Structure Theorems for Groups of Homeomorphisms of the Circle

Collin Bleak   Martin Kassabov *   Francesco Matucci†

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Abstract

In this partly expository paper, we study the set $\mathcal{A}$ of groups of orientation-preserving homeomorphisms of the circle S^1 which do not admit non-abelian free subgroups. We use classical results about homeomorphisms of the circle and elementary dynamical methods to derive various new and old results about the groups in $\mathcal{A}$. Of the known results, we include some results from a family of results of Beklaryan and Malyutin, and we also give a new proof of a theorem of Margulis. Our primary new results include a detailed classification of the solvable subgroups of R. Thompson's group T.

1 Introduction

In this paper we explore properties of groups of orientation preserving homeomorphisms of the circle S^1. In particular, we use a close analysis of Poincaré's rotation number, together with some elementary dynamical/analytical methods, to prove "alternative" theorems in the tradition of the Tits' Alternative. Our main result, Theorem 1.1, states that any group of orientation preserving homeomorphisms of the circle is either abelian or is a subgroup of a wreath product whose factors can be described in considerable detail. Our methods and the resulting Theorem 1.1 give us sufficient information to derive a short proof of Margulis' Theorem on the existence of an invariant probability measure on the circle in [21], and to classify the solvable subgroups of the group of orientation-preserving piecewise-linear homeomorphisms of the circle, and of its subgroup R. Thompson's group T.

Suppose G is a group of orientation preserving homeomorphisms of S^1. If one replaces the assumption in Theorem 1.1 that G has no non-abelian free

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subgroups with the assumption that G is a group for which the rotation number map $\mathrm{Rot} : G \to \mathbb{R}$ is a homomorphism, then many of the results within the statement of Theorem 1.1 can be found in one form or another in the related works of Beklaryan [1, 2, 3, 4, 5, 6]. However, the structure of the extension described by Theorem 1.1 is new.

A major stepping stone in the established theory of groups of homeomorphisms of the circle is the following statement (Lemma 1.8 below). *For groups of orientation-preserving homeomorphisms of the circle which do not admit non-abelian free subgroups, the rotation number map is a homomorphism.* As alluded in the next paragraph, we believe that the first proof of the statement comes as a result of combining a theorem of Beklaryan [3] with Margulis' Theorem in [21].

Although we arrived at Lemma 1.8 independently, our approach to its proof mirrors that of Solodov from his paper [28], which also states a version of the lemma as his Theorem 2.6. However, in Solodov's proof of his necessary Lemma 2.4, he uses a construction for an element with non-zero rotation number that does not actually guarantee that the rotation number is not zero (see Appendix A). Our own technical Lemmas 3.8 and 3.9 provide sufficient control to create such an element, and the rest of the approach goes through unhindered.

As also shown by Beklaryan, the results within Theorem 1.1 can be employed to prove Margulis' Theorem. Our proof of Theorem 1.1 (including Lemmas 3.8 and 3.9) and the consequential proof of Margulis' Theorem, both use only classical methods.

Due to the large intersection with known work and results, portions of this paper should be considered as expository. Many of the proofs we give are new, taking advantage of our technical Lemma 3.9. This lemma may have other applications as well. Further portions of this project, which trace out some new proofs of other well-known results, are given in the third author's dissertation [22].

We would like to draw the reader's attention to the surveys by Ghys [15] and Beklaryan [6] on groups of homeomorphisms of the circle, and to the book by Navas [24] on groups of diffeomorphisms of the circle, as three guiding works which can lead the reader further into the theory.

Statement and discussion of the main results

We use much of the remainder of the introduction to state and briefly discuss our primary results. Except for Lemma 1.8, Theorem 1.10 and parts of Theorem 1.1, our results are new.

1.1 The main structure theorem

Denote by $\mathrm{Homeo}_+(S^1)$ the maximal subgroup of $\mathrm{Homeo}(S^1)$ consisting of orientation preserving homeomorphisms of S^1 and let $\mathrm{Rot} : \mathrm{Homeo}_+(S^1) \to \mathbb{R}/\mathbb{Z}$ denote Poincaré's rotation number function. Although this function is not a homomorphism, we will denote by $\ker(\mathrm{Rot})$ its "kernel", i.e., the set of elements with rotation number equal to zero. Similarly, denote by $\mathrm{Homeo}_+(I)$ the maximal group of orientation-preserving homeomorphisms of the unit interval.

In order to state our first result, we note that by Lemma 1.8 the restriction of Rot to any subgroup of $\mathrm{Homeo}_+(S^1)$ which has no non-abelian free subgroups turns Rot into a homomorphism of groups. Also, note that throughout this article we use the expressions $C \wr T \simeq \left(\prod_{t\in T} C\right) \rtimes T$ and $C \wr_r T \simeq \left(\bigoplus_{t\in T} C\right) \rtimes T$ respectively to denote the unrestricted and restricted standard wreath products of groups C and T.

Theorem 1.1. *Let $G \in \mathrm{Homeo}_+(S^1)$, with no non-abelian free subgroups. Either G is abelian or there are subgroups H_0 and Q of $\mathrm{Homeo}_+(S^1)$, such that*

$$G \hookrightarrow H_0 \wr Q$$

where the embedding is such that the following hold.

1. *The group H_0 has the following properties.*
   (a) *Rot is trivial over H_0.*
   (b) *There is an embedding $H_0 \hookrightarrow \prod_{\mathfrak{N}} \mathrm{Homeo}_+(I)$, where $\mathfrak{N}$ is an index set which is at most countable.*
   (c) *The group H_0 has no non-abelian free subgroups.*
2. *The group $Q \cong G/\left(\ker(\mathrm{Rot}) \cap G\right)$ is isomorphic to a subgroup of $\mathbb{R}/\mathbb{Z}$, which is at most countable.*
3. *The subgroups $H_0, Q \leq \mathrm{Homeo}_+(S^1)$ generate a subgroup isomorphic to the restricted wreath product $H_0 \wr_r Q$. This subgroup can be "extended" to an embedding of the unrestricted wreath product into $\mathrm{Homeo}_+(S^1)$ where the embedded extension contains G.*

Remark 1.2. We note that if the kernel of the homomorphism Rot is trivial over G then G embeds in a pure group of rotations and so is abelian.

As mentioned in the introduction, most of points one and two above can be extracted from the results of Beklaryan in [2, 3, 4] under the assumption of the existence of a G-invariant probability measure on the circle (a property which Beklaryan shows to be equivalent to $\mathrm{Rot} : G \to \mathbb{R}/\mathbb{Z}$ being a homomorphism in [3]).

Theorem 1.1 attempts to provide an algebraic description of a dynamical picture painted by Ghys in [15]. We will quote a relevant statement below to clarify this comment. First though, we give a description of these same dynamics using the construction of a counter-example to Denjoy's Theorem in the C^1 category (there is a detailed, highly concrete construction of this counter-example in [29], and a detailed discussion of a family of counter-examples along these same lines in section 4.1.4 of [24]).

Denjoy's Theorem states that given a C^2 orientation-preserving circle homeomorphism $f : S^1 \to S^1$ with irrational rotation number α (in some sense, points are moved "on average" the distance α around the circle by f), then there is a homeomorphism $c : S^1 \to S^1$ so that $c \circ f \circ c^{-1}$ is a pure rotation of the circle by α.

We now discuss the counter-example: Take a rotation r of the circle by an irrational α (r is a circle map with real lift map $t \mapsto t + \alpha$, under the projection map $p(t) = e^{2\pi i t}$). The orbit of any point under iteration of this map is dense on the circle. Now, track the orbit of a particular point in the circle. For each point in the orbit, replace the point by an interval with decreasing size (as our index grows in absolute value), so that the resulting space is still homeomorphic to S^1. Now, extend r's action over this new circle so that it becomes a C^1 diffeomorphism $\tilde{r}$ of the circle which agrees with the original map r over points in the original circle, and which is nearly affine while mapping the intervals to each other.[1] The map $\tilde{r}$ still has the same rotation number as r, and cannot be topologically conjugated to a pure rotation because there are points whose orbits are not dense.

Let H_0 be any group of orientation-preserving homeomorphisms of the interval. Pick an element of H_0 to act on one of the "inserted" intervals above, and further elements in copies of H_0 (created by conjugating the original action of H_0 by powers of $\tilde{r}$) to act on the other "inserted" intervals. We have just constructed an element of $H_0 \wr \mathbb{Z}$, acting on (a scaled up version) of S^1.

While providing a useful picture, the above explanation does not really capture the full dynamical picture implied by Theorem 1.1; the group G may be any subgroup of the appropriate wreath product, so elements of the top group in the wreath product may not be available in G. Further, based on possible categorical restrictions on the group G, other restrictions on the wreath product may come into play.

Now let us relate this picture to Ghys' discussion in [15]. In a sentence near the end of the final paragraph of section 5 in [15], Ghys states the following.

> *... we deduce that [G] contains a non abelian free subgroup unless the restriction of the action of [G] to the exceptional minimal set is abelian and is semi-conjugate to a group of rotations ...*

Here, the complement of the exceptional minimal set of the action of G contains the region where our base group acts, and the top group acts essentially as (is semi-conjugate to) a group of rotations on the resultant circle which arises after "gluing together" the exceptional minimal set (using the induced cyclic ordering from the original circle).

[1] This can be done if the interval lengths are chosen carefully. However, by Denjoy's Theorem, no matter how one chooses lengths and the extension $\tilde{r}$, the result will fail to be a C^2 diffeomorphism of the circle.

1.2 Some embedding theorems

The theorems in this subsection follow by combining the results (see [7, 9, 23]) of the first author or of Navas on groups of piecewise-linear homeomorphisms of the unit interval together with Theorem 1.1.

Throughout this article, we will use $\mathrm{PL}_+(I)$ and $\mathrm{PL}_+(S^1)$ to represent the piecewise-linear orientation-preserving homeomorphisms of the unit interval $I := [0,1]$ and of the circle S^1, respectively.

In order to state our embedding results and to trace them as consequences of Theorem 1.1, we need to give some definitions and results from [7, 9]. Let $G_0 = 1$ and, for $n \in \mathbb{N}$, inductively define G_n as the direct sum of a countably infinite collection of copies of the group $G_{n-1} \wr_r \mathbb{Z}$:

$$G_n := \bigoplus_{\mathbb{Z}} (G_{n-1} \wr_r \mathbb{Z}) \,.$$

A result in [7] states that if H is a solvable group with derived length n, then H embeds in $\mathrm{PL}_+(I)$ if and only if H embeds in G_n. Using Theorem 1.1 and Remark 5.1 (see section 5), we are able to extend this result to subgroups of $\mathrm{PL}_+(S^1)$:

Theorem 1.3. *Suppose H is a solvable group with derived length n. The group H embeds in $\mathrm{PL}_+(S^1)$ if and only if one of the following holds,*

1. *H embeds in $\mathbb{R}/\mathbb{Z}$,*
2. *H embeds in G_n, or*
3. *H embeds in $G_{n-1} \wr_r K$ for some nontrivial subgroup K of $\mathbb{Q}/\mathbb{Z}$.*

The paper [9] also gives a non-solvability criterion for subgroups of $\mathrm{PL}_+(I)$. Let $W_0 = 1$ and, for $n \in \mathbb{N}$, we define $W_i = W_{i-1} \wr_r \mathbb{Z}$. Build the group

$$W := \bigoplus_{i \in \mathbb{N}} W_i.$$

The main result of [9] is that a subgroup $H \leq \mathrm{PL}_+(I)$ is non-solvable if and only if W embeds in H. Now again by using Theorem 1.1, we are able to give a Tits' Alternative type of theorem for subgroups of $\mathrm{PL}_+(S^1)$:

Theorem 1.4. *A subgroup $H \leq \mathrm{PL}_+(S^1)$ either*

1. *contains a non-abelian free subgroup on two generators, or*
2. *contains a copy of W, or*
3. *is solvable.*

As may be clear from the discussion of the counterexample to Denjoy's Theorem, it is not hard to produce various required wreath products as groups of homeomorphisms of the circle.

