & b \\ \vdots & & \ddots & & \vdots \\ \vdots & & & \ddots & \vdots \\ b & \dots & b & b & a \end{pmatrix} \tag{3.2}$$

Since it is a particular case of circulant matrix (see for instance [31]), it allows us to explicitly compute the eigenvalues of (3.1).

Proposition 3.1. *The Jacobian matrix of System (2.15) evaluated at the Nash equilibrium has eigenvalues*

$$\lambda_1 = 1 - \sigma'(0)(h'(Q^*_{-i}) + 1),$$

with multiplicity $N(1-\omega) - 1$, and

$$\lambda_2 = 1 - \sigma'(0)(h'(Q^*_{-i}) - Nh'(Q^*_{-i})(1-\omega) \\ \times (N\omega f'(Q^*_{\mathcal{N}}) + 1) + 1),$$

which is simple.

Proof. If the elements of the first row of a circulant matrix are $c_1, \dots, c_{N(1-\omega)}$, its eigenvalues are given by

$$\hat{\lambda}_m = \sum_{j=1}^{N(1-\omega)} c_j \rho_m^{j-1}, \quad m = 1, \dots, N(1-\omega)$$

where $\rho_m = \exp(2\pi i(m-1)/(N(1-\omega)))$ are the $N(1-\omega)$ distinct complex roots of unity [31], with i representing the imaginary unit. In (3.2) we have that $c_1 = a$ and $c_j = b$ for $j = 2, \dots, N(1-\omega)$, so that we can rewrite

$$\hat{\lambda}_m = a + b\sum_{j=2}^{N(1-\omega)} \rho_m^{j-1}, \; m = 1, \dots, N(1-\omega).$$

For $m = 1$, $\rho_m = 1$ and hence $\hat{\lambda}_1 = a + (N(1-\omega) - 1)b$. Conversely, if we consider $m > 1$, we have that $\rho_m \neq 1$ and so we can write

$$\hat{\lambda}_m = a + b\rho_m \sum_{j=2}^{N(1-\omega)} \rho_m^{j-2}.$$

For $m \neq 1$ we have

$$\rho_m \sum_{j=2}^{N(1-\omega)} \rho_m^{j-2} = \rho_m \frac{\rho_m^{N(1-\omega)-1} - 1}{\rho_m - 1} = \frac{\rho_m^{N(1-\omega)} - \rho_m}{\rho_m - 1}$$

where, since $\rho_m^{N(1-\omega)} = 1$, we can conclude that $\rho_m \sum_{j=2}^{N(1-\omega)} \rho_m^{j-2} = -1$ and hence $\hat{\lambda}_m = a - b$ for $m \geq 1$. Using (3.1a) and (3.1b), we can conclude. □

Let us then consider the case with $\omega = 1 - 1/N$. Since now $Q_{\mathcal{N},t} = q_{\mathcal{N},t}$, by (2.13) system (2.15) reduces to the one-dimensional equation

$$q_{N,t+1} = g(q_{N,t}) = q_{N,t} + \sigma(h(\omega N f(q_{N,t})) - q_{N,t}), \tag{3.3}$$

from which we obtain

$$g'(q^*_N) = 1 + \sigma'(0)(h'(\omega N f(q^*_N))\omega N f'(q^*_N) - 1).$$

We have the following result:

Proposition 3.2. *Setting $k = \sigma'(0)$, the local stability of the Nash equilibrium requires that*

$$z(\omega) = N(r-1)(kr - k + 4)\omega^2 + (4N + 4r - 2Nk \\ -8Nr + 2Nkr + 4)\omega + Nk - 4 < 0. \tag{3.4}$$

Proof. Firstly, we show that the equilibrium stability is guaranteed in both the one-dimensional and in the multi-dimensional case by imposing $\lambda_2 > -1$. In the one-dimensional framework, the local stability of (3.3) requires $|g'(q^*_N)| < 1$. We notice that $g'(q^*_N)$ is equal to (3.1a) and it is easy to see that, for $\omega = 1 - 1/N$, we also have $\lambda_2 = a$ in (3.1a), so that the equilibrium stability in the one-dimensional case is guaranteed by $|\lambda_2| < 1$. For the multi-dimensional case we need $-1 < \lambda_s < 1$, $s = 1, 2$. However, we shall show that it holds $\lambda_2 < \lambda_1 < 1$ for each $1/N \leq \omega < 1 - 1/N$ and that $\lambda_2 < 1$ for $\omega = 1 - 1/N$, so that the only stability condition to be imposed is $\lambda_2 > -1$ for each $1/N \leq \omega \leq 1 - 1/N$, which leads to (3.4).

To this end, we notice that, since

$$Q^*_{-i} = \omega N q^*_{\mathcal{R}} + (N(1-\omega) - 1)q^*_{\mathcal{N}} = \frac{c_{\mathcal{N}}(N-1)^2}{N^2(c_{\mathcal{N}} + c_{\mathcal{R}}\omega - c_{\mathcal{N}}\omega)^2}$$

and recalling (2.5), we have

$$h'(Q^*_{-i}) = \frac{N\omega(r-1) + 2 - N}{2(N-1)}.$$

Since for $\omega \leq 1 - 2/N$ by (2.6) it follows that $r < 2$, we have $h'(Q^*_{-i}) < 0$. Moreover, since $N \geq 2$, we have

$$h'(Q^*_{-i}) + 1 = \frac{N\omega(r-1) + N}{2(N-1)} > 0,$$

which allows concluding that $-1 < h'(Q^*_{-i}) < 0$. This means that $\lambda_1 < 1$. Since

$$Q^*_{\mathcal{N}} = \frac{(Nc_{\mathcal{R}}\omega - c_{\mathcal{N}}(N\omega - 1))(N-1)(1-\omega)}{N(c_{\mathcal{R}}\omega + c_{\mathcal{N}}(1-\omega))^2}$$

we have

$$f'(Q^*_{\mathcal{N}}) = \frac{1}{N\omega\sqrt{\eta}} - \frac{1}{N\omega}, \tag{3.5}$$

where

$$\eta = \left(\frac{c_{\mathcal{R}}\omega(N(2-\omega) - 1) - c_{\mathcal{N}}(1-\omega)(N\omega - 1)}{c_{\mathcal{N}}(1-\omega) + c_{\mathcal{R}}\omega}\right)^2.$$

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We notice that

$$c_{\mathcal{R}}\omega(N(2-\omega)-1)-c_{\mathcal{N}}(1-\omega)(N\omega-1)>0,$$

because

$$r>\frac{(1-\omega)(N\omega-1)}{\omega(N(2-\omega)-1)}$$

is satisfied thanks to the lower bound on r given by (2.6), as

$$\frac{N\omega-1}{N\omega}-\frac{(1-\omega)(N\omega-1)}{\omega(N(2-\omega)-1)}=\frac{(N\omega-1)(N-1)}{N\omega(2N-N\omega-1)}\geq 0,$$

so that we can rewrite (3.5) as

$$f'(Q^*_{\mathcal{N}})=-\frac{N+2r-2Nr-N\omega+Nr\omega}{N(\omega+N\omega+r\omega-N\omega^2-2Nr\omega+Nr\omega^2-1)}.$$

Moreover, from (3.5) it follows that

$$1+\omega Nf'(Q^*_{\mathcal{N}})=\frac{1}{\sqrt{\eta}}>0.$$

The last relation, together with $h'(Q^*_{-i})<0$, allows concluding that $\lambda_2<\lambda_1<1$, for each $\omega=1/N,\dots,1-2/N$.

Inserting the expressions of $f'(Q^*_{\mathcal{N}})$ and $h'(Q^*_{-i})$ in λ_2, after some algebraic manipulations, we find

$$\lambda_2=-\frac{Nk(r\omega-\omega+1)^2}{2(\omega+N\omega+r\omega-N\omega^2-2Nr\omega+Nr\omega^2-1)}+1,$$

which for $\omega=1-1/N$ gives

$$\lambda_2-1=\frac{-k(Nr-r+1)^2}{2(N-1)(rN-1)+1}<0.$$

Hence, $\lambda_2<1$ for all $\omega=1/N,\dots,1-1/N$ and thus the Nash equilibrium is locally asymptotically stable if $\lambda_2>-1$, which can be rewritten as

$$\frac{N(1-r^2)\omega^2-2(r(1-N)+1)\omega-(N-2)}{2N(1-r)\omega^2-2(N+r-2Nr+1)\omega+2}>1-\frac{2}{k}.$$

Rearranging the last inequality leads to (3.4). □

Solving (3.4) with respect to ω, we can obtain the stability of the Nash equilibrium on varying the oligopoly composition.

When $r\neq 1$ or $k\neq 4/(1-r)$, since stability is governed by a second degree inequality, we have, at least in principle, two stability thresholds. Such framework is investigated in Proposition 3.3, whose proof simply requires to solve (3.4) with respect to ω. We stress that when $r=1$ (3.4) reduces to a first degree inequality and thus we find a unique threshold $\bar{\omega}=(Nk-4)/(4N-8)$ for ω, above which the system is locally asymptotically stable and below which it is not stable. Similarly, when $k=4/(1-r)$ (which is possible only if $r<1$), we again have a first degree inequality, which gives the unique threshold $\bar{\bar{\omega}}=(N+r-1)/(N+Nr-2Nr^2+r^2-1)$, above which we again have that the system is locally asymptotically stable.

We shall return on the framework with $r=1$ after having analyzed the more interesting and richer scenario with different marginal costs. Due to the similarity with the case $r=1$, we will not consider the framework with $k=4/(1-r)$ anymore.

Let us introduce

$$\omega_1=\frac{-2N-2r+Nk+4Nr-Nkr-2-\sqrt{\Delta}}{N(r-1)(kr-k+4)}$$

$$\omega_2=\frac{-2N-2r+Nk+4Nr-Nkr-2+\sqrt{\Delta}}{N(r-1)(kr-k+4)}\tag{3.6}$$

and notice that $\omega_1\leq\omega_2$ provided that $\Delta>0$ and $N(r-1)(kr-k+4)>0$, namely if either $r>1$ or $r<1$ and $k>4/(1-r)$. Conversely, when $r<1$ and $k<4/(1-r)$, we have $\omega_2\leq\omega_1$.

