PRESENTATIONS FOR THE HIGHER DIMENSIONAL THOMPSON GROUPS nV

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Abstract. In his papers [2], [3] Brin introduced the higher dimensional Thompson groups nV which are generalizations to the Thompson group V of self-homeomorphisms of the Cantor set and found a finite set of generators and relations in the case $n = 2$. We show how to generalize his construction to obtain a finite presentation for every positive integer n. As a corollary, we obtain another proof that the groups nV are simple (first proved by Brin in [4]).

1. Introduction

The higher dimensional groups nV were introduced by Brin in his papers [2] and [3] and generalize Thompson's group V. We recall that the group V is a group of self-homeomorphisms of the Cantor set $\mathfrak{C}$ that is simple and finitely presented (the standard introduction to V is the paper by Cannon, Floyd and Parry [5]). The groups nV generalize the group V and act on powers of the Cantor set $\mathfrak{C}^n$. Brin shows in [2] that the groups V and $2V$ are not isomorphic and shows in [3] that the group $2V$ is finitely presented. Bleak and Lanoue [1] have recently showed that two groups mV and nV are isomorphic if and only if $m = n$.

In this paper we give a finite presentation for each of the higher dimensional Thompson groups nV. The argument extends to the ascending union ωV of the groups nV and returns an infinite presentation of the same flavor. As a corollary, we obtain another proof that the groups nV and ωV are simple. Our arguments follow closely and generalize those of Brin in [2]. [3] for the group $2V$.

This work arose during a Research Experience for Undergraduates (REU) program at Cornell University. The motivation for the project sprang from a commonly held opinion that the book-keeping required to generalize Brin's presentations to the groups nV would be overwhelming. One would expect from the similarity of the groups' constructions that all arguments for $2V$ would carry over to nV for all n. Standing in the way of this are the cross relations. Thus our paper has two kinds of arguments: those that verify the parts of [3] that carry over with no change to nV and those involving the cross relations that have to be modified to hold in nV (see Lemmas 6 and 20 and Remark 13 below).

Following a suggestion of Collin Bleak the authors have also explored an alternative generating set (see Section 8). An interesting project would be to find a set of relators for this alternative generating set in order to use a known procedure which

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significantly reduces the number of relations, and which has been successfully implemented in a number of papers by Guralnick, Kantor, Kassabov, Lubotzky (see for example [6]).

After a careful reading of Brin's original paper [3], it became clear what was needed to generalize his proof, and the current paper borrows heavily from Brin's. Brin was already aware that many of his arguments would probably extend (and he points out in several places in [2], [3] where it is evident that they do). We demonstrate how to deal with generators in higher dimensions and what steps are needed to obtain the same type of normalized words which are built for $2V$ in [3].

We also mention that Brin asks in [3] whether or not the group $2V$ has type F_∞ (that is, having a classifying space that is finite in each dimension). This has recently been answered by Kochloukova, Martinez-Perez and Nucinkis [7] who have shown that the groups $2V$ and $3V$ have type F_∞, therefore obtaining a new proof that these groups are finitely presented.

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2. The main ingredient and structure of this paper

Many arguments of Brin generalize word-by-word from $2V$ to nV. For this reason, we advise the reader to have a copy of Brin's papers [2], [3], as we will adapt some of their results and our results will be stated to appear as natural generalizations of those, including the general argument to show that what we will find is indeed a presentation.

The key observation which allows us to restate many results without proofs (or with little additional effort) is the following: many statements of Brin do not depend on dimension 2, except those which need to make use of the "cross relation" (relation (18) in Section 4 below) to rewrite a cut in dimension d followed by a cut in dimension d' as one in dimension d' followed by one in dimension d.

As a result, proofs which need to make use of this new relation require a slight generalization (for example, the normalization of words in the monoid across fully divided dimensions) while those which do not can be obtained directly using Brin's original proof. In any case, since statements need to be adapted to our context we sketch certain proofs to make it clear that they generalize directly. For example, we will show why Brin's proof that $2V$ is simple does not use the new relation (18) and therefore it lifts immediately to higher dimensions.

3. The monoid Π_n

In [2] section 4.5, Brin defines the monoid Π and $\widehat{nV}$ and observes that one can extend the definition for all n. Elements of Π_n are given by numbered patterns in X, where X is the union of the set $\{S_0, S_1, ...\}$ of unit n-cubes. Fix $n \in \mathbb{N}$ and fix an ordering on the dimensions d, $1 \leq d \leq n$. The monoid Π_n is generated by the elements $s_{i,d}$ and σ_i, and $s_{i,d}$ denotes the element which cuts the rectangle S_i in half across the d-th dimension (see figure 1) and σ_i is the transposition which

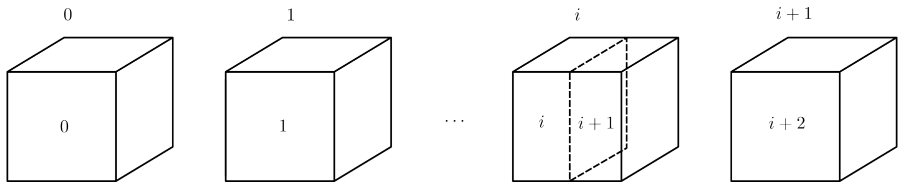

FIGURE 1. The generator $s_{i,d}$.

switches the rectangle labelled i with that labelled $i+1$, as defined for $2V$ (see figure 2).

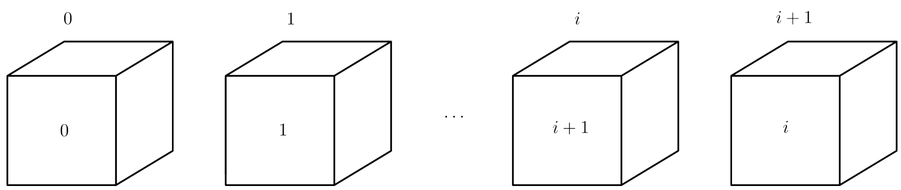

FIGURE 2. The generator σ_i.

After each cut, the numbering shifts as before. The following relations hold in Π_n.

(M1)	$s_{j,d'}s_{i,d} = s_{i,d}s_{j+1,d'}$	$i < j, 1 \leq d, d' \leq n$
(M2)	${\sigma_i}^2 = 1$	$i \geq 0$
(M3)	$\sigma_i\sigma_j = \sigma_j\sigma_i$	$\lvert i-j \rvert \geq 2$
(M4)	$\sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}$	$i \geq 0$
(M5a)	$\sigma_j s_{i,d} = s_{i,d}\sigma_{j+1}$	$i < j$
(M5b)	$\sigma_j s_{i,d} = s_{j+1,d}\sigma_j\sigma_{j+1}$	$i = j$
(M5c)	$\sigma_j s_{i,d} = s_{j,d}\sigma_{j+1}\sigma_j$	$i = j+1$
(M5d)	$\sigma_j s_{i,d} = s_{i,d}\sigma_j$	$i > j+1$
(M6)	$s_{i,d}s_{i+1,d'}s_{i,d'} = s_{i,d'}s_{i+1,d}s_{i,d}\sigma_{i+1}$	$i \geq 0, d \neq d'$

Note: Relations (M5b) and (M5c) are actually equivalent, using the fact that σ_i is its own inverse.

Remark 1. We observe that the proofs of results of Section 2 in [3] which use relations (M1) – (M5) do not depend on the fact that we are in dimension 2, except for the way they are formulated. For this reason, they generalize immediately to the case of the monoid Π_n and we do not reprove them. This includes every result up to and including Lemma 2.9 in [3].

On the other hand, Proposition 2.11 in [3] uses the cross relation (M6) and it requires us to make a choice on how we write elements to obtain some underlying pattern. Brin achieves this type of normalization by writing elements so that vertical cuts appear first, whenever possible. We generalize his argument by describing how to order nodes in forests (which represent cuts in some dimension).

The following definition is given inductively on the subtrees.

Definition 2. Given a forest F we say that a subtree T of some tree of F is *fully divided* across some dimension d if the root of T is labelled d or if both her left and right subtrees are fully divided across dimension d. A forest F is *normalized* if every subtree T satisfies the following condition: if T is fully divided across different the dimensions $d_1 < d_2 < \ldots < d_u$, then the root of T is labelled with d_1, the lowest among all possible dimensions over which T is fully divided.

Given a word w be the word in the generators $\{s_{i,d}, \sigma_i\}$, we define the *length* $\ell(w)$ of w to be the number of appearances in w of elements of $\{s_{i,d}\}$. It can easily be seen that the length of a word is preserved by relations (M1) – (M6).