Theorem 1.5. *For every countable subgroup K of $\mathbb{R}/\mathbb{Z}$ and for every $H_0 \leq \mathrm{Homeo}_+(I)$ there is an embedding $H_0 \wr K \hookrightarrow \mathrm{Homeo}_+(S^1)$.*

We recall the R. Thompson groups F and T. These are groups of homeomorphisms of the interval I and of the circle $\mathbb{R}/\mathbb{Z}$ respectively. In particular, they are the groups one obtains if one restricts the groups of orientation preserving homeomorphisms of these spaces to the piecewise-linear category, and insist that these piecewise linear elements (1) have all slopes as integral powers of two, (2) have all changes in slope occur at dyadic rationals, and (3) map the dyadic rationals to themselves.

Theorem 1.6. *For every $K \leq \mathbb{Q}/\mathbb{Z}$ there is an embedding $F \wr_r K \hookrightarrow T$, where F and T are the R. Thompson groups above.*

More generally, we have the following similar theorem.

Theorem 1.7. *For every $K \leq \mathbb{Q}/\mathbb{Z}$ there is an embedding $\mathrm{PL}_+(I) \wr_r K \hookrightarrow \mathrm{PL}_+(S^1)$.*

1.3 Useful Lemmas

Our proof of the following lemma sets the foundation upon which the other results in this article are built. As mentioned in the introduction, the standing proof of Lemma 1.8 is to quote Theorem 6.7 of [3], together with Margulis' Theorem (Theorem 1.10 below).

Lemma 1.8 (Beklaryan and Margulis, [3, 21])**.** *Let $G \leq \mathrm{Homeo}_+(S^1)$. Then the following alternative holds:*

1. *G has a non-abelian free subgroup, or*
2. *the map $\mathrm{Rot} : G \to (\mathbb{R}/\mathbb{Z}, +)$ is a group homomorphism.*

The heart of the proof of Lemma 1.8 is contained in the following lemma, which itself is proven using only on classical results (Poincare's Lemma and the Ping-pong Lemma). We mention the lemma below in this section as it provides a useful new technical tool.

In the statement below, if G is a group of homeomorphisms of the circle, and $g \in G$, then $\mathrm{Fix}(g)$ is the set of points of the circle which are fixed by the action of g and $G_0 = \{g \in G \mid \mathrm{Fix}(g) \neq \emptyset\}$.

Lemma 1.9 (Finite Intersection Property)**.** *Let $G \leq \mathrm{Homeo}_+(S^1)$ with no non-abelian free subgroups. The family $\{\mathrm{Fix}(g) \mid g \in G_0\}$ satisfies the finite intersection property, i.e., for all n-tuples $g_1, \ldots, g_n \in G_0$, we have $\mathrm{Fix}(g_1) \cap \ldots \cap \mathrm{Fix}(g_n) \neq \emptyset$.*

Another view of the above lemma is the following "generalization" of the Ping-pong lemma: let X be a collection of homeomorphisms of the circle such that

1. for all $g \in X$, $\mathrm{Fix}(g) \neq \emptyset$, and
2. for all $x \in S^1$ there is some $g \in X$ with $g(x) \neq x$,

then $\langle X \rangle$ contains embedded non-abelian free groups.

1.4 Some further applications

As mentioned above, our proof of Lemma 1.8 uses only elementary methods and classical results. Margulis' Theorem follows very simply with Lemma 1.8 in hand. We hope our approach provides a valuable new perspective on this theorem.

Theorem 1.10 (Margulis, [21])**.** *Let $G \leq \mathrm{Homeo}_+(S^1)$. Then at least one of the two following statements must be true:*

1. *G has a non-abelian free subgroup, or*
2. *there is a G-invariant probability measure on S^1.*

Finally, we mention a theorem which gives an example of how restricting the category gives added control on the wreath product of the main structure theorem. It may be that the following result is known, but we were not able to find a reference for it. The following application represents the only occasion where we rely upon Denjoy's Theorem.

Theorem 1.11. *Suppose G is a subgroup of $\mathrm{Homeo}_+(S^1)$ so that the elements of G are either*

1. *all piecewise-linear, each admitting at most finitely many breakpoints, or*
2. *all C^1 with bounded variation in the first derivative,*

and suppose there is $g \in G$ with $\mathrm{Rot}(g) \notin \mathbb{Q}/\mathbb{Z}$. Then G is topologically conjugate to a group of rotations (and is thus abelian) or G contains a non-abelian free subgroup.

Organization

The paper is organized as follows: Section 2 recalls the necessary language and tools which will be used in the paper; Section 3 shows that the rotation number map is a homomorphism under certain hypotheses; Section 4 uses the fact that the rotation number map is a homomorphism to prove Margulis' Theorem on invariant measures on the unit circle; Section 5 proves and demonstrates the main structure theorem.

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2 Background and Tools

In this section we collect some known results we will use throughout the paper. We use the symbol S^1 to either represent $\mathbb{R}/\mathbb{Z}$ (in order to have a well defined origin 0) or as the set of points in the complex plane with distance one from the origin, as is convenient. We begin by recalling the definition of rotation number. Given $f \in \mathrm{Homeo}_+(S^1)$, let $F : \mathbb{R} \to \mathbb{R}$ represent a lift of f via the standard covering projection $\exp : \mathbb{R} \to S^1$, defined as $\exp(t) = e^{2\pi i t}$.

Following [25, 26], we define the rotation number of an orientation-preserving homeomorphism of the circle. Consider the limit

$$\lim_{n\to\infty} \frac{F^n(t)}{n} \pmod 1. \tag{1}$$

It is possible to prove that this limit exists and that it is independent of the choice of t used in the above calculation (see [17]). Moreover, such a limit is independent of the choice of lift F, when considered $\pmod 1$.

Definition 2.1 (Rotation number of a function). Given $f \in \mathrm{Homeo}_+(S^1)$ and $F \in \mathrm{Homeo}(\mathbb{R})$ a lift of f, we say that

$$\lim_{n\to\infty} \frac{F^n(t)}{n} \pmod 1 := \mathrm{Rot}(f) \in \mathbb{R}/\mathbb{Z}$$

is the *rotation number* of f.

Definition 2.2. Given $f \in \mathrm{Homeo}_+(S^1)$, we define $\mathrm{Fix}(f)$ to be the set of points that are fixed by f, i.e. $\mathrm{Fix}(f) = \{s \in S^1 \mid f(s) = s\}$. A similar definition is implied for any $F \in \mathrm{Homeo}_+(\mathbb{R})$.

Since the rotation number is independent of the choice of the lift, we will work with a preferred lift of elements and of functions.

Definition 2.3 (The "hat" lift of a point and of a function). For any element $x \in S^1$ we denote by $\widehat{x}$ the lift of x contained in $[0, 1)$. For functions in $\mathrm{Homeo}_+(S^1)$ we distinguish between functions with or without fixed points and we choose a lift that is "closest" to the identity map. If $f \in \mathrm{Homeo}_+(S^1)$ and the fixed point set $\mathrm{Fix}(f) = \emptyset$, we denote by $\widehat{f}$ the lift to $\mathrm{Homeo}_+(\mathbb{R})$ such that $t < \widehat{f}(t) < t+1$ for all $t \in \mathbb{R}$. If $f \in \mathrm{Homeo}_+(S^1)$ and $\mathrm{Fix}(f) \neq \emptyset$, we denote by $\widehat{f}$ the lift to $\mathrm{Homeo}_+(\mathbb{R})$ such that $\mathrm{Fix}(\widehat{f}) \neq \emptyset$. The map $\widehat{f}$ can also be defined as the unique lift such that $0 \leq \lim_{n\to\infty} \frac{\widehat{f}^n(t)}{n} < 1$, for all $t \in \mathbb{R}$.

We will use these definitions for lifts of elements and functions in Lemma 2.4(4) and throughout the proof of Lemma 1.8. If we use this lift to compute the limit defined in (1), the result is always in $[0, 1)$. Proofs of the next three results can be found in [17] and [19].

Lemma 2.4 (Properties of the Rotation Number)**.** *Let $f, g \in \mathrm{Homeo}_+(S^1)$, $G \leq \mathrm{Homeo}_+(S^1)$ and n be a positive integer. Then:*

1. $\mathrm{Rot}(f^g) = \mathrm{Rot}(f)$.
2. $\mathrm{Rot}(f^n) = n \cdot \mathrm{Rot}(f)$.
3. *If G is abelian then the map*

$$\begin{array}{rccc} \mathrm{Rot}: & G & \longrightarrow & \mathbb{R}/\mathbb{Z} \\ & f & \longmapsto & \mathrm{Rot}(f) \end{array}$$

   is a group homomorphism.

4. *If* $\mathrm{Rot}(g) = p/q \pmod 1 \in \mathbb{Q}/\mathbb{Z}$ *and $s \in S^1$ is such that $g^q(s) = s$, then* $\widehat{g}^q(\widehat{s}) = \widehat{s} + p$.

Two of the most important results about the rotation number are stated below:

Theorem 2.5 (Poincaré's Lemma)**.** *Let $f \in \mathrm{Homeo}_+(S^1)$ be a homeomorphism. Then*

1. *f has a periodic orbit of length q if and only if* $\mathrm{Rot}(f) = p/q \pmod 1 \in \mathbb{Q}/\mathbb{Z}$ *and p, q are coprime.*
2. *f has a fixed point if and only if* $\mathrm{Rot}(f) = 0$.

We recall that *Thompson's group T* is the subgroup of elements of $\mathrm{PL}_+(S^1)$ such that for any such element all breakpoints occur at dyadic rational points, all slopes are powers of 2, and dyadic rationals are mapped to themselves. Moreover, recall that the subgroup of T consisting of all elements which fix the origin 0 is one of the standard representations of *Thompson's group F* (for an oft-cited introduction about Thompson's groups, see [12]). Ghys and Sergiescu prove in [16] that all the elements of Thompson's group T have rational rotation number. Liousse in [18] generalizes this result to the family of *Thompson-Stein groups* which are subgroups of $\mathrm{PL}_+(S^1)$ with certain suitable restrictions on rational breakpoints and slopes.

The following is a classical result proved by Fricke and Klein [14] which we will need in the proofs of section 3.

Theorem 2.6 (Ping-pong Lemma)**.** *Let G be a group of permutations on a set X, let g_1, g_2 be elements of G. If X_1 and X_2 are disjoint subsets of X and for all integers $n \neq 0$, $i \neq j$, $g_i^n(X_j) \subseteq X_i$, then g_1, g_2 freely generate the free group F_2 on two generators.*

We use the following theorem only to give an application of our main structure theorem. The version we give below is an expansion of Denjoy's original theorem. An elegant proof of the content of this statement is contained in the paper [19].

Theorem 2.7 (Denjoy [13])**.** *Suppose $f \in \mathrm{Homeo}_+(S^1)$ is piecewise-linear with finitely many breakpoints or is a C^1 homeomorphism whose first derivative has bounded variation. If the rotation number of f is irrational, then f is conjugate (by an element in $\mathrm{Homeo}_+(S^1)$) to a rotation. Moreover, every orbit of f is dense in S^1.*

3 The Rotation Number Map is a Homomorphism

Our main goal for this section is to prove Lemma 1.8, which states that the rotation number map is a homomorphism under certain assumptions. It is not true in general that the rotation number map is a group homomorphism. The example drawn in figure 1 below shows a pair of maps with fixed points (hence with rotation number equal to zero, by Poincaré's Lemma) and such that their product does not fix any point (thus has non-zero rotation number).

