

Article

Tail Dependence in Financial Markets: A Dynamic Copula Approach

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Abstract: This article is concerned with the study of the tail correlation among equity indices by means of dynamic copula functions. The main idea is to consider the impact of the use of copula functions in the accuracy of the model's parameters and in the computation of Value-at-Risk (VaR). Results show that copulas provide more sophisticated results in terms of the accuracy of the forecasted VaR, in particular, if they are compared with the results obtained from Dynamic Conditional Correlation (DCC) model.

Keywords: copula functions; Monte Carlo simulation techniques; risk measures

1. Introduction

One common mistake in portfolio theory is to assume that multivariate financial returns are normally distributed. In practice, this assumption does not hold given that financial returns are usually found to be leptokurtic and asymmetric: they exhibit greater dependence during market downturns or upturns.

For this reason, we will assume that the joint distribution of log-returns on the FTSE (Financial Times Stock Exchange) MIB (Milano Indice di Borsa) pairwise with the DAX 30 (German Exchange Index) and the IBEX 35 (Spanish Exchange Index), respectively, is described by a copula function, defined as

$$C(u, v) : [0, 1]^2 \mapsto [0, 1],$$

where u and v are the uniform transforms of the marginals.¹

The use of copulas is straightforward because of the theorem by Sklar (1959): It assesses that given F , joint distribution function, and given the marginal distributions F_1, F_2 , then there exists a copula function $C : [0, 1]^2 \mapsto [0, 1]$ such that, for all x, y in $\bar{\mathbb{R}} = [-\infty, +\infty]$,

$$F(x, y) = C(F_1(x_1), F_2(y)) = C(u, v),$$

meaning that any multivariate cumulative distribution function can be written in terms of a copula C , which describes the joint structure of the risk factors, and in terms of the cumulative distribution functions of the single risk factors.

The parameters characterizing the copula are assumed to be time-varying. An interesting approach in the context of copula theory is the one proposed by Patton (2006); he extended Sklar's theorem, allowing the dependence parameter to be conditional and time varying.

¹ Uniform transforms can be defined in the following way: if x has cumulative distribution function F_1 and y has cumulative distribution function F_2 , then $u = F_1(x)$ and $v = F_2(y)$ are standard uniform distributed.

In a further work, [Oh and Patton \(2018\)](#) improved the time-varying equation for copulas via the so-called generalized autoregressive score (GAS) model.

Another interesting methodology combines regime Markov switching models and copulas. It has been introduced by [Hamilton \(1989\)](#) and assumes that a time series' evolution is influenced by two different states of the economy, i.e., by a Markov chain with two states. [Jondeau and Rockinger \(2006\)](#) suggest that the time-varying parameter, i.e., the copula parameter, is driven by the following equation:

$$\delta_t = \delta_0(1 - S_t) + \delta_1 S_t,$$

where δ_0 is the value assumed by the dependence parameter during a low-dependence regime, δ_1 the value assumed during high-dependence regime, and S_t is the unobserved regime at time t . In this study, we will consider only the dynamics proposed by [Patton \(2006\)](#) and [Oh and Patton \(2018\)](#).

We want to introduce an approach to modeling dependence between financial returns allowing for two time-varying structures: we keep fixed one copula function across the two regimes characterizing equity markets, choosing between two copula forms.

An alternative to this approach is the one proposed by [Rodriguez \(2007\)](#), which assumes time variation in the functional form of the copula. We do not consider this possibility here.

The significance tests for the copula parameters are mainly settled on the Cramér–von Mises test statistic (t.s.). As in ([Genest et al. 2009](#)), we simulate from the hypothesized copula structure to get a distribution of fitted parameter from which we will obtain the desired error measure. Simulated pairs of observations from a given copula function are easy to obtain once the analytical form of the copula is known. As suggested by [McNeil et al. \(2015\)](#), we just need to rely on the inverse function of the copula density.

We are interested in comparing different dynamic copula models in terms of their ability to:

1. Describe the evolution of the dependence in the tails via the computation of tail indices².
2. Forecast capital losses, computing and then forecasting one popular risk measure, like Value-at-Risk (VaR).

Given the importance of extreme events for risk management purposes, we select the best model according to VaR forecasts tests proposed by [Kupiec \(1995\)](#) and [Christoffersen \(1998\)](#), for equally-weighted portfolios composed by the returns of the DAX 30 and IBEX 35 indices in pair with the FTSE MIB returns.

As will be seen later, given that the distribution of the portfolio is unknown, being that the portfolio itself a sum of Generalized Autoregressive Conditional Heteroscedasticity (GARCH)-distributed risk factors, we rely on Monte Carlo simulation technique to obtain such risk measures ([Christoffersen 2011](#)).

Once several simulated statuses of the portfolio for each time step t are computed, we test the goodness of the VaR forecasts considering the two aforementioned tests: the Unconditional Coverage Test introduced by [Kupiec \(1995\)](#) and the Conditional Coverage Test by [Christoffersen \(1998\)](#).

We want to prove that dynamic copula models are a better specification in covering capital losses with respect to “traditional” models, like the Dynamic Conditional Correlation (DCC), which takes into account only the linear correlation coefficient as the dependence measure describing the joint behaviour of the single risk factors.

Our work proceeds as follows. In Section 2, we describe the copula models adopted and the method for the computation of VaR; then, in Section 3, the results of the performed tests are provided. Section 4 contains a discussion of the results.

² We make use of the classical tail dependence indices for simplicity; as will be seen later, they can be easily obtained using a closed form formula, once a particular copula function is settled, but the reader should be aware of the shortcomings in the use of classical tail indices when moving apart from the “Gaussian world”. For details, see [Furman et al. \(2016\)](#), where the authors urge that the classical measures of tail dependence may underestimate the level of tail dependence in copulas.