We restate without proofs Lemmas 2.7, 2.8 and 2.9 from Brin [3] adapted to our case.

Lemma 3 (Brin, [3])**.** *If the numbered, labeled forest F comes from a word in $\{s_{i,d} \mid d, i \in \mathbb{N}\}$, then the leaves of F are numbered so that the leaves in F_i have numbers lower than those in F_j whenever $i < j$ and the leaves in each tree of F are numbered in increasing order under the natural left right ordering of the leaves.*

Lemma 4 (Brin, [3])**.** *If two words in the generators $\{s_{i,d}, \sigma_i \mid i \in \mathbb{N}, 1 \leq d \leq n\}$ lead to the same numbered, labeled forest, then the words are related by* (M1)–(M5)*.*

Lemma 5 (Brin, [3])**.** *If F is a numbered, labeled forest with the numbering as in Lemma 3, and if a linear order is given on the interior vertices (and thus of the carets) of F that respects the ancestor relation, then there is a unique word w in $\{s_{i,d} \mid d, i \in \mathbb{N}\}$ leading to F so that the order on the interior vertices of F derived from the order on the entries in w is identical to the given linear order on the interior vertices.*

The next lemma and corollary are used to prove results analogous to Lemma 2.10 and Proposition 2.11 from [3].

Lemma 6. *Let w be a word in the set $\{s_{i,d}, \sigma_i\}$ and suppose that the underlying pattern P has a fully divided hypercube S_i across dimension d. Then $w \sim w' = s_{i,d}a$ for some word $a \in \langle s_{i,d}, \sigma_i \rangle$.*

Proof. We use induction on $g := \ell(w)$. By using relations (M5a)–(M5d) as in Lemma 2.3 of [3] we can assume that $w = pq$ where $p \in \langle s_{i,d} \rangle$ and $q \in \langle \sigma_i \rangle$. This does not alter the length of w. If $g = 3$, then $p = p_1 p_2 p_3$. If $p_1 = s_{i,d}$ we are done, otherwise we have two cases: either $p_2 = s_{i+1,d}$ and $p_3 = s_{i,d}$ or $p_2 = s_{i,d}$ and $p_3 = s_{i+2,d}$. Up to using relation (M1), we can assume that $p_2 = s_{i+1,d}$ and $p_3 = s_{i,d}$ which is what to want to apply relation (M6) to p to get $w \sim w' = s_{i,d} s_{i+1,k} s_{i,k} q$.

Now assume the thesis true for all words of length less than g. We consider the word p and look at the labelled unnumbered tree F_i corresponding to S_i with root vertex u and children u_0 and u_1. Let T_r be the subtree of F_i with root vertex u_r, for $r = 0, 1$. We choose an ordering of the vertices of F_i which respects the ancestor relation and such that u corresponds to 1, u_0 corresponds to 2, the other interior nodes of T_0 correspond to the numbers from 3 to $j = \#(\text{interior nodes of } T_0)$ and u_2 corresponds to $j + 1$.

By Lemma 5, the word p is equivalent to

$$p \sim s_{i,k}(s_{i,m}p_0)(s_{f,l}p_1)$$

where $s_{i,m}p_0$ is the subword corresponding to the subtree T_0 and $s_{f,l}p_1$ is the subword corresponding to the subtree T_1 and with $p_0, p_1 \in \langle s_{i,d}\rangle$. We observe that $\ell(s_{i,m}p_0) < \ell(p) = g$ and $\ell(s_{f,l}\, p_1) < \ell(p) = g$ and that the underlying squares S_i for $s_{i,m}p_0$ and S_{i+1} for $s_{f,l}\, p_1$ are fully divided across dimension d. We can thus apply the induction hypothesis and rewrite

$$s_{i,m}p_0 \sim s_{i,d}\widetilde{p}_0\widetilde{q}_0 \quad \text{and} \quad s_{f,l}\, p_2 \sim s_{f,d}\widetilde{p}_1\widetilde{q}_1.$$

We restrict our attention to the subword $s_{i,d}\widetilde{p}_0\widetilde{q}_0 s_{f,d}$. Using the relations (M5a)–(M5d) we can move $\widetilde{q}_0$ to the right of $s_{f,d}$ and obtain

$$s_{i,d}\widetilde{p}_0\widetilde{q}_0 s_{f,d} \sim s_{i,d}\widetilde{p}_0 s_{g,d}\widetilde{q}$$

for some permutation word $\widetilde{q}$. Since the word $\widetilde{p}_0$ acts on the rectangle S_i and $s_{g,d}$ acts on the rectangle S_{i+1} we can apply Lemma 4 and 5 and put a new order on the nodes so that the node corresponding to $s_{i,d}$ is 1 and $s_{g,d}$ is 2. Thus we have that

$$s_{i,d}\widetilde{p}_0 s_{g,d}\widetilde{q} \sim s_{i,d}s_{i+2,d}\widetilde{p}\,\widetilde{q}$$

for some $\widetilde{p}$ word in the set $\{s_{i,d}\}$. Thus we have $w \sim w'' = s_{i,k}s_{i,d}s_{i+2,d}\widetilde{p}\,\widetilde{q}$ and so, by applying the cross relation (M6) to the first three letters of w'' we get

$$w \sim w'' \sim w' = s_{i,d}s_{i,k}s_{i+2,k}\widetilde{p}\,\widetilde{q} = s_{i,d}a$$

□

We have now proved Lemma 2.10 from [3], since in order for a tree in a forest to be non-normalized, one of the rectangles in the pattern corresponding to that tree must be fully divided across two different dimensions.

Lemma 7 (Brin, [3])**.** *If two different forests correspond to the same pattern in X, then at least one of the two forests is not normalized.*

Remark 8. Lemma 6 is used in our extension of Brin 2 Proposition 2.11 so that we can push dimension d under the root. This is explained better in the following Corollary.

Corollary 9. *Let w be a word in the generators $\{s_{i,d}, \sigma_i\}$ such that its underlying square S_i is fully divided across dimensions d and ℓ. Then*

$$w \sim w' = s_{i,d}s_{i,\ell}s_{i+2,\ell}a \sim w'' = s_{i,\ell}s_{i,d}s_{i+2,d}b$$

for some suitable words a, b in the generators $\{s_{i,d}, \sigma_i\}$.

Proof. This is achieved by a repeated application of the previous Lemma 6. We apply Lemma 6 to w and obtain $w \sim s_{i,d}a_1$. By construction, we notice that the underlying squares S_i and S_{i+1} of a_1 are fully divided across dimension ℓ, so we can apply the previous Lemma to a_1 to get $a_1 \sim s_{i,\ell}a_2$ and finally we apply it again to $a_2 \sim s_{i+2,\ell}a$. Hence $w \sim w' = s_{i,\ell}s_{i+2,\ell}a$. To get w'' we apply the cross relation (M6) to the subword $s_{i,\ell}s_{i,d}s_{i+2,d}$. □

Proposition 10. *A word w is related by* (M1) *through* (M6) *to a word corresponding to a normalized, labelled forest.*

Proof. We proceed by induction on the length of w. Let g be the length of w and assume the result holds for all words of length less than g. As before, write $w = pq$, where $p = s_{i_0} s_{i_1} \dots s_{i_{n-1}}$ (here, the i_j refers to the cube which is being cut; we omit the second index indicating dimension as it is unimportant for now). Write $w = s_{i_0} w'$; since the order of the interior vertices of the forest for p given by the order of the letters in p must respect the ancestor relation, we know that the interior vertex corresponding to s_{i_0} must be a root of some tree, T. As w' is a word of length less than g, we may apply our inductive hypothesis and assume that w' can be rewritten via relations (M1) through (M6) to obtain a corresponding normalized forest. The pattern P for w is obtained from the pattern P' for w' by applying the pattern of P' in unit square S_i to the rectangle numbered i in the pattern for s_{i_0}. The forest F for w is obtained from the forest F' for w' by attaching the i-th tree of F' to the i-th leaf of the forest for s_{i_0}. Since F' is normalized, it is seen that F has all interior vertices normalized except possibly for the root vertex of one tree, T.

Let u be the root vertex of T with label k and with children u_1 and u_2. Let T_1 and T_2 be the subtrees of T whose roots are u_1 and u_2, respectively. By hypothesis, T_1 and T_2 are already normalized. If T is not normalized already, then T must be fully divided across the dimension that u is labeled with, k, and some other dimension less than k. Let d be the minimal dimension across which T is fully divided. Since T_1 and T_2 are also fully divided across d, by Lemma 6, we may apply relations (M1) through (M6) to the subwords of w corresponding to T_1, T_2 until u_1 and u_2 are each labelled d. Now by lemma 2.9, we may assume $w = s_{i_0,k} s_{i_0,d} s_{i_0+2,d} w''$ where w'' is the remainder of w. We apply relation (M6) to obtain $w = s_{i_0,d} s_{i_0,k} s_{i_0+2,k} \sigma_{i_0} w''$. Now, we have normalized the vertex u, and we may now use the inductive hypothesis to renormalize the trees T_1 and T_2. The result is a normalized forest. □

The proof of the following result follows the same argument of Theorem 1 in [3], using Lemma 2.10 in [3] and Proposition 10 (to extend Proposition 2.11 in [3]).

