Università degli Studi di Milano-Bicocca
Scuola di Dottorato

Dipartimento di Fisica G. Occhialini
Corso di Dottorato in Fisica e Astrofisica XXXII ciclo
Curriculum in Fisica Teorica

Tesi di Dottorato

Supergravity solution classifications through bispinors

Dottorando:
Andrea Legramandi
Matricola 735841

Tutore:
Prof. Alessandro Tomasiello

Coordinatore:
Marta Calvi

Anno Accademico 2018-2019

Abstract

This thesis focuses on classification of supergravity solutions in ten and eleven dimensions. By imposing supersymmetry, supergravity reveals a plethora of elegant geometric structures which can be defined from the fermionic supersymmetry parameters. Such geometrical data are called bispinors and are the central topic of this thesis. In the first part we explore how it is possible to exploit bispinors in order to get a more elegant reformulation of background supersymmetry conditions. This discussion is performed in a general context without assuming any factorization of space-time. The bispinor framework allows to interpret many of the new supersymmetry equations as calibration conditions for sources, where a calibration is a differential form which detects branes with minimal energy. We also discuss the connection between calibrations and BPS bound and we provide a definition of central charges in purely gravitational terms. Aside from these formal results, probably the main achievement of the bispinor formalism is that it drastically simplifies the task of classifying supergravity solutions. After discussing how to apply these techniques to AdS_2 and $\mathbb{R}^{1,3}$ backgrounds, we perform a complete classification, in both type II supergravity and M-theory, of $\mathbb{R}^{1,3}$ solutions preserving $\mathcal{N} = 2$ supersymmetry with $\mathrm{SU}(2)$ R-symmetry geometrically realized by a round S^2 factor in the internal space. For the various cases of the classification, the problem of finding supersymmetric solutions can be reduced to a system of partial differential equations. These cases often accommodate systems of intersecting branes and higher-dimensional anti-de-Sitter solutions. Moreover we show that, using chains of dualities, all solutions can be generated from one of two master classes: an $\mathrm{SU}(2)$-structure in M-theory and a conformal Calabi–Yau in type IIB. In the last part of the thesis, we show that it is possible to relax some of the bispinor equations and generalize all the classification to a larger non-supersymmetric context.

Contents

INTRODUCTION

String theory [1, 2, 3] is so far the most successful setting which implements quantum gravity and gauge interactions in a consistent way. These features justify why it monopolized the attention of theoretical physics community in the last decades and, despite the enormous effort, only a small corner of possible string theory realizations has been explored so far.

Even if, in a perturbative limit, string is the only fundamental field, the theory is also populated by various higher dimensional extended objects which appears at a non-perturbative level. These are generically called branes [4] and can be considered as fundamental as the string itself. In string theory we can distinguish two different types of dynamical branes: D-brane and NS-brane. While the first kind can be perturbatively described in terms of open string, the second type cannot and must be understood via string dualities.

Since the string revealed not to be the only one fundamental object, one may ask if it can be substituted with a paradigm which contains just membranes. This is believed to happen and the resulting theory is called M-theory [5], which is another possible quantum gravity realization.

One of the main difference between string theory and M-theory lies in the number of space-time dimensions where they are well defined: the first one is consistent only in ten dimensions, while the second needs eleven. This feature is responsible for their richness, indeed one can think to realize some space dimensions as a small compact manifold, also called “internal” space, while leaving the “external” manifold non-compact. This mechanism is also used to embed our everyday physics in the string-theory context, indeed if the internal space is small enough the external-space theory will just have usual four-dimensional matter and interactions. For studying such realizations it is not necessary to take into account effects due to quantum physics or heavy states and it is enough to consider the low-energy classical limit of string and M-theory, which is respectively ten- and eleven-dimensional supergravity. As the name suggests, one of the key ingredient underlying supergravity is supersymmetry.

Supersymmetry is an incredibly useful tool for classifying supergravity backgrounds: when a solution saturates the Bogomol'nyi-Prasad-Sommerfield (BPS) bound it satisfies the equations of motion if it solves supersymmetry conditions, which are a first-order system of partial differential equations (PDEs). This system is the key to achieve a formal study of the geometry of the solution [6, 7, 8] : imposing supersymmetry reveals a plethora of elegant structures which can be constructed from the fermionic supersymmetry parameters. These structures are usually differential forms and they can be used to replace the first-order spinorial system with one which involves just closure conditions and other natural operations on forms [9, 10, 11]. In this reformulation the metric only appears indirectly, which makes the obtained equations easier to solve than the original ones. Moreover, in some cases they also have an elegant physical interpretation in terms of brane calibrations [12, 13].

In differential geometry, a calibration is a closed form that measures if a submanifold minimizes its volume [14]. In supergravity context a calibration measures if a brane minimizes its energy, therefore calibrated branes satisfies the BPS bound, i.e., they are supersymmetric. Calibration conditions are therefore equivalent to impose that part of the background supersymmetry is preserved, indeed it was conjectured that it is possible to reformulate the BPS equations entirely in terms of calibration conditions. A strong indication that this is the case can be found in [15] and it is part of the work this thesis is based on.

Requiring that supersymmetry is preserved dramatically simplifies the task of constructing explicit solutions, and saturation of BPS bound guarantees stability [16, 17]. However, many interesting theoretical and phenomenological problems such as realization of de Sitter (dS) vacua, holographic understanding of color confinement and studying of black hole thermodynamics require non-supersymmetric solutions. While models have been proposed, concrete constructions of non-supersymmetric backgrounds have been elusive so far and this task is complicated by the existence of several no-go theorems [18, 19], conjectures [20, 21] and swampland arguments [22, 23, 24].

Luckily, it is possible to evade the no-go theorems, which mostly affect the dS case, by adding quantum corrections and/or orientifolds, which are non-dynamical branes with negative tension, so one can still hope that string theory can be used to get a realistic description of our reality. The same no-go theorems apply if we are studying Minkowski vacua, where the cosmological constant is fine-tuned to vanish. Even if again orientifolds are necessary to get a proper solution with fluxes, supersymmetry can still be preserved, so one can use the powerful tools described above to find solutions which evade the no-go theorems [25]. Therefore a more general study of Minkowski backgrounds can be useful both as a laboratory for string theory dynamics and as intermediate construction for the dS case, as it was done in [26] (but also [27, 28] for instance).

Moreover, a Minkowski classification is also useful from a holographic perspective, indeed anti de-Sitter (AdS) solutions admit a description in terms of a foliation of Minkowski over a non compact interval. Even if supersymmetric AdS backgrounds admit a detailed classification, at least in high external-space dimensions [29, 30, 31, 32, 33], they always assume a global AdS factor from the start and therefore are not particularly useful for studying certain non-conformal behaviors such as RG flows, where the AdS vacuum corresponds to a conformal fixed point at one of the two ends of the flow.

A classification of four-dimensional external-space Minkowski solutions in string theory is performed in this thesis using the so-called pure spinor equations [11, 34]. These are a set of differential form equations which can be elegantly embedded in the context of generalized complex geometry [35, 36], which is a mathematical approach which consider G-bundle defined on the direct sum of the tangent and contangent space of the internal-space manifold. For M-theory it is not possible to find such an elegant description, still a system of BPS form-equations exists [37].

It is clear that these approaches are closely related to supersymmetry, and thus the study of general non-supersymmetric Minkowski compactifications is much more difficult. However, one can start by asking if there is a subset of non-supersymmetric solutions which share the same integrability properties of supersymmetric flux compactifications, i.e., if there exists a modification of the pure spinor equations which allows to solve the equations of motion but preserving a first order formalism. The answer to this question is in general unknown, but there are very special examples in type IIB string theory in which such a condition is satisfied [25, 38]. In this thesis, using the classification achieved in four-dimensional Minkowski solution, we will show how to break supersymmetry by directly modifying the pure spinor equations also in type IIA supergravity.

The thesis is divided in two parts. In the first one we deal with the more formal and general aspects of supergravity in ten and eleven dimensions, following mostly [15, 9, 10, 13]. We start with an introduction of them focusing on the solitonic objects that populate these theories and on the duality between them. In chapter 2 we introduce the bispinors method from an algebraic viewpoint; in particular we will derive the structure group defined by such objects on both the tangent bundle and the generalized tangent bundle and we will show how they transform after a string duality. In chapter 3 we apply what we have learned from the previous chapter to derive necessary and sufficient conditions to rewrite supersymmetry in terms of spinor bilinears; integrability is also discussed and a system of form equations which is invariant under the $\mathrm{Sl}(2,\mathbb{R})$-symmetry of type IIB is presented. Many of the form equations are interpreted in chapter 4 in terms of calibration conditions for fluxes. After a review of how a calibration is related to the BPS bound, calibration conditions for D-brane, fundamental string, M2- and M5-brane are discussed. Moreover, the calibrations for NS5- and NS9-branes are presented, together with a discussion on the KK-monopole calibration, which involves the definition of central charges in purely gravitational terms. Before concluding the first part, some applications of the systems presented in the third chapter are considered in turn in chapter 5. In particular, we focus on $\mathrm{AdS}_2 \times M_8$ solutions in type IIA supergravity, which are relevant for the classification of near-horizon backgrounds, and on $\mathbb{R}^{1,3} \times M_6$ solutions, which will be useful in the second part of the thesis.

In the second part, following [39, 40, 41], we perform a classification of $\mathbb{R}^{1,3} \times S^2$ solutions in both type II and M-theories. The classification is mainly based on the pure spinor equations derived in chapter 5, restricted to fit a round sphere in the internal space. After specializing spinor and fluxes to accommodate SO(3) isometries and discussing some properties of these solutions from a spinorial viewpoint in chapter 6, we start the classification for type II supergravity in chapter 7. In particular we discuss two master classes, one in IIA and the other in IIB, from which all the possible $\mathbb{R}^{1,3} \times S^2$ solutions can be generated using string dualities. In chapter 8 a similar classification is achieved but in M-theory; we will show that these solutions are actually linked to the ones of type II supergravity and they can actually generate some of them. In chapter 9 we will focus on backgrounds with an AdS factor, which can be derived from the $\mathbb{R}^{1,3} \times S^2$ classification and allow to make contact with many known solutions. In the last chapter we present, following [42], a method to break supersymmetry in all the classes contained in the $\mathbb{R}^{1,3} \times S^2$ classification of type II theories. In particular, similar to what it was done in [25], we will manage to solve the equations of motion while keeping a first order formalism.

Part I

General BPS configurations in supergravity

CHAPTER 1

INTRODUCTION TO SUPERGRAVITY

In this section we will introduce maximally supersymmetric ten- and eleven-dimensional supergravity, which are respectively the massless sector of type II string theory and M-theory. In ten dimensions maximal supergravity has two fermionic variations, and it comes in two types: type IIA and type IIB, depending on the chirality of supersymmetry generators; in particular we have that type IIB is chiral while IIA is not. Eleven-dimensional supergravity on the other hand is maximally supersymmetric with $\mathcal{N} = 1$ supersymmetry. Maximal supersymmetry is a strong constraint on the possible structure of these theories, indeed it fully determines them. Let's see a bit more in detail the properties of these theories.

1.1 Eleven-dimensional supergravity

In this section we will mainly adopt the conventions of [10], which are partially summarized in appendix A. Eleven-dimensional supergravity consists just of three fields: a metric g, a three-form potential A with four-form field strength $F = \mathrm{d}A$ and a gravitino Ψ_M. As customary in supergravity we set the all fermionic fields to zero[1], $\Psi_M = 0$, so that we have to deal just with bosonic configurations. The bosonic action is given by

$$S_M = \frac{1}{2\kappa_{11}^2} \int \left(\sqrt{-g} R \,\mathrm{d}^{11} x - \frac{1}{2} F \wedge *F - \frac{1}{6} A \wedge F \wedge F \right). \tag{1.1}$$

The get an eleven-dimensional supergravity solution one have to solve the following two equations of motion, obtained by varying the action respect to g_{MN} and

[1] This condition is actually a requirement if we are interested in vacuum solution, but for now we will keep the discussion general.

the potential A

$$R_{MN} - \frac{1}{2} F_M \cdot F_N + \frac{1}{12^2} g_{MN} F^2 = 0\,, \tag{1.2a}$$

$$d * F + \frac{1}{2} F \wedge F = 0\,, \tag{1.2b}$$

where $F_M = \iota_M F$, $F^2 = F \cdot F$ and the dot operator $\cdot$ together with the other form operators are defined in appendix A.1. Notice that also (1.2b) can be used to define a magnetic potential C associated with F, indeed we can rewrite it as

$$\mathrm{d}\left(*F + \frac{1}{2} A \wedge F \right) = 0\,, \tag{1.3}$$

which is locally satisfied if there exists C such that

$$\mathrm{d}C = *F + \frac{1}{2} A \wedge F\,. \tag{1.4}$$

As anticipated in the introduction, M-theory contains also non-perturbative membranes called Mk-branes where $k = 2, 5, 9$ is the number of their spatial directions. Membranes act as sources for the potential A and C, in particular the M2-brane is electrically charged respect to A and magnetically respect to C and vice-versa for the M5 brane; on the other hand the M9 arises as the $\mathbb{Z}_2$ fixed point of the Horava-Witten theory [43] and therefore is not charged under any fields and interacts just with the metric.

In presence of a brane we have to modify our action adding those of the sources, which is given by the ABJM action [44] for the M2-brane sitting on a ADE singularity, PST action for the M5 [45] and the action in [46, 47] for the M9. The result of this operation is that the equations of motion get modified and the potential A and C cannot be defined because F and $*F + \frac{1}{2} A \wedge F$ are not closed anymore near to the source, schematically

$$\begin{aligned} &\mathrm{d}F = \delta_5 \\ &d * F + \frac{1}{2} F \wedge F = \delta_8 \end{aligned} \tag{1.5}$$

where δ_k is a delta-like k-form localized on the transverse directions of the electrically-charged brane. We refer to the next section for a more detailed discussion in the D-brane setting.

Since the fermionic field is set to zero, in order to impose supersymmetry it is enough to set to zero the fermionic supersymmetry variation, which, using the properties of gamma-matrices in odd dimensions (appendix A.2), can be written in the following way

$$\nabla_M \epsilon - \frac{1}{12} \iota_M (*F + 2F)\, \epsilon = 0\,, \tag{1.6}$$

where ϵ is a Majorana spinor and all the forms are mapped to bispinors via Clifford map (refer to appendix A.3 for more details and definitions).

As anticipated in the introduction, solving supersymmetry constraint (1.6) implies, to a certain extent, the equations of motion. This can be made more precise in the following way; using the relation

$$\gamma^N \nabla_{[M}\nabla_{N]}\,\epsilon = \frac{1}{4} R_{MN}\gamma^N \epsilon \tag{1.7}$$

and substituting to the left-hand side the supersymmetry condition (1.6) it is possible to show that

$$(R_{MN} - \frac{1}{2} F_M \cdot F_N + \frac{1}{12^2} g_{MN} F^2)\gamma^M \epsilon = 0 \tag{1.8}$$

provided that the Bianchi identity $\mathrm{d}F = 0$ and the second equation of (1.2) are satisfied. It is important to stress that the condition (1.8) is not always enough to ensure that all the components of M-theory Einstein equation (1.2b) are set to zero, indeed it can happen that some of them must be imposed as extra-constraints (see for example [48]).

1.2 Ten-dimensional supergravity

In this section we will review both type IIA and type IIB supergravity following the conventions of [9, 15]. Even if they are two fundamentally different theories, it is possible to introduce them together by using the democratic formulation of supergravity [49]. In this formalism the fermionic sector of the theory is given by two gravitini and two dilatini

$$\psi_M^{1,2}\,, \qquad \lambda^{1,2} \tag{1.9}$$

which are Majorana–Weyl spinors. In IIA ψ_M^1 and λ^1 have positive chirality and ψ_M^2 and λ^2 have negative chirality while in IIB all the fields have the same (positive) chirality. The bosonic fields organize themselves in a common part, the NSNS sector (NS is a shorthand for Neveu–Schwarz) which is composed by the metric g, the dilaton ϕ and a two-form potential B, and the Ramond-Ramond (RR) sector, which can be recast as a formal sum of differential forms C_i with different degree i

$$C = \begin{cases} C_1 + C_3 + C_5 + C_7 + C_9 & \text{for IIA} \\ C_0 + C_2 + C_4 + C_6 + C_8 & \text{for IIB} \end{cases}. \tag{1.10}$$

All these potentials can be used to define the field strength in the following way

$$H = \mathrm{d}B\,, \qquad F = \mathrm{d}C - H \wedge C = \mathrm{d}_H C\,, \tag{1.11}$$

where now F is a polyform with even degree in IIA and odd degree in IIB. Actually type IIA supergravity admits also a constant zero form field F_0 which is called Romans mass, which did not arise from any potential because it does not carry any degrees of freedom. In massive type IIA supergravity (1.11) get modified as follows

$$F = \mathrm{d}_H C + e^{B\wedge} F_0 \,. \tag{1.12}$$

Notice that since H is closed the twisted external derivative d_H defines a cohomology $\mathrm{d}_H^2 = 0$, indeed the Bianchi identities for F, $\mathrm{d}_H F = 0$, is equivalent to the existence of the potential C. In the democratic formulation of supergravity however the RR fluxes are not all independent but are linked by a (anti)self-duality[2] relation

$$F = *\lambda(F) \tag{1.13}$$

which is not imposed by the equations of motion coming from the action we present here. This basically means that the democratic pseudo-action is nothing but a mnemonic tool to obtain the equations of motion; however, other approaches exist to get proper a proper lagrangian, for example [50]. Again, since we are mainly interested in the study of bosonic configurations, we set to zero all the fermionic fields. The remaining part reads:

$$S_{10} = \frac{1}{2\kappa_{10}^2} \int \mathrm{e}^{-2\phi} \sqrt{-g}\, \mathrm{d}^{10} x \left(R + 4(\mathrm{d}\phi)^2 - \frac{1}{2} H^2 - \frac{e^{2\phi}}{2} F^2 \right). \tag{1.14}$$

By varying S_{10} respect to the dilaton, the metric and B we get, after some manipulation, the following set of equations of motion

$$\nabla^2\phi - (\mathrm{d}\phi)^2 + \frac{1}{4} R - \frac{1}{8} H^2 = 0\,, \tag{1.15a}$$

$$\mathrm{d}\left(\mathrm{e}^{-2\phi} * H\right) - \frac{1}{2}(F, F)_8 = 0\,, \tag{1.15b}$$

$$R_{MN} + 2\nabla_M \nabla_N \phi - \frac{1}{2} H_M \cdot H_N - \frac{\mathrm{e}^{2\phi}}{2} \left(F_M \cdot F_N - \frac{1}{2} g_{MN} F^2 \right) = 0\,, \tag{1.15c}$$

which must be solved together with the Bianchi identities

$$\mathrm{d}H = 0\,, \qquad \mathrm{d}_H F = 0. \tag{1.16}$$

String theory contains a broader variety of objects compered to M-theory. The best understood are D-branes. D-brane is a shorthand for Dirichlet membrane, and is an extended objects upon which open strings can end with Dirichlet boundary conditions. A Dp-brane couples with the $p+1$-form potential C_{p+1} of the RR sector. This immediately implies that type IIA has just even-dimensional branes while type

[2] The field are self-dual or antiself-dual depending on the degree of F according to the definition of λ given in (A.4).

IIB odd-dimensional ones. Thanks to the self-duality relation (1.13) a Dp-brane can be electrically charged under C_{p+1} or a magnetically charge respect to C_{7-p}. Notice that type IIA admits a D8-brane which is charged under $F_{10} = *F_0$, which is a non-dynamical field strength, while IIB admit a self-dual $D3$-brane due to the relation $F_5 = *F_5$ and a space filling D9-brane, whoese potential is pure gauge. In the low-energy limit of string theory we are considering, the action of a Dp-brane wrapping a $p+1$-cycle $\mathcal{S}$ is given by a Dirac–Born–Infeld plus Chern–Simons terms

$$S_{D_p} = -\mu_{D_p} \int_{\mathcal{S}} \mathrm{d}\xi^{p+1}\, \mathrm{e}^{-\phi} \sqrt{-\det(g|_{\mathcal{S}} + \mathcal{F})} + \mu_{D_p} \int_{\mathcal{S}} C|_{\mathcal{S}} \wedge \mathrm{e}^{\mathcal{F}} \tag{1.17}$$

where $\mu_{D_p} > 0$ is the brane tension and $\mathcal{F}$ is the gauge invariant world-volume field-strength on the brane which satisfies $\mathrm{d}\mathcal{F} = -H|_{\mathcal{S}}$[3]. It is now clear that by adding this source to the supergravity action (1.14) and varying respect to C_{p+1} we get that the Bianchi identity (1.16) gets modified in a similar fashion to what happened in the M-theory case (1.5)

$$\mathrm{d}_H F_{8-p} = 2\kappa_{10}^2 \mu_{D_p} \delta_{9-p} \tag{1.18}$$

where the δ_{9-p} is defined such that

$$\int_{\mathcal{S}} C|_{\mathcal{S}} = \int C \wedge \delta_{9-p}\,. \tag{1.19}$$

Notice that thanks to the $\mathrm{e}^{\mathcal{F}}$ factor in the Chern–Simons term of (1.17) it is also possible to have Dp-branes which are charged under all the RR potential C with degree lower then p.

D-branes are not the only extended object which carry RR charges, indeed string theory allows also non-dynamical objects called O-planes. The “O” stays for orientifold, which is a quotient procedure that involves both a space-time involution and worldsheet parity. The fixed locus of the space-time involution becomes a source for the RR fields which is the O-planes itself. Their action is identical to the D-brane one except that it has not dynamical field $\mathcal{F}$:

$$\mathrm{d}_H F_{8-p} = 2\kappa_{10}^2 \mu_{O_p} \delta_{9-p}\,. \tag{1.20}$$

One of the most peculiar property of O-planes is that their tension has opposite sign respect to D-branes $\mu_{O_p} = -2^{p-4} \mu_{D_p}$ so, in a sense, they are source for anti-gravity. Their repulsive nature can be seen as the reason why they are necessary for compactification to dS and Minkowski external space.

[3]Notice that $\mathcal{F}$ cannot be just the B-field because it is not gauge invariant, so we need an extra one-form field a, which leaves on the branes, in order to compensate this freedom. In general we have $\mathcal{F} = B|_{\mathcal{S}} + \mathrm{d}a$

Obviously, also the fundamental string can be seen as an extended object of string theory. When it moves in a non-trivial background it couples with the NSNS three-form H, indeed its action is simply composed by the usual Nambu–Goto term plus a Wess–Zumino part:

$$S_{\mathrm{F1}} = -\mu_{\mathrm{F1}} \int_{\mathcal{S}} \mathrm{d}\xi^2 \sqrt{-\det(g|_{\mathcal{S}})} - \mu_{\mathrm{F1}} \int_N B|_{\mathcal{S}} \,. \tag{1.21}$$

Similarly to what we have seen in M-theory, it is also possible to define a magnetic potential for the fundamental string starting from the equation of motion of H. Using the Bianchi identity for F and the property $(A,B)_d = (-)^{d(d-1)/2}(B,A)_d$, we can rewrite $(F,F)_8 = -\mathrm{d}(F,C)_7 + F_0(e^{-B\wedge}C)_7$ (where F_0 is turned on just in IIA) and therefore the second equation of (1.15) reads

$$\mathrm{d}\left[(\mathrm{e}^{-2\phi} * H + \frac{1}{2}(F,C)_7 - \frac{1}{2}F_0(e^{-B\wedge}C)_7\right] = 0. \tag{1.22}$$

This means that we can locally define a potential $\widetilde{B}$ such that:

$$\mathrm{d}\widetilde{B} = \mathrm{e}^{-2\phi} * H + \frac{1}{2}(F,C)_7 - \frac{1}{2}F_0(\mathrm{e}^{-B\wedge}C)_7. \tag{1.23}$$

This differential form can also be seen as an electric potential for the "magnetic dual" of the fundamental string, which is a brane wrapping six space-time dimensions called NS5-brane. Differently from D-branes, the NS5-brane is a little more subtle: since it has not a direct definition in terms of open strings, but only as solitonic supergravity solution, its action can be understood just using string dualities. Dualities will be examined in depth in section 1.3 and 2.4, however let us anticipate that for IIA the NS5 action can be derived by dimensional reducing the M5-brane action of M-theory along a transversal direction as done in [51], while for IIB it can be obtained from the D5-action using an S-duality transformation [52]. Similarly to the D-brane case, there exists also a kind of O-planes that are charged under $\widetilde{B}$ but carry negative tension. These surfaces are called ONS5-planes and they are generated at the worldsheet level by a simultaneous action of the left fermionic number $(-)^{F_L}$ and a parity transformation along the transversal directions [53]. The last peculiar object we introduce in this section is the NS9-brane, which is a space-filling non-dynamical membrane which, similarly do the D9-brane, does not carry any charge.

After this digression on the classification of the string-theory objects, let us introduce the fermionic supersymmetry variations that must be set to zero in order to get a BPS solution. In the notation of [9], they read

$$\left(D_M - \frac{1}{4}H_M\right)\epsilon_1 + \frac{\mathrm{e}^{\phi}}{16}F\gamma_M\epsilon_2 = 0\,, \qquad \left(D - \frac{1}{4}H - \partial\phi\right)\epsilon_1 = 0\,, \tag{1.24a}$$

$$\left(D_M + \frac{1}{4}H_M\right)\epsilon_2 + (-)^{|F|}\frac{\mathrm{e}^{\phi}}{16}\lambda(F)\gamma_M\epsilon_1 = 0\,, \qquad \left(D + \frac{1}{4}H - \partial\phi\right)\epsilon_2 = 0\,, \tag{1.24b}$$

where the sign $(-)^{|F|} = (-)^{\deg(F)}$ is the only difference between IIA and IIB. Here ϵ_1 and ϵ_2 are a pair of Majorana–Weyl spinors and, while in IIB they have both positive chiralities, in type IIA we take the chirality of ϵ_1 to be positive while the one of ϵ_2 negative. Acting with γ^M on the two equations on the left and subtracting the ones on the right side it is possible to get other equations which were the original dilatino variations:

$$\left(\partial\phi - \frac{1}{2}H\right)\epsilon_1 + \frac{\mathrm{e}^{\phi}}{16}\gamma^M F\gamma_M\epsilon_2 = 0 \tag{1.25a}$$

$$\left(\partial\phi + \frac{1}{2}H\right)\epsilon_2 + (-)^{|F|}\frac{\mathrm{e}^{\phi}}{16}\gamma^M \lambda(F)\gamma_M\epsilon_1 = 0\,. \tag{1.25b}$$

Integrability conditions for a BPS solution are discussed in [38], let's review that argument here. Define, first of all, the following differential operators which act on a vector of spinors $\epsilon = (\epsilon_1, \epsilon_2)$

$$\begin{aligned}\mathcal{D}_M &= \left(D_M \otimes \mathbb{1} - \frac{1}{4}H_M \otimes \sigma_3\right) + \frac{\mathrm{e}^{\phi}}{16}\begin{pmatrix} 0 & F \\ (-)^{|F|}\lambda(F) & 0\end{pmatrix}\gamma_M\,, \\ \Delta &= \left(D \otimes \mathbb{1} - \partial\phi \otimes \mathbb{1} - \frac{1}{4}H \otimes \sigma_3\right)\,.\end{aligned} \tag{1.26}$$

Moreover, we can also use these operators to write the dilatino super-variation operator $\widetilde{\Delta} = \gamma^M \mathcal{D}_M - \Delta$. Now supersymmetry conditions can be rephrased as

$$\mathcal{D}_M \epsilon = 0\,, \qquad \Delta\epsilon = 0\,. \tag{1.27}$$

Using Bianchi identities for F and H it can be proven that

$$\begin{aligned}&[\mathcal{D}_N, \Delta]\,\epsilon - \left[\gamma^M, \mathcal{D}_N\right]\mathcal{D}_M\epsilon = \left(-\frac{1}{2}E_{NM}\gamma^M \otimes \mathbb{1} - \frac{1}{4}\delta H_{NM}\gamma^M \otimes \sigma_3\right)\epsilon \\ &\widetilde{\Delta}^2\epsilon - \left(D_M \otimes \mathbb{1} - 2\partial_M\phi \otimes \mathbb{1} - \frac{1}{4}H_M \otimes \sigma_3\right)\mathcal{D}^M\epsilon = -D\epsilon\end{aligned} \tag{1.28}$$

where D is the dilaton equation defined as in (1.15a), E_{MN} is the Einstein equation and δH_{NM} is the Hodge dual of (1.15b)

$$\delta H_{NM}\,\mathrm{d}x^N \wedge \mathrm{d}x^M = *\,\mathrm{e}^{2\phi}\left(\mathrm{d}\left(e^{-2\phi} * H\right) - \frac{1}{2}(F,F)_8\right)\,. \tag{1.29}$$

It is clear from (1.28) that setting to zero supersymmetry variations (1.27) implies the dilaton equation and at least some components of B-field and Einstein equations, but not necessarily all of them [54, 55] .

1.3 Dualities

String and M-theory enjoy a thick web of dualities which link apparently disconnected quantities like large and small scales, strong and weak coupling and quantities with different dimensions. The dualities we will discuss are T-duality, $\mathrm{Sl}(2,\mathbb{Z})$ symmetry and M-theory to IIA dimensional reduction. T-duality is a transformation which allows, under certain conditions, a map between type IIA and type IIB string theory. $\mathrm{Sl}(2,\mathbb{Z})$ symmetry, which in the supergravity limit becomes $\mathrm{Sl}(2,\mathbb{R})$ symmetry, is an equivalence of field configurations in type IIB, while M-theory dimensional reduction reproduces type IIA supergravity with the Romans mass turned off. In this section we will show how the various fields are mapped after one of these transformations.

1.3.1 T-duality

Type II theories with d commuting isometries are characterized by an $\mathrm{O}(d,d;\mathbb{Z})$ group of T-dualities. Any element of the $\mathrm{O}(d,d;\mathbb{Z})$ T-duality can be decomposed into a product of simple T-dualities along a given Abelian isometry[4], discrete diffeomorphisms and shifts of the B-field. We can then focus on the action of a single T-duality along a certain isometric direction, parameterized by a coordinate y where ∂_y is a compact vector field which is not just Killing but also a symmetry of the whole solution (i.e., its Lie derivative kills any fields). This kind of T-duality consists in a map between type IIA and IIB supergravity both endowed with an isometric compact direction, however the circle radii in these two theories are inversely proportional.

The T-duality rules for fields and supersymmetry parameters were first introduces by Buscher [57, 58] and Hassan [59], however in this section we will follow an approach more similar to [60], which revisits T-duality using a flat-index notation.

Let us then split the coordinates as $x^M = (x^m, y)$, with $m = 0, \ldots, 8$. We decompose the fields as:

$$\mathrm{d}s^2_{10} = \mathrm{d}s^2_{9,A} + e^{2C}(\mathrm{d}y + A_1)^2, \quad B = B_2 + B_1 \wedge \mathrm{d}y, \quad F = F_\perp + F_\parallel \wedge E^y, \tag{1.30}$$

where $E^y = e^C(\mathrm{d}y + A_1)$. Then a T-duality along one direction results in the following identifications of fields between type IIA and IIB supergravity

$$\begin{array}{lll} \mathrm{d}s^2_{9,B} = \mathrm{d}s^2_{9,A}, & \phi^B = \phi^A - C^A, & C^B = -C^A, \\ B_2^B = B_2^A + A_1^A \wedge B_1^A, & A_1^B = -B_1^A, & B_1^B = -A_1^A, \\ F_\perp^B = e^{C^A} F_\parallel^A, & F_\parallel^B = e^{C^A} F_\perp^A & \end{array} \tag{1.31}$$

where E^y is the vielbein one-form while superscripts A, B denote in which theory the field is sitting. One can check that the supersymmetry conditions (1.24) are invariant

[4] See [56] for non-Abelian case.

if also the spinors follow the transformation rules

$$\epsilon_1^B = \epsilon_1^A\,, \qquad \epsilon_2^B = -E^y \epsilon_2^A\,, \tag{1.32}$$

where now E^y must be interpreted using the Clifford map (A.16).

1.3.2 Sl$(2,\mathbb{R})$ duality

Type IIB supergravity enjoys a very natural reformulation in terms of Sl$(2,\mathbb{R})$ objects when the metric is rescaled according to Einstein frame. However, in order to give some technical details which will be useful later on, let's start by summarizing Sl$(2,\mathbb{R})$ symmetry in the formalism of type IIB supergravity we have seen so far. Given a generic element

$$\Lambda = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \mathrm{Sl}(2,\mathbb{R}) \tag{1.33}$$

the following transformation is a symmetry of the action:

$$\tau' = \frac{\alpha\tau + \beta}{\gamma\tau + \delta}, \qquad F_5' = F_5, \qquad g' = |\gamma\tau + \delta| g\,, \qquad \begin{pmatrix} C_2' \\ B' \end{pmatrix} = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \begin{pmatrix} C_2 \\ B \end{pmatrix} \tag{1.34}$$

where $\tau = C_0 + \mathrm{i}e^{-\phi}$. Let's now define S-duality as a particular involution inside Sl$(2,\mathbb{R})$ given by $\alpha = \delta = 0$

$$\tau' = -\tau^{-1} \qquad C_2' = B\,, \qquad B' = -C_2\,, \tag{1.35}$$

since the imaginary part of τ is the string coupling $e^{-\phi}$, the transformation rule for τ implies that we may move from a strong to a weak coupling regime.

From (1.34) it is possible to derive potential transformation rules. From $F_5' = F_5$ we get

$$\begin{aligned} dC_4' &= \mathrm{d}C_4 - dB \wedge C_2 + dB' \wedge C_2' \\ &= \mathrm{d}C_4 + \beta\delta B \wedge dB + \beta\gamma(B \wedge \mathrm{d}C_2 + dB \wedge C_2) + \alpha\gamma C_2 \wedge \mathrm{d}C_2 \end{aligned} \tag{1.36}$$

and thus

$$C_4' = C_4 + \beta\gamma B \wedge C_2 + \frac{1}{2}(\alpha\gamma C_2 \wedge C_2 + \beta\delta B_2 \wedge B_2). \tag{1.37}$$

Moreover, performing a Sl$(2,\mathbb{R})$-duality on the antiself-duality relation $F_7 = -\star F_3$

$$\mathrm{d}(C_6') = -\star' \mathrm{d}C_2' + C_0' \star' H' + H' \wedge C_4', \tag{1.38}$$

we get, using the fact that under a conformal transformation $g \to \alpha^2 g$ the Hodge dual of a k-form Ω_k transforms as $\star\Omega_k \to \alpha^{D-2k} \star \Omega_k$:

$$\begin{aligned}
&\mathrm{d}(C_6') = \gamma e^{-2\phi} \star H + (C_0\gamma+\delta)\,\mathrm{d}C_6 - \gamma C_0 C_4 \wedge \mathrm{d}B + \gamma C_4 \wedge \mathrm{d}_2 + \frac{1}{2}(\beta\delta^2 B^2 \wedge \mathrm{d}B \\
&+ \beta\gamma\delta B^2 \wedge \mathrm{d}C_2 + \beta\gamma\delta B \wedge C_2 \wedge \mathrm{d}B + \beta\gamma^2 B \wedge C_2 \wedge \mathrm{d}C_2 + \alpha\gamma\delta C_2^2 \wedge \mathrm{d}B + \alpha\gamma^2 C_2^2 \wedge \mathrm{d}C_2) \\
&= \gamma\,\mathrm{d}\widetilde{B} + \delta\,\mathrm{d}_6 + \frac{1}{2}\big(\gamma(C_0\,\mathrm{d}_6 + \mathrm{d}C_0 \wedge C_6 + C_4 \wedge \mathrm{d}C_2 + \mathrm{d}_4 \wedge C_2) + \beta\delta^2 B^2 \wedge \mathrm{d}B \\
&+ \beta\gamma\delta B^2 \wedge \mathrm{d}C_2 + \beta\gamma\delta B \wedge C_2 \wedge \mathrm{d}B + \beta\gamma^2 B \wedge C_2 \wedge \mathrm{d}C_2 + \beta\gamma^2 C_2^2 \wedge \mathrm{d}B + \alpha\gamma^2 C_2^2 \wedge \mathrm{d}C_2\big),
\end{aligned} \tag{1.39}$$

where the definition of $\widetilde{B}$ (1.23) was used. From the last line one can check that the correct transformation rule for C_6 is:

$$C_6' = \gamma\widetilde{B} + \delta C_6 + \frac{\gamma}{2}\big(C_0 C_6 + C_4 \wedge C_2 + \beta B \wedge C_2 \wedge (\delta B + \gamma C_2)\big) + \frac{1}{3}\big(\beta\delta^2 B^3 + \alpha\gamma^2 C_2^3)\big). \tag{1.40}$$

Another important ingredient we need is the transformation rule of spinors. Using the fact that we do not want (1.24) to vary, it is possible to check that the spinors transform under an U(1) subgroup of the original $\mathrm{Sl}(2,\mathbb{R})$ symmetry

$$\begin{pmatrix} \epsilon_1' \\ \epsilon_2' \end{pmatrix} = |\gamma\tau + \delta|^{\frac{1}{4}} \begin{pmatrix} \cos(\theta/2) & -\sin(\theta/2) \\ \sin(\theta/2) & \cos(\theta/2) \end{pmatrix} \begin{pmatrix} \epsilon_1 \\ \epsilon_2 \end{pmatrix} \tag{1.41}$$

where $\theta = \arg(\gamma\tau + \delta)$.

1.3.3 IIA/M-theory duality

If in M-theory we have a compact U(1) direction which is a symmetry of the whole solution, it is possible to perform a dimensional reduction along it and land to type IIA supergravity. Unfortunately it is not possible to turn on the Romans mass with this procedure, so that type IIA supergravity is not just a particular case of M-theory. On the other hand, when $F_0 = 0$, a solution in type IIA can be lifted to one in M-theory with an internal isometry, so again M-theory is broader than massless type IIA.

In this section we will use a hat to distinguish eleven-dimensional quantities from the ten-dimensional ones. To perform the dimensional reduction we will follow [61] and [3, Chap. 8], whose conventions are consistent with ours except that in IIA we have to map $\epsilon_{1,2} \to \epsilon_{2,1}$, $C_1 \to -C_1$ and $H \to -H$ (see appendix A.2 for more details). In particular, the metric and the supersymmetry parameter split as follows:

$$\begin{aligned}
&ds_{11} = e^{-\frac{2}{3}\phi}\,\mathrm{d}s_{10} + e^{\frac{4}{3}\phi}(\mathrm{d}x^{10} - C_1)^2 \\
&\epsilon = e^{-\frac{1}{6}\phi}\frac{\epsilon_1 + \epsilon_2}{\sqrt{2}}, \qquad \gamma_{10}\epsilon_1 = -\epsilon_1 \,.
\end{aligned} \tag{1.42}$$

while for $\widehat{A}$ and the associated field-strength we have:

$$\begin{aligned}\widehat{A} &= C_3 - B\wedge \mathrm{d}x^{10}\,,\\ \widehat{F} &= F_4 - H\wedge(\mathrm{d}x^{10} - C_1)\,.\end{aligned} \tag{1.43}$$

We can also express the M5 potential $\widehat{C}$ in terms of ten-dimensional quantities, starting from (1.4) and substituting

$$\begin{aligned}*_{11}\widehat{F} &= *_{11}(-e^{\frac{1}{3}\phi}\widehat{H}\wedge E^{10} + e^{\frac{4}{3}\phi}\widehat{F}_4) = -e^{-2\phi} *_{10} H - F_6\wedge C_1 + F_6\wedge \mathrm{d}x^{10}\\ \widehat{A}\wedge\widehat{F} &= C_3\wedge H\wedge C_1 + C_3\wedge F_4 - (B\wedge dC_3 + C_3\wedge H)\wedge \mathrm{d}x^{10}\,,\end{aligned} \tag{1.44}$$

we get, using (1.23) and (1.11)

$$\mathrm{d}\widehat{C} = \mathrm{d}\left(-\widetilde{B} - \frac{1}{2}C_5\wedge C_1 + C_5\wedge \mathrm{d}x^{10} - \frac{1}{2}B\wedge C_3\wedge \mathrm{d}x^{10}\right)\,, \tag{1.45}$$

so that we can take

$$\widehat{C} = -\widetilde{B} - \frac{1}{2}C_5\wedge C_1 + C_5\wedge \mathrm{d}x^{10} - \frac{1}{2}B\wedge C_3\wedge \mathrm{d}x^{10}\,. \tag{1.46}$$

1.3.4 Dualities and branes

As we have just seen, after a duality, fields rearrange themselves in a completely different manner, so it should not be surprising that also branes must be mapped from one to another. Understanding how this happens is of fundamental importance, since in many cases dualities provide the only shortcut to get a detailed description of many objects. So let's analyze these relations starting from T-duality.

The best known and understood mapping between branes under T-duality is the one of D-branes. As shown in (1.31), the orthogonal component of the RR-flux becomes, in the dual theory, a parallel component and viceversa, which means that under T-duality the orthogonal components increase their form-rank while parallel ones lessen it. This property reflects brane behavior, a Dp-brane becomes a D$(p-1)$-brane if the T-duality is performed longitudinally to the worldvolume and a D$(p+1)$-brane if it is orthogonal. This rule has a deeper explanation in the fact that T-duality exchanges Dirichlet with Neumann boundary conditions of the open string. The T-dual of a NS5-brane works in a slightly more delicate way: first of all since the NS5 electric potential is, roughly speaking, linked to the Hodge dual of B via (1.23), we must interpret the invariance of the orthogonal component of the B-field as the fact that the longitudinal part of a NS5-brane is mapped again into the longitudinal part of a NS5-brane of the dual theory. On the other hand, (1.31) exchanges the B-field longitudinal component with the metric fibration associated with the U(1) isometry. This means that the orthogonal part of a NS5-brane is T-dualized to an object which is a gravitational solitons [62], called Kaluza-Klein (KK) monopole. In type II theories such a solution is obtained as $\mathbb{R}^6\times$ a four-dimensional Gibbons–Hawking space [63], which implies that the KK monopole is a five-dimensional object (KK5). However, this issue is complicated by the fact that for the case of a single monopole the solution is actually completely smooth, so it is not clear on which submanifold a world-volume action should be based on. In spite of this difficulties, the existence of the KK-monopole in both type II supergravity and M-theory is guaranteed by the existence of corresponding central extension in the supersymmetry algebra for this object [64], and an attempt to write an effective action for it was made using dualities [52, 65].

Let's now move on and discuss $\mathrm{Sl}(2,\mathbb{R})$-duality. One of the most peculiar characteristics of type IIB supergravity is that it has a fundamental one-dimensional object, the string, and a solitonic one dimensional object, the D1-brane, which is not fundamental at least in a perturbative limit. One may wonder if this changes at the non-perturbative level. S-duality confirms that the answer to this question is yes and that the D1 is exactly as fundamental as the string, since (1.35) exchanges D1-brane and string potentials. Therefore a generic $\mathrm{Sl}(2,\mathbb{R})$ transforms a D1-brane and the string in a doublet, so it is customary to mix in general these two objects talking about (p,q) 1-branes. The same holds true for their magnetic dual, i.e. D5- and NS5-branes, or, more in general (p,q) 5-branes. On the other hand, since the F_5 field strength is always invariant, the D3-brane transforms into itself. Also D7-branes can be viewed as transforming in a doublet of (p,q) branes, even if in [66] it argued that there are actually three different eight-form potentials which transform as a $\mathrm{Sl}(2,\mathbb{R})$ triplet but with a constraint, such that they describe the same propagating degrees of freedom as the dilaton and C_0. D9-branes are even more peculiar objects, indeed it is possible to extend the supersymmetry multiplets adding bosonic spacetime fields which do not propagate any degrees of freedom. These "fake" degrees of freedom are perfect to describe ten-forms potential which couples to a space-filling branes, and it was found in [67] that there can be four types of ten-form potentials with two constraints between them. However, since these never explicitly appear in our discussion, we will simply need to distinguish the D9-brane and what we call the NS9-brane, i.e., the D9-brane after a S-duality transformation. Despite the fact that it is an exotic object, the NS9-brane is actually fundamental to justify the construction of type I string theory starting from IIB. In this scenario usually 32 D9-branes are necessary to compensate the O9-charge which arises from the worldsheet parity transformations, for this reason the fundamental string which end on the D9 carries SO(32) Chan–Paton factors. If we now try to describe everything in an S-dual setting, the D1-brane is the fundamental object, which now cannot end on a D9 but must end on its S-dual, so the presence of NS9 is necessary to make this construction consistent. NS9 exists also in type IIA, as we will show by T-dualizing the NS9 calibration form type IIB.

