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Wilson Loops in supersymmetric gauge theories

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Introduction

Quantum field theories have been for long time one of the most successful areas of particle physics in the last years, specially since the development of quantum electrodynamics (QED) and quantum chromodynamics (QCD). The understanding of electromagnetism in terms of photons and fermions have been completed after the inclusion of the renormalization framework that allowed to treat in a systematic way the divergences appearing when computing amplitudes associated to certain diagrams, and to contrast the computations with experimental results. In the search of new lagrangians that allow us to describe other forces present in nature, as the strong and weak nuclear forces, the extension of the concept of local gauge symmetry to other non-abelian Lie groups resulted in the appearance of the so-called Yang-Mills theories (in which QCD is included). These theories together with the ideas behind dimensional regularization and the Higgs model gave rise to the discover of what we now call the Standard Model, that describes the interactions of all known particles trough the electro-weak force and strong nuclear force within the framework of a relativistic quantum field theory and constitutes one of the theories with the highest level of experimental agreements. In the Standard Model, strong interactions are described by QCD which is a non-abelian gauge theory with quarks in the fundamental representation of $SU(3)$. The fact that the theory is non-abelian leads to the asymptotic freedom and the confinement of quarks.

Despite all its achievements, the knowledge of quantum field theories is still somewhat restricted. In fact, mathematically speaking functional integrals are objects that in most of the cases are hard to compute, and in general one works with them in terms of perturbative expansions. From a practical point of view, these expansions in power series of the coupling constant present the problem that the difficulty to compute them grows exponentially with the perturbative order. From a theoretical point of view, renormalization of gauge theories implies that these parameters are not constant any more and acquire a dependence with the energy scale. In QED this does not generates too much problems since at low energies (including the scale of the experiments that can be performed today) the electron charge is very small and only grows at very high energies, validating the use of a perturbative expansion. In non-abelian gauge theories like QCD the behaviour is the opposite, meaning that the analysis stops to be valid at relatively low energies, in particular close to the confinement scale of the colour degrees of freedom. One of the consequences in QCD is the fact that we are dealing with a confining theory. Under this situation, a set of operators result of particular utility to study QCD and other non-abelian gauge theories in the non-perturbative regime, these are the Wilson loops operators. In a Yang-Mils theory describing gauge bosons (A_μ) associated to the local symmetry of the lagrangian, together with other bosonic or fermionic fields, the Wilson loop is defined on a contour $\mathcal{C}$ as

$$W[\mathcal{C}] = \frac{1}{N}\mathrm{Tr}_{\mathcal{R}}\,\mathcal{P}\exp\left[-ig\oint_{\mathcal{C}} dx^\mu A_\mu(x)\right] \tag{1}$$

One of the most important properties of the Wilson loop is that it is a gauge invariant operator. It can be seen as the phase factor associated to the propagation of a massive quark in the fundamental representation of the gauge group. In this interpretation, one can consider a rectangular loop of width L and length T that simulates the propagation of the quark-antiquark pair placed at a distance L. In the limit $T \gg L$ the expectation value of the Wilson loop depends only on L and it is associated with the interaction energy $E(L)$ of the pair:

$$W \propto e^{-T\,E(L)} \tag{2}$$

This relation between the expectation value of the Wilson loop and the interaction energy of a quark and antiquark indicates that, in a theory like QCD, the analysis of these kind of operators allows us to study the confinement phenomenon.

Notwithstanding the great success of the standard model to describe particle interactions, the inclusion of gravity into this description is still an open problem. String theory turns out to be an alternative and it is the best candidate for a quantum theory of gravity. The main idea is that one has to replace the notion of point particles with one-dimensional strings, these strings have oscillation modes that describe the spectrum of the standard model and a massless spin-two particle that is identified with the graviton. A consistent string theory that includes bosonic and fermionic excitations requires the existence of extra spatial dimensions to the four known. Furthermore, in order for string theory to be free of inconsistencies it needs supersymmetry.

Supersymmetry is a space-time symmetry that maps particles of integer spin (bosons) into particles with half-integer spin (fermions), and viceversa. This symmetry is generated by an operator $\mathcal{Q}$ that acts as

$$\mathcal{Q}|\text{fermion}\rangle = |\text{boson}\rangle \qquad\qquad \mathcal{Q}|\text{boson}\rangle = |\text{fermion}\rangle \tag{3}$$

The generator $\mathcal{Q}$ has some properties that follow from (3):

- It changes the spin of a particle (meaning that $\mathcal{Q}$ transforms as a spin-1/2 particle) and hence its space-time properties. This is why supersymmetry is not an internal symmetry but a space-time symmetry.
- In a theory where supersymmetry is realized, each one-particle state has a super-partner. Therefore, in a supersymmetric world, instead of single particle states, one has to deal with super-multiplets of particle states.

A supersymmetric field theory is then a set of fields and a Lagrangian which exhibit such a symmetry. One can have theories with more than one supersymmetry generator: $\mathcal{Q}^I$ with $I = 1, \ldots, \mathcal{N}$. The number of supersymmetry generators, however, cannot be arbitrarily large, the reason is that any super-multiplet in 4 dimensions contains particles with spin at least as large as $\frac{\mathcal{N}}{4}$.

Albeit supersymmetry first appeared only as a theoretical tool in the context of string theory, later it was realized that it could be a symmetry of quantum field theory describing elementary particles. One of the reasons for this is that radiative corrections to certain quantities remain small as a result of some cancellations between boson and fermion contributions.

Supersymmetry could solve most of the current problems in theoretical physics (like the hierarchy problem, or the origin of dark matter). Still it can not be realized in nature, meaning that it must be broken at some energy scale since we do not measure all the particles that it predicts. However, models that enjoy supersymmetry are more constrained that the non-supersymmetric ones, thus they are easier to solve. Supersymmetric theories can then be used as toy models where certain analytical results can be obtained and might serve to get predictions on qualitative aspects of the realistic theories.

Due to the importance of supersymmetry and the Yang-Mills theories in the study of elementary interactions, it seems natural then to study supersymmetric Yang-Mills theories. In particular, we are going to study two types of these theories in four dimensions with gauge group $SU(N)$. One is the maximally supersymmetric one which has $\mathcal{N} = 4$ supersymmetries (for now on we will call this theory as $\mathcal{N} = 4$ SYM), this theory has been extensively studied in the last years due to its role in the AdS/CFT correspondence. The other one is a particular case of a Yang-Mills theory with $\mathcal{N} = 2$ supersymmetries that it is coupled to $N_f = 2N$ hypermultiplets in the fundamental representation of the gauge group (that we will call $\mathcal{N} = 2$ SCQCD). For these two theories, the β-function vanish meaning that the coupling constants do not run.

As is ordinary gauge theories, in $\mathcal{N} = 4$ SYM and $\mathcal{N} = 2$ SCQCD, supersymmetric Wilson loops (which are the supersymmetric generalization of (1) that includes additional couplings with some of the scalars of the theory) provide a rich class of observables that can be computed both by perturbation theory, or exactly by using supersymmetric localization. Localization has been proven to be one of the most powerful tools in obtaining non perturbative results in quantum supersymmetric gauge theories. The idea of this technique is that we can deform the action in some way that the integration domain of the partition function is reduced and the integral can be computed by saddle-point procedure. Using this method, a great number of new exact results have been derived for supersymmetric theories in different dimensions, mainly when formulated on spheres or ellipsoids.

An interesting quantity that plays an important role when probing quantum field theories is the so-called Bremsstrahlung function. It is defined as the energy lost by a heavy quark slowly moving in a gauge background.
When the theory is conformal invariant it is convenient to describe this heavy quark

by a Wilson operator: we think of a probe particle that suffers a transition from a velocity v_1 to v_2 in an infinitesimal angle, the worldline thus has a cusp and the vacuum expectation value of the Wilson operator develops a logarithmic UV divergence that depends on the coupling constant and on the angle of the cusp φ.

$$\langle W_\varphi \rangle \sim e^{-\Gamma_{\text{cusp}}(\varphi)\log\frac{\Lambda_{UV}}{\Lambda_{UV}}} \tag{4}$$

where Λ_{UV} and Λ_{UV} represent the UV and IR cut-off scales. The quantity that governs this divergence (Γ_{cusp}) is called the cusp anomalous dimension and it is related to a number of physical observables. In the small angle limit it is related to the amount of power radiated by a moving quark and it behaves as $\Gamma_{\text{cusp}} = -B\varphi^2$, where B is the Bremsstrahlung function of the theory. The cusp anomalous dimension can be generalized including an R-symmetry angle θ that controls the coupling of the scalars on the two sides of the cusp.

In order to use B for probing the theory at different scales, for instance for precision tests of the AdS/CFT correspondence, it is needed to go beyond the perturbative regime. This can be done by relating B to quantities that can be computed holographically and, in superconformal theories, by the use of localization techniques. Candidates for these quantities are circular BPS Wilson loops for which exact results can be obtained from a computable matrix model. For $\mathcal{N} = 4$ $SU(N)$ SYM theory, it was proved that B can be computed as a derivative of the vacuum expectation value of a 1/2 BPS circular Wilson loop with respect to the 't Hooft coupling [1] as we will see in chapter 4.
A similar formula to derive B for $\mathcal{N} = 2$ $SU(N)$ SCQCD from the expectation value of a 1/2 BPS circular Wilson loop was conjectured in [2]. However this formula has not been proven yet, but only checked for the $SU(2)$ case.

We will show in this thesis how we found the validity of the conjectured formula for general N up to three-loops performing a perturbative calculation of the cusp anomalous dimension for the supersymmetric Wilson operator, using HQET techniques [3].
The HQET formalism consists on performing first the integration on the contour parameters with a proper prescription for regularizing boundary divergences. In this way, the integrals reduce to ordinary massive momentum integrals, which can

be written as linear combinations of known Master Integrals by applying integrations by parts. The advantage of this computational framework is that it can be easily extended to higher loops. From the small angle expansion, we derive the corresponding Bremsstrahlung function at three loops, matching the matrix model prediction given in terms of derivatives of the Wilson loop on the ellipsoid.

There are other limits of the cusp anomalous dimension that can be considered, one of them is the large Minkowskian angles. In this limit the cusp anomalous dimension behaves linearly in the angle, and the function governing this behaviour is the so called light-like cusp anomalous dimension. This quantity can be used to check an universal behaviour of the cusp anomalous dimension: when expressed in terms of the light-like cusp replacing the coupling constant, the cusp anomalous dimension gives rise to an universal function that is independent of the number of fermion or scalar fields in the theory, this was shown to hold up to three-loops in $\mathcal{N} = 4$ Super Yang-Mills theories [4]. We compute this function and we found that the universal behaviour is also present up to three loops in $\mathcal{N} = 2$ SCQCD.

The structure of this thesis is the following. The chapter 1 is the introductory one, where we will study the gauge theories. We will introduce the supersymmetry and the superconformal symmetry and we will present the basic properties of the two theories of our interest: $\mathcal{N} = 4$ SYM and $\mathcal{N} = 2$ SCQCD. Chapter 2 is dedicated to the presentation of the Wilson loops which are the central objects of study in this thesis. After giving the definition and properties, we will give explicit examples of the computation of expectation values. In Chapter 3 we define the Wilson line with a cusp and we discuss the main features of its anomalous dimension in superconformal theories. In chapter 4 we will study the concept of supersymmetric localization, which provides a tool for computing the exact value of some observables. Chapter 5 is based on the original work and it is where we present the obtained results. At the end, we present the conclusions in 6. Notations are collected in appendix A and in appendix B we give some of the computational details.

Chapter 1

Supersymmetric gauge theories

In this chapter we are going to present the very basic ingredients that we need for the following chapters: the supersymmetric gauge theories. First we review the basic symmetries of a quantum field theory and in section 1.1.1 we introduce supersymmetry, an additional symmetry that mixes bosons and fermions. In section 1.1.2 and 1.1.3 we will discuss conformal symmetry and how to combine it with supersymmetry to enlarge the symmetry group. In section 1.1.4 we are going to study the representations of supersymmetry and how they are organized into multiplets. In section 1.1.5 we will discuss the quantum properties of a superconformal theory. In 1.2 and 1.3 we are going to present the two superconformal theories that we are going to study in the rest of the chapters, describing their field content and symmetries of the actions. Finally in section 1.4 we give an alternative way to derive the actions of these theories starting from superspace, which is an extension of the ordinary space-time that includes additional coordinates that anti-commute.

Gauge theories and supersymmetry

We are going to start by recalling the basic symmetries of a quantum field theory, these are

- **Poincarè symmetry:** semi-direct product of translations and Lorentz transformations. The algebra is:

$$\begin{aligned} [M_{\mu\nu}, M_{\rho\sigma}] &= -i\left(\eta_{\mu\rho}M_{\nu\sigma} - \eta_{\mu\sigma}M_{\nu\rho} + \eta_{\nu\sigma}M_{\mu\rho} - g_{\nu\rho}M_{\mu\sigma}\right) \\ [M_{\mu\nu}, P_\rho] &= i\left(P_\mu g_{\nu\rho} - P_\nu g_{\mu\rho}\right) \end{aligned} \tag{1.1}$$

- **Internal symmetries:** with generators T_a obeying a Lie algebra:

$$[T_a, T_b] = i\, f_{ab}{}^c\, T_c \tag{1.2}$$

- **Discrete symmetries:** C (charge conjugation), P (parity) and T (time reversal).

Supersymmetry

One may wonder if it is possible that the space-time symmetry groups (Poincarè) and internal symmetries of a theory can be included in a bigger group that contains both in a non-trivial way. With this we mean that the Poincarè generators (P_μ and $M_{\mu\nu}$) and the generators of internal symmetries (T_a, $a = 1, \ldots, N^2 - 1$) satisfy:

$$\begin{aligned} [M_{\mu\nu}, T_a] &\neq 0 \\ [P_\mu, T_a] &\neq 0 \end{aligned} \tag{1.3}$$

The answer to this question is given by the Coleman-Mandula theorem [5] that states that in a generic quantum field theory, under a number of assumptions (like locality, causality, positivity of energy, etc.), the only possible continuous symmetries of the S-matrix are those generated by Poincare group generators, P_μ and $M_{\mu\nu}$, plus some internal symmetry group G commuting with them. This theorem rules out the possibility of having a group satisfying (1.3), meaning that the most general symmetry group containing the Poincare group and an arbitrary internal symmetry group, is a direct product

$$\text{Poincarè} \times \text{Internal} \tag{1.4}$$

It is possible to evade the Coleman-Mandula theorem if we relax the assumption that the symmetry algebra (1.3) is a Lie algebra containing only commutators, which means that all the symmetries are bosonic. If we relax this and we allow to the algebra to have fermionic generators (that will satisfy anti-commutator relations) we could, in principle, include all the generators in a bigger algebra. An algebra containing generators that satisfy commutation and anti-commutation rules is called a *graded Lie algebra* or *superalgebra*, and the symmetries associated to it

are called *supersymmetries*. In [6] Haag, Loperzanski and Sohnius found a way to enlarge the Poincarè algebra to include such fermionic generators forming the *super-Poincarè* algebra.
The way this super-algebra is constructed goes as follows: beyond the usual bosonic generators, the only possibilities are $\mathcal{N}$ fermions generators of spin 1/2: Q^i_α, with $\alpha = 1,2$ and $i = 1, \ldots, \mathcal{N}$, we call these generators the *super-charges*.

The algebra is then extended:

$$\begin{aligned}
[M_{\mu\nu}, M_{\rho\sigma}] &= -i\left(g_{\mu\rho}M_{\nu\sigma} - g_{\mu\sigma}M_{\nu\rho} + g_{\nu\sigma}M_{\mu\rho} - g_{\nu\rho}M_{\mu\sigma}\right) \\
[M_{\mu\nu}, P_\rho] &= i\left(P_\mu g_{\nu\rho} - P_\nu g_{\mu\rho}\right) \qquad [T_a, T_b] = i\, f_{ab}{}^c T_c \\
\{Q^i_\alpha, Q^j_\beta\} &= C_{\alpha\beta} Z^{ij} \qquad \{Q^i_\alpha, \overline{Q}^j_{\dot\beta}\} = \delta^{ij}\sigma^\mu_{\alpha\dot\alpha} P_\mu \\
\left[Q^i_\alpha, M_{\mu\nu}\right] &= (\sigma_{\mu\nu})_\alpha{}^\beta Q^i_\beta \qquad \left[Q^i_\alpha, T_a\right] = (B_a)^i{}_j\, Q^j_\alpha
\end{aligned} \tag{1.5}$$

where, $Z^{ij} = Z^{ij}_a T^a$ are bosonic generators that satisfy $Z^{ij} = -Z^{ji}$ by construction and commute with all the other operators, they are called the *central charges*. The coefficients B^{ij}_a and Z^{jk}_b are related as: $B^{ij}_a A^{jk}_b = -A^{ij}_b (B^{jk}_a)^*$.
The supersymmetry algebra (1.5) is left invariant under a global rotation of the supercharges Q^i_α forming a group $U(1)_R$. In addition, when $\mathcal{N} > 1$, the different supercharges may be rotated into one another under a unitary transformation, belonging to $SU(\mathcal{N})_R$. These symmetries of the supersymmetry algebra are called R-symmetries.

Conformal symmetry

A conformal transformation is a transformation of the space-time coordinates that rescales the line element:

$$x_\mu \to x'_\mu \qquad \Rightarrow \qquad g_{\mu\nu}(x) \to \Omega(x) g_{\mu\nu}(x) \tag{1.6}$$

where $\Omega(x)$ is an arbitrary function of the coordinates. When $\Omega(x) =$ constant we have the scale transformations. In the general case, conformal transformations rescale the distances locally preserving the angles. Besides the ordinary scale transformations, other examples of conformal transformations are the Poincarè transformations (with $\Omega(x) = 1$). The conformal transformations are then a bosonic extension of the Poincarè transformations. In order to find all the possible conformal transformations we can write an infinitesimal change $x^\mu \to x^\mu + \epsilon^\mu(x)$ and impose (1.6) to find constraints on $\epsilon^\mu(x)$. Then $\epsilon^\mu(x)$ is given by

$$\epsilon^\mu(x) = a^\mu + \Lambda^\mu{}_\nu x^\nu + \lambda x^\mu + b^\mu x^2 - 2(b \cdot x)x^\mu \tag{1.7}$$

where each parameter corresponds to a different type of transformation, explicitly:

- **Poincarè transformations:** a^μ is the parameter of the translations and $\Lambda^\mu{}_\nu$ the parameter of the Lorentz transformations, with generators P_μ and $M_{\mu\nu}$ respectively.
- **Dilatations:** parametrized by λ and generated by the dilatation operator D.
- **Special conformal transformations:** parametrized by a vector b^μ with generators K_μ.

The set of conformal transformations in d-dimensions forms a group: $SO(d,2)$, which contains the Poincarè group as a subgroup. The conformal algebra is

$$\begin{aligned}
[M_{\mu\nu}, M_{\rho\sigma}] &= -i\,(g_{\mu\rho}M_{\nu\sigma} - g_{\mu\sigma}M_{\nu\rho} + g_{\nu\sigma}M_{\mu\rho} - g_{\nu\rho}M_{\mu\sigma}) \\
[M_{\mu\nu}, P_\rho] &= i\,(P_\mu g_{\nu\rho} - P_\nu g_{\mu\rho}) \qquad [M_{\mu\nu}, K_\rho] = (K_\mu g_{\nu\rho} - K_\nu g_{\mu\rho}) \\
[P_\mu, D] &= -i\,P_\mu \quad [K_\mu, D] = i\,K_\mu \quad [P_\mu, K_\nu] = 2i\,(M_{\mu\nu} - g_{\mu\nu}D)
\end{aligned} \tag{1.8}$$

Superconformal symmetry

Now we want to combine the conformal symmetry together with the supersymmetry. We will study theories that are invariant under supersymmetry and also under the conformal group, these are called *super-conformal field theories*. It turns out that when we want to extend the conformal algebra (1.8) by including the supersymmetry generators Q^i_α, the algebra does not close. For example, using Jacobi identities one can see that the commutator between the supercharges and the generators of the special conformal transformations is not contained in the algebra. To solve this issue, one needs to introduce another set of fermionic generators S^i_α commuting with K_μ that plays the same role of Q^i_α for P_μ and generate the *super-conformal symmetries*. All these generators enhance the conformal group to a supergroup which in 4 dimensions is $SU(2,2|\mathcal{N})$.

In $4d$ the algebra for the super-conformal symmetry is (including the equation 1.8)

$$
\begin{aligned}
\{Q^i_\alpha, \overline{Q}^j_{\dot\beta}\} &= \delta^{ij}\sigma^\mu_{\alpha\dot\alpha}P_\mu \quad \{Q^i_\alpha, Q^j_\beta\} = C_{\alpha\beta}Z^{ij} \qquad \{S^i_\alpha, \overline{S}^j_{\dot\beta}\} = \delta^{ij}\sigma^\mu_{\alpha\dot\alpha}K_\mu \\
\{Q^i_\alpha, S^j_\beta\} &= 2C_{\alpha\beta}\delta^{ij}\, D - i\,(\sigma^{\mu\nu})_\alpha{}^\gamma C_{\gamma\beta}\delta^{ij}M_{\mu\nu} + B^{ij}_a T^a - 4i\, C_{\alpha\beta}\delta^{ij}\, R \\
\left[Q^i_\alpha, M_{\mu\nu}\right] &= (\sigma_{\mu\nu})_\alpha{}^\beta Q^i_\beta \quad \left[Q^i_\alpha, D\right] = i\,\frac{i}{2}Q^i_\alpha \quad \left[Q^i_\alpha, K_\mu\right] = -\frac{1}{2}\,(\sigma^\mu)_{\alpha\dot\alpha}\overline{S}^{i\dot\alpha} \quad \left[Q^i_\alpha, T_a\right] = B^i_{aj}Q^j_\alpha \\
\left[S^i_\alpha, M_{\mu\nu}\right] &= (\sigma_{\mu\nu})_\alpha{}^\beta S^i_\beta \quad \left[S^i_\alpha, D\right] = -\frac{i}{2}S^i_\alpha \quad \left[S^i_\alpha, P_\mu\right] = \frac{1}{2}(\sigma^\mu)^{\alpha\dot\alpha}\overline{Q}^i_{\dot\alpha} \quad \left[S^i_\alpha, T_a\right] = -B^i_{aj}S^j_\alpha \\
\left[Q^i_\alpha, R\right] &= -i\left(\frac{4-\mathcal{N}}{4\mathcal{N}}\right)Q^i_\alpha \quad \left[S^i_\alpha, R\right] = i\left(\frac{4-\mathcal{N}}{4\mathcal{N}}\right)S^i_\alpha \quad \left[T_a, T_b\right] = i\, f_{ab}{}^c\, T_c
\end{aligned}
\tag{1.9}
$$

Multiplets in supersymmetric theories

In supersymmetric field theories, the fields belong to particular representations of the corresponding symmetry algebra. We are going to study field representations of the supersymmetry algebra with spin less or equal than 1. For $1 \leq \mathcal{N} \leq 4$ these consist of spin 1 vector particles (gauge fields), spin 1/2 Weyl fermions fields, and spin 0 scalar fields. These fields are restricted to enter into *multiplets* of the supersymmetry algebra. Two kind of multiplets can occur for $1 \leq \mathcal{N} \leq 4$: *gauge multiplets* or *matter multiplets*. For $\mathcal{N} = 3, 4$ the gauge multiplet which transforms in the adjoint representation is the only possible one, while for $\mathcal{N} = 1, 2$ we can also have matter multiplets (for $\mathcal{N} = 1$ is the chiral multiplet and for $\mathcal{N} = 2$ is the hypermultiplet) transforming in arbitrary representation ($\mathcal{R}$) of the gauge group. We study these multiplets in terms of their component fields [7]:

- $\mathcal{N} = 1$ **gauge multiplet** (A_μ, η_α): Consists of a gauge field A_μ and a Majorana fermion η_α (gaugino).