Theorem 11. *The monoid* Π_n *is presented by using the generators* $\{s_{i,d}, \sigma_i\}$ *and relations* (M1)–(M6).

4. Relations in nV

4.1. **Generators for nV.** The following generators are defined as in [2] and analogous arguments show why they are a generating set for nV.

$$X_{i,d} = (s_{0,1}^{i+1} s_{1,d}, s_{0,1}^{i+2}) \quad i \geq 0, 1 \leq d \leq n$$

$$C_{i,d} = (s_{0,1}^{i} s_{0,d}, s_{0,1}^{i+1}) \quad i \geq 0, 2 \leq d \leq n \qquad \text{(baker's maps)}$$

$$\pi_i = (s_{0,1}^{i+2} \sigma_1, s_{0,1}^{i+2}) \qquad i \geq 0 \qquad (\sigma_i \text{ defined as above})$$

$$\overline{\pi}_i = (s_{0,1}^{i+1} \sigma_0, s_{0,1}^{i+1}) \qquad i \geq 0$$

4.2. **Relations involving cuts and permutations.** In all the following relations (1) – (7) the reader can assume that $1 \leq d, d' \leq n$, unless otherwise stated.

$$
\begin{array}{llr}
(1) & X_{q,d}X_{m,d'} = X_{m,d'}X_{q+1,d} & m < q, \\
(2) & \pi_q X_{m,d} = X_{m,d}\pi_{q+1} & m < q \\
(3) & \pi_q X_{q,d} = X_{q+1,d}\pi_q\pi_{q+1} & q \geq 0 \\
(4) & \pi_q X_{m,d} = X_{m,d}\pi_q & m > q+1 \\
(5) & \overline{\pi}_q X_{m,d} = X_{m.d}\overline{\pi}_{q+1} & m < q \\
(6) & \overline{\pi}_m X_{m,1} = \pi_m\overline{\pi}_{m+1} & m \geq 0 \\
(7) & X_{m,d}X_{m+1,d'}X_{m,d'} = X_{m,d'}X_{m+1,d}X_{m,d}\pi_{m+1} & m \geq 0, d \neq d'
\end{array}
$$

4.3. **Relations involving permutations only.**

$$
\begin{array}{llr}
(8) & \pi_q\pi_m = \pi_m\pi_q & |m-q| > 2 \\
(9) & \pi_m\pi_{m+1}\pi_m = \pi_{m+1}\pi_m\pi_{m+1} & m \geq 0 \\
(10) & \overline{\pi}_q\pi_m = \pi_m\overline{\pi}_q & q \geq m+2 \\
(11) & \pi_m\overline{\pi}_{m+1}\pi_m = \overline{\pi}_{m+1}\pi_m\overline{\pi}_{m+1} & m \geq 0 \\
(12) & \pi_m^2 = 1 & m \geq 0 \\
(13) & \overline{\pi}_m^2 = 1 & m \geq 0
\end{array}
$$

4.4. **Relations involving baker's maps.** In all the following relations (14) – (18) the reader can assume that $2 \leq d \leq n$ and $1 \leq d' \leq n$, unless otherwise stated.

$$
\begin{array}{llr}
(14) & \overline{\pi}_m X_{m,d} = C_{m+1,d}\pi_m\overline{\pi}_{m+1} & m \geq 0, \\
(15) & C_{q,d}X_{m,d'} = X_{m,d'}C_{q+1,d} & m < q, \\
(16) & C_{m,d}X_{m,1} = X_{m,d}C_{m+2,d}\pi_{m+1} & m \geq 0, \\
(17) & \pi_q C_{m,d} = C_{m,d}\pi_q & m > q+1 \\
(18) & C_{m,d}X_{m,d'}C_{m+2,d'} = C_{m,d'}X_{m,d}C_{m+2,d}\pi_{m+1} & m \geq 0, 1 < d' < d \leq n
\end{array}
$$

Relations (1) through (17) are generalizations of those given in [2] and their proofs are completely analogous. The only new family of relations is (18) which we prove using relation (M6) from the monoid:

Proof.

$$
\begin{aligned}
C_{m,d}X_{m,d'}C_{m+2,d'} &= (s_{0,1}^m s_{0,d},\, s_{0,1}^{m+1})(s_{0,1}^{m+1}s_{1,d'},\, s_{0,1}^{m+2})(s_{0,1}^{m+2}s_{0,d'},\, s_{0,1}^{m+3}) \\
&= (s_{0,1}^m s_{0,d}s_{1,d'}s_{0,d'},\, s_{0,1}^{m+3}) \\
&= (s_{01}^m s_{0,d'}s_{1,d}s_{0,d}\sigma_1,\, s_{0,1}^{m+3}) \\
&= (s_{0,1}^m s_{0,d'},\, s_{0,1}^{m+1})(s_{0,1}^{m+1}s_{1,d},\, s_{0,1}^{m+2})(s_{0,1}^{m+2}s_{0,d},\, s_{0,1}^{m+3})(s_{0,1}^{m+3}\sigma_1,\, s_{0,1}^{m+3}) \\
&= C_{m,d'}X_{m,d}C_{m+2,d}\pi_{m+1}.
\end{aligned}
$$

□

Lemma 12 (Subscript Raising Formulas)**.** *We have that*

$$C_{r,d} \sim C_{r+1,d} X_{r,d} \pi_{r+1} X_{r,1}^{-1}$$

$$\overline{\pi}_r \sim \pi_r \overline{\pi}_{r+1} X_{r,1}^{-1} \sim X_{r,1} \overline{\pi}_{r+1} \pi_r$$

We observe that the first formula of Lemma 12 follows from relations (15) and (16), while the second is a generalization of the one found in [3].

4.5. **Secondary Relations for nV.**

$$X_{q,d}^{-1} X_{r,d} \sim \begin{cases} X_d X_d^{-1} & r \neq q \\ 1 & r = q \end{cases} \qquad (1 \leq d \leq n)$$

$$X_{q,d}^{-1} X_{r,d'} \sim \begin{cases} X_{d'} X_d^{-1} & r \neq q \\ w(X_{d'}) \pi w(X_d^{-1}) & r = q \end{cases} \qquad (1 \leq d, d' \leq n, d \neq d')$$

$$C_{q,d}^{-1} X_{r,d'} \sim \begin{cases} X_{d'} C_d^{-1} & r < q \\ w(X_1, \pi, X_d^{-1}) X_{d'} C_d^{-1} & r \geq q \end{cases} \qquad (2 \leq d \leq n, 1 \leq d' \leq n)$$

$$X_{r,d'}^{-1} C_{q,d} \sim \begin{cases} C_d X_{d'}^{-1} & r < q \\ C_d X_{d'}^{-1} w(X_d, \pi, X_1^{-1}) & r \geq q \end{cases} \qquad (2 \leq d \leq n, 1 \leq d' \leq n)$$

$$\pi_q X_{r,d} \sim \begin{cases} X_d w(\pi) \end{cases} \qquad (1 \leq d \leq n)$$

$$\overline{\pi}_q X_{r,1} \sim \begin{cases} X_1 \overline{\pi} & r < q \\ \pi \overline{\pi} & r = q \\ w(X_1) \overline{\pi} w(\pi) & r > q \end{cases}$$

$$\overline{\pi}_q X_{r,d} \sim \begin{cases} X_d \overline{\pi} & r < q \\ C_d \pi \overline{\pi} & r = q \\ w(X_1) X_d \overline{\pi} w(\pi) & r > q \end{cases} \qquad (2 \leq d \leq n)$$

$$\pi_q C_{r,d} \sim \begin{cases} C_d \pi & r > q+1 \\ C_d w(X_1^{-1}, \pi, X_d) & r \leq q+1 \end{cases} \qquad (2 \leq d \leq n)$$

$$\overline{\pi}_q C_{r,d} \sim \begin{cases} X_d \overline{\pi} \pi & r = q+1 \\ w(X_1) X_d \overline{\pi} w(\pi) & r > q+1 \\ w(X_d) C_d \pi \overline{\pi} w(\pi, X_1^{-1}) & r < q+1 \end{cases} \qquad (2 \leq d \leq n)$$

$$C_{q,d}^{-1} C_{r,d} \sim \begin{cases} w(X_1^{-1}, \pi, X_d) & q < r \\ 1 & q = r \\ w(X_1, \pi, X_d^{-1}) & q > r \end{cases} \qquad (2 \leq d \leq n)$$

$$C_{q,d}^{-1} C_{r,d'} \sim \begin{cases} X_{d'} C_{d'} \pi C_d^{-1} X_d^{-1} w(X_{d'}, \pi, X_1^{-1}) & q > r \\ X_{d'} C_{d'} \pi C_d^{-1} X_d^{-1} & q = r \\ w(X_1, \pi, X_{d'}^{-1}) X_d C_d \pi C_{d'}^{-1} X_{d'}^{-1} & q < r \end{cases} \qquad (1 \leq d' < d \leq n)$$