2. Model

2.1. Data Description

We consider daily log-returns of the stock indices IBEX 35, DAX 30, and FTSE MIB from 1 March 2006 to 8 June 2017. The data were downloaded from the Yahoo Finance website. The total number of observations is 2850: The first 2300 are used to fit the in-sample model, and the residual 550 are used for forecasting purposes. The daily returns we consider are those observed for simultaneous trading days occurring in all the Italian, German, and Spanish stock markets. For this reason, we dropped from the original data set all Italian, German, and Spanish closing days of the corresponding National Stock Exchange markets.

The period covered by our data sample comprises some main stock market crashes: first, the subprime crisis, during the period 2008–2009; and second, the explosion of the European debt crisis, from middle 2011 up to 2012, which was mainly driven by the Italian economic and political instability. For this reason, the period January–March 2013 will also be of particular interest, given the occurrence of the Italian Parliamentary election.

A few descriptive statistics of the returns are provided in Table 1.

Table 1. Descriptive statistics of the three stock indexes, FTSE (Financial Times Stock Exchange) MIB (Milano Indice di Borsa), IBEX 35 (Spanish Exchange Index), and DAX 30 (German Exchange Index).

	FTSE MIB	IBEX 35	DAX 30
Mean	−0.08599	−0.00012	0.00020
Median	0.00028	0.00062	0.00089
Maximum	0.10874	0.13484	0.10797
Minimum	−0.08599	−0.09586	−0.07433
St. Deviation	0.016844	0.01601	0.01463
Skewness	−0.06100	0.01222	0.01011
Kurtosis	6.83680	8.4853	8.65864

2.2. Estimation

The copula estimation method adopted will be the Inference For Margins (IFM) method. As suggested in Joe (1997), we first make an assumption on the marginal distributions of the single risk factors. Once the marginal distributions' set of parameters is fitted, we assume a particular analytic form for the copula describing their joint behaviour. This straightforward procedure is possible because of the particular structure of the copula likelihood

$$L(\theta) = \sum_{t=1}^T \log c(F_1(x_t), F_2(y_t)) + \sum_{t=1}^T (\log f_1(x_t) + \log f_2(y_t)), \quad (1)$$

where f_1 is the marginal density of x , f_2 the marginal density of y , and c is the copula density defined as

$$c(u, v) = \frac{\partial^2 C(u, v)}{\partial u \partial v}.$$

In Equation (1), the first sum refers to the joint distribution, whereas the second sum depends only on the marginals, meaning that IFM method consists of a two-steps optimization: first, two separate optimizations of the univariate likelihoods and then a maximization of the multivariate likelihood as a function of the dependence parameter vector. This procedure is computationally simpler than estimating all parameters simultaneously.

2.3. Marginal Distributions

In Table 1, there is evidence of sign of excess kurtosis. For this reason, we assume that the errors have a Student-t distribution. To capture the leverage effect, we also consider asymmetric GARCH specifications, such as the eGARCH model.

We selected an autoregressive moving average model (ARMA)(2,3)—eGARCH(1,1) with Student-t errors for FTSE MIB returns and an ARMA(2,2)—eGARCH(1,1) with Student-t errors for DAX 30 and IBEX 35 returns.

$$\begin{aligned}
 Y_t &= \sum_{i=1}^p \theta_i Y_{t-i} + \sum_{i=1}^q \phi_i z_{t-i} + \sigma_t \epsilon_t \\
 \log \sigma_t^2 &= \omega + \alpha \left(\frac{z_{t-1}}{\sqrt{\sigma_{t-1}^2}} - \sqrt{\frac{2}{\pi}} \right) + \beta \log \sigma_{t-1}^2 + \gamma \frac{z_{t-1}}{\sqrt{\sigma_{t-1}^2}} \\
 v_t &= \sqrt{\frac{v}{\sigma_t^2(v-2)}} \cdot \epsilon_t \sim \mathbb{T}_v, \quad z_t = \sigma_t \epsilon_t,
 \end{aligned}$$

where Y_t is the log-return at time t , z_t the residual at time t , and \mathbb{T}_v is a Student-t random variable with v degrees of freedom.

2.4. Joint Distribution

We consider two particular copula forms from the class of *parametric copulas*. These copula functions have the willing property of being completely described by a vector of parameters and by a *generator function*, like

$$C_{\phi}^A(u, v; \theta) = \phi^{-1}(\phi(u) + \phi(v)),$$

with $\phi : I \mapsto \overline{\mathbb{R}}^+$ called the generator function and θ the vector of the parameters.

The two functional forms for the copula are: a BB1 copula, as defined in Joe (1997), like

$$C_{BB1}(u, v; k, \gamma) = (1 + ((u^{-k} - 1)^\gamma + (v^{-k} - 1)^\gamma)^{1/\gamma})^{-1/k}, \tag{2}$$

with $k > 0$ and $\gamma \geq 1$ and a Symmetrized Joe-Clayton (SJC) copula,

$$C_{SJC}(u, v; k, \gamma) = 0.5 \left(C_{BB7}(u, v; k, \gamma) + C_{BB7}(1 - u, 1 - v; k, \gamma) + u + v - 1 \right), \tag{3}$$

with $k \geq 1$ and $\gamma > 0$. C_{BB7} is the BB7 Joe-Clayton copula defined as

$$C_{BB7}(u, v; k, \gamma) = 1 - (1 - ((1 - (1 - u)^k)^{-\gamma} + (1 - (1 - v)^k)^{-\gamma} - 1)^{-1/\gamma})^{1/k}.$$