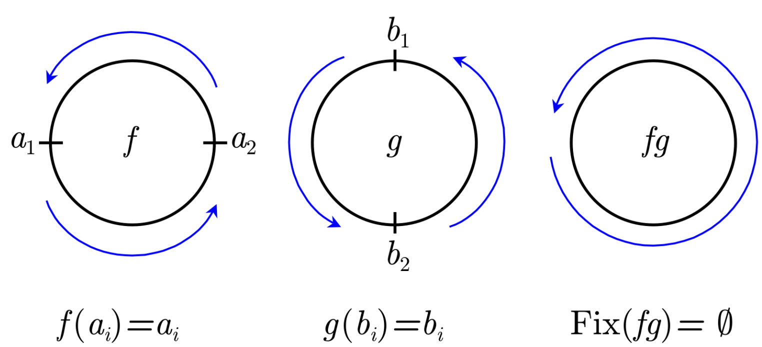

Figure 1: The rotation number map is not a homomorphism in general.

Definition 3.1. We define the (open) *support* of f to be the set of points which are moved by f, i.e., $\mathrm{Supp}(f) = S^1 \setminus \mathrm{Fix}(f)$.[2] A similar definition is implied for any $f \in \mathrm{Homeo}_+(\mathbb{R})$.

[2]Notice that this definition is a bit different from the definition in analysis, where supports are forced to be closed sets.

Our proof divides naturally into several steps. We start by showing how to use the Ping-pong Lemma to create free subgroups. This idea is well known (see for example Lemma 4.3 in [10]), but we give an account of it for completeness.

Lemma 3.2. *Let* $f, g \in \mathrm{Homeo}_+(S^1)$ *such that* $\mathrm{Fix}(f) \neq \emptyset \neq \mathrm{Fix}(g)$. *If the intersection* $\mathrm{Fix}(f) \cap \mathrm{Fix}(g) = \emptyset$*, then* $\langle f, g\rangle$ *contains a non-abelian free subgroup.*

Proof. Let $S^1 \setminus \mathrm{Fix}(f) = \bigcup I_\alpha$ and $S^1 \setminus \mathrm{Fix}(g) = \bigcup J_\beta$, for suitable families of pairwise disjoint open intervals $\{I_\alpha\}, \{J_\beta\}$. We assume $\mathrm{Fix}(f) \cap \mathrm{Fix}(g) = \emptyset$ so that $S^1 \subseteq (\bigcup I_\alpha) \cup (\bigcup J_\beta)$.

Since S^1 is compact, we can write $S^1 = I_1 \cup \ldots \cup I_r \cup J_1 \cup \ldots \cup J_s$, for suitable intervals in the families $\{I_\alpha\}, \{J_\beta\}$. Define $I = I_1 \cup \ldots \cup I_r$ and $J = J_1 \cup \ldots \cup J_s$. We observe that ∂I and ∂J are finite and that, since each $x \in \partial J$ lies in the interior of I, there is an open neighborhood U_x of x such that $U_x \subseteq I$. Let $X_g = \bigcup_{x \in \partial J} U_x$. Similarly we build an open set X_f. The neighborhoods used to build X_f and X_g can be chosen to be small enough so that $X_f \cap X_g = \emptyset$. If $x \in \partial J$, then the sequence $\{f^n(x)\}_{n \in \mathbb{N}}$ accumulates at a point of ∂I and so there is an $n \in \mathbb{N}$ such that $f^n(U_x) \subseteq X_f$. By repeating this process for each $x \in \partial J$ and $y \in \partial I$, we find an N big enough so that for all $m \geq N$ we have

$$f^m(X_g) \cup f^{-m}(X_g) \subseteq X_f, \quad g^m(X_f) \cup g^{-m}(X_f) \subseteq X_g.$$

If we define $g_1 = f^N, g_2 = g^N, X_1 = X_f, X_2 = X_g$, we satisfy the hypothesis of Theorem 2.6 since both of the elements g_1, g_2 have infinite order. Thus $\langle g_1, g_2\rangle$ is a non-abelian free subgroup of $\langle f, g\rangle$. □

Corollary 3.3. *Let* $f, g \in \mathrm{Homeo}_+(S^1)$ *such that* $\mathrm{Fix}(\widehat{f}) \neq \emptyset \neq \mathrm{Fix}(\widehat{g})$. *If* $\mathrm{Fix}(\widehat{f}) \cap \mathrm{Fix}(\widehat{g}) = \emptyset$*, then* $\langle f, g\rangle$ *contains a non-abelian free subgroup.*

Definition 3.4. If $G \leq \mathrm{Homeo}_+(S^1)$ is a group, as in the introduction we define the set of homeomorphisms with fixed points

$$G_0 = \{g \in G \mid \exists s \in S^1, g(s) = s\} = \{g \in G \mid \mathrm{Rot}(g) = 0\} \subseteq G.$$

Corollary 3.5. *Let* $G \leq \mathrm{Homeo}_+(S^1)$ *with no non-abelian free subgroups. The subset* G_0 *is a normal subgroup of* $\mathrm{Homeo}_+(S^1)$.

Proof. Let $f, g \in G_0$ then, by Lemma 3.2, they must have a common fixed point, hence $fg^{-1} \in G_0$ and G_0 is a subgroup of G. Moreover, if $f \in G, g \in G_0$ and $s \in \mathrm{Fix}(g)$, we have that $f^{-1}(s) \in \mathrm{Fix}(f^{-1}gf)$ and so that $f^{-1}gf \in G_0$ and therefore G_0 is normal. □

If f has no fixed points then the support of f is the whole circle S^1, otherwise the support can be broken into[3] open intervals upon each of which f acts as a one-bump function, that is $f(x) \neq x$ on each such interval.

Definition 3.6. Given $f \in \mathrm{Homeo}_+(S^1)$, we define an *orbital of* f as a connected component of the support of f. If $G \leq \mathrm{Homeo}_+(S^1)$ then we define an *orbital of* G as a connected component of the support of the action of G on S^1.

[3]possibly infinitely many

We note in passing that any orbital of G can be written as a union of orbitals of elements of G.

Lemmas 3.7, 3.8 , and 3.9 are highly technical lemmas from which one easily derives the useful Corollary 3.10. While Lemmas 3.7–3.9 are proven using elementary techniques, these Lemmas and the techniques involved in their proofs have no bearing on the remainder of the paper. Thus, the reader more interested in the global argument will not lose much by passing directly to Corollary 3.10 on an initial reading.

The following lemma is straightforward and can be derived using techniques similar to those of the first author in [8] or those of Brin and Squier in [10]. We omit its proof.

Lemma 3.7. *Let $H \leq \text{Homeo}_+(I)$ and let (a, b) be an interval such that $\text{Fix}(H) \cap (a, b) = \emptyset$. For every $\varepsilon > 0$, there is an element $w \in H$ such that w has an orbital containing $[a + \varepsilon, b - \varepsilon]$.*

The following will be used in the proof of Lemma 3.9.

Lemma 3.8. *Let $H \leq \text{Homeo}_+(I)$ and suppose that $(a_1, b_1), \ldots, (a_r, b_r)$ are orbitals of H. Let $\varepsilon > 0$ and suppose there is an element $f \in H$ such that $\text{Supp}(f) \supseteq \cup[a_i + \varepsilon, b_i - \varepsilon]$. Given any $g \in H$ there exists a positive integer M such that for all $m \geq M$, there exist positive integers K and N such that for all $n \geq N$, we have*

$$f^m g^n f^{-m} \cdot f^{-K} \left(\bigcup_{i=1}^{r} [a_i + \varepsilon, b_i - \varepsilon] \right) \cap \bigcup_{i=1}^{r} [a_i + \varepsilon, b_i - \varepsilon] = \emptyset.$$

Proof. We consider the set $\mathcal{J} = \{(s_1, t_1), \ldots, (s_r, t_r)\}$ of components of the support of f respectively containing the intervals $[a_i + \varepsilon, b_i - \varepsilon]$.

Fix an index i and let us suppose for now that $f(x) > x$ for all $x \in (s_i, t_i)$. We consider the possible fashions in which g can have support in (a_i, b_i), where the actions of g and f may interact.

There are three cases of interest.

1. There is an orbital (u_i, v_i) of g such that $s_i \in [u_i, v_i)$.

2. There is a non-empty interval (s_i, x_i) upon which g acts as the identity.

3. The point s_i is an accumulation point of a decreasing sequence of left endpoints $\{x_{i,j}\}_{j \in \mathbb{N}}$ of orbitals of g contained in (s_i, t_i).

In the first case, since f is increasing on (s_i, t_i), there exists a positive power M_i such that $f^m(v_i) > b_i - \varepsilon$ for all $m \geq M_i$. Hence any such conjugate $f^m g f^{-m}$ will have an orbital containing $(s_i, b_i - \varepsilon]$. For any $K_i > 0$ the set $W_{i,K_i} := f^{-K_i}([a_i + \varepsilon, b_i - \varepsilon])$ is a compact connected set inside $(s_i, b_i - \varepsilon)$, hence there exists an $N_i > 0$ such that for all $n > N_i$ we have

$$f^m g^n f^{-m}(W_{i,K_i}) \cap [a_i + \varepsilon, b_i - \varepsilon] = \emptyset.$$

In the second case we assume that g is the identity on an interval (s_i, x_i), for some $s_i < x_i < b_i$. There exists a non-negative power M_i such that $f^m(x_i) > b_i - \varepsilon$ for all $m \geq M_i$. Hence any conjugate $f^m g f^{-m}$ for $m \geq M_i$ will be the identity on the interval $(s_i, b_i - \varepsilon]$. In particular, if K is large enough so that $W_{i,K} \cap [a_i + \varepsilon, b_i - \varepsilon] = \emptyset$ we must have that for any $m > M_i$ and any integer $n \geq N_i$ (for any positive integer N_i) the product $f^m g^n f^{-m} f^{-K_i}$ will move $[a_i + \varepsilon, b_i - \varepsilon]$ entirely off of itself.

In the third case we assume that s_i is the accumulation point of a decreasing sequence of left endpoints $\{x_{i,j}\}_{j \in \mathbb{N}}$ of orbitals of g contained in (s_i, t_i). Given any positive integer M_i observe that if $m \geq M_i$, there exists an index j_m such that $x_m := f^m(x_{i,j_m}) < a_i + \varepsilon$. Let $N_i = 1$ and note that for any power $n \geq N_i$ the conjugate $f^m g^n f^{-m}$ fixes x_m. Now we choose K_i to be large enough so that $f^{-K_i}(b_i - \varepsilon) < x_m$. With these choices, the product $f^m g^n f^{-m} \cdot f^{-K_i}$ moves $[a_i + \varepsilon, b_i - \varepsilon]$ entirely off of itself.

We note in passing that in all three cases, K_i could always be chosen larger, with the effect (and only in the first case) that we might have to choose N_i larger.

If instead f is decreasing on the interval (s_i, t_i), similar (reflecting right and left) arguments based at the point t_i instead of s_i, will find products $f^m g^n f^{-m} \cdot f^{-K_i}$ in all three corresponding cases which move $[a_i + \varepsilon, b_i - \varepsilon]$ entirely off of itself.

Choose $M = \max\{M_1, \ldots, M_r\}$ and choose any $m \geq M$. Given this choice of m there are minimal positive choices of K_i for each index i as above. Set $K = \max\{K_1, \ldots, K_r\}$. For this choice of K we can find, for each index i, an integer N_i so that for all values of $n > N_i$, our product will move $[a_i + \varepsilon, b_i - \varepsilon]$ entirely off of itself. Now set $N = \max\{N_1, \ldots, N_r\}$. With these choices, we have that for all $n \geq N$ the product $f^m g^n f^{-m} \cdot f^{-K}$ moves every $[a_i + \varepsilon, b_i - \varepsilon]$ entirely off of itself for all indices i. □

Lemma 3.9. *Let $H \leq \mathrm{Homeo}_+(I)$, let $(a_1, b_1), \ldots, (a_r, b_r)$ be a finite collection of components of the support of H, and let $\varepsilon > 0$. Then there exists $w_\varepsilon \in H$ such that for all i*

$$w_\varepsilon\left([a_i + \varepsilon, b_i - \varepsilon]\right) \cap [a_i + \varepsilon, b_i - \varepsilon] = \emptyset. \tag{2}$$

Proof. We proceed by induction on the number r of intervals. The case $r = 1$ follows from Lemma 3.7. We now assume $r > 1$ and define the following family:

$$\mathcal{L} = \left\{ h \in H \;\middle|\; h\left(\bigcup_{i=1}^{r-1}[a_i + \varepsilon, b_i - \varepsilon]\right) \cap \bigcup_{i=1}^{r-1}[a_i + \varepsilon, b_i - \varepsilon] = \emptyset \right\}.$$

By the induction hypothesis the family $\mathcal{L}$ is non-empty. We also note in passing that the set $\mathcal{L}$ is closed under the operation of passing to inverses. We will now prove that there is an element w_ε in $\mathcal{L}$ with $w_\varepsilon\left([a_r + \varepsilon, b_r - \varepsilon]\right) \cap [a_r + \varepsilon, b_r - \varepsilon] = \emptyset$.