Proposition 3.3. *When the marginal costs of the rational firms are larger than those of the naive ones ($r>1$) and $\Delta>0$, the local stability at the Nash equilibrium requires that $\omega_1<\omega<\omega_2$. Otherwise, if $\Delta\leq 0$ the Nash equilibrium is always unstable. Conversely, when the marginal costs of the rational firms are smaller than those of the naive ones ($r<1$) and $\Delta>0$, the local stability at the Nash equilibrium requires that*

$$\omega_1<\omega<\omega_2\tag{3.7}$$

if $k>4/(1-r)$, or

$$\omega<\omega_2\ \vee\ \omega>\omega_1\tag{3.8}$$

if $k<4/(1-r)$. Otherwise, for $\Delta\leq 0$, the Nash equilibrium is always unstable if $k>4/(1-r)$ and always locally asymptotically stable if $k<4/(1-r)$.

We notice that, for $r>1$, condition $\Delta>0$ is equivalent to

$$k<k_\Delta=\frac{(4N^2-4N+1)r^2+2(-2N^2+N+1)r+N^2-2N+1}{2Nr(r-1)(N-1)}.$$

It is not difficult to see that, for any $r>1$ and $N\geq 3$, it holds that $k_\Delta>0$ and thus the previous condition is always fulfilled by some positive k. Conversely, when $r<1$ the sign of Δ is less clear and it is studied in Lemma 6.7.

In the remaining part of this section, we identify the occurrence (or the impossibility to occur) of some significant frameworks, depending on particular parameter settings. For the sake of clarity, we split the results concerning $r>1$ and $r<1$.

- **Case $r>1$**

In the next propositions, we prove the existence of unconditionally stable/unstable and stabilizing scenarios for oligopolies of any size. In particular, we show that these scenarios can occur for complete families of oligopolies, in which each heterogeneous composition has a strictly positive equilibrium, provided that the marginal cost ratio satisfies $r<1+1/(N-2)$.

We stress that results similar to Propositions 3.4–3.6 hold also for $N=3$ and $N=4$, but under different conditions on the parameters. For sake of brevity, we will not report such statements. Moreover, we remark that, for the sake of simplicity, we provide just sufficient conditions for the occurrence of the various scenarios, which are possible for other parameters conditions as well.

Proposition 3.4 (Unconditionally stable scenario)**.** *For any $N\geq 5$ and for sufficiently small values of k, there exist families $\mathcal{F}_{N,r}$ of oligopolies such that $\#(\mathcal{F}_{N,r})=N$ and for which the equilibrium is stable for any composition $\omega\in\mathcal{F}_{N,r}$. For such families the fraction of the firms with perfect foresight does not influence the stability of the steady state.*

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Proof. To have the unconditionally stable scenario we need that the whole interval of possible compositions be a subset of the stability interval, namely $\omega_1 < 1/N < 1 - 1/N < \omega_2$. From Lemmas 6.1 and 6.4, it is sufficient that $k < \min\{k_2, k_\Delta\} = k_2$ by (6.7), provided that $r < 1 + 1/(N-2)$. □

Proposition 3.5 (Unconditionally unstable scenario). *For any $N \geq 5$ and for sufficiently large values of k, there exist families $\mathcal{F}_{N,r}$ of oligopolies such that $\#(\mathcal{F}_{N,r}) = N$ and for which the equilibrium is unstable for all the compositions $\omega \in \mathcal{F}_{N,r}$. For such families the fraction of the firms with perfect foresight does not influence the stability of the steady state.*

Proof. To be in the unconditionally unstable scenario in principle we have three possibilities. We can have $\Delta < 0$, or equivalently $k > k_\Delta$. We could have $\omega_2 \leq 1/N$, which, however, from Lemma 6.3, is not possible if $r < 1 + 1/(N-2)$. Finally, we can have $\omega_1 \geq 1 - 1/N$ which, from (6.9) of Lemma 6.2, happens for $k_4 < k < \min\{k_3, k_\Delta\}$ and $r < 1 + 1/(N-2)$. □

Proposition 3.6 (Stabilizing scenario). *For any $N \geq 5$ and if k is neither too small nor too large, there exist families $\mathcal{F}_{N,r}$ of oligopolies such that $\#(\mathcal{F}_{N,r}) = N$ and for which the equilibrium is stable only if the number of rational firms is sufficiently large (namely $\omega > \bar{\omega}$, for suitable $\bar{\omega}$). For such families an increase in the fraction of rational firms leads to a local stabilization of the steady state.*

Proof. To have the stabilizing scenario we need $1/N < \omega_1 < 1 - 1/N < \omega_2$, inequalities that are considered in Lemma 6.1 in condition (6.2), in Lemma 6.2 in condition (6.8), in Lemma 6.4 in condition (6.15). Recalling (6.11), we have that the desired chain of inequalities holds true for $k_2 < k < k_4$, provided that $r < 1 + 1/(N-2)$. □

Theoretically, we may have two other possible frameworks: a destabilizing scenario, in which $\omega_1 < 1/N < \omega_2 < 1 - 1/N$, and a mixed one, in which $1/N < \omega_1 < \omega_2 < 1 - 1/N$. In both cases, suitably increasing the number of rational firms would lead to instability. However, through arguments similar to those used in the previous propositions, it can be proved that, for any $N \geq 5$, if we impose $\omega_2 < 1 - 1/N$, it is not possible to have $\#(F_{N,r}) > 3$, as the equilibrium strategies $q^*_{\mathcal{R}}$ of oligopolies with $\omega = 1/N, \ldots, 1 - 4/N$ would be actually null. This means that if we want to focus on dynamics involving strictly positive strategies, we should limit our attention to families consisting of subsets of $\mathcal{F}_{N,r} = \{1 - 3/N, 1 - 2/N, 1 - 1/N\}$. We have the following result, whose proof is a straightforward consequence of Lemmas 6.5 and 6.6.

Proposition 3.7 (Destabilizing scenario). *For each $N \geq 5$, it is possible to find suitable values of $r > 1$ and $k > 0$ so that $\mathcal{F}_{N,r} = \{1 - 2/N, 1 - 1/N\}$ and in which the Nash equilibrium is stable for $\omega = 1 - 2/N$ and unstable for $\omega = 1 - 1/N$.*

• **Case $r < 1$**

When the marginal cost of the rational firms is smaller than that of the naive ones, the classical stabilizing and unconditionally stable/unstable scenarios arise again. Since those situations have already been analytically investigated for $r > 1$ and are quite predictable, we avoid providing detailed evidence of their occurrence, which can be obtained by arguments similar to those used in the previous framework. We will only report some simulative results in Section 4. More interesting is to investigate the possibility to have a destabilizing or mixed scenario. The following proposition excludes such occurrence. First, we notice that to have a significant destabilizing scenario we necessarily need a family of at least two oligopolies ($\mathcal{F}_{N,r} = \{1/N, 2/N\}$) and, therefore, recalling the considerations about the equilibrium positivity before Subsection 2.1 and (2.6), we need $r > 1/2$. For the mixed scenario, we would actually need a family of at least three oligopolies ($\mathcal{F}_{N,r} = \{1/N, 2/N, 3/N\}$), so that $r > 1/2$ is indeed necessary (even if not sufficient).

Proposition 3.8. *Increasing ω has neither a destabilizing nor a mixed effect.*

Proof. To have that increasing ω leads to instability, we would need either $z(\omega)$ to be convex and $\omega_2 < 1 - 1/N$ or $z(\omega)$ to be concave and $\omega_2 > 1/N$. Both cases are not possible due to Lemmas 6.8 and 6.9. □

• **Discussion of the previous results**

We make some considerations about the previous propositions. First, we notice that the scenarios described are valid for any oligopoly with $N \geq 5$. We stress that in the unconditionally stable/unstable and stabilizing scenarios, both for $r < 1$ and $r > 1$, we can always consider complete families of oligopolies, as the equilibrium strategy remains positive when we vary ω between $1/N$ and $1 - 1/N$. Such scenarios are the same that occur when $r = 1$, i.e., when the marginal costs of the rational and naive firms coincide, and that we recovered in [22] using a linear demand function, even for different marginal costs.

Indeed, when $r = 1$ we recall that we found a unique stability threshold $\bar{\omega} = (Nk - 4)/(4N - 8)$ for ω. Hence, if $\bar{\omega} < 1/N$, the Nash equilibrium is locally asymptotically stable for all $\omega = 1/N, \ldots, 1 - 1/N$ and thus we are in the unconditionally stable scenario; if $1/N < \bar{\omega} < 1 - 1/N$, the Nash equilibrium is unstable for ω below $\bar{\omega}$ and locally stable above it, so that we are in the stabilizing scenario; finally, if $\bar{\omega} > 1 - 1/N$, the Nash equilibrium is unstable for all ω and thus we are in the unconditionally unstable scenario. No other frameworks may occur.

Conversely, when $r > 1$ in Proposition 3.7 we prove the existence of a scenario which is not possible with identical marginal costs or with a linear demand function. The situation described by this proposition is rather interesting and we want to stress further it through an example. Let us choose for instance $N = 10$ and let us consider suitable cost ratio r and k for which the conclusions of Proposition 3.7 are valid. Then we have that if we consider from 1 to 7 rational firms, all the rational firms production levels would converge to zero, so they would actually leave the market and the resulting oligopoly would simplify into homogeneous oligopolies of naive firms. Considering the oligopoly with eight rational firms and two naive ones, we would have a stable positive Nash equilibrium. However, replacing a naive firm with a rational one and thus considering nine rational firms and just a naive one, the equilibrium would become

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unstable, even if, with respect the former oligopoly, the degree of rationality of the firms is increased. Moreover, if we replaced the last naive firm with a rational one, the resulting homogeneous oligopoly would have a "stable" equilibrium. In fact, it would consist of rational firms only, which would choose in one shot the equilibrium strategy. We remark that, considering only strictly positive dynamics, the result in Proposition 3.7 is optimal with respect to the size of the family we can consider, as imposing the second threshold ω_2 to be smaller than $1 - 1/N$ requires conditions on the cost ratio r which are possible only for $\omega > 1 - 3/N$. We stress that such destabilizing scenario occurs in a situation of significant heterogeneity between the two groups of firms. In fact, they are different with respect to the informational endowment (perfect foresight versus static expectations), the technology ($c_{\mathcal{R}} > c_{\mathcal{N}}$) and the kind of adopted adjustment mechanism (function σ concerns naive firms only). To preserve the existence of the destabilizing scenario, a certain level of heterogeneity has to be kept: if we considered technological homogeneity ($r = 1$), increasing the fraction of rational firms would never destabilize the equilibrium. However, not every kind of technological heterogeneity leads to equilibrium destabilization, as the previous propositions show that frameworks obtained swapping the values of $c_{\mathcal{R}}$ and $c_{\mathcal{N}}$ (i.e., passing from $r > 1$ to $r < 1$) are *not equivalent*, as some scenarios obtained for $r = \hat{r} > 1$ cannot be reproduced by considering $r = 1/\hat{r} < 1$, even if the oligopoly composition ω were suitably modified. In fact, we find that only when rational firms are sufficiently disadvantaged an increase in their number may have a destabilizing effect. An explanation for such "lack of symmetry" may indeed lie in the effect of the inefficiency of the rational firms with respect to the naive ones. Such technological and rationality differences indeed induce sensibly different production levels between rational and naive agents and, if the less rational ones are sufficiently reactive, they may cause non convergent production trajectories. Of course, we need to further investigate the occurrence of such destabilizing behavior in more general settings, characterized by different demand and cost functions. We remark that this lack of symmetry occurs only for $N \geq 3$, as for the duopoly context it was shown in [20] that the equilibrium stability remains unchanged if the cost ratio is replaced by its reciprocal.

