ON FRACTIONAL MULTI-SINGULAR SCHRÖDINGER OPERATORS: POSITIVITY AND LOCALIZATION OF BINDING

VERONICA FELLI, DEBANGANA MUKHERJEE, AND ROBERTO OGNIBENE

Abstract. In this work we investigate positivity properties of nonlocal Schrödinger type operators, driven by the fractional Laplacian, with multipolar, critical, and locally homogeneous potentials. On one hand, we develop a criterion that links the positivity of the spectrum of such operators with the existence of certain positive supersolutions, while, on the other hand, we study the localization of binding for this kind of potentials. Combining these two tools and performing an inductive procedure on the number of poles, we establish necessary and sufficient conditions for the existence of a configuration of poles that ensures the positivity of the corresponding Schrödinger operator.

Keywords. Fractional Laplacian; Multipolar potentials; Positivity Criterion; Localization of binding.

MSC classification: 35J75, 35R11, 35J10, 35P05.

1. Introduction

Let $s \in (0,1)$ and $N > 2s$. Let us consider $k \geq 1$ real numbers $\lambda_1, \dots, \lambda_k$ (sometimes called *masses*) and k *poles* $a_1, \dots, a_k \in \mathbb{R}^N$ such that $a_i \neq a_j$ for all $i,j = 1, \dots, k$, $i \neq j$. The main object of our investigation is the operator

$$\mathcal{L}_{\lambda_1,\dots,\lambda_k,a_1,\dots,a_k} := (-\Delta)^s - \sum_{i=1}^{k} \frac{\lambda_i}{|x-a_i|^{2s}} \qquad \text{in } \mathbb{R}^N. \tag{1.1}$$

Here $(-\Delta)^s$ denotes the fractional Laplace operator, which acts on functions $\varphi \in C_c^\infty(\mathbb{R}^N)$ as

$$(-\Delta)^s \varphi(x) := C(N,s)\,\text{P.V.} \int_{\mathbb{R}^N} \frac{\varphi(x)-\varphi(y)}{|x-y|^{N+2s}}\,\mathrm{d}y = C(N,s) \lim_{\rho\to 0^+} \int_{|x-y|>\rho} \frac{\varphi(x)-\varphi(y)}{|x-y|^{N+2s}}\,\mathrm{d}y,$$

where P.V. means that the integral has to be seen in the principal value sense and

$$C(N,s) = \pi^{-\frac{N}{2}} 2^{2s} \frac{\Gamma\left(\frac{N+2s}{2}\right)}{\Gamma(2-s)} s(1-s),$$

with Γ denoting the usual Euler's Gamma function. Hereafter, we refer to an operator of the type $(-\Delta)^s - V$ as a *fractional Schrödinger operator* with potential V.

One of the reasons of mathematical interest in operators of type (1.1) lies in the criticality of potentials of order $-2s$, which have the same scaling rate as the s-fractional Laplacian.

We introduce, on $C_c^\infty(\mathbb{R}^N)$, the following positive definite bilinear form, associated to $(-\Delta)^s$

$$(u,v)_{\mathcal{D}^{s,2}(\mathbb{R}^N)} := \frac{1}{2} C(N,s) \int_{\mathbb{R}^{2N}} \frac{(u(x)-u(y))(v(x)-v(y))}{|x-y|^{N+2s}}\,\mathrm{d}x\,\mathrm{d}y \tag{1.2}$$

and we define the space $\mathcal{D}^{s,2}(\mathbb{R}^N)$ as the completion of $C_c^\infty(\mathbb{R}^N)$ with respect to the norm $\|\cdot\|_{\mathcal{D}^{s,2}(\mathbb{R}^N)}$ induced by the scalar product (1.2). Moreover, the following quadratic form is naturally associated to the operator $\mathcal{L}_{\lambda_1,\dots,\lambda_k,a_1,\dots,a_k}$

$$
\begin{aligned}
Q_{\lambda_1,\dots,\lambda_k,a_1,\dots,a_k}(u) :&= \frac{1}{2}C(N,s)\int_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}}\,\mathrm{d}x\,\mathrm{d}y - \sum_{i=1}^k \lambda_i \int_{\mathbb{R}^N} \frac{|u(x)|^2}{|x-a_i|^{2s}}\,\mathrm{d}x \\
&= \|u\|^2_{\mathcal{D}^{s,2}(\mathbb{R}^N)} - \sum_{i=1}^k \lambda_i \int_{\mathbb{R}^N} \frac{|u(x)|^2}{|x-a_i|^{2s}}\,\mathrm{d}x.
\end{aligned}
\tag{1.3}
$$