For ease of discussion, we denote the orbital (a_r, b_r) by A_r. Let $f \in \mathcal{L}$, if $[a_r + \varepsilon, b_r - \varepsilon] \subset \mathrm{Supp}(f)$ then there is some power n so that by setting

$w_\varepsilon = f^n$ we will have found the element we desire, thus, we assume below that $[a_r+\varepsilon, b_r-\varepsilon] \not\subseteq \mathrm{Supp}(f)$.

Define $\Gamma = \overline{\mathrm{Supp}(f)} \cap A_r$. There are three possible cases:

1. Neither a_r nor b_r are in Γ,
2. Exactly one of a_r and b_r is in Γ,
3. Both a_r and b_r are in Γ.

Throughout the cases below we will repeatedly construct a $g \in H$ which will always have an orbital (s,t) containing $[a_r+\varepsilon, b_r-\varepsilon]$ by evoking Lemma 3.7. We will specify other properties for g as required by the various cases.

Case 1: Possibly by inverting g we can assume that g is increasing on (s,t), and also by Lemma 3.7 we can assume that s is to the left of Γ and t is to the right of Γ (hence both s and t are fixed by f). Note that for any integers m and K and for all sufficiently large n, the product $f^m g^n f^{-m} \cdot f^{-K}$ has orbital (s,t) and sends $[a_r+\varepsilon, b_r-\varepsilon]$ to the right of $b_r-\epsilon$.

Case 2: We initially assume $a_r \in \Gamma$. There are two possible subcases.

(a) There is an orbital (a_r, x) of f, or

(b) a_r is the accumulation point of a decreasing sequence of left endpoints x_j of orbitals of f in (a_r, b_r).

In case (2.a), possibly by replacing f by its inverse, we can assume that f is decreasing on the orbital (a_r, x) with $x < b_r$. By Lemma 3.7 we can choose g so that $s \in [a_r, x)$ with $s < a_r+\varepsilon$, t is to the right of Γ, and g is increasing on its orbital (s,t) (by inverting g if necessary). For any positive integer M and for all $m \geq M$ we have that $f^m g^n f^{-m}$ is increasing on its orbital $(f^m(s), t) \supsetneq (s,t) \supsetneq [a_r+\varepsilon, b_r-\varepsilon]$. It is now immediate that for any positive integers $m \geq M$ and K and for all sufficiently large n, the product $f^m g^n f^{-m} \cdot f^{-K}$ moves $[a_r+\varepsilon, b_r-\varepsilon]$ entirely off of itself to the right.

In case (2.b) we choose an element x_j of the sequence $\{x_p\}$ such that $a_r < x_j < a_r+\varepsilon$. Moreover, we can choose g increasing so that $a_r < s < x_j$ and t is to the right of Γ. For any positive integer K the power f^{-K} fixes the interval $[x_j, \sup\Gamma] \supseteq [a_r+\varepsilon, b_r-\varepsilon]$ setwise. For any $M > 0$ and for any $m \geq M$ the conjugate $f^m g^n f^{-m}$ has orbital $(f^m(s), t) \supset [x_j, \sup\Gamma]$. Therefore, there exists an $N > 0$ so that for all $n \geq N$ the product $f^m g^n f^{-m} \cdot f^{-K}$ throws the interval $[a_r+\varepsilon, b_r-\varepsilon]$ off itself to the right.

If instead in Case 2 we have that b_r is the only endpoint contained in Γ similar arguments prove the existence of a suitable product $f^m g^n f^{-m} \cdot f^{-K}$ which moves the interval $[a_r+\varepsilon, b_r-\varepsilon]$ leftward entirely off of itself.

Case 3: We have two subcases.

(a) f has orbitals (a_r, x) and (y, b_r) with $x < y$, or

(b) at least one of a_r or b_r is the accumulation point of a monotone sequence sequence of endpoints x_j of orbitals of f in (a_r, b_r), or

In case (3.a) we have that f has orbitals (a_r, x) and (y, b_r) with $x < y$ (if f has (a_r, b_r) as an orbital, then there is a positive integer m such that $w_\varepsilon := f^m$ will satisfy our statement). We construct g so that it has an orbital (s, t) upon which it is increasing and where $s \in [a_r, x)$ and $t \in (y, b_r]$. Possibly by replacing f with its inverse, we can assume that f is decreasing on the orbital (a_r, x). We now have two subcases depending on whether f is increasing or decreasing on (y, b_r).

If f is increasing on (y, b_r), then for any positive integer M and for all $m \geq M$ the conjugate $f^m g f^{-m}$ will have an orbital containing (s, t). Given any $K > 0$ we can choose an positive integer N large enough so that, for all $n \geq N$, the element $f^m g^n f^{-m}$ moves both x and $a_r + \varepsilon$ to the right of $b_r - \varepsilon$. Under these conditions, the product $f^m g^n f^{-m} \cdot f^{-K}$ will move $a_r + \varepsilon$ leftward past $b_r - \varepsilon$.

Assume now that f is decreasing on (y, b_r). There exists an integer $j > 0$ such that $g^j(x) > y$ and so the support of the function $f^{(g^j)}$ contains the interval $(a_r, y]$. If J is the orbital of $f^{(g^j)}$ containing a_r, then $J \cup (y, b_r) = (a_r, b_r)$ and so there exist two positive integers k_1 and k_2 such that the support of the function $g^* := (f^{(g^j)})^{k_1} f^{k_2}$ contains the interval (a_r, b_r). For any positive integer M and for all $m \geq M$ the support of $f^m(g^*)f^{-m}$ contains (a_r, b_r), hence for any $K > 0$ we can select an integer $n \geq N$ large enough so that the product $f^m(g^*)^n f^{-m} \cdot f^{-K}$ moves the interval $[a_r + \varepsilon, b_r - \varepsilon]$ off itself.

In case (3.b) we initially assume that a_r is the accumulation point of a decreasing sequence of left endpoints x_j of orbitals of f in (a_r, b_r). Now, either f has a fixed point $y \geq b_r - \varepsilon$ or it has an orbital (y, b_r) with $y < b_r - \varepsilon$. In the second case we will assume f is increasing on its orbital (y, b_r) (possibly by replacing f by its inverse). In either case we choose g decreasing on (s, t) so that $t > y$ and $t > b_r - \varepsilon$. We also assume g is chosen so that s is to the left of a fixed point of f which is to the left of $a_r + \varepsilon$. Now by our choices it is easy to see that given any positive M and $m > M$ and any positive K we have

1. $f^{-K}(b_r - \varepsilon) < f^m(t)$,
2. $f^{-K}(a_r + \varepsilon) > f^m(s)$, and
3. there is positive N so that for all $n > N$ we have $f^m g^n f^{-m} \cdot f^{-K}(b_r - \varepsilon) < a_r + \varepsilon$.

A similar (reflected) argument can be made if b_r is the accumulation point of an increasing sequence of right endpoints x_j of orbitals of f in (a_r, b_r).

By Lemma 3.8 there exists an M_0 such that for all $m \geq M_0$ we can find a $K_0 > 0$ such that for all $k \geq K_0$ we can find an $N_0 > 0$ so that for all $n \geq N_0$ the product $f^m g^n f^{-m} \cdot f^{-k}$ has support containing $\bigcup_{i=1}^{r-1} [a_i + \varepsilon, b_i - \varepsilon]$. By the analysis in this proof we know we can choose an $M \geq M_0$ such that for any $m \geq M$ we can find a $K \geq K_0$ and $N \geq N_0$ (depending on K) so that for all $n \geq N$ the product $w_\varepsilon := f^m g^n f^{-m} \cdot f^{-K}$ throws $[a_r + \varepsilon, b_r - \varepsilon]$ entirely off of itself. □

We are finally in position to prove the Lemma 1.9 from our introduction.

Proof of Lemma 1.9. We argue via induction on n, with the case $n = 2$ being true by Lemma 3.2. Let $g_1, \ldots g_n \in G_0$ and define $H := \langle g_1, \ldots, g_{n-1} \rangle$.

Write $S^1 \setminus \mathrm{Fix}(H) = \bigcup I_\alpha$ and $S^1 \setminus \mathrm{Fix}(g_n) = \bigcup J_\beta$, for suitable families of open intervals $\{I_\alpha\}, \{J_\beta\}$.

We assume, by contradiction, that $\mathrm{Fix}(H) \cap \mathrm{Fix}(g_n) = \emptyset$, hence we have $S^1 \subseteq (\bigcup I_\alpha) \cup (\bigcup J_\beta)$. By the compactness of S^1 and there are indices r and s so that we can write $S^1 = I_1 \cup \ldots \cup I_r \cup J_1 \cup \ldots \cup J_s$.

Let $I_i = (a_i, b_i)$ and notice that $\mathrm{Fix}(H) \cap (\bigcup_{i=1}^r I_i) = \emptyset$, so we can apply Lemma 3.9 to build an element $w_\varepsilon \in H$ such that $\bigcup_{i=1}^r (a_i + \varepsilon, b_i - \varepsilon) \subseteq \mathrm{Supp}(w_\varepsilon)$. We choose $\varepsilon > 0$ to be small enough so that $\mathrm{Fix}(g_n) \subseteq \bigcup_{i=1}^r (a_i + \varepsilon, b_i - \varepsilon)$ thus implying $\mathrm{Fix}(w_\varepsilon) \cap \mathrm{Fix}(g_n) = \emptyset$. By Lemma 3.2 we can find a non-abelian free group inside $\langle w_\varepsilon, g_n \rangle$, contradicting the assumption on G. □

By compactness of S^1, the previous lemma immediately implies:

Corollary 3.10. *Let $G \leq \mathrm{Homeo}_+(S^1)$ with no non-abelian free subgroups. Then*

1. *G_0 admits a global fixed point, i.e.,* $\mathrm{Fix}(G_0) \neq \emptyset$, *and so*
2. *G_0 is a normal subgroup of G.*

Another application of the compactness is:

Claim 3.11. *Let $f \in \mathrm{Homeo}_+(S^1)$, then for any $0 < \varepsilon < 1$ there exists integer $n > 0$ and a point $x \in S^1$ such that the distance between x and $f^n(x)$ is less than ε, i.e., $\widehat{f}^n(\widehat{x}) = \widehat{x} + k + \delta$ for some integer k and $|\delta| < \varepsilon$.*

Proof. Let y be any point on S^1. The sequence $\{f^n(y)\}_n$ contains a converging subsequence $\{f^{n_i}(y)\}_i$. Therefore there exist $i < j$ such that distance between $f^{n_i}(y)$ and $f^{n_j}(y)$ is less the ε. Thus, we can take $x := f^{n_i}(y)$ and $n = n_j - n_i$. □

Lemma 3.12. *Given $f, g \in \mathrm{Homeo}_+(S^1)$ such that $\widehat{f} < \widehat{g}$, then there exists a function $h \in \mathrm{Homeo}_+(S^1)$ with rational rotation number and such that $\widehat{f} < \widehat{h} < \widehat{g}$.*

Proof. Let ε be the minimal distance between $\widehat{f}$ and $\widehat{g}$, i.e.,

$$\varepsilon = \frac{1}{2} \min_{t \in [0,1]} \left\{ |\widehat{f}(t) - \widehat{g}(t)| \right\}$$

and let $\widehat{h_0} := (\widehat{f} + \widehat{g})/2$. Choose x and n be the ones given by the claim for the function h_0 and the value $\varepsilon/3 > 0$, i.e., $|\widehat{h_0}^n(\widehat{x}) - \widehat{x} - k| < \varepsilon/3$ for some integer k. Consider the family of functions $\widehat{h_t}(s) := \widehat{h_0}(s) + t$ and their powers $\widehat{h_t}^n$. The monotonicity of $\widehat{h_t}$ implies that for any $t > 0$, we have

$$\widehat{h_t}^n(s) = \widehat{h_t}(\widehat{h_t}^{n-1}(s)) = \widehat{h_0}(\widehat{h_t}^{n-1}(s)) + t \geq \widehat{h_0}(\widehat{h_0}^{n-1}(s)) = \widehat{h_0}^n(s) + t.$$

Similarly we have $\widehat{h_t}^n(s) \leq \widehat{h_0}^n(s) + t$ if $t < 0$. The intermediate value theorem applied to the function $t \to \widehat{h_t}^n(\widehat{x})$ implies that there exists a t such that $|t| \leq \varepsilon/3$ and $\widehat{h_t}^n(\widehat{x}) - \widehat{x} = k$ is an integer, i.e., x is a periodic point for h_t. Hence h_t has rational rotation number. By construction $\widehat{h_t}$ is very close to $\widehat{h_0}$, therefore it is between $\widehat{f}$ and $\widehat{g}$. □

The proof of Lemma 1.8 involves observing that the element $(fg)^n$ can be rewritten $f^n g^n h_n$ for some suitable product of commutators $h_n \in [G, G]$; if we prove that $[G, G]$ has a global fixed point s we can compute the rotation number on s, so that $(fg)^n(s) = (f^n g^n)(s)$. The next lemma, together with Corollary 3.10, shows that this is indeed the case.

