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A family of models for Schelling binary choices

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HIGHLIGHTS

- We introduce a family of discrete dynamical systems to model Schelling binary choices.
- We study steady states and their relation to the equilibria predicted by Schelling.
- Local stability, possible destabilizations and chaos existence are analyzed.
- Using bifurcation theory, we study scenarios qualitatively described by Schelling.
- We provide examples and simulations that confirm the analytical results.

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ABSTRACT

We introduce and study a family of discrete-time dynamical systems to model binary choices based on the framework proposed by Schelling in 1973. The model we propose uses a gradient-like adjustment mechanism by means of a family of smooth maps and allows understanding and analytically studying the phenomena qualitatively described by Schelling. In particular, we investigate existence of steady states and their relation to the equilibria of the static model studied by Schelling, and we analyze local stability, linking several examples and considerations provided by Schelling with bifurcation theory. We provide examples to confirm the theoretical results and to numerically investigate the possible destabilizations, as well as the emergence of coexisting attractors. We show the existence of chaos for a particular example.

1. Introduction

A binary choice is an either-or situation, in which agents have exactly two possible choices, which we will indicate in what follows by R and L. It is a very simple setting which, however, allows describing a wide variety of different situations, ranging from the decision to drive or not on certain days, to take or not an antibiotic, to take part in a collective action or to stay inactive and so on. The criteria according to which an agent makes a decision or its opposite may be different and related to both the decision itself and its consequences, and the influence exerted by the choices of the other agents on his own choice. In particular, in the latter case, we have that the choice of each individual can be influenced by social interactions. For instance, in the context of statistical physics, models with local interactions may be found in Refs. [1,2], while in Ref. [3] the authors study the binary choices of a set of agents through mean-field analytical methods.

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A fundamental contribution about the effects of social interactions on binary choices is the seminal work by Schelling in Ref. [4], in which a systematic discussion of several binary choice contexts is presented, together with their qualitative study. A key assumption in Ref. [4] is that agents are supposed to be *rational*. The presence of rational agents is a fundamental aspect, and rational choice models are an important class of mathematical sociology models (see Coleman [5], Sorensen and Sorensen [6], and, more recently, Braun [7]). The work by Schelling in Ref. [4] provided a general framework suitable for the development of a wide range of models and in fact it has been taken into account in several works during the past decades. Without claiming to be complete, we can mention the influence of Ref. [4] on modeling resource management (Runge [8]), corruption diffusion (Andvig and Moene [9]), public opinion formation (Dodds and Watts [10]) and also nature sciences, as well as the diffusion of contagion (Dodds and Watts [11]).

Going back to the rationality assumption on agents, it allows studying possible equilibria in terms of the payoff functions associated to the individual choices. The payoff functions are assumed to depend only on the number (and not on the identity) of the agents that make a particular choice, so that those functions can be assumed to depend just on the fraction x of agents making, for instance, choice R. The agents opt for a choice or its opposite on the basis of the payoff. However, since, for each agent, the choice of the others influences his own payoff, he has to take into account also the effect of the others choice. This is called *externality*, and in Ref. [4] it is also taken into account for the classification of the different binary choices with respect to related payoffs. In the qualitative model by Schelling, *equilibria* correspond to the Nash equilibria and occur

- at the internal fractions of agents for which the payoff functions intersect (or, equivalently, the payoff differential is null) and no agents take advantage from changing its choice;
- in some extremal situations, in which all agents are making the same decision, provided that the payoff of such choice is larger than that of the opposite one.

For example, Schelling formalizes the multiperson prisoner's dilemma (MPD), for which the following conditions have to be satisfied:

1. each individual faces the same binary choice and payoffs;
2. each individual has a dominant choice independently of what the others do;
3. any individual is better off the more numerous are the agents who make their dominated choice.

In this situation, the payoff functions do not intersect and the equilibrium is represented by the configuration in which each agent chooses the dominated strategy. In this case, the externality is uniform, in the sense that when the fraction of agents making a particular choice is increased, it has the same effects on the payoffs of agents choosing both R and L. For example, imagine that both payoffs are described, as in the MPD, by increasing functions. If an agent chose R, then the payoffs of the other agents, who chose both R and L, would increase, while if an agent chose L, both R and L payoffs would decrease. Then, in the first case we speak of positive externality, while in the second one of negative externality.

Another framework that can be classified by means of the concept of externality is that of common goods. A concrete example of such situation is described by Schelling when he examines the decision to use or not a car with respect to the issue of traffic congestion. In this case, the two payoff functions have opposite monotonicity, intersect once and we have an internal stable equilibrium, different from the extremal situations in which all agents make the same decision. In such context, externality is contingent, as the effects of a given choice are not the same on both payoffs. For example, if the payoff of choice R is decreasing and that of choice L is increasing, choosing R increases the payoff of L and decreases that of R, while choosing L has an opposite effect.

A further example concerning externalities is given by the so-called network goods, in which the users of a good gain when additional users adopt it. This is somehow the opposite of the common goods example, since now the internal intersection for the payoff functions is no more a stable steady state, but it rather acts as a discriminating level between the extremal states in which all players make the same choice.

Such examples show that externalities are essentially connected with the monotonicity of the payoff functions. However, the monotonicity of the payoff functions alone is not sufficient to catch all the possible dynamic behaviors originating with binary choices. Schelling notices, for instance, that also the steepness of the payoff functions is relevant: "*... classification has to consider ... whether or not the externality favors more the choice that yields the externality. That is, with a Right choice yielding the positive externality, does it yield a greater externality to a Right choice or to the Left? Which curve is steeper?*" [4, p. 403].

In Ref. [4], it is also marginally considered the problem of how the situation changes when one of the payoff functions is kept fixed and the other is changed, for example rotated (see Ref. [4, p. 404]). Clearly, the static qualitative analysis of Ref. [4] does not allow to study what happens to the fractions dynamics when the payoff curves are varied, for example depending on a parameter that regulates their slope or position.

The aim of this work consists in providing a modeling framework based on a discrete dynamic model in order to analytically study the dynamics underlying the qualitative and essentially static setting in Schelling [4]. Taking into account a time-dependent model, we want to provide an explicit dynamic adjustment mechanism to validate and deepen the analysis by Schelling, focusing on the local properties of the equilibria, on their stability and on the causes of stability loss, together with the possible scenarios arising when equilibria become unstable. In our contribution we focus on social and economic issues that can be modeled by means of discrete-time processes, i.e., processes in which decisions do not change continuously in time, but rather require some time to be modified (for related examples, see for instance Ref. [12]). We also mention that in Ref. [4] a deep investigation of features such as the threshold rule and the critical mass, or of the effects of coalition formation, is present. These further aspects are not considered and studied in the present paper.

A model based on discrete dynamical systems for Ref. [4] was recently proposed by Bischi and Merlone in Ref. [13].[3] The model describes the evolution of agent fractions of a normalized population. The dynamic nature of the model is represented by the fact that, at time $t+1$, the fraction x_t of population that chose strategy R during the previous time period t can vary to x_{t+1}. Moreover, it is assumed that each agent makes its decision choosing at each time a strategy which is based on the payoffs of the last period only. The model proposed in Ref. [13] is given by

$$x_{t+1} = r(x_t) = \begin{cases} x_t + \delta_R g\big(\lambda\,(R(x_t) - L(x_t))\big)(1-x_t), & \text{for } R(x_t) \geq L(x_t),\\ x_t - \delta_L g\big(\lambda\,(L(x_t) - R(x_t))\big)x_t, & \text{for } R(x_t) < L(x_t), \end{cases} \tag{1.1}$$

where $R(x)$, $L(x)$ are the payoff functions, for $i \in \{R, L\}$, $\delta_i \in [0, 1]$ represent respectively how many agents decide to switch from R to L and vice versa, $\lambda \geq 0$ is the speed of reaction of the agents and $g : \mathbb{R}_+ \to [0, 1]$ is a continuous increasing real function.

In Ref. [13], payoff functions with both one and two intersections are considered, and some results about the steady states of (1.1) are provided. Then, the investigation is mainly focused on the study of the possible global dynamics, by means of several examples which refer to Ref. [4]. Convergence, chaotic behaviors, changes in the basins of attraction are considered, especially looking at the effects on stability of reaction speed variations. The main drawback of model (1.1) is that, being the map r not differentiable at its fixed points, most of the analytical results about stability in the literature on dynamical systems cannot be used and a general study becomes difficult to perform.

In this work we study a general family of adapting strategies, in order to investigate under which conditions it represents a modelization of Schelling qualitative framework. Our goal consists in providing a setting which allows explaining several aspects shown by Schelling in terms of the analytical properties of the resulting discrete dynamical systems. In particular, we shall focus on two aspects.

(A) We wish to study the effects on the dynamics of subjective aspects, focusing on what happens when the reaction of the agents to a particular payoff differential changes.
(B) We investigate the effects on the dynamics of changes in the economic and social context, which are represented by payoff differential variations.

To this end, differently from Ref. [13], we assume that the involved maps are sufficiently regular. In particular, since in our proofs we will need to compute up to the third derivatives, we may assume our maps to be of class $\mathcal{C}^3$. The models we focus on are based on a gradient-like dynamical adjustment mechanism by means of functions, depending on the previous period population fractions and on the payoff differential, which assess the (positive or negative) fraction of agents that, having chosen L at time t, decide to switch to R at time $t+1$. Such functions also depend on a parameter describing the speed of reaction of the agents to the payoff differential. To pursue goal (A), we will study the effect of varying the reaction speed γ, while we will investigate aim (B) by considering a family of payoff differentials, which depend on an exogenous parameter μ. This latter scenario represents a situation in which the payoff differential varies depending on an exogenous cause, which can for instance modify the reciprocal position of the payoff functions, as well as the resulting possible equilibria. Such scenario is qualitatively present in Ref. [4] too, but it has not been investigated in Ref. [13]. We will present several general situations in which the effect of the variation of the payoff differential can be read in terms of bifurcation theory.

We shall provide conditions under which the proposed family of maps is suitable to encompass the assumptions of Schelling in Ref. [4] for either setting (A) or (B). We prove general results about the existence of steady states and their relations to the equilibria considered in Ref. [4]. Then, we study the local stability conditions with respect to both the speed of reaction of the agents and payoff changes. We show that qualitative behaviors predicted explicitly or implicitly by Schelling can be understood in terms of bifurcation theory. Finally, we propose some examples of models belonging to the introduced family, to provide a confirmation of the analytical results and to cast a glance on the possible global dynamics.

The article is organized as follows. In Section 2 we introduce the model. In Section 3 we provide analytical results about local stability. In Section 4, we study some examples and perform simulations. Finally, in Section 5 we present some conclusions and future perspectives.

2. Binary choice models

We introduce a family of models in order to describe the aggregate outcome of repeated binary choice games. The setting we deal with is the same considered by Schelling in Ref. [4] and it is based on the following two assumptions (see in particular [4, p. 383]). First of all, we suppose that each agent is confronted with exactly two possible choices, that we will indicate with R (right) and L (left), so that, denoting by x_t the fraction of agents that at time t make, for example, choice R, we have the

[3] In Ref. [14], Bischi and Merlone propose a unified model which synthesizes both the Galam's rumor spreading model in Ref. [15] and the Schelling's binary choices model in Ref. [4].

Assumption 2.1. At each time t, each agent faces a purely binary choice, so that if x_t individuals play choice R, the remaining $1 - x_t$ are playing the opposite one L.

Moreover, as in Ref. [4], we assume that all the agents have the same impact on the others choices, in the sense that there is no ranking among players induced by sensitivity or influence, so that

Assumption 2.2. Each player's payoff only depends on the number of agents who make choice R or L and not on their identities.

The two previous assumptions by Schelling allow defining the payoff function $R : [0, 1] \to \mathbb{R}$ of agents playing choice R and, similarly, the payoff function $L : [0, 1] \to \mathbb{R}$ of agents playing the opposite choice L. In what follows, we will suppose that both $R(x)$ and $L(x)$ are sufficiently smooth to allow us to perform the needed computations.

In Ref. [4] several examples were discussed together with possible corresponding payoff functions, in order to identify the related equilibria. We stress that those payoff functions were however described just by their qualitative properties, such as monotonicity, and no analytic expression was provided.

Even if the original work by Schelling [4] was essentially static in nature, a dynamic behavior was implicitly present in his analysis. This was already noticed by Bischi and Merlone in Ref. [13], who summarized it in the following

Assumption 2.3. If $R(x_t) > L(x_t)$, the fraction x_t of agents which choose strategy R will increase, while it will decrease when $L(x_t) > R(x_t)$.

The previous assumption implicitly suggests a gradient-like dynamic behavior for the fractions evolution, as they increase towards the choice with the highest payoff. Introducing the payoff differential function $\chi : [0, 1] \to \mathbb{R}$, $\chi(x_t) = R(x_t) - L(x_t)$, the family of dynamical adjustments that we propose is given by

$$x_{t+1} = g(x_t) = x_t + f(x_t, \gamma\chi(x_t)), \tag{2.1}$$

where $f : [0, 1] \times \mathbb{R} \to \mathbb{R}$, $(x, y) \mapsto f(x, y)$, is a sufficiently smooth function which represents the variation in the fraction of the agents that, having chosen L at time t, switch to R at time $t + 1$. Notice that it may also be negative if many agents switch from R to L.

