ON THE ASYNCHRONOUS RATIONAL GROUP

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ABSTRACT. We prove that the asynchronous rational group defined by Grigorchuk, Nekrashevych, and Sushchanskiĭ is simple and not finitely generated. Our proofs also apply to certain subgroups of the asynchronous rational group, such as the group of all rational bilipschitz homeomorphisms.

In [9], Grigorchuk, Nekrashevych, and Sushchanskiĭ defined the group $\mathcal{R}$ of all asynchronous rational homeomorphisms of the Cantor set (denoted $\mathcal{Q}$ in [9]). This is a countable subgroup of the full homeomorphism group of the Cantor set, consisting of all homeomorphisms that can be defined by finite-state automata. As described in [9], such automorphisms are "repeating" in the sense that they have only finitely many different restrictions on cones.

Rational homeomorphisms first appeared in the study of Grigorchuk's group of intermediate growth [8] and other automata groups, including the Gupta–Sidki groups [10] and iterated monodromy groups [12], and $\mathcal{R}$ can be viewed as the "universal" group that contains all automata groups as subgroups. Many other groups are also known to embed into $\mathcal{R}$, including Thompson's groups F, T, and V [9], automorphism groups of full shifts [9], the matrix groups $\mathrm{GL}_n(\mathbb{Z})$ [6], solvable Baumslag-Solitar groups $BS(1, m)$ [2], generalized lamplighter

groups $(\mathbb{Z}/n) \wr \mathbb{Z}$ [14], and all Gromov hyperbolic groups [4]. Every finitely generated subgroup of $\mathcal{R}$ has solvable word problem [9], but there is no algorithm to determine whether a given element of $\mathcal{R}$ has finite order [3].

The purpose of this note is to prove the following theorem.

Theorem 1. $\mathcal{R}$ *is simple but not finitely generated.*

Our proof of simplicity uses the methods of Epstein [7], and resembles the proofs of simplicity for Thompson's group V, Brin's higher-dimensional Thompson groups nV [5], and Röver's group $V\mathcal{G}$ [13]. In each case, Epstein's methods prove that the commutator subgroup is simple, and some separate argument is required to show that the group is perfect. Indeed, there exist groups in the Thompson family, i.e. the Higman–Thompson groups $V_{n,r}$ for n odd, that are not perfect and hence not simple.

For the rational group $\mathcal{R}$, we use Epstein's argument to prove that $[\mathcal{R}, \mathcal{R}]$ is simple, and then a new trick involving the construction of the element
$$(f, (f, (f, (f, \ldots))))$$
from a rational homeomorphism $f \in \mathcal{R}$ to show that $\mathcal{R}$ is perfect. A similar idea was used by Anderson [1] to prove that the full group of homeomorphisms of the Cantor set is simple.

The proof that $\mathcal{R}$ is not finitely generated involves the primes that divide the lengths of cycles in the transducer for an element $f \in \mathcal{R}$. Specifically, we generate a sequence $\{f_p\}$ of elements of $\mathcal{R}$ (with p prime) that no finitely generated subgroup of $\mathcal{R}$ can contain. This contrasts, for example, with Thompson's group V, which is uniformly dense in the homeomorphism group of the Cantor set and contains embedded copies of all finite groups, but is nonetheless finitely generated. Even larger groups than V such as Brin's groups nV and Röver's group $V\mathcal{G}$ are also finitely generated, but it seems that $\mathcal{R}$ is too large to support a finite generating set.

The techniques of both of our proofs apply not just to $\mathcal{R}$ but also to certain interesting subgroups of $\mathcal{R}$. For example:

Theorem 2. *The group of all rational bilipschitz homeomorphisms of* $\{0, 1\}^\omega$ *is simple but not finitely generated.*

1. BACKGROUND AND NOTATION

Let $\{0, 1\}^\omega$ be the Cantor set of all infinite binary sequences, and let $\{0, 1\}^*$ be the set of all finite binary sequences, including the empty sequence ε.
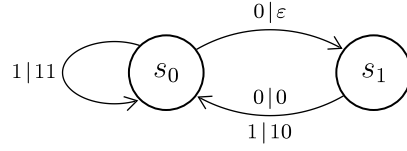
FIGURE 1. The state diagram for an asynchronous transducer.

