VIRTUALLY FREE PRO-P PRODUCTS

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ABSTRACT. It is shown that a finitely generated pro-p group G which is a virtually free pro-p product splits either as a free pro-p product with amalgamation or as a pro-p HNN-extension over a finite p-group. More precisely, G is the pro-p fundamental group of a finite graph of finitely generated pro-p groups with finite edge groups. This generalizes previous results of W. Herfort and the second author (cf. [2]).

1. Introduction

In 1965, J-P. Serre showed that a torsion free virtually free pro-p group must be free (cf. [7]). This motivated him to ask the question whether the same statement holds also in the discrete context. His question was answered positively some years later. In several papers (cf. [10], [11], [13]), J.R. Stallings and R.G. Swan showed that free groups are precisely the groups of cohomological dimension 1, and at the same time J-P. Serre himself showed that in a torsion free group G the cohomological dimension of a subgroup of finite index coincides with the cohomological dimension of G (cf. [8]).

One of the major tools for obtaining this type of result - the theory of ends - provided deep results also in the presence of torsion. The first result to be mentioned is 'Stallings' decomposition theorem' (cf. [12]). It generalizes the previously mentioned result to virtual free products.

Theorem 1.1 (J.R. Stallings)**.** *Let G be a finitely generated group containing a subgroup of finite index which is a non-trivial free product. Then G splits either as a free product with amalgamation or as an HNN-extension over a finite group.*

The purpose of this paper is to prove a pro-p analogue of Theorem 1.1.

Theorem A. *Let G be a finitely generated pro-p group containing an open subgroup which is a non-trivial free pro-p product. Then G splits either as a free pro-p product with amalgamation or as a pro-p HNN-extension over a finite p group.*

In the torsion free case Theorem A yields a splitting of G into a non-trivial free pro-p product.

Corollary B. *Let G be a finitely generated torsion free pro-p group which is a virtual free pro-p product. Then G is a non-trivial free pro-p product.*

In contrast to the proof of Theorem 1.1 which uses the theory of ends, the proof of Theorem A is accomplished by using purely combinatorial methods in pro-p group theory, and the description of finitely generated virtually free pro-p groups

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obtained by W. Herfort and the second author (cf. [2]). In fact, the techniques of pro-p groups acting on pro-p trees are used in order to obtain the following more conceptual version of Theorem A (cf. Thm. 3.6).

Theorem C. *Let G be a finitely generated pro-p group containing an open subgroup H which has a non-trivial decomposition as free product, i.e., there exists non-trivial closed subgroups $A, B \subsetneq H$ such that $H = A \amalg B$. Then G is isomorphic to the pro-p fundamental group of a finite graph of pro-p groups with finite edge stabilizers.*

Two achievements had caused dramatic advances in the combinatorial theory of groups; Bass-Serre theory of groups acting on trees and 'Stallings' decomposition theorem' of groups with infinitely many ends. The results of this paper contribute to the theory of pro-p groups acting on pro-p trees. Nevertheless, the absense of a 'Stallings' decomposition theorem' in the pro-p context is still overshadowing the combinatorial theory of pro-p groups.

2. Preliminaries

We will use the notion of graph as introduced by J-P. Serre in [9, §2.1].

2.1. **Finite graphs of pro-p groups.** Let Γ be a finite connected graph. A *graph of groups* $\mathcal{G}$ based on Γ is called a *finite graph of pro-p groups*, if all vertex groups $\mathcal{G}(v)$, $v \in V(\Gamma)$, and all edge groups $\mathcal{G}(e)$, $e \in E(\Gamma)$, are pro-p groups, and if all the group monomorphisms $\alpha_e\colon \mathcal{G}(e) \to \mathcal{G}(t(e))$ are continuous. So, if $(\mathcal{G}, \Gamma)$ is an (abstract) graph of groups such that all vertex and edge groups are finitely generated pro-p groups, then by a theorem of J-P. Serre (cf. [5, §4.8]), $(\mathcal{G}, \Gamma)$ is a finite graph of pro-p groups.