With respect to the mixed scenario, we remark that it can be proven that it is not possible to find a family of oligopolies in which compositions $\hat{\omega}$ and $\hat{\omega} + 2/N$ have stable positive equilibria and composition $\hat{\omega} + 1/N$ has an unstable positive equilibrium.

Conversely, if we consider also dynamics in which the production levels become null, it can be shown that more numerous families of oligopoly compositions may be taken into account for the occurrence of the destabilizing scenario. Moreover, in this case, also the mixed scenario is possible. We propose some simulative evidence of such cases in the next section. We stress however that the positivity issues are mainly connected to the particular form of the considered nonlinear demand function. It would be interesting to further investigate whether the previous scenarios occur for other nonlinear demand functions, too.

As concerns the role of function σ, we will get an insight of it in Section 5, where we shall show through simulations that if the same adjustment mechanism toward the best response is adopted by both the firms groups (while keeping their informational endowments different), the destabilizing scenario obtained for $r > 1$ can still occur.

4. Simulations

We now provide simulative evidence of the multiplicity of the possible dynamics studied in the previous section. We will focus on the scenarios obtained for the most interesting case with $r \neq 1$.

In particular, we shall also compare the model in (2.15) to a simplified framework, in which naive players are assumed to play the same strategy at each time t. Such scenario is realized if the initial strategies of naive players are identical, i.e., $q_{i,0} = q_{\mathcal{N},0}$, for $i = N\omega + 1, \ldots, N$. In this way, we have that $q_{i,t} = q_{\mathcal{N},t}$ for $i = N\omega + 1, \ldots, N$, and, acting as in Section 2, it is possible to obtain a one-dimensional equation which describes the evolution of the quantity $q_{\mathcal{N},t}$ chosen by a generic naive player

$$q_{\mathcal{N},t+1} = \sqrt{\tilde{\Delta}}\left(\frac{1}{\sqrt{c_{\mathcal{N}}}} - \sqrt{\tilde{\Delta}}\right), \tag{4.1}$$

where

$$\tilde{\Delta} = \frac{N\omega - 1 - 2c_{\mathcal{R}}N\omega q_{\mathcal{N},t} + \sqrt{(N\omega - 1)^2 + 4c_{\mathcal{R}}N^2\omega(1-\omega)q_{\mathcal{N},t}}}{2c_{\mathcal{R}}N\omega}.$$

The positive steady state of (4.1) is indeed the same as for (2.15) and it coincides with the Nash equilibrium. It is possible to prove that the steady state is locally asymptotically stable under condition (3.4), which means that from the local stability point of view, behaves in the same way as the steady state for (2.15).

In the next simulations, we will consider (4.1) and we will show that the stability thresholds are the same as for the model in (2.15), studied in the propositions of the previous section. To this end, we will assume that ω is a continuous parameter varying between $1/N$ and $1 - 1/N$. Moreover, we will assume that function σ is represented by the sigmoid function

$$\sigma(x) = a_2\left(\frac{a_1 + a_2}{a_1 e^{-\gamma x} + a_2} - 1\right), \tag{4.2}$$

with γ positive parameter representing the reaction speed and a_1, a_2 positive parameters playing the role of horizontal asymptotes, so that the possible output variations from t to $t+1$ can increase (resp. decrease) up to a_1 (resp. a_2). Function (4.2) indeed satisfies the requirements on σ specified in Section 2. We remark that a similar approach has been used in [32,33] in microeconomic frameworks, and in [34–36] in macroeconomic settings.

We notice that for (4.2) we have

$$k = \sigma'(0) = \gamma\frac{a_1 a_2}{a_1 + a_2}. \tag{4.3}$$

In all the following simulations we report the bifurcation diagrams obtained from the model in (4.1) on varying ω, setting $N = 10$. The remaining parameters a_1, a_2, γ are set using (4.3) to have the corresponding value of k, while the marginal costs are explicitly specified.

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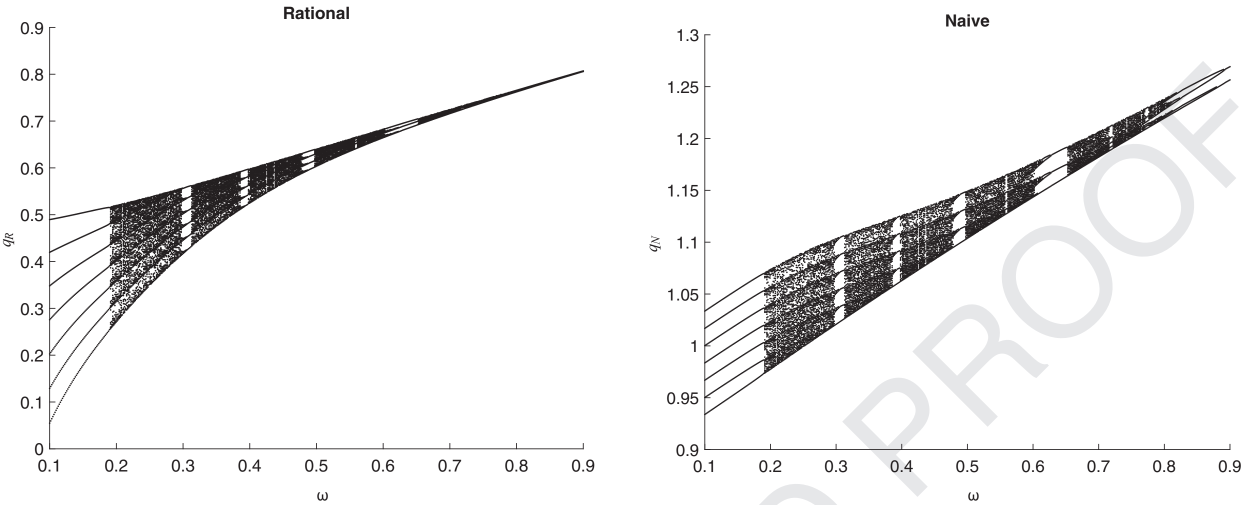

Fig. 1. Unconditionally unstable scenario. All the possible configurations corresponding to $\omega = 0.1, 0.2, \ldots, 0.9$ have unstable equilibrium.

Performing a simulation of (4.1) for the parameters choice $c_\mathcal{R} = 0.1063, c_\mathcal{N} = 0.1$ and $k = 0.5$, we find the unconditionally stable scenario, in which all the oligopoly configurations $\omega = 1/10, 2/10, \ldots, 9/10$ converge to the equilibrium, so that stability is independent from the fraction of the rational agents. This is in agreement with the stability thresholds $\omega_1 = 0.0277$ and $\omega_2 = 14.2086$ obtained from (3.4). Similarly, if we set $c_\mathcal{R} = 0.1, c_\mathcal{N} = 0.1063$ and still $k = 0.5$ we find the unconditionally stable scenario again, in agreement with the stability thresholds $\omega_2 = -11.9723$ and $\omega_1 = 0.0355$ and, since $r < 1$ and $k < 4/(1-r)$, the stability condition is (3.8).

The unconditionally unstable scenario is obtained considering for example $c_\mathcal{R} = 0.1063, c_\mathcal{N} = 0.1$ and $k = 4$ and it is reported in Fig. 1. We stress that in the rightmost part of the bifurcation diagram for the rational players, we have a period-two cycle. In this case all the possible configurations have unstable equilibrium, in agreement with the stability thresholds $\omega_1 = 1.2685$ and $\omega_2 = 10.6845$ given by Proposition 3.3 for the present values of r and k.

The same scenario occurs if we set $c_\mathcal{R} = 0.1, c_\mathcal{N} = 0.1063$ and still $k = 4$, in agreement with the stability thresholds $\omega_2 = -15.5984$ and $\omega_1 = 1.0422$ and, since $r < 1$ and $k < 4/(1-r)$, the stability condition is (3.8).

The stabilizing scenario is obtained considering for instance $c_\mathcal{R} = 0.1063, c_\mathcal{N} = 0.1$ and $k = 2$ and it is reported in Fig. 2, in which it is possible to see that the dynamics are unstable up to a certain threshold $\omega \approx 0.5$ and then there is convergence to the Nash equilibrium. In this case the stability thresholds given by Proposition 3.3 are $\omega_1 = 0.4516$ and $\omega_2 = 12.9019$. Similarly, if we set $c_\mathcal{R} = 0.1, c_\mathcal{N} = 0.1063$ and still $k = 2$, we find that increasing ω again leads to equilibrium stability. In this case the stability thresholds are $\omega_2 = -13.4984$ and $\omega_1 = 0.5153$ and, since $r < 1$ and $k < 4/(1-r)$, the stability condition is (3.8).

We stress that the previous qualitative scenarios (unconditionally stable/unstable and stabilizing) can be obtained also in the case of $r = 1$ with suitably different parameters choices.