Definition 3. An **asynchronous binary transducer** is a quadruple (S, s_0, t, o), where

(1) S is a finite set (the set of **internal states** of the transducer);
(2) s_0 is a fixed element of S (the **initial state**);
(3) t is a function $S \times \{0, 1\} \to S$ (the **transition function**); and
(4) o is a function $S \times \{0, 1\} \to \{0, 1\}^*$ (the **output function**).

Example 4. Figure 1 shows the **state diagram** for a certain asynchronous transducer. This is a directed graph with one node for each state $s \in S$, and a directed edge from s to s' if $t(s, \sigma) = s'$ for some $\sigma \in \{0, 1\}$. We label such an edge by the pair $\sigma \mid o(s, \sigma)$, with two labels in the case where $t(s, 0) = t(s, 1) = s'$. For example, the transducer in Figure 1 has two states s_0, s_1, with transition function

$$t(s_0, 0) = s_1, \quad t(s_0, 1) = s_0, \quad t(s_1, 0) = s_0, \quad t(s_1, 1) = s_0$$

and output function

$$o(s_0, 0) = \varepsilon, \quad o(s_0, 1) = 11, \quad o(s_1, 0) = 0, \quad o(s_1, 1) = 10. \qquad \square$$

If $T = (S, s_0, t, o)$ is a transducer, $s \in S$ is a state, and $\sigma_1 \sigma_2 \cdots$ is a binary sequence (finite or infinite), then the corresponding **sequence of states** $\{s_n\}$ is defined recursively by $s_1 = s$ and

$$s_{n+1} = t(s_n, \sigma_n)$$

for all $n \geq 1$. We define

$$t(s, \sigma_1 \cdots \sigma_n) = s_{n+1} \quad \text{and} \quad o(s, \sigma_1 \cdots \sigma_n) = o(s_1, \sigma_1) \cdots o(s_n, \sigma_n)$$

for each n, where the product on the right is a concatenation of binary sequences. This extends the functions t and o to functions

$$t \colon S \times \{0, 1\}^* \to S \quad \text{and} \quad o \colon S \times \{0, 1\}^* \to \{0, 1\}^*.$$

If $\sigma_1 \sigma_2 \cdots$ is an infinite binary sequence, we can also define

$$o(s, \sigma_1 \sigma_2 \cdots) = o(s_1, \sigma_1)\, o(s_2, \sigma_2) \cdots$$

where the product on the right is an infinite concatenation of binary sequences.

Definition 5. A homeomorphism $f\colon \{0,1\}^\omega \to \{0,1\}^\omega$ is **rational** if there exists an asynchronous binary transducer (S, s_0, t, o) such that $f(\psi) = o(s_0, \psi)$ for all $\psi \in \{0,1\}^\omega$.

In [9], Grigorchuk, Nekrashevych and Sushchanskiĭ proved that the set $\mathcal{R}$ of all rational homeomorphisms of $\{0,1\}^\omega$ forms a group under composition. This is the **(asynchronous) rational group** $\mathcal{R}$.

Note 6. Though we are restricting ourselves to a binary alphabet, there is a rational group $\mathcal{R}_A$ associated to any finite alphabet A with at least two symbols. It was proven in [9] that the isomorphism type of $\mathcal{R}_A$ does not depend on the size of A, so it suffices to consider only the binary case.