Let's now quickly move on the duality between M-theory and type IIA. As one can see from (1.43), the M-theory three-form potential contains both C_3 and B_2; this means that when we perform a dimensional reduction of a M2-brane along a longitudinal direction we get the fundamental string in type IIA, while if we perform the same operation but in an orthogonal direction we get the D2-brane. Similarly an M5-brane reduces into a NS5- and D4-branes if the reduction is performed orthogonally or longitudinally respectively. D6-branes are more peculiar since C_1 is pure geometry in M-theory (1.42), however we just introduced a purely geometric solitons, the KK-monopole, which can play the role of dual of a D6-brane in eleven dimension. In particular M-theory has six-dimensional KK6-monopole, which becomes a KK5-monopole or a D6-brane depending on the direction we are reducing on. We have seen that eleven-dimensional supergravity contains a M9-brane, which is not charged under any potential, in this case such a brane should reduce to the D8- and the NS9-brane in type IIA, which are both not charged under any potential.

CHAPTER 2

(BI-)SPINORIAL GEOMETRY

The presence of fields defined all over the D-dimensional spacetime manifold M has important implication from a topological and geometrical perspective. In such a situation the usual tangent frame bundle $\mathrm{Gl}(D)$ gets reduced to the stabilizer of the object we are considering, leading to a reduction of the structure group which is also called G-structure, where $\mathrm{G}\subseteq\mathrm{Gl}(D)$. One of the most classical examples is the presence of a metric tensor fields g, which leads the structure group from $\mathrm{Gl}(D)$ to $\mathrm{O}(D)$ and implies that the spacetime M must be paracompact. Similarly, if we have also a non-vanishing well-defined vector field $v \in TM$ we have to consider the common stabilizer of both g and v, which is $\mathrm{Stab}(g, v) = \mathrm{O}(D-1)$. In such a situation we also get that the Euler characteristic vanishes. It is natural to expect that something similar must occur also if we have spinors defined all over the manifold. This is exactly what happens when we have some amount of supersymmetry, indeed the spinorial parameters are well defined all over the manifold and therefore they lead to a structure group reduction. If the amount of supersymmetry is high enough it is even possible that the G-structure reduction completely constrains the supersymmetry conditions, see for example [68, 69, 70].

In this section we will see how to rephrase the geometrical information carried by spinors in terms of more familiar objects, like differential forms. We will then use them to recast the supersymmetry conditions (1.6),(1.24) in a new and more convenient way. In the first two sections we will mostly follow [9] Sec. 2.

2.1 Geometry of a ten-dimensional spinor

Let's start by analyzing a ten-dimensional spinor ϵ in its irreducible representation, which in ten dimensions is sixteen-dimensional Majorana–Weyl. In this situation one can choose the gamma matrices to be all real and then they satisfy

$$\gamma_M^t = \gamma_{\underline{0}}\gamma_M\gamma_{\underline{0}} \tag{2.1}$$

where we underlined the 0 to indicate that such an index must be interpreted as flat. More details on our gamma-matrices conventions can be found in appendix A.2.

In order to extract the geometrical content of ϵ more transparently, it is convenient to use its associated bispinor $\epsilon \otimes \overline{\epsilon} = \epsilon \otimes \epsilon^t \gamma_{\underline{0}}$. Using Fierz identity (A.15) it is possible to expand this bispinor on the antisymmetric products of k gamma matrices $\gamma^{M_1 \ldots M_k}$:

$$\epsilon \otimes \overline{\epsilon} = \sum_{k=0}^{10} \frac{1}{32\, k!} (\overline{\epsilon} \gamma_{M_k \ldots M_1} \epsilon)\, \gamma^{M_1 \ldots M_k} . \tag{2.2}$$

This bispinor can in turn be understood as a sum of forms of different degree using the Clifford map (A.16). If ϵ is chiral, only forms of even degree survive and moreover, using (A.19), we get that it must also obey a self-duality relation

$$\gamma(\epsilon \otimes \overline{\epsilon}) = \pm * \lambda(\epsilon \otimes \overline{\epsilon}) \tag{2.3}$$

where the chirality of the spinor is $\gamma \epsilon = \pm \epsilon$. Being ϵ also Majorana

$$\overline{\epsilon} \gamma_{M_k \ldots M_1} \epsilon = (\overline{\epsilon} \gamma_{M_k \ldots M_1} \epsilon)^t = -(-)^k (-)^{k(k-1)/2}\, \overline{\epsilon} \gamma_{M_k \ldots M_1} \epsilon , \tag{2.4}$$

which sets to zero all the degrees except for $k = 1, 5, 9$. Summing up all these information we get that the independent forms are

$$K_M = \frac{1}{32} \overline{\epsilon} \gamma_M \epsilon , \qquad \Omega_{M_1 \ldots M_5} = \frac{1}{32} \overline{\epsilon} \gamma_{M_1 \ldots M_5} \epsilon \tag{2.5}$$

and (2.2) reads:

$$\epsilon \otimes \overline{\epsilon} = K + \Omega \pm *K , \qquad *\Omega_5 = \pm \Omega_5 , \tag{2.6}$$

where again $\pm$ is the chirality of ϵ .

These forms enjoy some important algebraic properties . First of all using (A.24) we have that

$$K \epsilon = K_M \gamma^M \epsilon = \frac{1}{32} \gamma^M \epsilon \overline{\epsilon} \gamma_M \epsilon = -\frac{1}{4} (1 \pm \gamma) K \epsilon = -\frac{1}{2} K \epsilon \tag{2.7}$$

from which

$$K \epsilon = 0 . \tag{2.8}$$

This immediately implies that K is a null vector:

$$K^M K_M = -\frac{1}{2 \cdot 32} \overline{\epsilon} K \epsilon = 0 , \tag{2.9}$$

and moreover, applying K on the left and right of $\epsilon \otimes \overline{\epsilon}$ and using (A.17) one gets

$$K \wedge \Omega_5 = \iota_K \Omega_5 = 0 , \tag{2.10}$$

which allows us to rewrite the five-form as

$$\Omega_5 = K \wedge \Psi_4 \tag{2.11}$$

for some four-form Ψ_4.

As sketched in the introduction to this chapter, the presence of a spinor or of a form defined on all the space-time leads to a reduction of the structure group of the tangent bundle to their stabilizer. Let's determine the structure group defined by ϵ of positive chirality. For convenience, we choose a frame in which K is part of the vielbein $K = e_-$:

$$e_+ \cdot e_- = \frac{1}{2}\,, \quad e_\pm \cdot e_\pm = 0\,, \quad e_\pm \cdot e_\alpha = 0\,, \quad e_\alpha \cdot e_\alpha = 1\,, \tag{2.12}$$

with $\alpha = 1, \ldots, 8$. This index choice suggests a decomposition of the ten-dimensional Clifford algebra $\mathrm{Cl}(1,9) \simeq \mathrm{Cl}(1,1) \otimes \mathrm{Cl}(0,8)$ and therefore a spinor decomposition in

$$\epsilon = |\uparrow\rangle \otimes \eta\,, \tag{2.13}$$

where η is a eight-dimensional Majorana–Weyl spinor while $|\uparrow\rangle$ is the two-dimensional Majorana–Weyl component. In order to compute stab (ϵ) it is necessary to look at the infinitesimal action of a Lorentz transformation on ϵ:

$$\delta\epsilon = \omega_{MN}\gamma^{MN}\epsilon\,. \tag{2.14}$$

We have just seen that $K\epsilon = \gamma_-\epsilon = \gamma^+\epsilon = 0$ so that $\gamma^{+\alpha}\epsilon = 0$. Moreover an eight-dimensional Majorana–Weyl spinor η is annihilated by 21 out of 28 eight-dimensional gamma matrices, so we can write:

$$\mathrm{stab}(\epsilon) = \{\omega^{21}_{\alpha\beta}\gamma^{\alpha\beta}, \gamma^{+\alpha}\}\,. \tag{2.15}$$

The elements $\omega^{21}_{\alpha\beta}\gamma^{\alpha\beta}$ are in the adjoint representation of spin(7), so that they generate the Lie algebra Spin(7). Moreover because $[\gamma_{\alpha\beta}, \gamma^{+\delta}] = 2\delta^{\delta}_{[\alpha}\gamma^{+}_{\beta]}$ we have that

$$\mathrm{Stab}(\epsilon) = \mathrm{Spin}(7) \ltimes \mathbb{R}^8 = \mathrm{ISpin}(7)\,. \tag{2.16}$$

We expect that the same structure group can be deduced also using the forms generated by ϵ. Let's start from the stabilizer of K; since K is null

$$\mathrm{Stab}(K) = \mathrm{ISO}(8) = \mathrm{SO}(8) \ltimes \mathbb{R}^8\,. \tag{2.17}$$

Equation (2.10) says us that the four form Ψ_4 contains only components which are orthogonal to K different from K itself, i.e., using the decomposition (2.12), it has legs only along α directions. If we restrict our original spinor ϵ to this eight-dimensional subspace we obtain the Majorana–Weyl spinor η in eight dimensions, this is known to give rise to a Spin(7) structure. In fact Ψ_4 is nothing but Spin(7) four-form, which in a eight-dimensional space is usually obtained from $\eta \otimes \eta^t = (\eta^t\eta)(1 + \Psi_4 + \mathrm{Vol}_8)$. Then leaving Ψ_4 invariant reduces the SO(8) inside ISO(8) in Spin(7), so that we find again

$$\mathrm{Stab}(K, \Omega_5) = \mathrm{ISpin}(7)\,. \tag{2.18}$$

2.2 Geometry of two ten-dimensional spinors

As we have seen in 1.2 type II supergravity contains two fermionic parameters $\epsilon_{1,2}$; each of them defines an ISpin(7) structure. From (2.6) we get

$$\begin{aligned} \epsilon_1 \otimes \overline{\epsilon}_1 &\equiv K_1 + \Omega_1 + *K_1 \\ \epsilon_2 \otimes \overline{\epsilon}_2 &\equiv K_2 + \Omega_2 \mp *K_2 \qquad \text{for} \quad {}^{\text{IIA}}_{\text{IIB}}, \end{aligned} \tag{2.19}$$

but this time we can also define the mixed bispinor

$$\Phi = \epsilon_1 \otimes \overline{\epsilon}_2 \,, \tag{2.20}$$

which is a collection of forms with odd degree in IIB and even degree in IIA with the self-duality property $*\lambda(\Phi) = \Phi$. From (2.8), we see that

$$K_1 \Phi = \Phi K_2 = 0 \,. \tag{2.21}$$

If we define

$$K \equiv \frac{1}{2}(K_1 + K_2)^M \partial_M \,, \qquad \widetilde{K} \equiv \frac{1}{2}(K_1 - K_2)_M dx^M \,, \tag{2.22}$$

we can rewrite (2.23) using (A.17):

$$(\iota_K + \widetilde{K}\wedge)\Phi = 0 \,. \tag{2.23}$$

In the same spirit we define

$$\Omega \equiv \frac{1}{2}(\Omega_1 \pm \Omega_2) \,, \qquad \widetilde{\Omega} \equiv \frac{1}{2}(\Omega_1 \mp \Omega_2) \qquad \text{for} \quad {}^{\text{IIA}}_{\text{IIB}} . \tag{2.24}$$

Notice that $*\Omega = \widetilde{\Omega}$ in IIA while $*\Omega = \Omega, *\widetilde{\Omega} = \widetilde{\Omega}$ in IIB.

The vector K will play a key role in our discussion and in particular it can be seen that

$$K^2 \leq 0 \,. \tag{2.25}$$

The case $K^2 = 0$ is called the light-like case and implies $K_1 \propto K_2$, while the case where $K^2 = \frac{1}{2} K_1 \cdot K_2 < 0$ is called the timelike case. As we will immediately see, this distinction discerns the different cases in the classification of type II supergravity structure groups.

2.2.1 Structure group in TM

To evaluate the stabilizer of $\epsilon_{1,2}$ in $SO(1,9)$ we have then to look at the intersection of the two copies of ISpin(7). However, this intersection is not unique and various possibilities exist. Let's start from IIA. If we are in the light-like case we can use the vielbein basis (2.12) for both $K_{1,2}$ and then we are just considering two eight-dimensional spinors $\eta_{1,2}$ of opposite chirality. We have seen that the presence of a spinor reduces the structure group from $SO(1,9)$ to ISpin(7), the presence of another spinor with opposite chirality allows us to build a three-form $\eta_1^t \gamma_{\alpha\beta\delta} \eta_2$ on the eight-dimensional subspace orthogonal to $K_1 \propto K_2$. The subgroup of Spin(7) that preserves this form is G_2, so overall we have a $\mathrm{G}_2 \ltimes \mathbb{R}^8$ structure. If, on the other hand, we are in the timelike case, up to rescaling we can assume without loss of generality

$$K_1 = e^+ \quad \text{and} \quad K_2 = e^- \tag{2.26}$$

so that we can decompose the spinors as

$$\epsilon_1 = |\uparrow\rangle \otimes \eta_1 \,, \qquad \epsilon_2 = |\downarrow\rangle \otimes \eta_2 \,. \tag{2.27}$$

In this situation we are reduced to the common stabilizer of two eight-dimensional spinors with the same chirality, which is $\mathrm{Spin}(6) \cong \mathrm{SU}(4)$, but it can get enhanced to Spin(7) if η_1 and η_2 are proportional. Summarizing, we have found three possibilities:

$$\mathrm{G}_2 \ltimes \mathbb{R}^8 \,, \quad \mathrm{SU}(4) \,, \quad \mathrm{Spin}(7) \qquad \text{for IIA} \,. \tag{2.28}$$

In IIB conversely ϵ_1 and ϵ_2 have the same chirality. If we are in the null case again we write

$$\epsilon_1 = |\uparrow\rangle \otimes \eta_1 \,, \qquad \epsilon_2 = |\uparrow\rangle \otimes \eta_2 \,. \tag{2.29}$$

As seen before, the intersection of the stabilizers of two eight-dimensional spinors is $\mathrm{Spin}(6) \cong \mathrm{SU}(4)$ but could became Spin(7) if they are proportional. So we conclude that overall we have $\mathrm{SU}(4) \ltimes \mathbb{R}^8$ that can be enhanced to $\mathrm{Spin}(7) \ltimes \mathbb{R}^8$. When K_1 and K_2 are not proportional again we can assume the timelike condition (2.26) so that

$$\epsilon_1 = |\uparrow\rangle \otimes \eta_1 \,, \qquad \epsilon_2 = |\downarrow\rangle \otimes \eta_2 \tag{2.30}$$

where now η_1 and η_2 have opposite chiralities. As discussed above, the common stabilizer of two eight-dimensional spinors with opposite chiralities is G_2. Therefore for type IIB we got again three possibilities:

$$\mathrm{SU}(4) \ltimes \mathbb{R}^8 \,, \quad \mathrm{Spin}(7) \ltimes \mathbb{R}^8 \,, \quad \mathrm{G}_2 \qquad \text{for IIB} \,. \tag{2.31}$$

This multiplicity reflects directly on Φ, indeed if $K_1 \propto K_2$ from (2.23) we get that $\widetilde{K} \wedge \Phi = 0$ and then $\Phi = \widetilde{K} \wedge (\dots)$. Otherwise, in the time-like case $\Phi = \exp\big[-\frac{1}{K_1 \cdot K_2} K_1 \wedge K_2\big] \wedge (\dots)$. The remaining parts $(\dots)$ come from the eight-dimensional bilinear $\phi = \eta_1 \eta_2^t$ which is different for each structure group:

$$\begin{aligned}
&\Phi_{G_2 \ltimes \mathbb{R}^8} = \widetilde{K} \wedge \phi_{G_2} \,, \\
&\Phi_{SU(4)} = \exp\left[-\frac{1}{K_1 \cdot K_2} K_1 \wedge K_2\right] \wedge \phi_{SU(4)} \,, \qquad \text{for IIA} \\
&\Phi_{Spin(7)} = \exp\left[-\frac{1}{K_1 \cdot K_2} K_1 \wedge K_2\right] \wedge \phi_{Spin(7)} \,,
\end{aligned} \tag{2.32}$$

and

$$\begin{aligned} &\Phi_{SU(4)\ltimes\mathbb{R}^8} = \widetilde{K}\wedge\phi_{SU(4)}\,, \\ &\Phi_{Spin(7)\ltimes\mathbb{R}^8} = \widetilde{K}\wedge\phi_{Spin(7)}\,, \qquad\qquad \text{for IIB} \\ &\Phi_{G_2} = \exp\left[-\frac{1}{K_1\cdot K_2}K_1\wedge K_2\right]\wedge\phi_{G_2}\,. \end{aligned} \tag{2.33}$$

2.2.2 Structure group in $TM + T^*M$

The presence of more than one stabilizer for $\epsilon_{1,2}$ can be considered as an obstruction to achieve an unified description of string-theory BPS solutions, indeed nothing forbids the structure group to change from a point to another even inside the same solution, making difficult to understand which of (2.32) and (2.33) must be used a priori. One can react to this by enlarging the structure group defining it on the generalized tangent bundle $TM\oplus T^*M$ [35, 36].

Type II supergravity enjoys a deep connection with generalized complex geometry. $TM\oplus T^*M$ can be naturally endowed with a metric with half positive and half negative signature, so that the structure group is enlarge to $\mathrm{O}(10,10)$. As we have seen in section 1.3.1, this is the T-duality group in presence of ten-isometric directions. This is believed to be a reason why it is actually possible to reformulate type II supergravity in terms of manifestly $\mathrm{O}(10,10)$ covariant objects as done in [71].

The generalized complex geometry framework the metric and the B-field degrees of freedom are all encoded in an unique object $\mathcal{G}$ called generalized metric [35, section 6.4] which performs a reduction of the structure group from $\mathrm{O}(10,10)$ to $\mathrm{O}(9,1)\times\mathrm{O}(9,1)$. Another benefit of the generalized geometry approach is that the generalized spin bundle is nothing but the bundle of all the differential forms on M as it is shown in appendix A.1, and we can regard $\mathrm{Cl}(10,10)$ Clifford algebra as acting directly on differential forms via the usual gamma matrices product on the left and on the right as in (A.17). In this framework it results particularly easy to compute the stabilizer of differential forms if they derive from spinor bilinears. As we will see in a moment, the presence of a metric and a B field on M restricts the structure group to $\mathrm{o}(9,1)\oplus\mathrm{o}(9,1) = \mathrm{span}\{\overleftarrow{\gamma}^{MN}, \overrightarrow{\gamma}^{MN}\}$. If moreover we add as geometric data also the two spinors ϵ_1 and ϵ_2 we have a basis of the type (2.12) associated to both, so a subscript is needed to distinguish indices relative to ϵ_1 from the to ϵ_2 ones. The common stabilizer therefore reads:

$$\mathrm{stab}(g,B,\epsilon_1,\bar\epsilon_2) = \mathrm{span}\left\{\omega_{21}^{\alpha_1\beta_1}\overrightarrow{\gamma}_{\alpha_1\beta_1}, \overrightarrow{\gamma}_{-1\alpha_1}, \omega_{21}^{\alpha_2\beta_2}\overleftarrow{\gamma}_{\alpha_2\beta_2}, \overleftarrow{\gamma}_{-2\alpha_2}\right\} = \mathrm{ispin}(7)\oplus\mathrm{ispin}(7)\,. \tag{2.34}$$

Notice that we manage to collect all the possibilities in (2.28) and (2.31) in a single generalized G-structure.

Beside these advantages, by starting from the structure group $\mathrm{O}(10,10)$ instead of $\mathrm{O}(10)$ we have now lost any geometric information about how the metric is defined and for this reason we have to check that geometric data encoded in the bilinears are enough to include $(g,B,\epsilon_1,\bar{\epsilon}_2)$. The bilinear stabilizers read:

$$\begin{aligned}
\mathrm{stab}(\Phi) &= \mathrm{span}\left\{\begin{matrix} \omega_{21}^{\alpha_1\beta_1}\overrightarrow{\gamma}_{\alpha_1\beta_1}, \overrightarrow{\gamma}_{-_1\alpha_1}, \omega_{21}^{\alpha_2\beta_2}\overleftarrow{\gamma}_{\alpha_2\beta_2}, \overleftarrow{\gamma}_{-_2\alpha_2}, \overrightarrow{\gamma}_{-_1+_1}+\overleftarrow{\gamma}_{-_2+_2} \\ \overrightarrow{\gamma}_{-_1}\overleftarrow{\gamma}_{\alpha_2}, \overrightarrow{\gamma}_{-_1}\overleftarrow{\gamma}_{+_2}, \overrightarrow{\gamma}_{\alpha_1}\overleftarrow{\gamma}_{-_2}, \overrightarrow{\gamma}_{+_1}\overleftarrow{\gamma}_{-_2}, \overrightarrow{\gamma}_{-_1}\overleftarrow{\gamma}_{-_2} \end{matrix}\right\}, \\
\mathrm{stab}(\epsilon_i\bar{\epsilon}_i) &= \mathrm{span}\left\{\begin{matrix} \omega_{21}^{\alpha_i\beta_i}\overrightarrow{\gamma}_{\alpha_i\beta_i}, \overrightarrow{\gamma}_{-_i\alpha_i}, \omega_{21}^{\alpha_i\beta_i}\overleftarrow{\gamma}_{\alpha_i\beta_i}, \overleftarrow{\gamma}_{-_i\alpha_i}, \overrightarrow{\gamma}_{-_i+_i}+\overleftarrow{\gamma}_{-_i+_i} \\ \overrightarrow{\gamma}_{-_i}\overleftarrow{\gamma}_{\alpha_i}, \overrightarrow{\gamma}_{-_i}\overleftarrow{\gamma}_{+_i}, \overrightarrow{\gamma}_{\alpha_i}\overleftarrow{\gamma}_{-_i}, \overrightarrow{\gamma}_{+_i}\overleftarrow{\gamma}_{-_i}, \overrightarrow{\gamma}_{-_i}\overleftarrow{\gamma}_{-_i} \end{matrix}\right\}.
\end{aligned} \tag{2.35}$$

In the timelike case we already saw that it is allowed to choose $e_{+_1} \sim K_2$ and $e_{+_2} \sim K_1$ and therefore the common stabilizer reduces to

$$\mathrm{stab}_{K^2<0}(\Phi,\Omega_{1,2},K_{1,2}) \subseteq \mathrm{span}\left\{\omega_{21}^{\alpha_1\beta_1}\overrightarrow{\gamma}_{\alpha_1\beta_1}, \omega_{21}^{\alpha_2\beta_2}\overleftarrow{\gamma}_{\alpha_2\beta_2}\right\} = \mathrm{spin}(7)\oplus\mathrm{spin}(7) \tag{2.36}$$

which, being stricter then ispin(7) ⊕ ispin(7), it is enough to define $(g,B,\epsilon_1,\bar{\epsilon}_2)$.

However, as showed in [9], for the generic case this is not true anymore and we have to supplement the bilinears with some objects which cannot be defined through them; these are two sections of the generalized tangent bundle $(\overrightarrow{\gamma_{+_1}},\overleftarrow{\gamma_{+_2}})$, and it is easy to show that also in this case

$$\mathrm{stab}(\Phi,\overrightarrow{\gamma_{+_1}},\overleftarrow{\gamma_{+_2}}) = \mathrm{span}\left\{\omega_{21}^{\alpha_1\beta_1}\overrightarrow{\gamma}_{\alpha_1\beta_1}, \omega_{21}^{\alpha_2\beta_2}\overleftarrow{\gamma}_{\alpha_2\beta_2}\right\} = \mathrm{spin}(7)\oplus\mathrm{spin}(7). \tag{2.37}$$

2.3 Geometry of an eleven-dimensional spinor

Taking inspiration from the ten-dimensional case, let's now quickly review the geometrical aspects of a Majorana spinor in eleven dimensions. Following again the conventions of [10], we can use the supersymmetry generator ϵ to construct a vector

$$K = \frac{1}{2^5}\bar{\epsilon}\gamma^M\epsilon\,\partial_M \tag{2.38}$$

and a two- and five-form

$$\Omega = \frac{1}{2^5\cdot 2!}\bar{\epsilon}\,\gamma_{M_1M_2}\,\epsilon\,\mathrm{d}x^{M_1}\wedge\mathrm{d}x^{M_2}, \tag{2.39a}$$

$$\Sigma = \frac{1}{2^5\cdot 5!}\bar{\epsilon}\,\gamma_{M_1\ldots M_5}\,\epsilon\,\mathrm{d}x^{M_1}\wedge\cdots\wedge\mathrm{d}x^{M_5}. \tag{2.39b}$$

similarly to what we have seen in type II theories we always have $K \le 0$, so we will again distinguish the null from the timelike case. In the null case the structure group turns out to be [72] $(\mathrm{Spin}(7)\ltimes\mathbb{R}^8)\times\mathbb{R}$ while in the timelike case is SU(5). However, to carry on the analogy with type II theories will be enough to consider the timelike case. As we have seen in ten dimensions, all these bilinears are not independent but enjoy some algebraic relations

$$\iota_K\Omega = 0 \quad \iota_K\Sigma = \frac{1}{2}\Omega\wedge\Omega\,, \quad K^2\Omega\wedge\Sigma = \frac{1}{2}K\wedge\Omega\wedge\Omega\wedge\Omega\,, \quad \iota_K * \Sigma = -5\Omega^M\wedge\Sigma_M\,. \tag{2.40}$$

If we now define

$$\frac{1}{\sqrt{K^2}}\chi = \Sigma - \frac{1}{2K^2}K\wedge\Omega\wedge\Omega \tag{2.41}$$

we are able to write an holomorphic 5-form $\theta = \chi - \mathrm{i} *_{10}\chi$ on the ten-dimensional subspace orthogonal to K, so we have that θ and Ω defines an SU(5) structure on $K^\perp$. More generally, a real five-form χ, a real vector K and a real two-form Ω defines an SU(5) structure if they satisfy the following algebraic constraint [10, appendix E]:

$$\iota_K J = \iota_K\chi = 0\,, \qquad J\wedge\chi = J\wedge\iota_K *_{10}\chi = 0\,, \qquad \chi\wedge\iota_K *_{10}\chi = -\frac{2^4}{5!}\Omega^5\,, \tag{2.42}$$

which turns out to be true in our case thanks to (2.40). More details about this can be found in [10].

2.4 Bilinear dualities

Using the spinorial transformation rules we have showed in section 1.3 we can derive how bispinors transforms when we perform a string duality. This section is a follow up of 1.3 and therefore we will use the same notation.

2.4.1 T-duality

Assuming again the presence of an isometric direction y, we can decompose also bilinears in flat-index notation

$$\begin{aligned} &\Phi = \Phi_\perp + \Phi_\parallel\wedge E^y\,, \quad K = k_1 + k_0 E^y\,, \qquad \widetilde{K} = \widetilde{k}_1 + \widetilde{k}_0 E^y\,,\\ &\Omega = \omega_5 + \omega_4\wedge E^y\,, \quad \widetilde{\Omega} = \widetilde{\omega}_5 + \widetilde{\omega}_4\wedge E^y\,. \end{aligned} \tag{2.43}$$

Using (1.32) one gets

$$\Phi^B = \Phi^A E^y\,, \qquad (\epsilon_1\bar{\epsilon}_1)^B = (\epsilon_1\bar{\epsilon}_1)^A\,, \qquad (\epsilon_2\bar{\epsilon}_2)^B = (\epsilon_2\bar{\epsilon}_2)^A - 2e^{-2C^A}\iota^A_y(\epsilon_2\bar{\epsilon}_2)^A\wedge E^y \tag{2.44}$$

which implies the following transformation rules

$$\begin{array}{llll} \Phi^B_\perp = \Phi^A_\parallel\,, & & \Phi^B_\parallel = \Phi^A_\perp\,, & \\ k^B_1 = k^A_1\,, & k^B_0 = k^A_0\,, & \widetilde{k}^B_1 = \widetilde{k}^A_1\,, & \widetilde{k}^B_0 = \widetilde{k}^A_0\,, \\ \omega^B_5 = \widetilde{\omega}^A_5\,, & \omega^B_4 = \omega^A_4\,, & \widetilde{\omega}^B_5 = \omega^A_5\,, & \widetilde{\omega}^B_4 = \widetilde{\omega}^A_4\,. \end{array} \tag{2.45}$$

2.4.2 $\mathrm{Sl}(2,\mathbb{R})$ duality

From equation (1.41) we have seen that spinors don't transform under the whole $\mathrm{Sl}(2,\mathbb{R})$ group but under an U(1) subgroup. The same behavior is inherited by the spinor bilinears, which can be distinguish in two groups: the one that transforms as a singlet, as $K,\Phi_3,\widetilde{\Omega}$

$$K' = |\gamma\tau+\delta|K\,,\qquad \Phi_3' = |\gamma\tau+\delta|^2\Phi_3\,,\qquad \widetilde{\Omega}' = |\gamma\tau+\delta|^3\widetilde{\Omega}\,. \tag{2.46}$$

and the ones that transform as a doublet

$$\begin{aligned}(\widetilde{K}+\mathrm{i}\Phi_1)' &= |\gamma\tau+\delta|e^{\mathrm{i}\theta}(\widetilde{K}+\mathrm{i}\Phi_1) = (\gamma\tau+\delta)(\widetilde{K}+\mathrm{i}\Phi_1)\,,\\ (\Omega+\mathrm{i}\Phi_5)' &= |\gamma\tau+\delta|^3e^{\mathrm{i}\theta}(\Omega+\mathrm{i}\Phi_5) = |\gamma\tau+\delta|^2(\gamma\tau+\delta)(\Omega+\mathrm{i}\Phi_5)\,.\end{aligned} \tag{2.47}$$

2.4.3 IIA/M-theory duality

By using (1.42) it is possible to compute the relation between M-theory and IIA geometrical structures. Again, we will use the hat to distinguish the M-theory objects from the ten-dimensional ones. The result of the dimensional reduction is

$$\widehat{K} = K - e^{-\phi}\Phi_0\,\partial_{10}\,, \tag{2.48a}$$

$$\widehat{\Omega} = -e^{-\phi}\Phi_2 - \widetilde{K}\wedge(\mathrm{d}x^{10}-C_1)\,, \tag{2.48b}$$

$$\widehat{\Sigma} = e^{-2\phi}\Omega - e^{-\phi}\Phi_4\wedge(\mathrm{d}x^{10}-C_1)\,. \tag{2.48c}$$

Notice that not all the IIA forms appear on the right hand side of (2.48). However, it is possible to calculate them by reducing the Hodge-duals of $\widehat{\omega}$ and $\widehat{\Sigma}$, for example

$$\hat{*}\widehat{\Sigma} = -e^{-2\phi}\widetilde{\Omega}\wedge(\mathrm{d}x^{10}-C_1) - e^{-\phi}\Phi_6\,. \tag{2.49}$$

CHAPTER 3

REFORMULATION OF BPS CONDITIONS

In the previous section we have introduced all the main characters that will allow us to reformulate supergravity BPS conditions. To achieve this, one has basically to hit spinor bilinears with (1.6), (1.24), and, using proper Clifford algebra and differential-form identities, to recast the result in a new way. Quite often such a reformulation brings to light important geometrical structures and new interesting physical consequences which were hidden in the spinorial system, as we will see in this and the next chapter.

Even if finding such bispinor equations is in principle just a matter of performing correct computations, to prove that it is possible to replace the original BPS system with a new one made of differential form is a much harder task. This has been done in many situations assuming some spacetime factorization (see for example [37, 73, 74, 75, 76]). However, to extend this procedure without imposing any restriction is much more interesting since one could specialize such a result to any spacetime configuration without wondering if the reduced system is completely equivalent to the original BPS one. Achieving this goal turns out to be much more difficult and, even if some systems have been found [10, 48, 9], they are not always completely satisfactory.

In this chapter, after presenting some form equations which are necessary derived by supersymmetry conditions, we will introduce systems which are also sufficient to imply them. One of these systems is the main results in [15]. Since the M-theory case is less convoluted, we will deal with it in a separate section, while in the main part we will always refer to type II theories.

3.1 Differential form equations

This section is organized as follows: we will now present here a list of differential form equations, which is by no means exhaustive, and we will briefly discussed them; their proof is sketched in the subsections after the discussion. We can split the equations in two categories, the differential ones, which contain at least a bispinor derivative, and the algebraic ones which will be listed at the end of this section. The differential equations are:

$$\mathcal{L}_K g = 0\,, \tag{3.1a}$$
$$\mathrm{d}_H(\mathrm{e}^{-\phi}\Phi) = -(\iota_K + \widetilde{K}\wedge)F\,, \tag{3.1b}$$
$$e^{2\phi}\mathrm{d}(e^{-2\phi}K) = *\big(H\wedge\Omega + \tfrac{e^{\phi}}{4}\{\Phi,F\}_8\big)\,,\quad e^{2\phi}\,\mathrm{d}(e^{-2\phi}\widetilde{K}) = *\big(H\wedge\widetilde{\Omega} + \tfrac{e^{\phi}}{8}\{\Phi_M,F^M\}_8\big)\,, \tag{3.1c}$$
$$\mathrm{e}^{2\phi}\mathrm{d}(\mathrm{e}^{-2\phi}\Omega) = -\iota_K * H + \mathrm{e}^{\phi}(\Phi,F)_6,\quad \mathrm{e}^{2\phi}\mathrm{d}(\mathrm{e}^{-2\phi}\widetilde{\Omega}) = -*(\widetilde{K}\wedge H) - (-)^{|\Phi|}\tfrac{e^{\phi}}{2}\big(\Phi_M,F^M\big)_6, \tag{3.1d}$$
$$\mathrm{d}\big(e^{2\phi} * \widetilde{K}\big) = 0\,,\quad \mathrm{d}\big(e^{2\phi} * K\big) = 0\,, \tag{3.1e}$$
$$\mathrm{d}\widetilde{K} = \iota_K H\,,\quad \mathrm{d}K = \iota_{\widetilde{K}} H - \tfrac{e^{\phi}}{2} * (\Phi,F)_8\,, \tag{3.1f}$$
$$\mathrm{d}\widetilde{\Omega} = H_M\wedge\Omega^M - \tfrac{e^{\phi}}{4}\{\Phi_M,F^M\}_6,\quad \mathrm{d}\Omega = H_M\wedge\widetilde{\Omega}^M - \tfrac{e^{\phi}}{4}\big(\{\Phi,F\} + \tfrac{1}{4}\{\Phi_{MN},F^{MN}\}\big)_6 \tag{3.1g}$$
$$\mathrm{d}*K = 0\,,\quad \mathrm{d}*\widetilde{K} = -\tfrac{1}{8}(-)^{|\Phi|}\,e^{\phi}\big(\Phi,\gamma_M F\gamma^M\big)\,, \tag{3.1h}$$

We recall that $(,)_d$ is the d-dimensional Chevalley–Mukai pairing (A.7) while $\{,\}_d$ is an analog which is defined in (A.8), which up to the author knowledge, has not a mathematical interpretation yet. The sign $(-)^{|\Phi|}$ which appears in some equations is the only one difference between type IIA and type IIB description and by K we indicate both the vector and the corresponding one-form depending on the contest.

Using (3.1) we can make more concrete what we said in the introduction of this chapter, i.e. how differential form equations disclose some geometrical conditions which were hidden in the spinorial BPS formalism. First of all notice that (3.1a) tells us that K generate an isometry; we will show that more in general K is a symmetry for all the fields, which means that $\mathcal{L}_K$ annihilates all the fluxes. Combining the first of (3.1e) with the first of (3.1h) and taking the Hodge dual we get the following scalar condition

$$\mathcal{L}_K\phi = 0 \tag{3.2}$$

which tells us that the dilaton is K-invariant. If we now take the external derivatives of the first of (3.1f) it is immediate to find

$$\mathcal{L}_K H = 0 \tag{3.3}$$

where the closure of H was used. Before continuing, let's consider the following anti-commutator of differential form operators

$$\{\mathrm{d}_H, \iota_K + \widetilde{K}\wedge\} = (\mathrm{d}\widetilde{K} - \iota_K H)\wedge + \mathcal{L}_K = \mathcal{L}_K \tag{3.4}$$

where we used (3.1f) in the last step. Taking the twisted external derivatives d_H of (3.1b) and using the Bianchi identity $\mathrm{d}_H F = 0$ we finally get

$$\mathcal{L}_K F = 0\,. \tag{3.5}$$

In [77] it is proved that K is also a supersymmetry isometry, i.e. $\mathcal{L}_K\epsilon_{1,2} = 0$.

3.1.1 Proof of (3.1a)

(3.1a) already appeared in [78] and we will prove it providing all the details, this is a good exercise since the computation are similar for all the other equations.

$$
\begin{aligned}
D_N K_{1\,M} =& \frac{1}{32} D_N \bar{\epsilon}_1 \gamma_M \epsilon_1 + \frac{1}{32} \bar{\epsilon}_1 \gamma_M D_N \epsilon_1 \\
=& -\frac{1}{4\cdot 32} \bar{\epsilon}_1 \left[H_N, \gamma_M \right] \epsilon_1 - \frac{\mathrm{e}^{\phi}}{8\cdot 32} \bar{\epsilon}_1 \gamma_M F \gamma_N \epsilon_2 \\
=& \frac{1}{2} H_{NMR} K_1^R - \frac{4\,\mathrm{e}^{\phi}}{32^2} \bar{\epsilon}_1 \gamma_M F \gamma_N \epsilon_2 \,,
\end{aligned}
\tag{3.6}
$$

where in the first step we used (1.24) while in the second the gamma matrices algebra. Following the same procedure we can also get

$$
D_N K_{2\,M} = -\frac{1}{2} H_{NMR} K_2^R + \frac{4\,\mathrm{e}^{\phi}}{32^2} \bar{\epsilon}_1 \gamma_N F \gamma_M \epsilon_2 \;. \tag{3.7}
$$

Now, summing up this two equations it is immediate to verify that:

$$
D_{(N} K_{M)} = 0 \,, \tag{3.8}
$$

which is (3.1a).

3.1.2 Proof of (3.1b)

Equation (3.1b) makes its first appearance in [9], which we refer to for further details, even if its derivation was inspired by the pure spinor ones [34]. Let's now sketch some steps.

First of all, inverting (A.18), it is possible to re-express

$$
H\wedge = \frac{1}{8}\Big(\overrightarrow{H} + \overleftarrow{H}(-)^{\mathrm{deg}} + \overrightarrow{\gamma}^M \overleftarrow{H}_M + \overrightarrow{H}_M \overleftarrow{\gamma}^M (-)^{\mathrm{deg}} \Big) \,. \tag{3.9}
$$

From this result we can write, dropping the tensor product symbol to lighten up the notation,

$$
\begin{aligned}
2\,\mathrm{e}^{\phi}\,\mathrm{d}_H(\mathrm{e}^{-\phi}\,\Phi) =& (\overrightarrow{\gamma} + \overleftarrow{\gamma}(-)^{|\Phi|})(\nabla_M \Phi - \partial_M \phi\, \Phi) - 2H\wedge\Phi \\
=& \Big(D\epsilon_1 - \frac{1}{4} H\epsilon_1 - \partial\phi\,\epsilon_1 \Big)\bar{\epsilon}_2 + \gamma^M \epsilon_1 \Big(D_M \bar{\epsilon}_2 - \frac{1}{4}\bar{\epsilon}_2 H_M \Big) \\
& - \Big(D_M \epsilon_1 - \frac{1}{4} H_M \epsilon_1 \Big)\bar{\epsilon}_2 \gamma^M - \epsilon_1 \Big(D_M \bar{\epsilon}_2 \gamma^M - \bar{\epsilon}_2 \frac{1}{4} H - \bar{\epsilon}_2 \partial\phi \Big) \,.
\end{aligned}
\tag{3.10}
$$

Using equations in (1.24) and their transposed version this expression reads

$$
\begin{aligned}
2\,\mathrm{e}^{\phi}\,\mathrm{d}_H(\mathrm{e}^{-\phi}\,\Phi) &= \frac{\mathrm{e}^{\phi}}{16}\left(\gamma^M \epsilon_1\bar{\epsilon}_1\gamma_M F - (-)^{|F|} F\gamma^M\epsilon_2\bar{\epsilon}_2\gamma_M\right)\\
&= -2\,\mathrm{e}^{\phi}(\widetilde{K}\wedge + \iota_K)F
\end{aligned}
\tag{3.11}
$$

from which one gets (3.1b). We used (A.24) in the last step.

3.1.3 Proof of (3.1c)-(3.1e)

Since they comes from the external derivative of the same bilinear, (3.1c)-(3.1e) can be actually proved all together in one go. The discussion closely follows the one of [15]. Similarly to what we have seen in the previous section, let's start by rewriting

$$
\iota_H = \frac{1}{8}\left(\overrightarrow{H} - \overleftarrow{H}(-)^{\mathrm{deg}} + \overrightarrow{\gamma}^M\overleftarrow{H}_M - \overrightarrow{H}_M\overleftarrow{\gamma}^M(-)^{\mathrm{deg}}\right),
\tag{3.12}
$$

then:

$$
\begin{aligned}
2\,\mathrm{e}^{2\phi}\,\mathrm{d}(\mathrm{e}^{-2\phi}\epsilon_1\bar{\epsilon}_1) + 2\iota_H\epsilon_1\bar{\epsilon}_1 &= \left[\gamma^M, D_M(\epsilon_1\bar{\epsilon}_1) - 2\partial_M\phi\,\epsilon_1\bar{\epsilon}_1\right]\\
&\quad + \frac{1}{4}\left(H\epsilon_1\bar{\epsilon}_1 + \epsilon_1\bar{\epsilon}_1 H + \gamma^M\epsilon_1\bar{\epsilon}_1 H_M + H_M\epsilon_1\bar{\epsilon}_1\gamma^M\right) =\\
&= \left(D - \frac{1}{4}H - \partial\phi\right)\epsilon_1\bar{\epsilon}_1 + \gamma^M\epsilon_1\left(D_M\bar{\epsilon}_1 + \frac{1}{4}\bar{\epsilon}_1 H_M\right)\\
&\quad - \left(D_M - \frac{1}{4}H_M\right)\epsilon_1\bar{\epsilon}_1\gamma^M - \epsilon_1\left(D_M\bar{\epsilon}_1\gamma^M + \frac{1}{4}\bar{\epsilon}_1 H - \bar{\epsilon}_1\partial\phi\right)\\
&\quad - \left(\partial\phi - \frac{1}{2}H\right)\epsilon_1\bar{\epsilon}_1 + \epsilon_1\bar{\epsilon}_1\left(\partial\phi + \frac{1}{2}H\right).
\end{aligned}
\tag{3.13}
$$

If we now replace the supersymmetry equations (1.24), (1.25) we get

$$
\begin{aligned}
\mathrm{e}^{2\phi}\,\mathrm{d}(\mathrm{e}^{-2\phi}\epsilon_1\bar{\epsilon}_1) = -\iota_H\epsilon_1\bar{\epsilon}_1 &+ (-)^{|F|}\frac{\mathrm{e}^{\phi}}{32}\gamma^M\Phi\gamma_M\lambda(F) - (-)^{|F|}\frac{\mathrm{e}^{\phi}}{32}F\gamma^M\lambda(\Phi)\gamma_M\\
&- (-)^{|F|}\frac{\mathrm{e}^{\phi}}{32}\gamma^M F\gamma_M\lambda(\Phi) + (-)^{|F|}\frac{\mathrm{e}^{\phi}}{32}\Phi\gamma^M\lambda(F)\gamma_M\,.
\end{aligned}
\tag{3.14}
$$

The same procedure applied to $\epsilon_2\bar{\epsilon}_2$ leads to:

$$
\begin{aligned}
\mathrm{e}^{2\phi}\,\mathrm{d}(\mathrm{e}^{-2\phi}\epsilon_2\bar{\epsilon}_2) = \iota_H\epsilon_2\bar{\epsilon}_2 &- (-)^{|F|}\frac{\mathrm{e}^{\phi}}{32}\gamma^M\lambda(\Phi)\gamma_M F + (-)^{|F|}\frac{\mathrm{e}^{\phi}}{32}\lambda(F)\gamma^M\Phi\gamma_M\\
&+ (-)^{|F|}\frac{\mathrm{e}^{\phi}}{32}\gamma^M\lambda(F)\gamma_M\Phi - (-)^{|F|}\frac{\mathrm{e}^{\phi}}{32}\lambda(\Phi)\gamma^M F\gamma_M\,.
\end{aligned}
\tag{3.15}
$$

From the sum and the difference between (3.14) and (3.15) we get

$$\begin{aligned} \mathrm{e}^{2\phi}\,\mathrm{d}\left(\mathrm{e}^{-2\phi}\,\frac{\epsilon_1\bar{\epsilon}_1 \pm \epsilon_2\bar{\epsilon}_2}{2}\right) = -\iota_H\left(\frac{\epsilon_1\bar{\epsilon}_1 \mp \epsilon_2\bar{\epsilon}_2}{2}\right) - (-)^{|F|}\frac{\mathrm{e}^{\phi}}{64}\Big([\gamma^M F\gamma_M, \lambda(\Phi)]_\pm \\ + [F, \gamma^M \lambda(\Phi)\gamma_M]_\pm \mp [\gamma^M \lambda(F)\gamma_M, \Phi]_\pm \mp [\lambda(F), \gamma^M \Phi\gamma_M]_\pm\Big)\,, \end{aligned} \tag{3.16}$$

where $[\,,]_-$ indicates the usual commutator while $[\,,]_+$ is the anticommutator. The next step is to apply (A.24) and (A.18) to each commutator or anticommutator. After summing up all these terms one has to separate the two-, six- and ten-form part in order to get (3.1c)-(3.1e) .