- $\mathcal{N} = 1$ **chiral multiplet** (ϕ, ψ_α): A complex scalar ϕ and a Weyl fermion ψ_α.

- $\mathcal{N} = 2$ **gauge multiplet** $(A_\mu, \eta^\pm_\alpha, \phi)$: The gauge field A_μ, a Dirac fermion (which is a direct sum of left and right Weyl fermions $\eta^\pm_\alpha$) and a complex scalar ϕ.

- $\mathcal{N}=2$ **hypermultiplet** $(\psi_\alpha^\pm, \phi^\pm)$: Two Weyl fermions $\psi_\alpha^\pm$ (or equivalently 1 Dirac fermion) and 2 complex scalars $\phi^\pm$.
- $\mathcal{N}=4$ **gauge multiplet** $(A_\mu, \psi_\alpha^A, \phi^I)$: The gauge field A_μ, 4 Weyl fermions ψ_α^A (or equivalently 2 Dirac fermions), and 3 complex scalars ϕ^I (or 6 real).

It is also interesting to express the content of the multiplets in terms of the way their transform under the subalgebras or the corresponding supersymmetry algebra. So we can split the multiplets in the following way:

- $(\mathcal{N}=2 \text{ gauge}) \equiv (\mathcal{N}=1 \text{ gauge}) \oplus (\mathcal{N}=1 \text{ chiral})_{\text{adj}}$
- $(\mathcal{N}=2 \text{ hyper})_{\mathcal{R}} \equiv (\mathcal{N}=1 \text{ chiral})_{\mathcal{R}} \oplus (\mathcal{N}=1 \text{ chiral})_{\overline{\mathcal{R}}}$
- $(\mathcal{N}=4 \text{ gauge}) \equiv (\mathcal{N}=2 \text{ gauge}) \oplus (\mathcal{N}=2 \text{ hyper})_{\text{adj}}$

where the subscript "adj" in indicates that the multiplet transform in the adjoint representation, and $\overline{\mathcal{R}}$ indicates that it does it in the complex conjugate representation of $\mathcal{R}$.

Quantization

The fact that a symmetry is present in the classical equations of motion, does not imply that it will be also present at the quantum level. Ultraviolet (UV) divergences may appear when computing quantum corrections to classical quantities. However the final result must be finite (because it must correspond to the observables that one measures at experiments), this means that the divergences must be cancelled at one of the intermediate steps of the computation leaving the final result finite.
Since these divergences cannot affect the measurable quantities, we should be able to remove them by suitable redefinitions of the fields and couplings. When a theory happens to have this property, it is called to be *renormalizable.* This procedure naturally introduces a scale μ in the theory that in principle breaks the conformal invariance.
In order to control these divergences we first have to regularize them, this can be done by different ways (regularization schemes) for example by the introduction of an energy cut-off M that will be taken to infinity at the end, or by shifting the

spacetime dimension as $d = 4 - 2\epsilon$.
Imagine that we have a renormalizable theory with one coupling constant g, then we have the physical observables $\mathcal{O}$ which are generally given as power series of the coupling constant g, and are function of some momentum invariants $(p_1, \ldots, p_n)$ that characterize the process under consideration. In the computation of the quantum corrections UV divergences can appear, which we regulate by introducing a cut-off M on which $\mathcal{O}$ will also depend. We can write $\mathcal{O}$ schematically as:

$$\mathcal{O} = \mathcal{O}(g, M, p_1, \ldots, p_n) \tag{1.10}$$

If the theory is renormalizable, we are allowed to redefine the coupling constant as $g_R = f(g, M/\mu)$ in such a way that

$$\mathcal{O}(g(g_R, M/\mu), M, p_1, \ldots, p_n) = \widetilde{\mathcal{O}}(g_R, \mu, p_1, \ldots, p_n) \tag{1.11}$$

where μ is the energy scale of the process. In this way, the divergences that originally appear in $\mathcal{O}$ were reabsorbed in the renormalized coupling g_R. This redefinition of the coupling makes the physical quantities independent on the cut-off, so with this procedure UV divergent quantities can be made finite. Of course this redefinition of the coupling will be a function of the energy scale μ, the dependence (which is known as the *running*) of g with μ is governed by a quantity called β-function, which is defined as:

$$\beta(g) = \mu \frac{\partial g}{\partial \mu} \tag{1.12}$$

This is a very important quantity in the study of quantum field theories since it determines the conditions under we can safely take perturbation theory to be valid. When $\beta = 0$ it means that the coupling g does not flow, in these cases the theory is free of UV divergences and the classical conformal invariance is also present at the quantum level. Examples of such theories are $\mathcal{N} = 4$ SYM and $\mathcal{N} = 2$ SCQCD which we are going to discuss in the next sections.

$\mathcal{N} = 4$ Super Yang-Mills

$\mathcal{N} = 4$ super Yang-Mills (SYM) [8, 9, 10, 11] is a 4-dimensional field theory with conformal symmetry, both at quantum and classical level, with an additional local symmetry given by the invariance of the action under some gauge group G. For now on, we will take the gauge group to be $G = SU(N)$. All the fields in this theory transform in the adjoint representation of $SU(N)$. The field content is the following

- $N^2 - 1$ gauge bosons described by the field $A_\mu(x) = A^a_\mu(x)\, T^a$, with Lorentz index $\mu = 0, \dots, 3$.
- 3 complex scalar fields ϕ^I, with $I = 1, 2, 3$.
- 4 Weyl fermions ψ^A, with $A = 1, \dots, 4$ and spinor index $\alpha = 1, 2$.

The fields (A_μ, ψ^A, ϕ^I) constitute the gauge multiplet of $\mathcal{N} = 4$ SYM. Under the global R-symmetry group ($SU(4) \cong SO(6)$) the gauge field transform as a singlet, the fermions transform in the 4 representation and scalars in the 6 representation.

From the gauge field A_μ we can define a covariant derivative (as showed in the appendix) and the field strength $F_{\mu\nu}$. Having specified the field content of the theory, we can now write down the action of the theory [1], defined on the euclidean space $\mathbb{R}^4$.

$$
\begin{aligned}
S = \int & d^4x \; 2\,\mathrm{Tr}\Big[i\psi^{I\alpha}(\sigma^\mu)_\alpha{}^{\dot\beta}\mathcal{D}_\mu\bar\psi_{I\dot\beta} + i\eta^\alpha(\sigma^\mu)_\alpha{}^{\dot\beta}\mathcal{D}_\mu\bar\eta_{\dot\beta} - \frac{1}{4}F^{\mu\nu}F_{\mu\nu} + \phi^I\mathcal{D}^\mu\mathcal{D}_\mu\bar\phi_I \\
& + ig\sqrt{2}\,\bar\psi^{I\dot\alpha}\,[\bar\eta_{\dot\alpha},\phi_I] - ig\sqrt{2}\,[\bar\phi^I,\eta^\alpha]\psi_{\alpha I} + ig\frac{\sqrt{2}}{2}\epsilon_{IJK}[\phi^I,\psi^{\alpha J}]\psi^K_\alpha + ig\frac{\sqrt{2}}{2}\epsilon_{IJK}[\bar\phi^I,\bar\psi^{\dot\alpha J}]\bar\psi^K_{\dot\alpha} \\
& + g^2[\phi^I,\phi^J][\bar\phi^I,\bar\phi^J] - \frac{g^2}{2}[\phi^I,\bar\phi^I][\phi^J,\bar\phi^J]\Big] \quad , \qquad (1.13)
\end{aligned}
$$

where g is the coupling constant, $F_{\mu\nu}$ is the gauge field strength and D_μ is the usual covariant derivative defined as in the appendix A. The trace is taken on the gauge index and ensures the gauge invariance of the action.
This action can be seen as the dimensionally reduced action obtained from the $\mathcal{N} = 1$ SYM theory in 10-dimensions [9] given by

[1] In order to write the action in a way that will be more convenient for us later, we renamed one of the fermions as $\psi^4_\alpha \to \eta_\alpha$. In this way this fermion combines with the gauge bosons A_μ into the $\mathcal{N} = 1$ gauge multiplet, the fermion η_α is called the *gaugino*.

$$\mathcal{L}_{\mathcal{N}=1} = \frac{1}{2g^2}\left(\frac{1}{2}F_{MN}F^{MN} - \Psi\Gamma^M D_M \Psi\right) \tag{1.14}$$

In 10 dimensions with the signature $(-,+,\ldots,+)$, the fields are the gauge field A_M (with $M = 0,1,\ldots,9$) and a fermion field Ψ. With this conventions Ψ is a 16-component Majorana-Weyl fermion that takes values in the adjoint representation of $SU(N)$. We will identify the gauge fields $(A_0, A_5, \ldots, A_9)$ of the $\mathcal{N} = 1$ 10-dimensional theory with the 6 real scalars (that combine into the 3 complex ϕ^I) of the 4-dimensional theory. The matrices Γ^M are related with the 10-dimensional Dirac matrices (γ^M). The action (1.14) describes the low energy effective action coming from type-I superstring theory. After dimensional reduction from $10d \to 4d$ we obtain the action (1.13).

We can do some dimensional analysis on 1.13, taking the standard dimensions for the fields:

$$\begin{aligned} [A_\mu] &= [\phi^I] = 1 \\ \left[\lambda^i_\alpha\right] &= 3/2 \end{aligned} \tag{1.15}$$

we see that the coupling has mass dimension zero: $[g] = 0$, thus the theory is scale invariant. This, together with Poincarè invariance, makes it invariant under the full conformal group $SO(4,2) \cong SU(2,2)$ [2]. These considerations are all at the classical level, to see if these symmetries are also present at the quantum regime we should calculate the β-function. It turns out that this function is zero to all loops [12, 13, 14, 15], making the theory $\mathcal{N} = 4$ SYM conformal invariant also at quantum level.
The action (1.13) it is also invariant under the transformations generated by the Poincarè supercharges Q^A_α, $\overline{Q}^A_{\dot\alpha}$ thus enlarging the group to the superconformal group $SU(2,2|4)$. The supersymmetry transformations are characterized by 4 spinors ϵ^i_α (with $i = 0,1,2,3$) and are the following:

[2]This considerations are only valid in 4 dimensions.

$$
\begin{aligned}
\delta\phi^I &= -\epsilon_0^\alpha\,\psi_\alpha^I + \epsilon_\alpha^I\,\eta^\alpha - \epsilon^{IJK}\,\bar\epsilon_J^{\dot\alpha}\,\overline{\psi}_{K\dot\alpha}\\
\delta\bar\phi^I &= -\bar\epsilon_0^{\dot\alpha}\,\bar\psi_{\dot\alpha}^I + \bar\epsilon_{\dot\alpha}^I\,\bar\eta^{\dot\alpha} - \epsilon^{IJK}\,\epsilon_J^{\alpha}\,\psi_{K\alpha}\\
\delta\psi_\alpha^I &= -i\,\bar\epsilon_0^{\dot\alpha}\nabla_{\alpha\dot\alpha}\phi^I + \epsilon^{I\beta}f_{\beta\alpha} + \ldots\\
\delta\eta_\alpha &= -i\,\bar\epsilon_I^{\dot\alpha}\nabla_{\alpha\dot\alpha}\phi^I - \epsilon_0^\beta f_{\beta\alpha} + \ldots\\
\delta A_{\alpha\dot\alpha} &= -i\left(\epsilon_\alpha^0\,\bar\eta_{\dot\alpha} + \bar\epsilon_{\dot\alpha}^0\,\eta_\alpha - \epsilon_\alpha^I\,\bar\psi_{I\dot\alpha} + \bar\epsilon_{\dot\alpha}^I\,\psi_{\alpha I}\right)\,,
\end{aligned}
\tag{1.16}
$$

where $I = 1{,}2{,}3$ and we have introduced the spinorial way of writing the covariant derivatives and the fields, the conventions for doing this are listed in appendix A.

$\mathcal{N} = 2$ Superconformal QCD

We are going to study in this section a class of $\mathcal{N} = 2$ SYM with gauge group $SU(N)$ and N_f hypermultiplets transforming in the fundamental representation of the gauge group. These theories have a global symmetry group $U(N_f)\times SU(2)_R\times U(1)_R$, where $U(N_f)$ is the flavour symmetry and $SU(2)_R\times U(1)_R$ the R-symmetry group [3]. If the number of flavours is tuned to be $N_f = 2N$ the β-function is zero and theory becomes exactly superconformal at any value of g [17]. When $N_f = 2N$ the theory is called $\mathcal{N} = 2$ super-conformal QCD, we are going concentrate our study on this particular theory [4].

The field content of $\mathcal{N} = 2$ SCQCD consists of:

- $N^2 - 1$ gauge bosons described by the field $A_\mu(x)$, as in the $\mathcal{N} = 4$ SYM case.
- 1 complex scalar ϕ transforming in the adjoint representation of the gauge group.

[3] We are interested in superconformal field theories that admit a large N limit, a list of these kind of theories can be found in [16].

[4] $\mathcal{N} = 2$ SCQCD can be viewed as the limit of a two-parameter family of $\mathcal{N} = 2$ superconformal field theories. This family of $\mathcal{N} = 2$ theories have gauge group $SU(N)\times SU(N)$ and two fundamental hypermultiplets, they are governed by two coupling constants g and $\widehat{g}$ each of which is associated with a factor in the gauge group. For $\widehat{g}\to 0$ one recovers $\mathcal{N} = 2$ SCQCD plus a decoupled free vector multiplet in the adjoint representation of $SU(N)$. For $g = \widehat{g}$, we have instead Z_2 orbifold of $\mathcal{N} = 4$ SYM. Thus by tuning $\widehat{g}$ we interpolate continuously between $\mathcal{N} = 2$ SCQCD and the $\mathcal{N} = 4$ universality class. [18]

- 2 Weyl fermions η_α and ψ_α transforming in the adjoint representation of $SU(N)$
- $N_f = 2N$ pairs of scalar fields: q^I, $\widetilde{q}^I$ (with $I = 1,\ldots,N_f$) transforming in the fundamental and anti-fundamental representation of $SU(N)$
- $N_f = 2N$ pairs of Weyl fermions: λ^I_α, $\widetilde{\lambda}^I_\alpha$ (with $I = 1,\ldots,N_f$) transforming in the fundamental and anti-fundamental representation.

The action of this theory is given by:

$$\begin{aligned}
S = \int d^4x \Bigg\{ & 2\,\mathrm{Tr}\Big[i\psi^\alpha (\sigma^\mu)_\alpha^{\ \dot\beta} \mathcal{D}_\mu \bar\psi_{\dot\beta} + i\eta^\alpha (\sigma^\mu)_\alpha^{\ \dot\beta} \mathcal{D}_\mu \bar\eta_{\dot\beta} \\
& - \frac{1}{4} F^{\mu\nu} F_{\mu\nu} + \phi \mathcal{D}^\mu \mathcal{D}_\mu \bar\phi - \frac{g^2}{2} [\phi,\bar\phi][\phi,\bar\phi] + ig\sqrt{2}\,\bar\psi^{\dot\alpha} [\bar\eta_{\dot\alpha},\phi] - ig\sqrt{2}\,[\bar\phi,\eta^\alpha]\psi_\alpha \Big] \\
& + i\bar\lambda^I_{\dot\beta} (\sigma^\mu)_\alpha^{\ \dot\beta} \mathcal{D}_\mu \lambda^\alpha_I + i\tilde\lambda^{\alpha I} (\sigma^\mu)_\alpha^{\ \dot\beta} \mathcal{D}_\mu \bar{\tilde\lambda}_{\dot\beta I} + \bar q^I \mathcal{D}^\mu \mathcal{D}_\mu q_I + \tilde q^I \mathcal{D}^\mu \mathcal{D}_\mu \bar{\tilde q}_I \\
& + ig\sqrt{2}\,(\bar\lambda^{\dot\alpha I} \bar\eta_{\dot\alpha} q_I - \tilde q^I \bar\eta^{\dot\alpha} \bar{\tilde\lambda}_{\dot\alpha I}) - ig\sqrt{2}\,(\bar q^I \eta^\alpha \lambda_{\alpha I} - \tilde\lambda^{\alpha I} \eta_\alpha \bar{\tilde q}_I) + ig\sqrt{2}\,(\tilde\lambda^{\alpha I} \psi_\alpha q_I - \bar q^I \bar\psi^{\dot\alpha} \bar{\tilde\lambda}_{\dot\alpha I}) \\
& + ig\sqrt{2}\,(\tilde\lambda^{\alpha I} \phi \lambda_{\alpha I} - \bar\lambda^{\dot\alpha I} \bar\phi \bar{\tilde\lambda}_{\dot\alpha I}) + ig\sqrt{2}\,(\tilde q^I \psi^\alpha \lambda_{\alpha I} - \bar\lambda^{\dot\alpha I} \bar\psi_{\dot\alpha} \bar{\tilde q}_I) \\
& - g^2 \left[2\,\bar q^I \bar\phi \phi q_I + 2\,\tilde q^I \phi \bar\phi \bar{\tilde q}_I + (\bar q^I q_J)(\tilde q^J \bar{\tilde q}_I) \right] \\
& - \frac{g^2}{4} \Big[(\bar q^I q_J)(\bar q^J q_I) + (\tilde q^I \bar{\tilde q}_J)(\tilde q^J \bar{\tilde q}_I) - 2(\bar q^I \bar{\tilde q}_J)(\tilde q^J q_I) + 4\,\bar q^I [\phi,\bar\phi] q_I - 4\,\tilde q^I [\phi,\bar\phi] \bar{\tilde q}_I \Big] \Bigg\}
\end{aligned} \tag{1.17}$$

The action (1.17) is supersymmetric invariant and the transformation of the fields are characterized in this case by 2 spinors ϵ^i_α (with $i = 0,1$) and for the adjoint fields they are

$$\begin{aligned}
\delta\phi &= -\epsilon_0^\alpha\,\psi_\alpha + \epsilon^1_{\ \alpha}\,\eta^\alpha \\
\delta\psi_\alpha &= -i\,\bar\epsilon_0^{\dot\alpha} \nabla_{\alpha\dot\alpha}\phi + \epsilon^{1\beta} f_{\beta\alpha} + \ldots \\
\delta\eta_\alpha &= -i\,\bar\epsilon_1^{\dot\alpha} \nabla_{\alpha\dot\alpha}\phi - \epsilon_0^\beta f_{\beta\alpha} + \ldots \\
\delta A_{\alpha\dot\alpha} &= -i\left(\epsilon^0_\alpha\,\bar\eta_{\dot\alpha} + \bar\epsilon^0_{\dot\alpha}\,\eta_\alpha - \epsilon^1_\alpha\,\bar\psi_{\dot\alpha} + \bar\epsilon^1_{\dot\alpha}\,\psi_\alpha \right)\,.
\end{aligned} \tag{1.18}$$

In view of the discussion of section 1.1.4 we can describe the field content of $\mathcal{N} = 4$ SYM and $\mathcal{N} = 2$ SCQCD in a convenient way as follows:

- $\mathcal{N} = 4$ SYM $\equiv$ 1-($\mathcal{N} = 2$ gauge) $\oplus$ 1-($\mathcal{N} = 2$ hyper)$_{\mathrm{adj}}$

- $\mathcal{N}=2$ SCQCD $\equiv$ 1-($\mathcal{N}=2$ gauge) $\oplus$ N_f-($\mathcal{N}=2$ hyper)$_{\text{fund}}$

So we can see that both theories share the same $\mathcal{N}=2$ gauge multiplet and the difference is on the $\mathcal{N}=2$ matter multiplets. We will exploit this property in chapter 5 to compute the expectation value of some observables.

Superspace approach

An alternative way to derive the actions (1.13) and (1.17) is given by the $\mathcal{N}=1$ superspace formalism [19]. The superspace is defined as an extension of the ordinary space by including anti-commuting coordinates θ^α and $\overline{\theta}^{\dot{\alpha}}$. We can introduce a compact way to collectively denote a point in superspace and the partial derivatives as

$$\begin{aligned} z^A &= (x^{\alpha,\dot{\alpha}}, \theta^\alpha, \overline{\theta}^{\dot{\alpha}}) \\ \partial_A &= (\partial_{\alpha\dot{\alpha}}, \partial_\alpha, \overline{\partial}_{\dot{\alpha}}) \ , \end{aligned} \tag{1.19}$$

where we are using again the spinorial way to write the usual coordinates and derivatives (as explained in appendix A together with the conventions for raising and lowering indices). In superspace the supersymmetry algebra is obtained performing translations and rotations involving both the spacetime and the anti-commuting coordinates.
Representations of the supersymmetry algebra can also be described in superspace, to do this we introduce *superfields* which are functions of the space-time coordinates $F(x,\theta,\overline{\theta})$. These functions can be expanded in Taylor series of the anti-commuting coordinates. Since you can not have powers of θ higher than 2 because they vanish, the Taylor expansion is finite and a general superfield can be expanded in terms of a finite number of fields depending only on x^μ, this fields are called the *components* of the superfield.
A general superfield is not a reducible representation of supersymmetry, for this reason we need to impose constraints on them.

Chiral superfields

The simplest example of a constrained superfield which serves as an irreducible representation of supersymmetry is the chiral superfield defined as:

$$\overline{D}_{\dot{\alpha}}\Phi = 0 \tag{1.20}$$

where $\overline{D}_{\dot{\alpha}}$ is a fermionic covariant derivative defined in appendix A. The Taylor expansion of a chiral superfield is

$$\Phi(x,\theta,\overline{\theta}) = \phi(x)+\theta^\alpha\,\psi_\alpha(x)-\theta^2\,F(x)+\frac{i}{2}\,\theta^\alpha\overline{\theta}^{\dot{\alpha}}\,\partial_{\alpha\dot{\alpha}}\psi_\beta(x)\,C^{\alpha\beta}+\frac{\theta^2\overline{\theta}^2}{4}\Box\phi(x) \tag{1.21}$$

Their components are defined by:

$$\begin{aligned} \phi(x) &= \Phi|_{\theta=0} \\ \psi_\alpha(x) &= D_\alpha\Phi|_{\theta=0} \\ F(x) &= D^2\Phi|_{\theta=0} \end{aligned} \tag{1.22}$$

In the same way we can define *anti-chiral superfields* that are annihilated by D_α. Note that $\overline{\Phi}$ is anti-chiral if Φ is chiral.
So a chiral superfield describes the $\mathcal{N}=1$ chiral multiplet: a complex scalar and a fermion (the field F is auxiliary).