Note 7. A transducer (S, s_0, t, o) is said to be **synchronous** if the image of the output function o is the set $\{0,1\}$ of binary digits. The **synchronous rational group**, defined by Grigorchuk, Nekrashevych, and Sushchanskiĭ in [9], is the subgroup of $\mathcal{R}$ consisting of all homeomorphisms that can be defined by synchronous transducers. This paper is concerned with the larger asynchronous group, where the output function o takes values in the set $\{0,1\}^*$ of finite binary sequences.

Grigorchuk, Nekrashevych, and Sushchanskiĭ also gave a useful test for determining whether a homeomorphism is rational. If $\alpha \in \{0,1\}^*$ is any finite binary sequence, let I_α denote the subset of $\{0,1\}^\omega$ consisting of all sequences that start with α. Note then that $\{I_\alpha \mid \alpha \in \{0,1\}^*\}$ is a basis of clopen sets for the topology on $\{0,1\}^\omega$.

For a homeomorphism $f\colon \{0,1\}^\omega \to \{0,1\}^\omega$ and $\alpha \in \{0,1\}^*$, if β is the greatest common prefix of $f(I_\alpha)$ then define the **restriction** of f to α to be the unique map $f|_\alpha\colon \{0,1\}^\omega \to \{0,1\}^\omega$ such that

$$f(\alpha\gamma) = \beta f|_\alpha(\gamma)$$

for all $\gamma \in \{0,1\}^\omega$.

Theorem 8. *Let $f\colon \{0,1\}^\omega \to \{0,1\}^\omega$ be a homeomorphism. Then f is rational if and only if f has only finitely many different restrictions.*

Proof. This follows from [9, Theorem 2.5], since a homeomorphism cannot have any empty restrictions. $\qquad\square$

2. Simplicity

In [7], Epstein introduced a general framework for proving that a group G of homeomorphisms is simple. The first step is to use some variant of Epstein's commutator trick to prove that the commutator

subgroup $[G, G]$ is simple, and then one must give an independent proof that $G = [G, G]$.

We start by observing a few important properties of $\mathcal{R}$.

Definition 9. Let G be a group of homeomorphisms of a topological space X.

(1) We say that a homeomorphism h of X **locally agrees with G** if for every point $p \in X$, there exists a neighborhood U of p and a $g \in G$ such that $h|_U = g|_U$.

(2) We say that G is **full** if every homeomorphism of X that locally agrees with G belongs to G.

That is, G is full if one can determine whether a homeomorphism h lies in G by inspecting the germs of h. The word "full" here comes from the theory of étale groupoids, where G is full if and only if G is the "full group" of the étale groupoid consisting of all germs of elements of G.

Examples of full groups include the full homeomorphism group of any topological space, the full group of diffeomorphisms of any differentiable manifold, and the Thompson groups F, T, and V acting on the interval, the circle, and the Cantor set, respectively. Other Thompson-like groups such as Röver's group $V\mathcal{G}$ (see [13]) and Brin's higher-dimensional Thompson groups nV (see [5]) are also full.

Proposition 10. *Let G be a group of homeomorphisms of the Cantor set $\{0, 1\}^\omega$, and let h be a homeomorphism of $\{0, 1\}^\omega$. Then h locally belongs to G if and only if there exists a partition*

$$\{0, 1\}^\omega = I_{\alpha_1} \uplus \cdots \uplus I_{\alpha_n}$$

and elements $g_1, \ldots, g_n \in G$ such that h agrees with g_i on each I_{α_i}.