3.1.4 Proof of (3.1f)-(3.1h)

Equations (3.1f)-(3.1h) are somehow similar to (3.1c)-(3.1e), but they are derived starting from the equations corresponding to the gravitino variations only:

$$\begin{aligned} \mathrm{d}(\epsilon_1\bar{\epsilon}_1) &= \frac{1}{2}\left[\gamma^M, D_M(\epsilon_1\bar{\epsilon}_1)\right] = \frac{1}{2}\left[\gamma^M, \frac{1}{4}[H_M, \epsilon_1\bar{\epsilon}_1] + (-)^{|F|}\frac{e^{\phi}}{16}(F\gamma_M\lambda(\Phi) + \Phi\gamma_M\lambda(F))\right] \\ &= H_M\wedge\iota^M\epsilon_1\bar{\epsilon}_1 + (-)^{|F|}\frac{e^{\phi}}{32}(\gamma^M F\gamma_M\lambda(\Phi) + \gamma^M\Phi\gamma_M\lambda(F) - F\gamma_M\lambda(\Phi)\gamma^M - \Phi\gamma_M\lambda(F)\gamma^M) \end{aligned} \tag{3.17}$$

and

$$\mathrm{d}(\epsilon_2\bar{\epsilon}_2) = -H_M\wedge\iota^M\epsilon_2\bar{\epsilon}_2 + (-)^{|F|}\frac{e^{\phi}}{32}(\lambda(\Phi)\gamma^M F\gamma_M + \lambda(F)\gamma^M\Phi\gamma_M - \gamma_M\lambda(\Phi)\gamma^M F - \gamma_M\lambda(F)\gamma^M\Phi). \tag{3.18}$$

Taking the sum and the difference of the last two equations it results:

$$\begin{aligned} \mathrm{d}\left(\frac{\epsilon_1\bar{\epsilon}_1 \pm \epsilon_2\bar{\epsilon}_2}{2}\right) = H_M\wedge\iota_M\left(\frac{\epsilon_1\bar{\epsilon}_1 \mp \epsilon_2\bar{\epsilon}_2}{2}\right) + (-)^{|F|}\frac{e^{\phi}}{64}\Big([\gamma_M F\gamma^M, \lambda(\Phi)]_\pm \\ + [\gamma_M\Phi\gamma^M, \lambda(F)]_\pm - [F, \gamma_M\lambda(\Phi)\gamma^M]_\pm - [\Phi, \gamma_M\lambda(F)\gamma^M]_\pm\Big)\,. \end{aligned} \tag{3.19}$$

After similar manipulation to the one described in the previous section, one finally gets (3.1f)-(3.1h).

3.1.5 Algebraic equations for type IIB

The first thing one could complain about by looking at (3.1) is that not every of these differential equations are really "differential" according to our definition, indeed by combining them it is possible to eliminate the part containing the bispinor external derivative. This means that actually some equations can be seen as a combination of a differential plus an algebraic one. Here we will list some of the algebraic constraints which come from the BPS equations. Since this part will be mostly useful when we will deal with S-duality, all the computations are restricted to type IIB supegravity.

Combining for example the first of (3.1c) with the second of (3.1f), and the first (3.1g) with the second in (3.1d), we get the following relations:

$$\begin{aligned}&2F_1\wedge\Phi_7-F_3\wedge\Phi_5-e^{-\phi}H\wedge\Omega+e^{-\phi}\widetilde{K}\wedge *H+\Phi_1\wedge F_7+2\iota_K * \mathrm{d}\,e^{-\phi}=0\,,\\&e^{-\phi}H_M\wedge\Omega^M+F_{3\,M}\wedge\Phi_5^M-\iota_{\Phi_1}F_7+e^{-\phi}\iota_{\widetilde{K}} * H+2\,\mathrm{d}\,e^{-\phi}\wedge\widetilde{\Omega}-2F_{1\,M}\Phi_7^M=0\,.\end{aligned}\tag{3.20}$$

Since the dilatino equation (1.25) is already algebraic, we can use it to provide other supersymmetry constraints. We can proceed as follows: first of all we take the tensor product of (1.25a) with $\bar{\epsilon}_2$ and of the transpose of (1.25b) with ϵ_1. Now we want to combine these two. If we take, for instance, the difference between them, we don't have to consider also the sum because the two components have different left chirality, so the sum can be obtained by acting with $\overrightarrow{\gamma}$. The most interesting equations come from the ten-, eight-, four-form components. They read:

$$2\,\mathrm{d}\,e^{-\phi}\wedge\Phi_9+e^{-\phi}H\wedge\Phi_7-2\widetilde{K}\wedge F_9=0\,,\tag{3.21a}$$

$$2\,\mathrm{d}\,e^{-\phi}\wedge\Phi_7+e^{-\phi}H\wedge\Phi_5-e^{-\phi}\Phi_1\wedge *H-2\iota_K F_9-F_3\wedge\Omega-\widetilde{K}\wedge F_7=0\,,\tag{3.21b}$$

$$2\iota_{\mathrm{d}\,e^{-\phi}}\Phi_7+e^{-\phi}\iota_{\Phi_1} * H+e^{-\phi}H_M\wedge\Phi_5^M+2F_1\wedge\widetilde{\Omega}+\iota_{\tilde{k}}F_7-F_3^M\wedge\Omega_M=0\,,\tag{3.21c}$$

where we have taken the Hodge dual of the four-form part.

3.2 BPS-equivalent systems

In the previous section we have derived many differential-form equations which are necessarily implied by the supersymmetry conditions (1.24). However, it is not clear yet if it is possible to go the other way, namely, if it is possible to find a system which implies all the (1.24). From the preliminary discussion in section 2.2.2, in particular equation (2.37), one can conclude conclude that, since equations in (3.1) do not contain vectors e_{+_1},e_{+_2} nor any free index, it is not possible to have a system which implies the BPS conditions in full generality, even by using all of (3.1).

However, if we restrict ourselves to the timelike case, there are no limitations from (2.36) to use some of (3.1) to write a differential-form system which is equivalent to supersymmetry. Indeed in [15] it is shown that the following system

$$\mathrm{d}_H(\mathrm{e}^{-\phi}\Phi)=-(\iota_K+\widetilde{K}\wedge)F\,,\tag{3.22a}$$

$$\mathrm{e}^{2\phi}\,\mathrm{d}(\mathrm{e}^{-2\phi}\,\Omega)=-\iota_K * H+\mathrm{e}^{\phi}(\Phi,F)_6\,,\tag{3.22b}$$

$$\mathrm{e}^{2\phi}\,\mathrm{d}(\mathrm{e}^{-2\phi}\,\widetilde{\Omega})=- * (\widetilde{K}\wedge H)-\frac{1}{2}(-)^{|\Phi|}\,e^{\phi}\left(\Phi_M,F^M\right)_6\,,\tag{3.22c}$$

$$\mathcal{L}_K\phi=0\,,\qquad \mathrm{d} * \widetilde{K}=-\frac{1}{8}(-)^{|\Phi|}\,e^{\phi}\left(\Phi,\gamma_M\,F\,\gamma^M\right),\tag{3.22d}$$

is sufficient for supersymmetry for both IIA and IIB in the timelike case. While the timelike condition can be seen as a restriction, in the space of possible solutions the subset $K^2 < 0$ is actually the generic case while $K^2 = 0$ has measure zero. Even if the light-like case seems peculiar, it is actually of great significance since all the $\mathcal{N} = 1$ vacua with dimension grater than three fall in this class, as we will see in chapter 5. On the other hand the timelike case seems a more natural setting to describe stationary black-hole backgrounds, in which case, differently from vacua, there is not an abundance of known solutions.

In [9] it is possible to find a system which implies BPS conditions (1.24) also in the light-like case, it reads

$$\mathrm{d}_H(\mathrm{e}^{-\phi}\Phi) = -(\widetilde{K}\wedge + \iota_K)F\,, \tag{3.23a}$$

$$\mathcal{L}_K g = 0\,, \quad \mathrm{d}\widetilde{K} = \iota_K H\,, \tag{3.23b}$$

$$\left(e_{+_1}\Phi\, e_{+2}\,,\gamma^{MN}\left[(-)^{|F|+1}\,\mathrm{d}_H(\mathrm{e}^{-\phi}\,\Phi e_{+_2}) + \frac{1}{2}\mathrm{e}^{\phi}\,\mathrm{div}(\mathrm{e}^{-2\phi}\,e_{+_2})\Phi - F\right]\right) = 0\,, \tag{3.23c}$$

$$\left(e_{+_1}\Phi\, e_{+2}\,,\left[\mathrm{d}_H(\mathrm{e}^{-\phi}\,e_{+_1}\Phi) - \frac{1}{2}\mathrm{e}^{\phi}\,\mathrm{div}(\mathrm{e}^{-2\phi}\,e_{+_1})\Phi - F\right]\gamma^{MN}\right) = 0\,. \tag{3.23d}$$

Notice that in this case e_{+1}, e_{+2} appear in the equations together with Φ, as required by (2.37). Even if this system encompasses all the possible supersymmetric solutions, the last two equations are quite cumbersome and, as we will see in the next chapter, do not have clear physical interpretation, differently from (3.22).

The proof of the equivalence of these two systems to (1.24) can be found in [9] and [15] appendix B. It consists in rewriting (1.24) by expanding the intrinsic torsion on a spinor basis, so that the BPS conditions can be rewritten as some algebraic identities. The same reparameterization can be also used to re-express (3.22) and (3.23). To prove sufficiency one has to check if all the intrinsic torsion equations which comes from (1.24) are independently present in the differential-form systems. Even if the procedure is straightforward, the actual computation turns out to be quite convoluted, so we refer to the original papers for the proof.

3.2.1 Integrability

Thanks to the equivalence of (3.22), (3.23) to supersymmetry, we can switch between spinorial and differential-form description whenever we want, and in particular we can take advantage of the results obtained with the spinorial formalism, like the integrability conditions (1.28). As we discussed in section 1.2, to impose the BPS system is not enough to automatically solve all the equations of motion. First of all one has to check that Bianchi identities for H and F (1.16) are satisfied, moreover the integrability constraints (1.28) tell us that supersymmetry imposes the dilaton equation but not necessary all the components of the Einstein or B-field equations.

In the timelike case, for example, up to rescaling we can use the condition (2.26) for our choice of vielbein $e^a = (e^+, e^-, e^\alpha)$, and therefore we get $\gamma^+\epsilon_1 = \gamma^-\epsilon_2 = 0$, while $(\gamma^-\epsilon_1, \gamma^\alpha\epsilon_1)$ and $(\gamma^+\epsilon_2, \gamma^\alpha\epsilon_2)$ give two sets of linearly independent spinors. Hence from (1.28) we can get the following components of the equations of motion:

$$E_{++} = E_{--} = E_{M\alpha} = \delta H_{M\alpha} = 0\,, \tag{3.24}$$

together with

$$E_{+-} = \frac{1}{2}\delta H_{+-}\,. \tag{3.25}$$

Hence, to be sure that all the equations of motion are implied it remains to impose either $E_{+-} = 0$ or $\delta H_{+-} = 0$. The latter condition may be written as

$$K \wedge \tilde{K} \wedge \left[\mathrm{d}(e^{-2\phi} * H) - \frac{1}{2}(F, F)_8 \right] = 0\,, \tag{3.26}$$

while one can check that the first one is implied by

$$\nabla^2 e^{-2\phi} - e^{-2\phi} H^2 - \frac{1}{4}\sum_k k F_k^2 = 0\,, \tag{3.27}$$

which is a combination between the trace of the Einstein and the dilaton equation.

3.2.2 $\mathrm{Sl}(2,\mathbb{R})$-duality invariant system

Combining (3.22) with some of the equations in section 3.1 it is possible to obtain a system which is invariant under $\mathrm{Sl}(2,\mathbb{R})$ symmetry of type IIB. For example, combining the two-form part of (3.22a) with the first (3.1f) it is possible to get the following complex equation

$$\mathrm{d}(\mathrm{e}^{-\phi}\Phi_1) + \iota_K F_3 + \widetilde{K} \wedge F_1 + \mathrm{i}\,\mathrm{e}^{-\phi}(\mathrm{d}\widetilde{K} - \iota_K H) = 0 \tag{3.28}$$

which is $\mathrm{Sl}(2,\mathbb{R})$ invariant as one can check using (2.45) and (1.34). To make also the other equations invariant it is possible that one has to use the algebraic constraints in 3.1.5; for example the tenth-degree of (3.22a) must be summed with two times the algebraic equation (3.21a) to get that the following invariant combination:

$$\mathrm{e}^{-\phi}\,\mathrm{d}(\mathrm{e}^{-2\phi} * \widetilde{K}) + \mathrm{i}\left(\mathrm{d}(\mathrm{e}^{-3\phi}\Phi_9) - \mathrm{e}^{-2\phi}\,\widetilde{K} \wedge F_9\right) = 0 \tag{3.29}$$

which makes also use of (3.1e).

Before showing all the correct combinations which makes (3.22) invariant, it is better to express all the fields and bilinears using a formalism explicitly $\mathrm{Sl}(2,\mathbb{R})$ covariant, such that the new system is invariant at first sight. In this formalism all the objects are defined so that they transform just under the U(1) subgroup of $\mathrm{Sl}(2,\mathbb{R})$.

We will say that a field has charge q under U(1) if it transforms by a phase $e^{iq\theta}$ where $\theta = \arg(\gamma\tau + \delta)$.

For example, combining the three-form field-strengths in the complex one

$$G_3 = e^{\frac{1}{2}\phi}(F_3 - i\, e^{-\phi} H) \tag{3.30}$$

one can then check that G_3 has charge $q = -1$

$$G_3' = e^{-i\theta} G_3 \, . \tag{3.31}$$

As another example, the one-form $e^{\phi}\, d\tau$ has U-charge $q = -2$, which means

$$e^{\phi'}\, d\tau' = e^{-2i\theta}(e^{\phi}\, d\tau) \, . \tag{3.32}$$

Notice that the $U(1) \subseteq Sl(2,\mathbb{R})$ transformations are typically point-dependent, since τ is in general non-constant, so they do not commute with ordinary derivatives. A composite compatible connection however can be defined

$$Q = \frac{1}{2} e^{\phi} F_1 \tag{3.33}$$

which twists the covariant derivative as follows $D_M - iqQ_M$. In particular, also the exterior derivative gets modified

$$d_Q = d - iqQ \wedge \, . \tag{3.34}$$

In this reformulation it is convenient to use the Einsten-frame metric

$$g_E \equiv e^{-\frac{1}{2}\phi} g \tag{3.35}$$

so that Einstein-frame Hodge-operator $*_E$ commutes with the duality transformation; for instance, $*_E G_3$ has again charge -1.

By using the transformation rules of bispinors (2.47) it is easy to check that the Killing vector K, the three-form $e^{-\phi}\Phi_3 \equiv \Theta_3$ and the five-form $e^{-\frac{3}{2}\phi}\widetilde{\Omega} \equiv \widetilde{\Omega}_E$ are invariant under $Sl(2,\mathbb{Z})$ duality, while

$$\Theta_1 \equiv e^{-\frac{1}{2}\phi}(\widetilde{K} + i\Phi_1) \, , \qquad \Theta_5 \equiv e^{-\frac{3}{2}\phi}(\Omega + i\Phi_5) \, , \tag{3.36}$$

and their Hodge-duals, have charge $q = 1$. We have then reorganized all the fields in combinations transforming with definite U(1)-charges, summarized in table 3.1.

fields	$\mathrm{U}(1)_D$-charge
g_{E}, K, Θ_3, $\widetilde{\Omega}_E$, F_5	0
Θ_1, Θ_5	1
G_3	-1
$\mathrm{e}^{\phi}\,\mathrm{d}\tau$	-2

Table 3.1: U(1) charges of relevant fields.

So we have now all the ingredients to rewrite (3.22) in a $\mathrm{Sl}(2,\mathbb{R})$ invariant form:

$$\mathcal{L}_K\tau = 0\,, \qquad \mathrm{e}^{\phi}\,\mathrm{d}\tau\wedge *_{\mathrm{E}}\Theta_1 + \frac{\mathrm{i}}{2}G_3\wedge *_{\mathrm{E}}\Theta_3 = 0\,, \tag{3.37a}$$

$$\mathrm{d}_Q\Theta_1 - \frac{\mathrm{i}}{2}\mathrm{e}^{\phi}\,\mathrm{d}\bar\tau\wedge\overline{\Theta}_1 + \mathrm{i}\,\iota_K\overline{G}_3 = 0\,, \tag{3.37b}$$

$$\mathrm{d}\Theta_3 + \iota_K F_5 + \mathrm{Re}\big(\Theta_1\wedge G_3\big) = 0\,, \tag{3.37c}$$

$$\mathrm{d}_Q\Theta_5 + \frac{\mathrm{i}}{2}\mathrm{e}^{\phi}\,\mathrm{d}\bar\tau\wedge\overline{\Theta}_5 + \Theta_3\wedge\overline{G}_3 - \mathrm{i}\iota_K(*_{\mathrm{E}}\overline{G}_3) + \mathrm{i}\Theta_1\wedge F_5 = 0\,, \tag{3.37d}$$

$$\mathrm{d}*_{\mathrm{E}}\Theta_3 + \frac{1}{2}\mathrm{Re}\,(G_3\wedge\Theta_5 - *_{\mathrm{E}}\,G_3\wedge\Theta_1) = 0\,, \tag{3.37e}$$

$$\mathrm{d}_Q *_{\mathrm{E}}\Theta_1 - \frac{\mathrm{i}}{2}\mathrm{e}^{\phi}\,\mathrm{d}\bar\tau\wedge *_{\mathrm{E}}\overline{\Theta}_1 = 0\,, \tag{3.37f}$$

$$\mathrm{d}\widetilde{\Omega}_E + \frac{1}{4}g_{\mathrm{E}}^{MN}\,[\mathrm{Im}(\Theta_{5\,M}\wedge G_{3\,N}) - 2\Theta_{3\,M}\wedge F_{5\,N}] - 3*_{\mathrm{E}}\mathrm{Im}(\Theta_1\wedge G_3) = 0\,. \tag{3.37g}$$

From table 3.1 it is also easy to see that the system is manifestly $\mathrm{SL}(2,\mathbb{Z})$ invariant.

As showed at the beginning of this subsection, (3.37) contains more equations than (3.22). However, having used some algebraic constraints to modify the original ones, the equivalence with supersymmetry may not be guaranteed anymore. A conservative way to be sure that none of the supersymmetry data has been lost is to check that the algebraic constraints we have used are separately satisfied:

$$\begin{aligned}
&g_{\mathrm{E}}^{MN}(\overline{G}_{3\,M}\wedge\Theta_{5\,N}) - *_{\mathrm{E}}(\overline{G}_3\wedge\Theta_1) - 2\,\mathrm{e}^{\phi}\,\mathrm{d}\bar\tau\wedge\widetilde{\Omega}_E + 2\mathrm{i}*_{\mathrm{E}}(\mathrm{e}^{\phi}\,\mathrm{d}\bar\tau\wedge\Theta_3) = 0\,,\\
&\overline{G}_3\wedge\Theta_5 - \Theta_1\wedge *_{\mathrm{E}}\overline{G}_3 + 2\,\mathrm{e}^{\phi}\,\iota_K *_E \mathrm{d}\bar\tau + 2\mathrm{i}\,\mathrm{e}^{\phi}\,\mathrm{d}\bar\tau\wedge *_{\mathrm{E}}\Theta_3 = 0\,.
\end{aligned} \tag{3.38}$$

Again, by using the $\mathrm{U}(1)_D$-charges of table 3.1 one can easily check that (3.38) are manifestly invariant under $\mathrm{SL}(2,\mathbb{Z})$ dualities.

Notice that the large number of equations in (3.37)–(3.38) is due to the fact that the system lists separately each form degree, differently for example from (3.22), .

3.3 M-theory

The situation in M-theory resembles the one in ten dimensions. First of all again the BPS conditions can be rephrased in terms of the bilinears in section 2.3 and in the timelike case the differential-form equations just depend on external derivative of bispinors, without need of picking an explicit spacetime framing as in the lightlike case [48] (or as the type II null case). Necessary and sufficient conditions for supersymmetry in the timelike case are given by [10]

$$\mathcal{L}_K g = 0\,, \tag{3.39a}$$

$$\mathrm{d}K = \frac{2}{3}\iota_\Omega F + \frac{1}{2}\iota_\Sigma * F\,, \tag{3.39b}$$

$$\mathrm{d}\Omega = \iota_K F\,, \tag{3.39c}$$

$$\mathrm{d}\Sigma = \iota_K * F - \Omega \wedge F\,. \tag{3.39d}$$

From equation (3.39a) one can see that K is a Killing vector, as it was already suggested by the dimensional reduction formula to IIA (2.48a). Moreover, taking the external derivative of (3.39c) one immediately gets

$$\mathcal{L}_K F = 0 \tag{3.40}$$

which means that again K is a symmetry for all the fields.

Let's now discuss integrability in the timelike case. Once one has imposed F Bianchi identity and equation of motion, just the Einstein equation E_{MN} must be solved. Let's now rewrite (1.8) more compactly

$$E_{MN}\gamma^M \epsilon = 0\,. \tag{3.41}$$

contracting this with ϵ we immediately have that all the components E_{0N} of the Einstein equation are implied by supersymmetry constraints. Multiplying (3.41) with $E_{MP}\gamma^P$ one gets

$$E_{MN}E^N_M = 0 \qquad \text{no sum on M}\,, \tag{3.42}$$

therefore, since the contracted index N runs over spatial indices only, the previous equation is zero just if $E_{MN} = 0$. So in the timelike case all the components of the Einstein equation are set to zero by (3.39).

CHAPTER 4

CALIBRATION CONDITIONS

As anticipated, some equations in (3.22), or more in general (3.1), can be interpreted in terms of brane calibration. The concept of calibration is derived from Riemannian geometry [14] and it gives an alternative solution to the problem of finding surface with minimal area. In this context a calibration ω on a manifold M is defined as a p-form which satisfies the following two conditions:

$$\omega|_N \leq \mathrm{Vol}(N) \qquad \forall N \subseteq M \qquad \dim(N) = p\,, \tag{4.1a}$$

$$\mathrm{d}\omega = 0\,, \tag{4.1b}$$

where Vol(N) is the volume form of N. A submanifold N is said to be calibrated if the inequality (4.1a) is saturated on N. The idea of these definitions is that a calibrated submanifold has minimal volume in its homology class, indeed, given a deformation N' of N and a $p+1$-dimensional submanifold Γ whose boundary is $\partial\Gamma = N - N'$, we can write

$$\mathrm{vol}(N) - \mathrm{vol}(N') = \int_N \mathrm{Vol}(N) - \int_{N'} \mathrm{Vol}(N') \leq \int_N \omega - \int_{N'} \omega = \int_\Gamma \mathrm{d}\omega = 0\,. \tag{4.2}$$

Calibrations show up naturally in string theory dealing with solitonic supersymmetric objects (even if generalizations to non-supersymmetric cases are also possible [38]) that emerge from the supergravity algebra. They first appeared in compactification on Ricci-flat spaces, when all the background fields besides the metric are turned off [79, 80]. In such a situation a p-brane is described by the Dirac–Nambu–Goto action and therefore its energy is just given by the volume of the cycle wrapped by the brane. Physically, brane energy is expected to be subject to a BPS lower bound which is saturated by supersymmetric configurations, and therefore BPS branes are volume-minimizing, i.e. calibrated. This provides a natural geometric interpretation of the BPS bound with the calibrated submanifolds corresponding to the supersymmetric states which saturate the bound.

Clearly, more complicated backgrounds require a modification of the notion of calibration in both a physical and mathematical way, since the presence of fluxes must be taken into account and we are in general interested in a definition of calibration in the Lorentzian context. To face the first problem a notion of "generalized calibration" was introduce in [81], which means that (4.1b) get modified because of the presence of a flux F, schematically:

$$\mathrm{d}\omega = -\iota_K F \tag{4.3}$$

where K is the Killing vector associated to the supersymmetric configuration analog to the one defined in subsection 3.1. As one can see there are some similarities between (4.3) and some of the equations in (3.1) and (3.39), which are supersymmetry conditions for spacetime. It seems then that the interplay between branes and background geometry can be concretely realized in terms of calibrations, and it was even conjectured that it is possible to rephrase the BPS conditions (1.6) and (1.24) in terms of brane calibrations. This is true, for example, in the case of compactification over a four-dimensional spacetime [12]. We will devote this chapter to show that the system (3.22) admit, to some extent, such an interpretation.

We will mainly follow [13], whose approach also adapts the mathematical formalism to the Lorentzian context.

4.1 Calibration of a generic brane

Let's start by studying the case of a generic toy model of a p-brane. Let us suppose that the action of the brane wrapping a $(p+1)$-dimensional surface $\mathcal{S}$ is the sum of a Nambu–Goto and a Wess–Zumino term

$$S_p = -\mu_p \int_{\mathcal{S}} \mathrm{d}^{p+1}\xi \sqrt{-\det g|_S} + \mu_p \int_{\mathcal{S}} C\,, \tag{4.4}$$

where $\xi^\alpha = (\tau, \sigma^i)$ are the coordinates on $\mathcal{S}$, and that supersymmetry of the background imposes

$$\mathrm{d}\omega = -\iota_K F\,, \tag{4.5}$$

where $F = \mathrm{d}C$. For $p = 2$ this is exactly the description of a string moving in a non-trivial backgound (1.21) and for generic p it is an O-plane or a D-brane (1.17) with $\mathcal{F}$s turned of, so it is a good toy-model for understanding the role of calibrations avoiding technical difficulties.

A probe brane placed in a supersymmetric background does not automatically preserve supersymmetry. If we want this to happen we have to check that the κ-symmetry condition is satisfied

$$\gamma_p \epsilon = \epsilon \tag{4.6}$$

where γ_p can be seen as the chiral operator defined on the brane, $\gamma_p^2 = 1$. Let's now introduce the world-volume (or gauge-invariant) momentum conjugated to x^M:

$$P^M = -\sqrt{-h}\, h^{\tau\alpha} \partial_\alpha X^M \,, \qquad h \equiv g|_{\mathcal{S}} \,. \tag{4.7}$$

By construction P^M is timelike, so, taking a generic spinor χ, we always have

$$\overline{\chi} P \chi \geq 0 \tag{4.8}$$

since it is possible to choose the charge conjugation matrix such that it is parallel to the one-form associated to P. Now, considering $\chi = (\mathbb{1} - \gamma_p)\epsilon$, from the previous inequality one gets

$$-K^M P_M \, \mathrm{d}^p \sigma \geq \omega|_\Sigma \,, \tag{4.9}$$

where we have evaluated (4.8) on a space-like p-surface $\Sigma = \mathcal{S} \cap \mathcal{M}$ where $\mathcal{M}$ is a $(d-1)$-one. This inequality is the equivalent of (4.1a) in Lorentzian context, where $-K^M P_M \, \mathrm{d}^p \sigma$ plays the same role of the volume form.

Notice that the quantity we get by integrating (4.9)

$$-\int_\Sigma K^M P_M \, \mathrm{d}^n \sigma - \int_\Sigma \omega \geq 0 \tag{4.10}$$

is not a BPS bound (even if it can be shown to be the conserved charge related to the K isometry [13, section 3.2]) because P_M is not the canonical momentum since it is obtained from the Legendre transformation of the Nambu–Goto part of the action only, so the first term isn't the energy of the brane. Considering also the Wess–Zumino term we get that the canonical momentum reads $\mathcal{P}_M = P_M + \iota_M C|_\Sigma$, so we can modify (4.10) as following

$$-\int_\Sigma K^M \mathcal{P}_M \, \mathrm{d}^n \sigma \geq \int_\Sigma (\omega - \iota_K C) \,, \tag{4.11}$$

which is a full-flegded BPS bound. Indeed the quantity one the left-hand side is by definition the energy of the brane, while, by choosing gauge $\mathcal{L}_K C = 0$, we can write (4.5) as

$$\mathrm{d}\varphi = 0 \quad \text{with} \quad \varphi = \omega - \iota_K C \,. \tag{4.12}$$

Thus the right-hand side of (4.11) is a topological quantity, which can be interpreted as the brane central charge.

Up to now the p-brane was regarded as a probe, meaning that we were considering a regime where the back-reaction was neglected. Now, let us take a delta-like $D-(p+1)$ form $\delta_{D-(p+1)}$ localized on $\mathcal{S}$ (exactly as in (1.19)) as a source for the flux F so that it satisfies the following equation of motion for C, which mimics what we have seen in (1.5) or (1.20)

$$\mathrm{d} * F = \delta_{D-(p+1)} \,. \tag{4.13}$$

We can now rewrite the central charge in the following way

$$Z = \int_{\Sigma} (\omega - \iota_K C) = \int_{\mathcal{M}} (\omega - \iota_K C) \wedge \delta_{D-(p+1)} = \int_{\mathcal{M}} (\omega - \iota_K C) \wedge \mathrm{d} * F \,, \tag{4.14}$$

and, thanks to the calibration condition (4.12), we can use Stokes' theorem to find

$$Z = (-)^p \int_{\mathcal{B}} \varphi \wedge * F \,, \tag{4.15}$$

where $\mathcal{B} = \partial \mathcal{M}$. Assuming that $\mathcal{S} \cap \mathcal{B} = 0$, the integrated quantity is invariant under deformations of the boundary $\mathcal{B}$. Thanks to (4.15) we are able to compute the central charge on the space boundary, which means that we are defining it considering the source as an object in the entire bulk, instead of just using the theory on the brane world-volume. This is equivalent to promote the brane from a probe formalism to a back-reacting one. This novel reformulation of the central charge can also be applied to gravitational objects, such as KK-monopoles, where a priori it is not clear on which submanifold they are sitting since the solution can be smooth. We will use this result in the end of this chapter.

4.2 String and D-brane calibration

From the general discussion in the previous section we are now able to identify the calibration form for the fundamental string and the D-branes. Comparing (4.4) with the string action (1.21) and looking at equations (3.1) it is immediate to notice that the first of (3.1f),

$$\mathrm{d} \widetilde{K} = \iota_K H \,, \tag{4.16}$$

is the equivalent to the generalized calibration condition (4.5). By making the following gauge choice $\mathcal{L}_K B = 0$, which is allowed since we proved in section 3.1 that K is a symmetry for H, we get the closed one-form string calibration

$$\varphi_{\mathrm{F1}} = \widetilde{K} + \iota_K B \,. \tag{4.17}$$

For D-branes the argument is similar even if a bit more tricky. The best candidate for D-brane calibration condition is (3.22a), however there are many differences with respect to calibration condition (4.5). First of all we have the twisted external derivative d_H instead of the usual one. This is due to the fact that, in a generalized geometry perspective, if a B-field is turned on the spinors associated with the B-twisted generalized tangent bundle get modified with respect to the original one

$$C_{\mathrm{tw}} = \mathrm{e}^{-B} \wedge C \,, \qquad \Phi_{\mathrm{tw}} = \mathrm{e}^{-B} \wedge \Phi \,, \tag{4.18}$$

where $\mathrm{e}^{-B}\wedge$ is the effect of the twist. For such forms it is immediate to notice that

$$F_{\mathrm{tw}} = \mathrm{d}\, C_{\mathrm{tw}} = \mathrm{e}^{-B} \wedge \mathrm{d}_H C = \mathrm{e}^{-B} \wedge F \tag{4.19}$$

and therefore Bianchi identities become

$$\mathrm{d}\, F_{\mathrm{tw}} = 0 \quad \Longleftrightarrow \quad \mathrm{d}_H F = 0\,. \tag{4.20}$$

Moreover, in the right-hand side of (3.22a) not only the Killing vector K but also the one-form $\widetilde{K}$ appears. However, always using a generalized geometry approach, we can regard at $\iota_K + \widetilde{K}\wedge$ as a vector on the generalized tangent bundle which generates a symmetry together with the twisted external derivative. This was already proved in (3.4). In particular, by choosing $\mathcal{L}_K C = 0$, we can anti-commute $\iota_K + \widetilde{K}\wedge$ with d_H. Using all these ingredients we can get the following form for a Dp-brane calibration

$$\varphi_{\mathrm{D}p} = \left[\mathrm{e}^{-B\wedge}\left(e^{-\phi}\Phi - (\iota_K + \widetilde{K}\wedge) C\right)\right]_p\,. \tag{4.21}$$

The complete calibration for D-branes is actually given by the sum over all the degrees p, which allow to describe the energetics of D-branes supporting non-trivial fluxes [82, 12, 83, 13] and/or forming networks [84].

The derivation of the BPS bound for both string and D-branes can be argued along the line of section 4.1 using their actions and κ-symmetry operators. We refer to [13] for the details of this discussion.

4.3 M2- and M5-brane calibration

The M2- and M5-brane calibrations can be obtained from the equations in (3.39) as in [85]. The M2-brane calibration condition can be read from (3.39c); similarly to what we have seen in the previous section, by setting $\mathcal{L}_K A =$ it is easy to prove that the form

$$\varphi_{\mathrm{M2}} = \Omega + \iota_K A \tag{4.22}$$

is closed. For the M5-brane we have to start from (3.39d). The first step is to insert the definition of the M5 electric potential (1.4) in (3.39d)

$$\mathrm{d}(\Sigma + \iota_K C) = -\frac{1}{2}\left(\iota_K A \wedge F - A \wedge \iota_K F\right) - \omega \wedge F\,, \tag{4.23}$$

where we used $\mathcal{L}_K C = 0$. Now, from the following identity, which makes use of the M2-calibration condition (3.39c),

$$\omega \wedge F = \mathrm{d}(\omega \wedge A) - \iota_K F \wedge A\,, \tag{4.24}$$

we are able to rewrite the left hand side of (4.23) in a closed form. The result of this operation brings to the following equation for the M5-calibration:

$$\varphi_{\mathrm{M5}} = \Sigma + \iota_K C + A \wedge \Omega + \frac{1}{2} A \wedge \iota_K A\,. \tag{4.25}$$

4.4 NS5-brane calibration

Differently from D-branes and fundamental string, it is rather difficult to discuss the role of the NS5-brane calibration starting from its action since it was just defined in an indirect manner [51]. The intuition acquired by studying the other cases suggests that the proper supersymmetry condition we have to start from is (3.22b), so let's start with understanding if it is possible to trace it back to a closure condition.

Similarly to what we have seen with the M5 case, we have to insert the NS5 potential (1.23) in (3.22b); we will do all the computations in type IIA supergravity, since ones for IIB are identical with $F_0 = 0$:

$$\mathrm{d}\left(\mathrm{e}^{-2\phi}\,\Omega\right) = -\,\mathrm{e}^{-2\phi}\,\iota_K * H + (\mathrm{e}^{-\phi}\,\Phi, F)_6 = -\iota_K\,\mathrm{d}\,\widetilde{B} + \frac{1}{2}\iota_K\,[(F,C)_7 - F_0 C_{\mathrm{tw}7}] + (\mathrm{e}^{-\phi}\,\Phi, F)_6. \tag{4.26}$$

By recalling (3.22a), we can manipulate $(e^{-\phi}\Phi, F)_6$ a little bit

$$(\mathrm{e}^{-\phi}\,\Phi, F)_6 = \left[\mathrm{e}^{-\phi}\,\Phi\wedge\lambda(\mathrm{d}_H C + F_0\,\mathrm{e}^{B})\right]_6 = -\,\mathrm{d}(\mathrm{e}^{-\phi}\,\Phi, C)_5 - ((\iota_K + \widetilde{K}\wedge)F, C)_6 + F_0\,\mathrm{e}^{-\phi}\,\Phi_{\mathrm{tw}6} \tag{4.27}$$

so as to get

$$\begin{aligned}
\mathrm{d}\left[\mathrm{e}^{-2\phi}\,\Omega + (\mathrm{e}^{-\phi}\,\Phi, C)_5 - \iota_K\widetilde{B}\right] &= \frac{1}{2}(\iota_K F, C)_6 - \frac{1}{2}(\iota_K C, F)_6 - ((\iota_K + \widetilde{K}\wedge)F, C)_6 \\
&\quad + F_0\left(\mathrm{e}^{-\phi}\,\Phi_{\mathrm{tw}} - \frac{1}{2}\iota_K C_{\mathrm{tw}}\right)_6 \\
&= \frac{1}{2}\mathrm{d}((\iota_K + \widetilde{K}\wedge)C, C)_5 + F_0\varphi_{\mathrm{D}6}\,,
\end{aligned} \tag{4.28}$$

where we have chosen the gauge $\mathcal{L}_K\widetilde{B} = \mathcal{L}_K C = 0$ and used (3.4). $\varphi_{\mathrm{D}6}$ is the D6-brane calibration ((4.21) with $p = 6$), which, being locally exact, can be written as $\varphi_{\mathrm{D}6} = \mathrm{d}\sigma_5$ where σ_5 is a five-form. Bringing everything on the left-hand side we get that the following differential form

$$\varphi_{\mathrm{NS}5} = e^{-2\phi}\Omega + (e^{-\phi}\Phi, C)_5 - \iota_K\widetilde{B} - \frac{1}{2}\widetilde{K}\wedge(C,C)_4 - \frac{1}{2}(\iota_K C, C)_5 + F_0\sigma_5\,, \tag{4.29}$$

is closed. The expression of $\varphi_{\mathrm{NS}5}$ in type IIB supergravity is the same with $F_0 = 0$. One can already notice some similarities between (3.22b) and NS5-brane calibration condition proposed in [55] restricted to the assumption of a four-dimensional Minkowski external space. In this spirit, we can also notice that the influence of D6-calibration in (4.29) is necessary for an anomaly-free NS5 in Romans-mass background, indeed when $F_0 = n$ is turned on we have that n D6-brane must end on a NS5 [86] in order to compensate the Romans mass charge, as can be shown by integrating the Bianchi identity $\mathrm{d}F_2 - F_0 H = \delta_3$ on a three-dimensional sphere surrounding the NS5 worldvolume. The interpretation of $\varphi_{\mathrm{NS}5}$ as NS5 calibration will be confirmed by exploiting the duality relations we have seen in section 1.3.

4.4.1 NS5 calibration in IIB from S-duality

In section 1.3.4 we have showed how branes are related under the various string dualities, in particular Sl(2,$\mathbb{R}$) symmetry in type IIB mixes D5 with NS5-branes. Using this transformation it is possible also to simply exchange a D5 with a NS5; this is performed by the S-duality (1.35). In this section we will show that S-dualizing the D5-brane calibration (4.21) with $p = 5$ we exactly get (4.29). This would be a non-trivial check that our hypothesis for the NS5-calibration is correct, at least in type IIB.

In the particular case of an S-duality we have, besides (1.35):

$$
\begin{aligned}
C'_4 &= C_4 - B \wedge C_2 \,, \\
C'_6 &= \widetilde{B} + \frac{1}{2}(C_0 C_6 + C_4 \wedge C_2 - C_2 \wedge C_2 \wedge B) \,,
\end{aligned}
\tag{4.30}
$$

and we will also need (2.46) and (2.47) with $\delta = 0$ and $\gamma = 1$. Straightforwardly applying these transformation rules to the D5-calibration we get

$$
\begin{aligned}
\varphi'_{\mathrm{D5}} &= \left(\mathrm{e}^{-B} \wedge (\mathrm{e}^{-\phi}\Phi - (\iota_K + \widetilde{K}\wedge)C)\right)'_5 = \mathrm{e}^{-2\phi}\,\widetilde{\Omega} + \mathrm{e}^{-\phi}\, C_0 \Phi_5 - \mathrm{e}^{-\phi}\, C_2 \wedge \Phi_3 + \mathrm{e}^{-\phi}\, \Phi_1 \wedge C_4 \\
&- C_0 \widetilde{K} \wedge C_4 + \frac{1}{2} C_2 \wedge C_2 \wedge \widetilde{K} - \iota_K \widetilde{B} - \frac{1}{2}(C_0\, \iota_K C_6 + C_4 \wedge \iota_K C_2 - \iota_K C_4 \wedge C_2) = \varphi_{\mathrm{NS5}} \,.
\end{aligned}
\tag{4.31}
$$

4.4.2 NS5 and D4 calibrations from M-theory

Even if in type IIA we don't have an internal symmetry like S-duality for IIB to check our interpretation of (4.29), we can use the fact that the M5-brane can be dimensional reduced to a D4 and a NS5 brane in type IIA, as described in subsection 1.3.4. We expect the same mechanism to work also for calibrations.

Adopting the same notation we have used in 1.3.3 and 2.4.3 when dealing with dimensional reduction from M-theory to type IIA, we will use a hat to distinguish eleven- from ten-dimensional objects.