Parameter γ is a positive constant that represents the speed of reaction of the agents and the weight they assign to the payoff differential. We stress that, since (2.1) describes a very simple adaptive adjustment mechanism, it is natural to assume, as in Ref. [13], a reduced rationality degree for the agents, which, in making their choices, only know the (aggregate) choice of the other agents at time t and the resulting payoff.

We remark that model (2.1) complies with both the previous Assumptions 2.1 and 2.2. First of all, a one-dimensional model is enough, thanks to the purely binary nature of the choice established by Assumption 2.1. Then, the payoff differential χ depends on the agents fractions only, in agreement with Assumption 2.2. In order to satisfy Assumption 2.3, we need some further hypotheses on function f. First of all, to guarantee that $x_t \in [0, 1]$ for all t, and thus that $g(x_t)$ is a function from $[0, 1]$ into itself, we need

$$-x \le f(x, y) \le 1 - x, \quad \forall (x, y) \in [0, 1] \times \mathbb{R},$$

which guarantees that

$$-x_t \le f(x_t, \gamma\chi(x_t)) \le 1 - x_t, \quad \forall \chi : [0, 1] \to \mathbb{R},\ \gamma > 0,\ x_t \in [0, 1]. \tag{2.2}$$

Condition (2.2) says that if the fraction of agents making choice R increases (respectively decreases) from time t to time $t+1$, it can do at most of the fraction $1 - x_t$ (respectively x_t) of agents making choice L (respectively R) at time t. This is essentially the reason why the function f has to depend directly also on the fraction x_t and not only on the payoff differential $\chi(x_t)$. In particular, we highlight that condition (2.2) requires that when $x_t = 0$, then f has to be non-negative, while when $x_t = 1$, then f has to be non-positive.

To be consistent with Assumption 2.3, we require that, for a given pair (f, χ) and a reaction speed $\gamma > 0$

$$\begin{cases} f(x_t, \gamma\chi(x_t)) > 0, & \text{if } \chi(x_t) > 0, \\ f(x_t, \gamma\chi(x_t)) < 0, & \text{if } \chi(x_t) < 0, \end{cases} \tag{2.3}$$

for the internal fractions $x_t \in (0, 1)$ and, in order to encompass also the extremal Schelling equilibria in Ref. [4], for the extremal fractions we impose that

$$\begin{cases} f(\bar{x}_t, \gamma\chi(\bar{x}_t)) \ge 0, & \text{if } \chi(\bar{x}_t) > 0, \\ f(\bar{x}_t, \gamma\chi(\bar{x}_t)) \le 0, & \text{if } \chi(\bar{x}_t) < 0, \end{cases} \tag{2.4}$$

where $\bar{x}_t$ is either 0 or 1. Moreover, we assume that $f(x_t, \gamma\chi(x_t)) = 0$ for those $x_t \in [0, 1]$ for which $\chi(x_t) = 0$, which means that if the payoff differential is null, no agent modifies his choice. We notice that, thanks to (2.3), we have that, if both options are chosen by at least one agent (so that $x_t \neq 0, 1$), when the payoff differential is positive, then f is positive too and the fraction of agents that at time $t + 1$ chooses R increases; conversely, when the payoff differential is favorable to choice L, then f is negative and the agents that choose R at time $t + 1$ decrease.

Thanks to (2.3), we will prove that the only possible fractions equilibria of (2.1) with $x_t \in (0, 1)$ are those for which the profit differential is null, which is in accordance with the work by Schelling [4]. The difference between (2.3) and (2.4) lies in the possibility in (2.4) to have $f(\bar{x}_t, \gamma\chi(\bar{x}_t)) = 0$ even when $\chi(\bar{x}_t) \neq 0$, i.e., we only allow function f to possibly introduce equilibria $x = 0$ and $x = 1$ even if the profit differential does not vanish for such fractions. This is still consistent with Ref. [4]. In fact, when strategy R dominates L near $x = 1$, the equilibrium has to be $x = 1$, as well as when strategy L dominates R near $x = 0$, the equilibrium has to be $x = 0$, as shown in several examples therein (see, for instance, the multiperson prisoner's dilemma in Ref. [4, pp. 386–390] or the example in Ref. [4, p. 402]).

In the following result, we look for the possible steady states of (2.1) and their relation to the equilibria of the static setting in Ref. [4], which correspond to the fractions that make the payoff differential null or, possibly, to the extremal fractions $x = 0, 1$.

Proposition 2.1. *If $x^* \in (0, 1)$, then it is a steady state for (2.1) if and only if $\chi(x^*) = 0$. If $x^* = 0$, then it is a steady state for (2.1) if $\chi(0) \leq 0$ or if $\chi(0) > 0$ and $f(0, \gamma\chi(0)) = 0$. If $x^* = 1$, then it is a steady state for (2.1) if $\chi(1) \geq 0$ or if $\chi(1) < 0$ and $f(1, \gamma\chi(1)) = 0$.*

Proof. Firstly, we notice that steady states of (2.1) require that f vanishes and that, thanks to the hypotheses on f, we have $f(x^*, \chi(x^*)) = 0$ when $x^* \in [0, 1]$ is such that $\chi(x^*) = 0$.

Let us now show that if $x^* \in (0, 1)$ is a steady state of (2.1), then $\chi(x^*) = 0$. Indeed, if $x^* \in (0, 1)$ and $\chi(x^*) \neq 0$, from (2.3) we have that also $f(x^*, \gamma\chi(x^*)) \neq 0$, with $\gamma > 0$, so that the only possible internal steady states are those for which the payoff differential is null.

Let us finally consider $x^* = 0$. If $\chi(0) < 0$, then from (2.4) we have that $f(0, \gamma\chi(0)) \leq 0$, with $\gamma > 0$, and from (2.2) that $f(0, \gamma\chi(0)) \geq 0$, so that $f(0, \gamma\chi(0)) = 0$. Proceeding in a similar way for $x^* = 1$ allows concluding the proof. □

The previous result guarantees that the only possible internal steady states are those fractions for which the payoff differential vanishes. Moreover, if the payoff of choice R is larger than the payoff of choice L in some interval of the form $[a, 1]$, then $x^* = 1$ is a steady state. Similarly, if the payoff of choice L is larger than the payoff of choice R in some interval of the form $[0, b]$, then $x^* = 0$ is a steady state. We already noticed that both these situations are in agreement with Ref. [4]. However, we remark that from (2.4) extremal fractions can be steady states of (2.1) even if the previous conditions on payoffs are not satisfied. With this we want to allow for the possibility to include in (2.1) some classical choice adjustment mechanisms, based for instance on simple replicator or logistic dynamics (see also Section 4). As we will show in Proposition 3.3, such extremal $x^* = 0$ and $x^* = 1$ steady states, which are not Schelling equilibria, are always unstable and thus are reached only if the initial datum is exactly given by $x_0 = 0$ or $x_0 = 1$, respectively. Since the adjustment process we propose is based on a reduced degree of rationality, it seems perfectly acceptable to include those situations in which, although when $x_t = 0$ or $x_t = 1$, agreeing all the agents on the same choice, they could not be aware of the opposite choice payoff, any small perturbation in a single agent choice leads all the other agents to change their choices, too.

3. Local stability

Schelling in Ref. [4] discussed several situations in order to qualitatively study the possible equilibria of repeated binary choice games. As concerns the local analysis of the dynamical model (2.1), we will provide conditions for the local stability of steady states and we will investigate how they may possibly lose stability. In what follows, we split the study of the stability of the steady states into two parts, for both analytical and interpretative reasons. Indeed, stability of model (2.1) can be studied with respect to two factors, i.e., reaction speed and payoff differential variation. As regards the former, the stability or instability of the system is inherently related to the decision process of the agents, and in particular to their tendency to change their decisions and to weight the payoff differential. In regard to the latter, the stability or instability of (2.1) is connected to the economic or social changes that modify the payoffs: in fact, these also influence the nature of the possible loss of stability and the consequent global dynamics. Before proving stability results, we provide the following proposition.

Proposition 3.1. *Let x^* be such that $\chi(x^*) = 0$. Then*

$$\frac{\mathrm{d}}{\mathrm{d}x} f(x^*, 0)\chi'(x^*) \geq 0. \tag{3.1}$$

Proof. The conclusion is trivial if $\chi'(x^*) = 0$. Let us then suppose that $\chi'(x^*) > 0$. If $x^* \in (0, 1)$, then there exist intervals (a, x^*) and (x^*, b) in which we respectively have $\chi(x) < 0$ and $\chi(x) > 0$. From (2.3), we also have $f(x, \chi(x)) < 0$ for $x \in (a, x^*)$ and $f(x, \chi(x)) > 0$ for $x \in (x^*, b)$. This implies $\mathrm{d}f/\mathrm{d}x(x^*, 0) \geq 0$. For the extremal situations $x^* = 0$ and $x^* = 1$, the previous argument is still valid considering respectively right or left neighborhoods only. The case $\chi'(x^*) < 0$ can be handled similarly. □

In the following sections we will frequently make reference to several aspects of bifurcation theory. For the reader's convenience, we provide a brief graphical and interpretative summary of the bifurcations we will study. In Fig. 1 we report examples of flip, fold and pitchfork bifurcation diagrams, together with the one-dimensional maps, which depend on a Q4

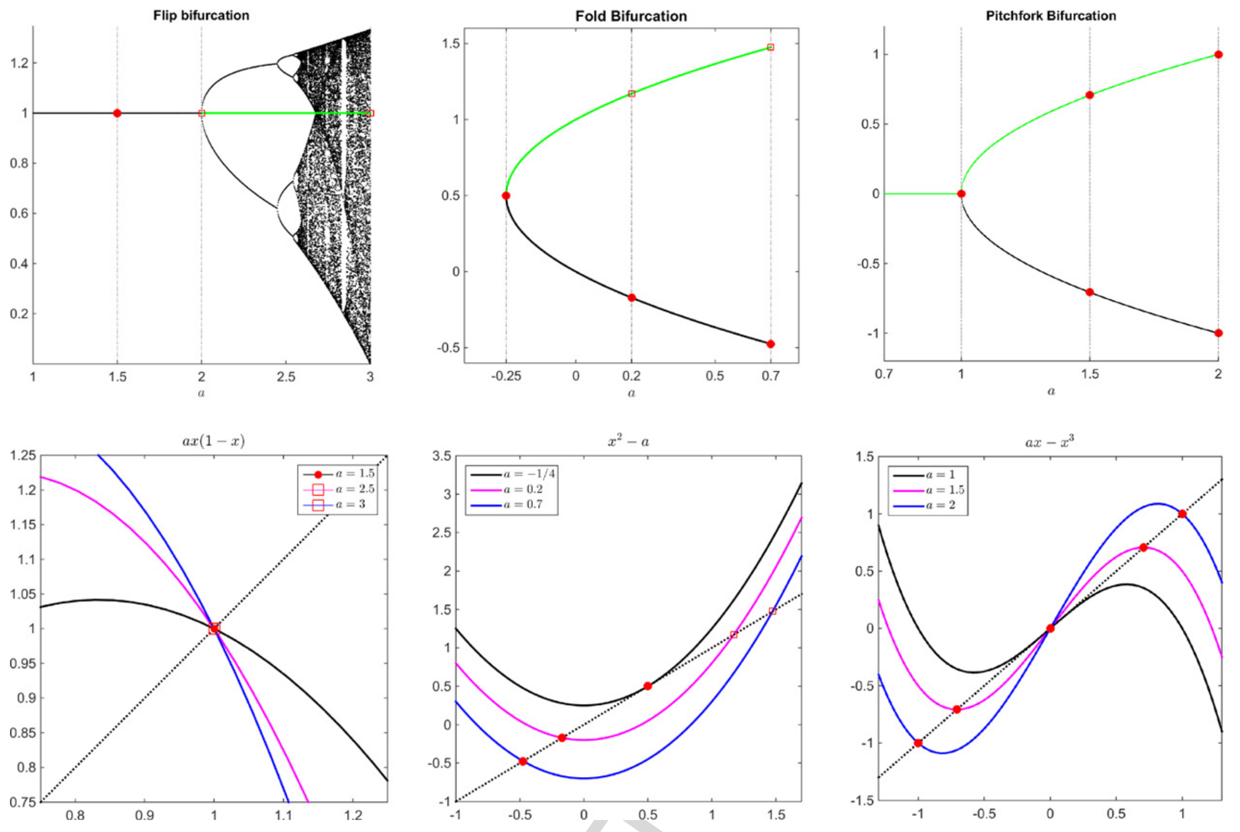

Fig. 1. Bifurcation diagrams (top row) of some one-dimensional maps (bottom row) depending on a parameter a. Stable steady states are indicated by filled dots, unstable ones by empty squares. In the fold bifurcation diagram the dynamics can only converge to the black points, while green points represent unstable steady states. In the pitchfork bifurcation diagram the dynamics can converge either to the black or green points, depending on the initial datum. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.) .

parameter a, they are generated from. In correspondence to a flip bifurcation (left column of plots in Fig. 1), the map becomes steeper and steeper at the steady state as the parameter increases. The initially stable steady state is then replaced by an initially stable period-2 cycle, which for increasing values of a incurs a cascade of period doublings leading to chaos. In correspondence to a fold bifurcation (middle column of plots in Fig. 1), on varying the parameter the map becomes tangent to the 45-degree line, so that a pair of stable/unstable steady states arises. In correspondence to a pitchfork bifurcation (right column of plots in Fig. 1), on varying the parameter a stable steady state becomes unstable and it is replaced by a pair of stable steady states. The dynamics can converge to both steady states, depending on the initial datum.