Proof. Clearly h locally belongs to G if it satisfies the given condition. For the converse, suppose that h locally belongs to G. Since the I_α form a basis for the topology on $\{0, 1\}^\omega$ and $\{0, 1\}^\omega$ is compact, there exists a finite cover $\{I_{\alpha_1}, \ldots, I_{\alpha_n}\}$ of $\{0, 1\}^\omega$ and elements $g_1, \ldots, g_n \in G$ so that h agrees with g_i on each I_{α_i}. Since any two I_{α_i} are either disjoint or one is contained in the other, we may assume that the cover $\{I_{\alpha_1}, \ldots, I_{\alpha_n}\}$ is a partition of $\{0, 1\}^\omega$. $\square$

Proposition 11. *The rational group $\mathcal{R}$ is full.*

Proof. Let h be a homeomorphism of $\{0, 1\}^\omega$, and suppose that h locally belongs to $\mathcal{R}$. By Proposition 10, there exists a partition

$$\{0, 1\}^\omega = I_{\alpha_1} \uplus \cdots \uplus I_{\alpha_n}$$

and elements $g_1, \ldots, g_n \in \mathcal{R}$ such that h agrees with g_i on each I_{α_i}. Then $h|_\alpha = g_i|_\alpha$ whenever α_i is a prefix of α, so all but finitely many restrictions of h are also restrictions of some g_i. Since each g_i is rational, each g_i has finitely many different restrictions by Theorem 8, and therefore h has finitely many different restrictions as well. Then $h \in \mathcal{R}$ by Theorem 8. $\square$

We also require a certain transitivity property.

Definition 12. Let G be a group of homeomorphisms of the Cantor set. We say that G is **flexible** if for every pair E_1, E_2 of proper, nonempty clopen subsets of X, there exists a $g \in G$ so that $g(E_1) \subseteq E_2$.

Examples of flexible groups include the full homeomorphism group of the Cantor set, Thompson's groups T and V, and many other Thompson-like groups, such as Röver's group $V\mathcal{G}$ and Brin's groups nV.

Proposition 13. *The rational group $\mathcal{R}$ is flexible.*

Proof. As observed in [9], $\mathcal{R}$ contains Thompson's group V, and since V is flexible $\mathcal{R}$ must be flexible as well. $\square$

Next we need some notation. Given two homeomorphisms $f, g \in \mathcal{R}$, let (f, g) denote the homeomorphism

$$(f, g)(\omega) = \begin{cases} 0 \, f(\zeta) & \text{if } \omega = 0\zeta, \\ 1 \, g(\zeta) & \text{if } \omega = 1\zeta. \end{cases}$$

Note that $(f, g)|_{0\alpha} = f|_\alpha$ and $(f, g)|_{1\alpha} = g|_\alpha$ for all $\alpha \in \{0, 1\}^*$, so by Theorem 8 (f, g) is rational.

Note also that $f = (f|_0, f|_1)$ for any $f \in \mathcal{R}$ such that $f(I_0) = I_0$ and $f(I_1) = I_1$. In particular, any element of $\mathcal{R}$ that is the identity on I_1 can be written as $(g, 1)$ for some $g \in \mathcal{R}$, and similarly any element of $\mathcal{R}$ that is the identity on I_0 can be written as $(1, g)$ for some $g \in \mathcal{R}$.

The following lemma lets us define the element

$$(f, (f, (f, (f, \ldots))))$$

for a given $f \in \mathcal{R}$.

Lemma 14. *Let $f \in \mathcal{R}$. Then there exists a $g \in \mathcal{R}$ such that $g = (f, g)$.*

Proof. Let (S, s_0, t, o) be a transducer for f, and consider the transducer (S', s_0', t', o') defined as follows:

    (1) S' is obtained from S by adding a new state s_0'.

    (2) t' agrees with t on $S \times \{0, 1\}$, and satisfies $t(s_0', 0) = s_0$ and $t(s_0', 1) = s_0'$.

    (3) o' agrees with o on $S \times \{0, 1\}$, and satisfies $o(s_0', 0) = 0$ and $o(s_0', 1) = 1$.

It is easy to check that (S', s_0', t', o') defines a rational homeomorphism g of $\{0, 1\}^\omega$, and that $g = (f, g)$. $\qquad\square$

We say that an element $f \in \mathcal{R}$ has **small support** if there exists a proper, nonempty clopen subset E of $\{0, 1\}^\omega$ such that f is the identity on the complement of E. In this case, we say that f is **supported** on E.