The M5 calibration was derived in (4.25)

$$
\widehat{\varphi}_{\mathrm{M5}} \equiv \widehat{\Sigma} + \iota_{\widehat{K}} \widehat{C} + \widehat{A} \wedge \widehat{\Omega} + \frac{1}{2} \widehat{A} \wedge \iota_{\widehat{K}} \widehat{A} \,. \tag{4.32}
$$

We can now reduce (4.32) to IIA, by taking the dimensional-reduction dictionary of 1.3.3 and 2.4.3 to the letter, we get that M5 calibration reads

$$
\begin{aligned}
\widehat{\varphi}_{\mathrm{M5}} = &\; \mathrm{e}^{-2\phi}\, \Omega + \mathrm{e}^{-\phi}\, \Phi_4 \wedge C_1 + C_3 \wedge \widetilde{K} \wedge C_1 - \mathrm{e}^{-\phi}\, C_3 \wedge \Phi_2 - \iota_K \widetilde{B} \\
& - \frac{1}{2} \iota_K C_1 C_5 - \frac{1}{2} \iota_K C_5 \wedge C_1 + \mathrm{e}^{-\phi}\, \Phi_0 C_5 + \frac{1}{2} C_3 \wedge \iota_K C_3 \\
& - \Big(\mathrm{e}^{-\phi}\, \Phi_4 + C_3 \wedge \widetilde{K} - \iota_K C_5 + B \wedge \iota_K C_3 + B \wedge \widetilde{K} \wedge C_1 \\
& \quad - \mathrm{e}^{-\phi}\, B \wedge \Phi_2 + \frac{1}{2} \mathrm{e}^{-\phi}\, \Phi_0 B \wedge B - \frac{1}{2} \iota_K C_1 B \wedge B \Big) \wedge dx^{10}
\end{aligned}
\tag{4.33}
$$

which is nothing but

$$\widehat{\varphi}_{\rm M5} = \varphi_{\rm NS5} - \varphi_{\rm D4} \wedge dx^{10}\,, \tag{4.34}$$

where $\varphi_{\rm NS5}$ is the (type IIA) NS5-calibration introduced in (4.29) with $F_0 = 0$ and $\varphi_{\rm D4}$ is the D4 calibration, as defined in (4.21).

4.4.3 NS5-calibration condition consistency from T-duality

The last non-trivial test before concluding this section is to prove that the longitudinal (respect to the isometric direction) part of $\varphi_{\rm NS5}$ is transformed into itself after a T-duality, which is what we expect from the discussion 1.3.4. For simplicity, we will perform this check on the calibration condition instead of T-dualizing the calibration itself. By decomposing (3.22b) using the notation of (1.30) and (2.43) we get that the longitudinal part reads:

$$\mathrm{d}(e^{-2\phi+C}\omega_4) = e^{-2\phi+C}\iota_{k_1} *_9 (\mathrm{d}B_2 - \mathrm{d}B_1 \wedge A_1) + e^{-\phi+C}((\Phi_\parallel, F_\perp)_6 - (\Phi_\perp, F_\parallel)_6)\,, \tag{4.35}$$

where m runs from 0 to 8. Using (1.31) and (2.45) it is immediate to see that this equation is invariant.

4.5 NS9-brane calibration

As we anticipated in section 1.2, the analysis of the central charges of type II theories reveals the existence of a nine-brane called NS9 [64, Sec. 6], which we defined in type IIB as the S-dual of the D9-brane (subsection 1.3.4). Since a nine-brane is space-filling, a calibration would not tell us where it should sit. Nevertheless, in IIB we can extend formally the calibrations for Dp-branes (4.21) to $p = 9$, and using S-duality we can infer the corresponding nine-form for the NS9.

From (3.29) we can see that the S-dual of the D9 calibration condition is the first equation of (3.1e)

$$\mathrm{d}(\mathrm{e}^{-2\phi} * \widetilde{K}) = 0, \tag{4.36}$$

which is present in both IIA and IIB. So $\mathrm{e}^{-2\phi} * \widetilde{K}$ may be interpreted as the NS9 calibration for type IIB. To check if the same form is the NS9 calibration for type IIA we use T-duality. Imposing an U(1) isometry and using the decomposition (2.45) we get

$$\mathrm{d}(e^{-2\phi} * \widetilde{K}) = \mathrm{e}^{-C}\,\mathrm{d}(\mathrm{e}^{-2\phi+C} * \widetilde{k}_1) \wedge E^y + \mathrm{e}^{-2\phi+C} * \widetilde{k}_1 \wedge \mathrm{d}A_1 - \mathrm{d}(\mathrm{e}^{-2\phi}\,\widetilde{k}_0) \wedge *_9 1. \tag{4.37}$$

Notice that the last two terms are zero because they are ten-forms on a nine-dimensional subspace, so we have just

$$\mathrm{d}(e^{-2\phi+C} * \widetilde{k}_1) = 0\,, \tag{4.38}$$

which is invariant under T-duality as one can check from (2.43). So $\mathrm{d}(e^{-2\phi} * \widetilde{K}) = 0$ in IIB transforms in the same equation in IIA and vice-versa. Therefore we define

$$\varphi_{\mathrm{NS9}} = \mathrm{e}^{-2\phi} * \widetilde{K} \tag{4.39}$$

for both IIA and IIB supergravity.

4.6 Gravitational calibrations and KK-monopole

There is a source we didn't consider so far, which is the KK-monopole. Even if this is a peculiar object for the reason explained in subsection 1.3.4, it appears in the superalgebra (and thus in the BPS bound) of every supergravity theory with $d \geq 5$ [64], so we expect that a central charge which is topologically conserved thanks to a calibration condition exists also for it.

This issue is complicated by the fact that for the KK-monopole no duality suggests explicitly a good candidate for its calibration condition among equations in (3.1) and moreover finding a new equation satisfying a closure condition turns out to be elusive so far. However, using the formalism developed in 4.1, we expect that it is possible to express at least the central charge of a KK-monopole in a similar way to equation (4.15). Even if this is in principle true, we have to find another approach to face this problem. Since the action for the KK-monopole is not well defined from first principles [52], we are forced to use the definition of central charges which comes from the supergravity algebra.

In order to deal with this issue M-theory is a more convenient arena compared to type II supergravity, so we will work in eleven dimensions.

4.6.1 Gravitational BPS bound in M-theory

Following [64], let's consider a family of backgrounds in M-theory defined via the supersymmetry conditions generated by ϵ which fixes the fields g, A

$$\left(\nabla_M - \frac{1}{12}\iota_M(*F + 2F)\right)\epsilon = \mathcal{D}_M\epsilon = 0\,. \tag{4.40}$$

Let's suppose that such a family tends to some asymptotic configuration $g^{(0)}, A^{(0)}$ such that

$$\mathcal{D}^{(0)}_M \epsilon|_{\mathcal{B}} = 0 \tag{4.41}$$

where $\mathcal{B}$ is the nine-dimensional spatial boundary. There are various way to define such a configuration as explained in [64], however we will avoid all these subtleties and we restrict ourselves to a more formal discussion.

Let's now define the following operator

$$T_M = \mathscr{D}_M - \mathscr{D}^{(0)}_M \tag{4.42}$$

which is nothing but some linear combination of tensors contracted with gamma matrices. By following [87, 64], the supercharge associated with a Killing spinor ϵ which satisfies the boundary conditions (4.41) takes the form

$$Q[\epsilon] = \int_{\mathscr{B}} \bar{\epsilon}\,\Gamma_{N_1\dots N_8}\psi_M \,\mathrm{d}x^M \wedge \mathrm{d}x^{N_1} \wedge \cdots \wedge \mathrm{d}x^{N_8} \tag{4.43}$$

and, up to normalization, we can write

$$\{Q(\epsilon), Q(\epsilon)\} = \int_{\mathscr{B}} \mathrm{d}x^M \wedge \bar{\epsilon}\,\Gamma_{(8)}\mathscr{D}_M \epsilon = \int_{\mathscr{B}} \mathrm{d}x^M \wedge \bar{\epsilon}\,\Gamma_{(8)}\, T_M \epsilon = \int_{\mathscr{B}} \mathrm{d}x^M \wedge (T_M \epsilon\bar{\epsilon})_8 \,. \tag{4.44}$$

this expression typically leads to the BPS bound since

$$\{Q(\epsilon), Q(\epsilon)\} = \mathscr{P}[K] - \sum_a Z_a \geq 0 \tag{4.45}$$

where Z_a are central charges. Our purpose now is to manipulate (4.44) trying to rewrite the central charges Z_a as in (4.15), with a corresponding calibrations φ_a and fluxes F_a.

From (1.6) we get that T_M reads

$$T_M = \frac{1}{4}\Delta\omega_M - \frac{1}{12}\iota_M (\Delta * F + 2\Delta F)\,, \tag{4.46}$$

where $\Delta F = F - F^{(0)}$ and $\Delta\omega_M = (\omega_M^{AB} - \omega_M^{AB\,(0)})\gamma_{AB}$ is a difference of spin connections, and hence a tensor at the boundary.

Plugging (4.46) into (4.44) and using (A.18) we obtain

$$\{Q(\epsilon), Q(\epsilon)\} = \frac{1}{4}\int_{\mathscr{B}} *(\mathrm{d}x^{AB} \wedge K) \wedge \Delta\omega_{AB} + \frac{1}{4}\int_{\mathscr{B}} \left[\Omega \wedge \Delta C - (\Sigma + \Omega \wedge A) \wedge \Delta F - *\Sigma \wedge \Delta\omega\right]. \tag{4.47}$$

Some comments are now in order. First of all we have used (1.2b)

$$\mathrm{d}\Delta * F + F \wedge \Delta F = 0\,, \tag{4.48}$$

to define the variation of the M5-brane potential

$$\Delta C = \Delta * F + A \wedge \Delta F\,. \tag{4.49}$$

Secondly, we have introduced the spin-connection three-form

$$\Delta\omega = \frac{1}{2}\Delta\omega_{MAB}\,\mathrm{d}x^M \wedge E^A \wedge E^B\,. \tag{4.50}$$

Notice now that first term is exactly the ADM momentum $P[K]$ as defined in [64, (3.2)], which we interpret as the equivalent of the gauge-invariant momentum (4.7), as one can see from the absence of form potentials. So the BPS bound which results from (4.47) is not written as (4.11) but must be interpreted as in (4.10). For this reason, it is not surprising that in the second and third term of (4.47) the M2 and M5 calibrations appear as in section 4.3 but without the contribution which comes from the Wess–Zumino part of the action.

Since we have correctly located the central charge contributions corresponding to all the M-theory solitons except for the KK-monopole, we are allowed to interpret the last term in (4.47) as the KK6 central charge. Indeed, following again [64], we can interpret the spin-connection three-form as the flux sourced by the KK6-monopoles, since its integral corresponds to the NUT charge in the case of a Taub–NUT solution. We are then led to recognize $*\Sigma$ as part of the calibration for KK6-monopoles in M-theory. However, identifying a topological central charge as in (4.15) is still difficult since the KK6-monopole appears to be mixed with the ADM momentum in the BPS bound (4.47).

In the following subsection we will see that this interpretation of the KK6 charge is consistent with dimensional reduction to IIA, which relates M-theory KK6-monopole to IIA KK5-monopole and D6-branes. This will allow us to identify (part of) the KK5 calibration in IIA.

4.6.2 Type II KK-monopole calibrating forms from dualities

We will now perform a dimensional reduction of the last term in (4.47).

Let us reintroduce the hat to distinguish M-theory quantities. M-theory KK6 calibrating form $\hat{*}\widehat{\Sigma}$ decomposes as in (2.49), while the associated geometric flux $\Delta\widehat{\omega}$ reduces to

$$\Delta\widehat{\omega} = \mathrm{e}^{-\frac{2}{3}\phi}\,\Delta\omega_{10} - \frac{1}{2}\,\mathrm{e}^{\frac{4}{3}\phi}\,\Delta F_2 \wedge (\mathrm{d}x^{10} - C_1). \tag{4.51}$$

From this we obtain

$$*_{11}\,\widehat{\Sigma} \wedge \Delta\widehat{\omega} = \left(\mathrm{e}^{-2\phi}\,\widetilde{\Omega} \wedge \Delta\omega_{10} + \frac{\mathrm{e}^{-\phi}}{2}\Phi_6 \wedge \Delta F_2\right) \wedge \mathrm{d}x^{10} + \ldots \tag{4.52}$$

where dots denote terms annihilated by $\iota_{\partial_{10}}$. Since $\mathrm{d}x^{10}$ must be a direction on the boundary to be shrunk, we have that all the terms that do not contain it cannot appear in the integral (4.47), and so can be neglected. Therefore we conclude that the last term in (4.47) reads

$$\int_{\widehat{\mathcal{B}}} \hat{*}\widehat{\Sigma} \wedge \Delta\widehat{\omega}_{11} = \int_{\mathcal{B}} \left(\mathrm{e}^{-2\phi}\,\widetilde{\Omega} \wedge \Delta\omega_{10} + \frac{1}{2}\,\mathrm{e}^{-\phi}\,\Phi_6 \wedge \Delta F_2\right), \tag{4.53}$$

where we have integrated off the S^1 of the M-theory nine-dimensional boundary $\widehat{\mathcal{B}} \simeq S^1 \times \mathcal{B}$ assuming that the circle radius was equal to $1/2\pi$. Both terms on the right-hand side of (4.53) are of the form (4.15). In the last term in (4.53), $e^{-\phi}\Phi_6$ is gauge-invariant contribution to the D6 calibration $\varphi_{\rm D6}$, see (4.21). On the other hand, in analogy with the M-theory case, we are led to identify

$$e^{-2\phi}\widetilde{\Omega} \tag{4.54}$$

with part of the type IIA KK5 calibration.

The analogous KK5 calibrating form for IIB can be obtained from T-duality, since, as discussed in 1.3.4, a transverse T-duality maps a KK5-monopole into a NS5-brane and vice-versa, while under a longitudinal one both KK5 and NS5 remain invariant. Using (2.43), it is immediate to notice that this is exactly what it happens if we identify $e^{-2\phi}\widetilde{\Omega}$ with the KK5 calibration in IIB as well.

Given this hint one may be immediately led to conclude that (3.22c) is the calibration condition for the KK5-monopole. However, since this equation cannot be recast as a closure condition, which would ensure that the monopole charge is topological, it is not possible to confirm such a statement. A proper definition of the canonical (instead of gauge-invariant) momentum in (4.47) and a study of how the second integral turns out to be topological may shed light on this issue.

CHAPTER 5

APPLICATIONS

So far we have discussed the general implication of the system (3.22) regarding geometrical properties of a supersymmetric solution and of the extended objects which leave inside it. However, we can also take advantage of the bispinorial formalism to classify solutions inside a given ansatz on the external geometry. In the present work we are mostly interested in vacuum solutions.

In this case we expect that the maximal amount of symmetry compatible with external spacetime dimensions is preserved. In order this to happen, a series of assumptions on fluxes and spinors are needed. For example the external metric must be a warped product

$$\mathrm{d}s^2_{10} = \mathrm{e}^{2A}\,\mathrm{d}s^2_{\mathrm{ext}} + \mathrm{d}s^2_{\mathrm{int}} \tag{5.1}$$

where A is a function with support on the internal space, fluxes must not contain any leg on the external space except for the volume form:

$$F = f_1 + \mathrm{Vol}_{\mathrm{ext}} \wedge f_2\,, \qquad H = H_1 + \mathrm{Vol}_{\mathrm{ext}} \wedge H_2\,, \tag{5.2}$$

and the external components of the spinors must be Killing respect to the external covariant derivative

$$\nabla_\mu \alpha = c\gamma_\mu \alpha \tag{5.3}$$

where c is a constant and μ points just in external directions.

The two main applications of (3.22) and (3.23) we will present here are the classification of near-horizion solutions and of flat-space backgrounds in four dimensions.

5.1 AdS$_2$ near-horizons

Supersymmetric solutions with a time-like Killing vector are suitable to describe static space-time and in particular can be used to study black holes. Up to now, a classification of backgrounds with singularities is far from being complete and the classification of vacuum solutions teaches us that approaching the problem using bispinors can be the key-ingredient to solve it. In this spirit, we present here a first step in this direction. Instead of looking at a full black-hole solution, we restrict ourselves to the near-horizon geometry which can be viewed as an $\mathrm{AdS}_2 \times M_8$ vacuum where M_8 is typically a fibration of a compact manifold M_6 over S^2. Classifications of black-hole horizons in a similar spirit were given in [88, 89].

5.1.1 $\mathrm{AdS}_2 \times M_8$ Ansatz

Following the logic we have presented in the introduction of this chapter, to preserve the isometry of AdS_2 we have to split the metric as

$$\mathrm{d}\,s^2_{10} = \mathrm{e}^{2A}\,\mathrm{d}\,s^2_{\mathrm{AdS}_2} + \mathrm{d}\,s^2_{M_8}\,, \tag{5.4}$$

while fluxes read

$$\begin{aligned} &H = H_3 + \mathrm{e}^{2A}\mathrm{Vol}(\mathrm{AdS}_2)\wedge H_1\,, \qquad *H = *_8 H_1 + \mathrm{e}^{2A}\mathrm{Vol}(\mathrm{AdS}_2)\wedge *_8 H_3\,,\\ &F = f + \mathrm{e}^{2A}\mathrm{Vol}(\mathrm{AdS}_2)\wedge *_8\lambda(f)\,, \end{aligned} \tag{5.5}$$

where we can see that the self-duality condition of F (1.13) is already satisfied.

Let's now consider spinors. (5.4) suggests the following gamma-matrix decomposition

$$\begin{aligned} \gamma^{(10)}_\mu &= e^A\sigma_\mu\otimes\mathbb{1}_{16} \qquad \mu = 0,1\,,\\ \gamma^{(10)}_m &= \sigma_3\otimes\gamma_m \qquad m = 2,\dots,9\,, \end{aligned} \tag{5.6}$$

where $\sigma_0 = \mathrm{i}\sigma_2$ are Pauli matrices and A is a function on M_8. The most general spinor ansatz reads

$$\begin{aligned} \epsilon_1 &= \alpha_+\otimes\eta_{1\,+} + \alpha_-\otimes\eta_{1\,-} = P_+(\alpha\otimes\eta_1)\\ \epsilon_2 &= \alpha_+\otimes\eta_{2\,\mp} + \alpha_-\otimes\eta_{2\,\pm} = P_\mp(\alpha\otimes\eta_2) \end{aligned} \tag{5.7}$$

where $\alpha = \alpha_+ + \alpha_-$ is a real Killing spinor on AdS_2, $P_\pm$ are the chiral projectors and $\eta_i = \eta_{i\,+} + \eta_{i\,-}$ are Majorana spinors on M_8, that we can take to be real. In general, not every spinor Ansatz in two dimensions leads to the timelike case; for example, if we set to zero α_- (or equivalently α_+) as done in [90] we will get a solution which is light-like and therefore we can not apply our system (3.22). However supersymmetry forbids such a situation in the case of AdS_2 vacua because the Killing spinor α cannot be chiral as showed in (5.3). From this observation it is easy to see that spinorial BPS conditions (1.24) kill all the possible solutions like the one in [90].

Since a near-horizon must be invariant under the isometry group $\mathrm{SO}(2,1)$, all the possible choices of the AdS_2 Killing spinor α are equivalent. For definiteness we take [91]

$$\alpha = \mathrm{e}^{r/2}\begin{pmatrix}1\\1\end{pmatrix}, \qquad \mathrm{d}\,s^2_{\mathrm{AdS}_2} = \mathrm{e}^{2r}\,\mathrm{d}\,t^2 + \mathrm{d}\,r^2\,; \tag{5.8}$$

we can use this for computing the spinor bilinears on AdS_2

$$\alpha \otimes \overline{\alpha} = -\mathrm{e}^{2r}\,\mathrm{d}t + \mathrm{e}^{2r}\,\mathrm{d}t \wedge \mathrm{d}r\,, \qquad \sigma_3 \alpha \otimes \overline{\alpha} = -\mathrm{e}^{r} + \mathrm{e}^{r}\,\mathrm{d}r\,. \tag{5.9}$$

Notice that α by itself it is enough to define a vielbein (i.e. an identity structure) on AdS_2.

On the other hand two Majorana spinors on M_8 do not determine an identity structure; using the discussion in subsection 2.2.1 we can see that the structure group is given by the intersection of two G_2-structure, so it turns out to be more convenient just to rename the eight-dimensional bilinears

$$\begin{array}{llll} \omega_1 = \eta_1 \eta_1^t\,, & \omega_2 = \eta_2 \eta_2^t\,, & \omega = (\omega_1 + \omega_2)/2\,, & \widetilde{\omega} = (\omega_1 - \omega_2)/2\,, \\ \omega_1^\gamma = \gamma \eta_1 \eta_1^t\,, & \omega_2^\gamma = \gamma \eta_2 \eta_2^t\,, & \omega_\gamma = (\omega_1^\gamma + \omega_2^\gamma)/2\,, & \widetilde{\omega}_\gamma = (\omega_1^\gamma - \omega_2^\gamma)/2\,, \\ \psi = \eta_1 \eta_2^t\,, & & \psi_\gamma = \gamma \eta_1 \eta_2^t & \end{array} \tag{5.10}$$

giving special names to the zero and one-form part, as we have done in ten dimensions:

$$\begin{array}{llll} a = (\omega)_0\,, & \widetilde{a} = (\widetilde{\omega})_0\,, & a_\gamma = (\omega_\gamma)_0\,, & \widetilde{a}_\gamma = (\widetilde{\omega}_\gamma)_0\,, \\ k_1 = (\omega_1)_1\,, & k_2 = (\omega_2)_1\,, & k = (\omega)_1\,, & \widetilde{k} = (\widetilde{\omega})_1 \end{array} \tag{5.11}$$

where the subscript 0 and 1 indicates to take the zero and one-form part only and γ is the chiral operator on M_8.

Now we have all the elements to explicitly compute the ten-dimensional bilinears of section 2.2.1 in terms of two- and eight-dimensional ones. We will report here the calculation for K and $\widetilde{K}$ in all the details, while for the other bilinears one can proceed by analogy:

$$\begin{aligned} 32K_1 =& (\alpha \otimes \eta_1)^t P_+^t \gamma_{\underline{0}}^{(10)} \gamma_M^{(10)} P_+ (\alpha \otimes \eta_1) E^M = (\alpha \otimes \eta_1)^t \gamma_{\underline{0}}^{(10)} \gamma_M^{(10)} P_+ (\alpha \otimes \eta_1) E^M \\ =& (\alpha \otimes \eta_1)^t \gamma_{\underline{0}}^{(10)} \gamma_M^{(10)} (\alpha \otimes \eta_1) E^M + (\alpha \otimes \eta_1)^t \gamma_{\underline{0}}^{(10)} \gamma_M^{(10)} \gamma (\alpha \otimes \eta_1) E^M \\ =& \frac{\mathrm{e}^A}{2} \left(\eta_1^t \eta_1\, \overline{\alpha} \sigma_\mu \alpha\, e^\mu + \eta_1^t \gamma \eta_1\, \overline{\alpha} \sigma_\mu \sigma_3 \alpha\, e^\mu \right) + \frac{1}{2} \overline{\alpha} \sigma_3 \alpha (16 k_1)\,, \end{aligned} \tag{5.12}$$

where e^μ is the vielbein on AdS_2. Using (5.9) we get:

$$32K_1 = -\mathrm{e}^{r+A} (\eta_1^t \eta_1 e^0 - \eta_1^t \gamma \eta_1 e^1) - \mathrm{e}^r (16 k_1)\,. \tag{5.13}$$

Performing the same steps for K_2

$$32K_2 = -\mathrm{e}^{r+A} (\eta_2^t \eta_2 e^0 + \eta_2^t \gamma \eta_2 e^1) - \mathrm{e}^r (16 k_2) \tag{5.14}$$

and from the sum and the difference of these expressions we can calculate K and $\widetilde{K}$:

$$K = -\frac{\mathrm{e}^{r+A}}{2} \left(a e^0 - \widetilde{a}_\gamma e^1 \right) - \frac{\mathrm{e}^r}{2} k\,, \qquad \widetilde{K} = -\frac{\mathrm{e}^{r+A}}{2} \left(\widetilde{a} e^0 - a_\gamma e^1 \right) - \frac{\mathrm{e}^r}{2} \widetilde{k}\,. \tag{5.15}$$

Following a similar logic for Ω and $\widetilde{\Omega}$ we get:

$$\begin{aligned}\Omega &= -\frac{e^r}{2}\left((\omega)_5 + e^A e^0 \wedge (\omega)_4 - e^A e^1 \wedge (\widetilde{\omega}_\gamma)_4 - e^{2A} e^0 \wedge e^1 \wedge (\widetilde{\omega}_\gamma)_3\right),\\ \widetilde{\Omega} &= -\frac{e^r}{2}\left((\widetilde{\omega})_5 + e^A e^0 \wedge (\widetilde{\omega})_4 - e^A e^1 \wedge (\omega_\gamma)_4 - e^{2A} e^0 \wedge e^1 \wedge (\omega_\gamma)_3\right),\end{aligned} \tag{5.16}$$

and, finally, Φ reads:

$$\Phi = -\frac{e^r}{2}\left[(\psi_\gamma)_+ - e^A e^0 \wedge (\psi_\gamma)_- + e^A e^1 \wedge (\psi)_- - e^{2A} e^0 \wedge e^1 \wedge (\psi)_+\right], \tag{5.17}$$

where the subscripts + and − indicate to take the even or the odd forms degree respectively.

5.1.2 Supersymmetry conditions

In the previous section we have calculated how all the ten-dimensional quantity reduces after an AdS$_2 \times M_8$ ansatz. Now we are ready to derive the supersymmetry conditions for this class by just plugging the expressions for fluxes and bispinors in (3.22). From (3.22a) we get:

$$d_{H_3}(e^{-\phi}\psi_\gamma)_+ = -(\widetilde{k} + \iota_k) f\,, \tag{5.18a}$$
$$d_{H_3}(e^{A-\phi}\psi_\gamma)_- = e^A(\widetilde{a} f + \widetilde{a}_\gamma *_8 \lambda(f))\,, \tag{5.18b}$$
$$d_{H_3}(e^{A-\phi}\psi)_- = -e^A(a_\gamma f - a *_8 \lambda(f)) - (e^{-\phi}\psi_\gamma)_+\,, \tag{5.18c}$$
$$d_{H_3}(e^{2A-\phi}\psi)_+ = e^{2A}(\widetilde{k} + \iota_k) *_8 \lambda(f) + 2(e^{A-\phi}\psi_\gamma)_- - H_1 \wedge (e^{2A-\phi}\psi_\gamma)_+\,, \tag{5.18d}$$

from (3.22b)

$$e^{2\phi} d(e^{-2\phi}\omega)_5 = -\iota_k *_8 H_1 + (e^\phi \psi_\gamma, f)_6\,, \tag{5.19a}$$
$$e^{2\phi-A} d(e^{-2\phi+A}\omega)_4 = -\widetilde{a}_\gamma *_8 H_3 - e^\phi(\psi_\gamma, f)_5\,, \tag{5.19b}$$
$$e^{2\phi-A} d(e^{-2\phi+A}\widetilde{\omega}_\gamma)_4 = -a *_8 H_3 - e^\phi(\psi, f)_5 + e^{-A}(\omega)_5\,, \tag{5.19c}$$
$$e^{2\phi-2A} d(e^{-2\phi+2A}\widetilde{\omega}_\gamma)_3 = \iota_k *_8 H_3 + e^\phi[(\psi, f)_4 + (\psi_\gamma, *_8\lambda(f))_4] - 2e^{-A}(\omega)_4\,, \tag{5.19d}$$

and in the end from (3.22c)

$$e^{2\phi} d(e^{-2\phi}\widetilde{\omega})_5 = -\iota_{\widetilde{k}} *_8 H_1 - \frac{e^\phi}{2}(\psi_\gamma^m, f_m)_6\,, \tag{5.20a}$$
$$e^{2\phi-A} d(e^{-2\phi+A}\widetilde{\omega})_4 = -a_\gamma *_8 H_3 - \frac{e^\phi}{2}[(\psi_\gamma^m, f_m)_5 - (\psi, *_8\lambda(f))_5]\,, \tag{5.20b}$$
$$e^{2\phi-A} d(e^{-2\phi+A}\omega_\gamma)_4 = -\widetilde{a} *_8 H_3 - \frac{e^\phi}{2}[(\psi^m, f_m)_5 + (\psi_\gamma, *_8\lambda(f))_5] + e^{-A}(\widetilde{\omega})_5\,, \tag{5.20c}$$
$$e^{2\phi-2A} d(e^{-2\phi+2A}\omega_\gamma)_3 = \iota_{\widetilde{k}} *_8 H_3 - \frac{e^\phi}{2}[(\psi^m, f_m)_4 + (\psi_\gamma^m, *_8\lambda(f)_m)_4] - 2e^{-A}(\widetilde{\omega})_4\,. \tag{5.20d}$$

Let's now reduce also last line of (3.22). In this case we have a scalar and a ten-form equation, so we will get just one equation on M_8 for each of them

$$\mathcal{L}_k\phi = 0\,, \quad \mathrm{e}^{-2A}\mathrm{d}(\mathrm{e}^{2A} *_8 \tilde{k}) = 2\,\mathrm{e}^{-A}\,a_\gamma \mathrm{Vol}_8 - \frac{\mathrm{e}^{\phi}}{4}\left[(\psi_\gamma, (3-\deg) *_8 \lambda(f)) - (\psi, (5-\deg) f)\right]. \tag{5.21}$$

As discusses in subsection 3.2.1, to find a solution we have also to impose the Bianchi identities for F and H (1.16) and we have to solve one equation between (3.26) and (3.27).

5.2 $\mathbb{R}^{1,3}$ vacuum

The BPS conditions for four-dimensional vacuum solutions where calculated in [73] and are the so-called pure spinor equations. In this section we will re-derive this result from (3.23) along the lines of [9].

5.2.1 $\mathbb{R}^{1,3} \times M_6$ Ansatz

Following the same procedure outlined in the beginning of this chapter, preserving Poincaré invariance imposes the following decomposition of the metric

$$\mathrm{d}s^2_{10} = e^{2A}\mathrm{d}s^2(\mathbb{R}^{1,3}) + \mathrm{d}s^2(M_6) \tag{5.22}$$

and of the RR fields

$$F = f + e^{4A}\mathrm{Vol}(\mathbb{R}^{1,3}) \wedge *_6 \lambda(f)\,, \tag{5.23}$$

where f is an internal form. Similarly H must be a three-form on M_6.

In four Lorentzian dimensions and in six Euclidean ones it is possible to impose Majorana or Weyl conditions, but not both. This means that, in order to have ten-dimensional spinors which are real and chiral, we have to adopt the following spinor ansatz:

$$\epsilon_1 = \zeta_+ \otimes \chi^1_+ + \zeta_- \otimes \chi^1_-\,, \qquad \epsilon_2 = \zeta_+ \otimes \chi^2_\mp + \zeta_- \otimes \chi^2_\pm\,. \tag{5.24}$$

where $\zeta_+ = \zeta_-^*$ and $\chi^i_+ = \chi^{i\,*}_-$, where the upper sign is for type IIA while the lower for IIB. The ten-dimensional gamma matrices can be decomposed as

$$\begin{aligned} \gamma^{(10)}_\mu &= e^A \gamma^{(4)}_\mu \otimes \mathbb{1}_8 \qquad \mu = 0, \ldots, 4\,, \\ \gamma^{(10)}_m &= \gamma^{(4)} \otimes \gamma_m \qquad m = 0, \ldots, 6\,, \end{aligned} \tag{5.25}$$

where the four-dimensional ones are taken to be real while the six-dimensional ones pure imaginary.

Let's now calculate bispinors in four and six dimensions. Using a bit of algebra one discovers that the four-dimensional ones read:

$$\zeta_+ \otimes \overline{\zeta}_+ = v + \mathrm{i} *_4 v\,, \qquad \zeta_+ \otimes \overline{\zeta}_+ = v \wedge w \tag{5.26}$$

where v is a real null vector, $w = w_1 + i w_2$ is a complex one and v, w_1, w_2 are all independent. For the six-dimensional bispinors it is convenient to adopt the following definition:

$$\begin{aligned} &\chi^1_+ \chi^2_+ = \phi_+\,, && \chi^1_+ \chi^2_- = \phi_-\,, \\ &\chi^i_+ \chi^{i\dagger}_+ = (1 - i *_6 \lambda)(\omega^i_0 + \mathrm{i}\omega^i_2)\,, && \chi^i_+ \chi^{i\dagger}_- = \omega^i_3 + \mathrm{i} *_6 \omega^i_3\,, \end{aligned} \tag{5.27}$$

where $\phi_\pm$ are complex self-dual forms while ω^i_k are real k-forms. $\phi_\pm$ are the so-called pure spinors; the name is due to the fact that by definition a spinor is pure if it is annihilated from half of the Clifford algebra. In six euclidean dimensions every spinor is pure, therefore also the bilinears $\phi_\pm$ can be considered as pure spinors on the generalized tangent bundle since they are annihilated by six generalized gamma matrices (which are the usual gamma matrices acting on the left and on the right of $\phi_\pm$ though (A.17)).

Again, we give names to the sum and difference of the forms generated by $\chi^i_+ \chi^{i\dagger}_+$:

$$\begin{aligned} \omega_k &= (\omega^1_k \pm \omega^2_k)/2\,, & \widetilde{\omega}_k &= (\omega^1_k \mp \omega^2_k)/2\,, \\ k_0 &= (\omega^1_0 + \omega^2_0)/2\,, & \widetilde{k}_0 &= (\omega^1_0 - \omega^2_0)/2\,. \end{aligned} \tag{5.28}$$

We have all the building blocks to re-express the ten-dimensional bilinears in terms of the ones defined on $\mathbb{R}^{1,3}$ and M_6

$$\begin{aligned} \Phi &= 2\mathrm{Re}\left(\mp(e^A v + \mathrm{i} e^{3A} *_4 v) \wedge \phi_\mp + e^{2A} v \wedge w \wedge \phi_\pm\right)\,, \\ K &= 2e^{-A} k_0 \partial_v\,, \qquad \widetilde{K} = 2e^A \widetilde{k}_0 v\,, \\ \widetilde{\Omega} &= 2\mathrm{Re}\left(-e^A v \wedge *_6 \widetilde{\omega}_2 - e^{3A} *_4 v \wedge \widetilde{\omega}_2 + e^{2A} v \wedge w \wedge \omega_3\right)\,. \end{aligned} \tag{5.29}$$

Since K is proportional to v which is a null vector, we have that the classification of $\mathbb{R}^{1,3}$ vacuum falls in the light-like case, and therefore we cannot use the system (3.22).

5.2.2 Supersymmetry conditions

Let's now derive the supersymmetry conditions for a four-dimensional Minkowski solution. Since we are in the null case, (3.23) must be adopt. Notice that the Killing spinor equation (5.3) is satisfied on $\mathbb{R}^{1,3}$ just for $c = 0$, therefore the external space spinors are constant and for this reason the external derivative annihilates all their bilinears (5.26).

In [9] it is shown that (3.23c) and (3.23d) are automatically solved if one chooses $e_{+1,2}$ to point in the external space directions. So we are left with just the first two equations, which split in the following system on the internal space

$$k_0 = \frac{c_+}{4}\,\mathrm{e}^{A}\,, \qquad \widetilde{k}_0 = \frac{c_-}{4}\,\mathrm{e}^{-A}\,, \tag{5.30a}$$

$$\mathrm{d}_H(\mathrm{e}^{2A-\phi}\,\phi_\pm) = 0\,, \tag{5.30b}$$

$$\mathrm{d}_H\,\mathrm{Re}(\mathrm{e}^{A-\phi}\,\phi_\mp) = \frac{c_-}{8}\,f\,, \tag{5.30c}$$

$$\mathrm{d}_H\,\mathrm{Im}(\mathrm{e}^{3A-\phi}\,\phi_\mp) = \frac{c_+}{8}\,\mathrm{e}^{4A}\,*_6\lambda(f)\,. \tag{5.30d}$$

In future applications we will often consider the equal norm case $c_- = 0$ while we will fix $c_+ = 1$. Moreover, in this situation it is also convenient to manifestly write the spinor norm in the spinor ansatz (5.24), which means $\chi^{1,2}_\pm \mapsto \mathrm{e}^{A/2}\,\chi^{1,2}_\pm$, so that we get normalized pure spinors which does not depend from the warping function A. With these hypothesis (5.30) reads:

$$\mathrm{d}_H(\mathrm{e}^{3A-\phi}\,\phi_\pm) = 0\,, \tag{5.31a}$$

$$\mathrm{d}_H\,\mathrm{Re}(\mathrm{e}^{2A-\phi}\,\phi_\mp) = 0\,, \tag{5.31b}$$

$$\mathrm{d}_H\,\mathrm{Im}(\mathrm{e}^{4A-\phi}\,\phi_\mp) = \frac{1}{8}\,\mathrm{e}^{4A}\,*_6\lambda(f)\,. \tag{5.31c}$$

We will extensively use this system in the next part of the thesis, when we will deal with the classification of $\mathbb{R}^{1,3}\times S^2$ backgrounds.

5.2.3 Sl$(2,\mathbb{R})$-invariant supersymmetry conditions

In the previous section we have seen that (3.23c) and (3.23d) do not play a role in the determination of supersymmetry constraints for four-dimensional vacuum, so it is possible to make the remaining part of (3.23) Sl$(2,\mathbb{R})$-covariant using the results in 3.2.2. The outcome of this operation is the following system

$$\mathcal{L}_K g_\mathrm{E} = 0\,, \tag{5.32a}$$

$$\mathrm{d}_Q\,\Theta_1 - \frac{\mathrm{i}}{2}\,\mathrm{e}^{\phi}\,\mathrm{d}\overline{\tau}\wedge\overline{\Theta}_1 + \mathrm{i}\,\iota_K\overline{G}_3 = 0\,, \tag{5.32b}$$

$$\mathrm{d}\Theta_3 + \iota_K F_5 + \mathrm{Re}\big(\Theta_1\wedge G_3\big) = 0\,, \tag{5.32c}$$

$$\mathrm{d}_Q\,\Theta_5 + \frac{\mathrm{i}}{2}\,\mathrm{e}^{\phi}\,\mathrm{d}\overline{\tau}\wedge\overline{\Theta}_5 + \Theta_3\wedge\overline{G}_3 - \mathrm{i}\iota_K(*_\mathrm{E}\,\overline{G}_3) + \mathrm{i}\Theta_1\wedge F_5 = 0\,, \tag{5.32d}$$

$$\mathrm{d}*_\mathrm{E}\Theta_3 + \frac{1}{2}\mathrm{Re}\,(G_3\wedge\Theta_5 - *_\mathrm{E}\,G_3\wedge\Theta_1) = 0 \tag{5.32e}$$

which must be supplemented with the algebraic constraint:

$$\overline{G}_3\wedge\Theta_5 - \Theta_1\wedge *_\mathrm{E}\overline{G}_3 + 2\,\mathrm{e}^{\phi}\,\iota_K *_\mathrm{E}\mathrm{d}\overline{\tau} + 2\mathrm{i}\,\mathrm{e}^{\phi}\,\mathrm{d}\overline{\tau}\wedge *_\mathrm{E}\Theta_3 = 0\,. \tag{5.33}$$

In the spirit of subsection 3.2.2, in order to make explicit the SL(2,$\mathbb{R}$)-invariant structure of the supersymmetry conditions it is convenient to use Einstein metric

$$g_{\rm E} = e^{2A_{\rm E}}\, g_4 + g_{6\,\rm E}\,, \qquad g_{\rm E} = e^{-\frac{\phi}{2}}\, g = e^{-\frac{\phi}{2}}\left(e^{2A}\, g_4 + g_6\right), \tag{5.34}$$

and to organize the components of the six-dimensional bilinears in terms of their $\mathrm{U}(1)_D$ charges. In particular we have five real neutral forms

$$\alpha_0 = e^{-\frac{1}{4}\phi}\,\mathrm{Im}(\phi_+)_0\,, \quad \alpha_2 = e^{-\frac{3}{4}\phi}\,\mathrm{Re}(\phi_+)_2\,, \quad c_0 = e^{-\frac{1}{4}\phi}\,\omega_0\,, \quad \alpha_1 = e^{-\frac{1}{2}\phi}\,\phi_-\,, \tag{5.35}$$

and three complex ones

$$\theta_0 = e^{-\frac{1}{4}\phi}(\widetilde{k}_0 + i\mathrm{Re}(\phi_+)_0)\,, \quad \theta_2 = e^{-\frac{3}{4}\phi}(\omega_2 + i\mathrm{Im}(\phi_+)_2)\,, \quad \theta_3 = e^{-\phi}(\widetilde{\omega}_3 + i\mathrm{Re}(\phi_-)_3)\,, \tag{5.36}$$

whose of $\mathrm{U}(1)_D$ charge equal to $+1$, as tabulated in 5.1.

fields	$\mathrm{U}(1)_D$-charge
α_0, α_1, α_2, c_0, f_5	0
θ_0, θ_2 , θ_3	1

Table 5.1: $\mathrm{U}(1)_D$ charges of relevant fields on the internal manifold.