A finite graph of pro-p groups $(\mathcal{G}, \Gamma)$ is said to be *reduced*, if for every geometric edge $\{e, \bar{e}\}$ which is not a loop neither $\alpha_e\colon \mathcal{G}(e) \to \mathcal{G}(t(e))$ nor $\alpha_{\bar{e}}\colon \mathcal{G}(e) \to \mathcal{G}(o(e))$ is an isomorphism. Any finite graph of pro-p groups can be transformed in a reduced finite graph of pro-p groups by the following procedure: If $\{e, \bar{e}\}$ is a geometric edge which is not a loop, we can remove $\{e, \bar{e}\}$ from the edge set of Γ, and identify $o(e)$ and $t(e)$ in a new vertex y. Let Γ' be the finite graph given by $V(\Gamma') = \{y\} \sqcup V(\Gamma) \setminus \{o(e), t(e)\}$ and $E(\Gamma') = E(\Gamma) \setminus \{e, \bar{e}\}$, and let $\mathcal{G}'$ denote the finite graph of pro-p groups based on Γ' given by $\mathcal{G}'(y) = \mathcal{G}(o(e))$ if α_e is an isomorphism, and $\mathcal{G}'(y) = \mathcal{G}(t(e))$ if α_e is not an isomorphism. This procedure can be continued until α_e is not surjective for all edges not defining loops. The resulting finite graph of pro-p groups $(\mathcal{G}_{\text{red}}, \Gamma_{\text{red}})$ is reduced.

2.2. **The fundamental pro-p group of a finite graph of finitely generated pro-p groups.** Let $(\mathcal{G}, \Gamma)$ be a finite graph of finitely generated pro-p groups. We define the *fundamental pro-p group* $G = \Pi_1(\mathcal{G}, \Gamma, v_0)$, $v_0 \in V(\Gamma)$, of $(\mathcal{G}, \Gamma)$ to be the pro-p completion of the usual fundamental group $\pi_1(\mathcal{G}, \Gamma, v_0)$ (cf. [9, §5.1]). In general, $\pi_1(\mathcal{G}, \Gamma, v_0)$ does not have to be residually p, but this will be the case in all of our considerations. In particular, edge and vertex groups will be subgroups of $\Pi_1(\mathcal{G}, \Gamma, v_0)$. Since $\mathcal{G}(e)$, $\mathcal{G}(v)$ are finitely generated, by a theorem of J-P. Serre (cf. [5, §4.8]), our definition is equivalent to the original definition of the fundamental group of a graph of groups in the category of pro-p groups (cf. [14]). Note that the previously mentioned reduction process does not change the fundamental pro-p group, i.e., one has a canonical isomorphism $\Pi_1(\mathcal{G}, \Gamma, v_0) \simeq \Pi_1(\mathcal{G}_{\text{red}}, \Gamma_{\text{red}}, w_0)$. So, if the pro-$p$ group G is the fundamental group of a finite graph of pro-p groups, we may assume that the finite graph of pro-p groups is reduced.

2.3. **The fundamental pro-p group of a finite graph of finite p-groups.** Let $(\mathcal{G},\Gamma)$ be a finite graph of finite p-groups. By [14, Thm. 3.10], every finite subgroup of $G=\Pi_1(\mathcal{G},\Gamma,v_0)$ is conjugate to a subgroup of a vertex group of $(\mathcal{G},\Gamma)$. Hence G has only finitely many finite subgroups up to conjugation. In particular, every maximal finite subgroup of G is G-conjugate to a vertex group of $(\mathcal{G},\Gamma)$, and the converse is true if $(\mathcal{G},\Gamma)$ is a reduced finite graph of finite p-groups.

3. Virtually free pro-p products

3.1. **Virtually free pro-p groups.** A pro-p group G will be called to be a *free pro-p product* if there exist non-trivial closed subgroups A and B such that $G=A\amalg B$. Otherwise we shall say that G is $\amalg$-indecomposable. The following properties are well known.

Proposition 3.1. *Let $H=\coprod_{i\in I}H_i\coprod F$ be a finitely generated pro-p group with a $\amalg$-decompositon, where H_i are non-trivial $\amalg$-indecomposable pro-p-groups, and F is a free pro-p group. Then*

(a) *I is finite, and H_i, $i\in I$, and F are finitely generated.*
(b) *Any finitely generated $\amalg$-indecomposable subgroup A of H is conjugate to a subgroup of H_i for some $i\in I$. Moreover, if $H=A\amalg B$ for some closed subgroup B of H, then A is conjugate to some H_i, $i\in I$.*
(c) *$H_i\cap H_j^h=1$ if either $i\neq j$ or $h\notin H_i$.*
(d) *For $K\subseteq H_i$, $K\neq\{1\}$, one has $N_H(K)\subseteq H_i$. In particular, if H_i is finite, so is $N_H(K)$.*

Proof. (a) is obvious. The first statement of (b) follows from the pro-p version of the Kurosh subgroup theorem [1, Thm. 4.4] and the second statement from [3, Thm. 4.3]. For (c) see Theorems 4.2 (a) and 4.3 (a) in [4]. In order to prove (d) take $h\in N_G(K)$. Then $K\subseteq H_i\cap H_i^h$, and, by (c), one has $h\in H_i$. □

From Proposition 3.1 one concludes the following properties for virtual free pro-p products.