Lemma 15. *Every element of $\mathcal{R}$ with small support is a commutator in $\mathcal{R}$.*

Proof. Let $f \in \mathcal{R}$ have small support. Since $\mathcal{R}$ is flexible by Proposition 13, we can conjugate f by an element of $\mathcal{R}$ so that it is supported on I_{01}. Then

$$f = \big((1, g), 1\big)$$

for some $g \in \mathcal{R}$. By Lemma 14, there exists an $h \in \mathcal{R}$ that satisfies the equation

$$h = (g, h).$$

Let $k = (1, h)$, and let $x_0 \in \mathcal{R}$ be the first generator for Thompson's group F, i.e. the homeomorphism $x_0 \colon \{0, 1\}^\omega \to \{0, 1\}^\omega$ satisfying

$$x_0(00\zeta) = 0\zeta, \qquad x_0(01\zeta) = 10\zeta, \qquad x_0(1\zeta) = 11\zeta$$

for all $\zeta \in \{0, 1\}^\omega$. (A trandsucer for x_0 is shown in Figure 2.) Then

$$x_0^{-1} k x_0 = x_0^{-1}(1, h)x_0 = x_0^{-1}\big(1, (g, h)\big)x_0 = \big((1, g), h\big) = fk$$

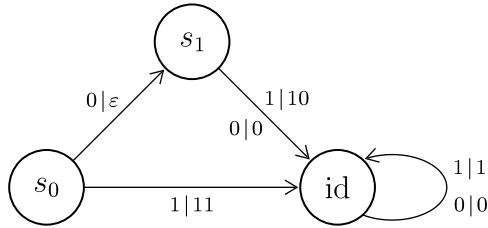and therefore $f = x_0^{-1} k x_0 k^{-1}$. $\qquad\square$

FIGURE 2. The transducer for the element x_0.

Proposition 16. *The elements of small support generate $\mathcal{R}$, and therefore $\mathcal{R} = [\mathcal{R}, \mathcal{R}]$.*

Proof. Let f be any non-identity element in $\mathcal{R}$. Then there exists a nonempty clopen set $E \subseteq \{0, 1\}^{\omega}$ such that $f(E)$ is disjoint from E and $E \cup f(E)$ is not the whole Cantor set. Let g be the homeomorphism of $\{0, 1\}^{\omega}$ that agrees with f on E, agrees with f^{-1} on $f(E)$, and is the identity elsewhere. Since $\mathcal{R}$ is full by Proposition 11, we know that $g \in \mathcal{R}$. Then g is supported on $E \cup f(E)$, and gf is the identity on E, so g and gf have small support. Then $f = g^{-1}(gf)$ is a product of elements of small support, and is therefore in $[\mathcal{R}, \mathcal{R}]$ by Lemma 15.   □

Note 17. In fact, the proof of Proposition 16 shows that every element of $\mathcal{R}$ is a product of at most two commutators. That is, the **commutator width** of $\mathcal{R}$ is at most two. It is an open question whether every element of $\mathcal{R}$ is in fact a commutator.   □

Theorem 18. *$\mathcal{R}$ is simple.*

Proof. Let N be a nontrivial normal subgroup of $\mathcal{R}$, and let f_0 be a nontrivial element of N. Then there exists a nonempty clopen set $E \subseteq \{0, 1\}^{\omega}$ such that $f_0(E)$ is disjoint from E. Since $\mathcal{R}$ is flexible by Proposition 13, there exists a $c \in \mathcal{R}$ so that $c(I_0) \subseteq E$. Then $f_1 = c^{-1} f_0 c$ is an element of N and has the property that $f_1(I_0)$ is disjoint from I_0.

Let $g, h \in \mathcal{R}$. Then $f_1(g, 1)^{-1} f_1^{-1}$ is supported on $f_1(I_0)$, so the element
$$g' = (g, 1) f_1(g, 1)^{-1} f_1^{-1}$$
of N agrees with $(g, 1)$ on I_0. It follows that $[g', (h, 1)] = ([g, h], 1)$, so the latter is in N.