Lemma 3.13. *Let $G \leq \mathrm{Homeo}_+(S^1)$ and let $f, g \in G$. Suppose one of the following two cases is true:*

1. *G has no non-abelian free subgroups and $\mathrm{Rot}(f) = \mathrm{Rot}(g) \in \mathbb{Q}/\mathbb{Z}$, or*
2. *$\mathrm{Rot}(f) = \mathrm{Rot}(g) \notin \mathbb{Q}/\mathbb{Z}$.*

Then $fg^{-1} \in G_0$.

Proof. (1) Assume $\mathrm{Rot}(f) = \mathrm{Rot}(g) = p/q \in \mathbb{Q}/\mathbb{Z}$ with p, q positive integers and that G has no non-abelian free subgroups.

In this case, f^q and g^q have fixed points in S^1. Now, $\widehat{f}^q(\widehat{x}) = \widehat{x} + p$ and $\widehat{g}^q(\widehat{y}) = \widehat{y} + p$ for any $x \in \mathrm{Fix}(f^q)$ and $y \in \mathrm{Fix}(g^q)$, by Lemma 2.4(4). In particular, f^q and g^q must have a common fixed point $s \in S^1$ by Lemma 3.2 (in the case that one of f^q or g^q is the identity map, then it is immediate that f^q and g^q have a common fixed point) and then for this s we must have $\widehat{f}^q(\widehat{s}) = \widehat{s} + p = \widehat{g}^q(\widehat{s})$.

Suppose now that $fg^{-1} \notin G_0$. In this case, either $\widehat{f} > \widehat{g}$ or $\widehat{f} < \widehat{g}$. We suppose without meaningful loss of generality that the latter is true. However, $f < q$ implies $\widehat{f}^q < \widehat{g}^q$, which is impossible as $\widehat{f}^q(\widehat{s}) = \widehat{s} + p = \widehat{g}^q(\widehat{s})$.

(2) Assume now that $\mathrm{Rot}(f) = \mathrm{Rot}(g) \notin \mathbb{Q}/\mathbb{Z}$.

Suppose $fg^{-1} \notin G_0$. Again, either $\widehat{f} < \widehat{g}$ or $\widehat{g} < \widehat{f}$. Without meaningful loss of generality we suppose that $\widehat{f} < \widehat{g}$. By Lemma 3.12 we can find a map $h \in$

$\mathrm{Homeo}_+(S^1)$ with $\widehat{f} < \widehat{h} < \widehat{g}$ where h has rational rotation number. However, this is impossible since $\widehat{f} < \widehat{h} < \widehat{g}$ guarantees us that $\mathrm{Rot}(f) \le \mathrm{Rot}(h) \le \mathrm{Rot}(g) = \mathrm{Rot}(f)$, so that all three rotation numbers must be equal.

In both (1) and (2), we ruled out the possibility that $fg^{-1} \notin G_0$, thus we must have that $fg^{-1} \in G_0$. $\square$

Corollary 3.14. *Let $G \le \mathrm{Homeo}_+(S^1)$ with no non-abelian free subgroups, then we have $[G,G] \le G_0$.*

The following Lemma is an easy consequence of the definition of lift of a map and Corollary 3.3 and we omit its proof (it can be found in [22]).

Lemma 3.15. *Let $G \le \mathrm{Homeo}_+(S^1)$ with no non-abelian free subgroups. Let $u, v \in G$ and $s \in S^1$ be a fixed point of the commutator $[u,v]$. Then $\widehat{s}$ is a fixed point for $[U,V]$, for any U lift of u and V lift of v in $\mathrm{Homeo}_+(\mathbb{R})$.*

We are now ready to give a proof the main result of this section.

Proof of Lemma 1.8. Let $f, g \in G$. We write the power $(fg)^n = f^n g^n h_n$ where h_n is a suitable product of commutators (involving f and g) used to shift the f's and g's leftward. Since $h_n \in [G,G] \le G_0$ for all positive integers n then, if $s \in S^1$ is a global fixed point for G_0, we have $h_n(s) = s$. Similarly, we observe that $(\widehat{f}\widehat{g})^n = \widehat{f}^n\, \widehat{g}^n\, H_n$ where H_n is a suitable product of commutators and H_n is a lift for h_n. By Lemma 3.15 we must have that $H_n(\widehat{s}) = \widehat{s}$ for all positive integers n. Thus we observe that:

$$(\widehat{f}\widehat{g})^n(\widehat{s}) = \widehat{f}^n\, \widehat{g}^n\, H_n(\widehat{s}) = \widehat{f}^n\, \widehat{g}^n(\widehat{s}).$$

We now find upper and lower bounds for $\widehat{f}^n\, \widehat{g}^n(\widehat{s})$. Observe that, for any two real numbers a, b we have that

$$\widehat{f}^n(a) + b - 1 < \widehat{f}^n(a) + \lfloor b \rfloor \le \widehat{f}^n(a+b) < \widehat{f}^n(a) + \lfloor b \rfloor + 1 \le \widehat{f}^n(a) + b + 1$$

where $\lfloor \cdot \rfloor$ denotes the floor function. By applying this inequality to $\widehat{f}^n\, \widehat{g}^n(\widehat{s}) = \widehat{f}^n(\widehat{s} + (\widehat{g}^n(\widehat{s}) - \widehat{s}))$ we get

$$\widehat{f}^n(\widehat{s}) + \widehat{g}^n(\widehat{s}) - \widehat{s} - 1 \le \widehat{f}^n(\widehat{s} + (\widehat{g}^n(\widehat{s}) - \widehat{s})) \le \widehat{f}^n(\widehat{s}) + \widehat{g}^n(\widehat{s}) - \widehat{s} + 1.$$

We divide the previous inequalities by n, and get

$$\frac{\widehat{f}^n(\widehat{s}) + \widehat{g}^n(\widehat{s}) - \widehat{s} - 1}{n} \le \frac{(\widehat{f}\widehat{g})^n(\widehat{s})}{n} \le \frac{\widehat{f}^n(\widehat{s}) + \widehat{g}^n(\widehat{s}) - \widehat{s} + 1}{n}.$$

By taking the limit as $n \to \infty$ of the previous expression, we immediately obtain $\mathrm{Rot}(fg) = \mathrm{Rot}(f) + \mathrm{Rot}(g)$. $\square$

Corollary 3.16. *Let $G \le \mathrm{Homeo}_+(S^1)$ with no non-abelian free subgroups. Then $\mathrm{Rot} : G \to \mathbb{R}/\mathbb{Z}$ is a group homomorphism and*

1. $\ker(\mathrm{Rot}) = G_0$,
2. $G/G_0 \cong \mathrm{Rot}(G)$.
3. *for all $f, g \in G$, $fg^{-1} \in G_0$ if and only if $\mathrm{Rot}(f) = \mathrm{Rot}(g)$.*

4 Applications: Margulis' Theorem

In this section we show how the techniques developed in Section 3 yield two results for groups of homeomorphisms of the unit circle. One of these results is Margulis' Theorem (Theorem 1.10) which states that every group G of orientation-preserving homomorphisms of the unit circle S^1 either contains a non-abelian free subgroup or admits a G-invariant probability measure on S^1.

Proof of Theorem 1.10. We assume that G does not contain free subgroups, so that the Rot map is a group homomorphism, by Lemma 1.8. The proof divides into two cases.

Case 1: G/G_0 is finite.
Let $s \in \mathrm{Fix}(G_0)$ and consider the finite orbit s^G. Then for every subset $X \subseteq S^1$ we assign:

$$\mu(X) = \frac{\# s^G \cap X}{\# s^G}.$$

This obviously defines a G-invariant probability measure on S^1.

Case 2: G/G_0 is infinite and therefore $\mathrm{Rot}(G)$ *is dense in* $\mathbb{R}/\mathbb{Z}$.

Fix $s \in \mathrm{Fix}(G_0)$ as an origin and identify S^1 with $[0,1]$. We regard s^G as a subset of $[0,1]$ and define the map $\varphi : s^G \to \mathrm{Rot}(G)$, given by $\varphi(s^g) = \mathrm{Rot}(g)$, for any $g \in G$. It is immediate that φ is well-defined and order-preserving on $s^G \subseteq [0,1]$. We take the continuous extension of this map, by defining the function:

$$\begin{array}{rccc} \overline{\varphi}: & [0,1] & \longrightarrow & [0,1] \\ & a & \longmapsto & \sup\{\mathrm{Rot}(g) \mid s^g \leq a, \;\; g \in G\}. \end{array}$$

By construction, the function $\overline{\varphi}$ is non-decreasing. Moreover, since the image of $\overline{\varphi}$ contains $\mathrm{Rot}(G)$, it is dense in $[0,1]$. Since $\overline{\varphi}$ is a non-decreasing function whose image is dense in $[0,1]$, $\overline{\varphi}$ is a continuous map. This allows us to define the Lebesgue-Stieltjes measure associated to $\overline{\varphi}$ on the Borel algebra of S^1 (see [20]), that is, for every half-open interval $(a,b] \subseteq S^1$ we define:

$$\mu((a,b]) := \overline{\varphi}(b) - \overline{\varphi}(a).$$

Since the Rot map is a homomorphism, it is straightforward to see that the measure μ is G-invariant. By definition, $\mu(S^1) = 1$ and $\mu(\{p\}) = 0$, for every point $p \in S^1$. $\square$

Next, we impose a categorical restriction on our group of homeomorphisms, so that Denjoy's Theorem applies. Under these conditions, the existence of an element with irrational rotation number yields an analog of the Tits' alternative — either the group is abelian or it contains a non-abelian free group.