Proof. We only prove the last set of secondary relations as it is the only one that does not immediately descend from the computations in Brin [3]. If $q > r$ we can apply the subscript raising formulas repeatedly for j times until $r + j = q$ and rewrite the product as

$$C_{q,d}^{-1}C_{r,d'} \sim C_{q,d}^{-1}C_{r+1,d'}X_{r,d'}\pi_{r+1}X_{r,1}^{-1} \sim \ldots \sim C_{q,d'}^{-1}C_{r+j,d'}w(X_{d'},\pi,X_1^{-1}).$$

We argue similarly if $q < r$. We now have to study the product $C_{q,d}^{-1}C_{q,d'}$. Without loss of generality we assume $d' < d$ and apply relation (18):

$$C_{q,d}^{-1}C_{q,d'} = X_{q,d'}C_{q+2,d'}\pi_{q+1}C_{q+2,d}^{-1}X_{q,d}^{-1},$$

which is what was claimed. Similar relations can be derived if $d' > d$. □

Remark 13. When using the the last two secondary relations, we alter a word in a way that does not increase the number of C's. This allows us to generalize the proof of Lemma 4.6 in Brin [3] thus rewriting a word of type $w(X, C, \pi, C^{-1}, X^{-1})$ in LMR form so that the number of C's does not increase (see Lemma 15 below). This observation lets us generalize Lemma 4.7 in Brin [3] (see Lemma 16 below). In fact, all our secondary relations are immediate generalizations of those in Brin [3] and the last one does not introduce appearances of $\overline{\pi}$ and therefore all the letters in the last secondary relations can be migrated to their needed position by means of the previous secondary relations, without altering the original argument of Lemma 4.7 in Brin [3]. Therefore even in the case of nV one is able to do the book-keeping without risk of creating extra letters which cannot be passed safely without recreating them, and hence we obtain an argument which terminates.

5. Presentations for nV

We now show how the relations above enable us to put our group elements into a normal form, starting with words in the generators of nV corresponding to elements from $\widehat{nV}$.

Lemma 14. *Let w be a word in $\{X_{i,d}, \pi_i, X_{i,d}^{-1} \mid 1 \leq d \leq n, i \in \mathbb{N}\}$. Then $w \sim LMR$ where L and R^{-1} are words in $\{X_{i,d}\}$ and M is a word in $\{\pi_i\}$.*

Proof. There is a homomorphism from $\widehat{nV}$ to nV given by $s_{i,d} \mapsto X_{i,d}$ and $\sigma_i \mapsto \pi_i$. This follows from the correspondence between the relations for $\widehat{nV}$ and nV as given below:

$$\begin{aligned} &(\mathrm{M1}) \to (1), && (\mathrm{M5}a) \to (2),\\ &(\mathrm{M2}) \to (12), && (\mathrm{M5}b), (\mathrm{M5}c) \to (3),\\ &(\mathrm{M3}) \to (8), && (\mathrm{M5}d) \to (4),\\ &(\mathrm{M4}) \to (9), && (\mathrm{M6}) \to (7). \end{aligned}$$

Hence, any word w as given above is the image under this homomorphism of a word w' in $\widehat{nV}$. Since $\widehat{nV}$ is the group of right fractions of the monoid Π_n, we can represent w' as pq^{-1} where p, q are words in $\{s_{i,d}, \sigma_i \mid 1 \leq d \leq n, i \in \mathbb{N}\}$. Now, as noted before in the proof of Lemma 6, we can assume p and q are in the form ab where $a \in \langle s_{i,d}\rangle$ and $b \in \langle \sigma_i \rangle$. Hence, we have written w' as lmr for $l, r^{-1} \in \langle s_{i,d}\rangle$ and $m \in \langle \sigma_i\rangle$ since elements of $\langle \sigma_i \rangle$ are their own inverse. Applying the homomorphism to w' puts w in the desired form. □

The following two results follow the original proofs of Lemma 4.6 and 4.7 in Brin [3] via Remark 13.

Lemma 15. *Let w be of the form $w(X, C, \pi, X^{-1}, C^{-1})$. Then $w \sim LMR$ where L and R^{-1} are words of the form $w(X, C)$ and M is of the form $w(\pi)$. Further the number of appearances of C in L will be no larger than the number of appearances of C in w and the number of appearances of C^{-1} in R will be no larger than the number of appearances of C^{-1} in w.*

Lemma 16. *Let w be a word in the generating set $\{X_{i,d}, C_{i,d'}, \pi_i, \overline{\pi}_i, , X_{i,d}^{-1}, C_{i,d'}^{-1} \mid 1 \leq d \leq n, 2 \leq d' \leq n, i \in \mathbb{N}\}$. Then $w \sim LMR$ where L and R^{-1} are words of the form $w(X, C)$ and M is of the form $w(\pi, \overline{\pi})$.*

Lemma 17. *Let w be a word in the generating set*

$$\{X_{i,d}, C_{i,d'}, \pi_i, \overline{\pi}_i, , X_{i,d}^{-1}, C_{i,d'}^{-1} \mid 1 \leq d \leq n, 2 \leq d' \leq n, i \in \mathbb{N}\}.$$

Then $w \sim LMR$ where

- *$L = C_{i_0,d_0} C_{i_1,d_1} \dots C_{i_g,d_g} q$ with $i_0 < i_1 < \cdots < i_g$ for $g \geq -1$ and q is a word in the set $\{X_{i,d} \mid 1 \leq d \leq n, i \in \mathbb{N}\}$*
- *$R^{-1} = C_{j_0,d_0'} C_{j_1,d_1'} \dots C_{j_m,d_m'} q'$ with $j_0 < j_1 < \cdots < j_m$ for $m \geq -1$ and q' is a word in the set $\{X_{i,d} \mid 1 \leq d \leq n, i \in \mathbb{N}\}$*
- *M is a word in the set $\{\pi_i, \overline{\pi}_i \mid i \in \mathbb{N}\}$*

Proof. By using the secondary relations, we can assume that $w \sim LMR$ where L and R^{-1} are words in $\{X_{i,d}, C_{i,d}\}$ and M is a word in $\{\pi_i, \overline{\pi}_i\}$ by analogous arguments used in Lemmas 4.6 and 4.7 of [3]. We then improve L using the subscript raising formula for the $C_{i,d}$ and relation (15) as in the proof of Lemma 4.8 of [3]. We notice that to adapt the quoted lemmas from [3] we need to make use of Remark 13 to make sure that the appearances of C's and $\overline{\pi}$'s do not increase. □

We define the notions of *primary* and *secondary tree* and of *trunk* exactly the same way that Brin does in [3]. The primary tree is the tree corresponding to the word t in Lemma 18 and any extension to the left is a secondary tree for L. The following extends Lemma 4.15 [3] adapted to our case. The proof is completely analogous.

Lemma 18. *Let $L = C_{i_0,d_0} C_{i_1,d_1} \cdots C_{i_g,d_g} X_{i_{n+1},d_{n+1}} \cdots X_{i_{l-1},d_{l-1}}$ where $i_0 < i_1 < \cdots < i_g$, where $2 \leq d_k \leq n$ for $k \in \{0, \dots, g\}$ and $1 \leq d_k \leq n$ for $k \in \{g+1, \dots, l-1\}$. Let m equal the maximum of*

$$\{i_j + g + 2 - j \mid g + 1 \leq j \leq l - 1\} \cup \{i_g + 1\}.$$

Then L can be represented as $L = (t, s_{0,1}^k)$ where t is a word in $\{s_{i,d}\}$ and k is the length of t, so that $k = m + l - g$, and so that the tree T for t is the primary tree for L and is described as follows. The tree T consists of a trunk Λ with a finite forest F attached. The trunk Λ has m carets and $m + 1$ leaves numbered 0 through m in the right-left order. If the carets in Λ are numbered from 0 starting at the top, then the label of the i-th caret is d_k if $i = i_k$ for k in $\{0, 1, \dots g\}$ and 1 otherwise.