Since $[\mathcal{R}, \mathcal{R}] = \mathcal{R}$ by Proposition 16, it follows that $(k, 1) \in N$ for all $k \in \mathcal{R}$. Since $\mathcal{R}$ is flexible, we can conjugate by elements of $\mathcal{R}$ to deduce that every element of $\mathcal{R}$ with small support lies in N. But such elements generate $\mathcal{R}$, and therefore $N = \mathcal{R}$.   □

Note that the argument in Lemma 15, Proposition 16, and Theorem 18 applies to many other groups as well. Indeed, we have proven the following.

Theorem 19. *Let G be a full, flexible group of homeomorphisms of the Cantor set. Suppose that:*
    (1) *For all $g \in G$ both $(1, g)$ and $(g, 1)$ lie in G, and every element of G supported on I_0 or I_1 has this form,*
    (2) *For all $g \in G$ there exists an $h \in G$ so that $h = (g, h)$, and*

(3) *The first generator x_0 of Thompson's group F lies in G.*

Then G is simple. □

Groups to which this theorem applies include:

- The group of all homeomorphisms of the Cantor set (see [1]).
- The group of all bilipschitz homeomorphisms of the Cantor set.
- The group of all bilipschitz elements of $\mathcal{R}$.

Note that the group of bilipschitz elements of $\mathcal{R}$ is a proper subgroup of $\mathcal{R}$, since for example it does not contain the homeomorphism whose transducer is shown in Figure 1.

Incidentally, an alternative to the proof of Theorem 18 above is the following remarkable theorem:

Matui's Theorem. *Let G be a full, flexible group of homeomorphisms of the Cantor set. Then $[G, G]$ is simple.*

Proof. See [11, Theorem 4.16]. The word "flexible" does not appear in Matui's work, but Matui does prove in [11, Proposition 4.11] that an essentially principal étale groupoid is purely infinite and minimal if and only if the associated full group is flexible as defined above. Technically, Matui assumes that the étale groupoid of germs is Hausdorff, but nothing in his proof requires this condition. □

In addition to the groups considered above, this theorem applies to many groups of interest, including the full homeomorphism group of the Cantor set, Thompson's groups V as well as the generalized groups $V_{n,r}$ (not all of which are simple), Röver's group $V\mathcal{G}$ (see [13]), and Brin's higher-dimensional Thompson groups nV (see [5]).

3. Finite Generation

For each prime p, let f_p be the element of $\mathcal{R}$ defined by the transducer shown in Figure 3. This homeomorphism switches every p'th digit of a binary sequence, leaving the remaining digits unchanged. The goal of this section is to prove the following theorem.

Theorem 20. *Let $\mathcal{M}$ be any submonoid of $\mathcal{R}$ that contains f_p for infinitely many primes p. Then $\mathcal{M}$ is not finitely generated.*

Since $\mathcal{R}$ is itself such a submonoid, it follows from this theorem that $\mathcal{R}$ is not finitely generated. (Note that a group is finitely generated if and only if it is finitely generated as a monoid.) This also proves that various other subgroups of $\mathcal{R}$ are not finitely generated, such as:

(1) The subgroup of $\mathcal{R}$ generated by Thompson's group V and all of the f_p.
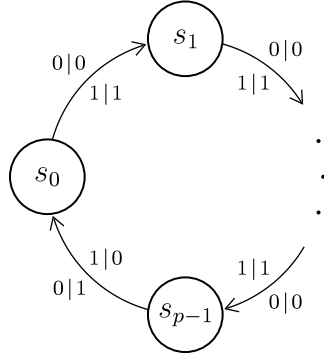
FIGURE 3. The transducer for f_p.