Vector superfield

A real scalar superfield ($V = V^\dagger$) is called the *vector superfield* and its expansion is given by

$$V(x,\theta,\overline{\theta}) = C+\theta^\alpha\,\chi_\alpha+\overline{\theta}^{\dot{\alpha}}\,\overline{\chi}_{\dot{\alpha}}-\theta^2\,M-\overline{\theta}^2\,M+\theta^\alpha\overline{\theta}^{\dot{\alpha}}\,A_{\alpha\dot{\alpha}}+i\,\overline{\theta}^2\theta^\alpha\,\eta_\alpha-i\,\theta^2\overline{\theta}^{\dot{\alpha}}\,\overline{\eta}_{\dot{\alpha}}+\theta^2\overline{\theta}^2\,D' \tag{1.23}$$

Turns out that the components C, χ_α and M can be removed by a gauge transformation. The gauge where all of these components are zero is called *Wess-Zumino gauge*. The superfield V is this gauge is given by

$$V(x,\theta,\overline{\theta}) = \theta^\alpha\overline{\theta}^{\dot{\alpha}}\,A_{\alpha\dot{\alpha}}+i\,\overline{\theta}^2\theta^\alpha\,\eta_\alpha-i\,\theta^2\overline{\theta}^{\dot{\alpha}}\,\overline{\eta}_{\dot{\alpha}}+\theta^2\overline{\theta}^2\,D'\,. \tag{1.24}$$

whose components are defined as:

$$\begin{aligned} A_{\alpha\dot{\alpha}}(x) &= \frac{1}{2}[\overline{D}_{\dot{\alpha}},D_\alpha]V|_{\theta=0} \\ \eta_\alpha(x) &= i\,\overline{D}^2 D_\alpha V|_{\theta=0} \\ \overline{\eta}_{\dot{\alpha}}(x) &= -i\,D^2\overline{D}_{\dot{\alpha}}V|_{\theta=0} \\ D'(x) &= \frac{1}{2}D^\alpha\overline{D}^2 D_\alpha V|_{\theta=0} \end{aligned} \tag{1.25}$$

This superfield describes in the correct way the $\mathcal{N}=1$ gauge multiplet, where the physical fields $A_{\alpha\dot{\alpha}}$ and λ_α are the gauge field and the gaugino, the field D is auxiliary.

The theories studied before can me conveniently described using this formalism.

$\mathcal{N}=4$ SYM from superspace

In $\mathcal{N}=1$ superspace formalism, the field content of $\mathcal{N}=4$ SYM is organized into one real vector superfield V (which give rise to the gauge multiplet) and 3 chiral superfields Φ^I ($I=1,2,3$) that forms the matter multiplet and are endowed with $SU(3)$ symmetry.

In this language, the superspace action for this theory is then:

$$\begin{aligned} S=&\int d^4x\ d^4\theta\,\mathrm{Tr}\big(e^{-gV}\bar{\Phi}_I e^{gV}\Phi^I\big)+\frac{1}{g^2}\int\ d^4x\ d^2\theta\,\mathrm{Tr}\big(W^\alpha W_\alpha\big)\\ &+\frac{ig}{3!}\int\ d^4x\ d^2\theta\ \epsilon_{IJK}\mathrm{Tr}\big(\Phi^I[\Phi^J,\Phi^K]\big)+h.c. \end{aligned}\tag{1.26}$$

where $W_\alpha = i\,\overline{D}^2(e^{-gV}\,D_\alpha\,e^{gV})$ is the superfield strength of V.

The component action (1.13) can be obtained projecting (1.26) down in components.

$\mathcal{N}=2$ SCQCD from superspace

We can do the same for $\mathcal{N}=2$ SCQCD. In this case the field content of the gauge multiplet is organized into 1 vector and 1 chiral superfield. The rest of the matter is described in terms of the quark chiral scalar superfields Q^I and $\widetilde{Q}^I$ ($I=1,\dots,N_f$), which transform respectively in the fundamental and anti-fundamental representation of $SU(N)$ and together form an $\mathcal{N}=2$ hypermultiplet. The superfield action is given by [20]:

$$
\begin{aligned}
S = & \int d^4x \; d^4\theta \Big[\mathrm{Tr}\big(e^{-gV}\bar{\Phi} e^{gV}\Phi\big) + \bar{Q}^I e^{gV} Q_I + \tilde{Q}^I e^{-gV} \bar{\tilde{Q}}_I \Big] \\
& + \frac{1}{g^2} \int \; d^4x \; d^2\theta \; \mathrm{Tr}\big(W^\alpha W_\alpha\big) \\
& + ig \int \; d^4x \; d^2\theta \; \tilde{Q}^I \Phi Q_I - ig \int \; d^4x \; d^2\bar{\theta} \; \bar{Q}^I \bar{\Phi} \bar{\tilde{Q}}_I
\end{aligned}
\tag{1.27}
$$

The component action (1.17) is obtained projecting (1.27).

Chapter 2

Supersymmetric Wilson loops

Wilson loops are one of the most studied gauge-invariant operators in supersymmetric gauge theories. Physically, they represent the phase factors associated to the propagation of a charged particle along a closed path and in some sense they measure the response of the gauge field to the insertion of an external point-like source passing around a closed contour.
They were proposed by Wilson in [21] in the context of lattice formulation of quantum chromodynamics but their study became interesting outside of this context by their relation to relevant observables in QFT (like the bremsstrahlung function or the quark potential, as we will see in the next chapters).
They form a complete set of observables, which is to say that you can write down all other observables in terms of them by performing algebraic operations [22]. They can be computed both at weak (by means of perturbation theory) and strong coupling (using AdS/CFT correspondence). In some cases with a lot of symmetries and specific contours, they can be evaluated **exactly** using supersymmetric localization, as we will see in chapter 4, this is very useful because it can be used to test some conjectures of the holographic duality.
In this chapter we are going to introduce these objects both in ordinary gauge theories and in the supersymmetric gauge theories we studied in chapter (1). After giving its definition we will study the conditions under which these objects preserve the original supersymmetries of the theory and give some examples.

The Wilson loop

The Wilson loop in ordinary gauge theory is defined as:

$$W[\mathcal{C}] = \frac{1}{\dim_{\mathcal{R}}} \mathrm{Tr}_{\mathcal{R}} \, \mathcal{P} \exp\left[-ig \oint_{\mathcal{C}} dx^{\mu} A_{\mu}(x)\right] \tag{2.1}$$

where $\mathcal{R}$ is the representation where the fields are taken and $\mathcal{C}$ is the curve that is parametrized by $x^{\mu}(\tau)$. The operator $\mathcal{P}$ is the *path ordering* operator and it is defined as:

$$\mathcal{P} \oint_{\mathcal{C}} dx_1^{\mu_1} \ldots \oint_{\mathcal{C}} dx_n^{\mu_n} \equiv \int_0^1 d\tau_1 \int_0^{\tau_1} d\tau_2 \ldots \int_0^{\tau_{n-1}} d\tau_n \; \dot{x}^{\mu_1}(\tau_1) \ldots \dot{x}^{\mu_n}(\tau_n) \tag{2.2}$$

Instead of (2.1) we could also consider gauge theories endowed with supersymmetry where the vector multiplet includes also scalars and fermions. In the next sections we will extend the definition of the Wilson loop for $\mathcal{N} = 4$ SYM theory and $\mathcal{N} = 2$ SCQCD. We will study their supersymmetric properties and give some examples.

Wilson loops in $\mathcal{N} = 4$ Super Yang-Mills

In $\mathcal{N} = 4$ SYM theory the Wilson loop (2.1) is generalized to a supersymmetric version that includes also couplings with the scalar fields [1]. The most general Wilson loop we can consider for this theory is

$$\begin{aligned} W[\mathcal{C}] &= \frac{1}{\dim_{\mathcal{R}}} \mathrm{Tr}_{\mathcal{R}} \, \mathcal{P} \exp\left[-ig \oint_{\mathcal{C}} d\tau \, \mathcal{L}(\tau)\right] \\ \text{with} \quad \mathcal{L}(\tau) &= \dot{x}^{\mu} A_{\mu} + |\dot{x}| \, \Theta_i \, \phi_R^i \end{aligned} \tag{2.3}$$

where ϕ_R^i are the six real scalar fields of the theory. This is known as the *Maldacena-Wilson loop* [23]. With our conventions, working instead with 3 complex fields, the connection reads

[1] The additional scalar couplings in 2.3 become natural if we think to the $\mathcal{N} = 4$ SYM as the dimensional reduction of the $\mathcal{N} = 1$ SYM in ten dimensions.

$$\mathcal{L}(\tau) = \dot{x}^\mu A_\mu + \frac{\sqrt{2}}{2}\,|\dot{x}|\,(n^I\,\phi^I + \overline{n}^I\,\overline{\phi}^I) \tag{2.4}$$

The couplings with the scalars can vary along the path but they must satisfy $\Theta^i\Theta_i = n^I\overline{n}_I = 1$ to ensure the cancellation of the 1-loop divergences [24]. To see this, we first write the connection (2.3) in a more convenient way as:

$$\mathcal{L}(\tau) = \dot{x}^\mu A_\mu + \frac{\sqrt{2}}{2}\,(\dot{y}^I\,\phi^I + \dot{\overline{y}}^I\,\overline{\phi}^I) \tag{2.5}$$

The expression (2.4) is recovered when $\dot{y}^I = |\dot{x}|\,n^I$. Let's compute the expectation value of the Wilson loop at linear order in the 't Hooft parameter $\lambda = g^2N$ and, in order to isolate the divergent terms, replace the propagator $1/|x_1 - x_2|^2 \to 1/(|x_1 - x_2|^2 + \epsilon^2)$:

$$\langle W[\mathcal{C}]\rangle = 1 - \frac{\lambda}{(2\pi)^2}\oint d\tau_1 \int_0^{\tau_1} d\tau_2\,\frac{\dot{x}^\mu(\tau_1)\,\dot{x}_\mu(\tau_2) + \frac{1}{2}(\dot{y}^I(\tau_1)\,\dot{\overline{y}}_I(\tau_2) + \dot{\overline{y}}_I(\tau_1)\,\dot{y}^I(\tau_2))}{|x(\tau_1) - x(\tau_2)|^2 + \epsilon^2} \tag{2.6}$$

The divergence arises when $|\tau_2 - \tau_2| \sim \epsilon$ in the limit $\epsilon \to 0$, so we can write:

$$\begin{aligned}\langle W[\mathcal{C}]\rangle &= \text{finite} + \frac{\lambda}{(2\pi)^2}\oint d\tau_1 \int_{-\frac{\epsilon}{|\dot{x}_1|}}^{\frac{\epsilon}{|\dot{x}_1|}} d\tau_2\,\frac{\dot{x}^\mu(\tau_1)\,\dot{x}_\mu(\tau_1) + \dot{y}^I(\tau_1)\,\dot{\overline{y}}_I(\tau_1)}{\epsilon^2} \\ &= \text{finite} + \frac{\lambda}{(2\pi)^2}\,\frac{2}{\epsilon}\oint d\tau_1\,\frac{\dot{x}^\mu(\tau_1)\,\dot{x}_\mu(\tau_1) + \dot{y}^I(\tau_1)\,\dot{\overline{y}}_I(\tau_1)}{|\dot{x}(\tau_1)|}\end{aligned} \tag{2.7}$$

Then the condition to remove this divergence is:

$$\dot{y}^I\,\dot{\overline{y}}_I = -\dot{x}^\mu\,\dot{x}_\mu \tag{2.8}$$

It can be shown that this also cancels higher order divergences [24].
For a time-like curve in Minkowski[2] space we have that $\dot{x}^\mu\,\dot{x}_\mu = -|\dot{x}|^2$, so we obtain the connection (2.4) with $n^I\overline{n}_I = 1$ that parametrizes a vector lying on S^5. As we

[2]We exclude the case when the curve is space-like, since it can be shown that a BPS Wilson loop in $\mathcal{N} = 4$ SYM must be time-like or null in order to satisfy spinor constraints [25], and null Wilson loops do not couple to the scalars.

will see, this condition is also needed to guaranty supersymmetry.

When we work in the Euclidean theory, as we will do in the next chapters when doing perturbation theory, an extra i factor is produced in the term $|\dot{x}|$ and the connection is

$$\mathcal{L}(\tau) = \dot{x}^\mu A_\mu + i\,\frac{\sqrt{2}}{2}\,|\dot{x}|\,(n^I\,\phi^I + \overline{n}^I\,\overline{\phi}^I) \tag{2.9}$$

Now we are going to study the supersymmetric properties of the Wilson loop. The supersymmetric variation of (2.4) gives:

$$\delta W[\mathcal{C}] = \frac{1}{\mathrm{dim}_\mathcal{R}} \mathrm{Tr}_\mathcal{R}\,\mathcal{P}\,\oint_\mathcal{C} d\tau\,\left(-ig\,\delta\mathcal{L}(\tau)\right)\,e^{-ig\oint d\tau \mathcal{L}(\tau)} \tag{2.10}$$

where

$$\delta\mathcal{L} = \sqrt{2}\,\dot{x}^{\alpha\dot{\alpha}}\delta A_{\alpha\dot{\alpha}} + \frac{\sqrt{2}}{2}\,|\dot{x}|\,\left(n^I\,\delta\phi_I + \overline{n}^I\,\delta\overline{\phi}_I\right) \tag{2.11}$$

Remembering that the fields transform according to (1.16), at linear order we can write:

$$\begin{aligned}
\delta\mathcal{L} &= \sqrt{2}\,\dot{x}^{\alpha\dot{\alpha}}\left[i\,(\epsilon^I_\alpha\,\overline{\psi}^I_{\dot{\alpha}} + \overline{\epsilon}^I_{\dot{\alpha}}\,\psi^I_\alpha) - i\,(\overline{\epsilon}^0_{\dot{\alpha}}\,\eta_\alpha + \epsilon^0_\alpha\,\overline{\eta}_{\dot{\alpha}})\right] \\
&+ \frac{\sqrt{2}}{2}\,|\dot{x}|\,n^I\left[-\eta^\alpha\,\epsilon^I_\alpha - \epsilon^{IJK}\,\overline{\epsilon}^{\dot{\alpha}}_J\,\overline{\psi}_{K\dot{\alpha}} - \epsilon^\alpha_0\,\psi^I_\alpha\right] \\
&+ \frac{\sqrt{2}}{2}|\,\dot{x}|\,\overline{n}^I\left[-\overline{\eta}^{\dot{\alpha}}\,\overline{\epsilon}^I_{\dot{\alpha}} - \epsilon^{IJK}\,\epsilon^\alpha_J\,\psi_{K\alpha} - \overline{\epsilon}^{\dot{\alpha}}_0\,\overline{\psi}^I_{\dot{\alpha}}\right]
\end{aligned} \tag{2.12}$$

Imposing that $\delta W[\mathcal{C}] = 0$ we find that:

$$\begin{aligned}
2i\,\dot{x}^{\alpha\dot{\alpha}}\,\epsilon^0_\alpha + |\dot{x}|\,\overline{n}^I\,\overline{\epsilon}^{\dot{\alpha}}_I &= 0 \\
2i\,\dot{x}^{\alpha\dot{\alpha}}\,\epsilon^I_\alpha - |\dot{x}|\,\epsilon^{IJK}\,n_J\,\overline{\epsilon}^{\dot{\alpha}}_K - |\dot{x}|\,\overline{n}^I\,\overline{\epsilon}^{\dot{\alpha}}_0 &= 0
\end{aligned} \tag{2.13}$$

These are the conditions for a **locally** supersymmetric Wilson loop, this is a local condition because at every point of the curve the equation is different, so the condition for supersymmetry preservation varies along the curve. We might wonder when this is also a global property: the Wilson loop will be **globally** supersymmetric if the equations (2.13) are the same for all τ, this puts restrictions on the shape of the contour and on the couplings n^I. When this happens, a fraction of the original supersymmetry is globally preserved and the number of linearly independent ϵ^I_α that satisfies (2.13) determines the number of conserved supercharges. For generic $x_\mu(\tau)$ and $n^I(\tau)$, (2.13) form an infinite set of equations for all the components of ϵ^I_α and ϵ^0_α. However, there exist examples of non-trivial solutions for these equations, like the straight line or the circle with n^I constant or the Zarembo generalizations [26] that we are going to see in the next section.

Examples of SUSY Wilson loops

The simplest examples where the condition (2.13) is preserved along the whole loop are the straight line and the circular loop (which are the only two globally supersymmetric trajectories that preserve the maximum amount of supersymmetry with n^I constant) and the Zarembo generalizations [26].

Zarembo loops

In [26] Zarembo proposed a way to classify the supersymmetric Wilson loops, this analysis selects a class of loops which satisfy the following condition on the scalar couplings

$$n^I = i\, M^I_\mu \,\frac{\dot{x}^\mu}{|\dot{x}|} \qquad \text{with} \;\; M^I_\mu \overline{M}^I_\nu = -g_{\mu\nu} \tag{2.14}$$

The connection (2.4) becomes:

$$\mathcal{L}(\tau) = \dot{x}^\mu(\tau) \left(A_\mu + i\,\frac{\sqrt{2}}{2}\, (M^I_\mu\, \phi^I - \overline{M}^I_\mu\, \overline{\phi}^I) \right) \tag{2.15}$$

This property reduces (2.13) to an algebraic set of equations that, if consistent, imply that the Wilson loop is globally supersymmetric. These equations are:

$$
\begin{aligned}
i\,\dot{x}^{\mu}\left[(\sigma_{\mu})^{\alpha\dot{\alpha}}\,\epsilon_{\alpha}-\overline{M}^{I}_{\mu}\,\bar{\epsilon}^{\dot{\alpha}}_{I}\right] &= 0\\
i\,\dot{x}^{\mu}\left[(\sigma_{\mu})^{\alpha\dot{\alpha}}\,\epsilon^{I}_{\alpha}-\epsilon^{IJK}\,M^{J}_{\mu}\,\bar{\epsilon}^{\dot{\alpha}}_{K}+\overline{M}^{I}_{\mu}\,\bar{\epsilon}^{\dot{\alpha}}\right] &= 0
\end{aligned}
\tag{2.16}
$$

For a generic curve we have 4 independent algebraic equations whose space of solutions has dimension 1 [26], this means that 1 out of the 16 supercharges is preserved by the Wilson loop defined by the connection (2.15), we say that the Wilson loop is 1/16 BPS.
If the contour has a special shape, the supersymmetry is enhanced. If we constrain the curve to lay in a 3-dimensional subspace (setting for example $\dot{x}^0 = 0$), we have 3 instead of 4 equations and the number of preserved supersymmetries doubles and the loop is 1/8 BPS. Going on, when the loop lies inside a dimension-2 subspace the Wilson loop is 1/4 BPS, and for a dimension-1 curve is 1/2 BPS being the straight line a particular case of this.

In [26] it has been checked up to order g^4 that the expectation value of the loop (2.15) is $\langle W[\mathcal{C}]\rangle = 1$, this is basically due to the equality of the gauge and scalar propagators. The fact that g^2 and g^4 terms vanish gives the indication that the expectation value of this kind of Wilson loops is free of quantum correction. This conjecture made by Zarembo in his paper was proven in [27] using the AdS/CFT correspondence.

Circular loop

The best known example of a Wilson loop which does not posses a trivial expectation value is the circular one with n^I constant, discussed in [28]. This Wilson loop preserve half of the supercharges, thus it is 1/2-BPS.
Let's compute perturbatively its expectation value. The circle is parametrized in the following way

$$
\begin{aligned}
x^{\mu}(\tau) &= (\sin\tau,\cos\tau,0,0) && \text{in Euclidean}\\
x^{\mu}(\tau) &= (\sinh\tau,\cosh\tau,0,0) && \text{in Minkowski}
\end{aligned}
\tag{2.17}
$$

The propagators that enters in the perturbative expansion are given by

$$\langle \mathcal{L}^a(\tau_1)\,\mathcal{L}^b(\tau_2)\rangle = \frac{\delta^{ab}}{(2\pi)^2}\,\frac{\dot{x}_1\cdot\dot{x}_2+|\dot{x}_1||\dot{x}_2|}{(x_1-x_2)^2} = \frac{\delta^{ab}}{(2\pi)^2}\,\frac{1-\cosh(\tau_1-\tau_2)}{2-2\cosh(\tau_1-\tau_2)} = \frac{\delta^{ab}}{8\pi^2}\,. \tag{2.18}$$

Since the propagator is coordinate independent, one can sum all the diagrams with no internal vertex (the so-called *ladder planar graphs*). This can be done by finding the appropriate recurrence relation and then summed up to all the orders. The problem of summing the ladder graphs on the circle can be mapped to a matrix-model, this has been conjectured in [29] and proven by Pestun in [30] using the supersymmetric localization techniques that we are going to explain in chapter 4.
Next we have to look for the diagrams with internal vertex. It is possible to show that at order g^4 the sum of all these diagrams equals to zero [28]. In [28, 29] was conjectured that this cancellation occurs at all order in perturbation theory, so the expectation value of the circular Wilson loop is given by the matrix model proposed in [29].

Wilson loops in $\mathcal{N}=2$ Super Conformal QCD

The supersymmetric Wilson loop in $\mathcal{N}=2$ SCQCD is defined analogously to the $\mathcal{N}=4$ case with the connection now:

$$\mathcal{L}(\tau) = \dot{x}^\mu A_\mu + \frac{\sqrt{2}}{2}|\,\dot{x}|\,(n\,\phi+\overline{n}\,\overline{\phi}) \tag{2.19}$$

Since $n\,\overline{n}=1$, the connection can be parametrized in the following way

$$\mathcal{L}(\tau) = \dot{x}^\mu A_\mu + \frac{\sqrt{2}}{2}|\,\dot{x}|\,(e^{i\theta(\tau)}\,\phi+e^{-i\theta(\tau)}\,\overline{\phi}) \tag{2.20}$$

The SUSY variation of the connection takes the form

$$\begin{aligned} \delta\mathcal{L} &= \sqrt{2}\,\dot{x}^{\alpha\dot{\alpha}}\left[i\,(\epsilon^1_\alpha\,\overline{\psi}_{\dot{\alpha}}+\overline{\epsilon}^1_{\dot{\alpha}}\,\psi_\alpha)-i\,(\overline{\epsilon}^0_{\dot{\alpha}}\,\eta_\alpha-\epsilon^0_\alpha\,\overline{\eta}_{\dot{\alpha}})\right] \\ &+ \frac{\sqrt{2}}{2}\,|\dot{x}|\,e^{i\theta}\left[-\eta^\alpha\,\epsilon^1_\alpha-\epsilon^\alpha_0\,\psi_\alpha\right]+\frac{\sqrt{2}}{2}\,|\dot{x}|\,e^{-i\theta}\left[-\overline{\eta}^{\dot{\alpha}}\,\overline{\epsilon}^1_{\dot{\alpha}}-\overline{\epsilon}^{\dot{\alpha}}_0\,\overline{\psi}_{\dot{\alpha}}\right] \end{aligned} \tag{2.21}$$

And the equations (2.13) for supersymmetry preservation now are

$$\begin{aligned} 2i\,\dot{x}^{\alpha\dot{\alpha}}\,\epsilon_\alpha + |\dot{x}|\,e^{-i\theta}\,\bar{\epsilon}_1^{\dot{\alpha}} &= 0 \\ 2i\,\dot{x}^{\alpha\dot{\alpha}}\,\bar{\epsilon}_{\dot{\alpha}}^1 - |\dot{x}|\,e^{i\theta}\,\epsilon^\alpha &= 0 \end{aligned} \tag{2.22}$$

In this case, the equations (2.22) can be solved very easily to give:

$$\epsilon^\alpha = 2i\,e^{-i\theta}\,\frac{\dot{x}^{\alpha\dot{\alpha}}}{|\dot{x}|}\,\bar{\epsilon}_{\dot{\alpha}}^1 \tag{2.23}$$

These constraints fix half of the parameters of the supersymmetry transformations. Again this condition should hold at every point of the path, so this is a **local** condition for supersymmetry. The conditions for the Wilson loop to be globally supersymmetric are the same as the ones considered for the $\mathcal{N} = 4$ SYM case.