The interesting feature of copula functions is the possibility to calculate upper and lower tail indices through closed formulas. In particular, the expressions for the upper and lower tail indices, for the BB1 copula, are given by

$$\lambda_U = 2 - 2^{1/\gamma} \quad \lambda_L = 2^{-1/\gamma k}, \tag{4}$$

and in the case of the SJC copula, by

$$\lambda_U = 2 - 2^{1/k} \quad \lambda_L = 2^{-1/\gamma}. \tag{5}$$

The parameters are assumed to be time-varying. The first dynamic we consider is the one proposed by Patton (2006):

$$\begin{cases} \delta_t = \omega + \beta \sum_{i=1}^{10} |u_{t-i} - v_{t-i}| + \alpha \delta_{t-1} \\ \theta_t = \Lambda(\delta_t), \end{cases} \quad (6)$$

where θ_t is the parameter of the copula under analysis and Λ is a given transformation. To keep the parameters in the right domain, we consider the following transformations:

- If $\theta_t \in [1, +\infty)$, then we will set $\Lambda(x) = x^2 + 1$.
- If $\theta_t \in (0, +\infty)$, we will set $\Lambda(x) = \exp(x)$.

Equation (6) contains an autoregressive term, $\alpha\theta_{t-1}$, and a forcing variable with coefficient β . The identification of a forcing variable for a time-varying copula parameter is somewhat difficult: Patton (2006) suggests the use of the mean absolute difference between u_t and v_t over the previous 10 observations.

An alternative definition for the dynamic equation of the copula parameters is the GAS model, proposed by Oh and Patton (2018).

The evolution equation for the copula parameter is given by

$$\delta_t = \omega + Bs_{t-1} + A\delta_{t-1}, \quad (7)$$

where

$$s_{t-1} = S_{t-1} \nabla_{t-1},$$

with

$$\nabla_{t-1} = \frac{\partial \log c(u_{t-1}; \delta_{t-1})}{\partial \delta_{t-1}} \quad (8)$$

being the score of the log-copula density function and S_{t-1} a scaling factor, i.e.,

$$S_{t-1} = H_{t-1}^p,$$

with H_{t-1} as the Hessian matrix and $p = \{0, 1, \frac{1}{2}\}$. For simplicity, we will consider $p = 0$.

As stated in Creal et al. (2013), the reason for using the score as a forcing variable is that it depends on the complete density, and not only on the first- or second-order moments of the observations u_t , so it may provide a more complete information about the improvement of the model's local fit in terms of the likelihood.

2.5. Value-at-Risk

The computation of VaR will be performed on an equally weighted portfolio, like

$$\Pi = 0.5R_1 + 0.5R_2,$$

with R_1 and R_2 the returns on the assets under analysis; in our case, R_1 is the return on FTSE MIB, and R_2 the return on DAX 30 and IBEX 35, respectively.

For the computation of portfolio VaR from copula models, we need to rely on Monte Carlo simulation techniques, given that the distribution of Π is hard to obtain, being itself a sum of GARCH-distributed risk factors. As suggested by Christoffersen (2011), Monte Carlo simulation essentially reverses the steps taken in model building:

1. First, generate a random sample $(u_{i,t}, v_{i,t})$ from the selected copula C_θ .
2. Second, create shocks from the copula probabilities using the marginal inverse cumulative distribution functions $z_{i,t} = T_v^{-1}(u_{i,t})$ on each asset.
3. Third, create returns $x_{i,t}, y_{i,t}$ from shocks using the dynamic volatility models.

Random samples in copula theory can be obtained considering the *conditional probability distribution*, defined as

$$c_u(v) = C_{V|U}(v|u) = P(V \leq v|U = u) = \lim_{\Delta u \rightarrow 0} \frac{C(u + \Delta u, v) - C(u, v)}{\Delta u} = \frac{\partial C(u, v)}{\partial u}.$$

The simulation algorithm goes through the following steps:

1. Generate two independent uniform (0, 1) rvs u and s .
2. Set $v = c_u^{-1}(s)$, where c_u^{-1} is the inverse of c_u .
3. The desired pair is (u, v) .

Once we have simulated N vectors of returns from the model, it is easy to compute the simulated portfolio returns using a given portfolio allocation. We consider a “rolling window” approach to compute one-step ahead forecasted VaR for the period February 2015–June 2017, just taking the α -quantile on each vector of simulated states of the portfolio for each time step t . In particular, we will consider $\alpha = 1\%$.

3. Results

The results of the estimation procedure are reported in Table 2.

Table 2. Fitted coefficients and standard errors.

	FTSE MIB	IBEX 35	DAX 30		FTSE MIB	IBEX 35	DAX 30
θ_1	−0.043168 (0.004125)	0.044679 (0.041526)	−0.026672 (0.005380)	ω	−0.094826 (0.001587)	−0.166808 (0.005469)	−0.169652 (0.002550)
θ_2	0.944940 (0.001412)	0.845535 (0.074948)	0.955586 (0.005338)	α	−0.144421 (0.013629)	−0.152084 (0.018368)	−0.178421 (0.018107)
ϕ_1	0.020101 (0.008237)	−0.009836 (0.044879)	0.056801 (0.000259)	β	0.989424 (0.000032)	0.980849 (0.0008379)	0.980906 (0.000058)
ϕ_2	−0.921542 (0.000008)	−0.848789 (0.083562)	−0.939253 (0.000007)	γ	0.094497 (0.014279)	0.111731 (0.0173419)	0.118315 (0.008174)
ϕ_3	0.053048 (0.008330)	-	-	ν	9.392484 (1.641448)	8.374261 (0.901715)	7.671721 (1.226093)

The choice of an eGARCH model with Student-t distribution for the marginals is justified by the results of Kolmogorov-Smirnov (KS) test, which are reported in Table 3.

Table 3. Kolmogorov-Smirnov (KS) test results.