The ten-dimensional multiplets in terms of the six-dimensional ones read:

$$\begin{aligned}
\Theta_1 &= 2\, e^{A_{\rm E}}\, \theta_0\, v\,,\\
\Theta_3 &= 2\left(e^{A_{\rm E}}\, v \wedge \alpha_2 - e^{3A_{\rm E}} *_4 v\alpha_0 + e^{2A_{\rm E}}\, v \wedge w_1 \wedge \mathrm{Re}\alpha_1 - e^{2A_{\rm E}}\, v \wedge w_2 \wedge \mathrm{Im}\alpha_1\right),\\
\Theta_5 &= 2\left(-e^{A_{\rm E}}\, v \wedge *_{\rm E}\theta_2 - e^{3A_{\rm E}} *_4 v \wedge \theta_2 + e^{2A_{\rm E}}\, v \wedge w_1 \wedge \theta_3 - e^{2A_{\rm E}}\, v \wedge w_2 \wedge *_{\rm E}\theta_3\right)
\end{aligned} \tag{5.37}$$

where $*_{\rm E}$ is a shorthand for $*_{6,E}$. We apply the same logic also to redefine fluxes:

$$G_3 = f_3 - i e^{-\phi}\, H\,, \qquad \tau = C_0 + i e^{-\phi}\,, \qquad F_5 = f_5 + e^{4A_{\rm E}}\,\mathrm{Vol}_4 \wedge *_{\rm E} f_5\,. \tag{5.38}$$

Now it is enough to substitute (5.37) in (5.32) to get the SL(2,$\mathbb{R}$) invariant conditions for four-dimensional vacua. Notice that to write the system in SL(2,$\mathbb{Z}$)-invariant form we have to expand all form degrees separately, in fact the total number of form equations is the same as in (5.30), even if in a much non-compact form. The result of this operation is the following system:

$$\mathrm{d}_Q(\mathrm{e}^{A_\mathrm{E}}\,\theta_0) - \frac{\mathrm{i}}{2}\,\mathrm{e}^{\phi+A_\mathrm{E}}\,\overline{\theta}_0\,\mathrm{d}\overline{\tau} = 0\,, \qquad c_0 = c_+\,\mathrm{e}^{A_\mathrm{E}}\,, \tag{5.39a}$$

$$\mathrm{d}(\mathrm{e}^{A_\mathrm{E}}\,\alpha_2) - \mathrm{e}^{A_\mathrm{E}}\,\mathrm{Re}(\theta_0 G_3) = 0\,, \qquad \mathrm{d}(\mathrm{e}^{2A_\mathrm{E}}\,\alpha_1) = 0\,, \tag{5.39b}$$

$$\mathrm{d}(\mathrm{e}^{3A_\mathrm{E}}\,\alpha_0) - c_0\,\mathrm{e}^{3A_\mathrm{E}}\,*_\mathrm{E} f_5 = 0\,, \tag{5.39c}$$

$$\mathrm{d}_Q(\mathrm{e}^{A_\mathrm{E}}\,*_\mathrm{E}\theta_2) + \frac{\mathrm{i}}{2}\,\mathrm{e}^{\phi+A_\mathrm{E}}\,\mathrm{d}\overline{\tau}\wedge *_\mathrm{E}\overline{\theta}_2 + \mathrm{e}^{A_\mathrm{E}}\,\alpha_2\wedge\overline{G}_3 + \mathrm{i}\,\mathrm{e}^{A_\mathrm{E}}\,\theta_0 f_5 = 0\,, \tag{5.39d}$$

$$\mathrm{d}_Q(\mathrm{e}^{2A_\mathrm{E}}\,\theta_3) + \frac{\mathrm{i}}{2}\,\mathrm{e}^{\phi+2A_\mathrm{E}}\,\mathrm{d}\overline{\tau}\wedge\overline{\theta}_3 + \mathrm{e}^{2A_\mathrm{E}}\,\mathrm{Re}\alpha_1\wedge\overline{G}_3 = 0\,, \tag{5.39e}$$

$$\mathrm{d}_Q(\mathrm{e}^{2A_\mathrm{E}}\,*_\mathrm{E}\theta_3) + \frac{\mathrm{i}}{2}\,\mathrm{e}^{\phi+2A_\mathrm{E}}\,\mathrm{d}\overline{\tau}\wedge *_\mathrm{E}\overline{\theta}_3 + \mathrm{e}^{2A_\mathrm{E}}\,\mathrm{Im}\alpha_1\wedge\overline{G}_3 = 0\,, \tag{5.39f}$$

$$\mathrm{d}_Q(\mathrm{e}^{3A_\mathrm{E}}\,\theta_2) + \frac{\mathrm{i}}{2}\,\mathrm{e}^{\phi+3A_\mathrm{E}}\,\mathrm{d}\overline{\tau}\wedge\overline{\theta}_2 - \mathrm{e}^{3A_\mathrm{E}}\,\alpha_0\overline{G}_3 + \mathrm{i}\,\mathrm{e}^{3A_\mathrm{E}}\,c_0 *_\mathrm{E}\overline{G}_3 = 0\,, \tag{5.39g}$$

$$\mathrm{d}(\mathrm{e}^{2A_\mathrm{E}}\,*_\mathrm{E}\mathrm{Im}\alpha_1) + \frac{\mathrm{e}^{2A_\mathrm{E}}}{2}\mathrm{Re}(\theta_\mathrm{E}\wedge G_3) = 0\,, \tag{5.39h}$$

$$\mathrm{d}(\mathrm{e}^{2A_\mathrm{E}}\,*_\mathrm{E}\mathrm{Re}\alpha_1) - \frac{\mathrm{e}^{2A_\mathrm{E}}}{2}\mathrm{Re}(*_\mathrm{E}\theta_3\wedge G_3) = 0\,, \tag{5.39i}$$

$$\mathrm{d}(\mathrm{e}^{3A_\mathrm{E}}\,*_\mathrm{E}\alpha_2) - \frac{\mathrm{e}^{3A_\mathrm{E}}}{2}\mathrm{Re}(\theta_2\wedge G_3) = 0\,, \tag{5.39j}$$

which must be supplemented with the algebraic constraint (5.33) which, in terms of the internal-space forms, reads:

$$\begin{aligned}
&\theta_3\wedge\overline{G}_3 + 2\mathrm{i}\,\mathrm{e}^{\phi}\,\mathrm{d}\overline{\tau}\wedge *_\mathrm{E}\mathrm{Im}\alpha_1 = 0\,,\\
&*_\mathrm{E}\,\theta_3\wedge\overline{G}_3 - 2\mathrm{i}\,\mathrm{e}^{\phi}\,\mathrm{d}\overline{\tau}\wedge *_\mathrm{E}\mathrm{Re}\alpha_1 = 0\,,\\
&\theta_2\wedge\overline{G}_3 - 2\mathrm{i}\,\mathrm{e}^{\phi}\,\mathrm{d}\overline{\tau}\wedge *_\mathrm{E}\alpha_2 - 2\,\mathrm{e}^{\phi}\,c_0 *_\mathrm{E}\mathrm{d}\overline{\tau} = 0\,.
\end{aligned} \tag{5.40}$$

Part II

BPS and non-BPS $\mathbb{R}^{1,3} \times S^2$ classification

CHAPTER 6

$\mathbb{R}^{1,3} \times S^2$ ANSATZ

6.1 Introduction and motivation

Compactifications to four-dimensional Minkowski space-time have always enjoyed great interest as starting point for semi-realistic models in string phenomenology [92, 93]. When the metric is the only non-trivial field, supersymmetry forces the internal space to be Ricci-flat and the study of such compactifications also led to discovery many developments at the interface between string theory and mathematics [94]. On the other hand, backgrounds with nontrivial fluxes are much harder to find and classify due to the back-reaction of non-trivial extra fields. Except for some special cases, where this back-reaction only introduces a conformal factor [25, 95, 96, 97], the geometry of the internal space is in general drastically deformed away from Ricci-flatness, e.g. [98, 99, 100, 101].

Following [39, 41, 40], we use a different approach: rather than making an ansatz on the metric and fluxes, we focus on a broad class and we let supersymmetry fixes the internal geometry. We will see that this method often provide a very detailed classification, since there are enough internal spinors to reduce the structure group to an identity structure, which means that we can automatically get a local expression for the metric.

In particular, the ansatz consists in imposing $\mathcal{N}=2$ supersymmetry with an SU(2) R-symmetry. The R-symmetry must be geometrically realized as a symmetry of the internal manifold, and for this reason we assume that the internal metric is factorized in a warped product $S^2 \times M_4$ where the warping function depends just on the M_4 coordinates.

We will see that preserved supersymmetry will be reduced to a system of PDEs and the classification will reproduce many known system of intersecting branes. In this context a particularly interesting role is played by AdS solutions, which often arise as near-horizon limits of such brane systems. In particular, we can obtain AdS_{d+1} solutions with $d \geq 4$ as a foliation of a d-dimensional Minkowski space over a non-compact interval (i.e., the Poincaré patch) and the R-symmetry is chosen such that it implements the superconformal $\mathrm{SU}(2)_R$-symmetry for the dual conformal field theory in $d = 4,5,6$ dimensions. These intersecting brane solutions are also useful to study more in general holographic RG-flows of the dual field theories, where in this case the AdS vacua correspond to conformal fixed points at one of the two ends of the RG-flow. Such non-conformal behaviors are impossible to detect starting from a class that contains an AdS factor from the start.

We begin by presenting the classification for type II supergravity and we will then focus on M-theory. We will see that all the solutions can be actually generated from two master classes using a web of dualities.

6.2 $\mathrm{SU}(2)_R$ preserving ansatz

The $\mathrm{SU}(2)_R$ ansatz we discussed before is realized by specializing the analysis in section 5.2 in order to preserve the S^2 isometries inside M_6. In this case the internal metric decomposes as

$$\mathrm{d}s_6^2 = \mathrm{e}^{2C}\,\mathrm{d}s^2(S^2) + \mathrm{d}s_4^2 \tag{6.1}$$

while fluxes must not have any leg along S^2 except for the volume form

$$f = F + e^{2C} G\wedge \mathrm{Vol}(S^2)\,, \qquad B = B_2 + \mathrm{e}^{2C} B_0 \mathrm{Vol}(S^2)\,, \qquad H = H_3 + \mathrm{e}^{2C} H_1 \wedge \mathrm{Vol}(S^2) \tag{6.2}$$

where C, F, G, B_2, B_0 are forms and functions on the unconstrained internal manifold M_4. The ten-dimensional spinor ansatz for $\mathcal{N} = 2$ solutions is simply given by the sum of two $\mathcal{N} = 1$ independent spinors as in (5.24):

$$\epsilon_1 = \sum_{b=1}^{2} \zeta_+^b \otimes \chi_{1+}^b + \text{m.c.}\,, \qquad \epsilon_2 = \sum_{b=1}^{2} \zeta_+^b \otimes \chi_{2\mp}^b + \text{m.c.}\,. \tag{6.3}$$

Here ζ^b are a $\mathrm{SU}(2)_R$ doublet and for this reason also χ_{i+}^b must be a doublet, so that the ten-dimensional spinors are overall invariant. As said before, the R-symmetry is realized on the internal manifold using the S^2 isometry group, which means that the internal spinors decompose as

$$\chi_{i+}^b = \xi_+^b \otimes \eta_{i+} + \xi_-^b \otimes \eta_{i-} + (\xi_-^b)^c \otimes \widetilde{\eta}_{i+} + (\xi_+^b)^c \otimes \widetilde{\eta}_{i-} + m.c.\,, \tag{6.4}$$

where ξ^b are a $\mathrm{SU}(2)$ doublet on S^2, $\eta_i, \widetilde{\eta}_i$ are spinors on M_4 and c is the charge conjugation, which in two dimensions changes the chirality.

It is convenient to consider S^2 as embedded in $\mathbb{R}^3$ so that it is the link of a cone. In this way, following [102], we have that covariantly constant spinors on $\mathbb{R}^3$ are Killing spinors on S^2 and they are conserved by the action of the isometry group of the link, which is $\mathrm{SU}(2)$. This argument suggests to use Killing spinors to build a doublet, and, since the space of Killing spinors in two dimensions is spanned by a (non-chiral) Killing spinor ξ and its Majorana conjugate ξ^c, we can take their linear combination and compute how they transform under the action of $\mathcal{L}_{K_i}$, where K_i are the SU(2) generators of S^2. After an explicit computation, as performed in [39, appendix A], one gets that the doublet ξ^b is given by

$$\xi^b = \begin{pmatrix} \xi \\ \xi^c \end{pmatrix} , \tag{6.5}$$

meaning that

$$\mathcal{L}_{K_i} \xi^b = \frac{\mathrm{i}}{2} (\sigma_i)^b_c \xi^c \tag{6.6}$$

where σ_i are Pauli matrices.

An important consequence of the doublet structure is the following: reducing the spinorial supersymmetry equations (1.24) on M_4 one gets that the equations which multiply ξ must be separated from the ones that multiply ξ^c, since they are linearly independent. This means that η_i and $\widetilde{\eta}_i$ in (6.4) never mix and therefore including $\widetilde{\eta}_i$ will only give more constraints instead of a generalization, so we can choose to set them to zero without losing generality. Another important advantage of having an R-symmetry in play is that, imposing that one internal spinor is preserved by supersymmetry, by means of a local transformation the second set of supercharges will be preserved too. So we can restrict ourselves to consider the $\mathcal{N} = 1$ problem defined by the internal spinors

$$\chi_{i+} = \xi_+ \otimes \eta_{i+} + \xi_- \otimes \eta_{i-} \tag{6.7}$$

for both type IIA and type IIB supergravity.

6.3 Spinorial analysis

Before using the full power of the bispinor equations (5.30), it is useful (even if in principle not necessary) to start with some preliminaries using the spinorial BPS system. This will be helpful to constrain the solutions and to highlight some features of the classification. The most stringent conditions come from the scalar (or zero-form) equations, which will suggest an useful reparametrization of the four-dimensional spinors $\eta^{1,2}$. Moreover we will discover that it is possible to restrict ourselves to (5.31). Even if the discussion is in principle similar, let's separate type IIA from type IIB.

6.3.1 Type IIA spinorial system

We begin our analysis by reducing (1.24) following [103]. Using the $\mathbb{R}^{1,3} \times M_6$ decomposition of section 5.2 we get that supersymmetry is implied by the following spinorial conditions on the six-dimensional manifold M_6 only

$$\Big[\partial A + \frac{e^{\phi}}{4}\big(f_0 + f_2\gamma^{(6)} + f_4 + i *_6 f_6\big)\Big]\chi = 0\,, \tag{6.8a}$$

$$\Big[\partial\phi + \frac{1}{2}H\gamma^{(6)} + \frac{e^{\phi}}{4}\big(5f_0 + 3f_2\gamma^{(6)} + f_4 - i *_6 f_6\big)\Big]\chi = 0\,, \tag{6.8b}$$

$$\Big[\nabla_m + \frac{1}{4}H_m\gamma^{(6)} + \frac{e^{\phi}}{8}\big(f_0 - f_2\gamma^{(6)} + f_4 - i *_6 f_6\big)\gamma^{(6)}_m\Big]\chi = 0\,, \tag{6.8c}$$

where $\chi = \chi_{1+} + \chi_{2-}$. Using (6.8a) and (6.8c) it is possible to prove that $d(e^{-A}|\chi|^2) = d(e^{A}\chi^\dagger\gamma\chi) = 0$, while including also (6.8b) one can find $d(e^{2A-\phi}\overline{\chi}\chi) = 0$. Using these relations we are able to fix

$$|\chi|^2 = 2c_+ e^{A}\,, \qquad \chi^\dagger\gamma^{(6)}\chi = 2c_- e^{-A}\,, \qquad \overline{\chi}\chi = c\, e^{-2A+\phi} \tag{6.9}$$

where $c_\pm$ are the same constants defined in (5.30), so again we can set $c_+ = 1$ without loss of generality, while $c_- = 0$ is the case of equal chiral-spinor norms. The third equation in (6.9) can be seen as a scalar product between χ_1 and χ_2, and in particular $c = 0$ corresponds to an orthogonal condition between the two spinors, which would lead to an orthogonal SU(2)-structure.

Using the properties of Hermitian gamma matrices it is possible to show by contracting (6.8a)-(6.8b) with χ and $\gamma^{(6)}\chi$ that

$$*_6 F_6 = F_0\chi^\dagger\gamma^{(6)}\chi = 0, \tag{6.10}$$

which come from the imaginary and real parts of the above inner products respectively. Since equal spinor norms is equivalent to $\chi^\dagger\gamma^{(6)}\chi = 0$, it is clear that it is possible to have the Romans mass turned on only when $c_- = 0$. This means that all the non-equal-norm classes can be derived from the M-theory classification, as it is performed in section 8.6. So for type IIA we can restrict ourselves to consider the system (5.31) instead of the general one.

Let's now impose the presence of the SU(2) R-symmetry, which imposes that we have to decompose χ in terms of spinors on S^2 and M_4. However, instead of using (6.7), let's formulate the ansatz in a more convenient way as

$$\chi = \xi \otimes \eta^1 + \sigma_3\xi \otimes \gamma\eta^2\,, \tag{6.11}$$

where σ_3 and γ are the chirality matrices in two and four dimensions and the Clifford algebra decomposition is given by:

$$\gamma^{(6)}_\mu = \sigma_\mu \otimes \mathbb{1}, \quad \mu = 1,2 \qquad\qquad \gamma^{(6)}_{m+2} = \sigma_3 \otimes \gamma_m, \quad m = 1,\dots,4 \tag{6.12}$$

with γ_m the four-dimensional gamma matrices. It is easy to check by projecting on the chiral components of χ that (6.11) is the same ansatz of (6.7) but with $\eta^1 \to \eta^1 + \eta^2$ and $\eta^2 \to \eta^1 - \eta^2$.

As a consequence of (6.9) we necessarily have

$$
\begin{aligned}
&|\eta^1|^2+|\eta^2|^2=2\,\mathrm{e}^{A}\,, &&\eta^{1\dagger}\gamma\eta^2+\eta^{2\dagger}\gamma\eta^1=0\,,\\
&\eta^{1\dagger}\eta^2+\eta^{2\dagger}\eta^1=2c_-\,\mathrm{e}^{-A}\,, &&\eta^{1\dagger}\gamma\eta^1+\eta^{2\dagger}\gamma\eta^2=0\,,
\end{aligned}
\tag{6.13}
$$

and

$$
\bar{\xi}\sigma_3\xi\,\overline{\eta^1}\gamma\eta^2=c\,\mathrm{e}^{-2A+\phi}\quad\Longrightarrow\quad\overline{\eta^1}\gamma\eta^2=c=0 \tag{6.14}
$$

since the right-hand side is charged under SU(2) while the left-hand side is invariant; so this last condition imposes an orthogonal SU(2)-structure in six dimensions.

Plugging (6.11) into (6.8a)-(6.8c), using the flux decomposition (6.2) and the fact that ξ is a Killing spinor, we are able to factorize out the S^2 dependence and get to

$$
\partial A\eta^1+\frac{\mathrm{e}^{\phi}}{4}\Big[\big((F_0+F_4)\gamma+\mathrm{i}G_0\big)\eta^2+\big(F_2\gamma+\mathrm{i}G_2\big)\eta^1\Big]=0\,, \tag{6.15a}
$$

$$
\partial A\eta^2-\frac{\mathrm{e}^{\phi}}{4}\Big[\big((F_0+F_4)\gamma+\mathrm{i}G_0\big)\eta^1+\big(F_2\gamma+\mathrm{i}G_2\big)\eta^2\Big]=0\,, \tag{6.15b}
$$

$$
(\partial\phi+\frac{\mathrm{i}}{2}H_1\gamma)\eta^1+\frac{1}{2}H_3\eta^2+\frac{\mathrm{e}^{\phi}}{4}\Big[\big((5F_0+F_4)\gamma+3\mathrm{i}G_0\big)\eta^2+\big(3F_2\gamma+\mathrm{i}G_2\big)\eta^1\Big]=0\,, \tag{6.15c}
$$

$$
(\partial\phi+\frac{\mathrm{i}}{2}H_1\gamma)\eta^2+\frac{1}{2}H_3\eta^1-\frac{\mathrm{e}^{\phi}}{4}\Big[\big((5F_0+F_4)\gamma+3\mathrm{i}G_0\big)\eta^1+\big(3F_2\gamma+\mathrm{i}G_2\big)\eta^2\Big]=0\,, \tag{6.15d}
$$

$$
(\partial C+\frac{\mathrm{i}}{2}H_1\gamma)\eta^1-\mathrm{i}\,\mathrm{e}^{-C}\gamma\eta^2+\frac{\mathrm{e}^{\phi}}{4}\Big[\big((F_0+F_4)\gamma-\mathrm{i}G_0\big)\eta^2+\big(F_2\gamma-\mathrm{i}G_2\big)\eta^1\Big]=0\,, \tag{6.15e}
$$

$$
(\partial C+\frac{\mathrm{i}}{2}H_1\gamma)\eta^2-\mathrm{i}\,\mathrm{e}^{-C}\gamma\eta^1-\frac{\mathrm{e}^{\phi}}{4}\Big[\big((F_0+F_4)\gamma-\mathrm{i}G_0\big)\eta^1+\big(F_2\gamma-\mathrm{i}G_2\big)\eta^2\Big]=0\,, \tag{6.15f}
$$

$$
\big(\nabla_m+\frac{\mathrm{i}}{4}H_{1\,m}\gamma\big)\eta_1+\frac{1}{4}H_{3\,m}\eta_2-\frac{\mathrm{e}^{\phi}}{8}\Big[\big((F_0+F_4)\gamma-\mathrm{i}G_0\big)\gamma_m\eta^2+(F_2\gamma-\mathrm{i}G_2)\gamma_m\eta^1\Big]=0\,, \tag{6.15g}
$$

$$
\big(\nabla_m+\frac{\mathrm{i}}{4}H_{1\,m}\gamma\big)\eta_2+\frac{1}{4}H_{3\,m}\eta_1+\frac{\mathrm{e}^{\phi}}{8}\Big[\big((F_0+F_4)\gamma-\mathrm{i}G_0\big)\gamma_m\eta^1+(F_2\gamma-\mathrm{i}G_2)\gamma_m\eta^2\Big]=0\,, \tag{6.15h}
$$

By taking inner products of (6.15a)-(6.15b) and (6.15e)-(6.15f) with $\{\eta^1,\,\eta^2,\,\gamma\eta^1,\,\gamma\eta^2\}$ and exploiting the identities

$$
\big(\overline{\eta^1}\gamma_{m_1\dots m_n}\eta^2\big)^t=-(-)^{\frac{n(n-1)}{2}}\,\overline{\eta^2}\gamma_{m_1\dots m_n}\eta^1,\quad \big(\overline{\eta^1}\gamma_{m_1\dots m_n}\gamma\eta^2\big)^t=-(-)^{\frac{n(n+1)}{2}}\,\overline{\eta^2}\gamma_{m_1\dots m_n}\gamma\eta^1 \tag{6.16}
$$

one gets that the following zero forms vanish

$$
|\eta^1|^2-|\eta^2|^2=\eta^{1\dagger}\eta^2-\eta^{2\dagger}\eta^1=0. \tag{6.17}
$$

We can now solve (6.13), (6.14), (6.17) without loss of generality by decomposing η^i in terms of a single spinor η and functions a, b which are complex and real respectively

$$e^{-\frac{1}{2}A}\eta^1=\sin\left(\frac{\alpha}{2}\right)\eta+\cos\left(\frac{\alpha}{2}\right)(a\gamma\eta+b\eta^c),\quad e^{-\frac{1}{2}A}\eta^2=\sin\left(\frac{\alpha}{2}\right)\eta-\cos\left(\frac{\alpha}{2}\right)(a\gamma\eta+b\eta^c) \tag{6.18}$$

where

$$|\eta|^2=1,\quad \eta^\dagger\gamma\eta=0,\quad a_1^2+a_2^2+b^2=1,\quad a=a_1+\mathrm{i}a_2\,,\quad \cos\alpha=-e^{-2A}c_-\,. \tag{6.19}$$

If one now projects (6.11) onto its chiral components and maps them back to (6.7) it becomes clear that we have re-derived the spinor decomposition of [39] but in a more general context where the six-dimensional chiral-spinor norms are not necessarily equal. As shown in [29], the spinor η can be used to define a vielbein $\{v_1, v_2, w_1, w_2\}$ on M_4 through

$$\eta^\dagger\gamma_m\eta\,\mathrm{d}x^m=v_1\,,\qquad \eta^\dagger\gamma_m\gamma\eta\,\mathrm{d}x^m=\mathrm{i}v_2\,,\qquad \overline{\eta}\gamma_m\gamma\eta\,\mathrm{d}x^m=w=w_1+\mathrm{i}w_2\,, \tag{6.20}$$

which is the identity structure on M_4.

Before moving to type IIB, let's go a bit further and define the following vector

$$k=\overline{\eta^2}\gamma^m\gamma\eta^1\partial_m\,. \tag{6.21}$$

By taking the inner products of (6.15a), (6.15b) with $\gamma\eta^2$, $\gamma\eta^1$ respectively and doing the same for (6.15c)-(6.15f) it is possible to show that Lie derivative with respect to k vanishes when acting on A, C, ϕ, which suggests that k can be associated to a symmetry for all the $\mathbb{R}^{1,3}\times S^2$ backgrounds. For example, it is possible to show that it is a Killing vector. In order to do so let's compute $\nabla_m k_n$

$$\begin{aligned}\nabla_m k_n=&-\frac{e^\phi}{4}\overline{\eta^2}\Big[\big((F_2)_{mn}+\mathrm{i}(G_2)_{mn}\gamma\big)+\frac{\gamma_{mn}^{\;\;pq}}{2}\big((F_2)_{pq}\\&+\mathrm{i}(G_2)_{pq}\gamma\big)\Big]\eta^1+\frac{1}{4}(H_3)_{mnp}\big(\overline{\eta^1}\gamma^p\gamma\eta^1+\overline{\eta^2}\gamma^p\gamma\eta^2\big)\\&+\frac{e^\phi}{8}\overline{\eta^1}\gamma_{mn}(F_0+\mathrm{i}G_0\gamma-F_4)\eta^1-\frac{e^\phi}{8}\overline{\eta^2}\gamma_{mn}(F_0+\mathrm{i}G_0\gamma-F_4)\eta^2\end{aligned} \tag{6.22}$$

and notice that $\nabla_m k_n$ is manifestly anti-symmetric, so $\nabla_{(m}k_{n)}=0$. k is a complex vector, so it defines two real isometric directions on the metric. In particular, k must parameterize some two-dimensional Lie group, and we refer to section 8.6 for the proof that k corresponds to a $U(1)\times U(1)$ Lie group under which spinors and form-fields are uncharged, which means that when k is not zero we have two directions which can be T-dualized.

In terms of (6.18) and (6.21) we have that the one-form associated to k reads

$$k=a\big(1+\cos\alpha\big)\big(bv_1+(a_1w_1-a_2w_2)\big)-\big(\cos\alpha\, w_1-\mathrm{i}w_2\big). \tag{6.23}$$

Notice that for generic values of α both real and imaginary components of k cannot vanish, so we generically have two real isometries. The exception is when $\cos\alpha = 0$, which is the case we are interested in since there the spinor norms are equal. In this case the number of isometries is controlled by the values of a_1, a_2, b, and in particular there is the following structure depending on how the spinor parameters are tuned

$$\begin{aligned} &\text{0 isometries:} \quad \cos\alpha = a_1 = b = 0\,, \\ &\text{1 isometries:} \quad \cos\alpha = a_1 = 0,\ b \neq 0\,, \\ &\text{2 isometries:} \quad \text{otherwise}\,, \end{aligned} \tag{6.24}$$

which is due to the fact that k is respectively zero or purely imaginary in the first two cases only, while for all other cases there are two real isometries.

This argument already suggests that, except for the case with zero isometries, it may be possible to generate all the other classes using dualities (for example T-duality, since k generates flavor symmetries).

6.3.2 Type IIB spinorial system

The discussion for type IIB supergravity mimics the one of type IIA, even if it is different in technical details. Again, what we want to do is to reparameterize the four-dimensional spinors using zero-form constraints and see if also in this case there are some symmetries at play.

By using the spinor ansatz (5.24) and the flux decomposition (5.23) we get that (1.24) and (1.25) become

$$\partial A \chi^1_+ - \frac{e^\Phi}{4}\Big(f_1 + f_3 + f_5\Big)\chi^2_+ = 0\,, \tag{6.25a}$$

$$\partial A \chi^2_+ - \frac{e^\Phi}{4}\Big(-f_1 + f_3 - f_5\Big)\chi^1_+ = 0\,, \tag{6.25b}$$

$$\partial \Phi \chi^1_+ - \frac{1}{2}H\chi^1_+ - e^\Phi\Big(f_1 + \frac{1}{2}f_3\Big)\chi^2_+ = 0\,, \tag{6.25c}$$

$$\partial \Phi \chi^2_+ + \frac{1}{2}H\chi^2_+ - e^\Phi\Big(-f_1 + \frac{1}{2}f_3\Big)\chi^1_+ = 0,, \tag{6.25d}$$

$$\Big(\nabla_a - \frac{1}{4}H_m\Big)\chi^1_+ + \frac{e^\Phi}{8}\Big(f_1 + f_3 + f_5\Big)\gamma^{(6)}_m\chi^2_+ = 0\,, \tag{6.25e}$$

$$\Big(\nabla_a + \frac{1}{4}H_m\Big)\chi^2_+ + \frac{e^\Phi}{8}\Big(-f_1 + f_3 - f_5\Big)\gamma^{(6)}_m\chi^1_+ = 0\,. \tag{6.25f}$$

Notice that in this case it is not possible to rewrite the system in terms of just one non-chiral spinor as in IIA since χ_1 and χ_2 always appear in the equations with the same chirality.

Using (6.25e) and (6.25f) together with (6.25a)-(6.25d) multiplied by γ_m, we get the following conditions on the zero-degree bilinears:

$$\begin{aligned}&|\chi^1_+|^2+|\chi^2_+|^2=2e^A, \quad |\chi^1_+|^2-|\chi^2_+|^2=2c_-e^{-A},\\ &\partial_m(e^{2A-\Phi}\chi^{2\dagger}_+\chi^1_+)-e^{2A-\Phi}\partial_m(\chi^{1\dagger}_+\chi^2_+)=ie^{3A}(\star F_5)_m-c_-e^A(F_1)_m,\end{aligned}\tag{6.26}$$

where, as in IIA, we have set $c_+=1$.

Assuming the presence of an SU(2) R-symmetry and adopting the Clifford algebra decomposition (6.12) one can show that (6.25a)-(6.25f) is equivalent to the following set of spinoral equations over the unconstrained part of the internal manifold:

$$\partial A\eta^1+\frac{e^\Phi}{4}\Big[(F_1\gamma+\mathrm{i}G_3)\eta^2-(F_3\gamma+\mathrm{i}G_1)\eta^1\Big]=0,\tag{6.27a}$$

$$\partial A\eta^2-\frac{e^\Phi}{4}\Big[(F_1\gamma+\mathrm{i}G_3)\eta^1-(F_3\gamma+\mathrm{i}G_1)\eta^2\Big]=0,\tag{6.27b}$$

$$\partial\Phi\eta_1-\frac{1}{2}(H_3+\mathrm{i}H_1\gamma)\eta^2+e^\Phi\Big[F_1\gamma\eta^2-\frac{1}{2}(F_3\gamma+\mathrm{i}G_1)\eta^1\Big]=0,\tag{6.27c}$$

$$\partial\Phi\eta_2-\frac{1}{2}(H_3+\mathrm{i}H_1\gamma)\eta^1-e^\Phi\Big[F_1\gamma\eta^1-\frac{1}{2}(F_3\gamma+\mathrm{i}G_1)\eta^2\Big]=0,\tag{6.27d}$$

$$\partial C\eta^1-\mathrm{i}e^{-C}\gamma\eta^2-\frac{\mathrm{i}}{2}H_1\gamma\eta^2+\frac{e^\Phi}{4}\Big[(F_1\gamma-\mathrm{i}G_3)\eta^2-(F_3\gamma-\mathrm{i}G_1)\eta^1\Big]=0,\tag{6.27e}$$

$$\partial C\eta^2-\mathrm{i}e^{-C}\gamma\eta^1-\frac{\mathrm{i}}{2}H_1\gamma\eta^1-\frac{e^\Phi}{4}\Big[(F_1\gamma-\mathrm{i}G_3)\eta^1-(F_3\gamma-\mathrm{i}G_1)\eta^2\Big]=0,\tag{6.27f}$$

$$\nabla_m\eta_1-\frac{1}{4}\Big((H_3)_m+\mathrm{i}(H_1)_m\gamma\Big)\eta^2+\frac{e^\Phi}{8}\Big[(F_1\gamma-\mathrm{i}G_3)\gamma_m\eta_2-(F_3\gamma-\mathrm{i}G_1)\gamma_m\eta_1\Big]=0,\tag{6.27g}$$

$$\nabla_m\eta_2-\frac{1}{4}\Big((H_3)_m+\mathrm{i}(H_1)_m\gamma\Big)\eta^1-\frac{e^\Phi}{8}\Big[(F_1\gamma-\mathrm{i}G_3)\gamma_m\eta_1-(F_3\gamma-\mathrm{i}G_1)\gamma_m\eta_2\Big]=0,\tag{6.27h}$$

where again we have used the spinorial ansatz (6.7) but this time with $\eta^1\to\eta^1+\gamma\eta^2$ and $\eta^2\to\eta^1-\gamma\eta^2$, in order to make contact with the previous discussion for type IIA.

From the zero-form conditions (6.26), imposing that the physical quantities are SU(2) singlets, we get the following equations for the four-dimensional spinors:

$$\begin{aligned}&|\eta^1|^2+|\eta^2|^2=2e^A, &&\eta^{1\dagger}\gamma\eta^2+\eta^{2\dagger}\gamma\eta^1=0,\\ &\eta^{1\dagger}\eta^2+\eta^{2\dagger}\eta^1=2c_-e^{-A}, &&\eta^{1\dagger}\gamma\eta^1+\eta^{2\dagger}\gamma\eta^2=0,\end{aligned}\tag{6.28}$$

which are the same for IIA (6.13), together with

$$|\eta^1|^2 - |\eta^2|^2 = \eta^{1\dagger}\eta^2 - \eta^{2\dagger}\eta^1 = 0. \tag{6.29}$$

From the difference between $\overline{\eta^2}$(2(6.27a)+2(6.27e)-(6.27c)) and $\overline{\eta^1}$(2(6.27b)+2(6.27f)-(6.27d)) we get another algebraic constraint:

$$\overline{\eta^2}\gamma\eta^1 = 0. \tag{6.30}$$

These scalar conditions suggest the following decomposition of η^i in terms of a single spinor η and complex functions a, b

$$e^{-\frac{1}{2}A}\eta^1 = \sin\left(\frac{\alpha}{2}\right)\eta + \cos\left(\frac{\alpha}{2}\right)(a\eta + b\gamma\eta^c), \quad e^{-\frac{1}{2}A}\eta^2 = \sin\left(\frac{\alpha}{2}\right)\eta - \cos\left(\frac{\alpha}{2}\right)(a\eta + b\gamma\eta^c) \tag{6.31}$$

where

$$|\eta|^2 = 1\,, \quad \eta^\dagger\gamma\eta = 0\,, \quad |a|^2 + |b|^2 = 1\,, \quad a = a_1 + \mathrm{i}a_2\,, \quad b = b_1 + \mathrm{i}b_2\,, \quad \cos\alpha = -c_- e^{-2A}. \tag{6.32}$$

Let's now consider the identity structure defined by η (6.20). Given the equation

$$\eta^c = \frac{1}{2}\overline{w}\hat{\gamma}\eta \tag{6.33}$$

we have that the phase of b in (6.31) can be fixed by rotating the vector w. In particular, we choose b to be real.

Let's now consider the vector

$$k = \overline{\eta^2}\gamma^m\gamma\eta^1\partial_m. \tag{6.34}$$

Exactly as in IIA, it is possible to prove that this vector generates a symmetry which parameterizes an $U(1)\times U(1)$ group. Since this is proved in detail in [41, section 3.2], we will omit to repeat here that technical (but straightforward) argument. Thanks to the parametrization of the internal spinors we adopted for η^1 and η^2, which is identical to the one in IIA (6.18), we have that the same discussion applies here, and in particular we get:

$$\begin{aligned} &0 \text{ isometries:} \quad \cos\alpha = a_1 = b = 0\,, \\ &1 \text{ isometries:} \quad \cos\alpha = a_1 = 0,\ b \neq 0\,, \\ &2 \text{ isometries:} \quad \text{otherwise}\,. \end{aligned} \tag{6.35}$$

Since the non-equal-norm constraint is unaffected by a T-duality transformation, we have that all the non-equal-norm cases $\cos\alpha \neq 0$ can actually be generated from type IIA classification, which is in turn a particular case of the M-theory class. We will return to this point in section 8.6, but for now it is enough to know that we can use (5.31) for the IIB classification without loss of generality.

6.4 Pure spinors

In the previous section we managed to find a convenient reparametrization for the spinors on M_4 which will be helpful to explicitly write the pure spinors $\Phi_\pm$ in terms of the identity structure on the internal manifold. In general, a pure spinor can always be written in the following way [35]

$$\Phi = \Omega_k \wedge \mathrm{e}^{\mathrm{i}J} \tag{6.36}$$

where J is a real form while k is a complex k-form with k that runs from 0 to 3.

In order to compute them in the case at hand, we have to use the expressions in [29, section 3] to parameterize the bilinears on M_4 and the one in [39, appendix A] for the bilinears on S^2. After a bit of manipulations we get that the pure spinors in type IIA can be written as

$$\begin{aligned}\Phi_+ &= \frac{1}{8}E_1 \wedge E_2 \wedge e^{\frac{1}{2}E_3\wedge\overline{E}_3},\\ \Phi_- &= \frac{1}{8}E_3 \wedge e^{\frac{1}{2}(E_1\wedge\overline{E}_1+E_2\wedge\overline{E}_2)}\end{aligned} \tag{6.37}$$

where the complex vielbein is given by the following definitions

$$\begin{aligned}&E_1 = b(\mathrm{e}^C\,\mathrm{d}y_3 - y_3 v_2) + \mathrm{i}(aw + bv_1)\,, \qquad E_2 = -\mathrm{e}^C\,\mathrm{d}(y_1 + \mathrm{i}y_2) + (y_1 + \mathrm{i}y_2)v_2\,,\\ &E_3 = \mathrm{i}\overline{a}(\mathrm{e}^C\,\mathrm{d}y_3 - y_3 v_2) + bw - \overline{a}v_1\,,\end{aligned} \tag{6.38}$$

while the ones in IIB read

$$\begin{aligned}\Phi_+ &= \frac{1}{8}\mathrm{e}^{\frac{1}{2}E_3\wedge\overline{E}_3} \wedge \left(\overline{a}\,\mathrm{e}^{\frac{1}{2}(E_1\wedge\overline{E}_1+E_2\wedge\overline{E}_2)} + \mathrm{i}bE_1 \wedge E_2\right),\\ \Phi_- &= \frac{1}{8}E_3 \wedge \left(\mathrm{i}b\,\mathrm{e}^{\frac{1}{2}(E_1\wedge\overline{E}_1+E_2\wedge\overline{E}_2)} + aE_1 \wedge E_2\right)\end{aligned} \tag{6.39}$$

with

$$E_1 = \mathrm{e}^C\,\mathrm{d}y_3 - y_3 v_2 + \mathrm{i}v_1\,, \quad E_2 = w\,, \quad E_3 = -(\mathrm{e}^C(\mathrm{d}y_1 + \mathrm{i}dy_2) - (y_1 + \mathrm{i}y_2)v_2). \tag{6.40}$$

Notice that, as anticipated in subsection 6.3.1, the pure spinors in IIA (6.37) parameterize an orthogonal SU(2)-structure while in IIB we have an intermediate [104], possibly dynamical, SU(2)-structure. This means that in this case it would be in principle possible that (6.39) varies from an SU(2) to an SU(3)-structure even from a point to another. However, after imposing supersymmetry, we will get that a and b have constant modulus, which means that the SU(2)-structure is in general not dynamical.

In (6.38) and (6.40) the y_μ are the embedding coordinates of S^2 in $\mathbb{R}^3$, which then satisfy $y_1^2 + y_2^3 + y_3^3 = 1$, $\{v_1, v_2, w\}$ is the vielbein on M_4 introduced in (6.20) and a, b

define a parameter space in our classification, meaning that depending on which of these is turned on we will get a different class. This was already suggested by the isometries analysis performed in the previous sections.

In the next chapter we will impose background supersymmetry by using the expression for the pure spinors (6.37), (6.39) in (5.31).

CHAPTER 7

CLASSIFICATION IN TYPE II SUPERGRAVITY

In this chapter we will present the classification of $\mathbb{R}^{1,3} \times S^2$ solutions in type II supergravity which was originally carried out in [39, 40]. We recall that solving (5.31) is not enough for a solution to exist, since we have also to impose Bianchi identities for fluxes (1.16). Thanks to (5.31c), we know that form fields electrically coupled to space filling branes already satisfy their equations of motion, which prevents from having sources that lay in the Minkowski directions, as required by preserved Poincaré invariance. To get the magnetic components of the flux we have to take the Hodge dual on the internal manifold, which would in general involve the technology of [105]. However, thanks to the identity structure, in our case it is possible to find a local expression for the metric and therefore directly compute the Hodge dual of every differential form. Once supersymmetry and Bianchi identities for fluxes have been imposed, it is not necessary to worry about further integrability conditions, as the one discussed in subsection 3.2.1, since it was proven in [106] that for the case of four-dimensional vacua they automatically follow.

Since in [41] it was discovered that many classes in the $\mathbb{R}^{1,3} \times S^2$ classification were not really fundamental but they can be derived using various solution generating techniques, in the next two sections we will focus just on the two classes (one in IIA and one in IIB) which cannot be derived using dualities, which we call master classes, while in the third section we will show how it is possible to generate all the solutions in [39, 40] from them.

7.1 IIB master class

The master class in type IIB supergravity is obtained by specializing (6.39) to $b = 0, a = -\mathrm{i}$; notice from (6.35) that with this choice of the parameters we have no extra isometries in the internal space. In the case at hand the solution falls in the so-called conformal Calabi–Yau class [97, 25]. Using the vielbein (6.40) to define

$$\Omega = \mathrm{e}^{3A-\phi} E_1 \wedge E_2 \wedge E_3 , \qquad J = \frac{\mathrm{i}}{2} \mathrm{e}^{2A-\phi} \left(E_1 \wedge \overline{E}_1 + E_2 \wedge \overline{E}_2 + E_3 \wedge \overline{E}_3 \right) \tag{7.1}$$

we get that supersymmetry conditions (5.31) are equivalent to imposing

$$\begin{aligned} &\mathrm{d} J = \mathrm{d}\Omega = 0 , && H \wedge \Omega = H \wedge J = 0 , \\ &*_6 f_1 = -\frac{1}{2} \mathrm{e}^{-4A} \mathrm{d}(\mathrm{e}^{\phi} J^2) , && *_6 f_3 = \mathrm{e}^{-\phi} H , \qquad *_6 f_5 = \mathrm{e}^{-4A} \mathrm{d}\, \mathrm{e}^{-4A-\phi} . \end{aligned} \tag{7.2}$$

We can see from $\mathrm{d}\Omega = 0$ that the internal manifold is complex while $\mathrm{d} J = 0$ is the integrability condition for a symplectic structure. Since as one can check from (7.1) the two structures are compatible $J \wedge \Omega = 0$ we have that the internal manifold is Kähler. The Calabi–Yau condition would require also $J^3 = \mathrm{i}\frac{3}{4}\Omega \wedge \overline{\Omega}$, which is satisfied by (7.1) up to a conformal factor $J^3 = \mathrm{i}\, \mathrm{e}^{-\phi} \frac{3}{4} \Omega \wedge \overline{\Omega}$.

Using the explicit expressions for the vielbein (6.40) and for fluxes (6.2) to impose the SU(2) isometries it is possible to characterize the solution even further and the first line of (7.2) reduces to:

$$\mathrm{d}(\mathrm{e}^{2A+2C-\phi}) + 2\, \mathrm{e}^{2A+C-\phi}\, v_2 = \mathrm{d}(\mathrm{e}^{A-\frac{1}{2}\phi}\, v_1) = \mathrm{d}(\mathrm{e}^{A}\, w) = 0, \tag{7.3a}$$

$$B_2 = -B_0 v_1 \wedge v_2, \quad \mathrm{d}(\mathrm{e}^{2C} B_0) \wedge w \wedge \overline{w} = \mathrm{d}(\mathrm{e}^{-\phi}) \wedge w \wedge \overline{w} = 0. \tag{7.3b}$$

We can solve (7.3a) in full generality by introducing local coordinates which provide a definition for the vielbein on M_4:

$$v_1 = \mathrm{e}^{-A+\frac{\phi}{2}}\, \mathrm{d} x_1 , \quad v_2 = -\, \mathrm{e}^{-A+\frac{\phi}{2}}\, \mathrm{d} x_2 , \quad w_1 = -\, \mathrm{e}^{-A}\, \mathrm{d} x_3 , \quad w_2 = -\, \mathrm{e}^{-A}\, d x_4 , \tag{7.4}$$

and we further have the condition

$$x_2^2 = \mathrm{e}^{2A+2C-\phi} \tag{7.5}$$

which can be used as a definition for the S^2 warping function C.

Equations (7.3b) determine part of the NSNS two-form potential and impose that

$$\mathrm{e}^{2C} B_0 = g(x_3, x_4) , \qquad \mathrm{e}^{-\phi} = f(x_3, x_4). \tag{7.6}$$

The ten-dimensional metric takes the form

$$\mathrm{d} s^2 = \mathrm{e}^{2A} \mathrm{d} s^2(\mathbb{R}^{1,3}) + \mathrm{e}^{-2A} \left(\frac{1}{f} \big(\mathrm{d} x_1^2 + \mathrm{d} x_2^2 + x_2^2\, \mathrm{d} s^2(S^2) \big) + (\mathrm{d} x_3^2 + \mathrm{d} x_4^2) \right), \tag{7.7}$$

we stress that x_2 can be considered as a radial coordinate so that the part of the metric spanned by (x_1, x_2, S^2) is warped $\mathbb{R}^4$ while the part spanned by (x_3, x_4) is warped

$\mathbb{R}^2$. The Calabi–Yau metric is given by $e^{2A}\,\mathrm{d}s^2(M_6)$, which can be viewed locally as a foliation of T^4 over T^2. At this stage it is not evident that this space is Ricci-flat, since f is unconstrained so far; we will see that imposing Bianchi identities it turns out to be harmonic.