Chapter 3

The cusp anomalous dimension

In this chapter we are going to study the behaviour of Wilson loops which suffers from discontinuities. We will study a particular type of discontinuous loops: those formed by the intersection of two straight lines. We will see that in these cases, the expectation value of the Wilson loop develops logarithmic divergences and these divergences are controlled by a quantity called *the cusp anomalous dimension*. This quantity is related to many physical observables in the theory and in some cases it can be systematically computed using perturbation theory or localization. One of the most interesting aspects of the cusp anomalous dimension is observed when we take the zero limit of the angle defining the cusp, in this case the cusp anomalous dimension is related to a quantity that determines the energy radiated by a moving particle: the *Bremsstrahlung function*.
We start in section 3.1 by reviewing some aspects of divergences in Wilson loops and we will see how the cusp anomalous dimension appears. Then in section 3.1.1 we will study various limits of the cusp anomalous dimension and we will see how certain physical observables can be obtained from it. In section 3.2 we will generalize the definition of this quantity to a form applicable to supersymmetric theories that have Wilson loop operators that include the extra coupling with scalars. We will see that in these cases it is possible to define another angle, apart from the geometrical one, that relates the coupling with the scalars at both sides of the cusp and study the various possible limits of it. In section 3.3 we will concentrate on the light-like cusp anomalous dimension and we will relate it to the

possibility of the existence of a universal function of gauge theories.

Cusped Wilson loops

We star by considering the ordinary Wilson loop in a generic gauge theory

$$W[\mathcal{C}] = \frac{1}{\dim_{\mathcal{R}}} \mathrm{Tr}_{\mathcal{R}}\, \mathcal{P} \exp\left[-ig \oint_{\mathcal{C}} dx^{\mu} A_{\mu}(x)\right] \tag{3.1}$$

The theory can have some UV divergences. As we saw in chapter 1, the fields and couplings are renormalized by counterterms, these potentially divergent terms are inherent to the lagrangian of the theory and don't depend on the contour of the Wilson loop. Let's assume that we already performed this renormalization for the fields and couplings and concentrate our discussion on the type of divergences that arise due to the particular shape of the loop.

When the loop is a smooth curve (meaning that it doesn't have discontinuities), these divergences can be factorized as [31]

$$\langle W[\mathcal{C}] \rangle = e^{-\frac{k}{a} L[\mathcal{C}]} \times \text{finite} \tag{3.2}$$

where $L[\mathcal{C}]$ is the length of the contour and a is the UV cut-off used to regularize small distances.

Now consider a loop with a cusp, which is a path formed by the intersection of two straight lines that form an angle φ as it is shown in figure 3.1

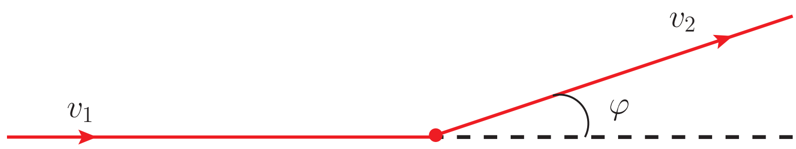


Figure 3.1: A cusped path $\mathcal{C}$ characterized by an angle φ.

The cusp is characterized by the euclidean angle φ defined as: $\cos\varphi = \frac{v_1 \cdot v_2}{|v_1||v_2|}$. Alternatively, working in Minkowski space: $\cosh\varphi_M = \frac{v_1 \cdot v_2}{|v_1||v_2|}$.

Due to the presence of the cusp new divergences appear. In [32] was shown that these divergences can be removed at any order by a multiplicative renormalization constant

$$W_R[\mathcal{C}] = Z^{-1}_{\text{cusp}}(\varphi) W[\mathcal{C}] \tag{3.3}$$

In [33, 32] a renormalization group equation was derived for the cusped Wilson loop:

$$\left(\mu \frac{\partial}{\partial\mu} + \beta(g_R) \frac{\partial}{\partial g_R} + \Gamma_{\text{cusp}}(\varphi, g_R)\right) W_R(\mu, \varphi, g_R) = 0 \tag{3.4}$$

where μ is the renormalization mass scale, g_R is the renormalized coupling constant, $\beta(g_R)$ is the β-function defined in section 1.1.5 and

$$\Gamma_{\text{cusp}} = \frac{\partial \log Z}{\partial \log \mu} \tag{3.5}$$

is the *cusp anomalous dimension*. Equation (3.4) together with the definition of the cusp anomalous dimension imply that the divergent part of the Wilson loop exponentiates as [31]

$$\langle W[\mathcal{C}_{\text{cusp}}]\rangle \propto e^{-\Gamma_{\text{cusp}}(\varphi, g)\log\frac{L}{a}} \tag{3.6}$$

where again L is the length of the contour and a is the short-distance cut-off.

Equation (3.6) provides the standard prescription for computing the cusp anomalous dimension at weak coupling. In fact, it is sufficient to compute the cusped Wilson operator order by order in g, using dimensional or cut–off regularization to tame UV divergences and introducing a suppression factor to mitigate divergences at large distances. In dimensional regularization, which consists on shifting the spacetime dimension as $d = 4 - 2\epsilon$, Γ_{cusp} can be read from the coefficient of the $1/\epsilon$ pole. We will see in chapter 5 explicitly how to do it.

Limits of the cusp anomalous dimension

The cusp anomalous dimension has an interesting dependence on the cusp angle. The following three limits are of physical importance:

- **large angles:** In the limit of large Minkowskian angles we obtain

$$\Gamma_{\text{cusp}}(g, \varphi_M) \xrightarrow{\varphi_M \to \infty} K(g)\, \varphi_M + O(\varphi_M^0) \tag{3.7}$$

  where $K(g)$ is the anomalous dimension of a null Wilson loop [32]. It was computed in perturbation theory [34, 35] in $\mathcal{N} = 4$ SYM. It can be determined exactly by an integral equation [36]. We are going to talk about it in more detail in section 3.3.

- **anti-parallel lines:** When $\varphi \sim \pi$ the cusp anomalous dimension develops a pole

$$\Gamma_{\text{cusp}}(g, \varphi) = -\frac{V(g)}{\pi - \varphi} \qquad \varphi \sim \pi \tag{3.8}$$

  In conformal gauge theories $V(g)$ is related to the the quark - anti quark potential.

- **small angles:** When $\varphi \to 0$ the cusp divergences disappear and the cusp anomalous dimension vanishes as

$$\Gamma_{\text{cusp}}(g, \varphi) \xrightarrow{\varphi \to 0} -\varphi^2 B(g) \tag{3.9}$$

  $B(g)$ is the Bremsstrahlung function of the theory and it is related to the energy radiated by a moving particle [1]. In some theories it can be computed exactly using localization, as we will see in chapter 4.

Generalized cusp anomalous dimension

Now we want to study the supersymmetric version of the Wilson loop that was defined in (2.4) and (2.19) for $\mathcal{N} = 4$ SYM and $\mathcal{N} = 2$ SCQCD respectively as:

$$\langle W[\mathcal{C}] \rangle = \frac{1}{\dim_{\mathcal{R}}} \mathrm{Tr}_{\mathcal{R}}\, \mathcal{P} \exp \left[-ig \oint_{\mathcal{C}} d\tau\, \mathcal{L}(\tau) \right] \tag{3.10}$$

with

$$\begin{aligned} \mathcal{L}_{\mathcal{N}=4} &= \dot{x}^\mu A_\mu + \frac{\sqrt{2}}{2} |\dot{x}| \, (n^I \phi^I + \overline{n}^I \overline{\phi}^I) \\ \mathcal{L}_{\mathcal{N}=2} &= \dot{x}^\mu A_\mu + \frac{\sqrt{2}}{2} |\, \dot{x}| \, (n\, \phi + \overline{n}\, \overline{\phi}) \end{aligned} \tag{3.11}$$

We can use the couplings n^I (and n) as directions in some internal space (S^5 for $\mathcal{N}=4$ and in S^1 for $\mathcal{N}=2$) and, instead of considering the configuration n^I constant along the whole loop, we can take $\overrightarrow{n}_1$ and $\overrightarrow{n}_2$, allowing a change when the direction v_1 changes to v_2. Then we can define an *internal angle*

$$\cos\theta = \frac{1}{2}\left[n_1^I\,\overline{n}_2^I + \overline{n}_1^I\,n_2^I\right] \tag{3.12}$$

as showed in picture 3.2

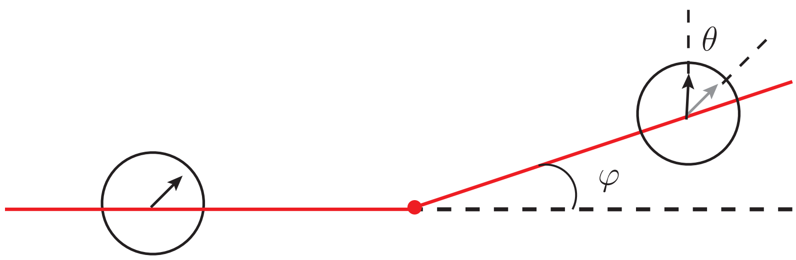


Figure 3.2: Change in the couplings with the scalars, characterized by an angle θ in the internal space.

Then the cusp anomalous dimension will depend on both angles: $\Gamma_{\text{cusp}}(g,\varphi,\theta)$. This quantity is called *generalized cusp anomalous dimension* and it shares some of the properties of the ordinary cusp anomalous dimension. For $\mathcal{N}=4$ SYM it has been computed at weak coupling in [24]

- in $\mathcal{N}=4$ SYM it characterizes the planar IR diverges that arise in the scattering of massive W-bosons on the Coulomb branch of $\mathcal{N}=4$ SYM [1].

- When $\theta=\pm\varphi$ the configuration is supersymmetric and the cusp anomalous dimension vanishes. In this case the Wilson loop is BPS, it is a particular case of the 1/4-BPS loops [26] that we saw in section 2.2.1. We will come back to this in section 3.2.

- In the limit $\varphi\to\pi$, the generalized quark-antiquark potential is obtained [37] and matches the usual one (3.8) when $\theta=0$.

BPS configurations

In general the Wilson loop defined by the angle 3.2 is not BPS, except in some trivial cases. For example, when the the coupling with the scalars doesn't change ($\theta = 0$) the Wilson loop is 1/2-BPS for $\varphi = 0$, which is the straight line studied in the previous chapter. But turns out that for the situation where $\theta \neq 0$ there exist a non-trivial case for which the loop becomes BPS as well, and it is when $\theta = \pm\varphi$. Around this point the cusp anomalous dimension behaves as

$$\Gamma_{\text{cusp}}(g, \varphi, \theta) \simeq -(\varphi^2 - \theta^2)\mathcal{H}(g, \varphi) \qquad \varphi \approx \pm\theta \tag{3.13}$$

The very interesting feature of this identity is that the function $\mathcal{H}(g, \varphi)$ has the following property:

$$\mathcal{H}(g, \varphi = 0) = B(g) \tag{3.14}$$

where $B(g)$ is the bremsstrahlung function for the ordinary loop with $\theta = 0$. This property trivially implies that

$$\Gamma_{\text{cusp}}(g, \varphi = 0, \theta) = \theta^2 B(g) \tag{3.15}$$

This means that we can compute $B(g)$ either by taking $\varphi \ll 1$ with $\theta = 0$, or $\theta \ll 1$ with $\varphi = 0$. Making a clever use of this fact, we can reduce a lot the computations when calculating the bremsstrahlung function.
Let's illustrate this with a 1-loop example: at this order, the contributions to the expectation value of the Wilson loop of the type (3.11) will be just the single gluon and scalar exchange:

Figure 3.3: Gluon and scalar exchange that contributes to the 1-loop order in the Wilson loop expectation value.

And will produce a factor

$$\langle W \rangle^{(1)} \propto \int d\tau_1 \int d\tau_2 \, \frac{\cos\varphi - \cos\theta}{|v_1^\mu \tau_1 - v_2^\mu \tau_2|^2} \tag{3.16}$$

As we saw before, the 1-loop term in the cusp anomalous dimension can be obtained directly from this expression after regularizing divergences. In order to

obtain $B(g)$, we will need to keep track of the coefficient of the quadratic term φ^2, but due to the properties (3.13) and (3.14), we can also derive $B(g)$ by setting $\varphi = 0$ and taking the θ^2 coefficient.
Therefore, from expression (3.16) only the term proportional to $\cos\theta$ (in this case, the scalar propagator contribution) will be necessary, so we can ignore all the diagrams that do not involve $\cos\theta$ factors (the gluon exchange in figure 3.3).

Light-like Cusp anomalous dimension

The light–like cusp can be used to check an interesting universal behavior of the cusp anomalous dimension that was found to hold up to three loops in QCD and in Yang–Mills theories with only adjoint matter [4, 38]. Precisely, when expressed in terms of the light-like cusp replacing the coupling constant, the cusp anomalous dimension gives rise to an universal function $\Omega(K,\varphi)$ that is independent of the number of fermion or scalar fields in the theory. This is done in the following way: Define a new effective coupling constant $a = K(g)$, the universal function is defined as

$$\Omega(a,\varphi) := \Gamma_{\mathrm{cusp}}(g,\varphi) \tag{3.17}$$

The expansion coefficients of the two functions are related to each other as

$$\begin{aligned} \Omega(a,\varphi) \;&=\; \frac{a}{K^{(1)}}\,\Gamma^{(1)}(\varphi) + \left(\frac{a}{K^{(1)}}\right)^2 \left[\Gamma^{(2)}(\varphi) - \frac{K^{(2)}\,\Gamma^{(1)}(\varphi)}{K^{(1)}}\right] \\ &+\; \left(\frac{a}{K^{(1)}}\right)^3 \left[\Gamma^{(3)}(\varphi) - \frac{K^{(3)}\,\Gamma^{(1)}(\varphi)}{K^{(1)}} - \frac{K^{(2)}\,\Gamma^{(2)}(\varphi)}{K^{(1)}} + \frac{(K^{(2)})^2\,\Gamma^{(1)}(\varphi)}{(K^{(1)})^2}\right] + \cdots \end{aligned} \tag{3.18}$$

This function has been computed for $\mathcal{N} = 4$ SYM, for QCD and for a general Yang-Mills theory containing fermions and scalars. Since the cusp anomalous dimension depends on the particle content of the theory, it is expected to find different results for $\Omega(a,\varphi)$. In [4] it was found that, at least to three loops, the function is the same in any gauge theory.

$$\Omega_{\mathcal{N}=4}(a,\varphi) = \Omega_{QCD}(a,\varphi) = \Omega_{YM}(a,\varphi). \tag{3.19}$$

This means that all the dependence on the particular theory stands in $K(g^2)$. Equation (3.17) is valid for any φ, in particular

$$\Omega(a,\varphi) = -\varphi^2 \widetilde{B}(a) + O(\varphi^3) \tag{3.20}$$

By construction this means that $\widetilde{B}(a) = B(g)$ is a universal function up to at least three loops.

We will come back to this point again at the end of chapter 5 when we will talk about the existence of a substitution rule relating $\mathcal{N} = 4$ SYM and $\mathcal{N} = 2$ SCQCD [39, 40, 41].

Chapter 4

Localization

In this chapter we will introduce the concept of localization. It was originally studied in the context of equivariant cohomology and topological theories [44, 45], then its utility was extended within quantum field theories.
This method allow us to compute certain functional integrals as partition functions or expectation values of some operators in supersymmetric theories. If we see the fields as coordinates in a super-manifold[1], the introduction of an operator that modifies the action in a controlled way permits to restrict the integration domain to a submanifold called the *localization locus*, which in general has lower dimensionality than the original one. The price to pay is the appearance of some determinants that in general are not easy to compute, except in some cases with a lot of symmetries.
The localization formula that we are going to derive provides a way to obtain non-perturbatively the exact results of local and non-local operators (such as Wilson loops [30] and their correlation functions). Having an exact expression for expectation values is very powerful since it can be used, for example, to test some aspects of dualities.
In this chapter we will start in section 4.1 by presenting the basic and intuitive idea of supersymmetric localization and explain how in some cases this procedure reduces the functional integral to a zero-dimensional one known as Matrix Model. In sections 4.2 and 4.3 we will see that this is the case for the theories $\mathcal{N} = 4$ SYM

[1] A supermanifold is a manifold with both bosonic and fermionic coordinates.

and $\mathcal{N} = 2$ SCQCD, and we will give the expectation value of a circular Wilson loop computed by this method. We will also see how we can use localization in a suitable limit to obtain the exact form of the Bremsstrahlung function.

Supersymmetric localization and Matrix Models

In this section we will explain the concept of localization [30, 46, 47]. Let's denote collectively the fields in the theory by Φ and the action by $S[\Phi]$. The euclidean partition function [2] of this theory is defined as

$$\mathcal{Z} = \int \mathcal{D}\Phi e^{-S[\Phi]} \tag{4.1}$$

Suppose that we have a fermionic symmetry of the action generated by $\mathcal{Q}$, this means that $\mathcal{Q}S[\Phi] = 0$. In general, $\mathcal{Q}^2$ is a bosonic symmetry of the theory.
Let's assume that we have an operator $\mathcal{O}$ that is also invariant under the action of $\mathcal{Q}$ ($\mathcal{Q}\,\mathcal{O} = 0$), and assume that the symmetry is not anomalous[3], the operator $\mathcal{O}$ could be a local operator or a non-local operator like a supersymmetric Wilson loop. We want to compute the expectation value of this operator, that is defined as

$$\langle \mathcal{O} \rangle = \int \mathcal{D}\Phi\, \mathcal{O}\, e^{-S[\Phi]} \tag{4.2}$$

The localization technique is based in the following idea: we can modify the action and the partition function in a certain way without affecting $\langle \mathcal{O} \rangle$. In fact, let $V[\Phi]$ be a fermionic functional constructed in such a way that $\mathcal{Q}^2 V = 0$ and the bosonic part of $\mathcal{Q}V$ being positive semi-definite (the need for this condition will be clearer later). We can consider a modification of the action given by

$$S[\Phi] \to S^{'}[\Phi, t] = S[\Phi] + t\, \mathcal{Q}V \tag{4.3}$$

where t is a real parameter. In this way, we get a new theory defined by the action $S^{'}[\Phi, t]$ which is still invariant under $\mathcal{Q}$. Note that we recover the original

[2] The same theory defined in Minkowski space-time we replace $-S[\Phi] \to iS[\Phi]$.

[3] An anomalous symmetry will produce a change of the integration measure $\mathcal{D}\Phi$ under the action of $\mathcal{Q}$, meaning that $\mathcal{Q}(\mathcal{D}\Phi) \neq \mathcal{D}\Phi$ and thus changing the value of the partition function.

theory for $t = 0$.

The path integral of the new theory is

$$\mathcal{Z}(t) = \int \mathcal{D}\Phi e^{-S[\Phi]-t\,\mathcal{Q}V[\Phi]} \tag{4.4}$$

We now compute the expectation value of $\mathcal{O}$ in the modified theory:

$$\langle\mathcal{O}\rangle(t) = \int \mathcal{D}\Phi\,\mathcal{O}\,e^{-S[\Phi]-t\,\mathcal{Q}V[\Phi]} \tag{4.5}$$

To see that the expectation value of $\mathcal{O}$ is the same in both theories, we take the derivative with respect to t of the above expression:

$$\frac{d}{dt}\langle\mathcal{O}\rangle(t) = -\int \mathcal{D}\Phi\,\mathcal{O}\,(\mathcal{Q}V)e^{-S[\Phi]-t\mathcal{Q}V[\Phi]} \tag{4.6}$$

We know that $\mathcal{Q}(S + t\mathcal{Q}V) = 0$ by construction and $\mathcal{Q}(\mathcal{O}) = 0$, so we can take $\mathcal{Q}$ outside the whole integrand:

$$\mathcal{O}\,(\mathcal{Q}V)\,e^{-S-t\,\mathcal{Q}V} = \mathcal{Q}\,\left(\mathcal{O}\,V\,e^{-S-t\,\mathcal{Q}V}\right) \tag{4.7}$$

with this, equation (4.6) is written as

$$\frac{d}{dt}\langle\mathcal{O}\rangle(t) = -\int \mathcal{D}\Phi\,\mathcal{Q}\,(\mathcal{O}\,V\,e^{-S[\Phi]-t\,\mathcal{Q}V[\Phi]}) \tag{4.8}$$

Finally, we can make use of the fact that the measure $\mathcal{D}\Phi$ is invariant under the action of $\mathcal{Q}$ to say that the integrand of (4.8) is a total derivative in the field space. So after conveniently fixing the boundary terms, the result is zero. Thus, the expectation value of the $\mathcal{Q}$-invariant operator $\mathcal{O}$ is the same if we compute it in any of the theories defined by the modified action (4.3):

$$\langle\mathcal{O}\rangle(t) = \langle\mathcal{O}\rangle \qquad \forall t \tag{4.9}$$

In particular the result will be the same if we compute it in the limit $t \to \infty$.

$$\langle\mathcal{O}\rangle = \lim_{t\to\infty}\langle\mathcal{O}\rangle(t) \tag{4.10}$$

This limit is particularly interesting because computations are much more easy to do, as we will see. In fact, is easy to see that due to the assumptions we made

on $\mathcal{Q}V$, the contributions to the functional integral for large t are exponentially suppressed, except at the points where $\mathcal{Q}V = 0$.
Therefore in the limit $t \to \infty$ the only points contributing to the integral are the ones that lay in the subset $\{\Phi_0 \,/\, \mathcal{Q}V[\Phi_0] = 0\}$ that we call the *localization locus* of $\mathcal{Q}V$. We say that the integral *localizes* in this set of points.