	FTSE MIB	IBEX 35	DAX 30
KS statistic	0.022156	0.015969	0.018821
p -value	0.2089	0.6005	0.3891

We tested for the null of uniform distribution for the probability transforms

$$U_t = T_\nu(v_t),$$

where v_t are standardized residuals and $T_\nu(\cdot)$ is the cumulative distribution function (cdf) of a Student-t distribution with ν degrees of freedom.

We do not reject the null of uniform distribution for the probability transforms of standardized residuals at all common significance levels.

The significance test adopted for the copulas is based on the Cramér–von Mises t.s., defined as:

$$S_n = \int_1 n \{C_n(\hat{U}) - C_{\theta_n}(\hat{U})\}^2 dC_n(\hat{U}) =$$

$$= \sum_{i=1}^n \{C_n(\widehat{U}_i) - C_{\theta_n}(\widehat{U}_i)\}^2,$$

with n being the sample size, $I = [0, 1]^2$, $\widehat{U}_i \in I$ uniform transform of log-returns, and θ_n the estimated vector of parameters of the copula to be tested. The Cramér–von Mises t.s. is a measure of the mean squared difference between the empirical copula, defined as

$$C_n(\cdot) = \frac{1}{T} \sum_{t=1}^T \mathbf{1}(u_t \leq \cdot),$$

and the hypothesized copula; one criterion is to select the copula C_θ that minimizes this quadratic distance between $C_{\widehat{\theta}}$, the estimated copula, and C_n .

A significance test suggested by Genest et al. (2009), and based on the Cramér–von Mises statistic, is then performed. We want to test the null

$$H_0 : C \in \mathcal{C}_\theta,$$

with \mathcal{C}_θ being the family of parametric copula under analysis.

The structure of the test is summarized in the following algorithm.

1. Compute C_n , the empirical copula, from the uniform transforms \widehat{U}_i and estimate the vector of copula parameters θ , say θ_n , via maximum likelihood.
2. Compute the t.s. S_n .
3. For some large N , repeat the following steps, $k = \{1, \dots, N\}$.
 - (a) Generate a random sample $\widehat{Y}_1^{(k)}, \dots, \widehat{Y}_n^{(k)}$ from copula C_{θ_n} , then compute their associated rank vectors $\widehat{R}_1^{(k)}, \dots, \widehat{R}_n^{(k)}$, $i = 1, \dots, n$.
 - (b) Compute $\widehat{U}_i^{(k)} = \widehat{R}_i^{(k)} / (n + 1)$ and let

$$C_n^{(k)}(\mathbf{u}) = \frac{1}{n} \sum_{i=1}^n \mathbf{1}(\widehat{U}_i^{(k)} \leq \mathbf{u}) \quad \mathbf{u} \in [0, 1]^2$$

be the empirical copula. Compute an estimate $\theta_n^{(k)}$ of θ from $\widehat{U}_1, \dots, \widehat{U}_n$ via maximum likelihood.

- (c) Compute an approximate realization of S_n by

$$S_n^{(k)} = \sum_{i=1}^n \{C_n^{(k)}(\widehat{U}_i^{(k)}) - C_{\theta_n^{(k)}}(\widehat{U}_i^{(k)})\}^2.$$

4. An approximation for the p -value of the test is given by

$$p_{value} = \frac{1}{N} \sum_{i=1}^N \mathbf{1}(S_n^{(k)} \geq S_n).$$

Kojadinovic et al. (2010) suggest using the following formula for the computation of p -values:

$$p_{value} = \frac{1}{N + 1} \left(\sum_{i=1}^N \mathbf{1}(S_n^{(k)} \geq S_n) + \frac{1}{2} \right),$$

in order to ensure that they are in the open interval $I = (0, 1)$.

The results of the previous procedure are shown in Tables 4 and 5.

Table 4. Fitted coefficients, Cramér–von Mises test statistic (t.s.), and log-likelihood for the BB1, and for the Symmetrized Joe-Clayton (SJC) copulas with dynamic as in Equation (6).

	BB1		SJC		
	FTSE MIB–IBEX 35	FTSE MIB–DAX 30	FTSE MIB–IBEX 35	FTSE MIB–DAX 30	
ω_U	−0.30000567 (0.03716655)	0.40884762 (−0.28192092)	ω_U	0.24838033 (0.24721904)	0.09810473 (0.2551627)
α_U	0.03950425 (0.29152632)	−0.94995981 (−0.19627552)	α_U	0.85179066 (0.85034764)	0.96073183 (0.8588131)
β_U	−4.94732447 (3.35765376)	−16.59315866 (7.20800344)	β_U	0.02490790 (0.02766444)	−0.31265168 (−0.4735605)
ω_L	1.28118967 (1.09177413)	0.01167368 (0.30021851)	ω_L	−0.04173219 (−0.04417824)	−0.07179274 (−0.1359612)
α_L	0.17951749 (0.46726049)	0.99249420 (0.83081313)	α_L	0.84924889 (0.84873280)	0.80251885 (0.5257496)
β_L	−2.58514184 (−2.39938572)	−0.03026523 (−0.06393856)	β_L	0.03121317 (0.02994828)	0.14441369 (−0.1443658)
S_n	0.2059648	0.2754048	S_n	0.01920454	0.3144344
p -value	0.0004995	0.0004995	p -value	0.6728272	0.2442557
Log-Lik	1576.834	1387.445	Log-Lik	1512.259	1343.717

Table 5. Fitted coefficients, Cramér–von Mises t.s., and log-likelihood for the BB1, and for the SJC copulas with generalized autoregressive score model (GAS) dynamic as in Equation (7).