Proof of Theorem 1.11. Let us suppose G contains no non-abelian free subgroups, and let $s \in \operatorname{Fix}(G_0)$. By Denjoy's Theorem there is a z in $\text{Homeo}_+(S^1)$ so that g^z is a pure rotation (by an irrational number). Thus, the orbits of g are dense in S^1 so in particular the orbit of s under the action of g is dense in S^1. Since $\operatorname{Fix}(G_0)$ must be preserved as a set by the action of G, we see that G_0 must be the trivial group. By Corollary 3.16, we have $G \cong \operatorname{Rot}(G) \leq \mathbb{R}/\mathbb{Z}$ and that G is contained in $C_{\text{Homeo}_+(S^1)}(g) \cong \mathbb{R}/\mathbb{Z}$. □

5 Structure and Embedding Theorems

5.1 Structure Theorems

We start the section with our main result which classifies the structure of subgroups of $\text{Homeo}_+(S^1)$ with no non-abelian free subgroups. We consider an orbit s^G of a point s of $\operatorname{Fix}(G_0)$ under the action of G (recall that $\overline{s^G} \subseteq \operatorname{Fix}(G_0)$), then we choose a fundamental domain D for the action of G on $S^1 \setminus \overline{s^G}$. Since the subset $S^1 \setminus \overline{s^G}$ is open, the fundamental domain will be given by a union of intervals. By restricting G_0 to this fundamental domain we get a group H_0 which acts as a set of homeomorphisms of a disjoint union of intervals. We will prove that if $G \leq \text{Homeo}_+(S^1)$ without non-abelian free subgroups then either G is abelian or G can be embedded into the wreath product $H_0 \wr (G/G_0)$.

Remark 5.1. Note that by Theorem 1.11 (a consequence of Denjoy's Theorem), if $G \leq \text{PL}_+(S^1)$ is non-abelian with no non-abelian free subgroups, then Q is isomorphic to a subgroup of $\mathbb{Q}/\mathbb{Z}$.

Proof of Theorem 1.1. If $G_0 = \{\text{id}_{S^1}\}$, then Corollary 3.16 implies $G \cong G/G_0 \cong \operatorname{Rot}(G) \leq \mathbb{R}/\mathbb{Z}$. Now suppose G_0 non-trivial, so that $\operatorname{Fix}(G_0) \neq S^1$ and define $P = G/G_0$. Let $s \in \operatorname{Fix}(G_0)$. Note that P acts on $\operatorname{Fix}(G_0)$ and consider the open subset $S^1 \setminus \overline{s^P}$, where s^P is the orbit of s under the action of P. The set $S^1 \setminus \overline{s^P}$ is a collection of at most countably many disjoint open intervals. We observe that P also acts on $S^1 \setminus \overline{s^P}$ thought of as a set whose elements are open intervals. We can define a fundamental domain for the action of P on $S^1 \setminus \overline{s^P}$ as the union $D = \bigcup_{i \in \mathfrak{N}} I_i$ of a collection $\{I_i\}_{i \in \mathfrak{N}}$ of at most countably many intervals I_i such that

$$k_1(D) \cap k_2(D) = \emptyset,\ k_1 \neq k_2,$$

$$S^1 \setminus \overline{s^P} = \bigcup_{k \in P} k(D).$$

We give proof of Claim 5.6 below, and leave the remaining claims to the reader.

Claim 5.2. *The fundamental domain D exists.*

Since $\overline{s^P} \subseteq \mathrm{Fix}(G_0)$ we have

$$S^1 \setminus \bigcup_{k \in P} k(D) \subseteq \mathrm{Fix}(G_0).$$

Claim 5.3. *Define $H_0 \leq \mathrm{Homeo}_+(S^1)$ to be the subgroup generated by functions f such that there exists a function $g_f \in G_0$ with f the restriction of g_f on D and the identity on $S^1 \backslash D$. Then $H_0 \hookrightarrow \prod_{i \in \mathfrak{N}} \mathrm{Homeo}_+(I_i)$, since $D = \bigcup_{i \in \mathfrak{N}} I_i$. Similarly for every $k \in G/G_0$, there is an embedding $H_0 \hookrightarrow \prod_{i \in \mathfrak{N}} \mathrm{Homeo}_+(k^{-1}(I_i))$.*

Remark 5.4. We will call the image group of this last embedding H_0^k.

It is important to notice that H_0 is not necessarily contained in G_0, since H_0 has its support in D, while an element of G_0 has support in $\bigcup_{k \in P} k(D)$.

Claim 5.5. *The conjugates of H_0 under P commute, and the group $\widetilde{H} := \langle H_0^s \mid s \in G \rangle \simeq \bigoplus_{k \in P} H_0^k$ is normalized by G. Moreover, the group $H := \prod_{k \in P} H_0^k$, thought of as a subgroup in $\mathrm{Homeo}_+(S^1)$, contains $\widetilde{H}$ and is also normalized by G.*

We define the following subgroup

$$E := \langle G, H \rangle \leq \mathrm{Homeo}_+(S^1)$$

and observe that, since G normalizes H by Claim 5.5, the group H is normal in E and we have the following exact sequence:

$$1 \to H \xrightarrow{i} E \xrightarrow{\pi} E/H \to 1$$

where i is the inclusion map and π is the natural projection $\pi : E \to E/H$. Notice that $E/H \cong G/(G \cap H)$ and $G \cap H = G_0$, by definition of G_0. Thus, $E/H \cong G/G_0 = P$, so we can rewrite the sequence as

$$1 \to H \xrightarrow{i} E \xrightarrow{\pi} P \to 1. \qquad (*)$$

Since G is a subgroup of E, the conclusion of the theorem will follow if we can show that $E \cong H_0 \wr P$, where H in the exact sequence $(*)$ above plays the role of the base group. In this case, the semi-direct product structure of E enables us to find a splitting $\phi : P \to E$ of the exact sequence $(*)$ so that if we set $Q = \mathrm{Im}(\phi) \cong P$ we will have the remaining points of our statement.

Claim 5.6. *The group $H \rtimes P \cong H_0 \wr P$ is the only extension of $\prod H_0^k = H$ by P, where P acts on H by permuting the copies of H_0.*

Proof. By a standard result in cohomology of groups (see Theorem 11.4.10 in [27]), if we can prove that $H^2(P, Z(\prod H_0^k)) = 0$ (where $Z(\prod H_0^k)$ denotes the center of $\prod H_0^k$), there can be only one possible extension of $\prod H_0^k$ by P. We observe that $H \rtimes P \simeq H_0 \wr P$ is one such extension, so it suffices to prove that $H^2(P, Z(\prod H_0^k)) = 0$. We use Shapiro's Lemma to compute this cohomology group (see Proposition 6.2 in [11]). We have

$$H^2(P, Z(\prod H_0^k)) = H^2(P, \prod Z(H_0)^k) =$$
$$= H^2(P, \mathrm{Coind}_{\{e\}}^P Z(H_0)) = H^2(\{e\}, Z(H_0)) = 0,$$

which completes the proof of the claim. $\square$

End of the proof of Theorem 1.1. Since $E \cong H_0 \wr P$, there is a splitting $\phi : P \to E$ of the exact sequence $(*)$ so that $E = \langle H, Q\rangle \cong H \rtimes Q$ where $Q = \mathrm{Im}(\phi) \cong P$. $\square$

Remark 5.7. We observe that the wreath product in the previous result is unrestricted; the elements of $\mathrm{Homeo}_+(S^1)$ can have infinitely many "bumps" and so the elements of G_0 can be non-trivial on infinitely many intervals. On the other hand, if we assume $G \leq \mathrm{PL}_+(S^1)$, this would imply that any element in G_0 is non-trivial only at finitely many intervals, and so G_0 can be embedded in the direct sum $\bigoplus$. This argument explains why the wreath product in Theorem 1.5 is unrestricted whereas the ones in Theorems 1.6 and 1.7 are restricted.

We now obtain structure results about solvable subgroups of $\mathrm{PL}_+(S^1)$. Following the first author in [7], we define inductively the following family of groups. Let $G_0 = 1$ and, for $n \in \mathbb{Z}_+$, we define G_n as the direct sum of infinitely many copies of the group $G_{n-1} \wr \mathbb{Z}$:

$$G_n := \bigoplus_{d \in \mathbb{Z}} (G_{n-1} \wr \mathbb{Z}).$$

We recall the following classification.

Theorem 5.8 (Bleak [7])**.** *Let H be a solvable group with derived length n. Then, H embeds in $\mathrm{PL}_+(I)$ if and only if H embeds in G_n.*

Using Theorem 1.1 and Remark 5.1, we are able to extend this result to obtain Theorem 1.3 from the introduction.

There is also a non-solvability criterion for subgroups of $\mathrm{PL}_+([0,1])$. Let $W_0 = 1$ and, for $n \in \mathbb{N}$, we define $W_i = W_{i-1} \wr \mathbb{Z}$. We build the group

$$W := \bigoplus_{i \in \mathbb{Z}} W_i.$$

The following is the non-solvability criterion mentioned above.

Theorem 5.9 (Bleak [9])**.** *Let $H \leq \mathrm{PL}_+([0,1])$. Then H is non-solvable if and only if it contains a subgroup isomorphic to W.*

Using this result and Theorem 1.1, one immediately derives a Tits' alternative for subgroups of $\mathrm{PL}_+(S^1)$; Theorem 1.4 from the introduction.

5.2 Embedding Theorems

We now turn to prove existence results and show that subgroups with wreath product structure do exist in $\mathrm{Homeo}_+(S^1)$ and in $\mathrm{PL}_+(S^1)$.

Remark 5.10. The same result is true for any H_0 that can be embedded in $\prod \mathrm{Homeo}_+(I_i)$ (following the notation of Theorem 1.1) and our proof can be extended without much effort, however we prefer to simplify the hypothesis in order to keep the proof cleaner. Alternatively, we can use the existence of embedding $\prod_{i \in K} \mathrm{Homeo}_+(I_i) \to \mathrm{Homeo}_+(I)$ if K is countable.

Proof of Theorem 1.5. We divide the proof into two cases: K infinite and K finite. If K is infinite, we enumerate the elements of $K = \{k_1, \ldots, k_n, \ldots\}$ and we choose the sequence:

$$\frac{1}{2}, \frac{1}{2^2}, \ldots, \frac{1}{2^n}, \ldots$$

We identify S^1 with the interval $[0,1]$ to fix an origin and an orientation of the unit circle. K is countable subgroup of $\mathbb{R}/\mathbb{Z}$, so it is non-discrete and therefore it is dense in S^1. Now define the following map:

$$\begin{array}{cccc} \varphi: & [0,1] = S^1 & \longrightarrow & [0,1] = S^1 \\ & x & \longmapsto & \sum_{k_i < x} \frac{1}{2^i} \end{array}$$

(where $k_i < x$ is written with respect to the order in $[0,1]$). It is immediate from the definition to see that the map is order-preserving and it is injective, when restricted to K.

For small enough $\varepsilon > 0$ we have

$$\varphi(k_1 + \varepsilon) = \sum_{k_i < k_1 + \varepsilon} \frac{1}{2^i}.$$

If we let $\varepsilon \to 0$, we then see that

$$\alpha := \varphi(k_1) < \varphi(k_1 + \varepsilon) \underset{\varepsilon \to 0}{\longrightarrow} \sum_{k_i \leq k_1} \frac{1}{2^i} = \alpha + \frac{1}{2}.$$

But now, as φ is non-decreasing, we must have $(\alpha, \alpha + \frac{1}{2}) \cap \varphi(K) = \emptyset$. More generally, it follows that:

$$\bigcup_{i \in \mathbb{N}} \left(\varphi(k_i), \varphi(k_i) + \frac{1}{2^i} \right) \cap \overline{\varphi(K)} = \emptyset$$

Claim 5.11. *The unit circle can be written as the disjoint union*

$$S^1 = \bigcup_{i \in \mathbb{N}} \left(\varphi(k_i), \varphi(k_i) + \frac{1}{2^i} \right) \cup \overline{\varphi(K)}.$$

Proof. Let $A := \bigcup_{i\in\mathbb{N}} \left(\varphi(k_i), \varphi(k_i) + \frac{1}{2^i}\right)$ and let $x_0 \notin A$. Let $\varepsilon > 0$ be given. We want to prove that we have $\varphi(K) \cap (x_0 - \varepsilon, x_0 + \varepsilon) \neq \emptyset$.

Suppose $(x_0 - \varepsilon, x_0 + \varepsilon) \cap A = \emptyset$, then we have

$$1 = m([0,1]) \geq m((x_0 - \varepsilon, x_0 + \varepsilon)) + m(A) = 2\varepsilon + \sum_{i=1}^{\infty} \frac{1}{2^i} = 2\varepsilon + 1 > 1$$

where m is the Lebesgue measure on $[0,1]$. In particular, we must have that $(x_0 - \varepsilon, x_0 + \varepsilon) \cap A$ is not empty.