3.1. Stability analysis with respect to speed of reaction

We want to study what happens when, considering the dynamics characterized by a particular function f, the reaction speed γ of the agents to a particular payoff χ is varied. In this scenario function f has to satisfy the hypotheses in Section 2 (in particular conditions (2.3) and (2.4)) for all $\gamma > 0$. This allows proving a consequence of Proposition 3.1.

Corollary 3.1. *Let f be a function for which Conditions (2.3) and (2.4) are fulfilled for any $\gamma > 0$. If x^* is such that $\chi(x^*) = 0$, then*

$$\partial_x f(x^*, 0)\chi'(x^*) \geq 0 \quad \text{and} \quad \partial_y f(x^*, 0) \geq 0. \tag{3.2}$$

Proof. The result is trivial for $\chi'(x^*) = 0$. If $\chi'(x^*) \neq 0$, from Proposition 3.1, we have that

$$\partial_x f(x^*, 0)\chi'(x^*) + \gamma(\chi'(x^*))^2 \partial_y f(x^*, 0) \geq 0, \tag{3.3}$$

which has to be valid for any $\gamma > 0$.

If $\partial_y f(x^*, 0) < 0$, considering $\gamma > -\partial_x f(x^*, 0)/(\chi'(x^*)\partial_y f(x^*, 0))$ would violate condition (3.3).

Similarly, if $\partial_x f(x^*, 0)\chi'(x^*) < 0$, then $\gamma < -\partial_x f(x^*, 0)/(\chi'(x^*)\partial_y f(x^*, 0))$ would violate (3.3). This concludes the proof. □

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We start our analysis by considering internal steady states.

Proposition 3.2. *System (2.1) is locally asymptotically stable at the steady state $x^* \in (0, 1)$ provided that*

$$-2 < \partial_x f(x^*, 0) + \gamma\chi'(x^*)\partial_y f(x^*, 0) < 0. \tag{3.4}$$

Proof. The proof is a straightforward consequence of the stability condition $|g'(x^*)| < 1$, where $g' = 1 + \partial_x f + \gamma\chi'\partial_y f$. □

The previous result says that stability essentially depends on the behavior of the payoff differential at the equilibrium itself, as well as on the reaction speed γ. In particular, if the payoff differential is not decreasing in x^*, as for example when $R(x) < L(x)$ and $R(x) > L(x)$ respectively in a left and a right neighborhoods I^- and I^+ of x^*, then the equilibrium is necessarily unstable. In fact, from $\chi'(x^*) \geq 0$ and (3.2), we have that both $\partial_x f(x^*, 0)$ and $\partial_y f(x^*, 0)$ are non-negative. This result is in agreement with Assumption 2.3, since, in this case, we have that, if $x \in I^-$, the fraction of agents choosing R must decrease and, conversely, when $x \in I^+$, the fraction of agents choosing R must increase.

When instead $\chi'(x^*) < 0$, from (3.2) we have that $\partial_x f(x^*, 0)$ is non-positive and $\partial_y f(x^*, 0)$ is non-negative. If $\partial_y f(x^*, 0) \neq 0$, then the right inequality in (3.4) is always satisfied and stability depends on the reaction speed γ, as increasing γ can lead to instability through a flip bifurcation.[4] We notice that if $\partial_y f(x^*, 0) = 0$, the reaction speed does not affect equilibrium stability. Since we are interested in the role of γ, in the examples considered in Section 4 it is always the case that $\partial_y f(x^*, 0) > 0$.

In the next proposition, we study the local stability of $x^* = 0, 1$.

Proposition 3.3. *Let $x^* = 0$ be a steady state for (2.1). If $\chi(0) = 0$, then it is locally asymptotically stable provided that*

$$-2 < \partial_x f(0, 0) + \gamma\chi'(0)\partial_y f(0, 0) < 0.$$

If $\chi(0) < 0$, then it is locally asymptotically stable independently of γ, while, if $\chi(0) > 0$, then it is unconditionally unstable.

Let $x^* = 1$ be a steady state for (2.1). If $\chi(1) = 0$, then it is locally asymptotically stable provided that

$$-2 < \partial_x f(1, 0) + \gamma\chi'(1)\partial_y f(1, 0) < 0.$$

If $\chi(1) > 0$, then it is locally asymptotically stable independently of γ, while if $\chi(1) < 0$, then it is unconditionally unstable.

Proof. The proof is similar for both extremal fractions, so we only detail the case $x^* = 0$. If $\chi(0) = 0$, the desired conclusion follows from the same arguments used for Proposition 3.2. If $\chi(0) < 0$ and $f(0, \gamma\chi(0)) = 0$, we have from (2.3) and the regularity assumptions that $f(x, \gamma\chi(x)) < 0$ in a suitably small interval $(0, a)$. Then, in such interval, we have $g(x) < x$ for all γ, and this guarantees unconditional local stability. If $\chi(0) > 0$ and $f(0, \gamma\chi(0)) = 0$, we have the opposite situation, and $f(x, \gamma\chi(x)) > 0$ in a suitably small interval $(0, a)$. This allows concluding. □

The previous proposition guarantees that if the payoff differential is null for extremal fractions, then the stability of such steady states is regulated by conditions similar to those obtained for internal fractions. If instead the payoff differential is favorable to choice R near $x = 1$ or to choice L near $x = 0$, then the corresponding extremal fraction is a stable equilibrium independently of the reaction speed, so that the dynamics always locally converge towards the extremal fraction. Conversely, the steady states that can be possibly introduced by function f and that are not equilibria in Ref. [4] are always unstable.

3.2. Stability analysis with respect to payoff differential

In the work by Schelling [4], a crucial concern is about the study of what happens when the relative position of the payoff functions varies. In Section "Prisoner's Dilemma" in Ref. [4], Schelling extends the two person prisoner's dilemma to Multiperson Prisoner's Dilemma (MPD) and describes several possible configurations arising from the choice of linear payoff functions (see p. 388). Later, in Section "Some Different Configurations", he examines other situations that, even if are not formally ascribable to MPD, show, at least locally, a similar behavior. As we already noticed in the Introduction, in such contexts a key role is played by the qualitative behavior of externality.

We then reconsider some examples inspired by Ref. [4], to briefly show possible scenarios that can arise when the payoff functions are shifted or rotated. To fix ideas, we keep R fixed and we vary L.

In the first situation we consider (see Fig. 2), we have two linear payoff functions.

Several examples of situations involving such payoff functions can be found in Ref. [4] (we refer in particular to Sections "Intersecting Curves" and "Contingent Externality", on pp. 401 and 404, respectively). More precisely, in Fig. 1 function L_1 is dominated by function R, and the only equilibrium is $x = 0$, as in the MPD. If the payoff of L is shifted up and slightly rotated, the initial uniform externality turns into a contingent externality and an internal equilibrium emerges, as in the case of R and L_2. Such steady state is stable, but if the payoff function of L is inclined like L_3 the equilibrium can become dynamically

[4] Referring to the notation introduced in the proof of Proposition 3.2, we recall that a flip bifurcation occurs at x^* when $g'(x^*) = -1$.

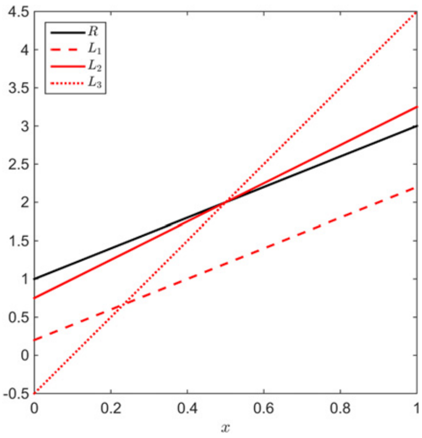

Fig. 2. All the payoff functions are linear. With R and L_1 we have a MPD situation, without internal equilibria; with R and L_2 we have an internal equilibrium; if the payoff of L is further rotated the equilibrium can become unstable, like in the case of R and L_3.

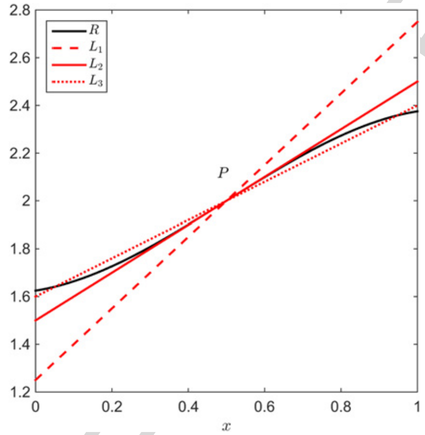

Fig. 3. The payoff functions of L are linear, while that for R presents a perturbation in the concavity with respect to the linear framework. For R and L_1 we have only the intersection P, which is an internal equilibrium. For R and L_2, the two payoff functions are tangent at P. For R and L_3, we have three internal equilibria: P has become unstable, while the new equilibria are locally asymptotically stable.

unstable (of course, as we saw in Proposition 3.2, stability depends also on γ and $\partial_y f$). In bifurcation theory, this is a well-known scenario corresponding to a flip bifurcation in which, as recalled at the beginning of Section 3, an initially locally stable equilibrium loses its stability and a stable period-2 cycle emerges.

However, even confining ourselves to monotonic payoff functions, a slight change in one of the payoffs can give rise to a totally different situation. Suppose for instance that function R changes its concavity, as in Fig. 3: notice that function R is only slightly different from the linear one considered in the previous example. The main dissimilarity between the two frameworks is that now the variation of payoff R is no longer constant, but it increases as we approach P. In this situation, as L rotates, two new equilibria join the initial one. In the case depicted in Fig. 3, the internal equilibrium is initially stable with R and L_1, then it becomes neutrally stable with R and L_2, and finally with R and L_3 it becomes unstable and two new locally stable equilibria emerge. Of course, increasing the reaction speed γ can lead the two new locally stable internal equilibria to instability. This phenomenon can be assimilated to the occurrence of a pitchfork bifurcation for a dynamical system which, as previously recalled, is characterized by the presence of an initially stable equilibrium that becomes unstable. At the same time, two new locally stable equilibria arise. We remark that even if the concavity of the payoff functions was not taken into account in the original work of Schelling, payoff functions reported in Fig. 3 are suitable modeling for several examples provided in Ref. [4]. To this end, we can mention the examples reported in Refs. [9] and [16]. We also underline that the opposite situation, characterized by an initially unique unstable equilibrium that gains stability and two unstable equilibria arise, is possible as well.

The third situation we want to point out is inspired by the example described in Ref. [4, p. 414], which Schelling reports to model a choice about opting for a particular communication system. Again we call attention to what happens when the intensity of the payoff function of L is varied (see Fig. 4).

The choice R stays for adopting the system, while, when L is chosen, the system is not adopted. If there are several reasons not to adopt that system (for instance, because of law restrictions or of heavy taxation) and the payoff function L_1 lies all

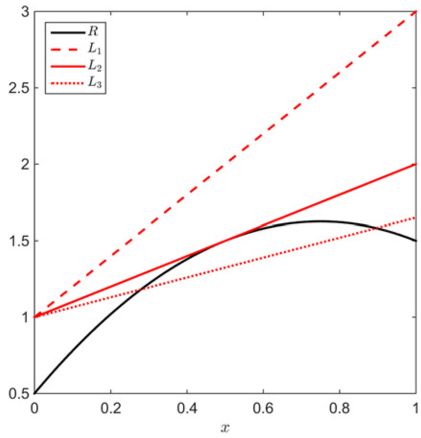

Fig. 4. The payoff of R is unimodal. With R and L_1 we have a MPD framework; with R and L_2 the payoff functions are tangent and the unique equilibrium is internal; if the payoff of L is further rotated, like in the case with R and L_3, then there are two internal equilibria.

above function R, then the only steady state is $x = 0$. If relative conditions for the communication systems become more favorable to its adoption, a new equilibrium arises when the two payoff functions are tangent, like in the case with R and L_2. If the payoff of L further decreases, like in the case of L_3, then there are two internal equilibria, the leftmost unstable and the rightmost possibly stable. Indeed, as we already noticed also referring to Assumption 2.3, an equilibrium x^* is necessarily unstable if, like for the leftmost equilibrium in our example, in a left neighborhood of x^* we have $L(x) > R(x)$ and in a right neighborhood of x^* we have $L(x) < R(x)$. The rightmost equilibrium can instead be stable, depending on the steepness of the functions, since the opposite scenario occurs. If this is the case, such behavior can be described in terms of bifurcation theory by means of a fold bifurcation which, as seen, consists in the sudden emergence of two new equilibria, a stable and an unstable one.