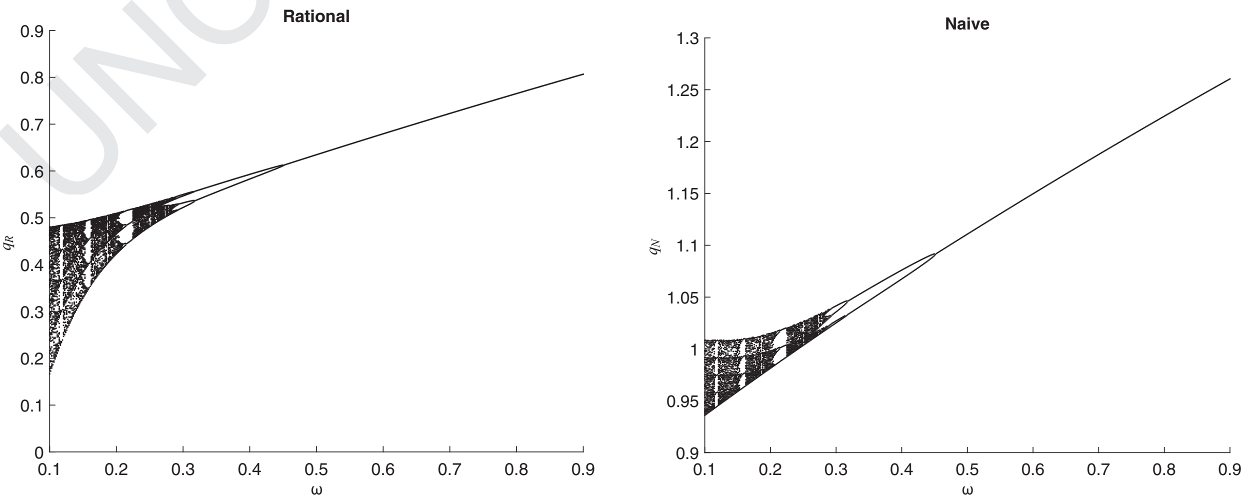

Fig. 2. Stabilizing scenario. When $\omega = 0.1, 0.2, 0.3, 0.4$ the equilibrium is unstable, while increasing the number of rational firms so that $\omega \geq 0.5$ we have a stable equilibrium.

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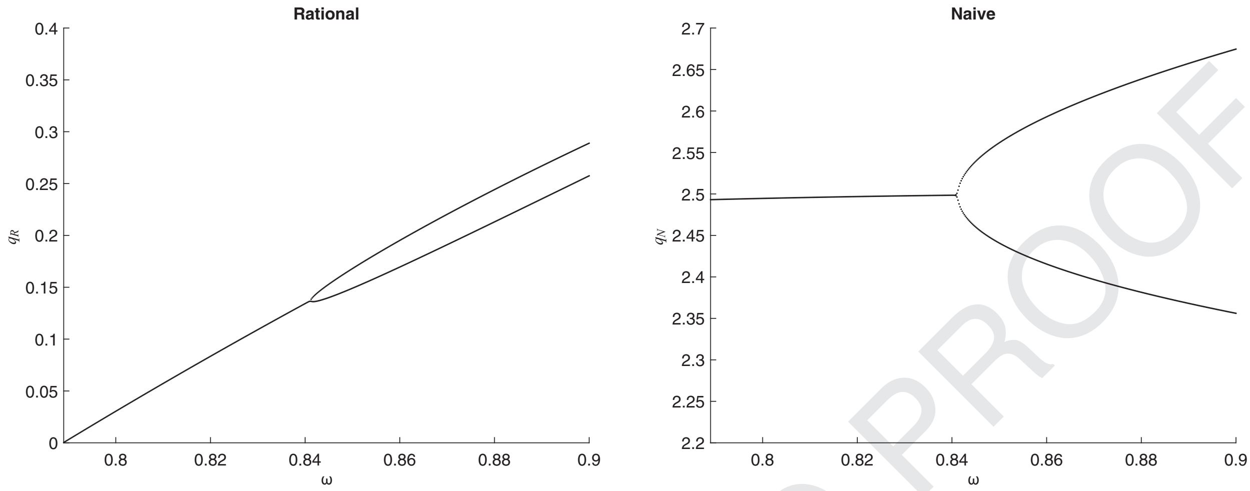

Fig. 3. Destabilizing scenario. For $\omega = 0.8$ the equilibrium is stable, while adding a rational firm so that $\omega = 0.9$ introduces instability.

The last scenario we consider is the destabilizing one. We saw that, to have strictly positive dynamics, such situation is possible only if $r > 1$ and for a family of oligopolies with compositions $\omega = 1 - 2/N, 1 - 1/N$. We report an example in Fig. 3, for a simulation obtained setting $c_{\mathcal{R}} = 0.19, c_{\mathcal{N}} = 0.1$ and $k = 2.04$. The thresholds, corresponding to this parameters choice, given by Proposition 3.3 are $\omega_1 = 0.3712$ and $\omega_2 = 0.8412$. We underline that ω_1 is inferior to 71/90, which is the positivity threshold for ω obtainable by (2.6) for the present marginal cost ratio $r = 1.9$. In all the simulations we performed concerning the destabilizing framework, we found for $\omega = 1 - 1/N$ only a period-two cycle, and thus we conjecture that more complex dynamics are not possible.

If we considered $c_{\mathcal{R}} = 0.1, c_{\mathcal{N}} = 0.19$ and still $k = 2.04$, the same oligopoly composition would have a negative equilibrium. In this case, the "symmetric" composition for positivity would actually be $\omega = 1/N, 2/N,$ for which we would however find an unconditionally unstable scenario. Indeed, we would have stability for $\omega < \omega_2 = -1.7273$ and for $\omega > \omega_1 = 0.6607,$ but for $\omega > 0.2$ the equilibrium is negative.

We remark that we thoroughly investigated via simulations the possible kinds of dynamics that arise in the destabilizing scenario when equilibrium loses its stability, but we always found a very low complexity level (namely, just period-two cycles). This is mainly connected to negativity issues, as the present economic context did not allow us to find sufficiently numerous families of oligopolies to let feasible, economically significant, complex behaviors arise. This aspect indeed requires further investigations, which need to take into account different demand and/or cost functions. However, we think that even such simple destabilized dynamics represent an interesting counterintuitive example which suggests that further reflection and effort should be devoted to deepen the analysis of the role of rationality and technological heterogeneity.

In Section 3 we remarked that the mixed scenario is actually impossible. An example is reported in Fig. 4, in which

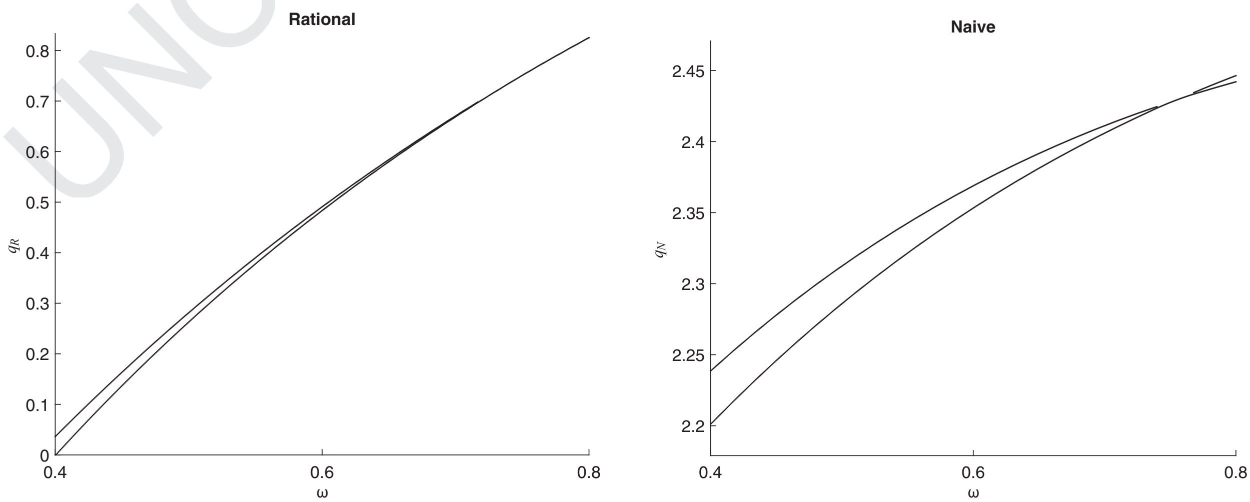

Fig. 4. The equilibrium is initially unstable, it becomes stable for $0.7439 < \omega < 0.7635$ and then it is again unstable. Hoverer, since $N = 5$, the closest oligopolies correspond to the choice of $\omega = 0.6$ and $\omega = 0.8$, at which the equilibrium is unstable.

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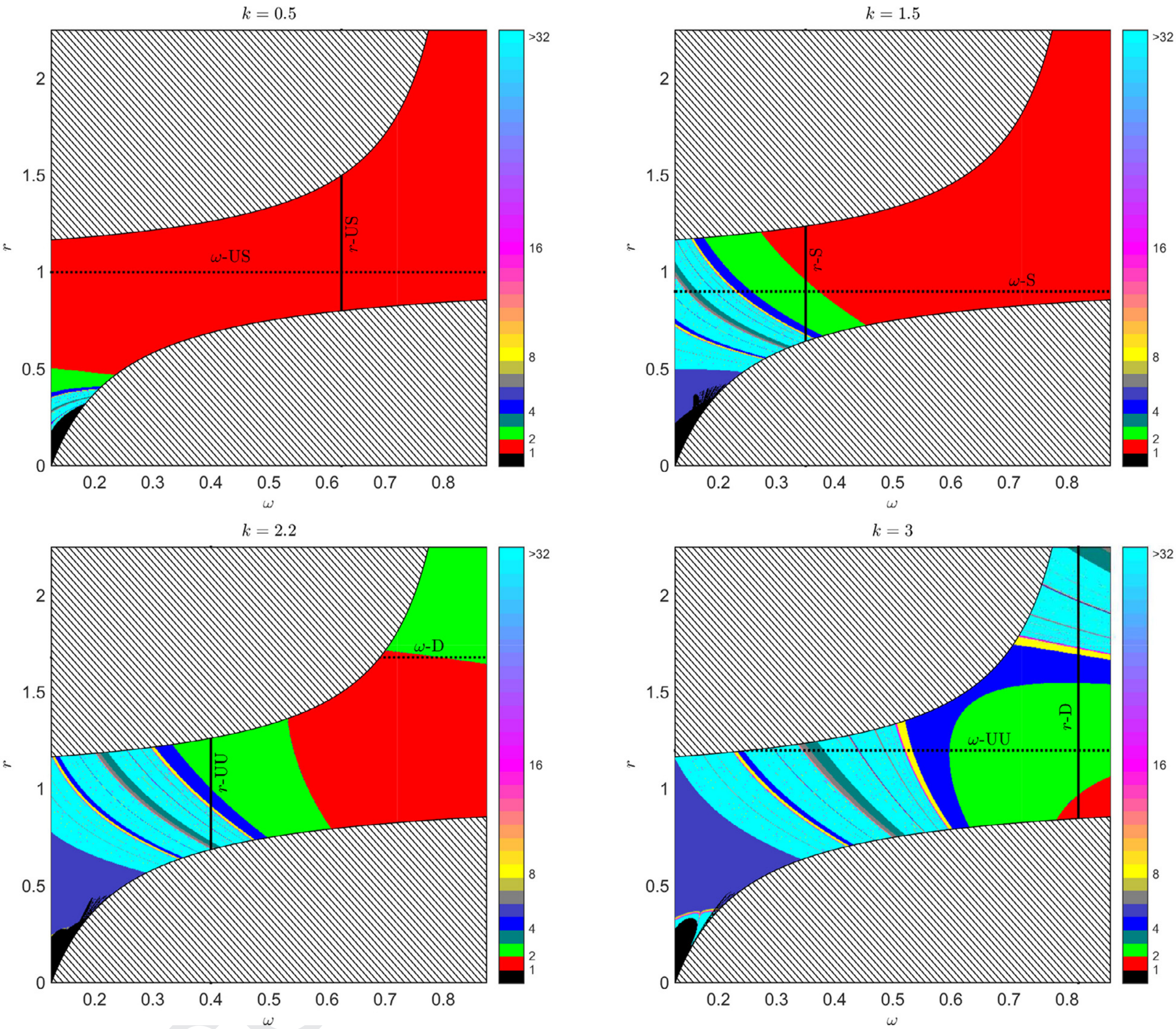