From the above, we know there is an index i with $k_i \in K$ so that

$$(x_0 - \varepsilon, x_0 + \varepsilon) \cap \left(\varphi(k_i), \varphi(k_i) + \frac{1}{2^i}\right) \neq \emptyset.$$

There are three cases of interest.

(a) $\varphi(k_i) \in (x_0 - \varepsilon, x_0 + \varepsilon)$.

In this case, as $\varepsilon > 0$ was arbitrary, we have shown that x_0 is in the closure of $\varphi(K)$.

(b) $\varphi(k_i) + \frac{1}{2^i} \in (x_0 - \varepsilon, x_0 + \varepsilon)$.

Let $\{k_{i_r}\} \subseteq K \subseteq [0,1]$ be a decreasing sequence converging to k_i. Then, $\lim_{r\to\infty} \varphi(k_{i_r}) = \varphi(k_i) + \frac{1}{2^i}$ and so there is an r such that $\varphi(k_{i_r}) \in (x_0 - \varepsilon, x_0 + \varepsilon)$, returning us to the previous case.

(c) $(x_0 - \varepsilon, x_0 + \varepsilon) \subseteq \left(\varphi(k_i), \varphi(k_i) + \frac{1}{2^i}\right)$.

This implies that $x_0 \in \left(\varphi(k_i), \varphi(k_i) + \frac{1}{2^i}\right) \subseteq A$, which contradicts our definition of x_0, so this case cannot occur.

In all possible cases above, we have that x_0 is in the closure of $\varphi(K)$, so our claim is proven. □

We can visualize the set $C := \overline{\varphi(K)}$ as a Cantor set. If we regard $[0,1]$ as S^1, then the group K acts on S^1 by rotations and so each $k \in K$ induces a map $k : C \to C$. Now we extend this map to a map $k : S^1 \to S^1$ by sending an interval $X_i := \left[\varphi(k_i), \varphi(k_i) + \frac{1}{2^i}\right] \subseteq S^1 \setminus C$ linearly onto the interval $k(X_i) := \left[\varphi(k_j), \varphi(k_j) + \frac{1}{2^j}\right]$, where $k_j = k + k_i$ according to the enumeration of K. Thus we can identify K as a subgroup of $\mathrm{Homeo}_+(S^1)$.

We squeeze the interval I into X_1 and regard the group H_0 as a subgroup of $\{g \in \mathrm{Homeo}_+(S^1) \mid g(x) = x, \forall x \notin X_1\} \cong \mathrm{Homeo}_+(X_1)$ (we still call H_0 this subgroup of $\mathrm{Homeo}_+(S^1)$).

We now consider the subgroup $H \leq \mathrm{Homeo}_+(S^1)$ whose elements are fixed away from all conjugates of X_1 (by the action of K), and restrict to elements of H_0^k over $k(X_1)$. Thus, H is the group we obtain spreading the action of H_0 over the circle through conjugation by elements of K (where these elements are allowed to be non-trivial even across infinitely many such conjugate intervals).

Since $\mathrm{supp}(H_0^k) \subseteq k(X_1)$ for any $k \in K$, the groups H_0^k have disjoint support hence they commute pairwise thus $H \cong \prod_{k \in K} H_0^k$. Moreover, the conjugation action of K on H permutes the subgroups H_0^k. If follows that

$$\langle H, K \rangle = H_0 \wr K \hookrightarrow \mathrm{Homeo}_+(S^1).$$

In case $K = \{k_1, \ldots, k_n\}$ is finite, then it is a closed subset of S^1. We define $X_i := (k_i, k_{i+1})$, for $i = 1, \ldots, n$, where $k_{n+1} := k_1$. We can copy the procedure of the infinite case, by noticing that $S^1 = \bigcup_{i=1}^n X_i \cup K$ and embedding H_0 into subgroups of $\mathrm{Homeo}_+(S^1)$ isomorphic with $\mathrm{Homeo}_+(X_i)$. $\square$

We now follow the previous proof, but we need to be more careful in order to embed Thompson's group T into $\mathrm{PL}_+(S^1)$ (see Section 2 for the definition of Thompson's groups T and F).

Proposition 5.12. *There is an embedding $\varphi : \mathbb{Q}/\mathbb{Z} \hookrightarrow T$ such that* $\mathrm{Rot}(\varphi(x)) =$ *x for every $x \in \mathbb{Q}/\mathbb{Z}$ and there is an interval $I \subseteq S^1$ with dyadic endpoints such that $\varphi(x)I$ and $\varphi(y)I$ are disjoint, for all $x, y \in \mathbb{Q}/\mathbb{Z}$ with $x \neq y$.*

Proof. Outline of the idea. We consider the set of elements $\{x_n = 1/n! \mid n \in \mathbb{N}\}$ of $\mathbb{Q}$ which are the primitive $n!$-th roots of 1 in $\mathbb{Q}$ with respect to addition. That is, $nx_n = x_{n-1}$ for each n. We want to send each x_n to a homeomorphism X_n of T with $\mathrm{Rot}(X_n) = 1/n!$ and such that $X_n^n = X_{n-1}$ and $(X_n)^{n!} = \mathrm{id}_{S^1}$. Then, as $\langle x_n \mid n \in \mathbb{N} \rangle = \mathbb{Q}/\mathbb{Z}$, we will have an embedding $\mathbb{Q}/\mathbb{Z} \hookrightarrow T$.

Notation for the proof. For every positive integer n we choose and fix a partition P_n of the unit interval $[0, 1]$ into $2n - 1$ intervals whose lengths are all powers of 2. To set up notation, we always assume we are looking at S^1 from the origin of the axes: from this point of view right will mean clockwise and left will mean counterclockwise and we will always read intervals clockwise. We are now going to use the partitions P_n of the unit interval to get new partitions of the unit circle.

Assume we have a partition of S^1 in $2m$ intervals, we define a "shift by 2" in T to be the homeomorphism X which permutes the intervals of the partition cyclically such that $\mathrm{Rot}(X) = 1/m$ and $X^m = \mathrm{id}_{S^1}$. In other words, "shift by 2" sends an interval V of the partition linearly to another interval W which is 2 intervals to the right of V.

Defining the maps X_n. We want to build a sequence of maps $\{X_n\}$ each of which acts on a partition of S^1 consisting of $2(n!)$ intervals $J_{n,1}, I_{n,1} \ldots, J_{n,n!}, I_{n,n!}$ ordered so that each is to the right of the previous. The map X_n will act as the "shift by 2" map on this partition. We define $X_1 = \mathrm{id}_{S^1}$. To build X_2, we cut S^1 in four intervals $I_{2,1}, J_{2,1}, I_{2,2}, J_{2,2}$ of length $1/4$, each one on the right of the previous one: X_2 is then defined to be the map which linearly shifts these intervals over by 2, thus sending the I's onto the I's and the J's onto the J's. The map X_2 is thus the rotation map by π. Assume now we have built X_n and we want to build X_{n+1}. Take the $2(n!)$ intervals of the partition associated to X_n and divide each of the intervals $I_{n,i}$ according to the proportions given by the partition P_{n+1}, cutting each $I_{n,i}$ into $2n+1 = 2(n+1)-1$ intervals. Leave all of the $J_{n,i}$'s undivided. We have partitioned S^1 into

$$n! + (2n+1)n! = 2[(n+1)!]$$

intervals with dyadic endpoints. Starting with $J_{n+1,1} := J_{n,1}$ we relabel all the intervals of the new partition by I's and J's, alternating them. The new piecewise linear map $X_{n+1} \in T$ is then defined by shifting all the intervals by 2 (see figure 2 to see the construction of the maps X_2 and X_3). We need to verify

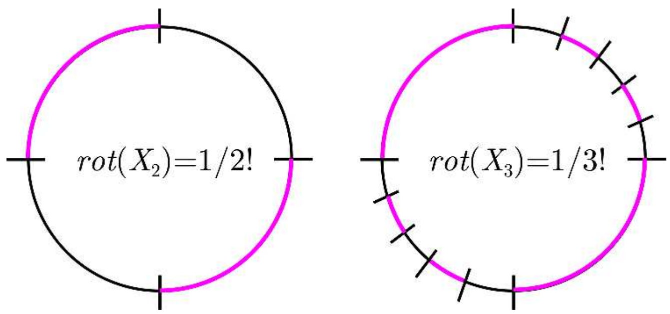

Figure 2: Building the map X_3 from X_2.

that $(X_{n+1})^{n+1} = X_n$. We observe that $Y_n := (X_{n+1})^{n+1} \in T$ shifts every interval linearly by $2n+2$. By construction Y_n sends $J_{n,i}$ linearly onto $J_{n,i+1}$, while it sends $I_{n,i}$ piecewise-linearly onto $I_{n,i+1}$. All the possible breakpoints of Y_n on the interval $I_{n,i}$ occur at the points of the partition P_{n+1}, but it is a straightforward computation to verify that the left and right slope coincide at these points, thus showing that Y_n sends $I_{n,i}$ linearly onto $I_{n,i+1}$.

Defining the embedding φ. To build the embedding $\varphi : \mathbb{Q}/\mathbb{Z} \to T$ we define $\varphi(x_n) := X_n$ and extend it to a group homomorphism by recalling that $\mathbb{Q}/\mathbb{Z} = \langle x_n \rangle$.

The map φ is easily seen to be injective. If $\varphi(x) = \mathrm{id}_{S^1}$ and $x = x_{i_1}^{m_{i_1}} \dots x_{i_\ell}^{m_{i_\ell}}$, then

$$\mathrm{id}_{S^1} = X_{i_1}^{m_{i_1}} \dots X_{i_\ell}^{m_{i_\ell}}.$$

Since $(X_{r+1})^{r+1} = X_r$ for any integer r, we can rewrite the product $X_{i_1}^{m_{i_1}} \dots X_{i_\ell}^{m_{i_\ell}}$ as $(X_n)^m$ for some suitable integers n, m. Since $\mathrm{id}_{S^1} = \varphi(x) = (X_n)^m$, we get that m is a multiple of $n!$ and we can rewrite x as $mx_n = (n!)x_n = 0$.

For every $x, y \in \mathbb{Q}/\mathbb{Z}, x \neq y$ the intervals $\varphi(x)(J_{2,1})$ and $\varphi(y)(J_{2,1})$ are disjoint. If we define $V = \varphi(y)(J_{2,1})$, then the two intervals can be rewritten as $\varphi(xy^{-1})(V)$ and V. Since φ is an embedding and $xy^{-1} \neq 1$, these intervals must be distinct. □

As an immediate consequence of the previous proposition, we get the following two results from the introduction.

Theorem 1.6 *For every $K \leq \mathbb{Q}/\mathbb{Z}$ there is an embedding $F \wr_r K \hookrightarrow T$, where F and T are the respective R. Thompson's groups.*

Proof. We prove it for the full group $K = \mathbb{Q}/\mathbb{Z}$. We apply the previous Theorem to build an embedding $\varphi : \mathbb{Q}/\mathbb{Z} \hookrightarrow T$. Moreover, by construction, the image $\varphi(\mathbb{Q}/\mathbb{Z})$ acts as permutations on the intervals $\{J_{n,i}\}_{n,i\in\mathbb{N}}$. Hence, we recover that

$$\mathrm{PL}_2(J_{2,1}) \wr \mathbb{Q}/\mathbb{Z} \hookrightarrow T.$$

where here $\mathrm{PL}_2(J_{2,1})$ is the subgroup of T which consists of elements which are the identity off of $J_{2,1}$, that is, a group isomorphic with F. □

Theorem 1.7 *For every $K \leq \mathbb{Q}/\mathbb{Z}$ there is an embedding $\mathrm{PL}_+(I) \wr_r K \hookrightarrow \mathrm{PL}_+(S^1)$.*

Proof. The proof of this result is similar to the one of Theorem 1.6, except that here we do not require the endpoints of the interval I to be dyadic. □

Remark 5.13. We remark that none of the proofs of the embedding results require the groups to have no non-abelian free subgroups, although we notice that this condition is automatically guaranteed in Theorems 1.6 and 1.7 because of the Brin-Squier Theorem (Theorem 3.1 in [10]). However, in Theorem 1.5 we may have non-abelian free subgroups inside $H_0 \leq \mathrm{Homeo}_+(I)$ and still build the embedding.