The following two lemmas are used in proving Proposition 13 which allows us to assume the trees corresponding to our group elements are in normal form.

Lemma 19. *Let $L = C_{i_0,d_0} C_{i_1,d_1} \cdots C_{i_g,d_g} u$ and $L' = C_{k_0,d_0'} C_{k_1,d_1'} \cdots C_{k_g,d_g'} u'$ where $i_0 < i_1 < \cdots < i_g$, where $k_0 < k_1 < \cdots < k_g$, where u is a word in the set $\{X_{i,d} \mid 1 \leq d \leq n, i \in \mathbb{N}\}$, and where u' is a word in the set $\{X_{i,d}, \pi_i \mid 1 \leq d \leq$*

$n, i \in \mathbb{N}\}$. *Assume that* L *is expressible as* $(t, s_{0,1}^p)$ *as an element of* $\widehat{nV}$ *with* t *a word in* $\{s_{i,d}\}$ *and* p *is the length of* t. *Let* m *be the number of carets of the trunk of the tree* T *corresponding to* t *and assume that* $m \geq k_g + 1$.

If $L \sim L'$, *then there is a word* u'' *in* $\{X_{i,d}\}$, *and there is a word* z *in* $\{\pi_i \mid i \leq p-2\}$ *so that setting* $L_1 = C_{k_0,d_0'}C_{k_1,d_1'}\cdots C_{k_g,d_g'}u''$ *and* $L_2 = L_1 z$ *gives that* $L \sim L_2$ *and* L_1 *is expressible as* $(t', s_{0,1}^p)$ *with* t' *a word in* $\{s_{i,d}\}$ *of length* p *so that the tree* T' *for* t' *is normalized except possibly at interior vertices in the trunk of the tree, and so that the trunk of* T' *has* m *carets.*

Proof. The homomorphism $\widehat{nV} \to nV$ given by $s_{i,d} \mapsto X_{i,d}$ and $\sigma_i \mapsto \pi_i$ allows us to write $u' \sim u''z'$ with u'' a word in $\{X_{i,d}\}$ and z' is a word in $\{\pi_i \mid i \in \mathbb{N}\}$ such that the forest F for u'' is normalized. The rest of the proof goes through as before, but we describe the slight modifications needed for our case. We write $L = (ts_{0,1}^k, s_{0,1}^{p+k}) = (\widehat{t}s_{1,0}^r x, s_{1,0}^{q+r}) = L_2$ as elements in $\widehat{nV}$ where x is a word in $\{\sigma_i\}$ and $p + k = q + r$. As before, we can conclude that the unnumbered patterns for $ts_{0,1}^k$ and $\widehat{t}s_{1,0}^r$ are identical.

In the tree for $ts_{0,1}^k$, let the left edge vertices be $a_0, a_1, \ldots, a_b$ reading from the top, so that a_0 is the root of the tree. Since we assume the trunk of the tree has m carets, we know $b = m + k$ and for $m \leq i < b$, the label for a_i is 1. Similarly, in the tree for $\widehat{t}s_{1,0}^r$, let the left edge vertices be $a_0', a_1', \ldots, a_b'$ reading from the top. Note that remark (*) in the proof of Theorem 4.21 in Brin [3] (which we are about to restate) remains true in our general case, by giving a new definition: for each left edge vertex, a_i, define the n-tuple $(x_1^i, \ldots, x_n^i)$ where x_k^i equals the number of left edge vertices above a_i with label k. (Note we are using i to denote an index, not an exponent). It follows that $x_1^i + x_2^i + \cdots + x_n^i$ is the total number of left edge vertices above a_i. Then we have:

(*) The rectangle corresponding to a left edge vertex a_i depends only on the n-tuple $(x_1^i, \ldots, x_n^i)$

In other words, for the rectangle labeled $''0''$ in any pattern, the order of the different cuts does not matter. This is because the rectangle labeled $''0''$ must contain the origin and its size in each dimension k will be $2^{-x_k^i}$. Hence, the analogous statement for our case follows, and we conclude that the n-rectangle R corresponding to a_m is identical to the n-rectangle R' corresponding to a_m' Since R is divided k times across dimension 1, so is R', and hence the tree below a_m' must consist of an extension to the left by k carets all labeled 1, and we can conclude that $r \geq k$. The rest of the proof follows exactly as before. □

Here, we define a notion of *complexity* to measure progress in the following lemma and proposition towards normalizing trees. If T is a labeled tree, let $a_0, a_1, \ldots, a_m$ be the interior, left edge vertices of T reading from top to bottom so that a_0 is the root. Let $b_0 b_1 \ldots b_m$ be a word in $\{1, 2, \ldots, n\}$ where $b_i = k$ if a_i is labelled k for $0 \leq i \leq m$. We say $b_0 b_1 \ldots b_m$ is the complexity of T. We impose the length-lex ordering on such words, that is if w_1 and w_2 are two such words, then we say $w_1 < w_2$ if w_1 is shorter than w_2 or if $w_1 = b_0^1 \ldots b_m^1$ and $w_2 = b_0^2 \ldots b_m^2$ are two such words of the same length, then $w_1 < w_2$ if when we take $j \in \{0, \ldots, m\}$ minimal where $b_j^1 \neq b_j^2$, we have $b_j^1 < b_j^2$. We will refer to this notion in the following lemma.

Lemma 20. *Let* $L = C_{i_0,d_0}C_{i_1,d_1}\cdots C_{i_g,d_g}u$ *where* $i_0 < i_1 < \cdots < i_g$ *and* u *is a word in the set* $\{X_{i,d}\}$. *Assume that the primary tree* T *for* L *is normalized except* *at one or more vertices in the trunk of T. Let m be the number of carets in the trunk of T. Then $L \sim L' = C_{k_0,c_0}C_{k_1,c_1}\cdots C_{k_g,c_g}u'$ where $k_0 < k_1 < \cdots < k_g$, where u' is a word in the set $\{X_{i,d}, \pi_s\}$, so that $m \geq k_g + 1$, and so that the complexity of the primary tree T' of L' is strictly less than the complexity of T.*

Proof. We want to use the relations to push a suitable instance of an $X_{u,v}$ in the word L to the left as far as possible in order to be able to apply a cross-relation. This operation normalizes a suitable vertex and decreases the complexity of the primary tree T.

Let Λ be the trunk of T. The interior vertices of Λ are the interior, left edge vertices of T and let these be $a_0, a_1, \cdots, a_{m-1}$. Let r be the highest value with $0 \leq r < m$ for which a_r is not normalized. Note that this is the lowest non-normalized interior vertex of Λ and that, since a_r is not normalized it is labelled $\ell \neq 1$ and must correspond to some $C_{i_j,\ell}$ and from Lemma 18, we have $i_j = r$.

Moreover, since it is not normalized, a_r must correspond to some hypercube S_{i_j} which is fully divided across dimension ℓ and some other dimension d, with $1 \leq d < \ell$.

By rewriting L as $(t, s_{0,1}^k)$ (which we can do by Lemma 18) and applying Corollary 9 to t, we can assume that the children of a_r, v_1 and v_2, are both labelled d. We divide our work in two cases, $d = 1$ and $d > 1$. We observe that the case $d = 1$ is entirely analog to the proof of Theorem 4.22 in Brin [3] while the case $d > 1$ is slightly different.