Fig. 5. Bifurcation diagrams in the (ω, r)-plane, for different values of k. Color red means that equilibrium is stable; hatched regions give non positive equilibrium; the remaining colors are used for attractors consisting of more than a single point. Black is used for parameter configurations for which strategies become unfeasible. Solid (resp. dotted) lines show possible scenarios – S = stabilizing, D = destabilizing, US = unconditionally stable, UU = unconditionally unstable – on increasing r (resp. ω). (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

we consider a smaller oligopoly size $N = 5$ and we set $k = 2.2156$, where we indeed have that the steady state is stable in an interval of values of ω surrounded by two unstable regions, but since oligopolies correspond just to $\omega = 0.4, 0.6, 0.8$, the scenario is actually unconditionally unstable since in these three cases the equilibrium is always unstable. We remark that also in this case, the thresholds $\omega_1 = 0.7439$ and $\omega_2 = 0.7635$ computed for the model in (2.15) are the same obtained by simulating the model in (4.1).

We stress that in all the previous simulations, equilibrium loses or regains stability by means of a flip bifurcation. In particular, focusing for instance on the case in which stability is lost, we may have a cascade of period doublings in which the trajectories oscillate among two or more values and which then lead to chaotic dynamics. In the unconditionally unstable scenario, dynamics can consist instead of either chaotic or periodic trajectories, with both possible scenarios of qualitative reduction or increase of the complexity level.

Moreover, all the simulations so far confirm that the stability conditions for the models in (2.15) and in (4.1) are the same.

In the previous bifurcation diagrams we focused on the role of the oligopoly composition ω. To investigate the effects of the remaining parameters, we report some two-parameters bifurcation diagrams. In both Figs. 5 and 6, in hatched regions the equilibrium is non positive, red regions correspond to stable parameters configurations, while the remaining colors are used for unstable parameters settings (different colors are associated to attractors with different number of elements). We remark that feasible dynamics are restricted to unhatched regions only, and thus in the following comments we focus on what happens in (horizontal or vertical) sections of the unhatched regions. In Fig. 5 we study

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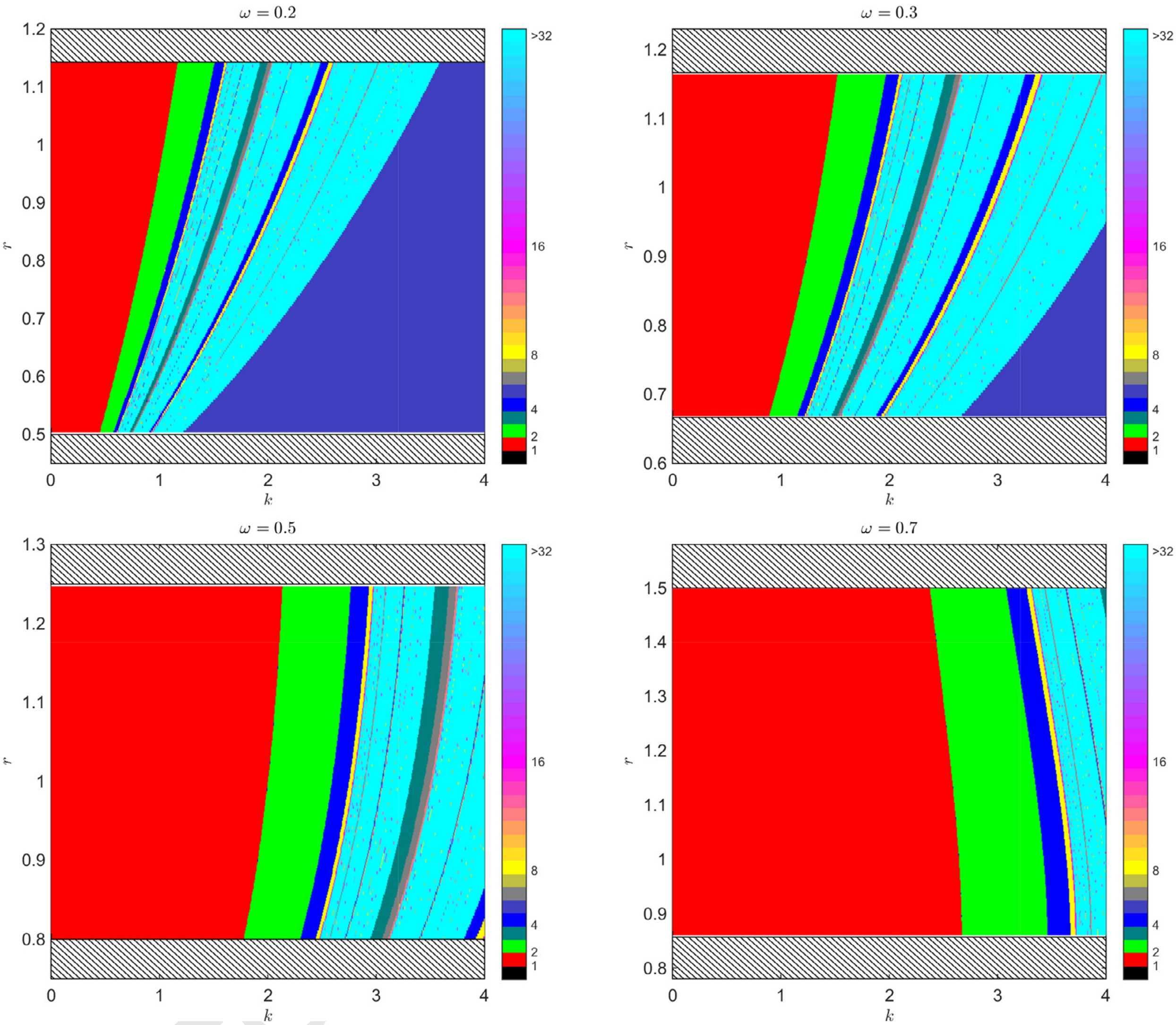

Fig. 6. Bifurcation diagrams in the (k, r)-plane, for different values of ω. Color red means that equilibrium is stable; hatched regions give non positive equilibrium; the remaining colors are used for attractors consisting of more than a single point. Black is used for parameter configurations for which strategies become unfeasible. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

the stability with respect to ω and r, for fixed values of k. Possible different scenarios on varying r and ω are pointed out using respectively solid and dotted lines. Moreover, the acronym x-S (resp. x-D, x-US, x-UU) stands for stabilizing (resp. destabilizing, unconditionally stable, unconditionally unstable) scenario on increasing parameter x (which can be either ω or r). We stress that unconditionally stable scenarios correspond to those (horizontal for ω or vertical for r) sections which pass through red regions only; unconditionally unstable scenarios correspond to those sections which never pass through red regions; stabilizing scenarios correspond to those sections which, starting in not red regions, then pass through red regions; destabilizing scenarios correspond to those sections which, starting in red regions, then pass through not red regions. On increasing ω, we can identify in Fig. 5 the four analytically studied kinds of scenarios. On increasing r, we find ambiguous behaviors too, since both unconditionally stable/unstable, stabilizing and destabilizing scenarios are possible. Conversely, from Fig. 6, we can see that, as predictable, increasing k is always destabilizing.

If we relax condition (2.6) on the strict positivity of the equilibrium and we take into account also dynamics involving null best responses, then we can show that the destabilizing scenario can occur for $\omega \in [1/N, 1 - 1/N]$, and a truly mixed scenario is also possible. We show the destabilization of the equilibrium in Fig. 7, in which we considered oligopolies of size $N = 10$, with $c_{\mathcal{R}} = 3$ and $k = 1.49$. As we can see, equilibrium is stable if $\omega \leq 0.7$ and increasing the number of rational players leads equilibrium to instability as it becomes unstable for $\omega > 0.79$, so that compositions with 2 and 1 rational players have unstable equilibrium. However, in all the oligopolies with more than 1 rational player, we have that $q_{\mathcal{R},t} = 0$, so that the rational firms do not produce anything and exit the market. The resulting oligopolies for $\omega \leq 0.8$ are actually homogeneous oligopolies of $N(1-\omega)$ naive firms.

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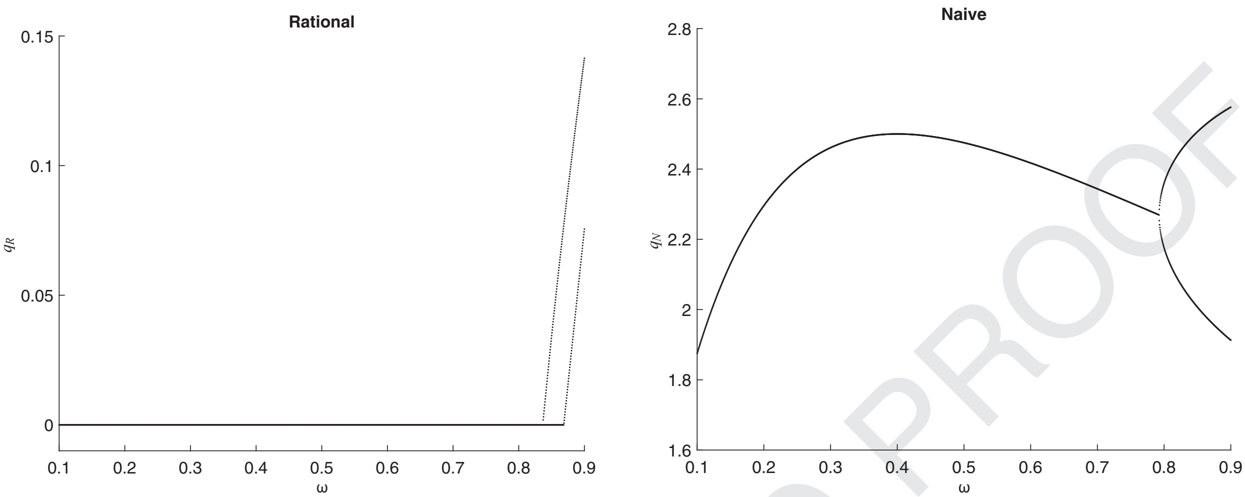

Fig. 7. Destabilizing scenario. Equilibrium is stable for compositions characterized by $\omega \leq 0.7$ and becomes unstable for $\omega = 0.8, 0.9$. However, the strategy of rational players is non null only for $\omega = 0.9$.