We observe that $Q_{\lambda_1,\dots,\lambda_k,a_1,\dots,a_k}$ is well-defined on $\mathcal{D}^{s,2}(\mathbb{R}^N)$ thanks to the validity of the following fractional Hardy inequality proved in [26]:

$$
\gamma_H \int_{\mathbb{R}^N} \frac{|u(x)|^2}{|x|^{2s}}\,\mathrm{d}x \le \|u\|^2_{\mathcal{D}^{s,2}(\mathbb{R}^N)} \quad \text{for all } u \in \mathcal{D}^{s,2}(\mathbb{R}^N), \tag{1.4}
$$

where the constant

$$
\gamma_H = \gamma_H(N,s) := 2^{2s}\frac{\Gamma^2\left(\frac{N+2s}{4}\right)}{\Gamma^2\left(\frac{N-2s}{4}\right)}
$$

is optimal and not attained.

One goal of the present paper is to find necessary and sufficient conditions (on the masses $\lambda_1,\dots,\lambda_k$) for the existence of a configuration of poles $(a_1,\dots,a_k)$ that guarantees the positivity of the quadratic form (1.3), extending to the fractional case some results obtained in [21] for the classical Laplacian. The quadratic form $Q_{\lambda_1,\dots,\lambda_k,a_1,\dots,a_k}$ is said to be *positive definite* if

$$
\inf_{u\in\mathcal{D}^{s,2}(\mathbb{R}^N)\setminus\{0\}} \frac{Q_{\lambda_1,\dots,\lambda_k,a_1,\dots,a_k}(u)}{\|u\|^2_{\mathcal{D}^{s,2}(\mathbb{R}^N)}} > 0.
$$

In the case of a single pole (i.e. $k=1$), the fractional Hardy inequality (1.4) immediately answers the question of positivity: the quadratic form $Q_{\lambda,a}$ is positive definite if and only if $\lambda < \gamma_H$. Hence our interest in multipolar potentials is justified by the fact that the location of the poles (in particular the shape of the configuration) could play some role in the positivity of (1.3). Furthermore, one could expect that some other conditions on the masses may arise when $k>1$. We mention that several authors have approached the problem of multipolar singular potentials, both for the classical Laplacian, see e.g. [5, 6, 10, 11, 19, 24] and for the fractional case, see [23].

A fundamental tool in our arguments is the well known Caffarelli-Silvestre extension for functions in $\mathcal{D}^{s,2}(\mathbb{R}^N)$, which allows us to study the nonlocal operator $(-\Delta)^s$ by means of a boundary value problem driven by a local operator in $\mathbb{R}^{N+1}_+ := \{(t,x) : t\in(0,+\infty), x\in\mathbb{R}^N\}$. We introduce the space $\mathcal{D}^{1,2}(\mathbb{R}^{N+1}_+;t^{1-2s})$, defined as the completion of $C_c^\infty(\overline{\mathbb{R}^{N+1}_+})$ with respect to the norm

$$
\|U\|_{\mathcal{D}^{1,2}(\mathbb{R}^{N+1}_+;t^{1-2s})} := \left(\int_{\mathbb{R}^{N+1}_+} t^{1-2s}|\nabla U|^2\,\mathrm{d}t\,\mathrm{d}x\right)^{1/2}.
$$

We have that there exists a well-defined and continuous trace map

$$
\mathrm{Tr}\colon \mathcal{D}^{1,2}(\mathbb{R}^{N+1}_+;t^{1-2s}) \to \mathcal{D}^{s,2}(\mathbb{R}^N) \tag{1.5}
$$

which is onto, see, for instance, [7]. Let us now consider, for $u\in\mathcal{D}^{s,2}(\mathbb{R}^N)$, the following minimization problem

$$
\min\left\{\int_{\mathbb{R}^{N+1}_+} t^{1-2s}|\nabla\Phi|^2\,\mathrm{d}t\,\mathrm{d}x \colon \Phi\in\mathcal{D}^{1,2}(\mathbb{R}^{N+1}_+;t^{1-2s}),\ \mathrm{Tr}\,\Phi = u\right\}. \tag{1.6}
$$

One can prove that there exists a unique function $U\in\mathcal{D}^{1,2}(\mathbb{R}^{N+1}_+;t^{1-2s})$ (which we call the *extension* of u) attaining (1.6), i.e.