In what follows, we aim to analytically study the possible dynamical behaviors of (2.1) due to variations in the payoff functions. To such end, for a fixed function f, we consider a family of payoff differentials, characterized by a real, exogenous parameter μ, which, for example, affects the position or the steepness of the payoff functions. Without loss of generality, for the sake of simplicity, we keep one of the payoff functions fixed and we allow the other one to vary depending on μ, so that the family of payoff differential can be represented by

$$\chi(x, \mu) = R(x) - L(x, \mu),$$

and we set

$$g(x, \mu) = x + f(x, \gamma\chi(x, \mu)).$$

Notice that, on varying μ, both the number and position of equilibrium fractions may change.

Still for simplicity, since all examples considered in Section 4 do satisfy such assumption, from now on we shall only focus on those families of payoff differentials for which, as μ varies, the set of the possible resulting equilibria contains the whole interval $(0, 1)$.

In order to focus on the effects on the agents' choices caused just by payoff differential variations, in this section we take γ as fixed and we assume that function f satisfies the hypotheses in Section 2 for *any* payoff differential of the family.

Under the assumptions above, we get the following consequence of Proposition 2.1.

Corollary 3.2. *Let x^* be such that $\chi(x^*) = 0$. Then we have*

$$\partial_x^n f(x^*, 0) = 0, \quad \forall n \geq 1, \tag{3.5}$$

and

$$\partial_y f(x^*, 0) \geq 0. \tag{3.6}$$

Proof. Since function f has to satisfy Condition (2.3) for any $\chi(x, \mu)$ and since equilibria x^*, as μ varies, assume all the values in $(0, 1)$, from Proposition 2.1, we have $f(x, 0) = 0$, $\forall x \in (0, 1)$. Hence, $\partial_x^n f(x, 0) = 0$, $\forall n \geq 1$, $\forall x \in [0, 1]$, and in particular $\partial_x^n f(x^*, 0) = 0$, $\forall n \geq 1$, as desired. Inequality (3.6) is obtained from (3.1) and (3.5). □

In the next propositions, we provide general sufficient conditions on (2.1) for the occurrence of flip, fold and pitchfork bifurcations. We will consider only isolated critical points belonging to $(0, 1)$. Moreover, we will focus on those situations in which agents are not neutral to payoff differential variations near the equilibrium, namely $\partial_y f(x^*, 0) \neq 0$, which allows rewriting condition (3.6) as

$$\partial_y f(x^*, 0) > 0. \tag{3.7}$$

According to Ref. [17, page 91], for the flip bifurcation at the steady state x^* we need that, for a suitable μ^{fl}, $\partial_x g(x^*, \mu^{fl}) = -1$, namely that the left inequality in (3.4) be violated. The remaining two conditions, i.e., $\partial_\mu g(x^*, \mu^{fl})\partial^2_{xx} g(x^*, \mu^{fl}) + 2\partial^2_{x\mu} g(x^*, \mu^{fl}) \neq 0$ and $3(\partial^2_{xx} g(x^*, \mu^{fl}))^2 + 2\partial^3_{xxx} g(x^*, \mu^{fl}) \neq 0$, are very nonrestrictive and can be easily satisfied by generic functions, as we will show in what follows. Since the particular expressions of such conditions in terms of χ are not very explanatory, we do not provide them explicitly.

Instead, the conditions for fold and pitchfork bifurcations are more interesting and we report them in Propositions 3.4 and 3.5, respectively.

Proposition 3.4. *Let μ^{fo} be such that $\partial_x \chi(x^*, \mu^{fo}) = 0$ and suppose that $\partial_\mu \chi(x^*, \mu^{fo}) \neq 0$ and $\partial^2_{xx} \chi(x^*, \mu^{fo}) \neq 0$. Then a fold bifurcation occurs for $\mu = \mu^{fo}$ at the steady state x^*.*

Proof. According to Ref. [18, Theorem 2.5, p. 76], we need $\partial_x g(x^*, \mu^{fo}) = 1$. Since

$$\partial_x g = 1 + \partial_x f + \gamma \partial_y f \partial_x \chi, \tag{3.8}$$

we have that such condition is satisfied thanks to the first hypothesis and to (3.5).

The second condition for a fold bifurcation requires $\partial_\mu g(x^*, \mu^{fo}) \neq 0$, but since

$$\partial_\mu g = \gamma \partial_y f \partial_\mu \chi, \tag{3.9}$$

thanks to (3.7) and the second hypothesis, it is fulfilled. The last condition in Ref. [18] requires $\partial^2_{xx} g(x^*, \mu^{fo}) \neq 0$. We have that

$$\partial^2_{xx} g = \partial^2_{xx} f + \gamma \partial^2_{xy} f \partial_x \chi + \gamma \left(\partial^2_{xy} f + \gamma \partial^2_{yy} f \partial_x \chi\right) \partial_x \chi + \gamma \partial_y f \partial^2_{xx} \chi, \tag{3.10}$$

which, thanks to (3.7), (3.5), the first and the third hypotheses, is indeed not null at (x^*, μ^{fo}). □

Proposition 3.5. *Let μ^p be such that $\partial_x \chi(x^*, \mu^p) = 0$ and let us suppose that $\partial_\mu \chi(x^*, \mu^p) = \partial^2_{xx} \chi(x^*, \mu^p) = 0$, $\partial^2_{x\mu} \chi(x^*, \mu^p) \neq 0$, $\partial^3_{xxx} \chi(x^*, \mu^p) \neq 0$. Then, a pitchfork bifurcation occurs for $\mu = \mu^p$ at the steady state x^*.*

Proof. According to Ref. [19], we need $\partial_x g(x^*, \mu^p) = 1$, which, recalling (3.8), is satisfied thanks to the first hypothesis and to (3.5).

The second condition requires $\partial_\mu g(x^*, \mu^p) = 0$. From (3.9), thanks to the second hypothesis, it is always satisfied. The third condition requires $\partial^2_{xx} g(x^*, \mu^p) = 0$. From (3.10), using (3.5), the first and the third hypotheses, it is indeed satisfied. For the fourth condition, we need $\partial^2_{x\mu} g(x^*, \mu^p) \neq 0$. Since

$$\partial^2_{x\mu} g = \gamma \left(\partial^2_{xy} f + \gamma \partial^2_{yy} f \partial_x \chi\right) \partial_\mu \chi + \gamma \partial_y f \partial^2_{x\mu} \chi,$$

from (3.7), the second and the fourth hypotheses, the fourth condition is valid. Finally, we need $\partial^3_{xxx} g(x^*, \mu^p) \neq 0$. We have

$$\begin{aligned}\partial^3_{xxx} g = {} & \partial^3_{xxx} f + \gamma \partial_x \chi + \gamma \partial^2_{xy} f \partial^2_{xx} \chi + \gamma \partial_x \chi \left[\partial^3_{xxy} f + \gamma \left(\partial^3_{xyy} f + \gamma \partial^3_{yyy} f \partial_x \chi\right) \partial_x \chi + \gamma \partial^2_{yy} f \partial^2_{xx} \chi + \gamma \partial^3_{xyy} f \partial_x \chi\right] \\ & + 2\gamma \left(\partial^2_{xy} f + \gamma \partial^2_{yy} f \partial_x \chi\right) \partial^2_{xx} \chi + \gamma \left(\partial^3_{xxy} f + \gamma \partial^3_{xyy} f \partial_x \chi\right) \partial_x \chi + \gamma \partial_y f \partial^3_{xxx} \chi\end{aligned}$$

in which, thanks to (3.5) and the first four hypotheses, all the terms but the last one vanish at (x^*, μ^p). Thanks to (3.7) and the last hypothesis, we can conclude. □

The two previous propositions say that, thanks to (3.5) (as well as to (3.7)), the appearance of a fold or pitchfork bifurcation in (2.1) is determined only by the properties of the payoff functions. Now we provide some families of payoff functions that satisfy the conditions in Propositions 3.4 and 3.5.

For the fold bifurcation, the functions we propose mimic the example reported in Fig. 4.

Corollary 3.3. *Let us consider $L(x, \mu) = x_0 + \mu x$ and $R(x) = x_0 + \mu^{fo} x - (x - x^*)^2 \hat{R}(x)$, with $\mu^{fo} \neq 0$ and $\hat{R}(x) > 0$. Then, for $\mu = \mu^{fo}$, a fold bifurcation occurs at $x^* \in (0, 1)$.*

Proof. It is sufficient to verify that the payoff functions R and L satisfy the conditions in Proposition 3.4. Since

$$\partial_x \chi(x, \mu) = \mu^{fo} - \mu - 2(x - x^*)\hat{R}(x) - (x - x^*)^2 \hat{R}'(x),$$

we have that the first condition $\partial_x \chi(x^*, \mu^{fo}) = 0$ is satisfied. The second condition is satisfied, since $\partial_\mu \chi(x^*, \mu^{fo}) = -x^* \neq 0$, as well as the last condition, since

$$\partial_{xx} \chi(x, \mu) = -2\hat{R}(x) - 4(x - x^*)\hat{R}'(x) - (x - x^*)^2 \hat{R}''(x)$$

gives $\partial^2_{xx} \chi(x^*, \mu^{fo}) = -2\hat{R}(x^*) \neq 0$, as needed. □

Like for the fold bifurcation, starting from the example shown in Fig. 3, we present a family of payoff functions that exhibit a pitchfork bifurcation. We stress that a sort of pitchfork bifurcation has been detected in Ref. [3], too, where a set of agents randomly chooses one of two competing shops which sell the same perishable products, selecting with a higher probability the store they are most satisfied with. Indeed, in Ref. [3, Fig. 2], for increasing value of the temperature parameter, which takes into account the influence of factors that are not explicitly described, the system, starting from a symmetric state where both stores maintain the same level of activity, reaches a state with broken symmetry where one of the two shops attracts more customers than the other.

Corollary 3.4. *Let us consider* $L(x, \mu) = x_0 + \mu(x - x^*)$ *and* $R(x) = x_0 + \mu^p(x - x^*) + (x - x^*)^3\hat{R}(x)$, *with* $\mu^p \neq 0$ *and* $\hat{R}(x) < 0$. *Then, for* $\mu = \mu^p$, *a pitchfork bifurcation occurs at* $x^* \in (0, 1)$.

Proof. It is sufficient to verify that the payoff functions R and L satisfy the conditions in Proposition 3.5. Since

$$\partial_x \chi(x, \mu) = \mu^p - \mu + (x - x^*)^3\hat{R}'(x) + 3\hat{R}(x)(x - x^*)^2$$

we have that the first condition $\partial_x \chi(x^*, \mu^p) = 0$ is satisfied. The second and the third conditions are satisfied, since

$$\partial^2_{xx}\chi(x, \mu) = 6(x - x^*)^2\hat{R}'(x) + 6\hat{R}(x)(x - x^*) + (x - x^*)^3\hat{R}''(x)$$

and $\partial_\mu \chi = x^* - x$. The fourth condition is satisfied as $\partial^2_{x\mu}\chi = -1$, while, noticing that

$$\partial^3_{xxx}\chi(x, \mu) = 6\hat{R}(x) + (x - x^*)^3\hat{R}'''(x) + 18(x - x^*)\hat{R}'(x) + 9(x - x^*)^2\hat{R}''(x),$$

thanks to the negativity of $\hat{R}$, we can infer that also the last condition is fulfilled. This concludes the proof. □

4. Examples

In this section, we provide some examples of f and we show simulative and graphical evidence of the analytical results of the previous section. We also use such examples to give an insight of the possible routes to chaos and to provide some brief considerations about global dynamics.

The first example can be considered as a smooth version of model (1.1). In particular, let $h_1 : \mathbb{R} \to [-1, 1]$ be a smooth strictly increasing function with $h_1(y) > 0$ for $y > 0$ and $h_1(y) < 0$ for $y < 0$. Let $h_2 : \mathbb{R} \to [0, 1]$ be a non-negative function, strictly decreasing for $y < 0$ and strictly increasing for $y > 0$, and such that $h_2(0) = h_2'(0) = 0$. Then, for any positive a and b such that $a + b = 1$, let us define the continuous function $f_1 : [0, 1] \times \mathbb{R} \to \mathbb{R}$,

$$f_1(x, \gamma\chi(x)) = \begin{cases} ax(1 - x)h_1(\gamma\chi(x)) + b(1 - x)h_2(\gamma\chi(x)), & \text{if } \chi(x) \geq 0, \\ ax(1 - x)h_1(\gamma\chi(x)) - b\,x\,h_2(\gamma\chi(x)), & \text{if } \chi(x) < 0, \end{cases} \tag{4.1}$$

where $\chi : [0, 1] \to \mathbb{R}$ is a suitable smooth map and γ a positive parameter.