	BB1–GAS		SJC–GAS		
	FTSE MIB–IBEX 35	FTSE MIB–DAX 30	FTSE MIB–IBEX 35	FTSE MIB–DAX 30	
ω_U	−1.31443861 (−0.89944562)	−0.003463384 (−0.46987585)	ω_U	0.25368733 (0.24816560)	0.28971776 (0.29316496)
α_U	−0.26458349 (0.05733857)	0.996577947 (0.26361708)	α_U	0.84285325 (0.84930233)	0.81958206 (0.80452288)
β_U	0.66099308 (0.48235972)	0.066533564 (0.01623878)	β_U	0.10302827 (0.03045566)	0.04663823 (0.04545464)
ω_L	0.06892752 (0.08599858)	0.080316329 (0.08656701)	ω_L	−0.03158979 (−0.04297152)	−0.03777936 (−0.05679195)
α_L	0.94315204 (0.93582273)	0.926766139 (0.92903208)	α_L	0.85087467 (0.84992074)	0.79993059 (0.80080747)
β_L	0.05274755 (0.05108619)	0.068472429 (0.06695990)	β_L	0.03762481 (0.01961471)	0.03008048 (0.04271727)
S_n	0.2372009	0.4471988	S_n	0.08591551	0.03982476
p -value	0.00075	0.000487	p -value	0.02147852	0.7787213
Log-Lik	1584.072	1413.833	Log-Lik	1542.525	1319.508

Small p -values can be observed in the case of a BB1 specification of the copula for both FTSE–IBEX and FTSE–DAX indices, whereas the null of an SJC copula distribution is accepted at all significance levels, provided that p -values in this case are equal to 0.67 for FTSE–IBEX indices (0.24 for FTSE–DAX, respectively). A similar result is obtained if a GAS dynamic is assumed for the copula parameters: the SJC–GAS copula specification is significant at 10% level in the case of the FTSE–IBEX and only at 1% level in the case of the FTSE–DAX. The null hypothesis of a BB1–GAS copula is rejected at all significance levels for both FTSE–DAX and FTSE-IBEX indices.

Value in brackets in Tables 4 and 5 refer to an error measure of the fitted coefficients obtained via Monte Carlo simulation technique; once we fitted the coefficients via maximum likelihood, we have simulated from the selected copula 1000 times with parameters equal to the fitted ones, and then we re-fitted the coefficients from the simulated pairs of observations. We collected all the “simulated parameters” to get the desired confidence intervals. In particular, the values in brackets represent the

mean of all the 1000 simulated parameters. A schematic representation of this procedure is given in the following algorithm:

1. Estimate the vector of parameters θ , say θ_n , via maximum likelihood.
2. For some large N , say $N = 1000$, repeat the following three steps, $k = \{1, \dots, N\}$.
 - Generate a random sample $\widehat{U}_1^{(k)}, \dots, \widehat{U}_n^{(k)}$ from copula C_{θ_n} .
 - Estimate $\theta_n^{(k)}$ of θ from $\widehat{U}_1, \dots, \widehat{U}_n$ via maximum likelihood.
 - Collect $\theta_n^{(k)}$.

The mean values of simulated parameters are close to the fitted values in the case of SJC copula, whereas in the case of BB1 copula, this happens only for few parameters. In the case of the SJC–GAS copula, a significant distance between fitted and mean value is observed only in the case of β_U for the FTSE–IBEX pair of indices. For the BB1–GAS parameters, only fitted values for $\omega_L, \alpha_L, \beta_U$, and β_L are close to the mean values of simulated parameters, for both pairs of indices.

The time evolution patterns of upper and lower correlations are provided in Figures 1–4, where the blue line refers to the tail indices computed considering a static copula approach (i.e., only a particular copula structure is selected without allowing for the parameters to be time-varying). We recall that the computation of tail indices uses formulas in Equations (4) and (5).

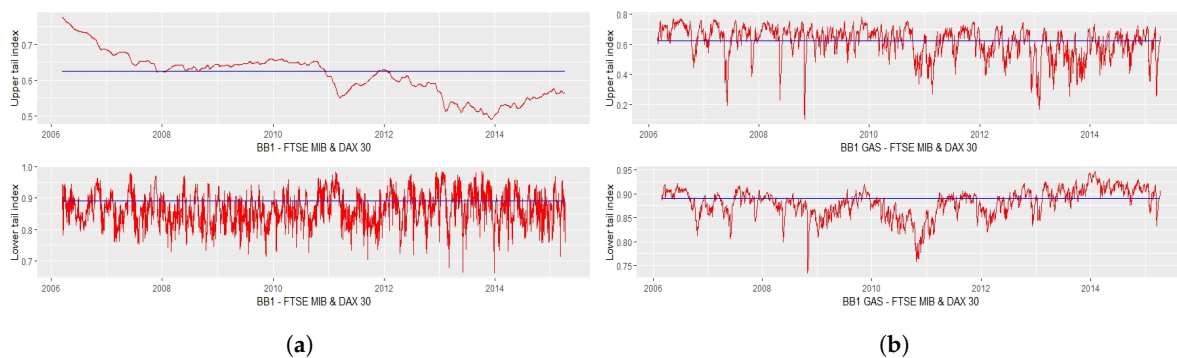


Figure 1. Upper and lower tail indices of FTSE MIB–DAX 30. (a) BB1 with dynamic Equation (6); (b) BB1 copula with dynamic Equation (7).