Now we want to evaluate the path-integral (4.4) in the limit $t \to \infty$. To do this, lets parametrize the fluctuations of the fields around the localization locus as

$$\Phi = \Phi_0 + \frac{1}{\sqrt{t}}\,\delta\Phi \tag{4.11}$$

Now expand the deformed action around these field configurations:

$$\begin{aligned} S + t\,\mathcal{Q}V &= S[\Phi_0] + t\left[(\mathcal{Q}V)[\Phi_0] + \int\left(\frac{\delta(\mathcal{Q}V)}{\delta\Phi}\bigg|_{[\Phi_0]}\frac{\delta\Phi}{\sqrt{t}}\right)\right. \\ &+ \left.\frac{1}{2}\int\!\!\int\left(\frac{\delta^{(2)}(\mathcal{Q}V)}{\delta\Phi\,\delta\Phi}\bigg|_{[\Phi_0]}\frac{(\delta\Phi)^2}{t}\right)\right] + O(t^{-1/2}) \\ &= S[\Phi_0] + \frac{1}{2}\int\!\!\int\left(\frac{\delta^{(2)}(\mathcal{Q}V)}{\delta\Phi\,\delta\Phi}\bigg|_{[\Phi_0]}(\delta\Phi)^2\right) + O(t^{-1/2}) \end{aligned} \tag{4.12}$$

where in the last line we have used the fact that Φ_0 are zeros and fixed points of $\mathcal{Q}V$. Therefore only the on-shell action S_0 and the second order variations of $\mathcal{Q}V$ around the fixed points enters in the limit $t \to \infty$. We say that the result is *1-loop exact*: meaning that the higher order terms in the expansion around Φ_0 vanish in the $t \to \infty$ limit.
Then we arrive to the localization formula for the path-integral

$$\mathcal{Z} = \int \mathcal{D}\Phi_0\, \mathcal{Z}_{\text{1-loop}}[\Phi_0]\, e^{-S[\Phi_0]} \tag{4.13}$$

where $\mathcal{Z}_{\text{1-loop}}[\Phi_0]$ is the 1-loop determinant that characterizes the fluctuations around Φ_0 and it is given by

$$\mathcal{Z}_{\text{1-loop}}[\Phi_0] = \text{SDet}\left[\frac{\delta^{(2)}(\mathcal{Q}V)}{\delta\Phi\,\delta\Phi}\right] \tag{4.14}$$

where "SDet[...]" is the ratio of the determinants of the operators appearing at quadratic orders in the bosonic and fermionic fluctuations in (4.12). The localization formula (4.13) is a very powerful tool that we can use to drastically simplify the path integral that one needs to compute to evaluate expectation values of $\mathcal{Q}$-invariant observables. Depending on the space-time dependence of the field configurations belonging to the localization locus, one may be left with the path integral of a lower-dimensional quantum field theory. In most of the cases cases, the fields on the locus are independent of the space-time coordinates and we get a finite-dimensional integral of a zero-dimensional quantum field theory such as a *matrix model* [47].

When we derived the formula (4.13) we haven't had into account the possible instanton contributions. To give a more accurate expression we should include their contribution to the path integral in the form of an extra factor in (4.13). The formula will be then

$$\mathcal{Z} = \int \mathcal{D}\Phi_0 \, \mathcal{Z}_{\text{1-loop}}[\Phi_0] \, \mathcal{Z}_{\text{inst}}[\Phi_0] \, e^{-S[\Phi_0]} \tag{4.15}$$

where $\mathcal{Z}_{\text{inst}}$ is the partition function of the instantons [48, 30].

We will apply these results to compute the **exact** expectation value of the 1/2-BPS circular Wilson loop in $\mathcal{N} = 4$ SYM theory and in $\mathcal{N} = 2$ SCQCD. [30, 49].

Matrix Models

Matrix models are the simplest examples of quantum field theories, they are quantum gauge theories in zero dimensions where the basic field is a Hermitian $N \times N$ matrix a. We will follow the review [50] and we take the action for a to be of the form:

$$S[a] = \frac{1}{2} \operatorname{Tr}(a^2) + \sum_{p \geq 3} \frac{g_p}{p} \operatorname{Tr}(a^p) \tag{4.16}$$

where g_p are p-dependent coupling constants.

The action has the gauge symmetry

$$a \to U \, a \, U^\dagger \tag{4.17}$$

where U is a $U(N)$ matrix. The partition function of this theory is

$$\mathcal{Z} = \frac{1}{\operatorname{Vol}(U(N))} \int [da] \, e^{-\frac{1}{g} S[a]} \tag{4.18}$$

where g is another coupling constant that gives the correct dimension to the propagators, each interaction vertex with p-legs gives a power g_p/g. The factor $\mathrm{Vol}(U(N))$ is the usual volume factor of the gauge group.

The measure of the path integral is

$$[da] = 2^{\frac{N(N-1)}{2}} \prod_{i=1}^{N} da_{ii} \prod_{1 \leq i \leq j \leq N} dRe\, a_{ij}\; dIm\, a_{ij} \tag{4.19}$$

The simplest example of a Matrix Model is the quadratic one, known as the Gaussian Matrix Model. It is defined by the partition function

$$\mathcal{Z} = \frac{1}{\mathrm{Vol}(G)} \int [da]\, e^{-\frac{1}{2g^2}\mathrm{Tr}(a^2)} \tag{4.20}$$

The expectation value of an observable $\mathcal{O}[a]$ in this model is given by

$$\langle \mathcal{O}[a] \rangle = \frac{\int [da]\, \mathcal{O}[a]\, e^{-\frac{1}{2g^2}\mathrm{Tr}(a^2)}}{\int [da]\, e^{-\frac{1}{2g^2}\mathrm{Tr}(a^2)}} \tag{4.21}$$

This model is exactly solvable, meaning that the expectation values (4.21) can be computed systematically [50].
The partition function of more general matrix models with action (4.16) can be evaluated by doing perturbation theory around the Gaussian point: one expands the exponential of $\sum_{p\geq 3}(g_p/g)\mathrm{Tr}(a^p)/p$ in (4.18), and computes the partition function as a power series in the coupling constants g_p. The evaluation of each term of the series involves the computation of Gaussian integrals of the form (4.21).

Matrix Model for $\mathcal{N} = 4$ Super Yang-Mills

In section 2.2.1 we mentioned that in order to compute the expectation value of the circular Wilson loop only certain type of diagrams (ladder graphs) have to be considered [28]. Moreover, since each propagator that enters in the loop contributes with a constant factor, the sum can be done by means of a zero-dimensional field theory, i.e: a matrix model. In [29] Drukker and Gross proposed that the value of the circular loop can be calculated exactly with a Gaussian Matrix Model. The conjecture then states that, at any order, the result is

$$\langle W[\mathcal{C}_{\bigcirc}]\rangle = \frac{\int [da]\,\mathrm{Tr}(e^{-2\pi a})\,e^{-\frac{8\pi^2}{g^2}\mathrm{Tr}(a^2)}}{\int [da]\,e^{-\frac{8\pi^2}{g^2}\mathrm{Tr}(a^2)}} \tag{4.22}$$

Since this matrix model is Gaussian, it can be solved explicitly. The final **exact** result turns out to be

$$\langle W[\mathcal{C}_{\bigcirc}]\rangle = \frac{1}{N}\,L^1_{N-1}(-\frac{g^2}{4})\,e^{\frac{g^2}{8}} \tag{4.23}$$

where L^i_j is a Laguerre polynomial, given by

$$L^i_j(x) = \frac{1}{n!}\,e^x\,x^{-i}\,\frac{d^j}{dx}(e^{-x}\,x^{i+j}) \tag{4.24}$$

In [30] Pestun proved this conjecture by using the localization technique that we saw before and generalized it for $\mathcal{N}=2$ theories. Looking at the formula (4.13), for the particular case of $\mathcal{N}=4$ SYM all the instanton contributions vanish and the 1-loop determinant is exactly 1 (meaning that fermionic and bosonic contributions cancel). The partition function is then

$$\mathcal{Z}_{\mathcal{N}=4} = \int_{S^4} [da]\,e^{-\frac{8\pi^2}{g^2}\mathrm{Tr}(a^2)} \tag{4.25}$$

The Bremsstrahlung function

In the previous chapter we have defined the bremsstrahlung function as the coefficient of the quadratic term in the small angle expansion of the cusp anomalous dimension.

$$\Gamma_{\mathrm{cusp}}(g,\varphi) \xrightarrow{\varphi\to 0} -\varphi^2 B(g) \tag{4.26}$$

In [1] it was proved that $B(g)$ can be obtained from the expectation value of the 1/2-BPS circular Wilson loop as a derivative with respect to the 't Hooft coupling $\lambda = g^2 N$:

$$B_{\mathcal{N}=4} = \frac{1}{2\pi^2}\,\lambda\,\partial_\lambda \log\langle W[\mathcal{C}_{\bigcirc}]\rangle \tag{4.27}$$

where $\langle W[\mathcal{C}_{\bigcirc}]\rangle$ is the expectation value of the circular Wilson that can be computed via the matrix model at all orders in λ and it is given by (4.23). In the large N limit the bremsstrahlung function is:

$$B_{\mathcal{N}=4} = \frac{\sqrt{\lambda}}{4\pi^2}\,\frac{I_2(\sqrt{\lambda})}{I_1(\sqrt{\lambda})} \;+\; O(1/N^2) \tag{4.28}$$

where $I_n(x)$ are the modified Bessel function of the first kind. We can expand this in power series of λ and we get:

$$B_{\mathcal{N}=4} = \frac{\lambda}{4\pi^2} \;-\; \frac{\lambda^2}{384\pi^2} \;+\; \frac{\lambda^3}{6144\pi^2} \;+\; O(\lambda^4) \tag{4.29}$$

which agrees with [37].

We can obtain an alternative expression for the bremsstrahlung function if we place the circular Wilson loop on the ellipsoid S_b:

$$x_0^2 + \frac{x_1^2 + x_2^2}{l^2} + \frac{x_3^2 + x_4^2}{\widetilde{l}^2} \tag{4.30}$$

The squashing parameter is $b = \sqrt{l/\widetilde{l}}$, and the case $b = 1$ corresponds to the sphere S^4. In this case, the expectation value of the circular Wilson loop given by the matrix model (4.22) is now a function of the squashing parameter b and becomes[4]:

$$\langle W_{\bigcirc}\rangle_b = \frac{\int [da]\,\mathrm{Tr}(e^{-2\pi b a})\; e^{-\frac{8\pi^2 N}{\lambda}\mathrm{Tr}(a^2)}}{\int [da]\, e^{-\frac{8\pi^2 N}{\lambda}\mathrm{Tr}(a^2)}} \;+\; O(b-1)^2 \tag{4.31}$$

Now, working in the ellipsoid, the expression (4.27) is equivalent to

$$B_{\mathcal{N}=4} = \frac{1}{4\pi^2}\,\partial_b \log\langle W_{\bigcirc}\rangle_b \Big|_{b=1} \tag{4.32}$$

It is easy to see that both expressions are equivalent. In (4.31) let's rescale $a = \sqrt{\lambda}\widehat{a}$:

$$\langle W_{\bigcirc}\rangle_b = \frac{\int [d\widehat{a}]\,\mathrm{Tr}(e^{-2\pi b\sqrt{\lambda}\widehat{a}})\; e^{-8\pi^2 N \mathrm{Tr}(\widehat{a}^2)}}{\int [d\widehat{a}]\, e^{-8\pi^2 N\mathrm{Tr}(\widehat{a}^2)}} \tag{4.33}$$

then we can see that $\langle W_{\bigcirc}\rangle_b$ depends on b only trough $b\sqrt{\lambda}$. So, noting that $\partial_b[\ldots] = \sqrt{\lambda}\,\partial_{(b\sqrt{\lambda})}[\ldots] = \frac{2\lambda}{b}\,\partial_\lambda[\ldots]$, is automatic to see that

$$\frac{1}{4\pi^2}\,\partial_b \log\langle W_{\bigcirc}\rangle_b \Big|_{b=1} = \frac{1}{2\pi^2}\,\lambda\,\partial_\lambda \log\langle W_{\bigcirc}\rangle_b \tag{4.34}$$

[4]We can obtain the Wilson loop on the ellipsoid by simply making the substitution $g \to gb$ in the S^4 model.

where we have used that $\langle W_{\bigcirc} \rangle|_{b=1}$ is the circular Wilson loop computed on the sphere.

Matrix Model for $\mathcal{N} = 2$ Super conformal QCD

In this section we present the matrix model for $\mathcal{N} = 2$ derived by Pestun in [30]. As in the $\mathcal{N} = 4$ case, the partition function of $\mathcal{N} = 2$ SCQCD on S^4 is computed using localization and it is [30]

$$\mathcal{Z}_{\mathcal{N}=2} = \int_{S^4} [da]\, e^{-\frac{8\pi^2}{g^2}\mathrm{Tr}(a^2)}\, \mathcal{Z}_{\text{1-loop}}(a)\, |\mathcal{Z}_{\text{inst}}(a, g^2)|^2 \tag{4.35}$$

Note that in this case, the matrix model is not Gaussian as in $\mathcal{N} = 4$. The last two terms are the one-loop partition function characterizing field fluctuations and the instanton contribution.
The instanton partition function $\mathcal{Z}_{\text{inst}}(a, g^2)$ can be found in [48] and it is a very complicated function of the eigenvalues of a and the Yang-Mills coupling. It is usually assumed that instantons are not important in the large-N limit.
The matrix $a = \mathrm{diag}(a_1, \ldots, a_N)$ is subject to the condition $\sum_{i=1}^{N} a_i = 0$. The one-loop partition function $\mathcal{Z}_{\text{1-loop}}(a)$ is thus a combination of terms that depend on $(a_i - a_j)^2$ and a_i^2. The explicit expression was computed by Pestun, and can be expressed through a single function $H(x)$ defined as

$$H(x) = e^{-(1+\gamma)x^2}\, G(1 + ix)\, G(1 - ix) \tag{4.36}$$

where $G(z)$ is the Barnes G-function.
The one-loop factor in the partition function is given by [30]

$$\mathcal{Z}_{\text{1-loop}} = \frac{\prod_{i,j=1}^{N} H(a_i - a_j)}{\prod_{i=1}^{N} H(a_i)^{2N}} \tag{4.37}$$

where the numerator is the contribution of the vector multiplet and the denominator is the contribution of $2N$ hypermultiplets [51].

The function $H(x)$ admits the following product representation

$$H(x) = \prod_{n=1}^{\infty} \left[\left(1 + \frac{x^2}{n^2}\right)^n e^{-\frac{x^2}{n}} \right] \tag{4.38}$$

and its logarithm can be expanded as

$$\log H(x) = -(1+\gamma)x^2 - \sum_{n=2}^{\infty} \zeta(2n-1)\frac{x^{2n}}{n} \tag{4.39}$$

So we can write the logarithm of (4.37) using (4.39):

$$\log \mathcal{Z}_{\text{1-loop}} = -\sum_{n=2}^{\infty} \frac{\zeta(2n-1)}{n} \left[\sum_{i<j} (a_i - a_j)^{2n} - 2N \sum_i a_i^{2n} \right] \tag{4.40}$$

where the x^2 part in $H(x)$ was removed due to conformal invariance and only happens when $N_f = 2N$. So the first non trivial contribution starts at $(\mathrm{Tr}(a^2))^2$ and we can schematically write

$$\log \mathcal{Z}_{\text{1-loop}} \;\cong\; \sum_{i<j=1}^{N} \log(a_i - a_j)^2 - \zeta(3) \left[\sum_{i,j=1}^{N} (a_i - a_j)^4 - N \sum_{i=1}^{N} a_i^4 \right] + O(a^6) \tag{4.41}$$

which can be simplified as:

$$\log \mathcal{Z}_{\text{1-loop}} = -\frac{3}{4}\zeta(3)\, \left(\mathrm{Tr}(a^2)\right)^2 \tag{4.42}$$

Then, to compute the expectation value of the Wilson loop

$$\langle W_{\bigcirc} \rangle = \frac{1}{\mathcal{Z}_{\mathcal{N}=2}} \int [da]\, \mathrm{Tr}(e^{-2\pi a})\, e^{-\frac{8\pi^2}{g^2}\mathrm{Tr}(a^2)}\, \mathcal{Z}_{\text{1-loop}} \tag{4.43}$$

In the weak coupling limit one obtains:

$$\langle W_{\bigcirc} \rangle = 1 + \frac{\lambda}{8} + \frac{\lambda^2}{192} + \lambda^3 \left(\frac{1}{9216} - \frac{3\zeta(3)}{512\pi^4} \right) \tag{4.44}$$

A conjecture for $B_{\mathcal{N}=2}$

In [2] a conjectured formula for the bremsstrahlung of $\mathcal{N} = 2$ SCQCD was proposed by Fiol, Gerchkovitz and Komargodski inspired by the $\mathcal{N} = 4$ result

$$B_{\mathcal{N}=2} = \frac{1}{4\pi^2}\, \partial_b \log \langle W_{\bigcirc} \rangle_b \Big|_{b=1} \tag{4.45}$$

where $\langle W_{\bigcirc}\rangle_b$ is the circular Wilson loop on the ellipsoid computed in [30, 52, 53, 54, 51, 55, 56], which is given by

$$\langle W_{\bigcirc}\rangle_b = \frac{\int [da]\,\mathrm{Tr}(e^{-2\pi b a})\, e^{-\frac{8\pi^2}{g^2}\mathrm{Tr}(a^2)}\, \mathcal{Z}_{\text{1-loop}}(a,b)}{\int [da]\, e^{-\frac{8\pi^2}{g^2}\mathrm{Tr}(a^2)}\, \mathcal{Z}_{\text{1-loop}}(a,b)} \tag{4.46}$$

Identity (4.45) has been explicitly checked up to three loop for gauge group $SU(2)$ by the authors of [2], while for $N > 2$ only a consistency check of its positivity has been given there. In chapter 5 we will extend this proof to the general $SU(N)$ case [5].

In the absence of the Wilson loop the ellipsoid is invariant $b \leftrightarrow b^{-1}$ and therefore the classical, one-loop and, instanton contributions start deviating from their S^4 expressions only at second order in $b-1$. The Wilson loop insertion in equation (4.46) is the only factor in the integrand that contains a term linear in $b-1$. Since in (4.45) we have to evaluate at $b = 1$, we note that after the derivative only the terms that where linear in $b-1$ in $\langle W_{\bigcirc}\rangle_b$ will contribute to the bremsstrahlung. So we conclude that $B_{\mathcal{N}=2}$ can be computed using just the 1-loop determinant and instanton factors of the round S^4 matrix model.

[5] In [2] the authors made another conjecture that states that $B_{\mathcal{N}=2} = 3h$, where h is the coefficient of the one–point correlation function for the stress–energy tensor in the presence of the Wilson line defect. This second conjecture was later proved in [57].

Chapter 5

The difference method and perturbative computations

In sections 4.2 and 4.3 we saw that it is possible to compute the bremsstrahlung function exactly as a derivative of the vacuum expectation value of a circular 1/2-BPS Wilson loop computed by a matrix model. While for $\mathcal{N} = 4$ SYM that relation was proved, for $\mathcal{N} = 2$ SCQCD it remained as a conjecture. This conjecture has been explicitly checked up to three loop for gauge group $SU(2)$, while for $N > 2$ only a consistency check of its positivity has been given in [2]. In this chapter we are going to extend this proof to the general $SU(N)$ case.

To do this we will consider a generalized Maldacena–Wilson operator like the one defined in chapter 2 along a cusped line with geometric angle φ and featured by an internal angle θ which rotates the couplings to the adjoint matter when moving through the cusp. The corresponding generalized cusp anomalous dimension turns out to be a function of both angles, as we saw in chapter 3.

For generic $SU(N)$ SCQCD we perform a genuine three–loop calculation of the cusped operator at generic angles and finite group rank N. From the $1/\epsilon$ pole of the dimensionally regularized result we then extract the generalized cusp at three loops and the corresponding bremsstrahlung function from its small angle expansion. We find a general result that, remarkably, coincides with the conjectured formula once we expand the matrix model up to that order.

The chapter is organized as follows. We first describe our computational strategy in Section 5.1, and recall the Matrix Model result in Section 5.1.1. Then the novel results are presented, where we report the diagrammatic approach to the three–loop calculation in section 5.1.2 and the evaluation of the Feynman integrals in section 5.2. In the last sections we give the main results and we provide an explicit check of the universal behaviour of the cusp anomalous dimension proposed in [38, 4], which should work up to three loops that we saw in 3.3. The results of these computations are summarized in [3].

The difference

We will compute the cusp anomalous dimension and the associated Bremsstrahlung function of $\mathcal{N}=2$ SCQCD by comparing them with the corresponding known quantities of $\mathcal{N}=4$ SYM. In fact, it is well–known that this trick drastically reduces the number of new diagrams to be computed, as we will see.
We review now the discussion of section 1.1.4 about the way fields organize into multiplets. We saw at the end of section 1.3 that in this language the field content of the $\mathcal{N}=2$ SCQCD theory with gauge group $SU(N)$ is organized into one vector and one chiral multiplets transforming in the adjoint representation of $SU(N)$, which form the $\mathcal{N}=2$ vector multiplet, together with $N_f=2N$ chiral multiplets building up $2N$ $\mathcal{N}=2$ hypermultiplets transforming in the fundamental representation of the gauge group.

Analogously, the $\mathcal{N}=4$ SYM theory is described by one vector multiplet plus a $SU(3)$ triplet of adjoint chiral multiplets. Together they build up the $\mathcal{N}=2$ vector multiplet, combining the $\mathcal{N}=1$ vector with one of the chiral multiplets in analogy with the $\mathcal{N}=2$ SCQCD case, plus one adjoint $\mathcal{N}=2$ hypermultiplet from the two remaining adjoint chiral multiplets.

Therefore, the two theories have the same $\mathcal{N}=2$ gauge sector, while the difference relies only in the matter content and entails the comparison of two of the adjoint chiral multiplets in $\mathcal{N}=4$ SYM as opposed to the pair of $2N$ fundamentals in $\mathcal{N}=2$ SCQCD [58]. This allows to drastically simplify the calculation of any observable $\mathcal{O}$ that is common to $\mathcal{N}=2$ SCQCD and $\mathcal{N}=4$ SYM theories if, instead of computing $\langle\mathcal{O}\rangle_{\mathcal{N}=2}$ directly, one computes the difference $\langle\mathcal{O}\rangle_{\mathcal{N}=2}-\langle\mathcal{O}\rangle_{\mathcal{N}=4}$. In fact, in the difference all the Feynman diagrams that are common to the two theories cancel, in particular the ones built with fields belonging to the gauge sector. The computational strategy of taking the difference was first introduced in [59], albeit working with a different description of the field content of the two theories.

In this context we consider a Maldacena–Wilson operator common to $\mathcal{N} = 2$ SCQCD and $\mathcal{N} = 4$ SYM theories

$$W = \frac{1}{N} \mathrm{Tr}\, \mathcal{P}\, e^{-ig \int_C d\tau \mathcal{L}(\tau)} \tag{5.1}$$

with Euclidean connection

$$\mathcal{L}(\tau) = \dot{x}^\mu A_\mu + i\, \frac{\sqrt{2}}{2}\, |\dot{x}|\, (\phi + \bar{\phi}) \tag{5.2}$$

where $\phi, \bar{\phi}$ are the adjoint scalars entering the $\mathcal{N} = 2$ vector multiplet shared by the two theories.

We consider a cusp contour $\mathcal{C}$ made by two infinite straight lines parametrized as

$$\begin{aligned} x^\mu(\tau_1) &= v_1^\mu \tau_1 \qquad\qquad 0 < \tau_1 < \infty \\ x^\mu(\tau_2) &= v_2^\mu \tau_2 \qquad\quad -\infty < \tau_2 < 0 \end{aligned} \tag{5.3}$$

The two lines form an angle φ, such that $\cos\varphi = v_1 \cdot v_2$ and $|v_1| = |v_2| = 1$.