First of all we notice that both the derivative of $b(1 - x)h_2(\gamma\chi(x))$ and of $-bxh_2(\gamma\chi(x))$ vanish when $\chi(x) = 0$, so f_1 is smooth. Furthermore, thanks to the hypotheses on the sign of h_1 and h_2, we have that (2.3) and (2.4) are both satisfied. We remark that for (4.1) the extremal fractions are not always equilibria for (2.1). Indeed, $x = 1$ is an equilibrium if $\chi(1) \geq 0$ and $x = 0$ is an equilibrium if $\chi(0) \leq 0$, while in the opposite cases they cannot be equilibria. Then, we notice that

$$-ax \leq ax(1 - x)h_1(\gamma\chi(x)) \leq a(1 - x),$$

and

$$\begin{cases} 0 \leq b(1 - x)h_2(\gamma\chi(x)) \leq b(1 - x), & \text{for } \chi(x) \geq 0, \\ -bx \leq -bxh_2(\gamma\chi(x)) \leq 0, & \text{for } \chi(x) < 0. \end{cases}$$

These inequalities allow concluding that f_1 satisfies (2.2). Since, with respect to y, both $b(1 - x)h_2(y)$ and $-xbh_2(y)$ are non-decreasing and $ax(1 - x)h_1(y)$ is strictly increasing, we also have that (3.7) is satisfied. Finally, an easy computation shows that (3.5) is also valid for f_1. A possible choice for h_1 and h_2 is given by

$$h_1(y) = \tanh(y), \qquad h_2(y) = 1 - \exp(-y^4) \tag{4.2}$$

(see Fig. 5(A)).

The second example we propose is a logistic map, with coefficient depending on the payoff differential, which can be obtained considering $a = 1$ and $b = 0$ in the previous example (4.1), namely

$$f_2(x, \gamma\chi(x)) = x(1 - x)h_1(\gamma\chi(x)). \tag{4.3}$$

In this case, $x = 0, 1$ are always steady states. Again we consider

$$h_1(y) = \tanh(y) \tag{4.4}$$

(see Fig. 5(B)).

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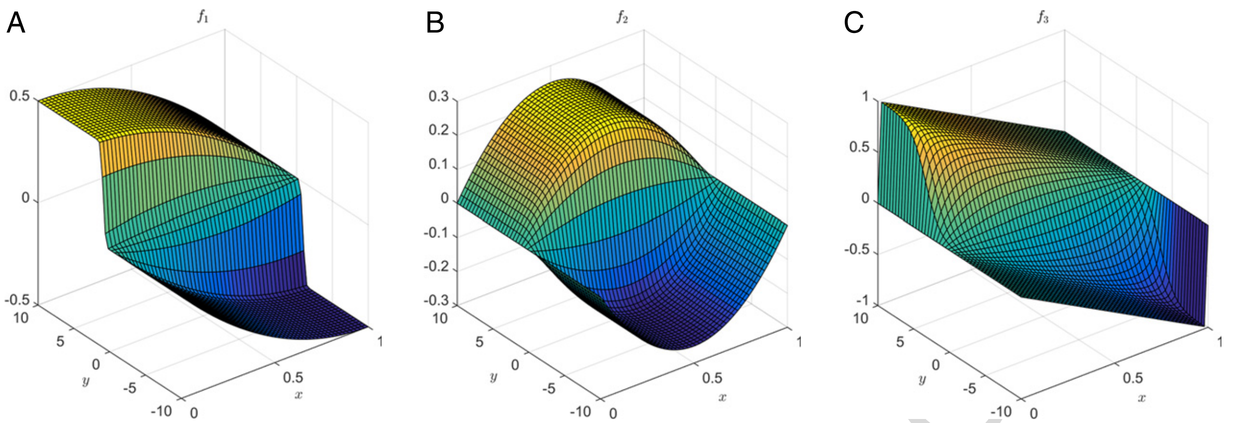

Fig. 5. In (A) the map in (4.1) with h_1 and h_2 given by (4.2). In (B) the map in (4.3) with h_1 given by (4.4). In (C) the map in (4.5) with h_3 given by (4.7). The three graphs are obtained considering $\gamma\chi(x)$ as a free variable y, i.e., not specifying γ and $\chi(x)$.

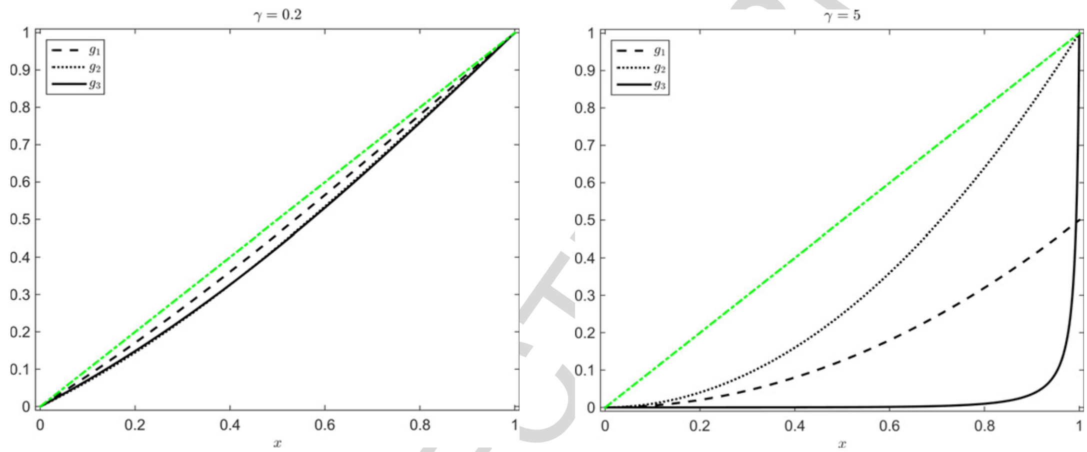

Fig. 6. Maps in (4.8)–(4.10) for $R(x) = 3x + 1$ and $L(x) = 2x + 3$ and for two different values of reaction speed γ, i.e., $\gamma = 0.2$ and $\gamma = 5$, respectively.

The last example we take into account is

$$f_3(x, \gamma\chi(x)) = x(1-x)\frac{1 - h_3(\gamma\chi(x))}{(1-x)h_3(\gamma\chi(x)) + x}. \tag{4.5}$$

If $h_3(y) \geq 0$, we have that condition (2.3) rewrites as Q5

$$\begin{cases} h_3(y) < 1, & \text{if } y > 0, \\ h_3(y) > 1, & \text{if } y < 0, \end{cases} \tag{4.6}$$

while $f_3(\hat{x}, y) = 0$ for $\hat{x} = 0, 1$ independently of h_3, so that (2.4) is automatically fulfilled and 0,1 are always equilibria. Thanks to the assumed positivity of h_3, we have that f_3 satisfies (2.2) also. We notice that, since

$$\partial_y f_3(x, y) = -x(1-x)\frac{h_3'(y)}{((1-x)h_3(y) + x)^2},$$

we have that (3.7) is valid if $h_3'(y) < 0$.

It can be proved that

$$\partial_x f_3(x, y) = \frac{h_3(y)}{((1-x)h_3(y) + x)^2} - 1$$

and for $n > 1$

$$\partial_x^n f_3(x, y) = \frac{n! h_3(y)(h_3(y) - 1)^{(n-1)}}{((1-x)h_3(y) + x)^{n+1}},$$

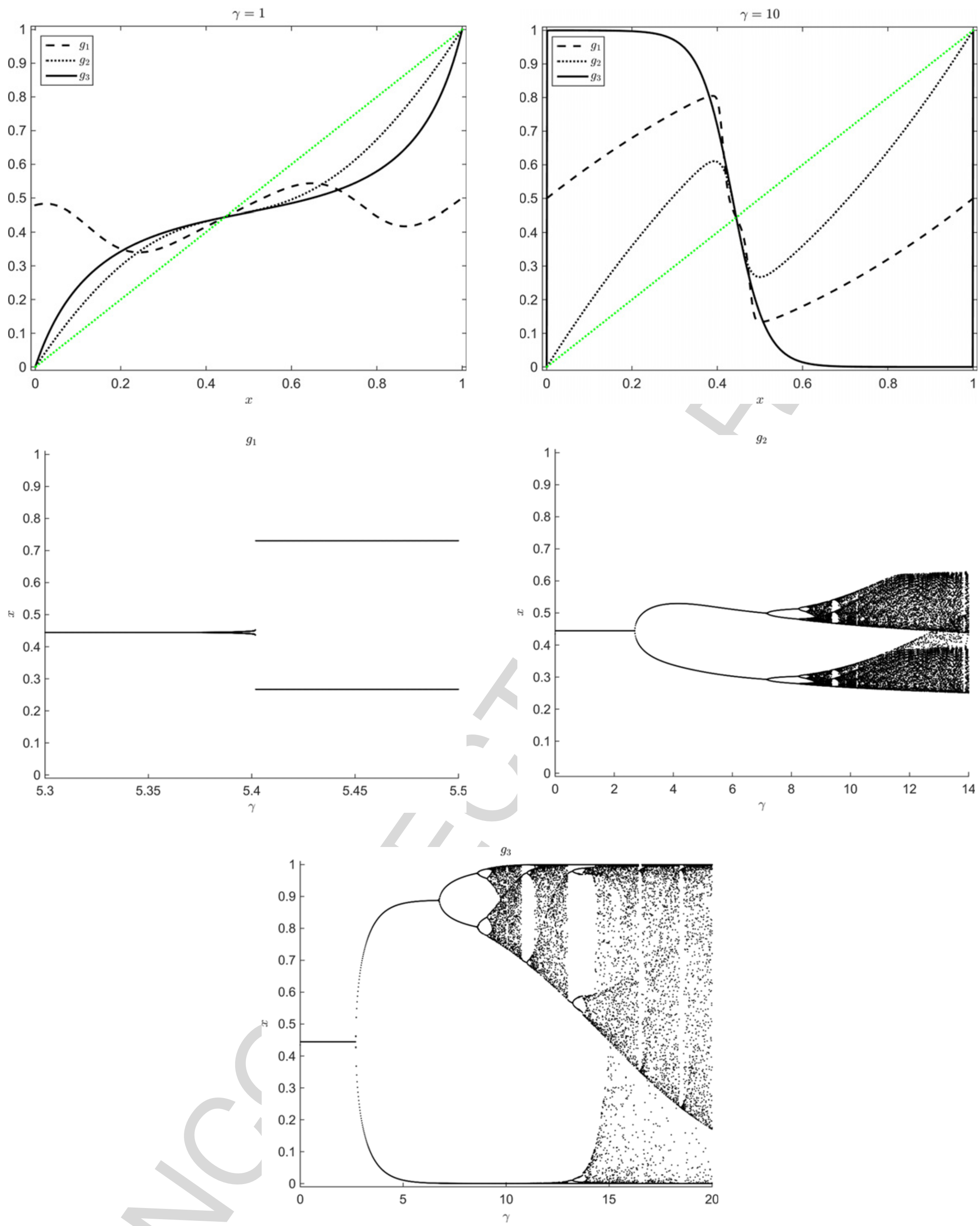

Fig. 7. Top: maps g_1, g_2 and g_3 when the payoff functions are $R(x) = -2x + 1$ and $L(x) = x - 1/3$, for two different values of γ (i.e., $\gamma = 1$ and $\gamma = 10$, respectively). Bottom: bifurcation diagrams with respect to $\gamma \in [0, 20]$, for the initial condition $x_0 = 0.1$.

so that, thanks to (4.6), for all $n \geq 1$, $\partial_x^n f_3(x^*, 0) = 0$ for those x^* for which $\chi(x^*) = 0$ and thus (3.5) is satisfied. In the following simulations we will consider

$$h_3(y) = \exp(-y) \tag{4.7}$$

(see Fig. 5(C)).