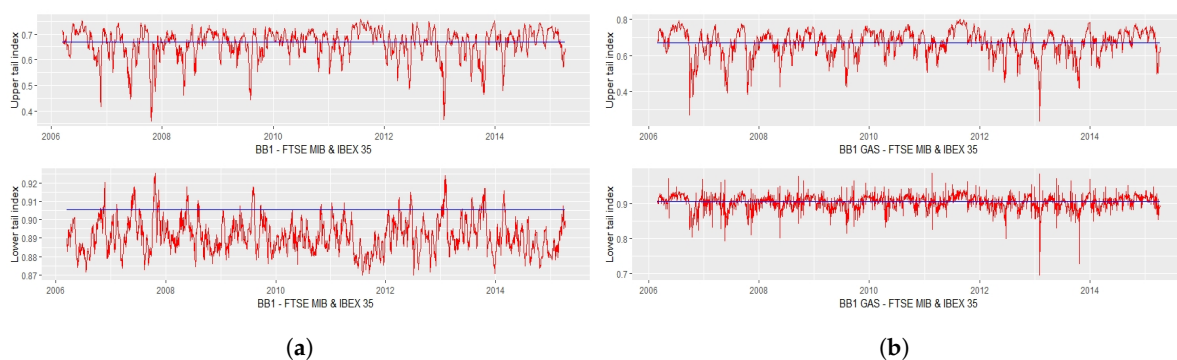


Figure 2. Upper and lower tail indices of FTSE MIB–IBEX 35; (a) BB1 copula with dynamic Equation (6); (b) BB1 copula with dynamic Equation (7).

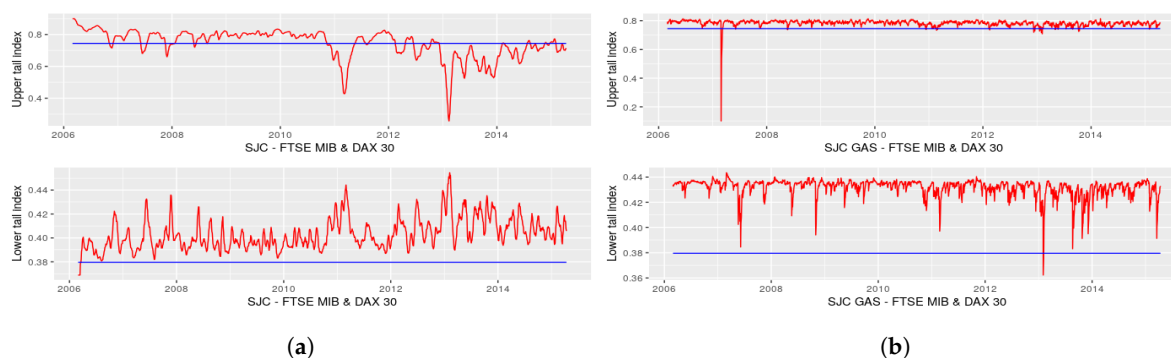


Figure 3. Upper and lower tail indices of FTSE MIB–DAX 30; (a) SJC copula with dynamic Equation (6); (b) SJC copula with dynamic Equation (7).

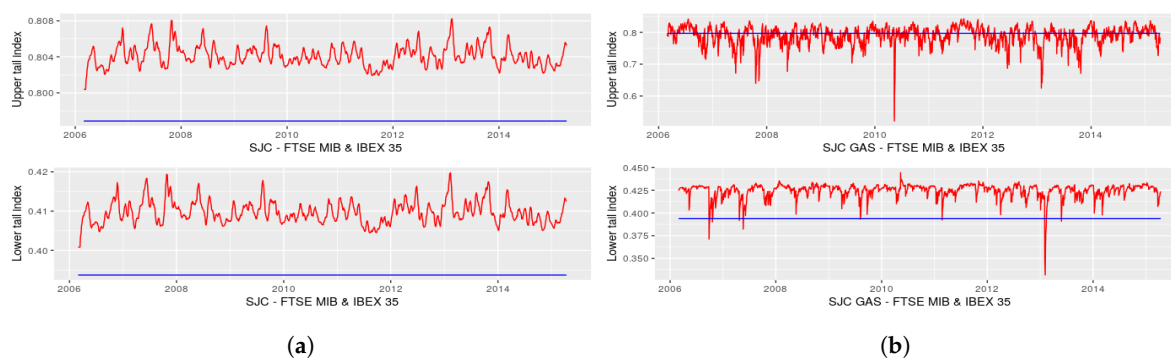


Figure 4. Upper and lower tail indices of FTSE MIB–IBEX 35; (a) SJC copula with dynamic Equation (6); (b) SJC copula with dynamic Equation (7).

The symmetric behaviour between upper and lower tail indices is evident in the case of a BB1 copula specification with dynamic Equation (6) for the Italian–Spanish pair of indices, as shown in Figure 2a; there are peaks in the lower tail index—and, consequently, drops in the upper tail index—in the last part of 2007 and from middle 2009 up to the beginning of 2011. The upward trend in the lower tail index starts again in the middle of 2011, just before the explosion of the European debt crisis, and continues until 2013, where it reaches another maximum level.

In the case of tail indices obtained from SJC copula, it can be observed, as shown in Figures 3a and 4a, that the values for the lower and for the upper tail indices are, in the case of the FTSE–IBEX pair, systematically above the values obtained from static SJC copula, meaning that the static approach seems to be more “conservative” with respect to the dynamic structure of Equation (6). Lower weight is given to lower tail indices in the case of SJC copula, with average values of 0.37 for the FTSE–IBEX pair and 0.38 for the FTSE–DAX pair.

Relevant peaks can be observed during periods of political instability in Italy: in particular, the period December 2011–January 2012, due to the beginning of president Monti’s technical government and during February 2013, on the occasion of the Italian Parliamentary election.