(2) The subgroup of all measure-preserving elements of $\mathcal{R}$.
(3) The subgroup of $\mathcal{R}$ generated by Thompson's group V and all synchronous rational functions.
(4) The subgroup of all bilipschitz elements of $\mathcal{R}$.

Note that (3) is a proper subgroup of (4). For example, the element $f \in \mathcal{R}$ satisfying $f = (x_0, f)$ (where x_0 is the element shown in Figure 2) is bilipschitz but does not lie in (3).

We now turn to the proof of Theorem 20. Given a binary asynchronous transducer $T = (S, s_0, t, o)$, recall that a state $s \in S$ is **accessible** if there exists an $\alpha \in \{0, 1\}^*$ such that

$$t(s_0, \alpha) = s.$$

A **cycle** in T is an ordered pair (c, γ), where $c \in S$ and $\gamma \in \{0, 1\}^*$ is a nonempty binary sequence satisfying

$$t(c, \gamma) = c.$$

Such a cycle corresponds to a directed cycle of edges in the state diagram for T. The length $|\gamma|$ of γ is called the **length** of the cycle. A cycle (c, γ) is **accessible** if c is an accessible state.

Definition 21. Let p be a prime.
(1) We say that a transducer T is **oblivious** to p if there exists an accessible cycle in T whose length is not a multiple of p.
(2) We say that a rational homeomorphism $f \in \mathcal{R}$ is **oblivious** to p if there exists a transducer for f that is oblivious to p.

Note that any transducer with fewer than p states is automatically oblivious to p.

Lemma 22. *If p is prime, then f_p is not oblivious to p.*

Proof. Let $T = (S, s_0, t, o)$ be any transducer for f_p, and let (c, γ) be an accessible cycle for T. Since c is accessible, there exists an $\alpha \in \{0, 1\}^*$ so that $t(s_0, \alpha) = c$. Fix a $\lambda \in \{0, 1\}^*$ so that $|\alpha| + |\gamma| + |\lambda|$ is a multiple of p, and let $\rho = 0^{p-1}1$. Then the infinite binary sequence

$$\alpha\gamma\lambda\rho^\omega$$

eventually has 1's in positions that are multiples of p, so $f_p(\alpha\gamma\lambda\rho^\omega)$ ends in an infinite sequence of 0's. In particular,

$$f_p(\alpha\gamma\lambda\rho^\omega) = \beta\delta\mu 0^\omega$$

where $\beta = o(s_0, \alpha)$, $\delta = o(c, \gamma)$, and $o(c, \lambda\rho^\omega) = \mu 0^\omega$ for some finite binary sequence $\mu \in \{0, 1\}^*$. If we now eliminate the trip around the cycle, we see that

$$f_p(\alpha\lambda\rho^\omega) = \beta\mu 0^\omega.$$

This ends with an infinite sequence of 0's, so $\alpha\lambda\rho^\omega$ must eventually have 1's in positions that are multiples of p. Then $|\alpha| + |\lambda|$ must be a multiple of p, and therefore $|\gamma|$ is a multiple of p. Since the cycle (c, γ) was arbitrary, we conclude that T is not oblivious to p. Since T was arbitrary, it follows that f_p is not oblivious to p. $\square$

Lemma 23. *Let p be a prime and let $f, f' \in \mathcal{R}$. If f is oblivious to p and f' has a transducer with fewer than p states, then $f'f$ is oblivious to p.*

Proof. Let $T = (S, s_0, t, o)$ and $T' = (S', s'_0, t', o')$ be transducers for f and f', where T is oblivious to p and T' has fewer than p states. Then there exists a transducer T'' for $f'f$ with state set $S \times S'$ and initial state (s_0, s'_0) whose transition and output functions t'', o'' satisfy

$$t''\big((s, s'), \alpha\big) = \big(t(s, \alpha), t'(s', \beta)\big) \qquad \text{and} \qquad o''\big((s, s'), \alpha\big) = o'(s', \beta)$$

for all $\alpha \in \{0, 1\}^*$, where $\beta = o(s, \alpha)$.