The ten-dimensional fluxes are then given by

$$B = g\mathcal{C}_2\,, \qquad F_1 = \partial_{x_4} f\,\mathrm{d}x_3 - \partial_{x_3} f\,\mathrm{d}x_4\,, \qquad \mathcal{C}_2 = \mathrm{Vol}(S^2) + \frac{\mathrm{d}x_1\wedge\mathrm{d}x_2}{x_2^2}\,, \tag{7.8a}$$

$$\begin{aligned}
&F_3 = B\wedge F_1 - \Big(\partial_{x_4}(fg)dx_3 - \partial_{x_3}(fg)dx_4\Big)\wedge\mathcal{C}_2\,,\\
&F_5 = B\wedge F_3 - \frac{1}{2}B\wedge B\wedge F_1 + x_2^2\Big(\partial_{x_2}\,e^{-4A}\,\mathrm{d}x_1 - \partial_{x_1}\,e^{-4A}\,\mathrm{d}x_2\Big)\wedge\mathrm{d}x_3\wedge\mathrm{d}x_4\wedge\mathrm{Vol}(S^2)\\
&\quad+\frac{1}{2}\Big(\partial_{x_4}(fg^2+x_2^4f^{-1}\,e^{-4A})\,\mathrm{d}x_3 - \partial_{x_3}(fg^2+x_2^4f^{-1}e^{-4A})dx_4\Big)\wedge\mathcal{C}_2\wedge\mathcal{C}_2 + \mathrm{Vol}_4\wedge\mathrm{d}(e^{4A}f)\,,
\end{aligned} \tag{7.8b}$$

where Vol_4 is the volume element of the Minkowski space. Imposing the closure condition for the twisted fluxes (4.20) we get that F_1 and F_3 yield

$$\triangle_2 f = 0\,, \qquad \triangle_2(fg) = 0\,, \qquad \triangle_2 = \partial_{x_3}^2 + \partial_{x_4}^2\,, \tag{7.9}$$

while F_5 leads to

$$\partial_{x_1}^2(e^{-4A}) + \frac{1}{x_2^2}\partial_{x_2}(x_2^2\partial_{x_2}(e^{-4A})) + \triangle_2(e^{-4A}f^{-1}) + \frac{1}{x_2^4}\triangle_2(fg^2) = 0\,. \tag{7.10}$$

7.1.1 Analysis of the solution

We managed to reduce the problem of finding a supersymmetric solution to a relatively small set of PDEs. It might seem that imposing these equations does not correspond to have any localized sources, since in string theory branes appear as a violation of Bianchi identities (1.20). However, it is possible to introduce sources by enlarging the space of solutions also to non-regular ones, where the singular loci determine where the branes are sitting. Following this method it can happen that not all singularities have a string-theory origin, so to rule out such solutions it is necessary to check if all fields behave correctly, meaning that they have to reproduce, at least in the near horizon limit, known behaviour of intersecting brane systems. Let's apply this method to analyze what kind of branes we can find in the IIB master class.

For the discussion of Bianchi identity solutions it is beneficial to rewrite the warp factor as

$$e^{-4A} = \frac{f}{x_2^2}\left[x_2^2 h - \big((\partial_{x_3}g)^2 + (\partial_{x_4}g)^2\big)\right] \tag{7.11}$$

where $h = h(x_1, x_2, x_3, x_4)$, so that (7.10) turns into

$$f\left(\partial^2_{x_1} h + \frac{1}{x_2^2}\partial_{x_2}(x_2^2\partial_{x_2} h)\right) + \Box_2 h = \frac{1}{x_2^2}\Box_2\left((\partial_{x_3} g)^2 + (\partial_{x_4} g)^2\right), \tag{7.12}$$

which makes the structure of a Laplace equation more evident. The generic case however remains quite complicated, so it is better to restrict to some special subclasses to get some insight.

For instance, let's impose that g is constant. In this case the NSNS potential B turns out to be closed, which means that it is pure gauge and therefore can be set to zero without loss of generality; this choice corresponds to $g = 0$. As a consequence, the right-hand side of (7.12) vanishes and the only non-trivial fluxes are F_1 and F_5:

$$\mathrm{d}s^2_{10} = \frac{1}{\sqrt{fh}}\mathrm{d}^2 s(\mathbb{R}^{1,3}) + \frac{\sqrt{h}}{\sqrt{f}}(\mathrm{d}x_1^2 + \mathrm{d}x_2^2 + x_2^2 ds^2(S^2)) + \sqrt{fh}(\mathrm{d}x_3^2 + \mathrm{d}x_4^2), \tag{7.13a}$$

$$F_1 = \partial_{x_4} f\,\mathrm{d}x_3 - \partial_{x_3} f\,\mathrm{d}x_4, \tag{7.13b}$$

$$F_5 = \mathrm{Vol}_4 \wedge \mathrm{d}h^{-1} + x_2^2\mathrm{Vol}(S^2) \wedge \left(\epsilon_{ij}\partial_{x_i} h\, dx_j\right) \wedge \left(f\,\mathrm{d}x_3 \wedge \mathrm{d}x_4 + \mathrm{d}x_1 \wedge \mathrm{d}x_2\right), \tag{7.13c}$$

$$f\left[\partial^2_{x_1} h + \frac{1}{x_2^2}\partial_{x_2}(x_2^2\partial_{x_2} h)\right] + \Box_2 h = 0, \qquad \Box_2 f = 0. \tag{7.13d}$$

The resulting ten-dimensional metric (7.13a) suggests a setup of intersecting D3-D7-branes [107, Eqs. (13)-(14)]. In particular, we have re-derived the D3-D7 case of the system used to study localized Dp-branes in the world volume of D(p+4)-branes [100, section 4.3], even if for the D3-D7 case it is not possible to use the techniques of that paper.

By turning on also g we have that F_3 and H do not vanish anymore and we may have in principle also D5- and NS-brane. However let's assume for simplicity $A = \phi = 0$, so that the metric is completely flat, and let's consider the relation $H = *_6 F_3$. If we want to impose the presence of a NS5 brane $\mathrm{d}H = \delta_4$ then we necessarily have, using the duality relation, that also $\mathrm{d}F_7 = \delta_8$, which is the equation of motion for a D1-brane instanton smeared on the external space-time. Even if such solutions can exist [108], the flat metric suggests there are not these kind sources and the solution to the second of (7.9) is regular. Of course the same argument applies also for the D5-brane and the fundamental string.

7.2 IIA master class

The master class for type IIA supergravity was first discussed in [39, appendix C] and it is obtained, analogously to type IIB, by setting $b = 0$ and $a = \mathrm{i}$. Again from (6.24) we have that this class does not contain any additional isometric direction.

As we have seen in the previous section, it is useful to reduce the pure spinor system (5.31) to a set of differential form equations on M_4 only. This can be done by using equations (6.37), (6.38) and imposing that nothing must depend on S^2. After a bit of algebra, one can show that all the supersymmetry conditions follow from

$$\mathrm{d}(\mathrm{e}^{A}\, w) = 0\,, \quad \mathrm{d}(\mathrm{e}^{2A+C-\phi}) + \mathrm{e}^{2A-\phi}\, v_2 = 0\,, \quad \mathrm{d}(\mathrm{e}^{-2A+\phi}(v_1 + B_0 v_2)) = 0\,, \tag{7.14a}$$

$$\mathrm{d}(\mathrm{e}^{-\phi}\, v_1) \wedge w \wedge \overline{w} = 0\,, \quad \mathrm{d}(\mathrm{e}^{2C-\phi}(B_0 v_1 - v_2)) \wedge w \wedge \overline{w} = 0\,, \quad B_2 = 0\,. \tag{7.14b}$$

Equations (7.14a) can be locally solved by defining the vielbein in terms of local coordinates

$$v_1 = \mathrm{e}^{2A-\phi}\,\mathrm{d}x_1 + B_0\,\mathrm{e}^{-2A+\phi}\,\mathrm{d}x_2\,, \qquad v_2 = -\,\mathrm{e}^{-2A+\phi}\,\mathrm{d}x_2\,, \qquad w = \mathrm{e}^{-A}(\mathrm{d}x_3 + i\,\mathrm{d}x_4)\,, \tag{7.15}$$

with

$$x_2 = \mathrm{e}^{2A+C-\phi}\,, \tag{7.16}$$

which implies the following form for the metric

$$\begin{aligned} \mathrm{d}s^2 = {} & \mathrm{e}^{2A}\,\mathrm{d}s^2(\mathbb{R}^{1,3}) + \mathrm{e}^{-4A+2\phi}\left(\mathrm{d}x_2^2 + x_2^2 ds^2(S^2)\right) + \mathrm{e}^{-2A}\left(\mathrm{d}x_3^2 + \mathrm{d}x_4^2\right) \\ & + \mathrm{e}^{4A-2\phi}\left(\mathrm{d}x_1 + B_0\,\mathrm{e}^{-4A+2\phi}\,\mathrm{d}x_2\right)^2 . \end{aligned} \tag{7.17}$$

In terms of these local coordinates (7.14b) imposes the BPS conditions

$$\begin{aligned} &\partial_{x_2}\left(\mathrm{e}^{2A-2\phi}\right) = \partial_{x_1}\left(\mathrm{e}^{-2A} B_0\right)\,, \\ &\partial_{x_2}\left(x_2^2\,\mathrm{e}^{-2A} B_0\right) = \partial_{x_1}\left(x_2^2\,\mathrm{e}^{-6A+2\phi}(1+B_0^2)\right)\,, \end{aligned} \tag{7.18}$$

but, differently for type IIB master class, place no restriction on the functional dependence on (x_3, x_4).

The fluxes are given by

$$\begin{aligned} F_0 = {} & 0\,, \qquad\qquad B = x_2^2\,\mathrm{e}^{-4A+2\phi}\, B_0 \mathrm{Vol}(S^2)\,, \\ F_2 = {} & \left(\partial_{x_4}(\mathrm{e}^{2A-2\phi})\,\mathrm{d}x_3 - \partial_{x_3}(\mathrm{e}^{2A-2\phi})\,\mathrm{d}x_4\right)\wedge \mathrm{d}x_1 - \partial_{x_1}(\mathrm{e}^{-4A})\,\mathrm{d}x_3\wedge\mathrm{d}x_4 \\ & + \left(\partial_{x_4}(\mathrm{e}^{-2A} B_0)\,\mathrm{d}x_3 - \partial_{x_3}(\mathrm{e}^{-2A} B_0)\,\mathrm{d}x_4\right)\wedge\mathrm{d}x_2\,, \\ F_4 = {} & B\wedge F_2 + x_2^2\Big[-\partial_{x_2}(\mathrm{e}^{-4A})\,\mathrm{d}x_3\wedge\mathrm{d}x_4 - \left(\partial_{x_4}(\mathrm{e}^{-2A} B_0)\,\mathrm{d}x_3 - \partial_{x_3}(\mathrm{e}^{-2A} B_0)\,\mathrm{d}x_4\right)\wedge\mathrm{d}x_1 \\ & + \left(\partial_{x_4}(\mathrm{e}^{-6A+2\phi}(1+B_0^2))\,\mathrm{d}x_3 - \partial_{x_3}(\mathrm{e}^{-6A+2\phi}(1+B_0^2))\,\mathrm{d}x_4\right)\wedge\mathrm{d}x_2\Big]\wedge\mathrm{Vol}(S^2)\,, \end{aligned} \tag{7.19}$$

and ensuring that they obey Bianchi identities imposes the following PDEs

$$\begin{aligned} &\partial^2_{x_3}(\mathrm{e}^{2A-2\phi}) + \partial^2_{x_4}(\mathrm{e}^{2A-2\phi}) + \partial^2_{x_1}(\mathrm{e}^{-4A}) = 0\,, \\ &\partial^2_{x_3}(\mathrm{e}^{-2A} B_0) + \partial^2_{x_4}(\mathrm{e}^{-2A} B_0) + \partial_{x_1}\partial_{x_2}(\mathrm{e}^{-4A}) = 0\,, \\ &\partial^2_{x_3}(x_2^2\,\mathrm{e}^{-6A+2\phi}(1+B_0^2)) + \partial^2_{x_4}(x_2^2\,\mathrm{e}^{-6A+2\phi}(1+B_0^2)) + \partial_{x_2}(x_2^2\partial_{x_2}(\mathrm{e}^{-4A})) = 0\,. \end{aligned} \tag{7.20}$$

7.2.1 Analysis of the solution

Since in this class the metric is not even diagonal, it is much harder to deduce the brane content compared to the IIB analog. In order to get some insight we will proceed by adopting a large ansatz which will allow us to get a diagonal metric.

Equations (7.14a) automatically give a definition of a diagonal vielbein except for the component v_1, which we diagonalize by hand with the introduction of an arbitrary function μ:

$$v_1 = \mathrm{e}^{\phi+\mu}\,\mathrm{d}x_1\,. \tag{7.21}$$

Thanks to this assumption we get an easy expression for the metric

$$\mathrm{d}s^2 = \mathrm{e}^{2A}\,\mathrm{d}s^2(\mathbb{R}^{1,3}) + \mathrm{e}^{-4A+2\phi}(\mathrm{d}x_2^2 + x_2^2\,\mathrm{d}s^2(S^2)) + \mathrm{e}^{2\phi+2\mu}\,\mathrm{d}x_1^2 + \mathrm{e}^{-2A}(\mathrm{d}x_3^2 + \mathrm{d}x_4^2). \tag{7.22}$$

but we have to derive from scratch BPS constraints and the fluxes. In order to do so it is useful to define the following functions

$$f = \mathrm{e}^{-2A+2\phi+\mu}\,, \qquad g = \mathrm{e}^{2C}\,B_0\,, \tag{7.23}$$

so that (7.14b) implies $f = f(x_1, x_2)$, $g = g(x_1, x_2)$ and $\mu = \mu(x_1, x_3, x_4)$. Moreover we have the PDEs

$$\partial_{x_1} g = -x_2^2 \partial_{x_2} f\,, \tag{7.24a}$$

$$\partial_{x_2} g = x_2^2\,\mathrm{e}^{-\mu}\,\partial_{x_1}(f\,\mathrm{e}^{-4A-\mu})\,, \tag{7.24b}$$

which determine H in terms of the other fields.

The fluxes read

$$B_2 = g\mathrm{Vol}(S^2), \tag{7.25a}$$

$$F_2 = (\partial_{x_4}\mathrm{e}^{\mu}\,\mathrm{d}s_3 - \partial_{x_3}\mathrm{e}^{\mu}\,\mathrm{d}s_4)\wedge\mathrm{d}s_1 - f^{-1}\partial_{x_1}\mathrm{e}^{-4A}\,\mathrm{d}s_3\wedge\mathrm{d}s_4, \tag{7.25b}$$

$$\begin{aligned} F_4 = {} & B_2\wedge F_2 + \mathrm{Vol}(S^2)\wedge\big(x_2^2 f(\partial_{x_3}\mathrm{e}^{-4A-\mu}\,\mathrm{d}s_4 - \partial_{x_4}\mathrm{e}^{-4A-\mu}\,\mathrm{d}s_3)\wedge\mathrm{d}s_2 \\ & - g(\partial_{x_4}\mathrm{e}^{\mu}\,\mathrm{d}s_3 - \partial_{x_3}\mathrm{e}^{\mu}\,\mathrm{d}s_4)\wedge\mathrm{d}s_1 + (x_2^2\partial_{x_2}\mathrm{e}^{-4A} + g f^{-1}\partial_{x_1}\mathrm{e}^{-4A})\,\mathrm{d}s_3\wedge\mathrm{d}s_4\big), \end{aligned} \tag{7.25c}$$

and the Bianchi identities for these impose:

$$\partial_{x_2}(f^{-1}\partial_{x_1}\mathrm{e}^{-4A}) = 0\,, \tag{7.26a}$$

$$\triangle\,\mathrm{e}^{\mu} + \partial_{x_1}(f^{-1}\partial_{x_1}\mathrm{e}^{-4A}) = 0\,, \tag{7.26b}$$

$$f\triangle\,\mathrm{e}^{-4A-\mu} + \frac{1}{x_2^2}\partial_{x_2}(x_2^2\partial_{x_2}\mathrm{e}^{-4A}) + \frac{1}{x_2^2}\partial_{x_2} g\, f^{-1}\partial_{x_1}\mathrm{e}^{-4A} = 0\,, \tag{7.26c}$$

where $\triangle = \partial_{x_3}^2 + \partial_{x_4}^2$.

Even if we are now provided with a diagonal metric, solving these PDEs in full generality is still hard, so we impose some further ansatz which will allow us to get some intersecting-brane systems.

The first issue we address is that the left-hand side of (7.24b) is independent of (x_3, x_4), while the right-hand side is not a priori. We can deal with this by making an ansatz for e^{-4A}; for example, a convenient one (even if it is not the only one, as showed in [41, appendic C]) is

$$\mathrm{e}^{-4A} = f^{-1} H(x_1, x_2) S(x_3, x_4)^2\,, \qquad \mathrm{e}^{\mu} = S(x_3, x_4)\,. \tag{7.27}$$

The discussion turns out to be different if we consider the warping A to depend or not from x_1, so let's analyze these two situations independently. We will see that the discussion suggests that the IIA master class describes a localized system of intersecting D4-D6-NS5-branes.

$\partial_{x_1} \mathrm{e}^{-4A} = 0$: Localized D4-NS5 smeared D6 system

To make e^{-4A} independent from x_1 we need to set $H = f\lambda(x_2)$ where λ is an arbitrary function. We can solve the (7.24a) by introducing an auxiliary function $h(x_1, x_2)$ such that

$$f = \partial_{x_1} h\,, \qquad g = -x_2^2 \partial_{x_2} h\,. \tag{7.28}$$

We are then left with the PDEs

$$\partial_{x_2}(x_2^2 \partial_{x_2} \lambda) = 0\,, \qquad \triangle_2 S = 0\,, \qquad \frac{1}{x_2^2}\partial_{x_2}(x_2^2 \partial_{x_2} h) + \lambda \partial_{x_1}^2 h = 0\,, \tag{7.29}$$

to solve, with physical fields given by

$$\mathrm{e}^{2A} = \frac{1}{S\sqrt{\lambda}}\,, \quad \mathrm{e}^{2\phi+2\mu} = \frac{\partial_{x_1} h}{\sqrt{\lambda}}\,, \quad \mathrm{e}^{-4A+2\phi} = \sqrt{\lambda}\partial_{x_1} h\,, \quad \mathrm{e}^{2C} B_0 = -x_2^2 \partial_{x_2} h\,. \tag{7.30}$$

When $S = 1$ this solution reproduces, up to two T-dualities along $\mathrm{d}x_3$ and $\mathrm{d}x_4$, the massless system of [98], which describes localized D6-NS5 brane intersection. On the other hand when S is not constant but $\partial_{x_1} h = 1$ we have a D4-D6 system where the D6 brane is smeared along x_1, since its harmonic function is $S = S(x_3, x_4)$, while the harmonic function of the D4 factorizes in λS. Therefore we can in general interpret this solution as a system of localized D4-NS5 with smeared D6.

$\partial_{x_1} \mathrm{e}^{-4A} \neq 0$: Localized D4-D6-NS5 system

We now allow e^{-4A} to depend on x_1. In this way we define H such that it absorbs f^{-1} in the definition (7.27). From (7.26a) we can fix f from H up to an arbitrary function

$G = G(x_1)$ as $f = \partial_{x_1} H/G$. Thanks to equation (7.26b) we have that G must be at least linear in x_1, $G = c_1 x_1 + c_2$. The remaining equations are:

$$\triangle_2 S = -c_1 S^2 , \qquad \frac{1}{x_2^2}\partial_{x_2}\left(x_2^2 \partial_{x_2} H\right) + \frac{1}{2}\partial^2_{x_1} H^2 = c_1 \frac{\partial_{x_1} H^2}{c_1 x_1 + c_2} , \tag{7.31}$$

and fields are defined as:

$$\begin{aligned} &\mathrm{e}^{2A} = \frac{1}{S\sqrt{H}} , \quad \mathrm{e}^{2\phi+2\mu} = \frac{1}{\sqrt{HG}}\partial_{x_1} H , \quad \mathrm{e}^{-4A+2\phi} = \frac{\sqrt{H}}{G}\partial_{x_1} H , \\ &\partial_{x_1}(\mathrm{e}^{2C} B_0) = \frac{x_2^2}{G}\partial_{x_1}\partial_{x_2} H , \qquad \partial_{x_2}(\mathrm{e}^{2C} B_0) = \frac{x_2^2}{2}\partial_{x_1}\left(\frac{\partial_{x_1} H^2}{G}\right) . \end{aligned} \tag{7.32}$$

Now, notice that if $c_1 = 0$ and $S = 1$ this solution is the massive system of localized D6-NS5 of [98, 109] up to T-dualities, so for generic S and $c_1 = 0$ we have an analog of the previous case but with a localized "massive" D4-NS5 system with a smeared D6. When $c_1 \neq 0$ the solution is more exotic and the physical interpretation less clear, but it points in the direction of a localized D4-D6-NS5-brane system.

7.3 Generating the other classes

All the possible $\mathbb{R}^{1,3} \times S^2$ solutions with equal spinor norm classified in [39, 40] fall into three distinct classes characterized by the isometry number of the internal four-dimensional manifold M_4, which corresponds to a precise choice of the parameters a, b, as already shown in (6.24) and (6.35). Specifically one has

$$\text{Case I: } M_4 = M_2 \times \Sigma_2 , \qquad \text{Case II: } \begin{array}{ccc} T^1 & \hookrightarrow & M_4 , \\ & & \downarrow \\ & & M_3 \end{array} \qquad \text{Case III: } \begin{array}{ccc} T^2 & \hookrightarrow & M_4 , \\ & & \downarrow \\ & & M_2 \end{array}$$

where Σ_2 is a two-dimensional Riemann surface spanned by $\mathrm{d}x_3$ and $\mathrm{d}x_4$. The master classes we discussed above belong to Case I and exhaust it. Cases II and III contain one and two constant parameters respectively and often reduce to locally five- and six-dimensional Minkowski backgrounds.

These higher dimensional flat space solutions are relatively easy to find. Let's start by considering the IIB master class 7.1; it is a straightforward application of subsection 1.3.1 to notice that by imposing that one of $\partial_{x_3}, \partial_{x_4}$ is a symmetry of the solution and T-dualizing along it we end up with the $\mathbb{R}^{1,4} \times S^2$ class in IIA of [39, section 4.2], while if we make both isometries we find, after two T-dualities, $\mathbb{R}^{1,5} \times S^2$ solutions of [40, section 4.1].

A similar story holds also for type IIA master class 7.1, indeed one can generate the $\mathbb{R}^{1,4} \times S^2$ solution in [40, section 4.2] by T-dualizing one direction among $\partial_{x_3}, \partial_{x_4}$,

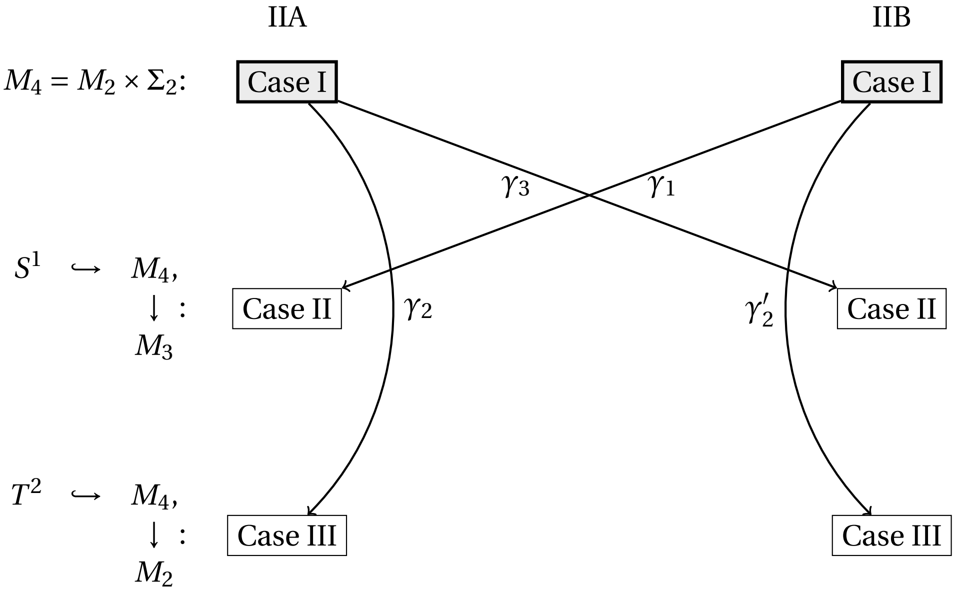

Figure 7.1: Depiction of the chains of dualities leading to the various $\text{Mink}_4 \times S^2$ class in type II with equal spinor norm. Cases I in IIA and IIB (shaded grey) are the fundamental master classes from which the other solutions are generated. Specifically γ_1 represent a transformation where one performs a diffeomorphism mixing the M_2 and Σ_2 factors in IIB Case I introducing a parameter, then imposes an isometry and T-dualizing on it. γ_2, γ_2' represent the following: impose $\Sigma_2 = T^2$ in Case I and T-dualize on both directions, this case is a $\mathbb{R}^{1,5}$ solution that we call case III_0, one then performs a formal U-duality followed by a T-s-T transformation with T^2, which generates a two-parameter deformation of case III_0 governed by the same PDEs, i.e. case III. γ_3 represents imposing an isometry in the Σ_2 factor of IIA case I, then T-dualizing to get to IIB. One then performs a formal U-duality on the spatial directions of $\mathbb{R}^{1,3}$.

and, as it was already explained in the subsection 7.2.1, by means of two T-duality it is possible to get all $\mathbb{R}^{1,5} \times S^2$ solutions of [98], which are exactly [39, section 4.1].

Even if these higher-dimensional Minkowski backgrounds do not have any parameter turned on, they are governed by (almost) the same PDEs of the rest of Case II and III. This indicates that a and b can be seen as parametric deformations of these solutions, which suggests that these parameters can be generated by some sort of duality. Two obvious candidates are formal U-dualities of the type in [110, 111] and T-s-T transformations [112], which are both solution generating techniques involving chains of string dualities and coordinate transformations.

In this section, following [41], we will see that it is possible to explain the origin of the parametric deformations and we will prove that Case II and III can be generated from the master classes. The map between the solutions is summarized in figure 7.1.

7.3.1 Generating Case II

Let's start with establishing how the parametric deformation of Case II in IIA and IIB is realized. We will see that for IIA we need to start again from the IIB master class while in IIB the generic solution is essentially the U-dual of the five-dimensional Minkowski sub-case we discussed before.

Case II in IIA

We start by performing a coordinate transformation on the solution in 7.1

$$x_1 \to x_1 - \frac{a_2\sqrt{f}}{b}x_4 \tag{7.33}$$

where $a_2\sqrt{f}/b$ is a constant and $b^2+a_2^2=1$. The NSNS sector takes the form

$$\begin{aligned}
&\mathrm{d}s^2 = \mathrm{e}^{2A}\mathrm{d}s^2(\mathbb{R}^{1,3}) + \mathrm{e}^{-2A}\left(\frac{1}{f}\big(b^2\,\mathrm{d}x_1^2+\mathrm{d}x_2^2+x_2^2\,\mathrm{d}s^2(S^2)\big)+\mathrm{d}x_3^2+\frac{1}{b^2}\left(\mathrm{d}x_4+\frac{ba_2}{\sqrt{f}}dx_1\right)^2\right),\\
&B = g\mathcal{C}_2 - \frac{a_2\sqrt{f}}{b}g\frac{\mathrm{d}x_2\wedge\mathrm{d}x_4}{x_2^2}, \qquad \mathcal{C}_2=\mathrm{Vol}(S^2)+\frac{\mathrm{d}x_1\wedge\mathrm{d}x_2}{x_2^2}.
\end{aligned} \tag{7.34}$$

If we now impose that ∂_{x_4} is a symmetry of the whole solution, we have that the PDEs governing the system (7.9), (7.10) become

$$\partial_{x_3}^2 f = 0, \quad \partial_{x_3}^2(fg)=0, \tag{7.35a}$$

$$\frac{1}{b^2}\partial_{x_1}^2(\mathrm{e}^{-4A}) + \frac{1}{x_2^2}\partial_{x_2}(x_2^2\partial_{x_2}(\mathrm{e}^{-4A})) + \partial_{x_3}^2(f^{-1}\mathrm{e}^{-4A}) + \frac{1}{x_2^4}\partial_{x_3}^2(fg^2)=0, \tag{7.35b}$$

which are exactly the same PDEs we have in [39, section 4.4]. If we now T-dualize on ∂_{x_4} applying the rule of section 1.3.1, the NSNS sector is mapped in

$$\begin{aligned}
&\mathrm{d}s^2 = \mathrm{e}^{2A}\mathrm{d}s^2(\mathbb{R}^{1,3}) + \mathrm{e}^{2A}b^2\left(\mathrm{d}x_4 - \frac{a_2 g\sqrt{f}}{b}\mathrm{d}x_2\right)^2 + \frac{\mathrm{e}^{-2A}}{f}\big(b^2\,\mathrm{d}x_1^2+\mathrm{d}x_2^2+f\,\mathrm{d}x_3^2+x_2^2\mathrm{d}s^2(S^2)\big),\\
&B = g\mathcal{C}_2 + \frac{a_2 b}{\sqrt{f}}\mathrm{d}x_1\wedge\left(\mathrm{d}x_4 - \frac{a_2 g\sqrt{f}}{bx_2^2}\mathrm{d}x_2\right), \qquad \mathrm{e}^{-\phi}=\frac{f}{b}\mathrm{e}^{-A},
\end{aligned} \tag{7.36}$$

which reproduces the NSNS sector of [39, section 4.4]. With an explicit computation it is possible to prove that also the RR-fluxes match.

Case II in IIB

To generate case II in IIB one starts from the five-dimensional Minkowski solution we discussed at the beginning of this section, which is obtained by T-dualizing IIA master class along ∂_{x_4} (∂_{x_3} is physically equivalent). Using (1.31) the NSNS sector is locally mapped to

$$\begin{aligned} \mathrm{d}s^2 &= e^{2A}\,\mathrm{d}s^2(\mathbb{R}^{1,3}) + e^{2A}\,\mathrm{d}x_4^2 + e^{-2A}\,\mathrm{d}x_3^2 + e^{-6A+2\phi}\left(\mathrm{d}x_2^2 + x_2^2\,\mathrm{d}s^2(S^2)\right) \\ &+ e^{6A-2\phi}\left(\mathrm{d}x_1 + B_0\,e^{-6A+2\phi}\,\mathrm{d}x_2\right)^2, \qquad B = x_2^2\,e^{-6A+2\phi}\,B_0\mathrm{Vol}(S^2) \end{aligned} \tag{7.37}$$

where all functions have support on (x_1, x_2, x_3) only, while the RR sector reads

$$\begin{aligned} &F_1 = \partial_{x_3}(e^{4A-2\phi})\,\mathrm{d}x_1 + \partial_{x_3}(e^{-2A}B_0)\,\mathrm{d}x_2 - \partial_{x_1}(e^{-4A})\,\mathrm{d}x_3\,, \\ &F_3 = B\wedge F_1 + x_2^2\Big(\partial_{x_2}(e^{-4A})\,\mathrm{d}x_3 - \partial_{x_3}(e^{-2A}B_0)\,\mathrm{d}x_1 - \partial_{x_3}(e^{-8A+2\phi}(1+B_0^2))\,\mathrm{d}x_2\Big)\wedge\mathrm{Vol}(S^2)\,, \\ &F_5 = 0. \end{aligned} \tag{7.38}$$

Now we will show how to introduce a parametric deformation c_0 via a formal U-duality. One begins by T-dualizing on the three space-like isometric directions contained in the external-space metric, which maps this solution to IIA with only (F_4, H) non trivial. Specifically, we have that H is unaffected while F_4 reads

$$F_4 = e^{-3A}\mathrm{Vol}_3\wedge F_1 \pm e^{A}\,\mathrm{d}t\wedge *_6 F_3 \tag{7.39}$$

where Vol_3 is the spatial volume form in $\mathbb{R}^{1,3}$ and the sign $\pm$ depends on the definition of Vol_3 and ont he order of the T-duality transformations. However we do not need to take care about this sign because we will T-dualize everything back to IIB in the same way. Before doing this, let's lift this solution to M-theory using 1.3.3 and let's perform the following change of coordinates which involve the M-theory extra dimension ∂_z

$$\mathrm{d}t \to b_1\,\mathrm{d}t + c_0\,\mathrm{d}z\,, \qquad \mathrm{d}z \to b_2\,\mathrm{d}t + \mathrm{d}z\,. \tag{7.40}$$

Here c_0, b_1, b_2 are all constants, c_0 is the deformation parameter, while b_1, b_2 should be non zero and satisfy $-b_1 + b_2 c_0 > 0$, but are otherwise arbitrary. The relevant part of the eleven-dimensional metric get mapped as follows:

$$\begin{aligned} -e^{-\frac{2}{3}\phi+2A}\,\mathrm{d}t^2 + e^{\frac{4}{3}\phi}\,\mathrm{d}z^2 \to &-e^{-\frac{2}{3}\phi+2A}\,\frac{(b_1 - b_2 c_0)^2}{1 - c_0^2\,e^{2A-2\phi}}\,\mathrm{d}t^2 \\ &+ e^{\frac{4}{3}\phi}(1 - c_0^2\,e^{2A-2\phi})^2\left(\mathrm{d}z + \frac{b_2 - b_1 c_0\,e^{2A-2\phi}}{1 - c_0^2\,e^{2A-2\phi}}\,\mathrm{d}t\right)^2. \end{aligned} \tag{7.41}$$

If we now reduce back to type IIA, always along ∂_z, we get that F_4 and H have mixed their components and we have now F_2 turned on, which has a leg in the time direction. The last step is to perform again three T-dualities along Vol_3 to get back to a four-dimensional Minkowski solution in IIB. These operations have left F_1 and the dilaton unchanged, while F_2 in IIA turns on F_5 in IIB. After rescaling $\mathrm{d}t \to -(b_1 - b_2 c_0)\,\mathrm{d}t$ all dependence on b_1, b_2 drops out of the solution and one finds that the new metric reads

$$\mathrm{d}s^2 = \frac{\mathrm{e}^{2A}}{\kappa_\perp}\,\mathrm{d}s^2(\mathbb{R}^{1,3}) + \kappa_\perp\,\mathrm{d}s^2(\mathrm{M}_6)\,, \tag{7.42}$$

where $\mathrm{d}s^2(\mathrm{M}_6)$ is the original internal metric before the U-duality and we have introduced $(\kappa_\parallel, \kappa_\perp)$ such that

$$\kappa_\parallel = c_0\,\mathrm{e}^{4A-\phi}\,, \qquad \kappa_\parallel^2 + \kappa_\perp^2 = 1\,. \tag{7.43}$$

The remaining fluxes after the U-duality become

$$\begin{aligned} H &\to \mathrm{d}B + \kappa_\parallel\,\mathrm{e}^{\phi} *_6 F_3\,, \\ F_3 &\to F_3 - \kappa_\parallel\,\mathrm{e}^{-\phi} *_6 H_3\,, \\ F_5 &\to \Big(1 + *\Big)\mathrm{Vol}_4 \wedge \mathrm{d}\left(\frac{\mathrm{e}^{4A-\phi}\kappa_\parallel}{\kappa_\perp^2}\right). \end{aligned} \tag{7.44}$$

It is then not hard to check that if we rescale

$$\mathrm{e}^{2A} \to \kappa_\perp\,\mathrm{e}^{2A}\,, \qquad B_0 \to \kappa_\perp B_0 \tag{7.45}$$

we get exactly the metric, dilaton, and fluxes of [40, section 4.4.2]. Notice that the PDEs are unaffected by the chain of dualities we used and they are just modified by (7.45). Thus we have shown that case II in IIB follows from case I in IIA by first T-dualizing to IIB, then performing a U-duality.

7.3.2 Generating case III

In this section we deal with the derivation of Case III in type IIA and IIB starting from the master systems. For these classes we have to generate a double parametric deformation, and this can be done by T-s-T transformations, since now we have two isometric directions, and by the same kind of U-duality we have seen in the previous section. Since the procedure is similar to the methods we have seen so far, we will only sketch the main steps.

Case III in IIB

Case III in IIB can be generated from the six-dimensional Minkowski case we have describe at the beginning of this section, which is nothing but the D5-brane in flat space. More precisely, this solution is given by the following metric

$$\mathrm{d}s^2 = \frac{1}{\sqrt{h}}\left(\mathrm{d}s^2(\mathbb{R}^{1,3}) + \mathrm{d}x_3^2 + \mathrm{d}x_4^2\right) + +\sqrt{h}\Big(\mathrm{d}x_1^2 + \mathrm{d}x_2^2 + x_2^2\,\mathrm{d}s^2(S^2)\Big)\,, \tag{7.46}$$

and fluxes

$$F_3 = x_2^2\Big(\partial_{x_2}h\,\mathrm{d}x_1 - \partial_{x_1}h\,\mathrm{d}x_2\Big)\wedge \mathrm{Vol}(S^2)\,, \qquad F_1 = F5 = H = 0\,, \tag{7.47}$$

where $h = \mathrm{e}^{-4A} = \mathrm{e}^{-2\Phi}$ is just a function of $(x1, x2)$ which satisfies a Laplace equation on the flat space generated by $\{x_1, x_2, S^2\}$.

The two parametric deformations can be obtained from this solution as follows:

1. Formal U-duality on spatial external directions (as the on in 7.3.1).
2. T-dualize on ∂_{x_4} and gauge transformation of the NSNS potential.
3. Shift $x_3 \to x_3 + \gamma x_4$.
4. T-dualize back on ∂_{x_4}.

One needs to supplement this by rescaling the dilaton, Minkowski and local coordinates, but after doing this carefully one is mapped to Case III in IIB.

Case III in IIA

Case III in IIA can be generated in a similar fashion. Again it is convenient to start from $\mathbb{R}^{1,5}$ solution without any parameter turned on. This solution is explicitly given by the following fields:

$$\mathrm{d}s^2 = \mathrm{e}^{2A}\big(\mathrm{d}s^2(\mathbb{R}^{1,3}) + \mathrm{d}x_3^4 + \mathrm{d}x_4^2\big) + \mathrm{e}^{-4A+2\Phi}\mathrm{d}x_1^2 + \mathrm{e}^{-8A+2\Phi}\big(\mathrm{d}x_2^2 + x_2^2\,\mathrm{d}s^2(S^2)\big)\,, \tag{7.48a}$$

$$B = -\frac{1}{m}x_2^2\partial_{x_2}\mathrm{e}^{-4A}\mathrm{Vol}(S^2)\,, \qquad \mathrm{e}^{2\Phi} = \frac{2}{m}\partial_{x_1}\mathrm{e}^{2A}\,, \qquad F_0 = m\,, \tag{7.48b}$$

$$F_2 = BF_0 - x_2^2\big(\partial_{x_2}(\mathrm{e}^{-4A}) - \mathrm{e}^{-2A}B_0\partial_{x_1}(\mathrm{e}^{-4A})\big)\mathrm{Vol}(S^2)\,, \qquad F_4 = 0\,. \tag{7.48c}$$

The warping A must satisfy the non-linear equation

$$\frac{1}{x_2^2}\partial_{x_2}(x_2^2\partial_{x_2}(\mathrm{e}^{-4A})) + \frac{1}{2}\partial_{x_1}^2(\mathrm{e}^{-8A}) = 0\,. \tag{7.49}$$

Now, it is possible to check that the following chain of dualities, boosts and shifts maps this to Case III in IIA

1. Four T-dualities performed on the spatial $\mathbb{R}^{1,3}$ directions and one direction in T^2, say x_4.
2. Lift to M-theory, boost along the M-theory isometry, reduce back to IIA.
3. Shift $x_3 \to x_3 + \gamma x_4$.
4. Four more T-dualities on the spatial external directions and x_4.

This completes the proof of the duality web depicted in figure 7.1.

CHAPTER 8

CLASSIFICATION IN M-THEORY

In the previous section we classified all the $\mathbb{R}^{1,3} \times S^2$ solutions in type II supergravity. Besides some considerations involving the spinorial system, most of the classification was carried on by exploiting pure spinor equations. Of course these do not apply to M-theory, so we will need an analog of them. In [37] a system of form equations which is equivalent to background supersymmetry for four-dimensional vacua is provided; even if in this case the system is completely general, differently for example from [74], supersymmetry turns out to be formulated in a rather cumbersome way. For this reason before using [37] in its full generality, we analyze again the spinorial system in order to find some simplification imposed by the S^2 ansatz.

8.1 $\mathbb{R}^{1,3} \times S^2$ ansatz

The treatment of the $\mathbb{R}^{1,3} \times S^2$ ansatz for M-theory is essentially similar to what we have seen in section 6.2, let's then quickly review here the argument in order to fix the notation. First of all we want to preserve the symmetries of the external space, which impose that the eleven-dimensional metric decomposes as a warped product

$$d s^2 = e^{2\Delta} d s^2(\mathbb{R}^{1,3}) + d s^2(M_7) , \tag{8.1}$$

where $e^{2\Delta}$ is a function with support on M_7 only and the four-form flux F is necessarily purely magnetic in the supersymmetric case, as showed in [113]. Since the SU(2) preserving ansatz requires $\mathcal{N} = 2$ supersymmetry, our eleven-dimensional spinor will decompose as

$$\epsilon = \sum_{a=1}^{2} \left(\zeta_+^a \otimes \chi^a + (\zeta_+^a)^c \otimes (\chi^a)^c \right) , \tag{8.2}$$

where both ζ_+^a and χ^a are charged under SU(2) R-symmetry, so that ϵ is invariant.

The R-symmetry is again realized by a S^2 factor in the internal geometry, so that metric and flux on M_7 become

$$\mathrm{d}s^2(M_7) = \mathrm{e}^{2C}\mathrm{d}s^2(S^2) + \mathrm{d}s^2(M_5)\,, \qquad F = F_4 + \mathrm{e}^{2C}\mathrm{Vol}(S^2)\wedge F_2, \tag{8.3}$$

where e^{2C} is a function depending on the coordinates on the remaining five-dimensional manifold M_5.

At the level of the spinors the SU(2) R-symmetry will be realized again by using the doublet (ξ, ξ^c) given in (6.6), which imposes the following decomposition:

$$\chi = \xi\otimes\eta^1 + \sigma_3\xi\otimes\eta^2\,, \tag{8.4}$$

where we dropped the doublet index since R-symmetry allows us to restrict to the $\mathcal{N} = 1$ case.

8.2 Spinorial analysis

As done for type II theories, we start the classification by deriving some conditions that will be useful in the next section.

The eleven-dimensional supersymmetry equation (1.6) can be rewritten using (A.20) and (A.22) as

$$D_M\epsilon + \frac{1}{24}(3F\gamma_M - \gamma_M F)\epsilon = 0. \tag{8.5}$$

By decomposing ϵ according to (8.2) (restricting to $\mathcal{N} = 1$) we get:

$$\left(\mathrm{e}^{-\Delta}\partial_a\mathrm{e}^{\Delta}\gamma^a + \frac{1}{6}F\right)\chi = 0\,, \tag{8.6a}$$

$$\left(D_a + \frac{1}{24}(3F\gamma_a - \gamma_a F)\right)\chi = 0\,. \tag{8.6b}$$

When $\chi^c = \chi$ we have that the spinor defines a G_2 structure on M_7. In particular, it is easy to check that in this case the solution has actually G_2 holonomy and $F = 0$ [114]. Although such solutions may exist, we need not consider them explicitly; indeed when we restrict to $M_7 = S^2\times M_5$ we have that the Killing spinors ξ, ξ^c on S^2 span a basis, and in particular cannot be equal. This means that imposing $\chi^c = \chi$ forces χ to contain two separable five-dimensional systems, one coupling to ξ and the other to ξ^c, which just impose additional constraints compared to the case that follows from the SU(3) structure $\chi^c \neq \chi$, where we can define χ as in (8.4) (i.e., containing just ξ and not ξ^c). Therefore solutions with G_2 holonomy can be considered as particular cases of the SU(3) system, so we can assume $\chi\neq\chi^c$ without loss of generality.

Now let's analyze the zero-form constraints on M_7, using (8.6b)

$$\partial_a(\chi^\dagger\chi) = D_a\chi^\dagger\chi + \chi^\dagger D_a\chi = \partial_a\Delta\chi^\dagger\chi\,, \tag{8.7}$$

we have that we can set without loss of generality

$$||\chi||^2 = 2\,\mathrm{e}^{\Delta}\,. \tag{8.8}$$

Moreover, by calculating in a similar way $\partial_a(\overline{\chi}\chi)$ we get the following condition

$$\overline{\chi}\chi = c\,\mathrm{e}^{-2\Delta} \tag{8.9}$$

where c is a constant.