It is interesting to underline the joint behaviour of the two tail indices: in the case of the FTSE–IBEX pair, we get the same pattern in the upper and in the lower tail indices, meaning that the correlation between Italian and Spanish equity markets moves in the same way in both the right and the left tail. Instead, from Figure 3a, it can be observed that an increase in the lower tail index corresponds to a decrease in the upper tail index for the FTSE–DAX pair, and vice versa. In this case, we would say that Italian and German tail correlation behave in a symmetric way during shock periods.

We now analyse the time evolution patterns of the tail indices, considering a BB1 copula and an SJC copula with dynamic equation as in (7).

The SJC–GAS model tends to provide more stable results: in particular, in Figure 3b it can be observed that the upper tail index for FTSE–DAX is flat around 0.8 with one only exception at the beginning of 2007, where the index reaches the minimum value of 0.15. Greater variability is provided

by the BB1–GAS model, as shown in Figures 1b and 2a; the minimum value for lower tail index for the FTSE–DAX is reached in the second half of 2008, with a subsequent sudden jump. Another shock is observed just before the beginning of 2011, with a persistent trend around the static value of 0.89. At some points in time, particularly during the second half of 2008, the upper tail index reaches values close to 0 for the FTSE–DAX pair, meaning that asymptotic independence in the right tail is observed during severe economic crashes, as in September 2008. The trend is more stable in the case of the FTSE–IBEX pair, considering always a BB1–GAS specification, with values always greater than 0.7 for the lower tail index and 0.3 for the upper t.i.

Fluctuations of the upper and lower tail indices are observed in the case of the BB1–GAS copula: as shown in Figure 1b, during the fall of 2008 for the FTSE–DAX and, particularly, at the beginning of 2013 for both FTSE–DAX and FTSE–IBEX pairs, there is a considerable shock in the upper and lower tail correlations.

The results for the p -values shown in Tables 4 and 5 suggest that the best copula specification is an SJC copula instead of a BB1 copula. For this reason, the SJC copula with dynamics given by Equations (6) and (7) will be taken into account for the computation of VaR.

In Figure 5, a graphical representation of the one-step-ahead forecasted VaR is provided.

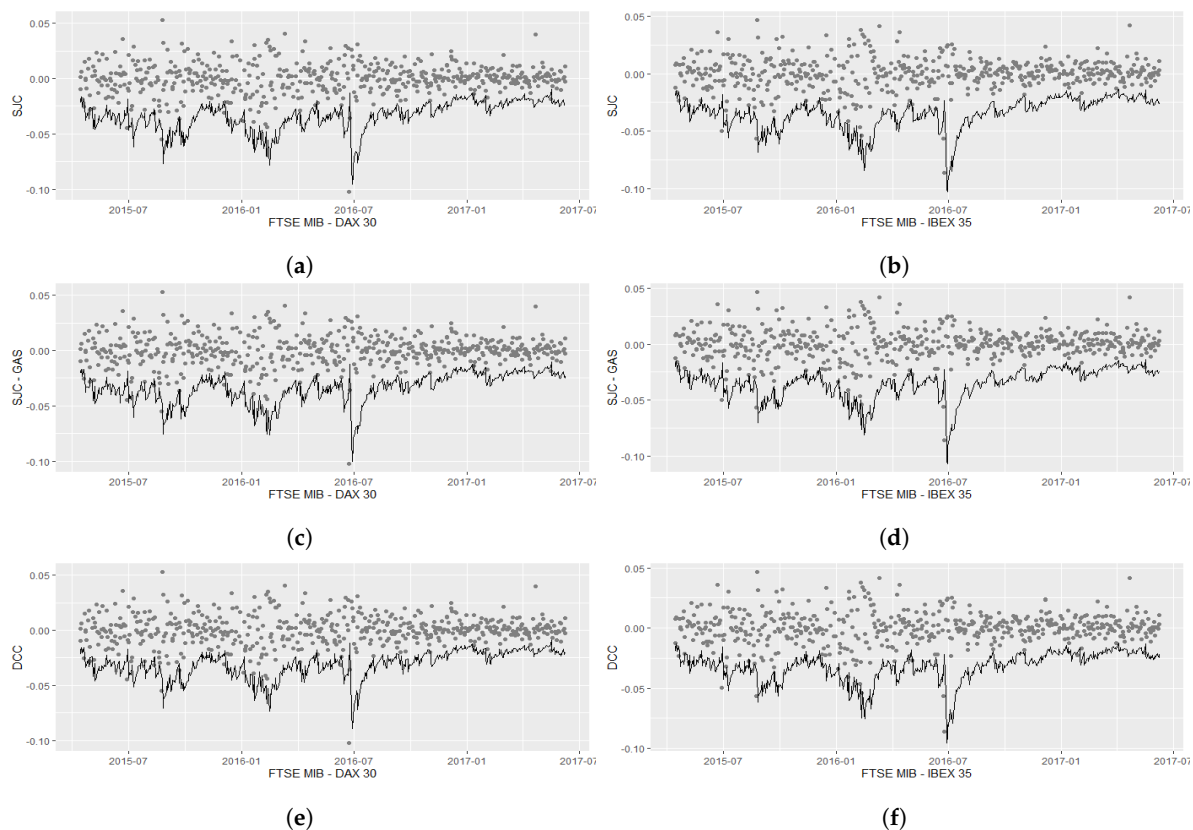


Figure 5. Value-at-Risk (VaR) forecasts time patterns considering an SJC copula with dynamic Equation (6) (a,b), an SJC–GAS copula (c,d), and a DCC model (e,f). (a,c,e) refer to the FTSE MIB–DAX 30 portfolio VaR, while (b,d,f) to the FTSE MIB–IBEX 35 portfolio.