Proposition 3.2. *Let $(\mathcal{G},\Gamma)$ be a reduced finite graph of finite p-groups, and suppose that $G=\Pi_1(\mathcal{G},\Gamma,v_0)$ contains an open, normal subgroup $H=F\amalg H_1\amalg\cdots\amalg H_s$, with H_i non-trivial finite and F free pro-p of rank r, $0\leq r<\infty$, such that $r+s\geq 2$. Then one has the following.*

(a) *For any edge e of Γ one has $\mathcal{G}(e)\cap H=\{1\}$; in particular, $|\mathcal{G}(e)|\leq|G:H|$.*
(b) *$|\mathbf{E}(\Gamma)|\leq 2(r+s)-1$ and $|V(\Gamma)|\leq 2(r+s)$, where $V(\Gamma)$ is the set of vertices of Γ, and $\mathbf{E}(\Gamma)$ is the set of geometric edges of Γ.*

Proof. Let $X=\pi_1(\mathcal{G},\Gamma,v_0)$ be the abstract fundamental group of the graph of groups and $Y=X\cap H$. Hence G and H are the pro-p completions of X and Y, respectively. Moreover, $|X:Y|=|G:H|$.
(a) Suppose that $\mathcal{G}(e)\cap H\neq\{1\}$. Since H is normal in G, $N_G(\mathcal{G}(e))$ normalizes $\mathcal{G}(e)\cap H$. We claim that $N_G(\mathcal{G}(e))$ is infinite. One has to distinguish two cases: Case 1: $\{e,\bar{e}\}$ is not a loop. In this case $N_G(\mathcal{G}(e))$ contains the infinite group $\langle N_{\mathcal{G}(v)}(\mathcal{G}(e)),N_{\mathcal{G}(w)}(\mathcal{G}(e))\rangle$, where $v=o(e)$, $w=t(e)$. Case 2: $\{e,\bar{e}\}$ is a loop. Let $v=t(e)=o(e)$, and let $z_e\in G$ be the stable letter associated with e. If $\mathcal{G}(e)=\mathcal{G}(v)$, then $N_G(\mathcal{G}(e))$ contains the infinite group $\langle z_e\rangle$. Otherwise $N_G(\mathcal{G}(e))$ contains the infinite group $\langle N_{\mathcal{G}(v)}(\mathcal{G}(e)),z_eN_{\mathcal{G}(v)}(\mathcal{G}(e))z_e^{-1}\rangle$.

Since $|G:H|<\infty$, the fact that $N_G(\mathcal{G}(e))$ is infinite implies that $N_H(\mathcal{G}(e)\cap H)=N_G(\mathcal{G}(e)\cap H)\cap H$ is infinite as well contradicting Proposition 3.1(d). Hence one has $\mathcal{G}(e)\cap H=\{1\}$ as required.
(b) It suffices to show the first inequality. By [9, §2.6, Ex. 3], one has

$$\begin{aligned} -\chi_X &= \sum_{e\in\mathbf{E}(\Gamma)}\frac{1}{|\mathcal{G}(e)|}-\sum_{v\in V(\Gamma)}\frac{1}{|\mathcal{G}(v)|} \\ &= -\frac{1}{|X:Y|}\cdot\chi_Y \qquad (3.1)\\ &= \frac{1}{|X:Y|}\cdot\Big(r+s-1-\sum_{1\le i\le s}\frac{1}{|H_i|}\Big), \end{aligned}$$

where χ_X denotes the Euler characteristic of the finitely generated virtually free group X. Thus one obtains

$$r+s-1\geq|X:Y|\Big(\sum_{e\in\mathbf{E}(\Gamma)}\frac{1}{|\mathcal{G}(e)|}-\sum_{v\in V(\Gamma)}\frac{1}{|\mathcal{G}(v)|}\Big). \tag{3.2}$$