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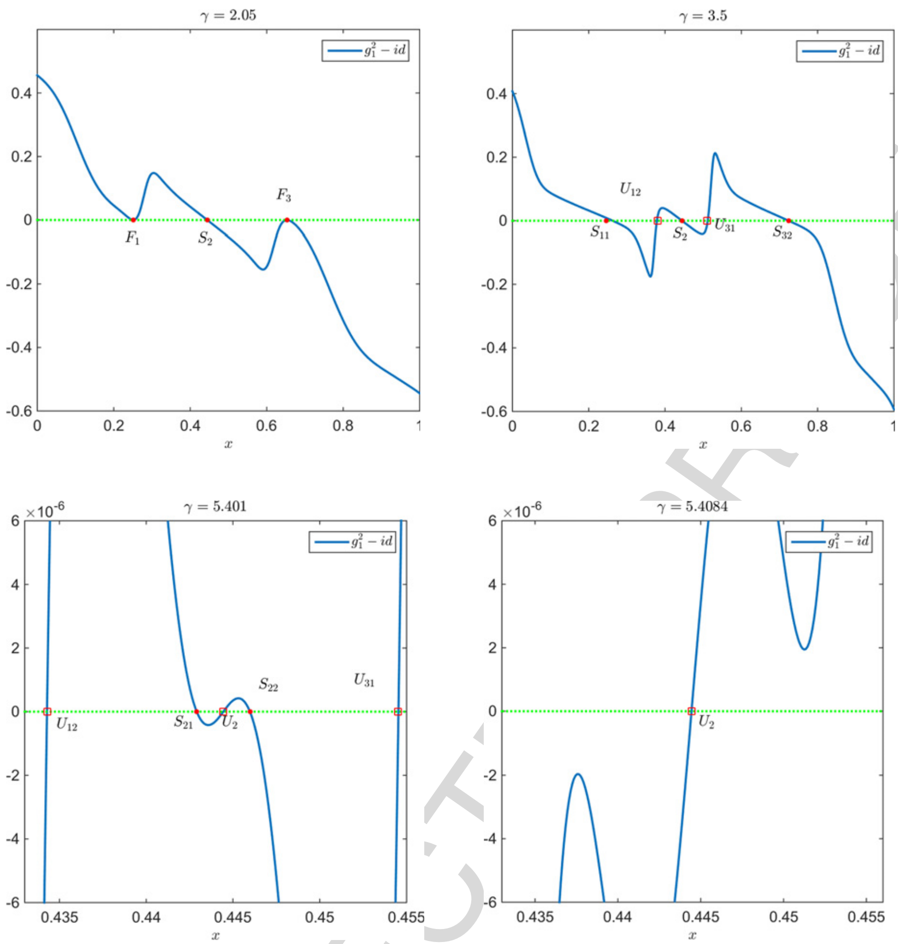

Fig. 8. Graph of $g_1^2 - id$ for different values of γ. Fold, stable and unstable fixed points are marked respectively with F, S and U. Point P_{ij} comes from a bifurcation of point Q_i. Stable fixed points are marked by red dots, unstable ones by red squares. In the second row, a blow-up near the region of $x^* = 4/9$ is plotted. Outside the plotted region, the shape of $g_1^2 - id$ remains similar to that of the first row of graphs. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.) .

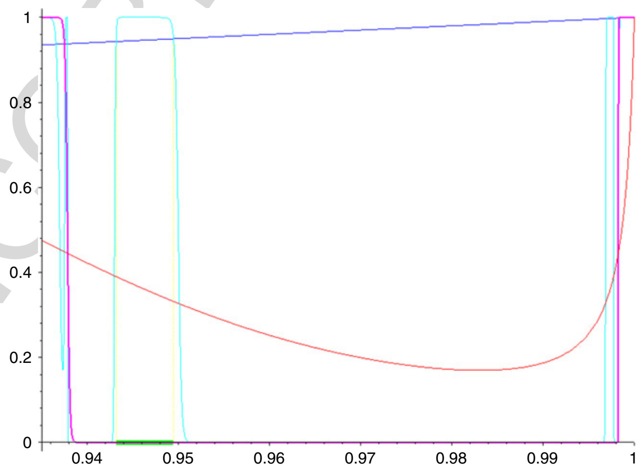

Fig. 9. The graph of the identity map (in blue) and of the first three iterates of g_3^2 (in red, magenta and cyan, respectively) for $\gamma = 20$. The interval highlighted in green on the horizontal axis is J from Remark 4.1. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.) .

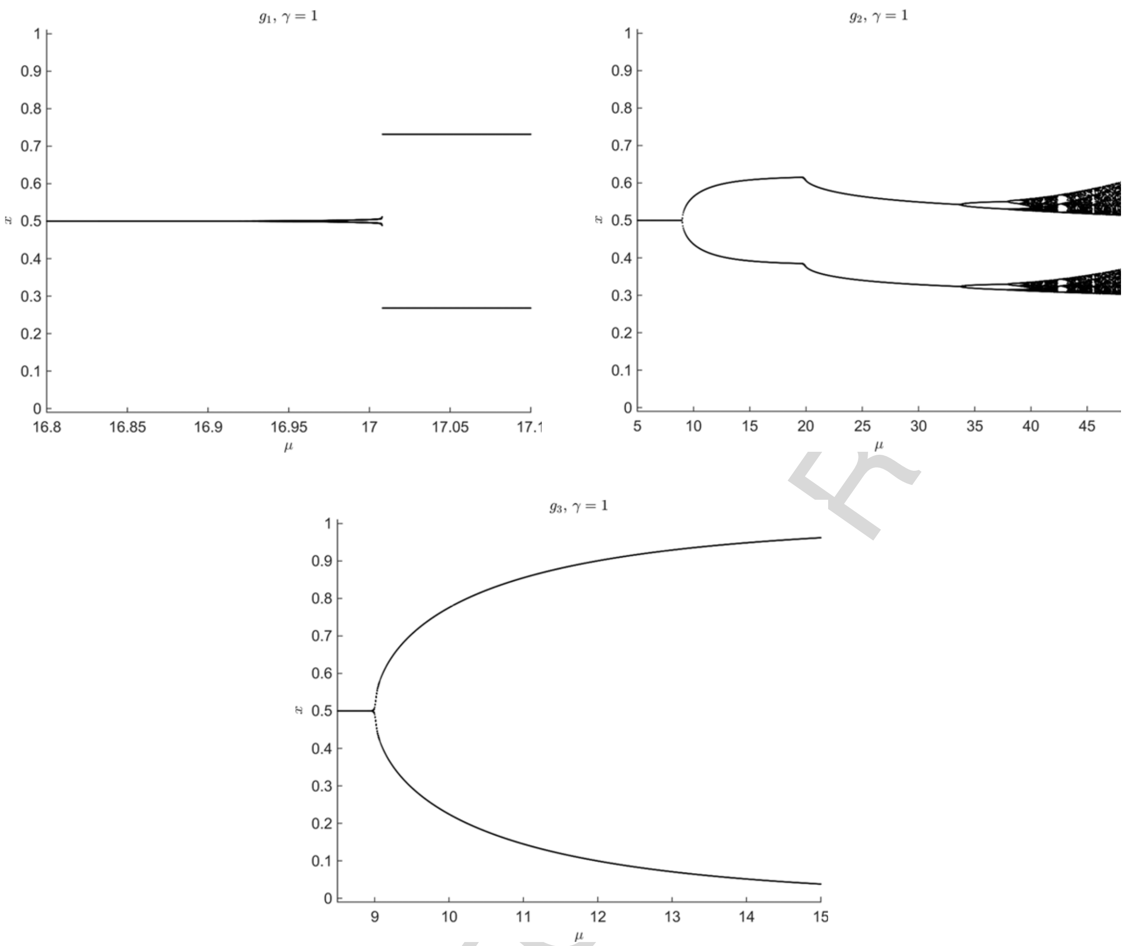

Fig. 10. Bifurcation diagrams for the maps g_1, g_2 and g_3 with $R = (x - 1/2) + 2$ and $L = \mu(x - 1/2) + 2$.

The resulting dynamical systems are then:

$$x_{t+1} = g_1(x_t) = \begin{cases} x_t + \dfrac{1}{2}(1 - x_t)\left(x_t \tanh(\gamma\chi(x_t)) + 1 - \exp(-(\gamma\chi(x_t))^4)\right), & \text{if } \chi(x_t) \geq 0, \\ x_t + \dfrac{1}{2}x_t\left((1 - x_t)\tanh(\gamma\chi(x_t)) - 1 + \exp(-(\gamma\chi(x_t))^4)\right), & \text{if } \chi(x_t) < 0, \end{cases} \tag{4.8}$$

coming from (4.1) and (4.2) with $a = b = \frac{1}{2}$;

$$x_{t+1} = g_2(x_t) = x_t + \tanh(\gamma\chi(x_t))x_t(1 - x_t), \tag{4.9}$$

coming from (4.3) and (4.4);

$$\begin{aligned} x_{t+1} &= g_3(x_t) = x_t + x_t(1 - x_t)\frac{1 - \exp(-\gamma\chi(x_t))}{(1 - x_t)\exp(-\gamma\chi(x_t)) + x_t} \\ &= \frac{x_t}{(1 - x_t)\exp(-\gamma\chi(x_t)) + x_t}, \end{aligned} \tag{4.10}$$

coming from (4.5) and (4.7). We remark that (4.10) belongs to the family of replicator functions: for more details see Refs. [20,21].

We stress that, for f_3 in (4.5) with h_3 in (4.7), as $\gamma \to +\infty$ (or as the payoff differential becomes very large), if $x \in (0, 1)$, then all the agents decide to adopt the most profitable strategy, namely

$$\begin{cases} \lim\limits_{\gamma\to+\infty} f_3(x, \gamma\chi) = 1 - x, & \text{if } \chi(x) > 0, \\ \lim\limits_{\gamma\to+\infty} f_3(x, \gamma\chi) = -x, & \text{if } \chi(x) < 0. \end{cases}$$

This is not true for f_1 and f_2.

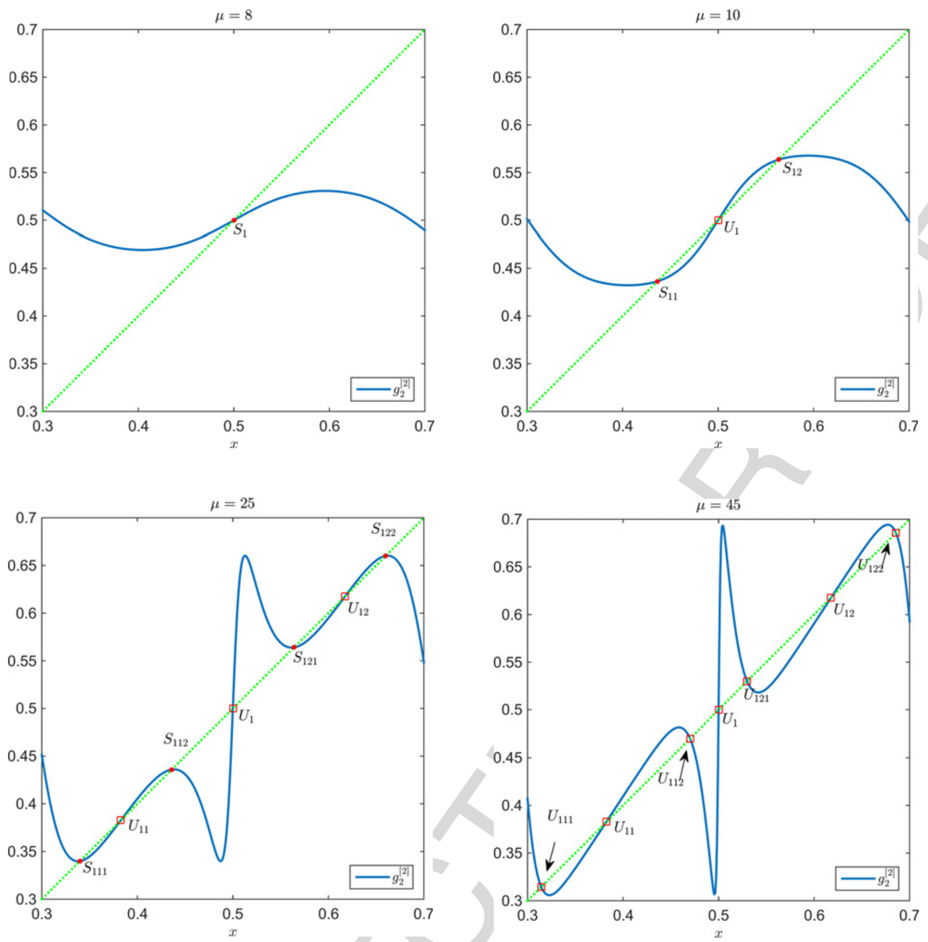

Fig. 11. The graph of the second iterate of g_2 for $\gamma = 1$ and several values of μ. Stable and unstable fixed points are labeled with S and U respectively. Point P_{ij} comes from a bifurcation of point Q_i and point P_{ijk} comes from a bifurcation of Q_{ij}.

In Fig. 6, we report the graph of the maps in (4.8)–(4.10) in a MPD situation, with payoff functions $R(x) = 3x + 1$ and $L(x) = 2x + 3$. All the maps vanish for $x = 0$, which is the equilibrium, while $g_2(1) = g_3(1) = 1 \neq g_1(1)$, as for g_2 and g_3 the extremal fractions are always steady states, even when they are not equilibria. For small values of γ all the functions look very similar, while, as γ increases, some stronger differences appear. In particular, the distance between $g_1(1)$ and 1 becomes more evident.