$$
\int_{\mathbb{R}^{N+1}_+} t^{1-2s}|\nabla U|^2\,\mathrm{d}t\,\mathrm{d}x \le \int_{\mathbb{R}^{N+1}_+} t^{1-2s}|\nabla\Phi|^2\,\mathrm{d}t\,\mathrm{d}x \tag{1.7}
$$

for all $\Phi \in \mathcal{D}^{1,2}(\mathbb{R}^{N+1}_+; t^{1-2s})$ such that $\operatorname{Tr}\Phi = u$. Furthermore, in [9] it has been proven that

$$\int_{\mathbb{R}^{N+1}_+} t^{1-2s}\nabla U \cdot \nabla \Phi \,\mathrm{d}t\,\mathrm{d}x = \kappa_s (u, \operatorname{Tr}\Phi)_{\mathcal{D}^{s,2}(\mathbb{R}^N)} \quad \text{for all } \Phi \in \mathcal{D}^{1,2}(\mathbb{R}^{N+1}_+; t^{1-2s}), \tag{1.8}$$

where

$$\kappa_s := \frac{\Gamma(1-s)}{2^{2s-1}\Gamma(s)}. \tag{1.9}$$

We observe that (1.8) is the variational formulation of the following problem

$$\begin{cases} -\operatorname{div}(t^{1-2s}\nabla U) = 0, & \text{in } \mathbb{R}^{N+1}_+, \\ -\lim\limits_{t\to 0} t^{1-2s}\dfrac{\partial U}{\partial t} = \kappa_s(-\Delta)^s u, & \text{on } \mathbb{R}^N. \end{cases} \tag{1.10}$$

In the classical (local) case, the problem of positivity of Schrödinger operators with multi-singular Hardy-type potentials was addressed in [21]. In that article, the authors tackled the problem making use of a *localization of binding* result that provides, under certain assumptions, the positivity of the sum of two positive operators, by translating one of them through a sufficiently long vector. This argument is based, in turn, on a criterion which relates the positivity of an operator to the existence of a positive supersolution, in the spirit of Allegretto-Piepenbrink Theory (see [4, 31]). As one can observe in [21], the strong suit of the local case is that the study of the action of the operator can be substantially reduced to neighbourhoods of the singularities. However, this is not possible in the fractional context due to nonlocal effects: in the present paper we overcome this issue by taking into consideration the Caffarelli-Silvestre extension (1.10), which yields a local formulation of the problem.

The equivalence between the fractional problem in $\mathbb{R}^N$ and the Caffarelli-Silvestre extension problem in $\mathbb{R}^{N+1}_+$ allows us to characterize the coercivity properties of quadratic forms on $\mathcal{D}^{s,2}(\mathbb{R}^N)$ in terms of quadratic forms on $\mathcal{D}^{1,2}(\mathbb{R}^{N+1}_+; t^{1-2s})$. We say that a function $V \in L^1_{\mathrm{loc}}(\mathbb{R}^N)$ satisfies the *form-bounded condition* if

$$\sup_{\substack{u\in\mathcal{D}^{s,2}(\mathbb{R}^N)\\ u\not\equiv 0}} \frac{\int_{\mathbb{R}^N} |V(x)|u^2(x)\,\mathrm{d}x}{\|u\|^2_{\mathcal{D}^{s,2}(\mathbb{R}^N)}} < +\infty. \tag{FB}$$

Let $\mathcal{H}$ be the class of potentials satisfying the form-bounded condition, i.e.

$$\mathcal{H} = \{V \in L^1_{\mathrm{loc}}(\mathbb{R}^N) : V \text{ satisfies } (FB)\}.$$

It is easy to understand that, if $V \in \mathcal{H}$, then $Vu \in (\mathcal{D}^{s,2}(\mathbb{R}^N))^\star$ for all $u \in \mathcal{D}^{s,2}(\mathbb{R}^N)$ and the quadratic form $u \mapsto \|u\|^2_{\mathcal{D}^{s,2}(\mathbb{R}^N)} - \int_{\mathbb{R}^N} Vu^2$ is well defined in $\mathcal{D}^{s,2}(\mathbb{R}^N)$. For all $V \in \mathcal{H}$ we define

$$\mu(V) = \inf_{u\in\mathcal{D}^{s,2}(\mathbb{R}^N)\setminus\{0\}} \frac{\|u\|^2_{\mathcal{D}^{s,2}(\mathbb{R}^N)} - \int_{\mathbb{R}^N} Vu^2\,\mathrm{d}x}{\|u\|^2_{\mathcal{D}^{s,2}(\mathbb{R}^N)}} \tag{1.11}$$

and observe that $\mu(V) > -\infty$.