Let's now go a bit further and impose also the spinor decomposition (8.4). The zero-form equations in terms of the five- and two-dimensional spinors read

$$(\eta_1^\dagger\eta_1 + \eta_2^\dagger\eta_2) + y_3(\eta_1^\dagger\eta_2 + \eta_2^\dagger\eta_1) = 2\,\mathrm{e}^{\Delta}\,; \tag{8.10}$$

to preserve the SU(2) R-symmetry the warp factor cannot depend on the embedding coordinates of the sphere y_i, so we have

$$||\eta_1||^2 + ||\eta_2||^2 = 2\,\mathrm{e}^{\Delta}, \qquad \mathrm{Re}(\eta_1^\dagger\eta_2) = 0. \tag{8.11}$$

On the other hand (8.9) becomes

$$-(y_1 - \mathrm{i}y_2)\overline{\eta_1}\eta_2 = c\,\mathrm{e}^{-2\Delta} \tag{8.12}$$

and then we must have

$$\overline{\eta_1}\eta_2 = 0, \qquad c = 0. \tag{8.13}$$

Given the normalization (8.8), it is better to rescale the seven-dimensional spinor as

$$\chi = \mathrm{e}^{\frac{\Delta}{2}}(\xi\otimes\eta^1 + \sigma_3\xi\otimes\eta^2) \tag{8.14}$$

instead of (8.4). This allows us to decompose the spinors on M_5 in a common basis in terms of a unit norm spinor η:

$$\eta_1 = \eta, \qquad \eta_2 = \mathrm{i}\cos\alpha\,\eta + \frac{1}{2}\sin\alpha\,\overline{w}\eta\,, \tag{8.15}$$

where w is a complex normalized vector on M_5. This is the most general parametrization consistent with (8.11), (8.13)[1].

[1] Actually it would be in principle possible to be slightly more general than this and demand that η_1 and η_2 do not have the same norm, however it is proved in [41] that this situation is actually forbidden by supersymmetry.

8.3 Supersymmetry conditions

The conditions found in the previous section considerably simplify supersymmetry equations, and in particular we can restrict ourselves to consider [33] (similar systems were previously obtained in [74, 115]), where necessary and sufficient geometric conditions were derived for the preservation of supersymmetry in the particular case $\overline{\chi}\chi = 0$. The system is given by the following equations:

$$\mathrm{d}(\mathrm{e}^{2\Delta} K) = 0\,, \tag{8.16a}$$
$$\mathrm{d}(\mathrm{e}^{4\Delta} J) = -\,\mathrm{e}^{4\Delta} *_7 F\,, \tag{8.16b}$$
$$\mathrm{d}(\mathrm{e}^{3\Delta}\Omega) = 0, \tag{8.16c}$$
$$\mathrm{d}(\mathrm{e}^{2\Delta} J\wedge J) = -2\,\mathrm{e}^{2\Delta} F\wedge K \tag{8.16d}$$

where

$$K_a = \chi^\dagger \gamma_a \chi\,, \quad J_{ab} = -\mathrm{i}\chi^\dagger \gamma_{ab}\chi\,, \quad \Omega_{abc} = -\mathrm{i}\chi^{c\dagger}\gamma_{abc}\chi \tag{8.17}$$

define an SU(3)-structure on the seven-dimensional internal space. Imposing these conditions and the Bianchi identity for F is enough to get a proper $\mathcal{N} = 1$ solution in M-theory [116].

The next step is to specialize (8.17) to the case $M_7 = S^2 \times M_5$, which can be done by making use of (8.14) together with of the bilinear product identity

$$\left[\xi^1 \otimes \eta^1\right] \otimes \left[\xi^2 \otimes \eta^2\right]^\dagger = (\eta^1 \otimes \eta^{2\dagger})_+ \wedge (\xi^1 \otimes \xi^{2\dagger}) + (\eta^1 \otimes \eta^{2\dagger})_- \wedge (\sigma_3 \xi^1 \otimes \xi^{2\dagger}) \tag{8.18}$$

where $\pm$ denotes the even/odd degree components of a form only. The bilinears generated from η are given in [30]. Differently from what we have seen in the analysis of type II theories, in M-theory a spinor on a five-dimensional manifold is not enough to define in general an identity structure, but just a SU(2)-structure. However, since we have already picked a complex vector w in (8.15), we are able to formally write the SU(2) structure in terms of a local vielbein:

$$\begin{aligned} &\eta\otimes\eta^\dagger = \frac{1}{4}(1+v)\wedge \mathrm{e}^{-\mathrm{i}j_2}\,, &&\eta\otimes\eta^{c\dagger} = \frac{1}{4}(1+v)\wedge\omega_2\,,\\ &\omega_2 = w\wedge u\,, && j_2 = \frac{\mathrm{i}}{2}(w\wedge\overline{w} + u\wedge\overline{u})\,, \end{aligned} \tag{8.19}$$

where

$$v\,, \quad w_1 = \mathrm{Re}\,w\,, \quad w_2 = \mathrm{Im}\,w\,, \quad u_1 = \mathrm{Re}\,u\,, \quad u_2 = \mathrm{Im}\,u \tag{8.20}$$

is the five-dimensional vielbein. Notice that just v can be directly defined in terms of the bilinears of η, and in order to determine w we need $\alpha \neq 0$ in (8.15). This will affect the classification of M-theory solutions, indeed we will get two different classes depending on whether α vanishes or not.

We have now all the elements to translate the SU(3)-structure (8.17) on M_7 in terms of the vielbein on M_5 and S^2. In order to do it, it is convenient to perform the following rotation:

$$w_2 \to \sin\alpha\, v + \cos\alpha\, w_2\,, \qquad v \to \cos\alpha\, v - \sin\alpha\, w_2\,. \tag{8.21}$$

In this new frame the vector K simply reads

$$K = -\,\mathrm{e}^{C} \cos\alpha\, \mathrm{d}\, y_3 + \sin\alpha\, w_1 + y_3 \cos\alpha\, v. \tag{8.22}$$

Let's now move on and examine J and Ω, which define and SU(3)-structure on the internal manifold orthogonal to K. Any SU(3)-structure in a six-dimensional space can be expressed in canonical form using a complex vielbein E^i as

$$J = \frac{\mathrm{i}}{2}\left(E^1 \wedge \overline{E}^1 + E^2 \wedge \overline{E}^2 + E^3 \wedge \overline{E}^3\right), \qquad \Omega = E^1 \wedge E^2 \wedge E^3\,, \tag{8.23}$$

where

$$E^1 = \cos\alpha\, w_1 + \mathrm{i} w_2 + \sin\alpha (\mathrm{e}^{C} \mathrm{d}\, y_3 - y_3\, v)\,, \qquad E^2 = u\,, \qquad E^3 = (y_1 + \mathrm{i} y_2)\, v - \mathrm{e}^{C}\, \mathrm{d}(y_1 + \mathrm{i} y_2)\,. \tag{8.24}$$

One can explicitly check that this complex vielbein leads to the factorized metric on $M_7 = S^2 \times M_5$

$$K^2 + E^i \overline{E}^i = \mathrm{e}^{2C}\, \mathrm{Vol}(S^2) + v^2 + w_1^2 + w_2^2 + u_1^2 + u_2^2\,. \tag{8.25}$$

Now we can start the classification by using the explicit expression for (K, J, Ω) in (8.16). As said before, we have to distinguish $\alpha = 0$ from the generic case.

8.4 Case A: $\alpha = 0$

In this case we have that $\eta_1 = \mathrm{i}\eta_2$ as one can check by looking at (8.15). Since the two spinors are proportional to one another they define a SU(2)-structure on M_5, which means that we will not get a local expression for the vielbein u, w by just imposing supersymmetry. (8.16) reduces to these conditions for the SU(2) structure:

$$\mathrm{d}\, \mathrm{e}^{2\Delta + C} + \mathrm{e}^{2\Delta}\, v = 0\,, \tag{8.26}$$

$$\mathrm{d}(\mathrm{e}^{\Delta}\, \omega_2) = 0, \tag{8.27}$$

while those involving the flux read

$$\mathrm{d}(\mathrm{e}^{4\Delta}\, j_2) - \mathrm{e}^{4\Delta} \star_5 F_2 = \mathrm{d}(\mathrm{e}^{-2\Delta}\, j_2) + \mathrm{e}^{-2\Delta}\, F_2 \wedge v = F_4 = 0. \tag{8.28}$$

We can solve (8.26) without loss of generality by introducing a local coordinate ρ such that

$$\rho = \mathrm{e}^{2\Delta + C}\,, \qquad v = -\,\mathrm{e}^{-2\Delta}\, d\rho. \tag{8.29}$$

Summarizing, we have that this class is determined by the following fluxes:

$$\begin{aligned} d s^2 &= e^{2\Delta} d s^2(\mathbb{R}^{1,3}) + e^{-4\Delta}\left(d\rho^2 + \rho^2 d s^2(S^2)\right) + d s^2(M_4), \\ F &= e^{-8\Delta}\rho^2 \mathrm{Vol}(S^2) \wedge \star_5 d(e^{4\Delta} j_2) \end{aligned} \tag{8.30}$$

where M_4 supports an SU(2)-structure which is complex thanks to (8.27).

8.4.1 Analysis of the solution

Without an explicit metric it is difficult to understand what kind of backgrounds this class can accommodate, so in this subsection we will specialize it by making some ansatz. We will see that this case is actually quite broad and may comprehend various system of localized M2-M5-branes, KK6-monopoles and Atiyah–Hitchin singularities, as showed in [41, section 5.2].

M5-brane case

It is easy to check for example that the supersymmetry condition (8.27) can be solved by imposing the presence of a space-filling M5-brane sitting on a dimension-two surface inside M_4. For example we can choose

$$w = e^{-2\Delta}(d x_1 + i d x_2)\,, \quad u = e^{\Delta}(d x_3 + i d x_4)\,, \tag{8.31}$$

with $\Delta = \Delta(\rho, x_1, x_2)$. Using this ansatz inside (8.28) we get

$$F_2 = -\rho^2 e^{-2C}\left(\partial_\rho e^{-6\Delta} d x_1 \wedge d x_2 + \partial_{x_2} e^{-6\Delta} d\rho \wedge d x_1 + \partial_{x_1} e^{-6\Delta} d x_2 \wedge d\rho\right). \tag{8.32}$$

The solution is then given by the following fluxes

$$d s^2 = e^{2\Delta} d s^2(\mathbb{R}^{1,5}) + e^{-4\Delta}\left(d x_1^2 + d x_2^2 + d\rho^2 + \rho^2 d s^2(S^2)\right), \tag{8.33}$$

$$F = -\rho^2\left(\partial_\rho e^{-6\Delta} d x_1 \wedge d x_2 + \partial_{x_2} e^{-6\Delta} d\rho \wedge d x_1 + \partial_{x_1} e^{-6\Delta} d x_2 \wedge d\rho\right) \wedge \mathrm{Vol}(S^2), \tag{8.34}$$

plus the Bianchi identity

$$\frac{1}{\rho^2}\partial_\rho\left(\rho^2 \partial_\rho e^{-6\Delta}\right) + \partial_{x_1}^2 e^{-6\Delta} + \partial_{x_2}^2 e^{-6\Delta} = 0\,, \tag{8.35}$$

which is a five-dimensional Laplace equation with a manifest SO(3) symmetry, so this solution describes a localized M5-brane with a rotational symmetry in its codimensions.

M2-KK-monopole system

Another easy way to solve (8.27)-(8.28) is to impose SU(2)-holonomy on M_4:

$$e^{\Delta}\,\omega_2 = \widetilde{\omega}_2 = e^{\Delta}\, j_2 = \widetilde{j}_2, \tag{8.36}$$

with $\widetilde{\omega}_2$, $\widetilde{j}_2$ closed. The local form of the solution is then

$$\begin{aligned} d s^2 &= H^{-2/3}\, d s^2(\mathbb{R}_{1,3}) + H^{1/3}\Big(d s^2(M_4) + H\big(d\rho^2 + \rho^2\, d s^2(S^2)\big)\Big), \\ F &= Q\,\widetilde{j}_2 \wedge \mathrm{Vol}(S^2)\,, \qquad H = 1 + \frac{Q}{\rho}. \end{aligned} \tag{8.37}$$

with M_4 any SU(2)-holonomy manifold. These solutions are all non-compact and the warp factor is not indicative of a simple brane set up. However, following [117], it looks like a particular KK6-M2 system.

8.5 Case B: $\alpha \neq 0$

Even if in this case α is generic, we will see that it actually factors out from all of the supersymmetry conditions and in the end nothing depend from its value as long as $\alpha \neq 0$. This condition implies that the two spinors η^1 and η^2 are linearly independent and therefore they define an identity structure, which is given by the one- and two-form supersymmetry constraints

$$d(e^{2\Delta + C}) + e^{2\Delta}\, v = d(e^{2\Delta}\, w) = d(e^{-\Delta}\, u) = 0\,, \tag{8.38}$$

which can be solved without loss of generality by introducing local coordinates

$$v = -e^{-2\Delta}\, d\rho\,, \quad w = e^{-2\Delta}(d x_1 + i\, d x_2)\,, \quad u = e^{\Delta}(d x_3 + i\, d x_4)\,, \quad \rho = e^{2\Delta + C}\,, \tag{8.39}$$

where

$$\rho = e^{2\Delta + C}\,. \tag{8.40}$$

The supersymmetry conditions on fluxes are given by:

$$d(e^{4\Delta}\, u_1 \wedge u_2) + e^{4\Delta} *_5 F_2 = 0\,, \tag{8.41}$$

$$d(e^{2C + 2\Delta}\, w_2 \wedge v) + e^{2C + 2\Delta}\, w_1 \wedge F_2 = 0\,, \tag{8.42}$$

$$\cos\alpha (d(e^{-2\Delta}(w_1 \wedge w_2) + e^{-2\Delta}\, F_2 \wedge v) = 0\,. \tag{8.43}$$

The definition of F_2 can be read from (8.41) and we can make use of (8.39) to take the Hodge dual:

$$F_2 = -\rho^2\, e^{-2C}\left(\partial_\rho\, e^{-6\Delta}\, d x_1 \wedge d x_2 + \partial_{x_2}\, e^{-6\Delta}\, d\rho \wedge d x_1 + \partial_{x_1}\, e^{-6\Delta}\, d x_2 \wedge d\rho\right). \tag{8.44}$$

We then plug this back into (8.42) which imposes

$$\partial_{x_3} \mathrm{e}^{2\Delta} = \partial_{x_4} \mathrm{e}^{2\Delta} = 0, \tag{8.45}$$

and therefore we get that ∂_{x_3}, ∂_{x_4} are isometries of the solution. This automatically solves (8.43) without imposing any further restriction on α, then nothing physical depends on it. Thus the local form of all solutions in this class is

$$\mathrm{d}s^2 = \mathrm{e}^{2\Delta}\,\mathrm{d}s^2(\mathbb{R}^{1,5}) + \mathrm{e}^{-4\Delta}\left(\mathrm{d}x_1^2 + \mathrm{d}x_2^2 + \mathrm{d}\rho^2 + \rho^2\,\mathrm{d}s^2(S^2)\right), \tag{8.46}$$

$$F = -\rho^2\left(\partial_\rho\,\mathrm{e}^{-6\Delta}\,\mathrm{d}x_1\wedge\mathrm{d}x_2 + \partial_{x_2}\mathrm{e}^{-6\Delta}\,\mathrm{d}\rho\wedge\mathrm{d}x_1 + \partial_{x_1}\mathrm{e}^{-6\Delta}\,\mathrm{d}x_2\wedge\mathrm{d}\rho\right)\wedge\mathrm{Vol}(S^2)\,. \tag{8.47}$$

Notice however that these are the same fluxes of (8.46), which means that, even if the spinors are different, class B is a particular case of class A.

8.6 Relation with type II classification

M-theory can be used to generate solutions in type IIA supergravity without Romans mass through dimensional reduction as showed in 1.3.3. Notice, for instance, that the IIA master class of section 7.2 can be derived from Class A of the M-theory classification of section 8.4 by imposing an U(1) isometry within the complex manifold M_4 and reducing along it.

A similar argument applies also for solutions in type IIA supergravity with non-equal spinorial norms, since it was proven in section 6.3.1 that they have $F_0 = 0$. We shall now argue that these solutions descend from Case B in M-theory.

Performing the dimensional reduction of M-theory along a generic direction e_{11}^{10} we have that the gamma-matrix corresponding to this vielbein becomes the chirality matrix in IIA (1.42). The non-equal-norm condition in IIA, in the conventions of section 6.3.1, is equivalent to $\chi^\dagger\gamma^{(6)}\chi \neq 0$ and, up to an irrelevant overall normalization, the six-dimensional non-chiral spinor in IIA and the seven-dimensional spinor in M-theory are the same (1.42). The non-equal-norm condition in M-theory therefore reads $\chi^\dagger\gamma_m^{(7)}\chi(e_{11}^{10})^m = K_m(e_{11}^{10})^m \neq 0$, which means that we can establish whether or not we reduce to a non-equal-norm class in IIA by studying the M-theory one-form K defined in (8.22). Specifically, we generate non-equal norms whenever K has a leg along a direction which can become an uncharged U(1). Of course we cannot compactify along $\mathrm{d}y_3$ in (8.22) since we want to preserve the S^2 isometries, nor along $v \sim \mathrm{d}\rho$, since in both case A and B ρ defines the S^2 warping e^C. This means that we can reduce K just along w_1, and w_1 is present in K only when $\sin\alpha \neq 0$. This proves that all non-equal-norm solutions in type IIA are generated from Case B in section 8.5. Moreover in this case we also have that $w \sim \mathrm{d}x_1 + \mathrm{i}\,\mathrm{d}x_2$ and therefore after the dimensional reduction we get that the IIA solution with non-equal norm must have a flavour U(1)×U(1) isometry group generated by $\mathrm{d}x_3$ and $\mathrm{d}x_4$.

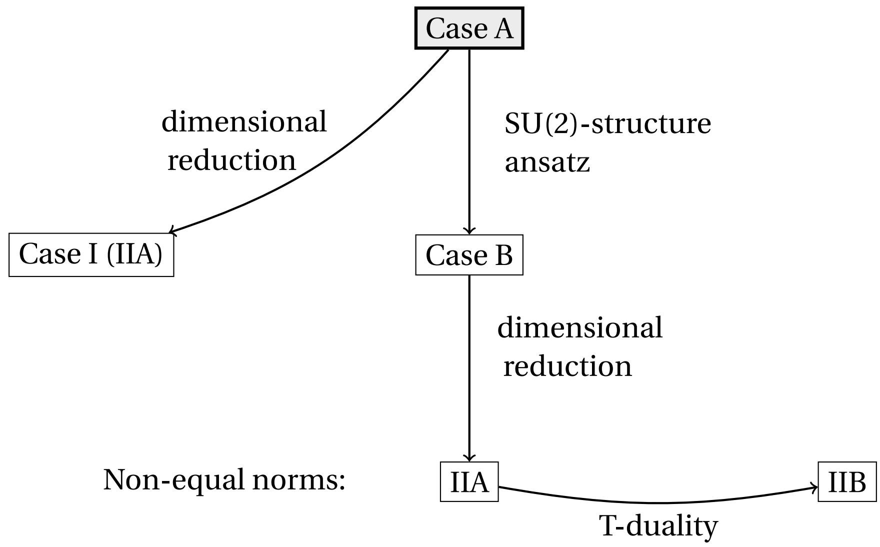

Figure 8.1: Depiction of the chains of dualities of the classes derived from the $\text{Mink}_4 \times S^2$ classification in M-theory. In subsection 8.4.1 it is proved that class B in M-theory is a particular case of class A and class B generates type IIA solutions with non-equal norms which can be T-dualized to the IIB ones. Case I in type IIA is generated from case A in M-theory via a dimensional reduction. Notice that the only fundamental classes in ten- and eleven-dimensional supergravity are Class A in M-theory and Case I in IIB, as showed in figure 7.1.

In [41, section 3.2] it is also argued that type IIB supergravity with non-equal norms has two uncharged $U(1)\times U(1)$ isometries, by T-dualizing along one of these one lands in type IIA supergravity with non-equal norms, that we just proved to descend from M-theory[2]. The situation is summarized in figure 8.1.

[2] In principle it would be also possible that type IIB solutions could follow by T-dualizing on the Hopf fiber of a squashed S^3, but it is easy to rule out this option [41].

CHAPTER 9

SOLUTIONS WITH AdS FACTOR

In this section we will recover solutions with an AdS factor starting from the Minkowski classification. This is useful for two reasons, first of all it shows that the classification we have performed so far is broad, since it contains various known and non-trivial system of intersecting branes. Secondly, understanding how to recover AdS solutions is a first step in the direction of finding backgrounds which are just asymptotically AdS. Such backgrounds are particularly interesting in holography since they point out the presence of RG flows. In particular, they can be used for studying both RG flows from two CFTs, where between the AdS factors we have a lower dimensional Minkwoski space [118, 119], or also RG flows between a CFT and a non-local solution [120], like little-string theory [121], which is dual to a flat-space geometry.

Let's now sketch the general procedure to get an higher-dimensional AdS background from a Minkowski one. Since we classified warped d-dimensional Minkowski solutions with $d \geq 4$, we can use the warping function to introduce the radial coordinate r of AdS_{d+1} in the following way:

$$e^{2A_d}\, \mathrm{d}s^2(\mathbb{R}^{1,d-1}) + \mathrm{d}s^2(M_{10-d}) = e^{2A_{d+1}}\, \mathrm{d}s^2(\mathrm{AdS}_{d+1}) + \mathrm{d}s^2(M_{9-d}), \tag{9.1}$$

where $e^{2A_d} = e^{2A_{d+1}+2r}$. In order to do this, we usually need to perform a change of coordinates to define r and to impose that the mixed terms in the metric containing $\mathrm{d}r$ vanish.

9.1 AdS_7

Let's start by showing how to recover the AdS_7 solutions of [31, 122] (see also [123] for a different approach) as particular examples of the $\mathbb{R}^{1,5}$ system in type IIA defined in (7.48). We don't have to worry about AdS_7 solutions in type IIB because there are no supersymmetric ones, as proved in [31].

As explained at the beginning of this chapter, what we have to do is to take $e^A = e^{r+A_7}$ in (7.48a) and to play with the remaining coordinates x_1, x_2. In order to be completely general, let's define the following change of variables:

$$x_1 = f(z)\,\mathrm{e}^{2r}\,, \qquad x_2 = \frac{2^6}{3^4}\alpha(z)\,\mathrm{e}^{4r}\,, \tag{9.2}$$

where the unusual coefficient in front of α is necessary to match the conventions of [124]. Imposing the presence of the AdS factor gives two conditions, one from the $\mathrm{d}r^2$ coefficient (which must be equal to the Minkowski warping) and the other from setting to zero the coefficient of $\mathrm{d}r\,\mathrm{d}z$. They read

$$\mathrm{e}^{4A_7} = -\frac{2^{13}}{3^8}\frac{\alpha\partial_z\alpha}{f\partial_z f}\,, \qquad \frac{f}{\partial_z\alpha}\partial_z\left(\frac{f\partial_z f}{\partial_z\alpha}\right) = -\frac{2^{10}}{3^8}m\,. \tag{9.3}$$

With these definitions, it turns out that equation (7.49) is automatically satisfied. We can solve the second condition in (9.3) by using reparametrization invariance to choose

$$f = -\frac{2^3}{3^4\pi}\partial_z\alpha\,, \tag{9.4}$$

which reduces (9.3) to the following ODE for α:

$$\partial_z^3\alpha = 2\,3^4\pi^3 m\,. \tag{9.5}$$

We also get the explicit expression for A_7 in terms of α:

$$\mathrm{e}^{4A_7} = -\frac{2^7\pi^2\alpha}{\partial_z^2\alpha}\,. \tag{9.6}$$

These definitions are all we need to make contact with the solution in [124, section 2.2.3], indeed it is straightforward to check, for example, that we get the same expression for the metric:

$$\mathrm{d}s^2 = \sqrt{2}\pi\left(\sqrt{-\frac{\alpha}{\partial_z^2\alpha}}\,\mathrm{d}s^2\mathrm{AdS}_7 + \sqrt{-\frac{\partial_z^2\alpha}{\alpha}}\left(\mathrm{d}z^2 + \frac{\alpha^2}{(\partial_z\alpha)^2 - 2\alpha\partial_z^2\alpha}\,\mathrm{d}s^2(S^2)\right)\right)\,. \tag{9.7}$$

9.2 AdS$_6$

In this section we will recover all supersymmetric AdS$_6$ solutions of type IIB supergravity, which were originally classified in [29] by using pure spinor equations and in [125] from the spinorial conditions. In both cases however a system of PDEs were obtained but not generally solved. In [32, 126, 127], exploiting Sl$(2,\mathbb{Z})$ symmetry of type IIB string-theory, all local solutions were given in terms of two holomorphic functions.

We will show that, starting from the $\mathbb{R}^{1,4}$ system in (7.37),(7.38), we will be able to give an alternative formulation of the AdS$_6 \times S^2$ solutions just in term of a Laplace equation in two variables.

Following [40, section 5.2.1], we start by performing the change of coordinates

$$x_1 = \tilde{x}_1 + \frac{1}{x_2}\,\mathrm{e}^{2C} B_0 \tag{9.8}$$

which redefines the metric as

$$\mathrm{d}s^2 = \mathrm{e}^{2A}\,\mathrm{d}s^2(\mathbb{R}^{1,4}) + \mathrm{e}^{-6A+2\phi}\left(\mathrm{d}x_2^2 + x_2^2\,\mathrm{d}s^2(S^2)\right) + \mathrm{e}^{-2A}\,\mathrm{d}x_3^2 + \mathrm{e}^{6A-2\phi}\left(\mathrm{d}\tilde{x}_1 + \frac{1}{x_2}\,\mathrm{e}^{2C} H_1\right)^2, \tag{9.9}$$

where H_1 was defined in (6.2). Now we change again parametrization

$$x_1 = \frac{1}{2}\,\mathrm{e}^{-3\rho} f(r,y)\,, \quad x_2 = \frac{8}{3}\,\mathrm{e}^{3\rho}\,y\,, \quad x_3 = 4\,\mathrm{e}^{\rho - \frac{1}{3}\Delta(r,y)}\,, \quad \mathrm{e}^{2A} = 2\sqrt{\frac{2}{3}}\,\mathrm{e}^{2\rho + \frac{1}{2}\phi + 2\lambda} \tag{9.10}$$

where e^{ρ} will be the AdS radial coordinate, Δ and λ are functions that depend just on $\{r, y\}$, and the powers of e^{ρ} in x_i are fixed such that they cancel those coming from the e^{A} factors in (9.9). If we demand that the NSNS three-form and the metric respect the isometry of AdS$_6$ we must impose

$$\mathrm{e}^{2C} H_1 = h_1(r,y)\,\mathrm{d}r + h_2(r,y)\,\mathrm{d}y\,, \tag{9.11}$$

and the following constraints:

$$\mathrm{e}^{\phi} = 6\frac{\mathrm{e}^{-2/3\Delta + 4\lambda}\,r^2 + f^2}{\mathrm{e}^{8\lambda} - y^2}\,, \qquad \mathrm{e}^{-8\lambda} = \frac{\partial_y \Delta}{y(1 + y\partial_y \Delta)}\,,$$
$$h_1 = -\frac{\mathrm{e}^{-8\lambda - \frac{2}{3}\Delta}\,y}{9f}\left(-3r + \mathrm{e}^{8\lambda + 2/3\Delta} f\partial_r f + r^2\partial_r \Delta\right), \quad h_2 = -\frac{4}{9}y(\partial_y f - f\partial_y \Delta)\,. \tag{9.12}$$

Up to this point we have just imposed conditions leading to AdS$_6$, however we have also to require that the BPS constraints are satisfied. These are simply given by the T-dual of (7.18) along x_4

$$\partial_{x_2}\mathrm{e}^{4A-\phi} = \partial_{x_1}\left(B_0\,\mathrm{e}^{-2A}\right), \qquad \frac{1}{x_2^2}\partial_{x_2}\left(x_2^2 B_0\,\mathrm{e}^{-2A}\right) = \partial_{x_1}\left(\mathrm{e}^{-8A+2\phi}(1 + B_0^2)\right) \tag{9.13}$$

and, in terms of the new coordinates, they read:

$$f = \mathrm{e}^{1/3\Delta}\,\frac{r\partial_r \Delta - 3}{1 + y\partial_y \Delta}\,, \qquad \partial_r^2\,\mathrm{e}^{1/3\Delta} = \frac{1}{3}\partial_y^2\,\mathrm{e}^{-\Delta}\,. \tag{9.14}$$

Thanks to all these definitions we have that Bianchi identities are automatically solved.

Notice that

$$\mathrm{e}^{\Delta} = \frac{c_1 r^3}{c_2 - y} \tag{9.15}$$

is a solution of (9.14) which can be T-dualized to the unique supersymmetric AdS$_6$ solution in IIA.

It is possible to make some further progress and rewrite the PDE in (9.14) in a more linear manner. This is done by performing, following [128, 129], another change of coordinates

$$\sigma = \mathrm{e}^{-1/3\Delta}\,, \qquad r = \partial_\eta V\,, \qquad y = \sigma^2 \partial_\sigma V\,, \tag{9.16}$$

where V must satisfies a four-dimensional Laplacian in cylindrical coordinates

$$\frac{1}{\sigma^2}\partial_\sigma(\sigma^2\partial_\sigma V) + \partial_\eta^2 V = 0\,. \tag{9.17}$$

Thanks to this transformation we get that the Toda equation in (9.14) is automatically solved. Now we can re-express all the fluxes in terms of V and the new coordinates. Another advantage we get from this parametrization is that it is actually possible to derive all the potential C_i of the RR-fluxes:

$$\begin{aligned}
\mathrm{d}s^2 &= \frac{2\sqrt{2}}{3^{3/4}}\,\mathrm{e}^{\frac{1}{2}\phi}\,\sigma\left(\frac{\Lambda\partial_\sigma V}{\partial_\eta^2 V}\right)^{1/4}\left[\mathrm{d}s^2(\mathrm{AdS}_6) + \frac{\partial_\sigma V \partial_\eta^2 V}{3\Lambda}\,\mathrm{d}s^2(S^2) + \frac{\partial_\eta^2 V}{3\sigma\partial_\sigma V}\big(\mathrm{d}\sigma^2 + \mathrm{d}\eta^2\big)\right],\\
\mathrm{e}^{\phi} &= \frac{6\sqrt{3}}{\sqrt{\frac{\Lambda\partial_\sigma V}{\partial_\eta^2 V}}\,\partial_\eta^2 V}\Big(3\big(\Lambda\sigma^2 + (\partial_\eta V)^2 + 2\sigma\partial_\eta V\partial_\sigma\partial_\eta V\big)\partial_\sigma V + \sigma\big((\partial_\eta V)^2 - 9(\partial_\sigma V)^2\big)\partial_\eta^2 V\Big),\\
B &= \frac{4}{3}\Big(\frac{\sigma\partial_\sigma V\big(\partial_\eta V\partial_\sigma\partial_\eta V + \sigma\left[(\partial_\sigma\partial_\eta V)^2 + (\partial_\eta^2 V)^2\right]\big)}{\Lambda} - V - \sigma\partial_\sigma V\Big)\mathrm{Vol}(S^2)\,,\\
C_0 &= -\frac{1}{18}\frac{3\partial_\sigma V(\partial_\eta V + \sigma\partial_\sigma\partial_\eta V) + \sigma\partial_\eta V\partial_\eta^2 V)}{\sigma(\partial_\eta V)^2\partial_\eta^2 V + 3\partial_\sigma V\big((\partial_\eta V + \sigma\partial_\sigma\partial_\eta V)^2 + (\sigma\partial_\eta^2 V)^2\big)}\,,\\
C_2 &= \frac{2}{27}\Big(\eta - \frac{\sigma\partial_\sigma V\partial_\sigma\partial_\eta V}{\Lambda}\Big)\mathrm{Vol}(S^2)\,, \qquad \Lambda = \sigma(\partial_\eta\partial_\sigma V)^2 + (\partial_\sigma V - \sigma\partial_\sigma^2 V)\partial_\eta^2 V\,.
\end{aligned} \tag{9.18}$$

9.3 AdS$_5$

In this section we will focus just on M-theory since the only AdS$_5$ solution in type IIB with an S^2 factor is AdS$_5 \times S^5$, while all the AdS$_5$ solutions in IIA can be uplifted to the M-theory one. AdS$_5$ backgrounds preserving $\mathcal{N} = 2$ supersymmetry in M-theory were classified in [130], and it was shown in [131] that they exhaust this class. We will prove that all these solutions can be embedded within M-theory class A of our classification (section 8.4).

Comparing (9.1) with the metric in (8.30) one can see that in order to realize an AdS$_5$ factor one has to impose

$$\rho = e^{2r}\, y, \qquad e^{2\Delta} = e^{2\lambda+2r}, \qquad v = -e^{-2\lambda}(2\,dr + dy). \tag{9.19}$$

The definition of v however generates a mixed $dr\,dy$ term in the metric which would break AdS isometries, so we need also the radial component r to point along one of the vielbein of the four-dimensional manifold M_4 with SU(2)-structure in such a way as to eliminate the dr cross term. Moreover, we have also to impose that all the AdS directions have a common warping factor and that the SU(2)-structure is charged under a U(1) isometry ∂_ψ. This last requirement is needed in order to realize the SU(2) × U(1) R-symmetry of the $\mathcal{N} = 2$ super-conformal algebra in four dimensions. Imposing all these constraints together with (8.27) is enough to get an identity structure on the internal space, which can be explicitly written as

$$u = \sqrt{\frac{-\partial_y D}{y}}\, e^{\lambda+\frac{1}{2}D}(dx_1 + i\,dx_2)\,, \quad w = 2\,e^{i\psi}\, e^{-2\lambda}\sqrt{\frac{y}{-\partial_y D}}\left(dr + \frac{1}{2}\partial_y D\, dy + i(d\psi + V)\right), \tag{9.20}$$

where

$$e^{-6\lambda} = \frac{-\partial_y D}{y(1-y\partial_y D)}\,, \qquad V = \frac{1}{2}(\partial_{x_2} D\, dx_1 - \partial_{x_1} D\, dx_2) \tag{9.21}$$

and provided that D satisfies the Toda equation on on (y, x_1, x_2)

$$(\partial^2_{x_1} + \partial^2_{x_2})D + \partial^2_y e^D = 0\,. \tag{9.22}$$

The flux F_2 can be derived from (8.28) and reads

$$F_2 = e^{4\lambda}\, y^{-2}\left(2(d\psi + V)\wedge d(y^3 e^{-6\lambda}) + 2y(1-y^2 e^{-6\lambda})\, dV - \partial_y e^D\, dx_1 \wedge dx_2\right), \tag{9.23}$$

and one can check that $F = e^{2C}\,\mathrm{Vol}(S^2)\wedge F_2$ obeys the Bianchi identity thanks to (9.22).

To recap, the solution in this class is given by the following fluxes:

$$\begin{aligned} ds^2 &= e^{2\lambda}\left(4\,ds^2(AdS_5) + y^2 e^{-6\lambda}\, ds^2(S^2)\right) + \frac{4}{(1-y\partial_y D)}\, e^{2\lambda}\left(d\psi + V\right)^2 \\ &\quad + \frac{-\partial_y D}{y}\, e^{2\lambda}\left(dy^2 + e^D(dx_1^2 + dx_2^2)\right), \\ F &= \mathrm{Vol}(S^2)\left(2(d\psi + V)\wedge d(y^3 e^{-6\lambda}) + 2y(1-y^2 e^{-6\lambda})\, dV - \partial_y e^D\, dx_1\wedge dx_2\right), \end{aligned} \tag{9.24}$$

provided the definitions (9.21). This is exactly the solution of [130, section 3].

CHAPTER 10

NON-SUPERSYMMETRIC SOLUTIONS

In this chapter we will argue that all the solutions presented in the type II supergravity classification can be generalized to a bigger class which has in general $\mathcal{N} = 0$ supersymmetry. One peculiar property of these new solutions is that they are obtained by relaxing some of the pure spinor equations, which means that it is possible to find non-supersymmetric backgrounds by solving a first order system of PDEs instead of a second order one (namely the equations of motion). This feature provides many advantages: first of all, as we have seen for the supersymmetric cases, a first order system is much easier to solve compared to a second order one, so one can run a classification of non-supersymmetric solutions exploiting the same methods we have seen in chapters 6 and 7. Moreover, by relaxing just some components of the pure spinor equations it is possible to preserve D-brane calibration conditions of the untouched terms, which would allow to detect sources with minimal energy also in a non-supersymmetric background. This is a useful property when one has to discuss non-perturbative stability in non-supersymmetric solutions, which is not protected by BPS conditions.

We will start by discussing how to break supersymmetry in the IIB master class 7.1 by reviewing [38]. Indeed, as said, this solution was first discovered by [25], which already pointed out a strategy to violate supersymmetry constraints. We will then discuss how to relax, in analogy with [25], supersymmetry in two classes of type IIA supergravity, as showed in [42]. We will see that from these three cases and thanks to the web of duality we pointed out (see figure 7.1), we are actually able to extend the break of supersymmetry to all the possible classes in the $\mathbb{R}^{1,3} \times S^2$ classification.

10.1 Supersymmetry breaking in type IIB

In order to make contact with the notation of [25], it is useful to use the $\mathrm{SL}(2,\mathbb{R})$ covariant formalism of subsection 3.2.2. Let's then define

$$G = f_3 - \mathrm{i}\,\mathrm{e}^{-\phi} H\,; \tag{10.1}$$

thanks to the condition $*_6 f_3 = \mathrm{e}^{-\phi} H$ in (7.2), we have that

$$*_6\, G = \mathrm{i}G\,, \tag{10.2}$$

this condition on G is called imaginary self-duality.

In presence of a SU(3) structure it is customary to classify forms in terms of a couple of integers (n,m) where n is the number holomorphic components while m the number of anti-holomorphic ones. We will also distinguish the forms which are annihilated by J by adding the superscript ^{0}; such forms are called primitive. For example if a (1,2)-form is primitive we have $\alpha^0_{(1,2)} \wedge J = 0$. In the case of three-forms, it is possible to show that they behave in a simple way under the action of the Hodge-star operator:

$$*_6\,\Omega = -\mathrm{i}\Omega\,, \qquad *_6\alpha^0_{(2,1)} = \mathrm{i}\alpha^0_{(2,1)}\,, \qquad *_6(\alpha_{(0,1)} \wedge J) = \mathrm{i}\alpha_{(0,1)} \wedge J\,, \tag{10.3}$$

where we recall that Ω is a $(3,0)$ form while J a $(1,1)$. Now, since $\{\overline{\Omega}, \alpha^0_{(2,1)}, \alpha_{(0,1)} \wedge J\}$ span the space of all possible imaginary-self-dual three-forms, we have that G must be a combination of these three. However, the supersymmetry constraints $H \wedge J = H \wedge \Omega = 0$ in (7.2) imply that the components proportional to $\overline{\Omega}$ and $\alpha_{(0,1)} \wedge J$ must be set to zero, which means that G is $(2,1)$ primitive in order to preserve supersymmetry.

In [25], it was discovered that the condition (10.2) is enough to make G disappear from all the equations of motion, which means that it is actually possible to get a proper solution even if G has $(0,3)$ or $\alpha_{(0,1)} \wedge J$ components. This is respectively equivalent to relaxing the six-form part of the first pure spinor equation (5.31a) and the five-form part of the second one (5.31b).

The next step is to introduce an explicit modification of the pure spinor equations in the framework of $\mathbb{R}^{1,3} \times S^2$ solutions and see how supersymmetry-breaking propagates to all the fluxes and equations. For example, let's focus on the following modified pure spinor system

$$\begin{aligned} &\mathrm{d}_H(\mathrm{e}^{3A-\phi}\Phi_-) = 0\,,\\ &\mathrm{d}_H(\mathrm{e}^{2A-\phi}\,\mathrm{Re}\Phi_+) = -\frac{c}{8}\mathrm{Vol}(S^2) \wedge \mathrm{d}x_2 \wedge \mathrm{d}x_3 \wedge \mathrm{d}x_4\,,\\ &\mathrm{d}_H(\mathrm{e}^{4A-\phi}\,\mathrm{Im}\Phi_+) = \frac{\mathrm{e}^{4A}}{8} \star \lambda(F)\,, \end{aligned} \tag{10.4}$$

where c is a constant which parameterizes the supersymmetry breaking. Since the pure spinors are not modified, we can use (6.39) and (6.40) for $b = 0, a = -\mathrm{i}$ and run a classification in an identical way as seen in section 7.1. The result of this operation is that the metric is not changed compared to (7.7), but now g (defined in (7.6)) can depend also from x_2, and in particular we have:

$$f\partial_{x_2} g = c x_2^2 \,. \tag{10.5}$$

This modification propagates also to F_3 thanks to (10.2), which now reads

$$F_3 = B \wedge F_1 - \big(\partial_{x_4}(fg)\,\mathrm{d}x_3 - \partial_{x_3}(fg)\,\mathrm{d}x_4\big) \wedge \mathcal{C}_2 - cf\,\mathrm{d}x_1 \wedge \mathrm{d}x_3 \wedge \mathrm{d}x_4 \,, \tag{10.6}$$

while other fluxes are unchanged. Also the Bianchi identities get modified, indeed we get that (7.10) reads

$$\partial^2_{x_1}(e^{-4A}) + \frac{1}{x_2^2}\partial_{x_2}(x_2^2\partial_{x_2}(e^{-4A})) + \triangle_2(e^{-4A}f^{-1}) + \frac{1}{x_2^4}\triangle_2(fg^2) = -c^2 \,. \tag{10.7}$$

Using the web of dualities in 7.1, we can extend this supersymmetry-breaking technique to case II in type IIA and case III in type IIB.

10.2 Supersymmetry breaking in type IIA

Let's now switch to type IIA supergravity. We will start by discussing how to break supersymmetry in case I in type IIA, which would be enough to cover all the possible cases described in the type II classification. However, we will see that the supersymmetry breaking technique we introduce for this case does not apply when we have six-dimensional Minkowski solutions, so we will find a variant for this class. For this section we will follow [42].

10.2.1 IIA master class

In analogy with the IIB case, in type IIA it is possible to relax in general the six-form part of (5.31b). This means that imposing the other components of the pure spinor equations and Bianchi identities it is enough to solve the equations of motion. To prove this is rather cumbersome and technical, so let's just sketch the procedure. Since the metric is written in local coordinates we always have an explicit expression for the Bianchi identities. Of course it is not possible to solve all of them in full generality, so what one can do is to use these equations in an algebraic way and see if the equations of motion can be obtained as linear combinations of them. For instance, we get the they are enough to solve the equation of motion for the NSNS three-form (1.15b), but they do not solve the dilaton and the Einstein equations. Since, as said, in doing all the simplifications we have used Bianchi identities from an algebraic point of view, it may happen that some further conditions can be obtained by combining the derivatives of them. Bianchi identities are second order conditions for the fields (A,ϕ,B_0) that determine the solution, so deriving them we will get third-order conditions. By combining these equations with appropriate coefficients such that all the functions which are derived three times vanish, we get a further "consistency" condition provided that H has some legs along x_3 and x_4. This consistency condition is enough to solve the remaining equations of motion.