In general, for $\gamma \to +\infty$, it is easy to see that g_1 pointwise converges to

$$\begin{cases} -1/2x^2 + x + 1/2, & \text{for } x \text{ such that } \chi(x) > 0, \\ 1/2x^2, & \text{for } x \text{ such that } \chi(x) < 0, \\ x, & \text{for } x \text{ such that } \chi(x) = 0, \end{cases}$$

while g_2 pointwise converges to

$$\begin{cases} 2x - x^2, & \text{for } x \text{ such that } \chi(x) > 0, \\ x^2, & \text{for } x \text{ such that } \chi(x) < 0, \\ x, & \text{for } x \text{ such that } \chi(x) = 0, \end{cases}$$

and g_3 pointwise converges to

$$\begin{cases} 1, & \text{for } x \text{ such that } \chi(x) > 0, \\ 0, & \text{for } x \text{ such that } \chi(x) < 0, \\ x, & \text{for } x \text{ such that } \chi(x) = 0 \text{ and } x \neq 0, 1. \end{cases}$$

The convergence provided by g_3 is in general the fastest one.

In Proposition 3.2, we analyzed the role of γ for the stability, proving that its increase leads equilibria to instability via a flip bifurcation. For example, let us consider the two linear payoff functions $R(x) = -2x + 1$ and $L(x) = x - 1/3$,

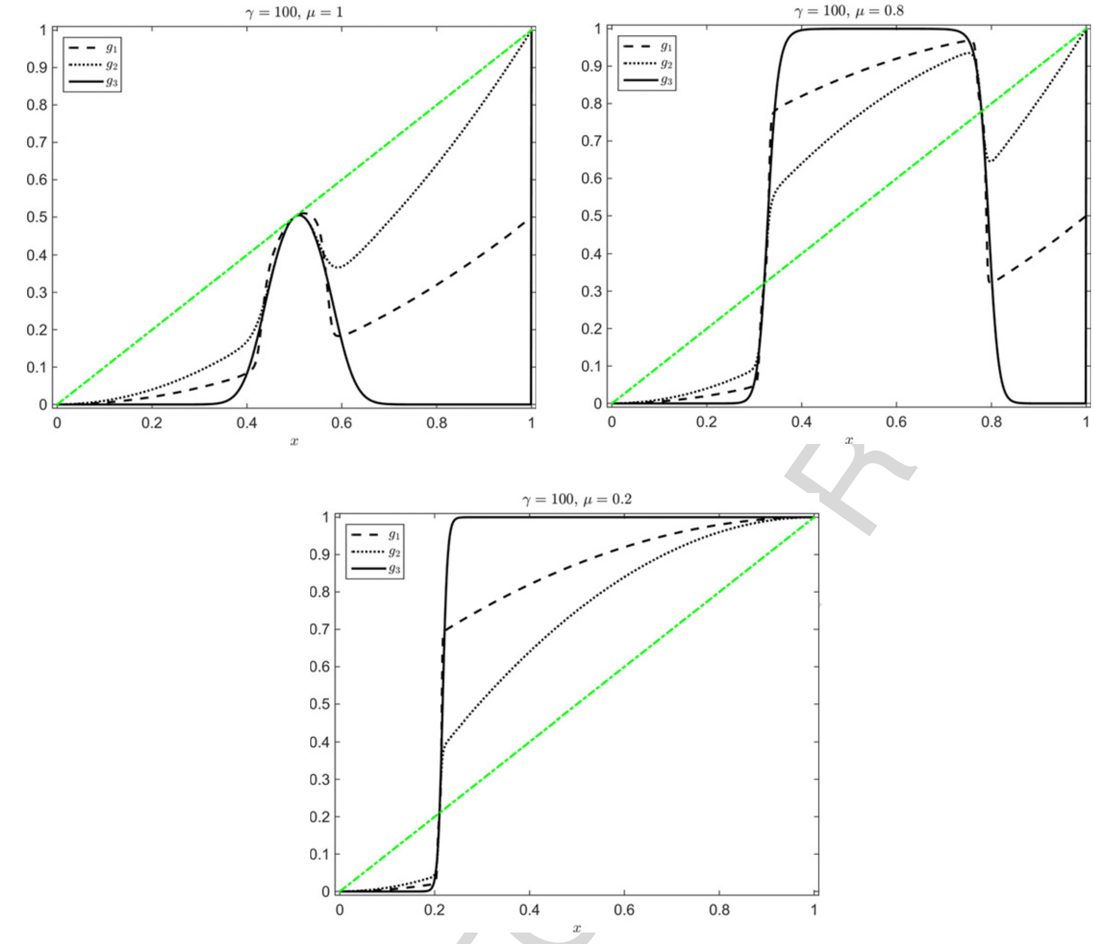

Fig. 12. Top: maps g_1, g_2 and g_3 when the payoff functions are $R(x) = -2x^2 + 3x + 0.5$ and $L(x) = \mu x + 1$, for $\gamma = 100$ and different values of μ.

which intersect at $x^* = 4/9$. Since $R(x) > L(x)$ for $x < x^*$ and $R(x) < L(x)$ for $x > x^*$, the intersection point is the equilibrium, which is conditionally stable, depending on γ and $\partial_y f(x^*, 0)$. In particular, we have that $\chi'(4/9) = -3$ and $\partial_y f_1(4/9, 0) = 10/81$, $\partial_y f_2(4/9, 0) = \partial_y f_3(4/9, 0) = 20/81$, so that the critical values of γ above which stability is lost are $\gamma_1 = 5.4$ and $\gamma_2 = \gamma_3 = 2.7$.

In Fig. 7, we report the graphs of g_1, g_2 and g_3 for two different values of γ and the corresponding bifurcation diagrams obtained with the initial condition $x_0 = 0.1$. As predicted by Proposition 3.2, the destabilization of the internal equilibrium occurs by means of a period doubling bifurcation at the values $\gamma = \gamma_i,\ i \in \{1, 2, 3\}$, just found, but the successive scenarios are quite different for the three maps. For map g_1, the bifurcation diagram shows the coexistence between different attractors. To better understand such phenomenon, for some significant values of γ, in Fig. 8 we report the graph of $g_1^2 - id$, where *id* denotes the identity map. We plot it in place of the second iterate of g_1 in order to make graphically visible the fixed points of g_1^2, which in $g_1^2 - id$ are the intersections with the horizontal axis. For small values of γ, the map g_1^2 has the only stable fixed point $x^* = 4/9$. When γ approaches 2.05, the second iterate has two fold bifurcations, which give rise to two couples of stable–unstable fixed points, so that we have coexistence of stable attractors. When γ reaches and exceeds the critical value 5.4, the second iterate of g_1 has a classical supercritical pitchfork bifurcation, which means a period doubling bifurcation of the original map g_1. However, the two stable fixed points that are just born very soon (at $\gamma \approx 5.408$) merge with the two unstable fixed points of g_1^2 coming from the pitchfork bifurcation, by means of a fold bifurcation. This means that above such value of γ the only stable attractor is the period-2 orbit that arose in the initial fold bifurcations of the second iterate, which, from simulations, seems stable when γ is further increased.

On the other hand, in both the diagrams related to (4.9) and (4.10), a cascade of flip bifurcations leads to chaos.

We now focus on the bifurcation diagram for $g_3(x)$ in Fig. 7 to show some conclusions about existence of chaos in the sense of Li–Yorke. Indeed in Remark 4.1 we illustrate how to apply the well-known result by Li and Yorke [22], using conditions $(T1)$ and $(T2)$ in Theorem 1 therein (from now on, Th1 LY), to the second iterate of the map

$$g_3(x) = \frac{x}{(1-x)e^{-\gamma(-3x+4/3)} + x} = \frac{x}{(1-x)e^{\gamma(3x-4/3)} + x}. \qquad (4.11)$$

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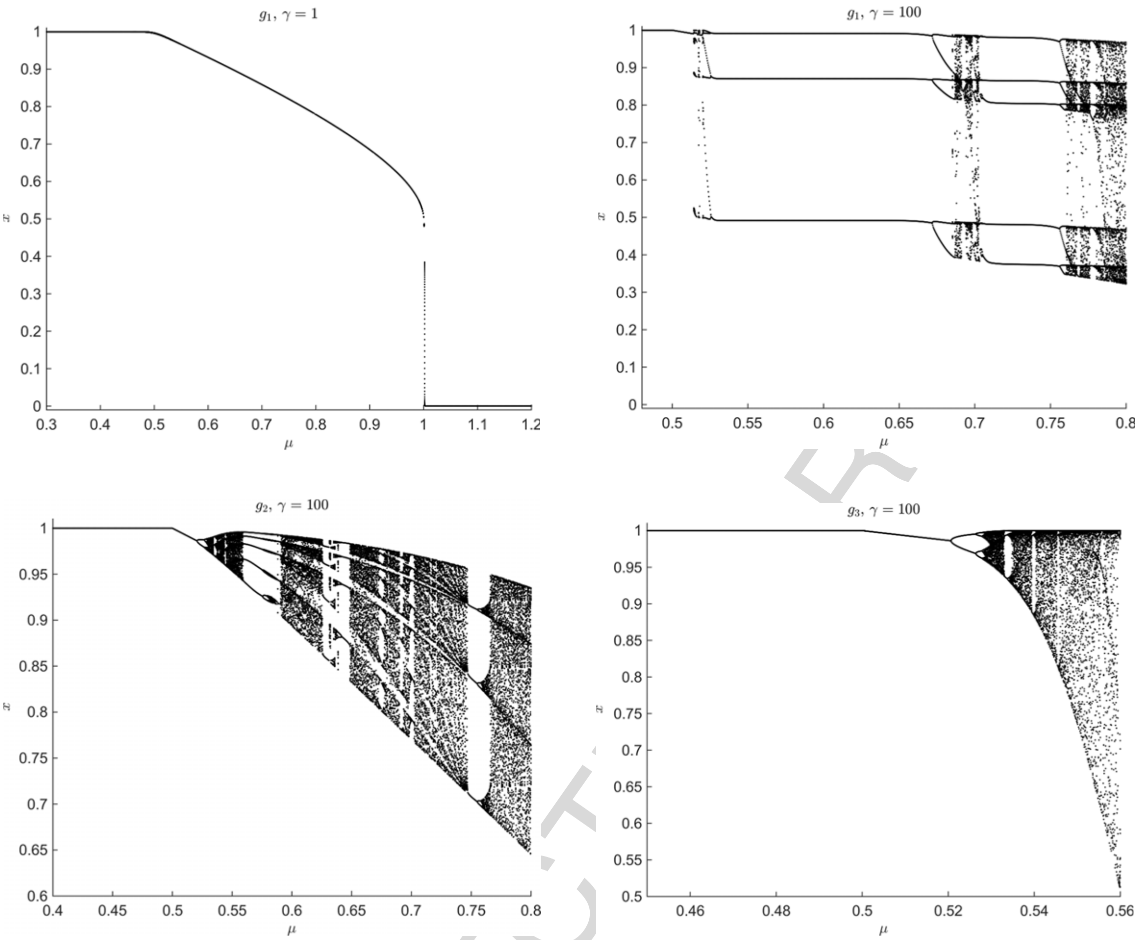

Fig. 13. Bifurcation diagrams of $g_1,\ g_2$ and g_3 with respect to μ for $R(x) = -2x^2 + 3x + 0.5$ and $L(x) = \mu x + 1$, starting from the initial condition $x_0 = 0.6$. For $\gamma = 1$ all the maps have the same bifurcation diagram (top left). For $\gamma = 100$ the bifurcation diagrams are different.

We recall that if the map F in Th1 LY has a period-three orbit, then that theorem applies. Moreover, as observed in Ref. [22], Th1 LY can be generalized to the case in which $F : J \to \mathbb{R}$ is a continuous function that does not map the interval J onto itself.

Remark 4.1. For the map g_3 in (4.11), fix $\gamma = 20$ and set $J = [0.94315, 0.94944]$. Then for any point $x \in J$ it holds that $y = g_3^2(x),\ z = g_3^4(x)$ and $w = g_3^6(x)$ satisfy $w \geq x > y > z$ and thus Conditions $(T1)$ and $(T2)$ in Th1 LY do hold true for g_3^2. In particular,[5] for any $x \in \text{int}(J)$ it holds that $g_3^6(x) > x$, while for $x \in \partial(J)$ it holds that $g_3^6(x) = x$, that is, the extreme points of J are period-three points for g_3^2.

We show that the chain of inequalities $w \geq x > y > z$ is satisfied on J by plotting in Fig. 9 the graphs of the identity map in blue, of g_3^2 in red, of g_3^4 in magenta and of g_3^6 in cyan. A direct inspection of that picture shows that it is possible to apply Th1 LY to g_3^2 on J and thus we immediately get the desired conclusions.

We stress that the computer simulation in Fig. 7 confirms the presence of chaos for g_3, theoretically verified in Remark 4.1 when $\gamma = 20$. Moreover we remark that, although in Remark 4.1 we have fixed a particular parameter value for γ, by continuity, the same result still holds, suitably modifying the interval J, also for small variations of γ. Hence, by Remark 4.1 the existence of Li–Yorke chaos for the map g_3^2 follows when γ lies in a neighborhood of 20. Notice that, from the fact that g_3^2 is Li–Yorke chaotic interesting features can be deduced for the map g_3 as well. For instance, by $(T1)$ in Th1 LY it follows that g_3 has periodic points of any even period.