A A counterexample to a construction of Solodov

Write the unit circle S^1 as $\mathbb{R}/\mathbb{Z}$ and define the intervals

$$J_1 = \left[0, \frac{1}{2}\right], \qquad J_2 = \left[\frac{1}{4}, \frac{3}{4}\right], \qquad J_3 = \left[\frac{1}{2}, 0\right], \qquad J_4 = \left[\frac{3}{4}, \frac{1}{4}\right]$$

and the intervals

$$R_1 = \left[\frac{1}{8}, \frac{1}{4}\right], \qquad R_2 = \left[\frac{3}{8}, \frac{1}{2}\right], \qquad R_3 = \left[\frac{5}{8}, \frac{3}{4}\right], \qquad R_4 = \left[\frac{7}{8}, 0\right].$$

which are written making use of the local ordering of S^1. Finally, let A_1, A_2, A_3, A_4 be the left endpoints of R_1, R_2, R_3, R_4, that is $A_1 = \frac{1}{8}, A_2 = \frac{3}{8}, A_3 = \frac{5}{8}, A_4 = \frac{7}{8}$ (see figure 3). We notice that $A_1 \in J_4 \cap J_1, A_2 \in J_1 \cap J_2, A_3 \in J_2 \cap J_3, A_4 \in J_3 \cap J_4$.

We want to build maps a, b in Thompson's group T which act as in the figure. Let $f : \left[0, \frac{1}{2}\right] \to \left[0, \frac{1}{2}\right]$ be following piecewise linear map:

$$f(t) := \begin{cases} 4t & t \in \left[0, \frac{3}{32}\right] \\ t + \frac{9}{32} & t \in \left[\frac{3}{32}, \frac{1}{8}\right] \\ \frac{1}{4}t + \frac{3}{8} & t \in \left[\frac{1}{8}, \frac{1}{2}\right]. \end{cases}$$

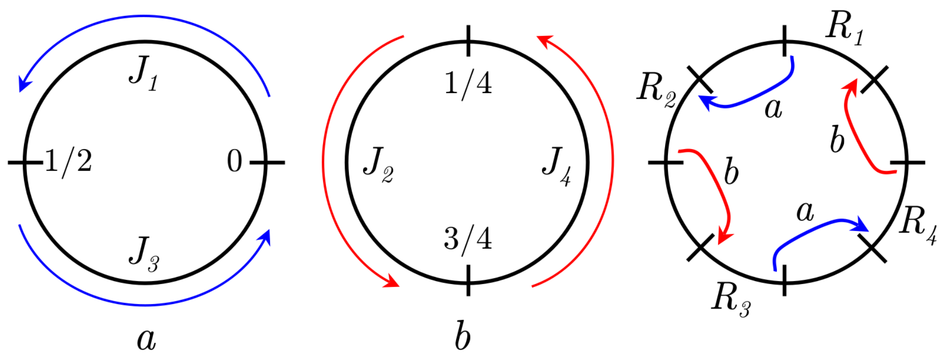

Figure 3: Construction and behavior of the maps a and b.

It is immediate to verify that $f(R_1) = \left[\frac{13}{32}, \frac{7}{16}\right] \subseteq \mathring{R}_2$ (where $\mathring{R}_2$ denotes the interior of R_2). Now we define a to be equal to f on $\left[0, \frac{1}{2}\right]$ and to "act like f" on $\left[\frac{1}{2}, 1\right]$, that is we conjugate f by a rotation by half a circle

$$a(t) := \begin{cases} f(t) & t \in \left[0, \frac{1}{2}\right] \\ \rho_{\frac{1}{2}} f \rho_{\frac{1}{2}}^{-1}(t) & t \in \left[\frac{1}{2}, 1\right], \end{cases}$$

where $\rho_{\frac{1}{2}}(t) = t + \frac{1}{2} \pmod 1$ is the required rotation map. The homeomorphism a is in the standard copy of Thompson's group F, within Thompson's group T. To build b, we conjugate a by a rotation by a quarter of the circle. More precisely, we define

$$b(t) := \rho_{\frac{1}{4}} a \rho_{\frac{1}{4}}^{-1}(t) \pmod 1 \qquad t \in [0, 1].$$

where $\rho_{\frac{1}{4}}(t) = t + \frac{1}{4} \pmod 1$ is the required rotation map. The homeomorphism b is in Thompson's group T.

We now follow Solodov's construction from the proof of Lemma 2.4 in [28] to find an element of $\langle a, b\rangle$ which has rotation number non-zero. Of course, for this choice of a and b, such elements are easy to find, but we are testing here the actual construction employed by Solodov.

By construction, we observe that a and b are in Thompson's group T with $\mathrm{Supp}(a) = \mathring{J}_1 \cup \mathring{J}_3, \mathrm{Supp}(b) = \mathring{J}_2 \cup \mathring{J}_4$. We also note that a and b satisfy the following inclusions (see figure 3)

$$a(R_1) \subseteq \mathring{R}_2, \qquad b(R_2) \subseteq \mathring{R}_3, \qquad a(R_3) \subseteq \mathring{R}_4, \qquad b(R_4) \subseteq \mathring{R}_1.$$

It is easy to see that our choice of a, b yields that $\mathrm{Supp}\langle a, b\rangle = S^1$ and $a(A_1) > A_2, b(A_2) > A_3, a(A_3) > A_4, b(A_4) > A_1$.

Moreover, we observe that the element $baba$ (composing right-to-left) sends A_1 'Around the circle and past itself', which is effectively the condition created by Solodov's construction, and which is used to verify that such a constructed element would have non-zero rotation number. However, we observe that *baba* also sends the closed interval R_1 inside the open interval $\mathring{R}_1$, therefore *baba* fixes some point in R_1 by the contraction mapping lemma, and so *baba* has rotation number zero. It can be verified that the attracting fixed point for *baba* detected by the previous argument is $\frac{1}{6} \in R_1$.

References

[1] L. A. Beklaryan. Invariant and projectively invariant measures for groups of orientation-preserving homeomorphisms of $\mathbf{R}$. *Dokl. Akad. Nauk*, 332(6):679–681, 1993.

[2] L. A. Beklaryan. On the classification of the orientation-preserving homeomorphism groups of $\mathbf{R}$. I. Invariant measures. *Mat. Sb.*, 187(3):23–54, 1996.

[3] L. A. Beklaryan. On the classification of the orientation-preserving homeomorphism groups of $\mathbf{R}$. II. Projective invariant measures. *Mat. Sb.*, 187(4):3–28, 1996.

[4] L. A. Beklaryan. On the classification of the orientation-preserving homeomorphism groups of $\mathbf{R}$. III. ω-projectively invariant measures. *Mat. Sb.*, 190(4):43–62, 1999.

[5] L. A. Beklaryan. On analogues of the Tits alternative for groups of homeomorphisms of the circle and the line. *Mat. Zametki*, 71(3):334–347, 2002.

[6] L. A. Beklaryan. Groups of homeomorphisms of the line and the circle. Topological characteristics and metric invariants. *Uspekhi Mat. Nauk*, 59(4(358)):3–68, 2004.

[7] Collin Bleak. An algebraic classification of some solvable groups of homeomorphisms. *J. Algebra*, 319(4):1368–1397, 2008.

[8] Collin Bleak. A geometric classification of some solvable groups of homeomorphisms. *J. London Math. Soc.*, 78(2):352–372, 2008.

[9] Collin Bleak. A minimal non-solvable group of homeomorphisms. *Groups, Geometry, and Dynamics*, pages 1–30, to appear.

[10] Matthew G. Brin and Craig C. Squier. Groups of piecewise linear homeomorphisms of the real line. *Invent. Math.*, 79(3):485–498, 1985.

[11] Kenneth S. Brown. *Cohomology of groups*, volume 87 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1994. Corrected reprint of the 1982 original.

[12] J.W. Cannon, W.J. Floyd, and W.R. Parry. Introductory notes on Richard Thompson's groups. *Enseign. Math. (2)*, 42(3-4):215–256, 1996.

[13] A. Denjoy. Sur les courbes définies par des équations différentielles á la surface du tore. *J. Math. Pures et Appl.*, 11:333–375, 1932.

[14] Robert Fricke and Felix Klein. *Vorlesungen über die Theorie der automorphen functionen*, volume 2. B. G. Teubner, Leipzig, 1912.

[15] Étienne Ghys. Groups acting on the circle. *Enseign. Math. (2)*, 47(3-4):329–407, 2001.

[16] Étienne Ghys and Vlad Sergiescu. Sur un groupe remarquable de difféomorphismes du cercle. *Comment. Math. Helv.*, 62(2):185–239, 1987.

[17] Michael-Robert Herman. Sur la conjugaison différentiable des difféomorphismes du cercle à des rotations. *Inst. Hautes Études Sci. Publ. Math.*, 49:5–233, 1979.

[18] Isabelle Liousse. Rotation numbers in Thompson-Stein groups and applications. *Geom. Dedicata*, 131:49–71, 2008.

[19] R. S. MacKay. A simple proof of Denjoy's theorem. *Math. Proc. Cambridge Philos. Soc.*, 103(2):299–303, 1988.

[20] Paul Malliavin. *Integration and probability*, volume 157 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1995. With the collaboration of Hélène Airault, Leslie Kay and Gérard Letac, Edited and translated from the French by Kay, With a foreword by Mark Pinsky.

[21] Gregory Margulis. Free subgroups of the homeomorphism group of the circle. *C. R. Acad. Sci. Paris Sér. I Math.*, 331(9):669–674, 2000.

[22] F. Matucci. *Algorithms and classification in groups of piecewise-linear homeomorphisms*. PhD thesis, Cornell University, 2008. `arXiv:math.GR/0807.2871v1`.

[23] Andrés Navas. Quelques groupes moyennables de difféomorphismes de l'intervalle. *Bol. Soc. Mat. Mexicana (3)*, 10(2):219–244 (2005), 2004.

[24] Andrés Navas. *Groups of Circle Diffeomorphisms*. Sociedade Brasileira de Matemática, 2007. English translation of the original *Grupos de difeomorfismos del círculo*.

[25] H. Poincaré. Mémoire sur les courbes définies par une équation différentielle. *Journal de Mathématiques*, 7:375–422, 1881.

[26] H. Poincaré. Mémoire sur les courbes définies par une équation différentielle. *Journal de Mathématiques*, 8:251–296, 1882.

[27] Derek J. S. Robinson. *A course in the theory of groups*, volume 80 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, second edition, 1996.

[28] V. V. Solodov. Homeomorphisms of the circle and foliations. *Izv. Akad. Nauk SSSR Ser. Mat.*, 48(3):599–613, 1984.

[29] Itiro Tamura. *Topology of foliations: an introduction*, volume 97 of *Translations of Mathematical Monographs*. American Mathematical Society, Providence, RI, 1992. Translated from the 1976 Japanese edition and with an afterword by Kiki Hudson, With a foreword by Takashi Tsuboi.

Collin Bleak
School of Mathematics and Statistics,
University of St Andrews,
Mathematical Institute,
North Haugh,
St Andrews, Fife KY16 9SS,
Scotland
E-mail address: collin@mcs.st-and.ac.uk

Martin Kassabov
Department of Mathematics, Cornell University,
590 Malott Hall, Ithaca, NY 14853,
U.S.A.
E-mail address: kassabov@math.cornell.edu

Francesco Matucci
Department of Mathematics, University of Virginia,
325 Kerchof Hall, Charlottesville, VA 22904,
U.S.A.
E-mail address: fm6w@virginia.edu