Case 1: $d = 1$. In this case, the left child v_1, which is in the trunk Λ, is labelled 1. In the case that $j < n$ we observe that $i_{j+1} > r + 1 = i_j + 1$, since the interior vertex of the trunk corresponding to $C_{i_{j+1},d_{j+1}}$ is not labelled 1 (otherwise, $a_r = a_{i_j}$ would not be the lowest non-normalized interior vertex). Since the right child v_2 is an interior vertex not on the trunk, there must be a letter $X_{q,1}$ corresponding to it. By Lemma 5 we can assume that $X_{q,1}$ occurs as the first letter of u, that is $u = X_{q,1}u''$. Hence

$$L = C_{i_0}\cdots C_{i_{j-1}}\underline{C_{i_j,\ell}}C_{i_{j+1}}\cdots C_{i_g}\underline{X_{q,1}}u''$$

where we have omitted all the dimension subscripts of the baker's maps $C_{i,d}$ (except for one map) since they are not important for the argument. The subword $C_{i_0}\cdots C_{i_j,\ell}\cdots C_{i_g}X_{q,1}$ is a trunk with a single caret labelled 1 attached at the caret i_j of the trunk on its right child. By a careful observation of the right-left ordering it is evident that $q = i_j$. By using relation (15) repeatedly on L we can move $X_{q,1} = X_{i_j,1}$ to the left and rewrite the word L as

$$C_{i_0}\cdots C_{i_{j-1}}\underline{C_{i_j,\ell}X_{i_j,1}}C_{i_{j+1}+1}\cdots C_{i_g+1}u'',$$

since $i_0 < i_1 < \ldots < i_g$ and $i_{j+1} > i_j + 1$. Combining relations (15) and (16) on the product $C_{i_j,\ell}X_{i_j,1}$ we rewrite L as

$$C_{i_0}\cdots C_{i_{j-1}}\underline{C_{i_j+1,\ell}X_{i_j,\ell}\pi_{i_j+1}}C_{i_{j+1}+1}\cdots C_{i_g+1}u''.$$

Now we apply (17) to commute π_{i_j+1} back to the right without affecting the indices of the baker's maps. This is possible since $i_{j+1} > i_j+1$ and therefore $i_{j+1}+1 > i_j+2$. Now we apply (15) repeatedly to the word

$$C_{i_0}\cdots C_{i_{j-1}}\underline{C_{i_j+1,\ell}X_{i_j,\ell}}C_{i_{j+1}+1}\cdots C_{i_g+1}\underline{\pi_{i_j+1}}u''$$

to bring $X_{i_j,\ell}$ back to the right decreasing the indices of the the baker's maps by 1

$$C_{i_0}\cdots C_{i_{j-1}}\underline{C_{i_j+1,\ell}}C_{i_{j+1}}\cdots C_{i_g}\underline{X_{i_j,\ell}\pi_{i_j+1}}u''.$$

By setting $u' = X_{i_j,\ell}\pi_{i_j+1}u''$ in the previous equation and relabelling the indices with k_i's, we obtain the word $L' = C_{k_0,c_0}C_{k_1,c_1}\cdots C_{k_g,c_g}u'$ whose primary tree T' is the same as T up until the vertex a_r, which is now labelled $d = 1$ instead of ℓ. Thus, $L \sim L' = C_{k_0,c_0}C_{k_1,c_1}\cdots C_{k_g,c_g}u'$ and the complexity of the primary tree T' of L' is strictly less than the complexity of T.

The only thing we still need to prove in this case is that $m \geq k_g + 1$. However, it has been observed above that $i_j = r < m-1$ so $i_j + 2 \leq m$. This gives the result in the case that $j = n$. If $j < n$, then $k_g = i_g$ and $m \geq i_g + 1$ by Lemma 18.

Case 2: $1 < d < \ell$. We observe that a_r corresponds to $C_{i_j,\ell}$ and that v_1 corresponds to $C_{i_k,d}$. By Lemma 18, we have $r + 1 = i_k$ which implies $i_k = i_j + 1 = i_{j+1}$. In fact, if $i_j + 1 < i_{j+1}$, there would be a vertex labelled 1 on the trunk between the vertices i_j and i_{j+1} (and this is impossible since $d > 1$). Let $X_{i_j,d}$ correspond to the right child v_2. Arguing as in the case $d = 1$ we have

$$L = C_{i_0}\cdots C_{i_{j-1}}\underline{C_{i_j,\ell}C_{i_j+1,d}}C_{i_{j+2}}\cdots C_{i_g}\underline{X_{q,d}}u''$$

We apply relation (15) as before to move $X_{q,d} = X_{i_j,d}$ to the left while increasing the subscript of each baker's map by 1:

$$C_{i_0}\cdots C_{i_{j-1}}\underline{C_{i_j,\ell}X_{i_j,d}C_{i_j+2,d}}C_{i_{j+2}+1}\cdots C_{i_g+1}u''.$$

By using the cross relation (18) on the underlined portion, we read it as

$$C_{i_0}\cdots C_{i_{j-1}}\underline{C_{i_j,d}X_{i_j,\ell}C_{i_j+2,\ell}\pi_{i_j+1}}C_{i_{j+2}+1}\cdots C_{i_g+1}u''$$

Since $i_{j+2} > i_{j+1}$, then $i_{j+2} + 1 > i_{j+1} + 1$, hence π_{i_j+1} and the baker's maps to its right commute, so the word becomes

$$C_{i_0}\cdots\underline{C_{i_j,d}X_{i_j,\ell}C_{i_j+2,\ell}}C_{i_{j+2}+1}\cdots C_{i_g+1}\underline{\pi_{i_j+1}}u''.$$

We apply (15) repeatedly and move $X_{i_j,\ell}$ back to the right to obtain

$$L \sim C_{i_0}\cdots\underline{C_{i_j,d}C_{i_j+1,\ell}}C_{i_{j+2}}\cdots C_{i_g}X_{i_j,\ell}\pi_{i_j+1}u'',$$

where the product $C_{i_j,d}C_{i_j+2,\ell}$ has been underlined to stress that the new trunk has the vertices labelled d and ℓ which are now switched. Thus the complexity of the tree has been lowered. In this second case, the new sequence $k_0 < \ldots < k_g$ is exactly equal to the initial one $i_0 < \ldots < i_g$. By the definition of m (given in Lemma 18) applied on the initial word L, we have that $m \geq i_g + 1$ and so, since $k_g = i_g$, we are done. $\square$

Remark 21. As observed in the proof above, the case $d = 1$ is equivalent to Theorem 4.22 in [3], though the proof in there leads to a condition that is equivalent to lowering the complexity. When the index in some $C_{i_j,d}$ goes up by 1, this corresponds to switching the vertices with labels d and 1 in the primary tree and thus lowering the complexity by making more vertices normalized.

Proposition 22. *Let w be a word in the generating set*

$$\{X_{i,d}, C_{i,d'}, \pi_i, \overline{\pi}_i, , X_{i,d}^{-1}, C_{i,d'}^{-1} \mid 1 \leq d \leq n, 2 \leq d' \leq n, i \in \mathbb{N}\}.$$

Then $w \sim LMR$ as in Lemma 17 and when expressed as elements of $\widehat{nV}$ we have $L = ts_{0,1}^{-p}$, $R^{-1} = ys_{0,1}^{-p}$, and $M = s_{0,1}^{p}us_{0,1}^{-p}$ where t, y are words in $\{s_{i,d} \mid 1 \leq d \leq$

$n, i \in \mathbb{N}\}$, u is a word in $\{\sigma_j \mid 0 \leq j \leq p-1\}$, and the lengths of t and y are both p. Further, we may assume the trees for t and y are normalized, and if u can be reduced to the trivial word using relations (2) – (4), then M can be reduced to the trivial word using relations (13)–(17).

Proof. The proof of the first conclusion is exactly the same as the proof of lemma 4.19 of [4]. In order to assume the trees for t and y are normalized, we alternate applying Lemmas 19 and 20. We have L expressed as $(t, s_{0,1}^p)$, where p is the length of t and the number of carets in the trunk of the tree T for t is m. Setting $L = L'$ certainly gives that $L \sim L'$ and $m \geq k_g + 1$ by Lemma 18, so we have satisfied the hypotheses of Lemma 19. Therefore, $L \sim L_1 z$ where L_1 expressed as $(t', s_{0,1}^p)$ where the trunk of the tree T' for t' has m carets. Since we set $L = L'$, we see that the trunks of T and T' are identical and the only way in which the two trees differ is that T' is normalized off the trunk. Since z is a word in $\{\pi_i\}$, z can be absorbed into M without disrupting the assumptions on M, namely M can still be written in the form $M = s_{0,1}^p u s_{0,1}^{-p}$ as above. We now replace L with L_1 and proceed to use Lemma 20.

Since the tree for L is now normalized off the trunk, we satisfy the hypotheses of Lemma 20 and write $L \sim L'$ where the tree for L' has complexity lower than the tree for L and $m \geq k_g + 1$. Hence, we can now apply Lemma 19 again and obtain $L \sim L_1 z$ and let z be absorbed into M. We apply this process over and over, decreasing the complexity of the tree associated to L each time. Since there are only finitely many linearly ordered complexities, eventually this process will terminate, at which point the tree for L will be normalized. We can apply the same procedure to the inverse of LMR to normalize the tree for R. The last statement regarding M follows immediately from Lemma 4.18 of [3].

□

Theorem 23. *Let w be a word in the generating set*

$$\{X_{i,d}, C_{i,d'}, \pi_i, \overline{\pi}_i, , X_{i,d}^{-1}, C_{i,d'}^{-1} \mid 1 \leq d \leq n, 2 \leq d' \leq n, i \in \mathbb{N}\}$$

that represents the trivial element of nV. Then $w \sim 1$ using the relations in (1)–(18). Hence, we have a presentation for nV.