Now, since T is oblivious to p, there exists an accessible cycle (c, γ) for T whose length is not a multiple of p. Since c is accessible, there exists an $\alpha \in \{0, 1\}^*$ such that $t(s_0, \alpha) = c$. Then

$$t(s_0, \alpha\gamma^n) = c$$

for all $n \geq 0$. Let $\beta = o(s_0, \alpha)$ and $\delta = o(c, \gamma)$, so

$$o(s_0, \alpha\gamma^n) = \beta\delta^n$$

for all $n \geq 0$. Since T' has fewer than p states, by the pigeonhole principle there exist numbers j, k with $0 \leq j < k < p$ such that

$$t'(s'_0, \beta\delta^j) = t'(s'_0, \beta\delta^k).$$

Let $c' = t'(s_0', \beta\delta^j)$, and observe that

$$t'(c', \delta^{k-j}) = c'.$$

Then (c, c') is an accessible state for T'' since

$$t''\big((s_0, s_0'), \alpha\gamma^j\big) = \big(t(s_0, \alpha\gamma^j), t'(s_0', \beta\delta^j)\big) = (c, c').$$

Moreover

$$t''\big((c, c'), \gamma^{k-j}\big) = \big(t(c, \gamma^{k-j}), t'(c', \delta^{k-j})\big) = (c, c')$$

so $\big((c, c'), \gamma^{k-j}\big)$ is an accessible cycle in T''. But neither $|\gamma|$ nor $k - j$ is a multiple of p, so the length $|\gamma^{k-j}| = (k - j)\,|\gamma|$ of this cycle is not a multiple of p. We conclude that T'' is oblivious to p, so $f'f$ is oblivious to p. $\qquad\square$

Proof of Theorem 20. Let $\mathcal{M}$ be a submonoid of $\mathcal{R}$ that contains infinitely many f_p. For each $n \in \mathbb{N}$, let $\mathcal{M}_{\leq n}$ be the submonoid of $\mathcal{M}$ generated by all elements of $\mathcal{M}$ that can be represented by transducers with n or fewer elements. This gives us an ascending sequence of monoids

$$\mathcal{M}_{\leq 1} \subseteq \mathcal{M}_{\leq 2} \subseteq \mathcal{M}_{\leq 3} \subseteq \cdots$$

with $\bigcup_{n\in\mathbb{N}} \mathcal{M}_{\leq n} = \mathcal{M}$. If p is prime, then by Lemma 23 every element of $\mathcal{M}_{\leq p-1}$ is oblivious to p. By Lemma 22, it follows that $f_p \notin \mathcal{M}_{\leq p-1}$ for each p, and therefore $\mathcal{M}$ is an ascending union of proper submonoids. $\qquad\square$

Note 24. Although our proof uses submonoids of $\mathcal{R}$, there is a slight modification that gives an ascending sequence of proper subgroups. Specifically, for each $n \in \mathbb{N}$ let $\mathcal{R}_{\leq n}$ be the subgroup generated by all $f \in \mathcal{R}$ for which both f and f^{-1} can be represented by transducers with n or fewer states. Clearly

$$\mathcal{R}_{\leq 1} \subseteq \mathcal{R}_{\leq 2} \subseteq \mathcal{R}_{\leq 3} \subseteq \cdots$$

and $\bigcup_{n\in\mathbb{N}} \mathcal{R}_{\leq n} = \mathcal{R}$. But if p is prime then every element of $\mathcal{R}_{\leq p-1}$ is oblivious to p. Since $f_p \in \mathcal{R}_{\leq p}$ for each p (with $f_p^{-1} = f_p$) it follows that $\mathcal{R}_{\leq p-1}$ is properly contained in $\mathcal{R}_{\leq p}$ for each prime p. $\qquad\square$

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