Lemma 1.1. *Let $V \in \mathcal{H}$. Then*

$$\mu(V) = \inf_{\substack{U\in\mathcal{D}^{1,2}(\mathbb{R}^{N+1}_+;t^{1-2s})\\ U\not\equiv 0}} \frac{\displaystyle\int_{\mathbb{R}^{N+1}_+} t^{1-2s}|\nabla U|^2\,\mathrm{d}t\,\mathrm{d}x - \kappa_s\int_{\mathbb{R}^N} V|\operatorname{Tr}U|^2\,\mathrm{d}x}{\displaystyle\int_{\mathbb{R}^{N+1}_+} t^{1-2s}|\nabla U|^2\,\mathrm{d}t\,\mathrm{d}x}. \tag{1.12}$$

In the present paper we will focus our attention on the following class of potentials

$$\Theta := \Bigg\{ V(x) = \sum_{i=1}^k \frac{\lambda_i \chi_{B'(a_i,r_i)}(x)}{|x-a_i|^{2s}} + \frac{\lambda_\infty \chi_{\mathbb{R}^N\setminus B'_R}(x)}{|x|^{2s}} + W(x) \colon r_i, R > 0,\ k \in \mathbb{N},$$
$$a_i \in \mathbb{R}^N,\ a_i \neq a_j \text{ for } i \neq j,\ \lambda_i, \lambda_\infty < \gamma_H,\ W \in L^{N/2s}(\mathbb{R}^N)\cap L^\infty(\mathbb{R}^N) \Bigg\},$$

where, for any $r > 0$ and $x \in \mathbb{R}^N$, we denote

$$B'(x,r) := \{y \in \mathbb{R}^N : \ |y - x| < r\} \quad \text{and} \quad B'_r := B'(0,r).$$

We observe that, when considering a potential $V \in \Theta$, it is not restrictive to assume that the sets $B'(a_i, r_i)$ and $\mathbb{R}^N \setminus B'_R$ appearing in its representation are mutually disjoint, up to redefining the remainder W.

It is easy to see that, for instance,

$$\sum_{i=1}^{k} \frac{\lambda_i}{|x - a_i|^{2s}} \in \Theta, \quad \text{when } \lambda_i < \gamma_H \text{ for all } i = 1, \ldots, k \text{ and } \sum_{i=1}^{k} \lambda_i < \gamma_H.$$

We observe that any $V \in \Theta$ satisfies the form-bounded condition, i.e. $\Theta \subset \mathcal{H}$, thanks to the fractional Hardy and Sobolev inequalities stated in (1.4) and (2.1) respectively.

Our first main result is a criterion that provides the equivalence between the positivity of $\mu(V)$ for potentials $V \in \Theta$ and the existence of a positive supersolution to a certain (possibly perturbed) problem. This criterion is reminiscent of the Allegretto-Piepenbrink Theory, developed in 1974 in [4, 31] (see also [2, 3, 29, 33]). As far as we know, the result contained in the following lemma is new in the nonlocal framework; nevertheless, some tools from the Allegretto-Piepenbrink Theory have been used in [25, 28] to prove some Hardy-type fractional inequalities.

Lemma 1.2 (Positivity Criterion). *Let* $V = \sum_{i=1}^{k} \frac{\lambda_i \chi_{B'(a_i,r_i)}(x)}{|x-a_i|^{2s}} + \frac{\lambda_\infty \chi_{\mathbb{R}^N \setminus B'_R}(x)}{|x|^{2s}} + W(x) \in \Theta$ *and let* $\tilde{V} \in L^\infty_{\mathrm{loc}}(\mathbb{R}^N \setminus \{a_1, \ldots, a_k\})$ *be such that* $V \leq \tilde{V} \leq |V|$ *a.e. in* $\mathbb{R}^N$. *The following two assertions hold true.*