We also allow for two different scalar couplings on the two lines of the contour, characterized by a relative internal angle θ. Precisely, on the two rays we choose

$$\mathcal{L}_1(\tau) = v_1^\mu A_\mu + \frac{i}{\sqrt{2}} (\phi\, e^{i\theta/2} + \bar{\phi}\, e^{-i\theta/2}) \tag{5.4}$$

$$\mathcal{L}_2(\tau) = v_2^\mu A_\mu + \frac{i}{\sqrt{2}} (\phi\, e^{-i\theta/2} + \bar{\phi}\, e^{i\theta/2}) \tag{5.5}$$

Even if our computation is entirely done in the component formalism, it is worth to mention that in $\mathcal{N} = 1$ superspace some effective rules to evaluate diagrammatic difference $\langle O \rangle_{\mathcal{N}=2} - \langle O \rangle_{\mathcal{N}=4}$ have been derived [68] and later formalized [39] in the context of the calculation of the $SU(2{,}1|2)$ spin chain Hamiltonian of $\mathcal{N} = 2$ SCQCD. In that case it was shown that the only source of diagrams potentially contributing to the difference is given by graphs containing chiral loops cut by an adjoint line (either vector or chiral). This rule was found later to be valid also in the context of the computation of the adjoint scattering amplitudes of

$\mathcal{N}=2$ SCQCD [65]. In particular, topologies containing "empty" chiral loops are constrained to produce the same result for the two models. In fact, for such type of diagrams computing the difference is only a matter of counting the number of possible realizations of the loop in terms of adjoint and/or fundamental superfields. As a consequence of the condition $N_f = 2N$, the two models turn out to give the same result.

One might wonder whether similar rules survive when reducing the theory to components and if they can be easily applied to the computation of the cusp anomalous dimension. As we are going to show in the next sections, this turns out to be the case for diagrams involving only minimal gauge matter-couplings up to three loops, due to the fact that the actions in components display the same flavour structure of their $\mathcal{N}=1$ superspace versions. Of course, possible complications in taking too seriously the parallel with the superfield rules may arise when considering higher order diagrams involving superpotential vertices. In this case we expect the component diagramatics to follow different rules with respect to the superspace version.

Matrix Model result

We start this section by remembering the formulas for the bremsstrahlung function that we saw in the previous chapter:

$$B = \frac{1}{4\pi^2}\partial_b \log\langle W_{\bigcirc}\rangle_b\Big|_{b=1} \tag{5.6}$$

where it is understood that B is applicable to both $\mathcal{N}=4$ SYM and $\mathcal{N}=2$ SCQCD and where here $\langle W_{\bigcirc}\rangle_b$ is the 1/2 BPS circular Wilson loop of the form (5.1, 5.2) defined on the maximal latitude or the maximal longitudinal circles of the ellipsoid. For $\mathcal{N}=4$ SYM this equation was proved in [1], where $\langle W_b\rangle$ can be computed exactly by the matrix model [28, 29, 30]

$$\langle W_b\rangle = \frac{\int da\ \mathrm{tr}(e^{-2\pi ba})\, e^{-\frac{8\pi^2 N}{\lambda}\mathrm{tr}(a^2)}}{\int da\ e^{-\frac{8\pi^2 N}{\lambda}\mathrm{tr}(a^2)}} + \mathcal{O}\left((b-1)^2\right) \tag{5.7}$$

and turns out to be a function of the squashing parameter $b=\sqrt{l/\tilde{l}}$.

In order to prove identity (5.6) for $\mathcal{N}=2$ SCQCD we begin by recalling the evaluation of its right hand side, where the matrix model in this case is given by

[51, 52, 53, 54, 55, 56]

$$\langle W_b \rangle = \frac{\int da \mathrm{Tr}\, e^{-2\pi ba}\, e^{-\frac{8\pi^2}{g^2}\mathrm{Tr}(a^2)} Z_{1-loop}(a,b)|Z_{inst}(a,b)|^2}{\int da\, e^{-\frac{8\pi^2}{g^2}\mathrm{Tr}(a^2)} Z_{1-loop}(a,b)|Z_{inst}(a,b)|^2} \tag{5.8}$$

According to conjecture the only terms in the matrix model which can contribute to B are the ones linear in $(b-1)$. As we mention in chapter 4, the classical, one-loop and instanton contributions start deviating from their S^4 counterparts only at second order in $(b-1)$, it follows that $\langle W_b \rangle$ in (5.8) can be computed using the one–loop determinant and instanton factors of the round S^4 matrix model [2].

Assuming prescription (5.6) to be true for any N and expanding the two matrix models (5.7, 5.8) up three loops, we obtain the general prediction for the difference of the Bremsstrahlung function in the two theories

$$B_{\mathcal{N}=2} - B_{\mathcal{N}=4} = -\frac{3\zeta(3)}{1024\pi^6}\,\frac{(N^2-1)(N^2+1)}{N}\,g^6 + \mathcal{O}(g^8) \tag{5.9}$$

For $N = 2$ this expression reduces to

$$B_{\mathcal{N}=2} - B_{\mathcal{N}=4} = -\frac{45}{2048\pi^6}\,\zeta(3)\,g^6 + \mathcal{O}(g^8) \tag{5.10}$$

which has been already checked in [2] against a three–loop perturbative calculation. In the next section we generalize the proof of equation (5.9) to any finite N.

The perturbative result

In order to check equation (5.9) we perform a perturbative three–loop calculation of its left hand side along the lines described in section 5.1, that is by extracting the difference of the two bremsstrahlung functions from the small angle limit of the difference of the corresponding cusp anomalous dimensions, $\Gamma_{\mathcal{N}=2} - \Gamma_{\mathcal{N}=4}$.

As a first step we have to evaluate $W_{\mathcal{N}=2} - W_{\mathcal{N}=4}$. At order $\mathcal{O}(g^2)$ the only diagrams are the single gluon and single adjoint scalar exchanges, for which the result is the same in both theories. At this order the difference is therefore zero. This property extends to all the diagrams built with tree level n–point functions inserted into the Wilson line, since in this case contributions from the hypermultiplets do not appear. The next order is $\mathcal{O}(g^4)$, where the only non–tree diagrams are the exchange of one–loop corrected propagators. However, it has been shown that in the difference they still cancel since the contribution from a loop of $2N$ fundamental fields is the same as the one from the loop of one adjoint field [59].

The first non–trivial contribution starts at $\mathcal{O}(g^6)$ where the contributing diagrams correspond to the insertion of two–loop corrected gauge/scalar propagators and one–loop corrected cubic vertices. Here we analyse them separately, and postpone the evaluation of the corresponding integrals to section 5.2.

Two–loop propagator diagrams

We begin by considering the diagrams with two–loop corrections to the vector and adjoint scalar propagators. Taking the difference between the $\mathcal{N} = 2$ and $\mathcal{N} = 4$ propagators, the diagram topologies which survive are the ones listed in figure 5.1. Here we neglect topologies that would produce vanishing cusp integrals.

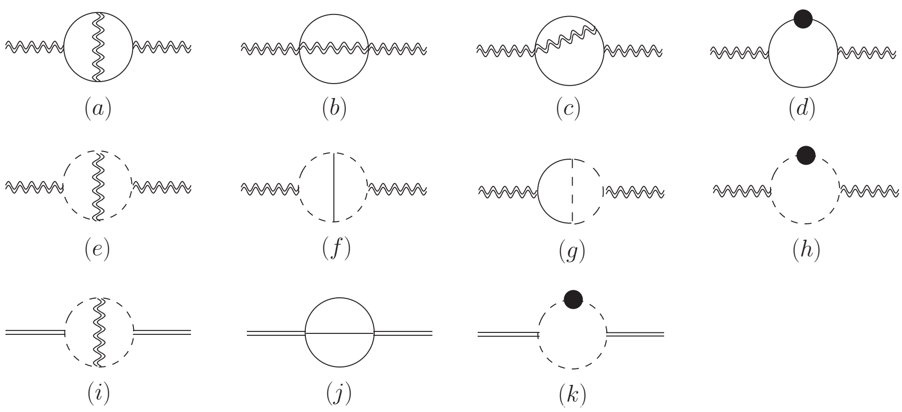


Figure 5.1: Diagram topologies that contribute to the difference of the $\mathcal{N} = 2$ and $\mathcal{N} = 4$ propagators at two loops.

For simplicity we are not depicting the insertion of the diagrams into the Wilson loop contour. We use a double solid, curly and dashed line to represent the adjoint scalar, vector and fermion fields, respectively. Instead, we use simple solid and dashed lines for scalars and fermions, collectively representing diagrams which admit different specific realizations in terms of both adjoint and fundamental fields of the two models. For instance, in figure $1(a)$ the simple solid loop stands generically for one of the following realizations: In $\mathcal{N} = 4$ SYM it indicates any of the three adjoint scalar fields ϕ^I, $I = 1, 2,3$, whereas in $\mathcal{N} = 2$ SCQCD it corresponds to either the adjoint scalar ϕ or one of the two fundamental sets of fields $q_I, \bar{\tilde{q}}_I$

with $I = 1, \ldots ,2N$. The same happens for diagrams $(b), (c), (d)$. For diagrams $(e), (h), (i), (k)$ involving a simple fermionic loop we have a parallel counting, this time in terms of the adjoint fermion fields ψ and the fundamentals $\lambda, \tilde{\lambda}$.

We see that, excluding diagrams $(f), (g), (j)$, we are only dealing with minimal gauge–matter couplings, so that the superfield difference selection rules of [39] still hold and we are left only with diagrams with matter loops cut by an adjoint line. Instead, diagrams $(f), (g), (j)$ involve interaction vertices from the potential. Consequently, the list of possible field realizations cannot exactly parallel the superfield counting any more. For instance, diagram (g) produces non-vanishing contributions to the difference which include the gaugino field η, while diagram (j) requires a careful analysis of all possible flavour realizations stemming from the quartic vertices.

It is interesting to note that diagrams $(d), (h), (k)$, which are generated by the 1–loop corrected fermion and scalar propagators, do not have a correspondent in $\mathcal{N} = 1$ superspace. In fact, in a $\mathcal{N} = 1$ superspace setup the 1-loop corrections to the superfield propagators are exactly vanishing for both $\mathcal{N} = 4$ SYM and $\mathcal{N} = 2$ SCQCD. However, in the component formulation this is no longer the case and the one-loop corrections turn out to be divergent. This is not in contradiction with conformal invariance and can be interpreted as a consequence of working in the susy-breaking Wess–Zumino gauge [69]

One–loop three–point vertex diagrams

In principle, other contributions at order g^6 may come from the insertion into the cusp of the one–loop corrections to the three–point vertices [59, 2]. The diagram topologies potentially contributing to the difference between the $\mathcal{N} = 2$ and $\mathcal{N} = 4$ three–point vertices are depicted in figure 5.2. Again we neglect topologies that would produce vanishing cusp integrals.

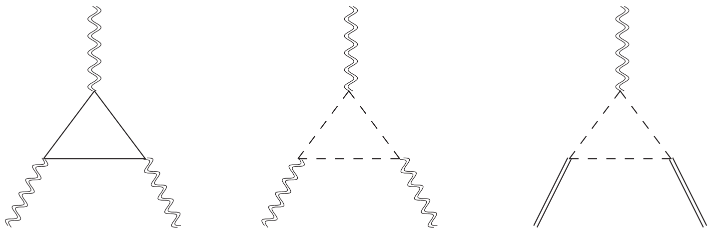


Figure 5.2: One–loop corrections to the three–point vertices that potentially contribute to the difference.

In [2, 59] it has been proved that in the $SU(2)$ case the contribution to the difference $W_{\mathcal{N}=2} - W_{\mathcal{N}=4}$ is vanishing at the conformal point, due to the fact that for algebraic reasons the result from two adjoint scalars running into the loop is identical to the result from $2N$ fundamentals. However, as argued in [59], the result cannot be immediately generalized to $SU(N)$, since for $N > 2$ extra contributions from the adjoint scalar loop may arise, which are proportional to the symmetric structure constants d_{abc} (see equation (A.4) in the appendix A).

Here we perform a detailed analysis of these diagrams and prove that for symmetry reasons contributions proportional to d_{abc} can never appear. Therefore, we conclude that diagrams in figure 5.2 never contribute to the difference $W_{\mathcal{N}=2} - W_{\mathcal{N}=4}$, for any $SU(N)$ gauge group, so generalizing the result of [2, 59].

We illustrate how the cancellation works by focusing on an explicit example, that is the first vertex topology in figure 5.2 that corresponds to the scalar loop corrections to the three–gluon vertex.

In the $\mathcal{N} = 4$ SYM case we have the three adjoint scalars ϕ^I, $I = 1,2,3$ running into the loop. Using Feynman rules in appendix A the corresponding expression reads

$$V_{\mathcal{N}=4} = 3\, N\, g^3\, \mathrm{Tr}\big(T^a[T^b, T^c]\big) \int d^d z_{1/2/3}\, f^{\mu\nu\rho}(z_1, z_2, z_3) \times A^a_\mu(z_1) A^b_\nu(z_2) A^c_\rho(z_3) \tag{5.11}$$

where the factor 3 stems from the sum over all possible flavour loops, the two terms building up the colour trace commutator correspond to the two possible orientations of the adjoint loop cycles, and $f^{\mu\nu\rho}$ is a function of the vertex points

$z_{1/2/3}$ that can be expressed in momentum space as

$$f^{\mu\nu\rho}(z_1,z_2,z_3) = \int \frac{d^d(q+k_2)}{(2\pi)^d} \int \frac{d^d(q-k_1)}{(2\pi)^d} \int \frac{d^dq}{(2\pi)^d}\, e^{iq(z_1-z_2)} e^{i(q+k_2)(z_2-z_3)} e^{i(q-k_1)(z_3-z_1)} \times \frac{(2q-k_1)^\mu (2q+k_2)^\nu (2q+k_2-k_1)^\rho}{q^2(q-k_1)^2(q+k_2)^2} \tag{5.12}$$

In $\mathcal{N}=2$ SCQCD the same kind of diagram topology can be constructed using either the single adjoint scalar ϕ or the two fundamental sets of fields $q_I, \bar{\tilde{q}}_I$ with $I = 1,\ldots,2N$. The adjoint loop will give exactly the same result as in (5.11), without the factor 3. Instead, the two sets of fundamental loops yield

$$V^{fund}_{\mathcal{N}=2} = 2 \times 2N\, g^3\, \mathrm{Tr}\big(T^aT^bT^c\big)\, f^{\mu\nu\rho}(z_1,z_2,z_3) \times A^a_\mu(z_1)A^b_\nu(z_2)A^c_\rho(z_3) \tag{5.13}$$

where now we have a single possible colour orientation and the integral is still given in (5.12). Taking the difference we obtain

$$V_{\mathcal{N}=2} - V_{\mathcal{N}=4} = \big\{4N\mathrm{Tr}\big(T^aT^bT^c\big) - 2N\mathrm{Tr}\big(T^a[T^b,T^c]\big)\big\} \times\, g^3 \int d^dz_{1/2/3}\, f^{\mu\nu\rho}(z_1,z_2,z_3)\, \times A^a_\mu(z_1)A^b_\nu(z_2)A^c_\rho(z_3) \tag{5.14}$$

where for $SU(2)$ the colour structure inside the bracket is identically vanishing, whereas for $SU(N)$ it is nothing but the totally symmetric d^{abc} tensor (see equation (A.4)).

It is now easy to see that, independently of the gauge group, this expression always vanishes. In fact, the string $d_{abc}A^a_\mu(z_1)A^b_\nu(z_2)A^c_\rho(z_3)$ is symmetric under the exchange of any pair of gauge fields, but it is contracted with $f^{\mu\nu\rho}$ which is antisymmetric under any exchange

$$f^{\mu\nu\rho}(z_1,z_2,z_3) = -f^{\nu\mu\rho}(z_2,z_1,z_3) \qquad \text{etc...} \tag{5.15}$$

An alternative reasoning goes as follows. Independently of the gauge group, once we insert the vertex correction (5.14) into the cusp contour, for symmetry reasons the two colour trace structures of the commutator term $\mathrm{Tr}\big(T^a[T^b,T^c]\big)$ sum up to $2\mathrm{Tr}\big(T^aT^bT^c\big)$, so that the difference in (5.14) vanishes identically. This can be loosely summarized stating that each adjoint empty loop counts as twice a fundamental loop contribution, thus producing a vanishing counting.

It is easy to realize that similar symmetry arguments hold for all the other topologies in figure 5.2. We then conclude that, against previous expectations, there are no contributions to $W_{\mathcal{N}=2} - W_{\mathcal{N}=4}$ coming from one–loop three–point vertices, for generic $SU(N)$ gauge group.

HQET methods for solving integrals

According to the previous discussion, the only non–trivial contributions to the difference $W_{\mathcal{N}=2} - W_{\mathcal{N}=4}$ come from the insertion of diagrams in figure 5.1. In this section we focus on the evaluation of the corresponding loop integrals.

We can focus only on insertions which connect the two lines of the cusped Wilson loop (1PI diagrams in the HQET context) since the ones where the two insertion points lie both on the same ray can be factorized out and do not contribute to the evaluation of B [60].

The most efficient way to compute the corresponding loop integrals is the so–called HQET method [61, 4]. Working in momentum space, it consists in integrating first on the contour parameters with a proper prescription for regularizing boundary divergences. This reduces the integrals to ordinary massive momentum integrals, which can be written as linear combinations of known Master Integrals by applying integrations by parts.

The full list of results for diagrams of figure 5.1 can be found in appendix B. Here we briefly illustrate the procedure by computing for instance the integral corresponding to diagram (e). In the $\mathcal{N}=4$ SYM case the fermionic loop, represented in our notation with a simple dashed line, can be constructed with any of the three adjoint fermions ψ^I, with $I = 1, 2,3$. In the $\mathcal{N}=2$ SCQCD case, instead, the loop can be realized either with the adjoint fermion ψ or with one of the two sets of fundamental fermions $\lambda_I, \tilde{\lambda}^I$, with $I = 1, ...,2N$. Taking the difference of $\mathcal{N}=2$ and $\mathcal{N}=4$ propagators and inserting it in the Wilson line, the corresponding integral reads (we neglect a factor $\frac{g^6(N^2-1)(N^2+1)}{2N}$)

$$I^{(e)} = -\int_0^\infty d\tau_1 \int_{-\infty}^0 d\tau_2\, v_1^\mu\, v_2^\nu\, \mathrm{tr}(\sigma^\mu\sigma^\rho\sigma^\xi\sigma^\tau\sigma^\nu\sigma^\sigma\sigma_\xi\sigma^\eta)$$
$$\int \frac{d^dk_{1/2/3}}{(2\pi)^{3d}}\, e^{ik_3\cdot(x_1-x_2)}\, \frac{(k_1-k_3)_\rho(k_1)_\eta(k_2-k_3)_\tau(k_2)_\sigma}{k_1^2\,k_2^2\,k_3^4\,(k_1-k_2)^2\,(k_1-k_3)^2\,(k_2-k_3)^2} \tag{5.16}$$

where we work in $d = 4-2\epsilon$ dimensions and we have defined $x_1^\mu \equiv v_1^\mu\tau_1$, $x_2^\mu \equiv v_2^\mu\tau_2$.

Now the trick consists in changing the order of contour and momentum integrals and perform first the contour ones. This amounts to first compute

$$\begin{aligned}
&\int_0^{\infty} d\tau_1 \; e^{ik_3\cdot v_1\tau_1} e^{\delta\tau_1} \int_{-\infty}^{0} d\tau_2 \; e^{-ik_3\cdot v_2\tau_2} e^{-\delta\tau_2} \\
= \quad & \frac{1}{(ik_3\cdot v_1+\delta)} \frac{1}{(ik_3\cdot v_2+\delta)}
\end{aligned} \tag{5.17}$$

where, following the prescription of [4], a damping factor $e^{\delta\tau}$ with $\delta < 0$ has been introduced for each contour integral in order to make them well defined at infinity.

Since the final result is expected to be independent of the IR regulator δ, we conveniently choose $\delta = -1/2$. Absorbing the i factor in a redefinition of the velocity, $v = i\,\widetilde{v}$, we are left with

$$I^{(e)} = 4\int \frac{d^dk_{1/2/3}}{(2\pi)^{3d}} \frac{\mathrm{tr}(\sigma^\mu\sigma^\rho\sigma^\xi\sigma^\tau\sigma^\nu\sigma^\sigma\sigma_\xi\sigma^\eta)}{(1+2k_3.\widetilde{v}_1)(1+2k_3.\widetilde{v}_2)} \times \frac{\tilde{v}_{1\mu}\,\tilde{v}_{2\nu}\,(k_1-k_3)_\rho(k_1)_\eta(k_2-k_3)_\tau(k_2)_\sigma}{k_1^2k_2^2k_3^4(k_1-k_2)^2(k_1-k_3)^2(k_2-k_3)^2} \tag{5.18}$$

Now, using the σ-matrix algebra we can reduce the numerator to a linear combination of scalar products of momenta and external velocities which can be written in terms of inverse propagators. Therefore, we end up with a sum of momentum integrals of the form B.1. These integrals are not all independent and, using integration by parts, performed with the Mathematica package FIRE [62, 63], they can be expressed in terms of a finite set of Master Integrals [4, 38]. For our example, after the FIRE reduction we obtain

$$\begin{aligned}
I^{(e)} = \Bigg[\Bigg(&\frac{32(3d-7)(-480+964d-796d^2+335d^3-71d^4+6d^5)}{(d-5)(d-4)^3(d-3)(d-1)} \\
&+\frac{32(3d-7)(-4736+8360d-5494d^2+1663d^3-222d^4+9d^5)\cos\varphi}{3(d-5)(d-4)^3(d-3)(d-1)} \\
&+\frac{32(3d-7)(-2720+4736d-3196d^2+1036d^3-159d^4+9d^5)\cos^2\varphi}{3(d-5)(d-4)^3(d-3)(d-1)}\Bigg)\times I_1 \\
&-\Bigg(\frac{16(1+\cos\varphi)^2(80-54d+9d^2)(96-140d+81d^2-21d^3+2d^4)}{(d-5)(d-4)^3(d-3)(d-1)} \\
&-\frac{16(1+\cos\varphi)^2(80-54d+9d^2)(272-392d+202d^2-43d^3+3d^4)\cos\varphi}{3(d-5)(d-4)^3(d-3)(d-1)}\Bigg)\times I_2 \\
&-\frac{12(d-3)(10-\cos\varphi(d-8)-3d)(8-5d+d^2)}{(d-5)(d-4)^2(d-1)}\times I_3 \\
&-\frac{2(1+\cos\varphi)(\cos\varphi(d-8)-3(d-4))(80-74d+25d^2-3d^3)}{(d-5)(d-4)^2(d-1)}\times I_4\Bigg]
\end{aligned} \tag{5.19}$$

where the Master Integrals I_i are defined in Appendix B.

This technique is known as "Heavy Quark Effective Theory" (HQET) due to its relation with the theory of scattering of heavy particles. The propagator-like integrals that we obtain with the method described above formally coincide with the integrals describing the propagation of heavy quarks. The direction v^μ of the Wilson line is the velocity of the quark, whereas the damping factor δ corresponds to the introduction of a residual energy for the particle. In the presence of a cusp, the Bremsstrahlung function controls the energy radiated by the heavy particle undertaking a transition from a velocity v_1 to v_2 in an infinitesimal angle φ.