As $(\mathcal{G},\Gamma)$ is reduced, for every edge e in a maximal subtree T of Γ the edge group $\mathcal{G}(e)$ is isomorphic to a proper subgroup of $\mathcal{G}(t(e))$. Hence, $|\mathcal{G}(t(e))|\geq 2|\mathcal{G}(e)|$. Let $E^+(T)$ be an orientation of T such that every vertex of Γ except $v_0\in V(\Gamma)$ is the terminus of precisely one edge of T, and let $f\in E(T)$ be an edge satisfying $t(f)=v_0$. Taking into account that $|\mathbf{E}(T)|=|V(\Gamma)|-1$, one concludes from (a) that

$$r+s-1\geq\frac{1}{2}\cdot\sum_{e\in\mathbf{E}(\Gamma)\setminus\{f,\bar f\}}\frac{|X:Y|}{|\mathcal{G}(e)|}\geq\frac{1}{2}\cdot(|\mathbf{E}(\Gamma)|-1). \tag{3.3}$$

This yields the claim. □

From Proposition 3.2 one concludes the following straightforward fact.

Corollary 3.3. *Let $(\mathcal{G},\Gamma)$ be a reduced finite graph of finite p-groups, and suppose that $G=\Pi_1(\mathcal{G},\Gamma,v_0)$ contains a free open subgroup H of rank $r\geq 2$. Then there exist finitely many reduced finite graphs of finite p-groups $(\mathcal{G}',\Gamma')$ up to isomorphism such that $G\simeq\Pi_1(\mathcal{G}',\Gamma',w_0)$.*

Let $G=\Pi_1(\mathcal{G},\Gamma,v_0)$ be the pro-p fundamental group of a finite graph of finite p-groups, and let U be an open and normal subgroup of G. Then, by construction, $\widetilde{U}=\mathrm{cl}(\langle\, U\cap\mathcal{G}(v)^g\mid g\in G,\ v\in V(\Gamma)\,\rangle)$ is a closed normal subgroup of G. By [6, Prop. 1.10], one has a natural decomposition of $G/\widetilde{U}$ as the pro-p fundamental group $G/\widetilde{U}=\Pi_1(\mathcal{G}_U,\Gamma,v_0)$ of a finite graph of finite p-groups $(\mathcal{G}_U,\Gamma)$, where the vertex and edge groups satisfy $\mathcal{G}_U(x)=\mathcal{G}(x)\widetilde{U}/\widetilde{U}$, $x\in V(\Gamma)\sqcup E(\Gamma)$. Thus we have a morphism $\eta\colon(\mathcal{G},\Gamma)\longrightarrow(\mathcal{G}_U,\Gamma)$ of graphs of groups such that the induced homomorphism on the pro-p fundamental groups coincides with the canonical projection $\varphi_U\colon G\longrightarrow G/\widetilde{U}$.

Lemma 3.4. *Let $G=\Pi_1(\mathcal{G},\Gamma,v_0)$ be the pro-p fundamental group of a finite graph of finite p-groups, and let H be an open normal subgroup of G that decomposes as a free pro-p product $H=\coprod_{1\le i\le s}H_i\coprod F$ of finite p-groups H_i and a free pro-p group F. Let $U\subseteq H$ be an open normal subgroup of G such that $U\cap H_i\neq H_i$ for every $i\in\{1,\dots,s\}$. If $(\mathcal{G},\Gamma)$ is reduced, then $(\mathcal{G}_U,\Gamma)$ is reduced.*

Proof. Suppose on the contrary that there exists an edge e in Γ which is not a loop such that for $v = t(e)$ one has $\mathcal{G}(v)\widetilde{U} = \mathcal{G}(e)\widetilde{U} \subseteq G/\widetilde{U}$. Then, by the second isomorphism theorem,

$$\mathcal{G}(v) = \mathcal{G}(e)(\widetilde{U} \cap \mathcal{G}(v)). \tag{3.4}$$