Proof. Using the Proposition 22, we can assume

$$w \sim LMR = (ts_{0,1}^{-p})(s_{0,1}^p u s_{0,1}^{-p})(s_{0,1}^p y^{-1}) = tuy^{-1}$$

where t, y are words in $\{s_{i,d} \mid 1 \leq d \leq n, i \in \mathbb{N}\}$, u is a word in $\{\sigma_j \mid 0 \leq j \leq p-1\}$, and the trees associated to t and y are normalized. By assumption, $tuy^{-1} = (tu, y)$ is the trivial element of $\widehat{nV}$ and so tu and y represent the same numbered patterns in Π_n. Furthermore, t and y must give the same unnumbered pattern, while u enacts a permutation on the numbering. Since the forests for t and y are normalized and give the same pattern, the forests are identical with the same labeling by Lemma 7. The numbering on the leaves for both forests follows the left-right ordering, hence t and y give the same numbered patterns, which implies that u enacts the trivial permutation and $M \sim 1$ by Proposition 22.

We now wish to show that $L \sim R^{-1}$. By Lemma 17, we have

- $L = C_{i_0,d_0} C_{i_1,d_1} \dots C_{i_g,d_g} q$
- $R^{-1} = C_{j_0,d'_0} C_{j_1,d'_1} \dots C_{j_m,d'_m} q'$

Since we know that the trunks of the trees corresponding to L and R^{-1} are identical with the same labeling, the sequences $(i_0, i_1, \ldots, i_g)$ and $(j_0, j_1, \ldots, j_m)$ are identical and $d_k = d'_k$ for each $k \in \{0, 1, \ldots, n = m\}$. Hence, the subwords $C_{i_0,d_0}C_{i_1,d_1} \ldots C_{i_g,d_g}$ and $C_{j_0,d'_0}C_{j_1,d'_1} \ldots C_{j_m,d'_m}$ are the same and it remains to show that $q \sim q'$. This follows from Lemma 4 and the homomorphism from $\widehat{nV}$ to nV as before.

□

6. Finite Presentations

6.1. **Finite Presentation for $\widehat{nV}$.** We now give a finite presentation for $\widehat{nV}$, using analogous arguments found in [3] to show that the full set of relations is the result of only finitely many of them.

Theorem 24. *The group $\widehat{nV}$ is presented by the $2n+2$ generators $\{s_{i,d}, \sigma_i \mid i \in \{0,1\}, 1 \le d \le n\}$ and the $5n^2 + 7n + 6$ relations given below:*

(M1)	$s_{1,1}^{-1} s_{1+k,d'} s_{1,1} = s_{2+k,d'}$	$k = 1,2$
	$s_{i,d}^{-1} s_{i+k,d'} s_{i,d} = s_{i+k+1,d'}$	$i = 0,1, k = 1,2; 2 \le d \le n$
(M2)	${\sigma_i}^2 = 1$	$i = 0,1$
(M3)	$\sigma_i \sigma_{i+k} = \sigma_{i+k} \sigma_i$	$i = 0,1, k = 2,3$
(M4)	$\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}$	$i = 0,1$
(M5a)	$\sigma_{k+1} s_{1,1} = s_{1,1} \sigma_{k+2}$	$k = 1,2$
	$\sigma_{i+k} s_{i,d} = s_{i,d} \sigma_{i+k+1}$	$i = 0,1, k = 1,2; 2 \le d \le n$
(M5b/M5c)	$\sigma_i s_{i,d} = s_{i+1,d} \sigma_i \sigma_{i+1}$	$i = 0,1$
(M5d)	$\sigma_i s_{i+k,d} = s_{i+k,d} \sigma_i$	$i = 0,1, k = 2,3$
(M6)	$s_{i,d} s_{i+1,d'} s_{i,d'} = s_{i,d'} s_{i+1,d} s_{i,d} \sigma_{i+1}$	$i = 0,1, d \ne d'$

Proof. First, recall our generating set is $\{s_{i,d}, \sigma_i \mid i \in \mathbb{N}, 1 \le d \le n\}$. When $i < j$, relations (M1) and (M5a) give $s_{i,1}^{-1} x_j s_{i,1} = x_{j+1}$ where $x_j = s_{j,d}$ (for some d) or σ_j. Hence, we can use $s_{i,d} = s_{0,1}^{1-i} s_{1,d} s_{0,1}^{i-1}$ and $\sigma_i = s_{0,1}^{1-i} \sigma_1 s_{0,1}^{i-1}$ as definitions for $i \ge 2$. Therefore, $\widehat{nV}$ is generated by $\{s_{i,d}, \sigma_i \mid i \in \{0,1\}, 1 \le d \le n\}$, which gives a generating set of size $2n+2$ for each n.

We treat relations (M1) through (M6) in the same way as they are treated in [3]. Relations involving only one parameter, such as (M2), (M4), and (M6), are obtained for $i \ge 2$ by setting $i = 1$ and conjugating by powers of $s_{0,1}$, therefore the only necessary relations to include are when $i = 0$ and $i = 1$. As before, (M2) and (M4) follow from: $\sigma_0^2 = 1$, $\sigma_1^2 = 1$, $\sigma_0\sigma_1\sigma_0 = \sigma_1\sigma_0\sigma_1$, and $\sigma_1\sigma_2\sigma_1 = \sigma_2\sigma_1\sigma_2$, or 4 relations for each n. Relation (7) follows from 2 relations for each pair of distinct dimensions, giving $2\binom{n}{2} = n(n-1)$ relations for each n.

Relation (M3) is treated the same way as in [3] for each n. Hence, for all i, j, (M3) follows from the 4 relations: $\sigma_0\sigma_2 = \sigma_2\sigma_0$, $\sigma_0\sigma_3 = \sigma_3\sigma_0$, $\sigma_1\sigma_3 = \sigma_3\sigma_1$, $\sigma_1\sigma_4 = \sigma_4\sigma_1$.

For relation (M1), which can be rewritten as $s_{i,d}^{-1} s_{i+k,d'} s_{i,d} = s_{i+k+1,d'}$ for $k > 0$, we have two cases: the case where $d = 1$ and the case where $d \ne 1$. If $d = 1$, then the case $i = 0$ follows by definition, and by the same induction argument used in [3] implies that the relation for all i,k follows from the cases where $i=1$ and $k=1,2$, hence we need only 2 relations per dimension. If $d\neq 1$, we do not get the case $i=0$ by definition and we must include $i=0,1$ and $k=1,2$, i.e. 4 relations per each pair of dimensions. There are $n-1$ choices for d, as $d\neq 1$, and n choices for d', so this case yields $4n(n-1)$ relations. Hence, in total (M1) can be obtained for all i,k by $2n+4n(n-1)=4n^2-2n$ relations.

For relation (M5b), $\sigma_i s_{i,d}=s_{i+1,d}\sigma_i\sigma_{i+1}$, there is only a single parameter to deal with, hence the relation for $i\geq 2$ can be obtained from the cases where $i=0,1$ by conjugating by $s_{0,1}$ as before. Relation (M5c) is actually equivalent to (M5b), hence for each n we only need $2n$ relations for (M5b), (M5c). We treat (M5a) $\sigma_{i+k}s_{i,d}=s_{i,d}\sigma_{i+k+1}$ for $k>0$ the same way as for (M1), hence 2 relations are required for $d=1$ and 4 for $d\neq 1$ for a total of $4n-2$ relations. And lastly, (M5d) $\sigma_i s_{i+k,d}=s_{i+k,d}\sigma_i$ can be obtained in the same way as the second case of (M1) where the relation for all i,k is obtained by $i=0,1,k=2,3$, i.e. $4n$ relations. □

6.2. **Finite Presentation for nV.** We can now prove the following:

Theorem 25. *The group nV is presented by the $2n+4$ generators $\{X_{i,d},\pi_i,\overline{\pi}_i \mid i\in\{0,1\},1\leq d\leq n\}$, the $5n^2+7n+6$ relations obtained from the homomorphism $\widehat{nV}\to nV$, and the additional $5n^2+3n+4$ relations given below for a total of $10n^2+10n+10$ relations.*