Since we are eventually interested in computing the cusp anomalous dimension that in dimensional regularization can be read from the $1/\epsilon$ pole of $\log\langle W\rangle$, it is convenient to expand the master integrals in powers of ϵ. Defining the new variable $x = e^{i\varphi}$, where φ is the geometric angle of the cusp, for the $I^{(e)}$ we find

$$
\begin{aligned}
I^{(e)} = {}& \frac{1}{\epsilon^3}\,\frac{2\left(1-x^2-2(1+x^2)\log[x]\right)}{9(x^2-1)} \\
&+\frac{2}{\epsilon^2}\left(\frac{4-\pi^2-(4+\pi^2)x^2+3(1+x^2)\log[x]^2}{9(x^2-1)}-\frac{2\log[x]\left(5+x(5x-3)+6(1+x^2)\log[1+x]\right)+12(1+x^2)\mathrm{Li}_2[-x]}{9(x^2-1)}\right) \\
&+\frac{1}{\epsilon}\left(\frac{80-7\pi^2+12\pi^2x-(80+33\pi^2)x^2}{18(x^2-1)}-\frac{\log[x]\left(101-(48-101x)x+7\pi^2(1+x^2)\right)}{9(x^2-1)}\right. \\
&+\frac{6\log^2[x]\left(5-(3-5x)x-(1+x^2)\log[x]\right)}{9(x^2-1)}-\frac{12\left(3\pi^2(1+x^2)+2(5+x(-3+5x))\log[x]-3(1+x^2)\log[x]^2\right)\log[1+x]}{9(x^2-1)} \\
&+\frac{144(1+x^2)(\log[-x]-\log[x])\log[1+x]^2}{18(x^2-1)}-\frac{48(5+x(-3+5x))\mathrm{Li}_2[-x]-144(1+x^2)\mathrm{Li}_3[-x]-288(1+x^2)\mathrm{Li}_3[1+x]}{18(x^2-1)} \\
&\left.+\frac{96\left(-2-x^2+\log[x]+x^2\log[x]\right)\zeta[3]}{18(x^2-1)}\right)+\mathcal{O}(\epsilon^0)
\end{aligned}
\tag{5.20}
$$

The expansions of the integrals corresponding to the rest of the diagrams in figure 5.1 are listed in appendix B. We note that the expansions may contain higher order poles in ϵ, up to $1/\epsilon^3$.

Result

Applying the HQET procedure to every single diagram and summing the results for the integrals as listed in appendix B, we can distinguish the contribution coming from the insertion of diagrams $(a)-(h)$ (insertion of a gauge propagator)

$$
\left[\langle W_{\mathcal{N}=2}\rangle-\langle W_{\mathcal{N}=4}\rangle\right]\Big|^{(3L)}_{gauge} = g^6\,\frac{(N^2-1)(N^2+1)}{2048\,\pi^6 N}\zeta(3)\,\frac{-1+x^2+2(1+x^2)\log x}{(x^2-1)\epsilon} \tag{5.21}
$$

from the contribution arising from diagrams $(i)-(k)$ (insertion of an adjoint scalar propagator)

$$\big[\langle W_{\mathcal{N}=2}\rangle - \langle W_{\mathcal{N}=4}\rangle\big]\Big|^{(3L)}_{scalar} = -g^6\, \frac{(N^2-1)(N^2+1)}{2048\,\pi^6 N}\zeta(3)\,\cos\theta\,\frac{4x\log x}{(x^2-1)\epsilon} \quad (5.22)$$

It is remarkable that, although individually the integrals corresponding to the various topologies in figure 5.1 exhibit up to cubic poles in ϵ, in the sum they all cancel and only a simple pole survives. This has a simple physical explanation and represents a non–trivial consistency check of our calculation. In fact, according to equation (3.6), which in dimensional regularization reads $\langle W\rangle \sim \exp\left(\Gamma(g^2)/\epsilon\right)$, higher order ϵ-poles in the Wilson loop expansion only come from the exponentiation of $\frac{\Gamma(g^2)}{\epsilon}$. Since the difference $\langle W_{\mathcal{N}=2}\rangle - \langle W_{\mathcal{N}=4}\rangle$ is identically vanishing up to two loops, at three loops we expect to find only simple poles. Taking into account that the exponentiation works also when we turn off the scalar coupling, both the gauge and the scalar contributions have to display the higher order poles cancellation, independently.

Now, summing the two contributions and defining $\xi = \dfrac{1+x^2-2x\cos\theta}{1-x^2}$, we obtain

$$\langle W_{\mathcal{N}=2}\rangle - \langle W_{\mathcal{N}=4}\rangle \;=\; g^6\,\frac{\zeta(3)}{2048\,\pi^6}\frac{(N^2-1)(N^2+1)}{N}\times(1-2\xi\log x)\,\frac{1}{\epsilon}+O(g^8) \quad (5.23)$$

The presence of the IR regulator $e^{\delta\tau}$ inside the contour integrals breaks gauge invariance. As a consequence, gauge–dependent spurious divergences survive, which need to be eliminated prior computing the cusp anomalous dimension. As explained in details in [32, 60], this can be done by introducing a multiplicative renormalization constant Z_{open}, which in practice corresponds to remove the value at $\varphi=\theta=0$

$$\langle\widetilde{W}(\varphi,\theta)\rangle \equiv Z^{-1}_{open}\langle W(\varphi,\theta)\rangle = \frac{\langle W(\varphi,\theta)\rangle}{\langle W(0{,}0)\rangle} \quad (5.24)$$

We then obtain the IR–divergence free difference, which reads

$$\langle\widetilde{W}_{\mathcal{N}=2}\rangle - \langle\widetilde{W}_{\mathcal{N}=4}\rangle \;=\; -g^6\,\frac{\zeta(3)}{1024\,\pi^6}\frac{(N^2-1)(N^2+1)}{N}\times\xi\log x\,\times\frac{1}{\epsilon}+O(g^8) \quad (5.25)$$

Recalling that in dimensional regularization with $d = 4 - 2\epsilon$ we need to rescale $g \to g\mu^{-\epsilon}$ where μ is a mass scale, and using definition (3) we can easily read the difference of the two cusp anomalous dimensions from the aforementioned $1/\epsilon$ pole, obtaining

$$\Gamma_{\mathcal{N}=2} - \Gamma_{\mathcal{N}=4} \;=\; g^6 \, \frac{3\zeta(3)}{512\,\pi^6} \frac{(N^2 1)(N^2+1)}{N} \, \xi \log x + O(g^8) \tag{5.26}$$

This equation represents the most complete result for the three–loop deviation of $\Gamma_{\mathcal{N}=2}$ from $\Gamma_{\mathcal{N}=4}$. In particular, it is valid for any finite θ, φ and N.

Remarkably, we find that at $\theta = \pm\varphi$ equation. (5.26) vanishes, suggesting that at these points the cusped Wilson loop of $\mathcal{N} = 2$ SCQCD might become 1/2 BPS as in the $\mathcal{N} = 4$ SYM case.

Now, re-expressing ξ and x in terms of the original θ, φ variables and taking the small angle limit, $2\xi \log x \underset{\varphi,\theta \ll 1}{\sim} (\varphi^2 - \theta^2)$, we obtain the difference of the corresponding Bremsstrahlung functions

$$B_{\mathcal{N}=2} - B_{\mathcal{N}=4} = -g^6 \, \frac{3\zeta(3)}{1024\,\pi^6} \frac{(N^2-1)(N^2+1)}{N} + \mathcal{O}(g^8) \tag{5.27}$$

This result remarkably coincides with prediction (5.9) from the matrix model. We have then found confirmation at three loops that conjecture (5.6) proposed in [2] is valid for any $SU(N)$ gauge group.

If we insert the known value of $B_{\mathcal{N}=4}$ [34], in the large N limit we find

$$B_{\mathcal{N}=2} = \frac{g^2 N}{16\pi^2} - \frac{g^4 N^2}{384\pi^2} + \frac{g^6 N^3}{512\pi^2} \left(\frac{1}{12} - \frac{3\zeta(3)}{2\,\pi^4} \right) + \mathcal{O}(g^8) \tag{5.28}$$

Light–like cusp

Given our previous results, it is interesting to study the limit of large Minkowskian angles. To this end we substitute $\varphi = i\varphi_M$, that is $x = e^{-\varphi_M}$, and send $x \to 0$. Remember that in this limit the cusp anomalous dimension behaves linearly in the angle

$$\Gamma_{\text{cusp}}(g^2, \varphi) \underset{\varphi \to \infty}{\sim} K(g^2)\varphi_M + \mathcal{O}(\varphi_M^0) \tag{5.29}$$

The function $K(g^2)$ is the light-like cusp anomalous dimension that we discuss in section 3.3.

Using the large–N exact results for the $\mathcal{N}=4$ SYM case previously found in the literature ([34] [38]), for $\mathcal{N}=2$ SCQCD we obtain

$$K_{\mathcal{N}=2}(g^2) = \frac{g^2 N}{8\pi^2} - \frac{g^4 N^2}{384\pi^2} + \frac{g^6 N^3}{512\pi^2}\left(\frac{11}{180} - \frac{3\zeta(3)}{\pi^4}\right) + \mathcal{O}(g^8) \tag{5.30}$$

We discussed in section 3.3 that when Γ_{cusp} is expressed in terms of $K(g^2)$ replacing the coupling constant it gave rise to a universal function $\Omega(K(g^2)),\varphi)$ that was independent of the number of fermion or scalar fields in the theory. This universal behaviour was present up to three loops in $\mathcal{N}=4$ SYM and in QCD.
It is easy to prove that up to three loops the universal behaviour is also present in $\mathcal{N}=2$ SCQCD. With respect to the $\mathcal{N}=4$ SYM case, the cusp anomalous dimension gets the additional $\zeta(3)$ term in equation (5.26) at three loops, which produces a corresponding term in the light–like cusp expansion (5.30). Then one can invert (5.30) to express the coupling g^2 as a perturbative expansion in K and substitute this expansion back in the full cusp $\Gamma_{\text{cusp}}(g^2,\varphi)$ to obtain the function $\Omega(K,\phi)$. The additional $\zeta(3)$ terms coming from the genuine $\Gamma_{\text{cusp}}(g^2,\varphi)$ and from the substitution of the expansion $g^2(K)$ trivially cancel, producing the same universal function as derived in $\mathcal{N}=4$ SYM. In [42, 43] it was shown that at four loops the universality is in general violated.

Substitution rules

It has been suggested in [64] and then substantiated in [39] that the closed $SU(2,1|2)$ subsector of $\mathcal{N}=2$ SCQCD inherits integrability from $\mathcal{N}=4$ SYM, since its Hamiltonian can be essentially obtained from the $\mathcal{N}=4$ SYM one by substituting the coupling constant g^2 with an effective coupling $f(g^2)$. The explicit form of $f(g^2)$ was first derived in [41] by comparing the exact results available from localization for circular BPS Wilson loops in $\mathcal{N}=4$ SYM and $\mathcal{N}=2$ SCQCD. In the conventions of [41] the first few orders in the weak coupling expansion read [1]

$$f(g^2) = g^2 - 12\zeta(3)g^6 + 120\zeta(5)g^8 + \Big(-1120\zeta(7) + 80\zeta(2)\zeta(5) + 288\zeta(3)^2\Big)g^{10} + \dots \tag{5.31}$$

[1] In order to compare with our results we should substitute $g^2 \to \frac{g^2 N}{16\pi^2}$.

It is an interesting open problem to first understand the origin of this substitution rule and at the same time to test to what extent it is universal when applied to other observables that can be entirely built using fields from the $\mathcal{N}=2$ vector multiplet.

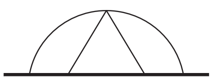
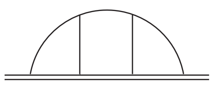
In [40] it was conjectured that the effective coupling $f(g^2)$ could be interpreted as the relative finite renormalization of the gluon propagator of the two models, enforcing the argument presented in [39]. This proposal was supported by some diagrammatic checks of the coefficients of (5.31) up to order g^8 [40]. Nevertheless, first in [40] and then in [41], it was noticed that this interpretation can be hardly extended at higher orders, because of the presence of terms that cannot be generated by purely massless two-point integrals. The $\zeta(2)\zeta(5)g^{10}$ contribution in expansion (5.31) is the first example of such kind of terms, which ask for a clear interpretation.

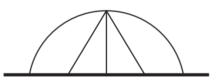
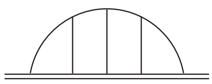
Concerning the generalization of the substitution rule to other physical quantities, in [41] a similar analysis was applied for extracting $f_B(g^2)$ from the comparison of the Bremsstrahlung functions of the two models, as computed from the Wilson loop expectation values on the ellipsoid. In this case the effective coupling $f_B(g^2)$ slightly differs from the one in (5.31) starting at order g^{10}. In [41] the discrepancy was explained as a consequence of scheme dependence in the choice of the relative infrared regulators. Once again, a term containing $\zeta(2)\zeta(5)$ appears, which cannot be explained if $f_B(g^2)$ has to be interpreted as the finite relative renormalization of gauge propagators, without resorting to coupling the model to curved space [41]. Moreover, computing the Bremsstrahlung function from the two–point function of the stress energy tensor and the 1/2 BPS Wilson loop, in [55] it was argued that in general for $\mathcal{N}=2$ theories with a single gauge group the substitution rule may be not working.

The validity of the substitution rule is made even more obscured by the results on the direct computation of purely adjoint scattering amplitudes in $\mathcal{N}=2$ SCQCD. In fact, it has been shown [65] that the amplitude/Wilson loop duality is broken already at two loops, displaying a qualitatively different functional dependence on the kinematic variables with respect to the $\mathcal{N}=4$ SYM amplitude. This arises even deeper questions about the integrability of the $SU(2,1|2)$ subsector, if the amplitude/Wilson loop duality has to be considered as direct consequence of integrability, like in $\mathcal{N} = 4$ SYM [2].

One way to shed some more light on the validity of the substitution rule and, in particular, on the actual origin of discrepancy terms of the form $\zeta(2)\zeta(5)$ would be a direct computation of the difference of the Bremsstrahlung functions at higher orders, along the lines introduced in section 5.2. Our $SU(N)$ computation, preceded by the ones in [59, 2], confirms the validity of the substitution rule (5.31) up to order g^6. From the diagrammatic analysis it is also clear how to associate the $\zeta(3)g^6$ term to diagrams containing propagator corrections.

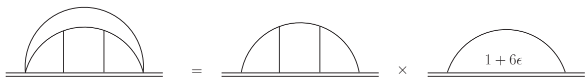
At higher orders the situation is more intricate, but the use of the HQET techniques seems to be promising. At first, the HQET integrals arise naturally as massive integrals, due to the presence of the heavy quark contour propagators. Indeed using inversion transformations it is easy to map massive on shell propagator type integrals to HQET integrals, a procedure which has been used for QCD/HQET matching [66, 67]. We consider for instance, as candidates for the production of $\zeta(2)\zeta(3)$ or $\zeta(2)\zeta(5)$ terms, the massive propagator integrals introduced in [41]. Following for instance [61], inversion relations can be used to map such integrals to corresponding HQET versions

 $=$  $\sim \quad -5\zeta(5) + 12\zeta(2)\zeta(3)$

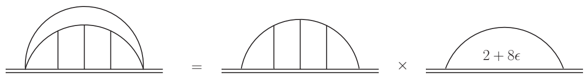
 $=$  $\sim \quad -14\zeta(7) - 12\zeta(3)\zeta(4) + 36\zeta(2)\zeta(5)$

Here we indicate the massive propagators with a thick solid line and the Wilson loop contour with a double line. In this way we are left with two examples of finite three and four loop HQET integrals containing $\zeta(2)\zeta(3)$ or $\zeta(2)\zeta(5)$ terms. Now the result of the integrals in our examples is finite, thus they cannot directly produce contributions to the cusp at three and four loops. Nevertheless, it is easy to embed these HQET topologies in higher order diagrams, producing poles potentially contributing to the cusp anomalous dimension. For example we could proceed as follows

[2]See the conclusions in [41] for a discussion on possible ways out.



$$\sim \big(-5\zeta(5) + 12\zeta(2)\zeta(3)\big)\frac{1}{\epsilon}$$



$$\sim \big(-14\zeta(7) - 12\zeta(3)\zeta(4) + 36\zeta(2)\zeta(5)\big)\frac{1}{\epsilon}$$

The integrals can be evaluated by factorization, reducing them to the product of our initial integrals and a one loop HQET bubble with a non-trivial index on the heavy line.

Therefore, we conclude that in the HQET formalism terms such as $\zeta(2)\zeta(3)$ and $\zeta(2)\zeta(5)$ can arise quite naturally in the expansion of the Bremsstrahlung function from the standard flat space computation of the cusped Wilson loop expectation value. In particular, there is no need to introduce mass regulators, beside the usual IR cut-off δ that eventually drops out from the final result. It is also natural to expect that some of these terms survive once taking the difference between $\mathcal{N} = 2$ SCQCD and $\mathcal{N} = 4$ SYM, as predicted by the matrix model results.

The interesting point is to understand whether these terms in the difference can be interpreted in terms of the substitution of an effective coupling given by the finite different renormalization of the two point functions of the models, as advocated in [39, 40]. Since we are working in component formalism, we expect that this claim should imply that at least some of these terms should originate from propagator type insertions into the cusp line. At first glance, integrals such as the ones discussed above seem to originate from topologies which could hardly be associated to propagator type diagrams. However, without an explicit derivation we cannot draw any definite conclusion and therefore a direct calculation of the Bremsstrahlung function at four and five loops is mandatory.

Chapter 6

Conclusions

The objective of this thesis was to study Wilson loops operators in $\mathcal{N} = 4$ SYM and $\mathcal{N} = 2$ SCQCD. In particular we have studied a special type of loop, which is the one formed by two straight lines that meet at a point, namely the cusped Wilson loop. In this case we have seen that UV logarithmic divergences appeared in the computation of the expectation value and the quantity governing this divergences is the cusp anomalous dimension. We dedicated chapter 3 to study how this quantity arises, some of its properties and various limits that lead to physical observables. We concentrated in two particular limits: one when the angle between the two lines is small and the other when the Minkowskian version of this angle tends to infinity. In the first case, we saw that the cusp anomalous dimension behaves quadratically with the angle and the coefficient controlling this dependence is the Bremsstrahlung function of the theory, which is related by the energy radiated by an accelerating particle [1]. The other limit (large angles) leads to a linear behaviour, governed by the light-like cusp anomalous dimension that give rise to a definition of a universal function that is independent on the number of fermions and scalars [4, 38].

In chapter 4 we studied in detail the appearance of one of the first exact results in the study of supersymmetric gauge theories: the possibility of computing the expectation value of a circular Wilson loop using a matrix model due the existence of a localization process. In [29] it was made one of the first concrete proposals in this direction, they claimed that for the circular Wilson loop the problem of finding the exact result reduces to a pure combinatorial analysis of a zero-dimensional

theory and they proposed a Gaussian matrix model for this.
From the perturbative side, in [28] the authors demonstrated that in the computation of the expectation value of the circular Wilson loop the contributions of diagrams with internal vertices automatically cancel, therefore it was only necessary to sum the ladder graphs. This conjecture was later proved by Pestun in [30] and the proof was based on the concept of localization. This technique was first introduced in the context of topological theories and then extended to quantum field theories. The idea is basically that, in certain supersymmetric field theories, there are situations in which the partition function and expectation values only receive non-vanishing contributions from a subspace of the field configurations, and in some case this allows us to obtain exact results for some observables.

The exact results of the circular Wilson loops obtained from localization can be used for probing the theory at different scales, aimed for instance at performing precision tests of the AdS/CFT correspondence. A physical quantity that can be obtained using this results is the Bremsstrahlung function that appeared in the small angle limit of the cusp anomalous dimension. In [1] the authors derived a formula to compute the Bremsstrahlung function of $\mathcal{N} = 4$ SYM as a derivative of the expectation value of the circular 1/2 BPS Wilson loop (obtained from the matrix model) with respect to the 't Hooft coupling. A similar formula for $\mathcal{N} = 2$ SCQCD was conjectured in [2], this formula has not been proven except for the $SU(2)$ case. The main purpose of this thesis was the check the conjecture for general $SU(N)$. In chapter 5 we performed a three–loop calculation of the cusp anomalous dimension for a generalized Maldacena–Wilson operator, using HQET formalism. This approach has the great advantage that, by applying a clever chain of integration by parts, all the integrals can be expressed in terms of a linear combination of a basis of known three-loop HQET Master Integrals. In addition, it provides a promising framework where we can attempt higher–loop calculations and speculate about the origin of some unexpected terms in the higher order expansion of the B function [41], which can be shown to arise naturally in the HQET context.
We obtained an expression for the cusp anomalous dimension that is valid at generic geometric and internal angles and finite gauge group rank N. For equal and opposite angles this expression vanishes, proving that at these points the cusp becomes BPS. From its small angle expansion we derive the corresponding Bremsstrahlung function at three loops, matching the matrix model prediction given in terms of

derivatives of the Wilson loop on the ellipsoid.
The computational framework we have set up in section 5.2 can be arguably extended to higher loops where some very non–trivial checks can be performed, especially on the existence of a universal behaviour shared by $\mathcal{N} = 2$ SCQCD and $\mathcal{N} = 4$ SYM.