As $(\mathcal{G}, \Gamma)$ is reduced, and thus $\mathcal{G}(e) \neq \mathcal{G}(v)$, one has $\widetilde{U} \cap \mathcal{G}(v) \neq \{1\}$. From Proposition 3.1(a) one deduces that $\widetilde{U} \cap \mathcal{G}(v)$ is contained in some H_i^g for $1 \leq i \leq s$ and $g \in G$. If $N_G(\widetilde{U} \cap \mathcal{G}(v))$ would be infinite, so would be $N_H(\widetilde{U} \cap \mathcal{G}(v))$ contradicting Proposition 3.1(d). Hence $N_G(\widetilde{U} \cap \mathcal{G}(v))$ is finite and equal to $\mathcal{G}(v)$. In particular, for $y \in \mathcal{G}(v)$ one concludes that $H_i^{gy} \cap H_i^g \neq \{1\}$. Hence, by Proposition 3.1(c), $H_i^{gy} = H_i^g$ and thus $\mathcal{G}(v) \subseteq N_G(H_i^g)$. The maximality of $\mathcal{G}(v)$ and the finiteness of $N_G(H_i^g)$ (cf. Prop. 3.1(d)) imply that $\mathcal{G}(v) = N_G(H_i^g)$. By construction, $\mathcal{G}(e)H_i^g$ is a finite subgroup of G containing $\mathcal{G}(v)$ (cf. (3.4)). As $\mathcal{G}(v)$ is a maximal finite subgroup of G, this implies that

$$\mathcal{G}(e)(\widetilde{U} \cap \mathcal{G}(v)) = \mathcal{G}(v) = \mathcal{G}(e)\, H_i^g. \tag{3.5}$$

Since $\widetilde{U} \cap \mathcal{G}(v) \subseteq H_i^g$, and as $\mathcal{G}(e) \cap H_i^g = \{1\}$ (cf. Prop. 3.2(a)), one concludes that $\widetilde{U} \cap \mathcal{G}(v) = H_i^g$. Hence $H_i \subseteq \widetilde{U} \subseteq U$ contradicting the hypothesis. □

The proof of the structure theorem for virtual free pro-p products (cf. Thm. 3.6) in the subsequent subsection is based on the following result due to W. Herfort and the second author.

Theorem 3.5. (cf. [2, Thm. 1.1]) *Let G be a finitely generated pro-p group with a free open subgroup F. Then G is the pro-p fundamental group of a finite graph of finite p-groups whose orders are bounded by $|G : F|$.*

3.2. **Virtual free pro-p products.** The following theorem gives a description of the structure of virtual free pro-p products.

Theorem 3.6. *Let G be a finitely generated pro-p group containing an open subgroup H which has a non-trivial decomposition as free product, i.e., there exists non-trivial closed subgroups $A, B \subsetneq H$ such that $H = A \amalg B$. Then G is isomorphic to the pro-p fundamental group of a finite graph of pro-p groups with finite edge stabilizers.*

Proof. By replacing H by the core of H in G and applying the Kurosh subgroup theorem for open subgroups (cf. [5, Thm. 9.1.9]), we may assume that H is normal in G. Refining the free decomposition if necessary and collecting free factors isomorphic to $\mathbb{Z}_p$ we obtain a free decomposition

$$H = F \amalg H_1 \amalg \cdots \amalg H_s, \tag{3.6}$$

where F is a free subgroup of rank t, and the H_i are $\amalg$-indecomposable finitely generated subgroups which are not isomorphic to $\mathbb{Z}_p$ (cf. Prop. 3.1(a)). By hypothesis, $s + t \geq 2$. By construction, one has for all $g \in G$ and for all $i \in \{1, \ldots, s\}$ that H_i^g is a free factor of H. Since H_i is indecomposable, we deduce from Proposition 3.1(b) that the indecomposable non-free subgroup H_i^g of H equals H_j^h for some $j \in \{1, \ldots, s\}$. Thus $\{H_i^g \mid g \in G,\ 1 \leq i \leq s\} = \{H_i^h \mid h \in H,\ 1 \leq i \leq s\}$.
Step 1: Let $\mathcal{B}$ be a basis of neighbourhoods of $1_G \in G$ consisting of open normal subgroups U of G which are contained in H with $H_i \not\subseteq U$ for every $i = 1, \ldots s$. For

$U \in \mathcal{B}$ put

$$\widetilde{U} = \mathrm{cl}(\langle U \cap H_i^g \mid g \in G,\ 1 \le i \le s \rangle) = \mathrm{cl}(\langle U \cap H_i^h \mid h \in H,\ 1 \le i \le s \rangle). \tag{3.7}$$

Then $\widetilde{U}$ is a closed normal subgroup of H, and

$$H/\widetilde{U} = F \amalg H_1\widetilde{U}/\widetilde{U} \amalg \cdots \amalg H_s\widetilde{U}/\widetilde{U} \tag{3.8}$$

(cf. [3, Prop. 1.18]). The group $G/\widetilde{U}$ contains the open normal subgroup $H/\widetilde{U}$ which is a finitely generated, virtually free pro-p group (since $U/\widetilde{U}$ is free pro-p by Theorem 2.6 in [14]), and thus $G/\widetilde{U}$ is a finitely generated, virtually free pro-p group.