(I) *Assume that there exist some* $\varepsilon > 0$ *and a function* $\Phi \in \mathcal{D}^{1,2}(\mathbb{R}^{N+1}_+; t^{1-2s})$ *such that* $\Phi > 0$ *in* $\overline{\mathbb{R}^{N+1}_+} \setminus \{(0, a_1), \ldots, (0, a_k)\}$, $\Phi \in C^0\left(\overline{\mathbb{R}^{N+1}_+} \setminus \{(0, a_1), \ldots, (0, a_k)\}\right)$, *and*

$$\int_{\mathbb{R}^{N+1}_+} t^{1-2s} \nabla \Phi \cdot \nabla U \, \mathrm{d}t \, \mathrm{d}x \geq \kappa_s \int_{\mathbb{R}^N} (V + \varepsilon \tilde{V}) \operatorname{Tr} \Phi \operatorname{Tr} U \, \mathrm{d}x, \tag{1.13}$$

for all $U \in \mathcal{D}^{1,2}(\mathbb{R}^{N+1}_+; t^{1-2s})$, $U \geq 0$ *a.e. in* $\mathbb{R}^{N+1}_+$. *Then*

$$\mu(V) \geq \varepsilon/(\varepsilon + 1). \tag{1.14}$$

(II) *Conversely, assume that* $\mu(V) > 0$. *Then there exist* $\varepsilon > 0$ *(not depending on* $\tilde{V}$*) and* $\Phi \in \mathcal{D}^{1,2}(\mathbb{R}^{N+1}_+; t^{1-2s})$ *such that* $\Phi > 0$ *in* $\mathbb{R}^{N+1}_+$, $\Phi \in C^0\left(\overline{\mathbb{R}^{N+1}_+} \setminus \{(0, a_1), \ldots, (0, a_k)\}\right)$, $\Phi \geq 0$ *in* $\overline{\mathbb{R}^{N+1}_+} \setminus \{(0, a_1), \ldots, (0, a_k)\}$, *and* (1.13) *holds for every* $U \in \mathcal{D}^{1,2}(\mathbb{R}^{N+1}_+; t^{1-2s})$ *satisfying* $U \geq 0$ *a.e. in* $\mathbb{R}^{N+1}_+$. *If, in addition, we assume that* V *and* $\tilde{V}$ *are locally Hölder continuous in* $\mathbb{R}^N \setminus \{a_1, \ldots, a_k\}$, *then* $\Phi > 0$ *in* $\overline{\mathbb{R}^{N+1}_+} \setminus \{(0, a_1), \ldots, (0, a_k)\}$.

In order to use statement (I) to obtain positivity of a given Schrödinger operator with potential in Θ, it is crucial to exhibit a weak supersolution to the corresponding Schrödinger equation, i.e. a function satisfying (1.13), which is *strictly positive* outside the poles. Nevertheless, the application of maximum principles to prove positivity of solutions to singular/degenerate extension problems is more delicate than in the classic case, due to regularity issues (see the Hopf type principle proved in [8, Proposition 4.11] and recalled in Proposition A.2 of the Appendix). For this reason, in order to apply the above criterion in Sections 6 and 7, we will develop an approximation argument introducing a class of more regular potentials (see (6.2)).

The following theorem, whose proof heavily relies on Lemma 1.2, fits in the theory of *Localization of Binding*, whose aim is study the lowest eigenvalue of Schrödinger operators of the type

$$-\Delta + V_1 + V_2(\cdot - y), \quad y \in \mathbb{R}^N,$$

in relation to the potentials V_1 and V_2 and to the translation vector $y \in \mathbb{R}^N$. The case in which V_1 and V_2 belong to the Kato class has been studied in [32], while Simon in [35] analyzed the case of compactly supported potentials; singular inverse square potentials were instead considered in [21]. Our result concerns the fractional case and provides sufficient conditions on the potentials and on the length of the translation for the positivity of the corresponding fractional Schrödinger operator.

Theorem 1.3 (Localization of Binding)**.** *Let*

$$V_1(x) = \sum_{i=1}^{k_1} \frac{\lambda_i^1 \chi_{B'(a_i^1, r_i^1)}(x)}{|x - a_i^1|^{2s}} + \frac{\lambda_\infty^1 \chi_{\mathbb{R}^N \setminus B'_{R_1}}(x)}{|x|^{2s}} + W_1(x) \in \Theta,$$

$$V_2(x) = \sum_{i=1}^{k_2} \frac{\lambda_i^2 \chi_{B'(a_i^2, r_i^2)}(x)}{|x - a_i^2|^{2s}} + \frac{\lambda_\infty^2 \chi_{\mathbb{R}^N \setminus B'_{R_2}}(x)}{|x|^{2s}} + W_2(x) \in \Theta,$$

and assume $\mu(V_1), \mu(V_2) > 0$ *and* $\lambda_\infty^1 + \lambda_\infty^2 < \gamma_H$*. Then there exists* $R > 0$ *such that, for every* $y \in \mathbb{R}^N \setminus \overline{B'_R}$,

$$\mu(V_1(\cdot) + V_2(\cdot - y)) > 0.$$

Combining the previous theorem with an inductive procedure on the number of poles k, we obtain a necessary and sufficient condition for positivity of the operator (1.1).