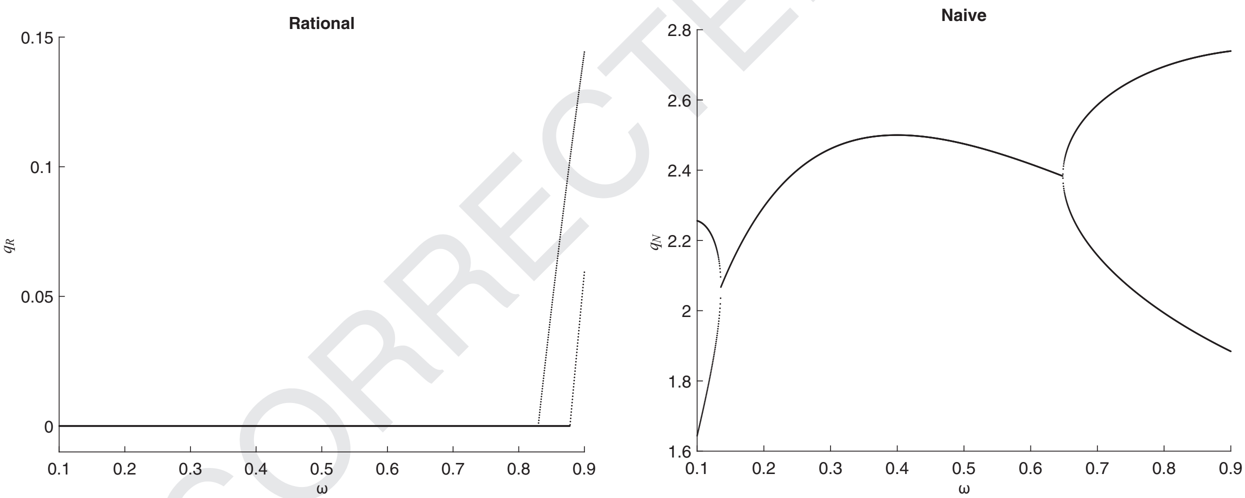

Fig. 8. Mixed scenario. Equilibrium is unstable for both suitably small and large values of ω and it is stable for intermediate compositions. However, the strategy of rational players is non null only for $\omega = 0.9$.

Similarly, taking $k = 1.7$ in the previous configuration, the scenario becomes mixed (see Fig. 8), as for $\omega = 0.1$ the equilibrium is unstable, for $\omega = 0.2, \ldots, 0.6$ it is stable and then it becomes unstable again for $\omega = 0.7, 0.8, 0.9$. However, as in the scenario considered in 7, $q_{\mathcal{R},t} = 0$ for $\omega \leq 0.8$.

5. Reducing heterogeneity: homogeneous adjustment mechanisms

In the previous sections we showed that increasing the fraction of rational firms, for suitable parameters configurations, may destabilize the equilibrium. As already noticed, in principle this could be induced by the heterogeneity in the adjustment mechanisms used by the two groups of firms, as naive firms move toward their best response more cautiously. To disentangle the effect of function σ, in this section we introduce the same function σ also in the adjustment mechanism of the rational players, and we investigate the dynamics of the resulting theoretical model. In this respect, we notice that it will be no more true that all the rational players are necessarily identical. This means that the resulting model would consist of an N-dimensional system. Since we just want to give a simulative insight of this scenario, we assume that the rational (resp. naive) players are identical, so that the resulting model actually consists of two equations. Recalling that function f, defined in (2.10), (resp. function h, defined in (2.11)) is the best response of the generic rational (resp. naive) player with respect to the in this case, identical strategies of the other players, introducing function σ in (2.10), we obtain

$$\begin{cases} q_{\mathcal{R},t+1} = q_{\mathcal{R},t} + \sigma\left(f(Q_{\mathcal{N},t+1}) - q_{\mathcal{R},t}\right), \\ q_{\mathcal{N},t+1} = q_{\mathcal{N},t} + \sigma\left(h(Q_t - q_{\mathcal{N},t}) - q_{\mathcal{N},t}\right), \end{cases}$$

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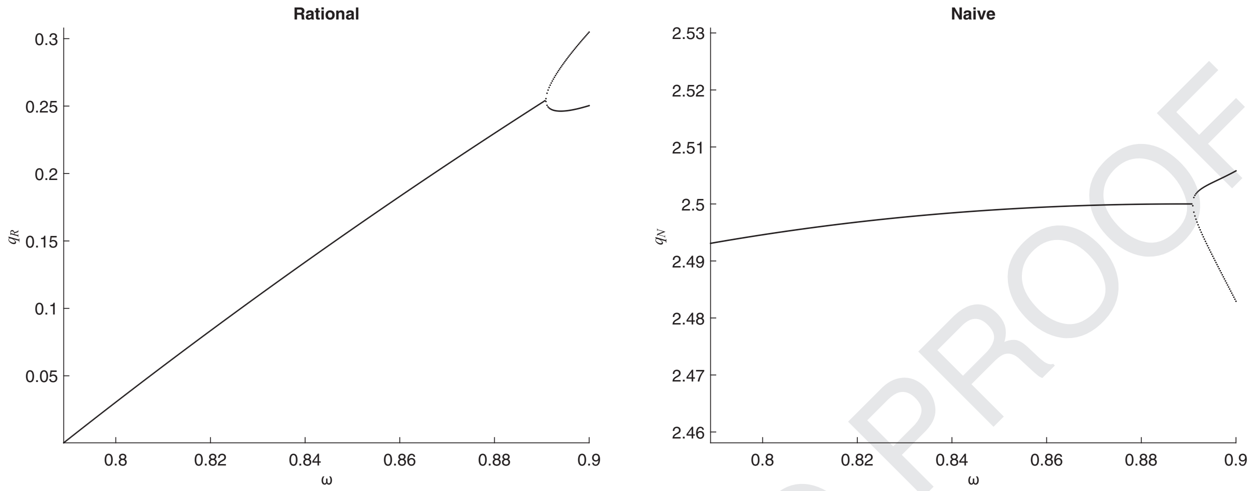

Fig. 9. Destabilizing scenario with homogeneous partial adjustment mechanisms. For $\omega = 0.8$ the equilibrium is stable, while adding a rational firm so that $\omega = 0.9$ introduces instability.

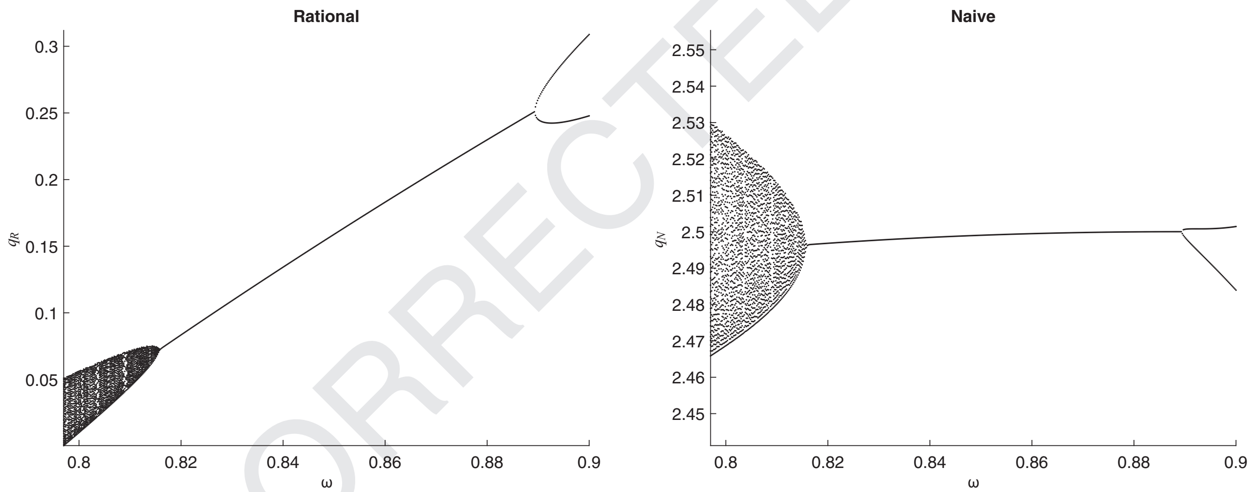

Fig. 10. Double stability threshold with homogeneous partial adjustment. For both $\omega = 0.8$ and $\omega = 0.9$ the equilibrium is unstable.

where

$$Q_{\mathcal{N},t+1} = (1-\omega)Nq_{\mathcal{N},t+1},$$
$$Q_t - q_{\mathcal{N},t} = \omega N q_{\mathcal{R},t} + ((1-\omega)N - 1)q_{\mathcal{N},t}.$$

The previous system can be rewritten as

$$\begin{cases} q_{\mathcal{R},t+1} = f_{\mathcal{R}}(q_{\mathcal{R},t}, q_{\mathcal{N},t+1}) = q_{\mathcal{R},t} \\ \qquad + \sigma(f((1-\omega)Nq_{\mathcal{N},t+1}) - q_{\mathcal{R},t}), \\ q_{\mathcal{N},t+1} = f_{\mathcal{N}}(q_{\mathcal{R},t}, q_{\mathcal{N},t}) = q_{\mathcal{N},t} \\ \qquad + \sigma(h(\omega N q_{\mathcal{R},t} + ((1-\omega)N - 1)q_{\mathcal{N},t}) - q_{\mathcal{N},t}), \end{cases}$$

which, inserting the second equation into the first one, becomes

$$\begin{cases} q_{\mathcal{R},t+1} = f_{\mathcal{R}}(q_{\mathcal{R},t}, f_{\mathcal{N}}(q_{\mathcal{R},t}, q_{\mathcal{N},t})) = q_{\mathcal{R},t} \\ \qquad + \sigma(f((1-\omega)Nf_{\mathcal{N}}(q_{\mathcal{R},t}, q_{\mathcal{N},t})) - q_{\mathcal{R},t}), \\ q_{\mathcal{N},t+1} = f_{\mathcal{N}}(q_{\mathcal{R},t}, q_{\mathcal{N},t}) = q_{\mathcal{N},t} \\ \qquad + \sigma(h(\omega N q_{\mathcal{R},t} + ((1-\omega)N - 1)q_{\mathcal{N},t}) - q_{\mathcal{N},t}). \end{cases} \tag{5.1}$$