Appendix A

Conventions

For $SU(N)$ gauge group we take the generators normalized as

$$\mathrm{Tr}(T^a T^b) = \frac{1}{2}\delta^{ab} \tag{A.1}$$

and obey the following identity

$$(T^a)_i{}^j\,(T^a)_k{}^l = \frac{1}{2}\left(\delta_i^l\,\delta_j^k - \frac{1}{N}\delta_i^j\,\delta_k^l\right) \tag{A.2}$$

whereas the structure constants can be read from

$$[T^a, T^b] = i f^{abc} T^c \tag{A.3}$$

$$\{T^a, T^b\} = \frac{1}{N}\delta^{ab} + d^{abc} T^c \tag{A.4}$$

Spinor and vector indices are raised and lowered according to

$$\psi^\alpha = C^{\alpha\beta}\psi_\beta \qquad \psi_\alpha = \psi^\beta C_{\beta\alpha} \qquad \overline{\psi}^{\dot\alpha} = C^{\dot\alpha\dot\beta}\overline{\psi}_{\dot\beta} \qquad \overline{\psi}_{\dot\alpha} = \overline{\psi}^{\dot\beta} C_{\dot\beta\dot\alpha} \tag{A.5}$$

where the matrices $C_{\alpha\beta}$ are

$$C^{\alpha\beta} = C^{\dot\alpha\dot\beta} = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix} \qquad C_{\alpha\beta} = C_{\dot\alpha\dot\beta} = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

Coordinates, fields and derivatives can be written using spinor notation as

$$\begin{aligned}
\text{coordinates:}\quad & x^\mu = (\sigma^\mu)_{\alpha\dot\beta}\, x^{\alpha\dot\beta} \qquad x^{\alpha\dot\beta} = \tfrac{1}{2}(\sigma_\mu)^{\alpha\dot\beta}\, x^\mu \\
\text{derivatives:}\quad & \partial_\mu = \tfrac{1}{2}(\sigma^\mu)^{\alpha\dot\beta}\,\partial_{\alpha\dot\beta} \qquad \partial_{\alpha\dot\beta} = (\sigma_\mu)_{\alpha\dot\beta}\,\partial^\mu \\
\text{fields:}\quad & V^\mu = \tfrac{\sqrt{2}}{2}(\sigma^\mu)_{\alpha\dot\beta}\, V^{\alpha\dot\beta} \qquad V^{\alpha\dot\beta} = \tfrac{\sqrt{2}}{2}(\sigma_\mu)^{\alpha\dot\beta}\, V^\mu
\end{aligned} \tag{A.6}$$

Pauli σ matrices satisfy

$$\sigma_\mu^{\alpha\dot\alpha}\sigma^\nu_{\alpha\dot\alpha} = 2\delta_\mu^\nu \qquad \sigma^\mu_{\alpha\dot\alpha}\sigma_\mu^{\beta\dot\beta} = 2\delta_\alpha^\beta\delta_{\dot\alpha}^{\dot\beta} \tag{A.7}$$

Superspace spinor covariant derivatives are defined as

$$D_\alpha = \partial_\alpha + \frac{i}{2}\,\overline{\theta}^{\dot\beta}\partial_{\alpha\dot\beta} \qquad \overline{D}_{\dot\alpha} = \overline{\partial}_{\dot\alpha} + \frac{i}{2}\,\theta^\beta\partial_{\beta\dot\alpha} \tag{A.8}$$

The covariant derivatives are defined as

$$\begin{aligned} \mathcal{D}^\mu q^I &= \partial^\mu q^I - ig\,A^\mu q^I \\ \mathcal{D}^\mu\phi &= \partial^\mu\phi - ig\,[A^\mu,\phi] \end{aligned} \tag{A.9}$$

The Fourier transform is defined as

$$\int \frac{d^{4-2\epsilon}p}{(2\pi)^{4-2\epsilon}}\,\frac{e^{ip\cdot(x-y)}}{(p^2)^s} = \frac{\Gamma(2-s-\epsilon)}{4^s\,\pi^{2-\epsilon}\,\Gamma(s)}\,\frac{1}{(x-y)^{2(2-s-\epsilon)}} \tag{A.10}$$

From the actions (1.13) and (1.17) the propagators in momentum space read

$$\langle A_\mu^a(x)A_\nu^b(y)\rangle = \delta^{ab}\int \frac{d^{4-2\epsilon}p}{(2\pi)^{4-2\epsilon}}e^{ip\cdot(x-y)}\,\frac{\delta_{\mu\nu}}{p^2} \tag{A.11}$$

$$\langle\bar\phi^a(x)\phi^b(y)\rangle = \delta^{ab}\int \frac{d^{4-2\epsilon}p}{(2\pi)^{4-2\epsilon}}e^{ip\cdot(x-y)}\,\frac{1}{p^2} \tag{A.12}$$

$$\langle\bar q^I(x)q_J(y)\rangle = \langle\bar{\tilde q}_J(x)\tilde q^I(y)\rangle = \delta^I_J\int \frac{d^{4-2\epsilon}p}{(2\pi)^{4-2\epsilon}}e^{ip\cdot(x-y)}\,\frac{1}{p^2} \tag{A.13}$$

$$\langle\psi^{\alpha a}(x)\bar\psi^b_{\dot\beta}(y)\rangle = \langle\eta^{\alpha a}(x)\bar\eta^b_{\dot\beta}(y)\rangle = \delta^{ab}\int \frac{d^{4-2\epsilon}p}{(2\pi)^{4-2\epsilon}}e^{ip\cdot(x-y)}\,\frac{(-p^\mu)(\sigma_\mu)^\alpha{}_{\dot\beta}}{p^2} \tag{A.14}$$

$$\langle\lambda^\alpha_J(x)\bar\lambda^I_{\dot\beta}(y)\rangle = \langle\tilde\lambda^{\alpha I}(x)\bar{\tilde\lambda}_{\dot\beta J}(y)\rangle = \delta^I_J\int \frac{d^{4-2\epsilon}p}{(2\pi)^{4-2\epsilon}}e^{ip\cdot(x-y)}\,\frac{(-p^\mu)(\sigma_\mu)^\alpha{}_{\dot\beta}}{p^2} \tag{A.15}$$

The vertices entering the three–loop diagrams can be read directly from actions (1.13) and (1.17).

Appendix B

Result of diagrams

The three-loop Master Integrals introduced in section 5.2 are defined as follows

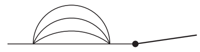
$$G_{a_1,\dots,a_{12}} = \int \frac{d^d k_{1/2/3}}{(2\pi)^{3d}} \frac{1}{P_1^{a_1} \dots P_{12}^{a_{12}}} \tag{B.1}$$

With

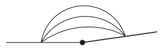
$$\begin{aligned}
P_1 &= 1 + 2\widetilde{v}_1 \cdot k_1 & P_7 &= k_1^2 \\
P_2 &= 1 + 2\widetilde{v}_2 \cdot k_1 & P_8 &= k_2^2 \\
P_3 &= 1 + 2\widetilde{v}_1 \cdot k_2 & P_9 &= k_3^2 \\
P_4 &= 1 + 2\widetilde{v}_2 \cdot k_2 & P_{10} &= (k_1 - k_2)^2 \\
P_5 &= 1 + 2\widetilde{v}_1 \cdot k_3 & P_{11} &= (k_2 - k_3)^2 \\
P_6 &= 1 + 2\widetilde{v}_2 \cdot k_3 & P_{12} &= (k_1 - k_3)^2
\end{aligned}$$

For specialized sets of $a_1, \dots, a_{12}$ indices these integrals can be computed analytically by *Mathematica* packages. Actually, for our purposes only the ϵ expansion of the result is necessary. Omitting a common factor $\frac{e^{-3\epsilon\gamma_E}}{(4\pi)^{3d/2}}$ and stopping the expansion at the required order, the Master Integrals that enter our calculation read

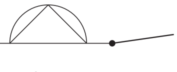
$$I_1 \equiv G_{0,0,0,0,1,0,0,1,0,1,0,1} =$$



$$= -\frac{1}{720\epsilon} - \frac{137}{7200} - \frac{12019+325\pi^2}{72000}\epsilon - \frac{874853+44525\pi^2-71000\zeta[3])}{720000}\epsilon^2 + \mathcal{O}(\epsilon^3) \tag{B.2}$$

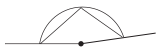
$$I_2 \equiv G_{0,0,0,0,1,1,0,1,0,1,0,1} =$$ 

$$
\begin{aligned}
&= \frac{x\Big(-1-8x+8x^3+x^4+12x^2\log(x)\Big)}{144(-1+x)(1+x)^5\epsilon} \\
&+\frac{x\Big(-7+x\Big(-59+3\pi^2x+x^2(59+7x)\Big)-9x^2\log(x)(-6+\log(x)-4\log(1+x))+36x^2\mathrm{Li}_2(-x)\Big)}{72(-1+x)(1+x)^5} \\
&+\epsilon\left[\frac{x\Big((-1+x)(1+x)(499+x(4400+499x))+\pi^2(-13+x(-104+x(216+13x(8+x))))\Big)}{576(-1+x)(1+x)^5}\right. \\
&+\frac{x^3\Big(\log(x)\Big(207+7\pi^2-6(9-\log(x))\log(x)\Big)+3\Big(\pi^2+(6-\log(x))\log(x)\Big)\log(1+x)-6(\log(-x)-\log(x))\log(1+x)^2\Big)}{48(-1+x)(1+x)^5} \\
&\left.+\frac{3x^3(3\mathrm{Li}_2(-x)-\mathrm{Li}_3(-x)-2\mathrm{Li}_3(1+x)+\zeta(3))}{2(-1+x)(1+x)^5}\right] \\
&+\epsilon^2\left[\frac{x\Big(1128\pi^4x^2+10\pi^2(-91+x(-767+x(621+13x(59+7x))))\Big)}{2880(-1+x)(1+x)^5}\right. \\
&+\frac{x^3\log(x)\Big(648+42\pi^2-\log(x)\Big(207+7\pi^2+3(-12+\log(x))\log(x)\Big)\Big)}{32(-1+x)(1+x)^5} \\
&+\frac{x^3\Big(54\pi^2+\log(x)\Big(207+7\pi^2+6(-9+\log(x))\log(x)\Big)\Big)\log(1+x)}{8(-1+x)(1+x)^5} \\
&+\frac{x^3\Big(72\Big(\pi^2-6\log(-x)-(-6+\log(x))\log(x)\Big)\log(1+x)^2+96(-\log(-x)+\log(x))\log(1+x)^3\Big)}{32(-1+x)(1+x)^5} \\
&+\frac{x\Big(-3671-33586x+33586x^3+3671x^4+284\zeta(3)\Big)}{576(-1+x)(1+x)^5} \\
&+\frac{x^2\Big(18(207+13\pi^2)x\mathrm{Li}_2(-x)-648x((3+2\log(1+x))\mathrm{Li}_3(-x)+6\mathrm{Li}_3(1+x)-\mathrm{Li}_4(-x)+4\mathrm{Li}_4(1+x)+2\mathrm{S}_{2,2}(-x))\Big)}{144(-1+x)(1+x)^5} \\
&\left.+\ \frac{x^2(568\zeta(3)+x(1944-71x(8+x)-204\log(x)+1296\log(1+x))\zeta(3))}{144(-1+x)(1+x)^5}\right]+\mathcal{O}(\epsilon^3)
\end{aligned}
\tag{B.3}
$$

$$I_3 \equiv G_{1,0,0,0,0,0,0,1,1,1,0,1} =$$ 

$$
= -\frac{1}{18\epsilon^2}-\frac{2}{3\epsilon}-\frac{16}{3}-\frac{13\pi^2}{72}-\frac{656+39\pi^2-65\zeta[3]}{18}\epsilon+\mathcal{O}(\epsilon^2)
\tag{B.4}
$$

$$
I_4 \equiv G_{1,1,0,0,0,0,0,1,1,1,0,1} =
$$



$$
\begin{aligned}
&= \frac{x\left(-1+x^2+2x\log(x)\right)}{3(-1+x)(1+x)^3\epsilon^2} + \frac{x\left(-13+x\left(\pi^2+13x\right)+x\log(x)(14-3\log(x)+12\log(1+x))+12x\mathrm{Li}_2(-x)\right)}{3(-1+x)(1+x)^3\epsilon} \\
&+ \frac{x^2\left(\log(x)\left(132+7\pi^2+6(-7+\log(x))\log(x)\right)+12\left(3\pi^2+(14-3\log(x))\log(x)\right)\log(1+x)+72(-\log(-x)+\log(x))\log(1+x)^2\right)}{6(-1+x)(1+x)^3} \\
&+ \frac{x\left(444(-1+x^2)+\pi^2(-13+x(28+13x))\right)}{12(-1+x)(1+x)^3} + \frac{4x^2(7\mathrm{Li}_2(-x)-3\mathrm{Li}_3(-x)-6\mathrm{Li}_3(1+x)+3\zeta(3))}{(-1+x)(1+x)^3} \\
&+ \epsilon\Bigg[\frac{x\left(188\pi^4x+15940\left(-1+x^2\right)+\pi^2\left(-845+660x+845x^2\right)\right)}{60(-1+x)(1+x)^3} \\
&+ \frac{x^2\log(x)\left(1048+98\pi^2-3\log(x)\left(132+7\pi^2+\log(x)(-28+3\log(x))\right)\right)}{12(-1+x)(1+x)^3} \\
&+ \frac{x^2\left(42\pi^2+\log(x)\left(132+7\pi^2+6(-7+\log(x))\log(x)\right)\right)\log(1+x)}{(-1+x)(1+x)^3} \\
&+ \frac{x^2\left(72\left(3\pi^2-14\log(-x)+(14-3\log(x))\log(x)\right)\log(1+x)^2+288(-\log(-x)+\log(x))\log(1+x)^3\right)}{12(-1+x)(1+x)^3} \\
&+ \frac{20x^2\left(\left(396+39\pi^2\right)\mathrm{Li}_2(-x)-36((7+6\log(1+x))\mathrm{Li}_3(-x)+14\mathrm{Li}_3(1+x)-3\mathrm{Li}_4(-x)+12\mathrm{Li}_4(1+x)+6\mathrm{S}_{2,2}(-x))\right)}{60(-1+x)(1+x)^3} \\
&+ \frac{x(1300\zeta(3)+20x(252-65x-22\log(x)+216\log(1+x))\zeta(3))}{60(-1+x)(1+x)^3}\Bigg] + \mathcal{O}(\epsilon^2)
\end{aligned}
$$

We can now write the contribution of every single diagram in figure 5.1 in terms of these Master Integrals. Omitting a common factor $\frac{g^6(N^2-1)(N^2+1)}{2N}$, from the insertion of corrected gauge propagators we have

$$
\begin{aligned}
(a) = &\Bigg[\frac{4(-7+3d)\left(9d^5(1+x(8+x(-2+x(8+x))))-64(35+x(241+x(25+x(241+35x))))\right)}{3(-5+d)(-4+d)^3(-3+d)(-1+d)x^2} \\
&+ \frac{4(-7+3d)\left(-3d^4(47+x(388+x(-102+x(388+47x))))+2d^3(425+x(3574+x(-830+x(3574+425x))))\right)}{3(-5+d)(-4+d)^3(-3+d)(-1+d)x^2} \\
&+ \frac{4(-7+3d)\left(8d(469+x(3698+x(-190+x(3698+469x))))-4d^2(631+x(5264+x(-850+x(5264+631x))))\right)}{3(-5+d)(-4+d)^3(-3+d)(-1+d)x^2}\Bigg] I_1 \\
&+ \Bigg[\frac{(80+9(-6+d)d)(1+x)^4\left(224-308d+160d^2-37d^3+3d^4\right)}{3(-5+d)(-4+d)^3(-3+d)(-1+d)x^3} \\
&+ \frac{(80+9(-6+d)d)(1+x)^4\left(6(-4+d)(-3+d)(4+(-8+d)d)x+(-2+d)(-112+d(98+d(-31+3d)))x^2\right)}{3(-5+d)(-4+d)^3(-3+d)(-1+d)x^3}\Bigg] I_2 \\
&+ \Bigg[\frac{12(-3+d)\left(d\left(1+6x+x^2\right)-4\left(2+5x+2x^2\right)\right)}{(-5+d)(-4+d)^2(-1+d)x}\Bigg] I_3 \\
&+ \Bigg[\frac{(1+x)^2\left(3d^2\left(1-6x+x^2\right)+80\left(1-3x+x^2\right)-2d\left(17-66x+17x^2\right)\right)}{(-5+d)(-4+d)^2(-1+d)x^2}\Bigg] I_4 \qquad \text{(B.5)}
\end{aligned}
$$

$$
\begin{aligned}
(b) = & \left[\frac{8(-7+3d)\left(\frac{1}{x}+x\right)(3d(1+x(6+x))-2(5+x(28+5x)))}{(-4+d)(-3+d)x}\right] I_1 \\
& + \left[\frac{2(80+9(-6+d)d)(1+x)^4\left(\frac{1}{x}+x\right)}{(-4+d)(-3+d)x^2}\right] I_2
\end{aligned}
\tag{B.6}
$$

$$
\begin{aligned}
(c) = & \left[-\frac{4(-7+3d)(9d^3(1+x(8+x(-2+x(8+x))))-16(35+x(241+x(25+x(241+35x)))))}{(-5+d)(-4+d)^2(-3+d)x^2}\right. \\
& \left.+\frac{4(-7+3d)(3d^2(35+x(268+x(-30+x(268+35x))))-d(418+2x(1522+x(10+x(1522+209x)))))}{(-5+d)(-4+d)^2(-3+d)x^2}\right] I_1 \\
& + \left[-\frac{(80+9(-6+d)d)(1+x)^4\left(3d^2(1+x)^2+8(7+x(9+7x))-d(25+x(42+25x))\right)}{(-5+d)(-4+d)^2(-3+d)x^3}\right] I_2
\end{aligned}
\tag{B.7}
$$

$$
\begin{aligned}
(d) = & \left[-\frac{2(-7+3d)(-160\left(1+x^2\right)(5+x(28+5x))+9d^3(1+x(8+x(-2+x(8+x)))))}{3(-5+d)(-4+d)^2(-3+d)x^2}\right. \\
& \left.-\frac{2(-7+3d)(-6d^2(19+x(134+x(6+x(134+19x))))+8d(65+x(406+x(100+x(406+65x)))))}{3(-5+d)(-4+d)^2(-3+d)x^2}\right] I_1 \\
& + \left[-\frac{(-10+3d)(-8+3d)(1+x)^4\left(3d^2(1+x)^2+80\left(1+x^2\right)-4d(7+x(6+7x))\right)}{6(-5+d)(-4+d)^2(-3+d)x^3}\right] I_2
\end{aligned}
\tag{B.8}
$$

$$
\begin{aligned}
(e) = & \left[\frac{8(-7+3d)\left(9d^5(1+x(2+x(10+x(2+x))))-3d^4(53+x(148+x(390+x(148+53x))))\right)}{3(-5+d)(-4+d)^3(-3+d)(-1+d)x^2}\right. \\
& + \frac{8(-7+3d)(-32(85+x(296+x(350+x(296+85x))))+16d(296+x(1045+x(1315+x(1045+296x)))))}{3(-5+d)(-4+d)^3(-3+d)(-1+d)x^2} \\
& \left.\frac{8(-7+3d)\left(2d^3(518+x(1663+x(3046+x(1663+518x))))-4d^2(799+x(2747+x(3986+x(2747+799x))))\right)}{3(-5+d)(-4+d)^3(-3+d)(-1+d)x^2}\right] I_1 \\
& + \left[\frac{2(80+9(-6+d)d)(1+x)^4\left(272-392d+202d^2-43d^3+3d^4\right)}{3(-5+d)(-4+d)^3(-3+d)(-1+d)x^3}\right. \\
& \left.+\frac{2(80+9(-6+d)d)(1+x)^4\left(-6(-4+d)(-3+d)(8+d(-7+2d))x+(-2+d)(-136+d(128+d(-37+3d)))x^2\right)}{3(-5+d)(-4+d)^3(-3+d)(-1+d)x^3}\right] I_2 \\
& + \left[\frac{6(-3+d)(8+(-5+d)d)(-4(2+x)(1+2x)+d(1+x(6+x)))}{(-5+d)(-4+d)^2(-1+d)x}\right] I_3 \\
& + \left[\frac{(-10+3d)(8+(-5+d)d)(1+x)^2\left(-8+d-6(-4+d)x+(-8+d)x^2\right)}{2(-5+d)(-4+d)^2(-1+d)x^2}\right] I_4
\end{aligned}
\tag{B.9}
$$

$$
\begin{aligned}
(f) = \Bigg[& \frac{4(-7+3d)\big(9d^3(1+x)^4-3d^2(33+x(132+x(166+33x(4+x))))\big)}{3(-5+d)(-4+d)(-3+d)(-1+d)x^2} \\
& + \frac{4(-7+3d)(4d(83+x(316+x(370+x(316+83x))))-4(85+x(296+x(350+x(296+85x)))))}{3(-5+d)(-4+d)(-3+d)(-1+d)x^2}\Bigg] I_1 \\
& + \left[\frac{(80+9(-6+d)d)(1+x)^4\big(34-23d+3d^2-6(-4+d)(-3+d)x+(-2+d)(-17+3d)x^2\big)}{3(-5+d)(-4+d)(-3+d)(-1+d)x^3}\right] I_2 \\
& + \left[\frac{6(-3+d)^2(-4(2+x)(1+2x)+d(1+x(6+x)))}{(-5+d)(-4+d)(-1+d)x}\right] I_3 \\
& + \left[\frac{(-3+d)(-10+3d)(1+x)^2\big(-8+d-6(-4+d)x+(-8+d)x^2\big)}{2(-5+d)(-4+d)(-1+d)x^2}\right] I_4
\end{aligned}
\tag{B.10}
$$

$$
\begin{aligned}
(g) = \Bigg[& \frac{8(-7+3d)\big(9d^3(1+x(8+x(-2+x(8+x))))+8(5+x(-62+x(100+x(-62+5x))))\big)}{3(-5+d)(-4+d)(-3+d)(-1+d)x^2} \\
& + \frac{8(-7+3d)\big(-3d^2(23+x(196+x(-54+x(196+23x))))+2d(59+x(646+x(-290+x(646+59x))))\big)}{3(-5+d)(-4+d)(-3+d)(-1+d)x^2}\Bigg] I_1 \\
& + \left[\frac{2(80+9(-6+d)d)(1+x)^4\big(3d^2(1+x)^2-4(1+(-18+x)x)-d(13+x(42+13x))\big)}{3(-5+d)(-4+d)(-3+d)(-1+d)x^3}\right] I_2 \\
& + \left[\frac{24(-3+d)(-4(2+x)(1+2x)+d(1+x(6+x)))}{(-5+d)(-4+d)(-1+d)x}\right] I_3 \\
& + \left[\frac{2(-10+3d)(1+x)^2\big(-8+d-6(-4+d)x+(-8+d)x^2\big)}{(-5+d)(-4+d)(-1+d)x^2}\right] I_4
\end{aligned}
\tag{B.11}
$$

$$
\begin{aligned}
(h) = \Bigg[& -\frac{64(-7+3d)\big(9d^3(-1+x)^2x-2d(1+(-12+x)x)(8+x(-17+8x))\big)}{3(-5+d)(-4+d)^2(-3+d)x^2} \\
& - \frac{64(-7+3d)\big(3d^2(1+x(-25+x(52+(-25+x)x)))+4(5+x(-62+x(100+x(-62+5x))))\big)}{3(-5+d)(-4+d)^2(-3+d)x^2}\Bigg] I_1 \\
& + \left[-\frac{16(80+9(-6+d)d)(1+x)^4\big(-2+d+3(-4+d)(-3+d)x+(-2+d)x^2\big)}{3(-5+d)(-4+d)^2(-3+d)x^3}\right] I_2
\end{aligned}
\tag{B.12}
$$

Similarly, from the insertion of corrected scalar propagators, omitting a common factor $\frac{g^6(N^2-1)(N^2+1)}{2N}\cos\theta$ we have

$$
\begin{aligned}
(i) = & \left[-\frac{32(-7+3d)(16+d(-11+2d))(3d(1+x(6+x))-2(5+x(28+5x)))}{(-4+d)^3(-3+d)x}\right] I_1 \\
& - \left[\frac{8(-10+3d)(-8+3d)(16+d(-11+2d))(1+x)^4}{(-4+d)^3(-3+d)x^2}\right] I_2 \\
& - \left[\frac{96(-3+d)^2}{(-4+d)^2}\right] I_3 - \left[\frac{8(-3+d)(-10+3d)(1+x)^2}{(-4+d)^2 x}\right] I_4
\end{aligned}
\tag{B.13}
$$

$$
\begin{aligned}
(j) = & \left[-\tfrac{16(-7+3d)(3d(1+x(6+x))-2(5+x(28+5x)))}{(-4+d)(-3+d)x}\right] I_1 \\
& - \left[\tfrac{4(80+9(-6+d)d)(1+x)^4}{(-4+d)(-3+d)x^2}\right] I_2
\end{aligned} \tag{B.14}
$$

$$
\begin{aligned}
(k) = & \left[-\tfrac{64(-2+d)(-7+3d)(3d(1+x(6+x))-2(5+x(28+5x)))}{(-4+d)^2(-3+d)x}\right] I_1 \\
& - \left[\tfrac{16(-2+d)(80+9(-6+d)d)(1+x)^4}{(-4+d)^2(-3+d)x^2}\right] I_2
\end{aligned} \tag{B.15}
$$

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