Step 2: By Theorem 3.5, $G/\widetilde{U}$ is isomorphic to the pro-p fundamental group $\Pi_1(\mathcal{G}_U, \Gamma_U, v_U)$ of a finite graph of finite p-groups. Although neither the finite graph Γ_U nor the finite graph of finite p-groups $\mathcal{G}_U$ are uniquely determined by U (resp. $\widetilde{U}$), the index U in the notation shall express that both these objects are depending on U. Using the procedure described in subsection 2.2 we may assume that $(\mathcal{G}_U, \Gamma_U)$ is reduced. Hence from now on we may assume that for every $U \in \mathcal{B}$ the vertex groups of $G/\widetilde{U} = \Pi_1(\mathcal{G}_U, \Gamma_U, v_U)$ are representatives of the $G/\widetilde{U}$-conjugacy classes of maximal finite subgroups. Note that by Proposition 3.2(a), one has $\mathcal{G}_U(e) \cap H/\tilde{U} = 1$.

Step 3: As explained before Lemma 3.4, for $V \subseteq U$ both open and normal in G the decomposition $G/\widetilde{V} = \Pi_1(\mathcal{G}_V, \Gamma_V, v_V)$ gives rise to a natural decomposition of $G/\widetilde{U}$ as the fundamental group $G/\widetilde{U} = \Pi_1(\mathcal{G}_{V,U}, \Gamma_V, v_V)$ of a graph of groups $(\mathcal{G}_{V,U}, \Gamma_V)$. Moreover, by Lemma 3.4, if $(\mathcal{G}_V, \Gamma_V)$ is reduced, then $(\mathcal{G}_{V,U}, \Gamma_V)$ is reduced. Thus in this case one has a morphism $\eta\colon (\mathcal{G}_V, \Gamma_V) \longrightarrow (\mathcal{G}_{V,U}, \Gamma_V)$ of reduced graph of groups such that the induced homomorphism on the pro-p fundamental groups coincides with the canonical projection $\varphi_{UV}\colon G/\widetilde{V} \longrightarrow G/\widetilde{U}$.

Step 4: By Proposition 3.2, the number $|V(\Gamma_U)|+|\mathbf{E}(\Gamma_U)|$ is bounded by $4(r+s)-1$. So we have only finitely many graphs Γ_U up to isomorphism, when U runs. It follows that there is a finite graph Γ such that Γ_U is isomorphic to Γ for infinitely many U's. Therefore, by passing to a cofinal system $\mathcal{C}$ of $\mathcal{B}$ if necessary, we may assume that $\Gamma_U = \Gamma$ for each $U \in \mathcal{C}$. Then, by Corollary 3.3, the number of isomorphism classes of finite reduced graphs of finite p-groups $(\mathcal{G}'_U, \Gamma)$ which are based on Γ satisfying $G/\widetilde{U} \simeq \Pi_1(\mathcal{G}', \Gamma, v_0)$ is finite. Suppose that Ω_U is a set containing a copy of every such isomorphism class. For $V \in \mathcal{C}$, $V \subseteq U$, one has a map $\omega_{V,U}\colon \Omega_V \to \Omega_U$ (cf. Step 3). Hence $\Omega = \varprojlim_{U \in \mathcal{C}} \Omega_U$ is non-empty. Let $(\mathcal{G}'_U, \Gamma)_{U \in \mathcal{C}} \in \Omega$. Then $(\mathcal{G}', \Gamma)$ given by $\mathcal{G}'(x) = \varprojlim \mathcal{G}'_U(x)$ if x is either a vertex or an edge of Γ, is a reduced finite graph of finitely generated pro-p groups satisfying $G \simeq \Pi_1(\mathcal{G}', \Gamma, v_0)$. By Proposition 3.2(a), $\mathcal{G}'(e)$ is finite for every edge e of Γ. This yields the claim. □

Proof of Theorem A. By Theorem C, G is the fundamental pro-p group of a finite graph of pro-p groups $(\mathcal{G}, \Gamma)$. Let e be an edge of Γ. If by removal of an edge e the graph Γ becomes disconnected, G splits as a free amalgamated pro-p product over the edge group G_e. Otherwise it splits as a pro-p HNN-extension over G_e.

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