In the next set of simulations, we show the destabilization due to payoff variations. First, we consider a situation similar to that depicted in Fig. 2, obtained taking $R = (x - 1/2) + 2$ and $L = \mu(x - 1/2) + 2$. The resulting maps $g_1,\ g_2$ and g_3 are similar

[5] Given an interval I, we denote its interior by $\text{int}(I)$ and its boundary by $\partial(I)$.

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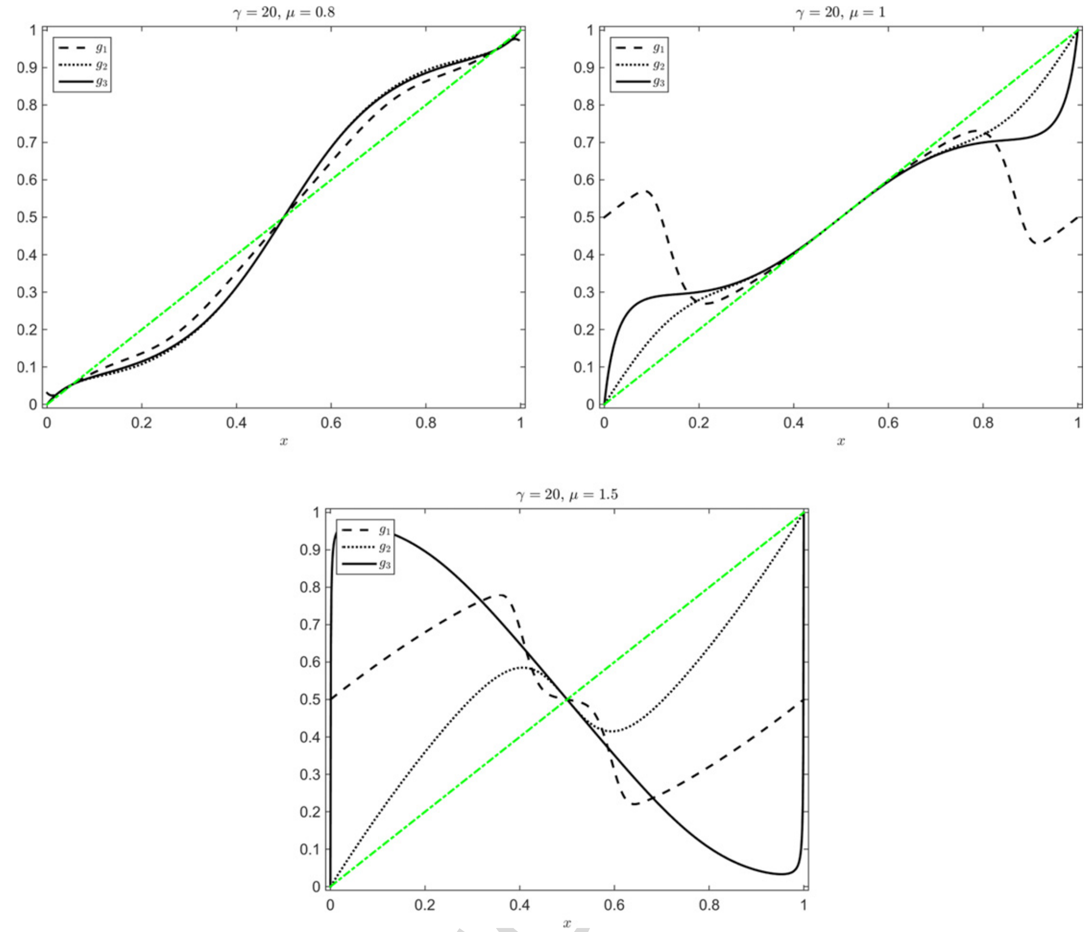

Fig. 14. Maps g_1, g_2 and g_3 when the payoff functions are $R(x) = -(x - 1/2)^3 + (x - 1/2) + 2$ and $L(x) = \mu(x - 1/2) + 2$, for $\gamma = 20$ and different values of μ.

to those reported in Fig. 7. The period doubling occurs when $\mu = (\gamma + 16)/\gamma$ for g_1 and when $\mu = (\gamma + 8)/\gamma$ for g_2 and g_3. The bifurcation diagrams for $\gamma = 1$ are reported in Fig. 10. For g_1, we have a situation similar to that of the flip bifurcation due to γ, with an initial destabilization through a period-2 cycle quickly replaced by a different coexisting period-2 orbit. The bifurcation diagram for g_3 suggests that, when the equilibrium loses its stability, a period-2 cycle emerges, which remains stable also when increasing μ. Conversely, for g_2 the behavior is quite different.

Each point of the initial period-2 cycle has for $\mu \approx 20$ a pitchfork bifurcation, and it is replaced by a new point which is initially stable for the second iterate. Each of such points then loses stability via a flip bifurcation, and, increasing μ further, the chaotic regime emerges. This is evident looking at the plot of the second iterate of g_2 reported in Fig. 11 for some values of μ.

Now we focus on the occurrence of a fold bifurcation considering a situation similar to that in Fig. 4. To this end, we take $R(x) = -2x^2 + 3x + 0.5$ and $L(x) = \mu x + 1$, obtained by setting $x_0 = 1$, $\mu^{fo} = 1$, $x^* = 1/2$ and $\hat{R} = 2$ in Corollary 3.3, and we consider $\gamma = 1$ and $\gamma = 100$. In Fig. 12 we report the graph of the corresponding maps for $\gamma = 100$ and three different values of μ. We remark that when there is more than a single equilibrium, in particular when the payoff functions have multiple intersections, the basin of attraction of each stable equilibrium can consist of the union of several disjoint intervals.

In Fig. 13, we show the bifurcation diagrams for g_1, g_2 and g_3 with respect to μ, starting from $x_0 = 0.6$, for $\gamma = 1$ and $\gamma = 100$. In the former case, the diagram is the same for both g_1, g_2 and g_3 and we observe that if μ is greater than 1, the only equilibrium is $x^* = 0$, as choice L is dominant. When $\mu = 1$, the new equilibrium $x^* = 1/2$ arises, coexisting with $x^* = 0$. When $1/2 \leq \mu < 1$, payoff functions intersect in two distinct points internal to $[0, 1]$, corresponding to an unstable and a stable equilibria. Finally, when $\mu < 1/2$, the intersection corresponding to the stable equilibrium is outside the interval $[0, 1]$, and $x^* = 0, 1$ are the stable equilibria. When γ increases, the situation may change, as the internal steady state can become unstable. This is indeed what happens in Fig. 13 when we take $\gamma = 100$. More precisely, for $\gamma = 100$ maps g_2 and g_3 display a flip bifurcation, while the dynamics of g_1 are more complicated and are characterized by the presence of coexisting attractors. We do not investigate this case further.

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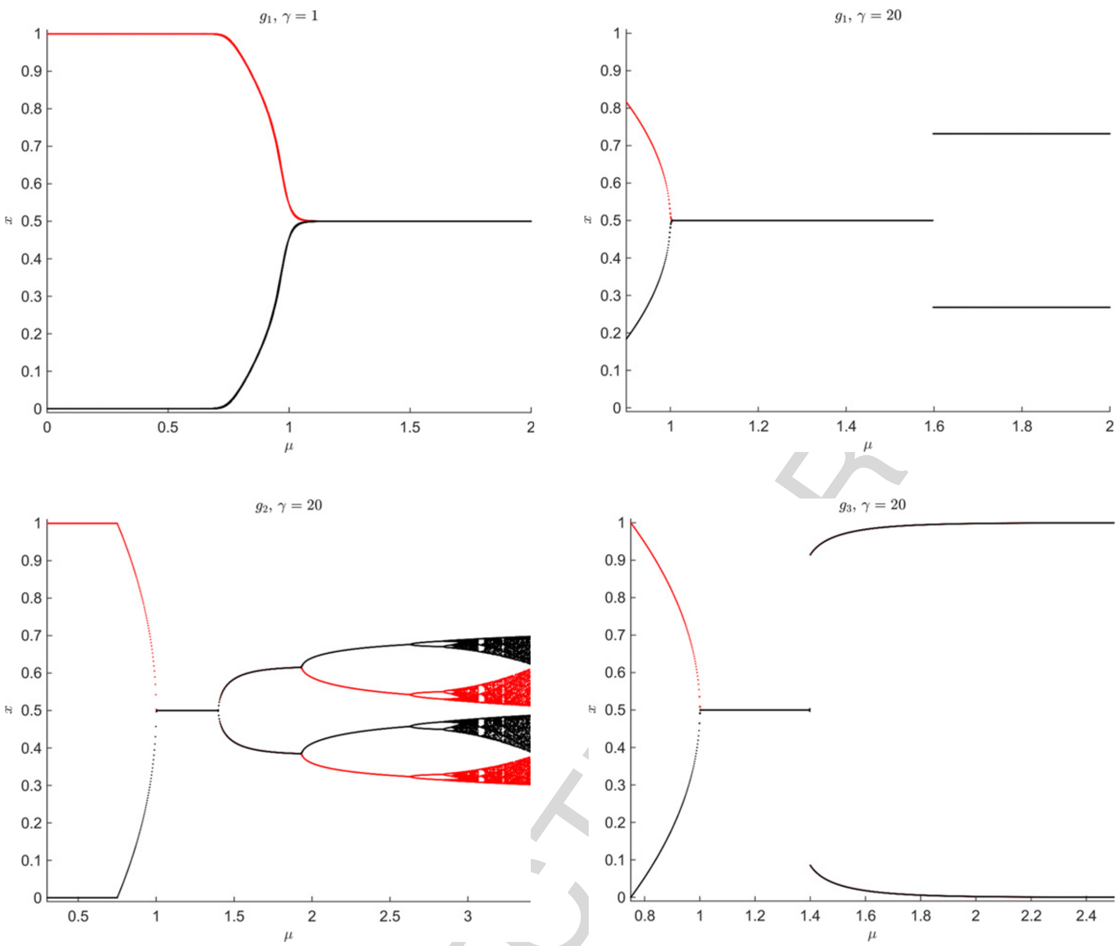

Fig. 15. Bifurcation diagrams of g_1, g_2 and g_3 with respect to μ, for $R(x) = -(x-1/2)^3 + (x-1/2) + 2$ and $L(x) = \mu(x-1/2) + 2$. For $\gamma = 1$ the bifurcation diagram, starting from the initial condition $x_0 = 0.45$, is the same for all the maps (top left). For $\gamma = 20$ the bifurcation diagrams are different. In particular, we depict in black the diagram obtained for the initial condition $x_0 = 0.45$, while in red that for $x_0 = 0.55$. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.) .

The last example we consider concerns the pitchfork bifurcation and refers to the framework shown in Fig. 3. The expressions of the payoff functions are $R(x) = -(x-1/2)^3 + (x-1/2) + 2$ and $L(x) = \mu(x-1/2) + 2$, which correspond to the choice of $x_0 = 2$, $x^* = 1/2$, $\mu^p = 1$, $\hat{R} = -1$ in Corollary 3.4. In Fig. 14, we report the corresponding maps for $\gamma = 20$ and three different values of μ. In Fig. 15, we show the bifurcation diagrams for g_1, g_2 and g_3 with respect to μ, starting from the initial condition $x_0 = 0.45$ (in black) and from $x_0 = 0.55$ (in red), for $\gamma = 1$ and $\gamma = 20$. In the former case the diagram is the same for g_1, g_2 and g_3 and we observe that if μ is larger than 1, both $x = 0$, $1/2$ and 1 are stable equilibria. Decreasing μ below 1, the internal equilibrium loses its stability and two stable equilibria emerge, that are stable for $\mu \in (3/4, 1)$, while at $\mu = 3/4$ they contemporaneously leave $[0, 1]$. For large values of γ, also this new stable equilibria can become unstable. This is indeed what happens to g_2 in Fig. 15 when we take $\gamma = 20$. We remark that, instead, for increasing values of μ, we can identify behaviors that are similar to those described for the previously considered bifurcations.

5. Conclusions

In the present paper we introduced a family of models to describe Schelling binary choices in a dynamic setting. We analyzed the general model, studying the possible steady states and their connection with the equilibria in Ref. [4]. We then studied several cases examined in Ref. [4], in terms of the local stability properties of the models. We showed that both reaction speed and payoff functions can lead to instability, and that the ways the equilibria lose their stability can be linked to the scenarios qualitatively described by Schelling. Finally, we presented three particular models and we provided numerical confirmation of the analytical results, together with an investigation of the possible global behaviors of the model.

Future research should focus on extending the model in order to take into account more complex frameworks, considering not only purely binary choices, introducing possible differences in the influence of the various agents and studying the effect of their spatial distribution, in view of analyzing both global and local interactions.

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