In Fig. 9 we report the bifurcation diagram obtained considering the same marginal costs as in the destabilizing scenario in Section 4 ($c_{\mathcal{R}} = 0.19$, $c_{\mathcal{N}} = 0.1$) and $k = 1.94$, using as σ function (4.2). As we can see, the results are qualitatively similar to those in Fig. 3, and this suggests that the destabilizing role of ω cannot be ascribed to the presence of the sigmoidal function in the adjustment mechanism of naive firms only. Moreover, we notice that System (5.1) exhibits two stability thresholds. This can be argued for instance by increasing k in the previous simulation. For instance, setting $k = 1.97$ we find the bifurcation diagram reported in Fig. 10, which shows that equilibrium is unstable for ω outside the interval (0.8160, 0.8895). We notice that in this case the leftmost destabilization occurs through a Neimark–Sacker bifurcation, giving rise to quasi-periodic dynamics. However such last scenario is not a mixed one, as for the significant values of $\omega = 0.8, 0.9$ the equilibrium is always unstable. We finally stress that, for suitable parameter configurations, also model (5.1) is able to provide the classical stabilizing or unconditionally stable/unstable scenarios.

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6. Conclusions

In this work we studied a discrete dynamical system of variable dimension which models the oligopolistic competition between heterogeneous (rational and naive) firms in a market characterized by isoelastic demand function. Analyzing the local stability of the equilibrium, we showed that, among the possible scenarios, increasing the global degree of rationality in the oligopoly, namely, considering larger fractions of rational players in an oligopoly of generic fixed size, may destabilize the equilibrium, provided that marginal costs are unfavorable to the rational firms. Indeed, also technological heterogeneity plays a fundamental role in this destabilization, as in the case of identical marginal costs we proved that the destabilizing scenario is not possible. However, technological heterogeneity alone is not sufficient to explain the occurrence of the destabilization, since such scenario never occurs when naive firms face larger marginal costs. As the destabilizing scenario is very surprising and counterintuitive, we are going to pursue further this research strand considering more general economic contexts, varying the demand and cost functions, in order to better understand under which conditions such scenario may occur. In particular, since the present demand and cost functions gave unimodal best response functions, we aim to investigate whether a similar destabilizing scenario is possible also for monotonic (nonlinear) best response functions. Moreover, since in the present economic setting the destabilization gave rise only to a periodic attractor characterized by a very low complexity level (period-two cycle), we are going to study whether in different economic contexts more complex, maybe chaotic, dynamics are possible. Nevertheless, we showed that even apparently expected behaviors (like increasing the rationality degree necessarily leads to stability improvement) may require more cautious investigations, especially when other degrees of heterogeneity (for example in technology) are involved. Finally, the approach we used to model the oligopoly allowed us considering both identical and distinct agents, which we compared with respect to the local stability of the equilibrium, focusing on the variation of the oligopoly composition. We wish to extend the comparison also to global dynamic properties, studying the differences between the case in which agents are assumed to be distinct (and then the model consists of a multidimensional dynamical system) and the case in which agents are identical (and the model consists of a single equation), for instance on varying either the marginal costs ratio or the sigmoidal function.

Acknowledgments

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Appendix A

Let us introduce the following constants

$$k_1 = \frac{2-2N-6r+4Nr}{(r-1)(N+r-1)},$$

$$k_2 = \frac{8r(N-1)}{(N+r-1)^2},$$

$$k_3 = \frac{2(N+r-3)}{(r-1)(r(N-1)+1)},$$

$$k_4 = \frac{4(N^2r-Nr-N+2)}{(r(N-1)+1)^2},$$

$$k_5 = \frac{2(N+3r-5)}{(r-1)(Nr-2r+2)},$$

$$k_6 = \frac{4N^2r-8N-4Nr-8r+24}{(Nr-2r+2)^2}.$$

Setting $\alpha = -2N-2r+Nk+4Nr-Nkr-2$ and $\beta = N(r-1)(kr-k+4)$ we may rewrite $\omega_{1,2}$ in (3.6) as

$$\omega_{1,2} = \frac{\alpha \pm \sqrt{\Delta}}{\beta}.$$

We prove some auxiliary results. Lemmas 6.1–6.6 concern the framework with $r > 1$, while the subsequent ones concern the framework with $r < 1$.

Lemma 6.1. *If $1 < r < 1 + 1/(N-2)$, a sufficient condition to have $w_1 < 1/N$ is*

$$k < k_2, \tag{6.1}$$

while a sufficient condition to have $w_1 > 1/N$ is

$$k_2 < k < k_\Delta. \tag{6.2}$$

Proof. To have $\omega_1 < 1/N$, we need $N\alpha - \beta - N\sqrt{\Delta} < 0$, which is true in particular if

$$\begin{cases} N\alpha - \beta > 0, \\ (N\alpha - \beta)^2 - N^2\Delta < 0, \\ k < k_\Delta. \end{cases} \tag{6.3}$$

Solving $N\alpha - \beta > 0$, we find

$$-(r-1)(N+r-1)k + (2-2N-6r+4Nr) > 0, \tag{6.4}$$

so, since $-(r-1)(N+r-1) < 0$ for $r > 1$, the first condition in (6.3) is equivalent to $k < k_1$. For the second inequality we have

$$(N\alpha - \beta)^2 - N^2\Delta = N^2(r-1)(k(r-1)+4) \times ((N+r-1)^2k - 8r(N-1)) < 0 \tag{6.5}$$

which requires $k < k_2$. System (6.3) is then solved for $k < \min\{k_\Delta, k_1, k_2\}$ and since all k_1, k_2 and k_Δ are positive for every $r > 1$ and $N > 2$, it is always possible to find some k that satisfy such condition. Moreover, since under the imposed conditions

$$k_1 - k_\Delta = \frac{(r-N+1)(N+r-2Nr-1)(N-r+2Nr-1)}{2Nr(N-1)(r-1)(N+r-1)} \geq 0 \tag{6.6}$$

and

$$k_2 - k_\Delta = -\frac{(r-N+1)^2(N+r-2Nr-1)^2}{2Nr(N-1)(r-1)(N+r-1)^2} < 0, \tag{6.7}$$

the relation can be further simplified into (6.1).

To have instead $\omega_1 > 1/N$, we need $N\alpha - \beta - N\sqrt{\Delta} > 0$, which requires

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$$\begin{cases} N\alpha - \beta > 0, \\ (N\alpha - \beta)^2 - N^2\Delta > 0, \\ k < k_\Delta. \end{cases}$$

Thanks to the first part of the proof, we have that the first two inequalities of the previous system are solved for $k_2 < k < k_1$. Since we proved that $k_\Delta \le k_1$, this is in agreement with (6.2). By (6.7) the interval is nonempty. □

Lemma 6.2. *If $1 < r < 1 + 1/(N-2)$, a sufficient condition to have $w_1 < 1 - 1/N$ is that $N \ge 5$ and*

$$k < k_4, \tag{6.8}$$

while a sufficient condition to have $w_1 > 1 - 1/N$ is that $N \ge 5$ and

$$k_4 < k < \min\{k_3, k_\Delta\}. \tag{6.9}$$

Proof. For $w_1 > 1 - 1/N$ we need $N\alpha - (N-1)\beta - N\sqrt{\Delta} > 0$, namely

$$\begin{cases} N\alpha - (N-1)\beta > 0, \\ (N\alpha - (N-1)\beta)^2 - N^2\Delta > 0, \\ k < k_\Delta. \end{cases} \tag{6.10}$$

The first condition is equivalent to $-N(r-1)(r(N-1)+1)k + 2N(N+r-3) > 0$ and hence requires $k < k_3$. The second condition is equivalent to $N^2(r-1)(k(r-1)+4)((r(N-1)+1)^2k - 4(N^2r - Nr - N + 2)) > 0$ and it is satisfied for $k > k_4$. This means that we need $k_4 < k < \min\{k_\Delta, k_3\}$. Since

$$k_4 - k_\Delta = -\frac{(2N^2r^2 - 3N^2r - 3Nr^2 + 4Nr + N + r^2 - 1)^2}{2Nr(N-1)(r-1)(Nr-r+1)^2} < 0, \tag{6.11}$$

the previous conditions are consistent if $k_4 < k_3$, or equivalently

$$k_4 - k_3 = \frac{2((2N^2 - 3N + 1)r^2 + (4N - 3N^2)r + N - 1)}{(r-1)(Nr-r+1)^2} < 0, \tag{6.12}$$

which is fulfilled if and only if

$$q(r) = (2N^2 - 3N + 1)r^2 + (4N - 3N^2)r + N - 1 < 0. \tag{6.13}$$

The last condition is satisfied for $1 < r < 1 + 1/(N-2)$. Indeed, for all $N \ge 5$ $q(r)$ is a convex parabola with

$$q(1) = 2(-N^2 + 2N) < 0$$

and

$$q(1 + 1/(N-2)) = \frac{-2(N-1)^2(N^2 - 5N + 3)}{(N-2)^2} < 0.$$

Conversely, to have $w_1 < 1 - 1/N$ we need $N\alpha - (N-1)\beta - N\sqrt{\Delta} < 0$, for which it is sufficient

$$\begin{cases} N\alpha - (N-1)\beta > 0, \\ (N\alpha - (N-1)\beta)^2 - N^2\Delta < 0, \\ k < k_\Delta. \end{cases}$$

Such system requires $k < k_3$, $k < k_4$ and $k < k_\Delta$, which means $k < \min\{k_3, k_4, k_\Delta\} = k_4$ by (6.11) and (6.12). □

Lemma 6.3. *If $1 < r < 1 + 1/(N-2)$, we always have that $w_2 \ge 1/N$, provided that $k \le k_\Delta$.*

Proof. To have $w_2 < 1/N$, we need $N\alpha - \beta + N\sqrt{\Delta} < 0$, that is

$$\begin{cases} N\alpha - \beta < 0, \\ N^2\Delta - (N\alpha - \beta)^2 < 0, \\ k < k_\Delta. \end{cases} \tag{6.14}$$