(5)	$\overline{\pi}_{k+1}X_{1,1}=X_{1,1}\overline{\pi}_{k+2}$	$k=1,2$
	$\overline{\pi}_{m+k}X_{m,d}=X_{m,d}\overline{\pi}_{m+k+1}$	$m=0,1,k=1,2,2\leq d\leq n$
(10)	$\overline{\pi}_{m+k}\pi_m=\pi_m\overline{\pi}_{m+k}$	$m=0,1,k=2,3$
(11)	$\pi_m\overline{\pi}_{m+1}\pi_m=\overline{\pi}_{m+1}\pi_m\overline{\pi}_{m+1}$	$m=0,1$
(13)	$\overline{\pi}_m^2=1$	$m=0,1$
(6)	$\overline{\pi}_mX_{m,1}=\pi_m\overline{\pi}_{m+1}$	$m=0,1$
(14)	$\overline{\pi}_mX_{m,d}=C_{m+1,d}\pi_m\overline{\pi}_{m+1}$	$m=0,1,d\neq 1$
(15)	$C_{k+1,d}X_{1,1}=X_{1,1}C_{k+2,d}$	$k=1,2$
	$C_{m+k,d}X_{m,d'}=X_{m,d'}C_{m+k+1,d}$	$m=0,1,k=1,2;2\leq d,d'\leq n$
(16)	$C_{m,d}X_{m,1}=X_{m,d}C_{m+2,d}\pi_{m+1}$	$m=0,1;2\leq d\leq n$
(17)	$\pi_mC_{m+k,d}=C_{m+k,d}\pi_m$	$m=0,1,k=2,3$
(18)	$C_{m,d}X_{m,d'}C_{m+2,d'}=C_{m,d'}X_{m,d}C_{m+2,d}\pi_{m+1}$	$m=0,1;1<d'<d\leq n$

Proof. We can use the relations in nV to write

$$X_{i,d}=X_{0,1}^{1-i}X_{1,d}X_{0,1}^{i-1},$$
$$\pi_i=X_{0,1}^{1-i}\pi_1X_{0,1}^{i-1},$$
$$\overline{\pi}_i=X_{0,1}^{1-i}\overline{\pi}_1X_{0,1}^{i-1}$$

for $i\geq 2$ and $1\leq d\leq n$. We can also use the relations for nV as in Proposition 6.2 of [2] to write

$$C_{m,d}=(\overline{\pi}_mX_{m,d}\overline{\pi}_{m+1}\pi_m)(X_{m,d}\pi_{m+1}X_{m,1}^{-1})$$

for $m \geq 0$ and $2 \leq d \leq n$, which we use as a definition. Hence, the $C_{m,d}$ are not needed to generate nV.

The homomorphism $\widehat{nV} \to nV$ given by $s_{i,d} \mapsto X_{i,d}$ and $\sigma_i \mapsto \pi_i$ implies that the work done for the relations for $\widehat{nV}$ carries over to relations (1)–(4), (7)–(9), and (12) (see Lemma 14). Relations (10)-(11) and (13)-(6) are exactly the same as those from $2V$ and can be treated as in [3], contributing a total of 10 relations to our finite set.

Relation (5) can be treated in a manner similar to (M1) from $\widehat{nV}$, where 2 relations are needed for dimension 1 and 4 for all others, contributing a total of $4(n-1)+2$ relations. Relations (14) and (16) include only one parameter and hence can be obtained from the cases where $i = 0, 1$ as before, contributing $2(n-1)$ relations apiece. And (17) requires 4 relations for each $d \neq 1$, hence adding an additional $4(n-1)$ relations.

For relation (15), we have two cases: for $d' = 1$, all cases follow from when $i = 0, 1$, giving us $2(n-1)$ relations since $2 \leq d \leq n$. For $d' \neq 1$, 4 relations are required for each pair $d, d' \in \{2, ..., n\}$, contributing $4(n-1)(n-1)$ relations. And lastly, since (18) involves only one parameter in the first component, we only need 2 relations for each $1 < d' < d \leq n$, the number of such pairs being $\frac{(n-1)(n-2)}{2}$. $\square$

Remark 26. Since ωV is an ascending union of the nV's, a word $w \in \{X_{i,d}, \pi_i, \overline{\pi}_i \mid i \in \{0,1\}, d \in \mathbb{N}\}$ such that $w =_{\omega V} 1$ must be contained in some nV (for some $n \in \mathbb{N}$) and so we can use the same ideas and the relations inside nV to transform w into the empty word. Therefore, the following result is an immediate consequence of Theorem 25.

Corollary 27. *The group ωV is generated by the set $\{X_{i,d}, \pi_i, \overline{\pi}_i \mid i \in \{0,1\}, d \in \mathbb{N}\}$ and satisfies the family of relations in Theorem 25 with the only exception that the parameters $d, d' \in \mathbb{N}$.*

7. Simplicity of nV and ωV

Brin proved in [4] that the groups nV and ωV are simple by showing that the baker's map is a product of transpositions and following the outline of an existing proof that V is simple.

We reprove Brin's simplicity result verify that Brin's original proof that $2V$ is simple (Theorem 7.2 in [2]) generalizes using the generators and the relations that have been found.

Theorem 28. *The groups nV, $n \leq \omega$, equal their commutator subgroups.*

Proof. The goal is to show that the generators $X_{m,i}, \pi_m, \overline{\pi}_m$ are products of commutators. We write $f \simeq g$ to mean that $f = g$ modulo the commutator subgroup. We also observe that the arguments below are independent of the dimension i.

From relation (1) we see that $X_{q,i}^{-1} X_{0,1}^{-1} X_{q,i} X_{0,1} = X_{q,i}^{-1} X_{q+1,i}$ for $q \geq 1$ and so $X_{q+1,i} \simeq X_{q,i}$. Therefore $X_{q,i} \simeq X_{1,i}$, for $q \geq 1$. Using relation (2) and arguing similarly, we see that $\pi_q \simeq \pi_1$, for $q \geq 1$.

From relation (3) we see that $\pi_0 X_{0,i} \pi_0^{-1} X_{0,i}^{-1} = X_{1,i} \pi_1 X_{0,i}^{-1}$ so that $X_{0,i} \simeq X_{1,i}\pi_1$. Also, by relation (3), $X_{2,i} \simeq X_{1,i}$ and the fact that $\pi_2 \simeq \pi_1$, we see $\pi_1 X_{1,i} = X_{2,i}\pi_1\pi_2 \simeq X_{1,i}\pi_1\pi_1 = X_{1,i}$. Therefore $\pi_1 \simeq 1$ and so $X_{0,i} \simeq X_{1,i}$.

Relation (9) and $\pi_1 \simeq 1$ give that $\pi_0^2 \simeq \pi_0\pi_1\pi_0 = \pi_1\pi_0\pi_1 \simeq \pi_0$ which implies $\pi_0 \simeq 1$.

By relation (6) and the fact that $\pi_1 \simeq 1$ and $\overline{\pi}_1 \simeq \overline{\pi}_0$ we get $\overline{\pi}_1 X_{1,1} = \pi_1 \overline{\pi}_2 \simeq \overline{\pi}_1$. Hence $X_{0,1} \simeq X_{1,1} \simeq 1$.

Now, relation (6) and $X_{0,1} \simeq 1$ give that $\overline{\pi}_0 \simeq \overline{\pi}_0 X_{0,1} = \overline{\pi}_1$. Relation (11) and $\pi_0 \simeq 1$ lead to $\overline{\pi}_1 \simeq \pi_0 \overline{\pi}_1 \pi_0 = \overline{\pi}_1 \pi_0 \overline{\pi}_1 \simeq \overline{\pi}_1^2$. Therefore $\overline{\pi}_0 \simeq \overline{\pi}_1 \simeq 1$.

Finally, by relation (7) and $X_{0,1} \simeq X_{1,1} \simeq 1 \simeq \pi_1$ we get $X_{1,i} X_{0,i} \simeq X_{0,1} X_{1,i} X_{0,i} = X_{0,i} X_{1,1} X_{0,1} \pi_1 \simeq X_{0,i}$ which implies $X_{0,i} \simeq X_{1,i} \simeq 1$. We have thus proved that all the generators of nV are in the commutator subgroup. The case of ωV is identical: each generator lies in some nV and can be written as a product of commutators within that subgroup. □

From Section 3.1 in [2] (which generalizes to nV and ωV as observed by Brin in [3] and [4]) the commutator subgroup of nV and ωV are simple, therefore Theorem 28 implies the following result.

Theorem 29. *The groups nV, $n \leq \omega$, are simple.*

8. An alternative generating set

Observe that, for any $n \in \mathbb{N}$, we have $(n-1)V \times V \leq nV$. It can be shown that another generating set for nV is given by taking a generating set for $(n-1)V \times V$ and adding an involution which swaps two disjoint subcubes of $[0,1]^n$, one of which has the origin as one of its vertices and the other one which contains the vertex $(1, \ldots, 1)$. This second generating set has the advantage of taking the generators of $(n-1)V$ and adding only the generators of V plus another one. This leads to a smaller generating set which was suggested to us by Collin Bleak. It seems feasible that a good set of relations exist for this alternative generating set.

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