As an example, let's modify the pure spinor equations in the following way

$$\begin{aligned}
&\mathrm{d}_H(\mathrm{e}^{3A-\phi}\,\Phi_+) = 0\,,\\
&\mathrm{d}_H(\mathrm{e}^{2A-\phi}\,\mathrm{Re}\Phi_-) = \frac{c}{8}\,\mathrm{e}^{6A-2\phi}\,\mathrm{Vol}(M_6)\,,\\
&\mathrm{d}_H(\mathrm{e}^{4A-\phi}\,\mathrm{Im}\Phi_-) = \frac{\mathrm{e}^{4A}}{8}\star\lambda(F)\,,
\end{aligned}\tag{10.8}$$

where $\mathrm{Vol}(M_6)$ is the internal-space volume. Solving (10.8) using the pure spinors defined in (6.37),(6.38) with $b=0, a=\mathrm{i}$, we get the following modification of (7.18):

$$\partial_{x_2}(\mathrm{e}^{2A-2\Phi}) = \partial_{x_1}(\mathrm{e}^{-2A}\,B_0), \tag{10.9a}$$

$$\frac{1}{x_2^2}\partial_{x_2}(x_2^2\,\mathrm{e}^{-2A}\,B_0) = \partial_{x_1}(\mathrm{e}^{-6A+2\Phi}(1+B_0^2)) - c, \tag{10.9b}$$

which provide a new definition for $H=\mathrm{d}B$ where $B = x_2^2\,\mathrm{e}^{-4A+2\phi}\,B_0\mathrm{Vol}(S^2)$. The last equation affect also the RR-fields

$$\begin{aligned}
F_2 =\;& (\partial_{x_4}\mathrm{e}^{2A-2\phi}\,\mathrm{d}x_3 - \partial_{x_3}\mathrm{e}^{2A-2\phi}\,\mathrm{d}x_4)\wedge\mathrm{d}x_1 + \big(\partial_{x_4}(\mathrm{e}^{-2A}B_0)\,\mathrm{d}x_3\\
&-\partial_{x_3}(\mathrm{e}^{-2A}B_0)\,\mathrm{d}x_4\big)\wedge\mathrm{d}x_2 - \big(\partial_{x_1}(\mathrm{e}^{-4A}) - c\,\mathrm{e}^{2A-2\phi}\big)\,\mathrm{d}x_3\wedge\mathrm{d}x_4\,,\\
F_4 =\;& B\wedge F_2 - x_2^2\Big(\big(\partial_{x_4}(\mathrm{e}^{-2A}B_0)\,\mathrm{d}x_3 - \partial_{x_3}(\mathrm{e}^{-2A}B_0)\,\mathrm{d}x_4\big)\wedge\mathrm{d}x_1\\
&+\big(\partial_{x_4}(\mathrm{e}^{-6A+2\phi}(1+B_0^2))\,\mathrm{d}x_3 - \partial_{x_3}(\mathrm{e}^{-6A+2\phi}(1+B_0^2))\,\mathrm{d}x_4\big)\wedge\mathrm{d}x_2\\
&-\big(\partial_{x_2}(\mathrm{e}^{-4A}) - c\,\mathrm{e}^{2A-2\phi}\big)\,\mathrm{d}x_3\wedge\mathrm{d}x_4\Big)\wedge\mathrm{Vol}(S^2)\,,
\end{aligned}\tag{10.10}$$

while again $F_0 = 0$. The Bianchi identities (7.20) get modified as following:

$$\begin{aligned}
&\partial^2_{x_3}(\mathrm{e}^{2A-2\phi}) + \partial^2_{x_4}(\mathrm{e}^{2A-2\phi}) + \partial^2_{x_1}(\mathrm{e}^{-4A}) = c\partial_{x_1}\mathrm{e}^{2A-2\phi}\,,\\
&\partial^2_{x_3}(\mathrm{e}^{-2A}B_0) + \partial^2_{x_4}(\mathrm{e}^{-2A}B_0) + \partial_{x_1}\partial_{x_2}(\mathrm{e}^{-4A}) = c\partial_{x_2}\mathrm{e}^{2A-2\phi}\,,\\
&\partial^2_{x_3}(x_2^2\,\mathrm{e}^{-6A+2\phi}(1+B_0^2)) + \partial^2_{x_4}(x_2^2\,\mathrm{e}^{-6A+2\phi}(1+B_0^2)) + \partial_{x_2}(x_2^2\partial_{x_2}(\mathrm{e}^{-4A})) = c\partial_{x_2}(x_2^2\,\mathrm{e}^{-2A}B_0)\,.
\end{aligned}\tag{10.11}$$

10.2.2 Massive $\mathbb{R}^{1,5}$ case

The approach showed for the IIA master class can be applied to case II of the IIB classification, but of course doesn't work if we have a six-dimensional Minkowski factor (as in Case III) since it requires to impose that ∂_{x_3} and ∂_{x_4} are Killing directions, and therefore H cannot have any leg along them.

Let's then find a new way to break supersymmetry starting from scratch and deriving the $\mathbb{R}^{1,5} \times S^2$ solution just by imposing the pure spinor equation (5.31c) to find fluxes and the one- and two-form constraints coming from (5.31a),(5.31b) (which are useful to get a local expression for the metric). What one has to do is basically to use (6.37),(6.38) with $a = 1, b = 0, F_0 \neq 0$, and, after a straightforward procedure similar to the one performed in [39, section 4.1] to derive (7.48), one gets that the RR-fluxes read

$$\begin{aligned} F_0 =& 2\,\mathrm{e}^{-2\phi}\,\partial_{x_1}\mathrm{e}^{2A} + \mathrm{e}^{12A-4\phi}\,\mathrm{BPS2} = m \\ F_2 =& -x_2^2\partial_{x_2}\mathrm{e}^{-4A}\mathrm{Vol}(S^2) + \mathrm{e}^{12A-4\phi}\,y_3\,\mathrm{d}x_3\wedge\mathrm{d}x_4\mathrm{BPS1} \\ F_4 =& 0. \end{aligned} \tag{10.12}$$

where

$$\begin{aligned} \mathrm{BPS1} =& \partial_{x_2}\mathrm{e}^{-6A+2\phi} - \partial_{x_1}\left(B_0\,\mathrm{e}^{-8A+2\phi}\right), \\ \mathrm{BPS2} =& \frac{1}{x_2^2}\partial_{x_2}\left(x_2^2 B_0\,\mathrm{e}^{-8A+2\phi}\right) + \partial_{x_1}\mathrm{e}^{-10A+2\phi} \end{aligned} \tag{10.13}$$

are the supersymmetric conditions, which must be set to zero if one wants to solve the rest of the pure spinor equations (i.e., preserve supersymmetry). Now what we will try to do is to solve the Bianchi identities without imposing the BPS conditions (10.13), in order to see if there is a way to evade them. Actually, it is immediate to notice from the Bianchi identity for F_2 that also in this case we have to set BPS1$=0$, since when the external derivative hits embedding coordinate of the sphere y_3 produces a term that cannot be compensated in other ways. However, we do not have to set BPS2$=0$ and we can define $\partial_{x_2}B_0$ by imposing that $F_0 = m$ is constant. This will imply the following change in the NSNS three form:

$$H = x_2^2\mathrm{Vol}(S^2)\wedge\left(\partial_{x_2}\mathrm{e}^{-6A+2\phi}\,\mathrm{d}x_1 - \mathrm{e}^{-4A}\,\partial_{x_1}\mathrm{e}^{-6A+2\phi}\,\mathrm{d}x_2 + m\,\mathrm{e}^{-12A+4\phi}\,\mathrm{d}x_2\right). \tag{10.14}$$

Let's now move on the Bianchi identity for F_2. These can be used to define ϕ in terms of A up to an arbitrary function $c(x_1)$

$$e^{2\phi} = \frac{e^{6A}}{m}\left(c - \partial_{x_1}e^{-4A}\right). \tag{10.15}$$

Notice that if $c = 0$ then we have the same definition of ϕ and H as in the supersymmetric case (7.48), so the function c controls the supersymmetry breaking. From the Bianchi we also get a PDE for the warping function which is a generalization of (7.49):

$$\frac{1}{x_2^2}\partial_{x_2}\left(x_2^2\partial_{x_2}\mathrm{e}^{-4A}\right) + \frac{1}{2}\partial_{x_1}^2\mathrm{e}^{-8A} = -c\left(c - 2\partial_{x_1}e^{-4A}\right). \tag{10.16}$$

With these conditions the equation of motion for H is automatically solved; however, even if all the Bianchi identities are satisfied, the Einstein and the dilaton equations are not automatically implied in this case, but only if c is a constant. This is another sign that this solution cannot be derived from the previous subsection.

Now, doing some reverse engineering, we are able to find how the pure spinor equations are modified for this case. The result of this operation is

$$\begin{aligned} &d_H(\mathrm{e}^{3A-\phi}\,\Phi_+) = 0\,, \\ &d_H(\mathrm{e}^{2A-\phi}\,\mathrm{Re}\Phi_-) = \frac{c}{8}\,\mathrm{e}^{8A-2\phi}\,\mathrm{Vol}(M_4)\,, \\ &d_H(\mathrm{e}^{4A-\phi}\,\mathrm{Im}\Phi_-) = \frac{\mathrm{e}^{4A}}{8}\star\lambda(F)\,; \end{aligned} \tag{10.17}$$

notice that, as in the previous case, we are changing just the second condition of (5.31b), but now this equation cannot be relaxed in general but with the precise factor given in (10.17).

APPENDIX A

FORMS, SPINORS AND CLIFFORD MAP

In this section we will review some of the conventions we adopted in the main text. To make contact with the results in the literature, we used [10] conventions for eleven dimensions and [9, 15] for type II theories. Notice that these hide some subtleties, and an extra care is needed when one deals with dimensional reduction from M-theory to type IIA. However the main part of what we will present here applies for generic dimensions and space-time signature.

A.1 Form conventions

Let's consider a differential form C of degree c defined over a d-dimensional space

$$C = \frac{1}{c!} C_{M_1 \dots M_c} \, \mathrm{d} x^{M_1} \wedge \dots \wedge \mathrm{d} x^{M_c} \,, \tag{A.1}$$

we define the contraction operator along a generic direction N as

$$\iota_N C = C_N = \frac{1}{(c-1)!} C_{N M_1 \dots M_{c-1}} \, \mathrm{d} x^{M_1} \wedge \dots \wedge \mathrm{d} x^{M_{c-1}} \,. \tag{A.2}$$

Using this formula multiple times, we can also consider the contraction of C respect to another differential form B of lower degree $b \le c$:

$$\iota_B C = B^{N_1 \dots N_b} \iota_{N_1 \dots N_b} C = (-)^{b(b-1)/2} B^{N_1 \dots N_b} \frac{1}{(c-b)!} C_{N_1 \dots N_b M_1 \dots M_{c-b}} \, \mathrm{d} x^{M_1} \wedge \dots \wedge \mathrm{d} x^{M_{c-b}} \tag{A.3}$$

where the sign $(-)^{b(b-1)/2}$ is due to the permutation of the b indices. Since this operation appears frequently, it is useful to give it a name:

$$\lambda(C) = (-)^{c(c-1)/2} C \,. \tag{A.4}$$

Combining the last two definitions, we can introduce also the dot operator

$$B \cdot C = \iota_{\lambda(B)} C = B^{N_1 \dots N_b} \frac{1}{(c-b)!} C_{N_1 \dots N_b M_1 \dots M_{c-b}} \,\mathrm{d}x^{M_1} \wedge \dots \wedge \mathrm{d}x^{M_{c-b}} \,. \tag{A.5}$$

The contraction operator ι_M and the usual wedge product $\mathrm{d}x^N \wedge$ satisfies a remarkable algebraic identity

$$\{\mathrm{d}x^M, \mathrm{d}x^N\} = \{\iota_M, \iota_N\} = 0 \qquad \{\iota_N, \mathrm{d}x^M\} = \delta^M_N \tag{A.6}$$

which allows to interpret them as gamma-matrices over the $\mathrm{O}(d,d)$ metric of the generalized tangent bundle. Using this perspective differential forms are nothing but the spinor representation of the Clifford algebra over the generalized tangent bundle, as explained in section 2.2.2.

Using the definition of (A.4) we can introduce the Chevalley–Mukai pairing of degree k between two forms B and C:

$$(A, B) = (A \wedge \lambda(B))_k \tag{A.7}$$

where the subscript k denotes keeping the k-form degree only. Analogously, we define the following bracket

$$\{B, C\} = (B \wedge \lambda[(2d - c)C])_k \,. \tag{A.8}$$

Notice that these two brackets have the opposite symmetry properties, while $(A, B)_k = (-)^{k(k-1)/2}(B, A)_k$, we have $\{A, B\}_k = -(-)^{k(k-1)/2}\{B, A\}_k$.

The last operator we have to introduce is the Hodge star, which is defined as

$$* \, C_{M_1 \dots M_{d-c}} = \frac{1}{(d-c)!} \sqrt{-g}\, \epsilon_{M_1 \dots M_{d-c} N_1 \dots N_c} \, C^{N_1 \dots N_c} \,, \tag{A.9}$$

where the Levi-Civita symbol is fixed to

$$\epsilon_{1 \dots d} = 1 \,. \tag{A.10}$$

A.2 Clifford algebra

Let's start from the definition of the chiral operator, which in generic dimensions can be written as

$$\gamma = c \gamma^0 \dots \gamma^{d-1} \tag{A.11}$$

where c is a constant such that $c^2 = (-)^s (-)^{d(d-1)/2}$ (s is the metric signature). For example, when $d = 10$ we will choose $c = 1$, even if also $c = -1$ was a possible choice. An important property of the chiral operator which will be useful later is the following:

$$\gamma \gamma_{M_1 \dots M_k} = c \frac{(-)^{k(k-1)/2}}{(d-k)!} \epsilon_{N_1 \dots N_{d-k} M_1 \dots M_k} \gamma^{N_1 \dots N_{d-k}} \,, \tag{A.12}$$

where $\gamma_{M_1\dots M_k}$ is the anti-symmetrized product of k gamma matrices, also called a k-vector.

The chiral operator is just defined in even dimensions, in odd dimensions we have that k-vectors of degree grater that $(d-1)/2$ are actually determined by the lower degree ones. For example in eleven dimensions we fix, consistently with [10],

$$\gamma_{0\dots 10} = 1 \tag{A.13}$$

so that we have $\gamma_{10} = \gamma_{0\dots 9} = -\gamma^{0\dots 9} = -\gamma$ where γ is the ten-dimensional chiral operator according to our definition. This is the reason why we have a misleading minus sign in equation (1.42).

The next formula we want to introduce is the Fierz identity. It consists simply in expanding a byspinor C on the basis defined by $\{\gamma^{M_1\dots M_k}\}_{k=0}^{\widetilde{d}}$:

$$C = \sum_{k=0}^{\widetilde{d}} \frac{1}{2^{\lfloor \frac{d}{2} \rfloor}} \frac{1}{k!} \mathrm{Tr}\left(C\gamma_{M_1\dots M_k}\right)\gamma^{M_k\dots M_1} \tag{A.14}$$

where $\lfloor\ \rfloor$ is the floor function while $\widetilde{d}$ is d in even dimensions and $\lfloor d/2 \rfloor$ in odd ones. The $2^{\lfloor \frac{d}{2} \rfloor}$ is nothing but the dimension of the gamma-matrices representation. If in particular C is a bispinor $\epsilon_1 \otimes \epsilon_2$ we can rewrite the trace as:

$$\epsilon_1 \otimes \bar{\epsilon}_2 = \sum_{k=0}^{\widetilde{d}} \frac{1}{2^{\lfloor \frac{d}{2} \rfloor}} \frac{1}{k!} \bar{\epsilon}_2 \gamma_{M_1\dots M_k} \epsilon_1 \gamma^{M_k\dots M_1} \,. \tag{A.15}$$

For the definition of the charge conjugation matrix and other standard spinor operators, when it is not specified in the main text, we are adopting the conventions of [2] appendix B.

A.3 Clifford map

Clifford map is the mathematical formalization of the Feynman slash operator: it is a map between Clifford algebra and exterior algebra

$$\gamma^{M_1\dots M_k} \quad \mapsto \quad \mathrm{d}x^{M_1} \wedge \cdots \wedge \mathrm{d}x^{M_k} \,. \tag{A.16}$$

In even dimensions this map is an algebraic isomorphism, while in odd dimensions it cannot be because the a basis in the Clifford algebra is determined by just an half of the k-vectors . This mapping is reinforced by the fact that basically every Clifford algebra operation can be rewritten in terms of differential-form ones and vice versa, as we will now see. In the rest of the section, as in the main text, we will identify differential forms and Clifford algebra k-vectors.

The first gamma-matrix operation we will rewrite in terms of some familiar form operations is Clifford multiplication of a single gamma matrix γ^M with a form C:

$$\overrightarrow{\gamma}^M C = \gamma^M C = (dx^M \wedge + \iota^M) C\,, \qquad \overleftarrow{\gamma}^M C = C\gamma^M = (-)^c (dx^M \wedge - \iota^M) C\,. \tag{A.17}$$

By iteration of this formula we have

$$\begin{aligned} \overrightarrow{\gamma}^{M_1 \dots M_k} &= \sum_{i=0}^{k} \binom{k}{i} dx_{[N_1 \dots N_i} \iota_{\iota_{i+1} \dots \iota_k]}\,, \\ \overleftarrow{\gamma}^{M_1 \dots M_k} &= (-)^{k(k-1)/2} \sum_{i=0}^{k} \binom{k}{i} dx_{[N_1 \dots N_i} \iota_{\iota_{i+1} \dots \iota_k]} (-)^{k \deg + i}\,, \end{aligned} \tag{A.18}$$

where the operation deg is defined as $\deg(C) = c$.

Using (A.12) we can also rewrite, in even dimensions:

$$\gamma C = * \lambda(C)\,. \tag{A.19}$$

So it is immediate to notice that the self-duality relation of the RR fields (1.13) can be written as $\gamma F = F$. On the other hand, in odd dimensions we have (A.13) and therefore the following identification

$$C = \lambda * C \tag{A.20}$$

which holds just if C is interpreted as an element of the Clifford algebra.

We can also combine the operators we have seen so far, from

$$\overrightarrow{\gamma}^M \overrightarrow{\gamma} = - \overrightarrow{\gamma}\, \overrightarrow{\gamma}^M\,, \qquad \overleftarrow{\gamma}^M \overrightarrow{\gamma} = \overrightarrow{\gamma}\, \overleftarrow{\gamma}^M\,, \tag{A.21}$$

it follows that

$$dx^M \wedge \overrightarrow{\gamma} = - \overrightarrow{\gamma}\, \iota^M\,, \qquad \iota^M \overrightarrow{\gamma} = - \overrightarrow{\gamma}\, dx^M\,, \tag{A.22}$$

while using the definition of λ we also get

$$\lambda(dx^M \wedge C_k) = (-)^k dx^M \wedge \lambda(C_k)\,, \quad \lambda(\iota^M C_k) = -(-)^k \iota^M \lambda(C_k)\,, \quad \lambda(\gamma^M C_k) = \lambda(C_k) \gamma^M\,. \tag{A.23}$$

Using the relation

$$\gamma^M C \gamma_M = (-)^c (d - 2c) C \tag{A.24}$$

we can also re-express the two Mukai pairings (A.7) and (A.8) in terms of Clifford algebra operations when their degree is $k = d$; we have for the first one

$$(B, C)_d = (-)^s \frac{(-1)^{\deg(B)}}{2^{\lfloor \frac{d}{2} \rfloor}} \mathrm{Tr}(*BC)\,. \tag{A.25}$$

where again s is the metric signature, while for the second pairing

$$\{B, C\}_d = (-)^s \frac{(-)^d}{2^{\lfloor \frac{d}{2} \rfloor}} \mathrm{Tr}(*B \gamma^M C \gamma_M)\,. \tag{A.26}$$

Bibliography

[1] M. B. Green, J. H. Schwarz, and E. Witten, *SUPERSTRING THEORY. VOL. 1: INTRODUCTION.* Cambridge Monographs on Mathematical Physics. 1988.

[2] J. Polchinski, *String Theory.* Cambridge monographs on mathematical physics. Cambridge Univ. Press, Cambridge, 1998.

[3] K. Becker, M. Becker, and J. H. Schwarz, *String theory and M-theory: A modern introduction.* Cambridge University Press, 2006.

[4] J. Polchinski, "Dirichlet-Branes and Ramond–Ramond Charges," *Phys. Rev. Lett.* **75** (1995) 4724–4727, hep-th/9510017.

[5] E. Witten, "String theory dynamics in various dimensions," *Nucl. Phys.* **B443** (1995) 85–126, hep-th/9503124. [,333(1995)].

[6] J. P. Gauntlett, D. Martelli, S. Pakis, and D. Waldram, "G structures and wrapped NS5-branes," *Commun.Math.Phys.* **247** (2004) 421–445, hep-th/0205050.

[7] U. Gran, J. Gutowski, and G. Papadopoulos, "The Spinorial geometry of supersymmetric IIB backgrounds," *Class.Quant.Grav.* **22** (2005) 2453–2492, hep-th/0501177.

[8] M. Blau, J. M. Figueroa-O'Farrill, C. Hull, and G. Papadopoulos, "A New maximally supersymmetric background of IIB superstring theory," *JHEP* **01** (2002) 047, hep-th/0110242.

[9] A. Tomasiello, "Generalized structures of ten-dimensional supersymmetric solutions," *JHEP* **1203** (2012) 073, 1109.2603.

[10] J. P. Gauntlett and S. Pakis, "The geometry of $D = 11$ Killing spinors," *JHEP* **04** (2003) 039, hep-th/0212008.

[11] M. Graña, R. Minasian, M. Petrini, and A. Tomasiello, "Generalized structures of $\mathcal{N} = 1$ vacua," *JHEP* **11** (2005) 020, hep-th/0505212.

[12] L. Martucci and P. Smyth, "Supersymmetric D-branes and calibrations on general $\mathcal{N} = 1$ backgrounds," *JHEP* **11** (2005) 048, hep-th/0507099.

[13] L. Martucci, "Electrified branes," *JHEP* **02** (2012) 097, 1110.0627.

[14] R. Harvey and H. B. Lawson, "Calibrated geometries," *Acta Math.* **148** (1982) 47–157.

[15] A. Legramandi, L. Martucci, and A. Tomasiello, "Timelike structures of ten-dimensional supersymmetry," *JHEP* **04** (2019) 109, 1810.08625.

[16] G. Gibbons and C. Hull, "A Bogomolny Bound for General Relativity and Solitons in N=2 Supergravity," *Phys.Lett.* **B109** (1982) 190.

[17] G. W. Gibbons, C. M. Hull, and N. P. Warner, "The Stability of Gauged Supergravity," *Nucl. Phys.* **B218** (1983) 173.

[18] J. M. Maldacena and C. Núñez, "Supergravity description of field theories on curved manifolds and a no-go theorem," *Int. J. Mod. Phys.* **A16** (2001) 822–855, hep-th/0007018.

[19] M. P. Hertzberg, S. Kachru, W. Taylor, and M. Tegmark, "Inflationary constraints on type IIA string theory," arXiv:0711.2512 [hep-th].

[20] N. Arkani-Hamed, L. Motl, A. Nicolis, and C. Vafa, "The String landscape, black holes and gravity as the weakest force," *JHEP* **06** (2007) 060, hep-th/0601001.

[21] H. Ooguri and C. Vafa, "Non-supersymmetric AdS and the Swampland," *Adv. Theor. Math. Phys.* **21** (2017) 1787–1801, 1610.01533.

[22] G. Obied, H. Ooguri, L. Spodyneiko, and C. Vafa, "De Sitter Space and the Swampland," 1806.08362.

[23] H. Ooguri and C. Vafa, "Non-supersymmetric AdS and the Swampland," 1610.01533.

[24] H. Ooguri, E. , G. Shiu, and C. Vafa, "Distance and de Sitter Conjectures on the Swampland," *Phys. Lett.* **B788** (2019) 180–184, 1810.05506.

[25] S. B. Giddings, S. Kachru, and J. Polchinski, "Hierarchies from fluxes in string compactifications," *Phys. Rev.* **D66** (2002) 106006, hep-th/0105097.

[26] S. Kachru, R. Kallosh, A. Linde, and S. P. Trivedi, “De Sitter vacua in string theory,” *Phys. Rev.* **D68** (2003) 046005, hep-th/0301240.

[27] V. Balasubramanian, P. Berglund, J. P. Conlon, and F. Quevedo, “Systematics of Moduli Stabilisation in Calabi–Yau Flux Compactifications,” *JHEP* **03** (2005) 007, hep-th/0502058.

[28] C. P. Burgess, R. Kallosh, and F. Quevedo, “De Sitter string vacua from supersymmetric D terms,” *JHEP* **10** (2003) 056, hep-th/0309187.

[29] F. Apruzzi, M. Fazzi, A. Passias, D. Rosa, and A. Tomasiello, “AdS_6 solutions of type II supergravity,” *JHEP* **1411** (2014) 099, 1406.0852.

[30] F. Apruzzi, M. Fazzi, A. Passias, and A. Tomasiello, “Supersymmetric AdS_5 solutions of massive IIA supergravity,” *JHEP* **06** (2015) 195, 1502.06620.

[31] F. Apruzzi, M. Fazzi, D. Rosa, and A. Tomasiello, “All AdS_7 solutions of type II supergravity,” *JHEP* **1404** (2014) 064, 1309.2949.

[32] E. D’Hoker, M. Gutperle, A. Karch, and C. F. Uhlemann, “Warped $AdS_6 \times S^2$ in Type IIB supergravity I: Local solutions,” *JHEP* **08** (2016) 046, 1606.01254.

[33] J. P. Gauntlett, D. Martelli, J. Sparks, and D. Waldram, “Supersymmetric AdS_5 solutions of M theory,” *Class.Quant.Grav.* **21** (2004) 4335–4366, hep-th/0402153.

[34] M. Graña, R. Minasian, M. Petrini, and A. Tomasiello, “A scan for new $\mathcal{N} = 1$ vacua on twisted tori,” *JHEP* **05** (2007) 031, hep-th/0609124.

[35] M. Gualtieri, “Generalized complex geometry,” math/0401221. Ph.D. Thesis (Advisor: Nigel Hitchin).

[36] N. Hitchin, “Generalized Calabi–Yau manifolds,” *Quart. J. Math. Oxford Ser.* **54** (2003) 281–308, math.dg/0209099.

[37] A. Lukas and P. M. Saffin, “M-theory compactification, fluxes and AdS_4,” *Phys. Rev.* **D71** (2005) 046005, hep-th/0403235.

[38] D. Lüst, F. Marchesano, L. Martucci, and D. Tsimpis, “Generalized non-supersymmetric flux vacua,” *JHEP* **11** (2008) 021, 0807.4540.

[39] N. T. Macpherson and A. Tomasiello, “Minimal flux Minkowski classification,” *JHEP* **09** (2017) 126, 1612.06885.

[40] F. Apruzzi, J. C. Geipel, A. Legramandi, N. T. Macpherson, and M. Zagermann, "Minkowski$_4 \times S^2$ solutions of IIB supergravity," *Fortsch. Phys.* **66** (2018), no. 3, 1800006, 1801.00800.

[41] A. Legramandi and N. T. Macpherson, "Mink$_4 \times S^2$ Solutions of 10 and 11 dimensional supergravity," *JHEP* **07** (2019) 134, 1811.11224.

[42] A. Legramandi and A. Tomasiello, "Breaking supersymmetry with pure spinors," 1912.00001.

[43] P. Horava and E. Witten, "Heterotic and type I string dynamics from eleven dimensions," *Nucl. Phys.* **B460** (1996) 506–524, hep-th/9510209.

[44] O. Aharony, O. Bergman, D. L. Jafferis, and J. Maldacena, "$\mathcal{N} = 6$ superconformal Chern–Simons-matter theories, M2-branes and their gravity duals," *JHEP* **10** (2008) 091, 0806.1218.

[45] P. Pasti, D. P. Sorokin, and M. Tonin, "Covariant action for a $D = 11$ five-brane with the chiral field," *Phys. Lett.* **B398** (1997) 41–46, hep-th/9701037.

[46] E. Bergshoeff and J. P. van der Schaar, "On M nine-branes," *Class. Quant. Grav.* **16** (1999) 23–39, hep-th/9806069.

[47] E. Eyras and Y. Lozano, "Brane actions and string dualities," *Lect. Notes Phys.* **525** (1999) 356, hep-th/9812225. [,356(1998)].

[48] J. P. Gauntlett, J. B. Gutowski, and S. Pakis, "The geometry of $D = 11$ null Killing spinors," *JHEP* **12** (2003) 049, hep-th/0311112.

[49] E. Bergshoeff, R. Kallosh, T. Ortin, D. Roest, and A. Van Proeyen, "New Formulations of $D = 10$ Supersymmetry and D8–O8 Domain Walls," *Class. Quant. Grav.* **18** (2001) 3359–3382, hep-th/0103233.

[50] G. Dall'Agata, K. Lechner, and M. Tonin, "Action for IIB supergravity in 10-dimensions," *Lect. Notes Phys.* **525** (1999) 416, hep-th/9812170. [,416(1998)].

[51] I. A. Bandos, A. Nurmagambetov, and D. P. Sorokin, "The Type IIA NS5-brane," *Nucl. Phys.* **B586** (2000) 315–330, hep-th/0003169.

[52] E. Eyras, B. Janssen, and Y. Lozano, "Five-branes, K K monopoles and T duality," *Nucl. Phys.* **B531** (1998) 275–301, hep-th/9806169.

[53] A. Sen, "Stable nonBPS bound states of BPS D-branes," *JHEP* **08** (1998) 010, hep-th/9805019.

[54] U. Gran, J. Gutowski, and G. Papadopoulos, "The G(2) spinorial geometry of supersymmetric IIB backgrounds," *Class. Quant. Grav.* **23** (2006) 143–206, hep-th/0505074.

[55] P. Koerber and D. Tsimpis, "Supersymmetric sources, integrability and generalized- structure compactifications," arXiv:0706.1244 [hep-th].

[56] Y. Lozano, "NonAbelian duality and canonical transformations," *Phys. Lett.* **B355** (1995) 165–170, hep-th/9503045.

[57] T. H. Buscher, "A Symmetry of the String Background Field Equations," *Phys. Lett.* **B194** (1987) 59–62.

[58] T. H. Buscher, "Path Integral Derivation of Quantum Duality in Nonlinear Sigma Models," *Phys. Lett.* **B201** (1988) 466–472.

[59] S. Hassan, "Supersymmetry and the systematics of T-duality rotations in type II superstring theories," *Nucl.Phys.Proc.Suppl.* **102** (2001) 77–82, hep-th/0103149.

[60] Ö. Kelekci, Y. Lozano, N. T. Macpherson, and E. Ó Colgáin, "Supersymmetry and non-Abelian T-duality in type II supergravity," *Class. Quant. Grav.* **32** (2015), no. 3, 035014, 1409.7406.

[61] P. Saffin, "Type IIA Killing spinors and calibrations," *Phys.Rev.* **D71** (2005) 025018, hep-th/0407156.

[62] D. Tong, "NS5-branes, T duality and world sheet instantons," *JHEP* **0207** (2002) 013, hep-th/0204186.

[63] C. M. Hull and P. K. Townsend, "Unity of superstring dualities," *Nucl. Phys.* **B438** (1995) 109–137, hep-th/9410167. [,236(1994)].

[64] C. M. Hull, "Gravitational duality, branes and charges," *Nucl. Phys.* **B509** (1998) 216–251, hep-th/9705162.

[65] E. Eyras and Y. Lozano, "The Kaluza-Klein monopole in a massive IIA background," *Nucl. Phys.* **B546** (1999) 197–218, hep-th/9812188.

[66] P. Meessen and T. Ortin, "An SL(2, $\mathbb{Z}$) multiplet of nine-dimensional type II supergravity theories," *Nucl. Phys.* **B541** (1999) 195–245, hep-th/9806120.

[67] E. A. Bergshoeff, M. de Roo, S. F. Kerstan, T. Ortin, and F. Riccioni, "IIB nine-branes," *JHEP* **06** (2006) 006, hep-th/0601128.

[68] J. M. Figueroa-O'Farrill and G. Papadopoulos, "Maximally supersymmetric solutions of ten-dimensional and eleven-dimensional supergravities," *JHEP* **03** (2003) 048, hep-th/0211089.

[69] U. Gran, J. Gutowski, and G. Papadopoulos, "Classification of IIB backgrounds with 28 supersymmetries," *JHEP* **01** (2010) 044, 0902.3642.

[70] U. Gran, J. Gutowski, and G. Papadopoulos, "M-theory backgrounds with 30 Killing spinors are maximally supersymmetric," *JHEP* **03** (2010) 112, 1001.1103.

[71] A. Coimbra, C. Strickland-Constable, and D. Waldram, "Supergravity as Generalised Geometry I: Type II Theories," 1107.1733.

[72] R. L. Bryant, "Pseudo-Riemannian metrics with parallel spinor fields and vanishing Ricci tensor," *ArXiv Mathematics e-prints* (Apr., 2000) arXiv:math/0004073.

[73] M. Graña, R. Minasian, M. Petrini, and A. Tomasiello, "Supersymmetric backgrounds from generalized Calabi–Yau manifolds," *JHEP* **08** (2004) 046, hep-th/0406137.

[74] P. Kaste, R. Minasian, and A. Tomasiello, "Supersymmetric M-theory compactifications with fluxes on seven-manifolds and G-structures," *JHEP* **07** (2003) 004, hep-th/0303127.

[75] M. Haack, D. Lüst, L. Martucci, and A. Tomasiello, "Domain walls from ten dimensions," *JHEP* **0910** (2009) 089, 0905.1582.

[76] D. Lüst, P. Patalong, and D. Tsimpis, "Generalized geometry, calibrations and supersymmetry in diverse dimensions," *JHEP* **1101** (2011) 063, 1010.5789.

[77] J. M. Figueroa-O'Farrill, E. Hackett-Jones, and G. Moutsopoulos, "The Killing superalgebra of ten-dimensional supergravity backgrounds," *Class. Quant. Grav.* **24** (2007) 3291–3308, hep-th/0703192.

[78] E. J. Hackett-Jones and D. J. Smith, "Type IIB Killing spinors and calibrations," *JHEP* **11** (2004) 029, hep-th/0405098.

[79] K. Becker, M. Becker, D. R. Morrison, H. Ooguri, Y. Oz, and Z. Yin, "Supersymmetric cycles in exceptional holonomy manifolds and Calabi-Yau 4 folds," *Nucl. Phys.* **B480** (1996) 225–238, hep-th/9608116.

[80] B. S. Acharya, J. M. Figueroa-O'Farrill, and B. J. Spence, "Branes at angles and calibrated geometry," *JHEP* **04** (1998) 012, hep-th/9803260.

[81] J. Gutowski, G. Papadopoulos, and P. K. Townsend, "Supersymmetry and generalized calibrations," *Phys. Rev.* **D60** (1999) 106006, hep-th/9905156.

[82] P. Koerber, "Stable d-branes, calibrations and generalized Calabi–Yau geometry," *JHEP* **08** (2005) 099, hep-th/0506154.

[83] P. Koerber and L. Martucci, "D-branes on AdS flux compactifications," *JHEP* **0801** (2008) 047, 0710.5530.

[84] J. Evslin and L. Martucci, "D-brane networks in flux vacua, generalized cycles and calibrations," *JHEP* **07** (2007) 040, hep-th/0703129.

[85] E. J. Hackett-Jones, D. C. Page, and D. J. Smith, "Topological charges for branes in M theory," *JHEP* **10** (2003) 005, hep-th/0306267.

[86] A. Hanany and A. Zaffaroni, "Chiral symmetry from type IIA branes," *Nucl. Phys.* **B509** (1998) 145–168, hep-th/9706047.

[87] C. M. Hull, "The Positivity of Gravitational Energy and Global Supersymmetry," *Commun. Math. Phys.* **90** (1983) 545.

[88] U. Gran, J. Gutowski, and G. Papadopoulos, "IIB black hole horizons with five-form flux and KT geometry," *JHEP* **05** (2011) 050, 1101.1247.

[89] U. Gran, J. Gutowski, and G. Papadopoulos, "IIB black hole horizons with five-form flux and extended supersymmetry," *JHEP* **09** (2011) 047, 1104.2908.

[90] D. Rosa, "Generalized geometry of two-dimensional vacua," 1310.6357.

[91] H. Lu, C. N. Pope, and J. Rahmfeld, "A construction of Killing spinors on S^n," *J. Math. Phys.* **40** (1999) 4518–4526, hep-th/9805151.

[92] P. Candelas, G. T. Horowitz, A. Strominger, and E. Witten, "Vacuum configurations for superstrings," *Nucl. Phys.* **B258** (1985) 46–74.

[93] B. R. Greene, K. H. Kirklin, P. J. Miron, and G. G. Ross, "A Three Generation Superstring Model. 1. Compactification and Discrete Symmetries," *Nucl. Phys.* **B278** (1986) 667–693.

[94] A. Strominger, S.-T. Yau, and E. Zaslow, "Mirror symmetry is T-duality," *Nucl. Phys.* **B479** (1996) 243–259, hep-th/9606040.

[95] K. Dasgupta, G. Rajesh, and S. Sethi, "M theory, orientifolds and G-flux," *JHEP* **08** (1999) 023, hep-th/9908088.

[96] K. Becker and M. Becker, "M-theory on eight-manifolds," *Nucl. Phys.* **B477** (1996) 155–167, hep-th/9605053.

[97] M. Graña and J. Polchinski, "Supersymmetric three-form flux perturbations on AdS_5," *Phys. Rev.* **D63** (2001) 026001, hep-th/0009211.

[98] Y. Imamura, "1/4 BPS solutions in massive IIA supergravity," *Prog.Theor.Phys.* **106** (2001) 653–670, hep-th/0105263.

[99] D. Andriot, "New supersymmetric vacua on solvmanifolds," *JHEP* **02** (2016) 112, 1507.00014.

[100] D. Youm, "Partially localized intersecting BPS branes," *Nucl. Phys.* **B556** (1999) 222–246, hep-th/9902208.

[101] E. Malek and H. Samtleben, "Ten-dimensional origin of Minkowski vacua in N=8 supergravity," *Phys. Lett.* **B776** (2018) 64–71, 1710.02163.

[102] J. M. Figueroa-O'Farrill, "On the supersymmetries of Anti-de Sitter vacua," *Class. Quant. Grav.* **16** (1999) 2043–2055, hep-th/9902066.

[103] G. B. De Luca, G. L. Monaco, N. T. Macpherson, A. Tomasiello, and O. Varela, "The geometry of $\mathcal{N} = 3$ AdS_4 in massive IIA," *JHEP* **08** (2018) 133, 1805.04823.

[104] D. Andriot, "New supersymmetric flux vacua with intermediate SU(2) structure," *JHEP* **08** (2008) 096, 0804.1769.

[105] A. Tomasiello, "Reformulating Supersymmetry with a Generalized Dolbeault Operator," *JHEP* **02** (2008) 010, arXiv:0704.2613 [hep-th].

[106] D. Lüst and D. Tsimpis, "Supersymmetric AdS_4 compactifications of IIA supergravity," *JHEP* **02** (2005) 027, hep-th/0412250.

[107] K. Behrndt, E. Bergshoeff, and B. Janssen, "Intersecting D-branes in ten dimensions and six dimensions," *Phys. Rev.* **D55** (1997) 3785–3792, hep-th/9604168.

[108] P. Koerber and L. Martucci, "From ten to four and back again: how to generalize the geometry," *JHEP* **08** (2007) 059, arXiv:0707.1038 [hep-th].

[109] B. Janssen, P. Meessen, and T. Ortin, "The D8-brane tied up: String and brane solutions in massive type IIA supergravity," *Phys.Lett.* **B453** (1999) 229–236, hep-th/9901078.

[110] J. Maldacena and D. Martelli, "The unwarped, resolved, deformed conifold: fivebranes and the baryonic branch of the Klebanov–Strassler theory," *JHEP* **01** (2010) 104, 0906.0591.

[111] E. Caceres, C. Nunez, and L. A. Pando-Zayas, "Heating up the Baryonic Branch with U-duality: A Unified picture of conifold black holes," *JHEP* **03** (2011) 054, 1101.4123.

[112] O. Lunin and J. M. Maldacena, "Deforming field theories with $U(1) \times U(1)$ global symmetry and their gravity duals," *JHEP* **0505** (2005) 033, hep-th/0502086.

[113] P. Candelas and D. J. Raine, "Spontaneous Compactification and Supersymmetry in $d = 11$ Supergravity," *Nucl. Phys.* **B248** (1984) 415.

[114] M. J. Duff, B. E. W. Nilsson, and C. N. Pope, "Kaluza-Klein supergravity," *Phys. Rept.* **130** (1986) 1–142.

[115] G. Dall'Agata and N. Prezas, "$\mathcal{N} = 1$ geometries for M theory and type IIA strings with fluxes," *Phys. Rev.* **D69** (2004) 066004, hep-th/0311146.

[116] J. Bellorin and T. Ortin, "A Note on simple applications of the Killing Spinor Identities," *Phys. Lett.* **B616** (2005) 118–124, hep-th/0501246.

[117] L. Jarv and C. V. Johnson, "Orientifolds, M theory, and the ABCD's of the enhancon," *Phys. Rev.* **D62** (2000) 126010, hep-th/0002244.

[118] G. B. De Luca, A. Gnecchi, G. Lo Monaco, and A. Tomasiello, "Holographic duals of 6d RG flows," *JHEP* **03** (2019) 035, 1810.10013.

[119] L. Girardello, M. Petrini, M. Porrati, and A. Zaffaroni, "Novel local CFT and exact results on perturbations of $\mathcal{N} = 4$ superYang Mills from AdS dynamics," *JHEP* **12** (1998) 022, hep-th/9810126.

[120] D. Israel, C. Kounnas, and M. P. Petropoulos, "Superstrings on NS5 backgrounds, deformed AdS(3) and holography," *JHEP* **10** (2003) 028, hep-th/0306053.

[121] M. Berkooz, M. Rozali, and N. Seiberg, "Matrix description of M theory on T^4 and T^5," *Phys. Lett.* **B408** (1997) 105–110, hep-th/9704089.

[122] F. Apruzzi, M. Fazzi, A. Passias, A. Rota, and A. Tomasiello, "Six-Dimensional Superconformal Theories and their Compactifications from Type IIA Supergravity," *Phys. Rev. Lett.* **115** (2015), no. 6, 061601, 1502.06616.

[123] E. Malek, H. Samtleben, and V. Vall Camell, "Supersymmetric AdS_7 and AdS_6 vacua and their minimal consistent truncations from exceptional field theory," 1808.05597.

[124] S. Cremonesi and A. Tomasiello, "6d holographic anomaly match as a continuum limit," *JHEP* **05** (2016) 031, 1512.02225.

[125] H. Kim, N. Kim, and M. Suh, "Supersymmetric AdS_6 Solutions of Type IIB Supergravity," *Eur. Phys. J.* **C75** (2015), no. 10, 484, 1506.05480.

[126] E. D'Hoker, M. Gutperle, and C. F. Uhlemann, "Warped $AdS_6 \times S^2$ in Type IIB supergravity II: Global solutions and five-brane webs," *JHEP* **05** (2017) 131, 1703.08186.

[127] E. D'Hoker, M. Gutperle, and C. F. Uhlemann, "Warped $AdS_6 \times S^2$ in Type IIB supergravity III: Global solutions with seven-branes," *JHEP* **11** (2017) 200, 1706.00433.

[128] H. Lin and J. M. Maldacena, "Fivebranes from gauge theory," *Phys. Rev.* **D74** (2006) 084014, hep-th/0509235.

[129] D. Gaiotto and J. Maldacena, "The Gravity duals of $\mathcal{N} = 2$ superconformal field theories," *JHEP* **1210** (2012) 189, 0904.4466.

[130] H. Lin, O. Lunin, and J. M. Maldacena, "Bubbling AdS space and 1/2 BPS geometries," *JHEP* **0410** (2004) 025, hep-th/0409174.

[131] E. O Colgain, J.-B. Wu, and H. Yavartanoo, "On the generality of the LLM geometries in M-theory," *JHEP* **04** (2011) 002, 1010.5982.