From the proof of the first inequality of System (6.3) in Lemma 6.1, we have that the first inequality of (6.14) requires $k > k_1$, which, however, is not compatible with $k < k_\Delta$ by (6.6). □

Lemma 6.4. *If $1 < r < 1 + 1/(N-2)$ and $N \ge 5$, a sufficient condition for $w_2 > 1 - 1/N$ is*

$$k < k_\Delta. \tag{6.15}$$

Proof. To have $\omega_2 > 1 - 1/N$ we need $N\alpha - (N-1)\beta + N\sqrt{\Delta} > 0$, for which it is sufficient to have

$$\begin{cases} N\alpha - (N-1)\beta > 0, \\ k < k_\Delta. \end{cases} \tag{6.16}$$

The first condition has been solved in (6.10) and gives $k < k_3$, which means that System (6.16) is fulfilled if $k < \min\{k_\Delta, k_3\}$. Since

$$k_\Delta - k_3 = \frac{(N - r + 2Nr - 1)((2N^2 - 3N + 1)r^2 + (4N - 3N^2)r + N - 1)}{2Nr(N-1)(r-1)(Nr-r+1)} < 0 \tag{6.17}$$

holds true if and only (6.13) is satisfied, for $N \ge 5$ and $1 < r < 1 + 1/(N-2)$, we have that the solution simplifies as in (6.15). □

Lemma 6.5. *For $N \ge 4$, there exists $1 < r < 2$ such that if*

$$\max\{k_3, k_4, k_5\} < k < \min\{k_6, k_\Delta\}$$

then $1 - 2/N < w_2 < 1 - 1/N$.

Proof. To have $w_2 < 1 - 1/N$ we need $N\alpha - (N-1)\beta + N\sqrt{\Delta} < 0$ and thus

$$\begin{cases} N\alpha - (N-1)\beta < 0, \\ N^2\Delta - (N\alpha - (N-1)\beta)^2 < 0, \\ k < k_\Delta. \end{cases}$$

Recalling the proof of Lemma 6.2, the first and the second inequalities are solved by $k > k_3$ and $k > k_4$ respectively, and hence the system is fulfilled for $\max\{k_3, k_4\} < k < k_\Delta$. To check that the previous interval is nonempty, we notice that from (6.11) we have $k_4 < k_\Delta$, while by (6.17) we have $k_3 < k_\Delta$ only if $(2N^2 - 3N + 1)r^2 + (4N - 3N^2)r + N - 1 > 0$. The last inequality is satisfied for example when $r > \tilde{r}(N)$. It is possible to analytically prove that $\tilde{r}(N) < 3/2$ for $N \ge 3$. We only give graphical evidence in Fig. 11 for $N \ge 4$.

To have $\omega_2 > 1 - 2/N$ we need $N\alpha - (N-2)\beta + N\sqrt{\Delta} > 0$, for which it is sufficient that

$$\begin{cases} N\alpha - (N-2)\beta < 0, \\ N^2\Delta - (N\alpha - (N-2)\beta)^2 > 0, \\ k < k_\Delta. \end{cases}$$

UNCORRECTED PROOF

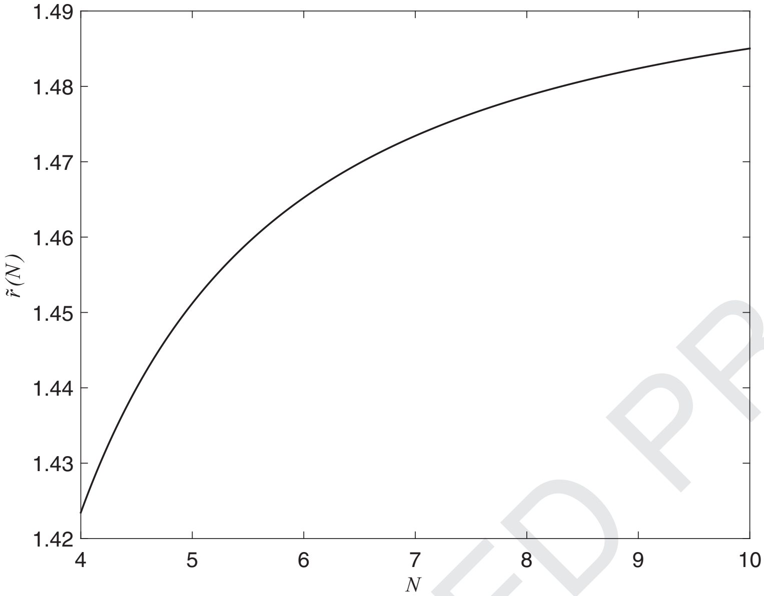

Fig. 11. Plot of function $\tilde{r}(N)$.

UNCORRECTED PROOF

The first inequality is equivalent to $-N((r-1)(Nr-2r+2)k-2N-6r+10)<0$ which is solved for $k>k_5$, while the second inequality is equivalent to $-N^2(r-1)(k(r-1)+4)((Nr-2r+2)^2k+8N+8r+4Nr-4N^2r-24)>0$ which is solved for $k<k_6$, so that the system above requires $k_5<k<\min\{k_6,k_\Delta\}$.

Combining conditions for $\omega_2<1-1/N$ and for $\omega_2>1-2/N$ we find $\max\{k_3,k_4,k_5\}<k<\min\{k_6,k_\Delta\}$. It is possible to choose $1<r<2$ so that this interval is nonempty. In fact we have

$$\lim_{r\to 2^-} k_6-k_4=\frac{2(2N-3)}{(2N-1)^2}>0$$

$$\lim_{r\to 2^-} k_6-k_3=\frac{2N}{(2N-1)}>0$$

$$\lim_{r\to 2^-} k_4-k_5=\frac{4N^3-20N^2+23N-9}{(2N-1)^2(N-1)}>0, \quad \text{for} \quad N\geq 4$$

and thus in a suitable left neighborhood of $r=2$ it holds that $k_6>k_4>k_5$ and $k_6>k_3$. Moreover by (6.11) we have $k_4<k_\Delta$ and by the proof of Lemma 6.5 $k_3<k_\Delta$ for $r>\tilde{r}(N)$. □

Lemma 6.6. *For $N\geq 4$, there exists $1<r<2$ such that if*

$$k_5<k<k_\Delta$$

then $w_1<1-2/N$.

Proof. To have $w_1<1-2/N$ we need $N\alpha-(N-2)\beta-N\sqrt{\Delta}<0$ for which it is sufficient to have

$$\begin{cases} N\alpha-(N-2)\beta<0,\\ k<k_\Delta. \end{cases}$$

Recalling the second half of the proof of Lemma 6.5, we can conclude. □

Lemma 6.7. *Let $r<1$ and*

$$r_1=\frac{(N-1)(2N+1-2\sqrt{2N})}{(2N-1)^2}$$

$$r_2=\frac{(N-1)(2N+1+2\sqrt{2N})}{(2N-1)^2}.$$

We have that $\Delta>0$ if $0<r\leq r_1$, if $r_2\leq r<1$, or if $r_1<r<r_2$ and $k>k_\Delta$. In particular, for $N>2$, it holds that $0<r_1<1/2<r_2<1$.

Proof. We that $\Delta>0$ can be rewritten as $a_1k+a_2>0$ where $a_1=2Nr(N-1)(1-r)>0$ and $a_2=r^2(2N-1)^2+2r(-2N^2+N+1)+(N-1)^2$. We have that $a_2\geq 0$ provided that $r\leq r_1$ or $r\geq r_2$, while if $r_1<r<r_2$ we need $k>-a_2/a_1=k_\Delta$. Straightforward computations allow proving the last desired properties for r_1 and r_2. □

Lemma 6.8. *Let $r<1$ and $k>4/(1-r)$, then $\omega_2\geq 1-1/N$.*

Proof. To have $\omega_2<1-1/N$, recalling that $\beta>0$, we would need $N\alpha-\beta(N-1)+N\sqrt{\Delta}<0$. This necessarily requires that $N\alpha-\beta(N-1)<0$, i.e. $N(1-r)(r(N-1)+1)k+2N(N+r-3)<0$, which is indeed impossible. □

Lemma 6.9. *Let $1/2<r<1$ and $k<4/(1-r)$, then $\omega_2\leq 1/N$.*

Proof. To have $\omega_2>1/N$, recalling that $\beta<0$, we would need $N\alpha-\beta+N\sqrt{\Delta}<0$, namely

$$\begin{cases} N\alpha-\beta<0,\\ N^2\Delta-(\beta-N\alpha)^2<0,\\ \Delta>0. \end{cases} \tag{6.18}$$

The first condition (recall (6.4)) requires that

$$2-2N-6r+4Nr<0. \tag{6.19}$$

If $r>1/2$ we need $N<(3r-1)/(2r-1)$, but since $N\geq 3$, this requires $(3r-1)/(2r-1)\geq 3$, i.e. $r\leq 2/3$. Conversely, if $r\leq 1/2$, condition (6.19) is always satisfied, since it is indeed true for $r=1/2$ and if $r<1/2$, we need $N>(3r-1)/(2r-1)$, but this condition is satisfied because $(3r-1)/(2r-1)<2$. Under either of the two previous conditions on r, $N\alpha-\beta<0$ when $k<k_1$.

The second condition requires (recall (6.5)) $k < k_2$. Recalling Lemma 6.7, system (6.18) then requires either

$$\begin{cases} r_2 \le r \le \dfrac{2}{3}, \\ k < \min\left\{k_1, k_2, \dfrac{4}{1-r}\right\} \end{cases} \tag{6.20}$$

or

$$\begin{cases} \dfrac{1}{2} < r < \min\left\{r_2, \dfrac{2}{3}\right\}, \\ k_\Delta < k < \min\left\{k_1, k_2, \dfrac{4}{1-r}\right\}. \end{cases} \tag{6.21}$$

Regarding (6.20), it can be easily proven that $r_2 < 2/3$ provided that $N \ge 21$. However, $k_1 > 0$ for $r < (N-1)/(2N-3)$, but for $N \ge 3$ we have $(N-1)/(2N-3) < r_2$, so (6.20) is impossible. Regarding (6.21), we notice that in order to have $k_1 > k_\Delta$ we need (see (6.6) and recall $r < 1$) $N + r - 2Nr - 1 > 0$, or, equivalently, $r < (N-1)/(2N-1) < 1/2$, which is impossible. Hence, also (6.21) cannot be verified. □

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