To evaluate the VaR forecasts, we use the Unconditional Coverage test proposed by Kupiec (1995) and the Conditional Coverage test by Christoffersen (1998). It is possible to construct an indicator sequence $I_t, t = 1, \dots, T$, defined as

$$I_t = \begin{cases} 0 & \text{if } r_t > VaR_{\alpha,t} \\ 1 & \text{otherwise} \end{cases} ,$$

with r_t being the return of the portfolio at time t ; this is a straightforward way to count the number of violations.

The Unconditional Coverage test is a test for the null that the fraction of violations π_{obs} is significantly different from the assumed fraction π_{exp} . The t.s. is a likelihood ratio statistic given by

$$LR_{uc} = -2 \log \left(\frac{\pi_{exp}^{n_1} (1 - \pi_{exp})^{n_0}}{\pi_{obs}^{n_1} (1 - \pi_{obs})^{n_0}} \right) \sim \chi_1^2,$$

where $\pi_{exp} = \alpha$, n_1 is the observed number of violations, and $n_0 = T - n_1$ is the number of returns with indicator 0.

We can easily estimate π_{obs} from $\widehat{\pi}_{obs} = n_1/T$. χ_1^2 is a chi-square distribution with one degree of freedom.

We do not want the VaR violations to come in clusters; if the VaR violations are clustered, then we can predict that if today is a violation, then tomorrow is more than $\alpha \cdot 100\%$ likely to be a violation. Christoffersen (1998) proposed a combined test for both unconditional coverage and independence in the sequence I_t , with H_0 being the null hypothesis of serial independence and a violation rate of α of the violations I_t . The test statistic in this case is given by

$$LR_{cc} = -2 \log \left(\frac{\pi_{exp}^{n_1} (1 - \pi_{exp})^{n_0}}{\pi_{01}^{n_{01}} (1 - \pi_{01})^{n_{00}} \pi_{11}^{n_{11}} (1 - \pi_{11})^{n_{10}}} \right) \sim \chi_2^2,$$

where n_{ij} is the number of returns with indicator i followed by returns with indicator j , with $i, j = 0, 1$ and $\pi_{ij} = P(I_t = j | I_{t-1} = i) = n_{ij} / \sum_j n_{ij}$. In this case, the chi-square distribution χ_2^2 has two degrees of freedom.

Under H_0 , $\pi_{01} = \pi_{11} = \alpha$, meaning that, if the exceedances are independent, the probability of observing a new violation of the VaR at t is the same, independent of what happened in the earlier instant.

The two tests were performed on three multivariate models: an SJC copula with dynamic equation as in (6), an SJC copula with GAS dynamic, and a DCC model. Results are provided in Table 6. P -values are always equal to 0 if we consider a DCC model describing the joint behaviour of FTSE MIB and DAX 30, IBEX 35, respectively.

Table 6. Test statistic (t.s.) and, in brackets, p -values for Kupiec test and Christoffersen test, considering 1%–VaR on an equally weighted portfolio.

FTSE MIB—IBEX 35		
	Kupiec Test (Unconditional Coverage)	Christoffersen Test (Conditional Coverage)
SJC	1.006591 (0.3157209)	3.670008 (0.1596129)
SJC - GAS	0.3804048 (0.5373867)	3.774803 (0.1514649)
DCC	72.16992 (0)	77.56573 (0)
FTSE MIB—DAX 30		
	Kupiec Test (Unconditional Coverage)	Christoffersen Test (Conditional Coverage)
SJC	4.304982 (0.03800091)	5.927579 (0.05162293)
SJC—GAS	5.80171 (0.0160106)	7.142801 (0.02811646)
DCC	64.669 (0)	74.18366 (0)

4. Concluding Remarks

In Tables 4 and 5, small p -values can be observed in the case of a BB1 specification of the copula for both FTSE–IBEX and FTSE–DAX indices, whereas the null of an SJC copula distribution with the same dynamic is accepted at all significance levels, provided that p -values in this case are equal to 0.67 for FTSE–IBEX indices (0.24 for FTSE–DAX, respectively), considering the dynamic as in Equation (6). p -values for the SJC–GAS copula are significant at 10% level in the case of the FTSE–IBEX and only at 1% level in the case of the FTSE–DAX. The null hypothesis of a BB1–GAS copula is rejected at all significance levels for both FTSE–DAX and FTSE–IBEX indices.

SJC copula always provides higher p -values, meaning that it should be considered as the best specification in terms of the goodness of the fitted parameters.

The results of the Kupiec test and of the Christoffersen test are provided in Table 6: for the 1%–VaR the null of the expected number of violations close to 1% is not rejected at all common significance levels for the FTSE–IBEX portfolio and at 1% level for the FTSE–DAX, considering both dynamic copulas under analysis. The null hypothesis of the Kupiec test is rejected at all common significance levels for 1%–VaR if it is computed considering a DCC dynamic.

P -values for the conditional coverage test are always equal to 0 considering a DCC dynamic, whereas they are greater than 10% for the FTSE–IBEX portfolio with both the SJC–GAS copula and SJC copula with dynamic as in (6). In the case of the FTSE–DAX portfolio, we get a p -value for the SJC–GAS copula, greater than 1% and a p -value greater than 5% for the SJC copula with dynamic as in (6).

The DCC joint structure should be rejected if the 1%–VaR is considered, whereas both the SJC copula with dynamic as in (6) and the SJC–GAS copula perform well in the VaR forecasts tests considered.

We conclude saying that dynamic copulas, in comparison with the DCC model, not only provide a more sophisticated measure of the correlation (providing a specific measure for the dependence in the left and in the right tail), but they are also preferable in covering capital losses.

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Abbreviations

The following abbreviations are used in this manuscript:

rv	random variable
SJC	Symmetrized Joe-Clayton
GAS	Generalized Autoregressive Score
t.i.	tail index
cdf	cumulative distribution function
t.s.	test statistic
DCC	Dynamic Conditional Correlation

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