Theorem 1.4. *Let* $(\lambda_1, \dots \lambda_k) \in \mathbb{R}^k$*. Then*

$$\lambda_i < \gamma_H \quad \text{for all } i = 1, \dots, k, \quad \text{and} \quad \sum_{i=1}^{k} \lambda_i < \gamma_H \tag{1.15}$$

is a necessary and sufficient condition for the existence of a configuration of poles $(a_1, \dots, a_k)$ *such that the quadratic form* $Q_{\lambda_1,\dots,\lambda_k,a_1,\dots,a_k}$ *associated to the operator* $\mathcal{L}_{\lambda_1,\dots,\lambda_k,a_1,\dots,a_k}$ *is positive definite.*

Besides the interest in the existence of a configuration of poles making $Q_{\lambda_1,\dots,\lambda_k,a_1,\dots,a_k}$ positive definite, one can search for a condition on the masses $\lambda_1, \dots, \lambda_k$ that guarantees the positivity of this quadratic form for every configuration of poles; in this direction, an answer is given by the following theorem (we refer to [22, Proposition 1.2] for an analogous result in the classical case of the Laplacian with multipolar inverse square potentials).

Theorem 1.5. *Let* $t^+ := \max\{0, t\}$*. If*

$$\sum_{i=1}^{k} \lambda_i^+ < \gamma_H, \tag{1.16}$$

then the quadratic form $Q_{\lambda_1,\dots,\lambda_k,a_1,\dots,a_k}$ *is positive definite for all* $a_1, \dots, a_k \in \mathbb{R}^N$*. Conversely, if*

$$\sum_{i=1}^{k} \lambda_i^+ > \gamma_H$$

then there exists a configuration of poles $(a_1, \dots, a_k)$ *such that* $Q_{\lambda_1,\dots,\lambda_k,a_1,\dots,a_k}$ *is not positive definite.*

Finally, it is natural to ask whether $\mu(V)$, defined as an infimum in (1.12), is attained or not. In the case of a single pole, it is known that the infimum is not achieved, see e.g. [25]; however, when dealing with multiple singularities, the outcome can be different. Indeed, for V in the class Θ, we have that $\mu(V) \le 1 - \frac{1}{\gamma_H} \max_{i=1,\dots,k,\infty} \lambda_i$, see Lemma 5.1, and the infimum is attained in the case of strict inequality, as established in the following proposition.

Proposition 1.6. *If* $V \in \Theta$ *is such that*

$$\mu(V) < 1 - \frac{1}{\gamma_H} \max\{0, \lambda_1, \dots, \lambda_k, \lambda_\infty\}, \tag{1.17}$$

then $\mu(V)$ *is attained.*

The paper is organized as follows. In Section 2 we recall some known results about spaces involving fractional derivatives and weighted spaces in $\mathbb{R}^{N+1}_+$ and we prove some estimates needed in the rest of the article. In Section 3 we prove Theorem 1.5. In Section 4 we prove the positivity criterion, i.e. Lemma 1.2, while in Section 5 we look for upper and lower bounds of the quantity $\mu(V)$. In Section 6 we investigate the persistence of the positivity of $\mu(V)$, when the potential V is subject to a perturbation far from the origin or close to a pole. Section 7 is devoted to the proof of Theorem 1.3, that is the primary tool used in the proof of Theorem 1.4, pursued in Section 8. Finally, in Section 9 we prove Proposition 1.6.

Notation. We list below some notation used throughout the paper:

- $B'(x,r) := \{y \in \mathbb{R}^N \colon |x-y| < r\}$, $B'_r := B'(0,r)$ for the balls in $\mathbb{R}^N$;
- $\mathbb{R}^{N+1}_+ := \{(t,x) \in \mathbb{R}^{N+1} \colon t > 0,\ x \in \mathbb{R}^N\}$;
- $B_r^+ := \{z \in \mathbb{R}^{N+1}_+ \colon |z| < r\}$ for the half-balls in $\mathbb{R}^{N+1}_+$;
- $\mathbb{S}^N := \{z \in \mathbb{R}^{N+1} \colon |z| = 1\}$ is the unit N-dimensional sphere;
- $\mathbb{S}^N_+ := \mathbb{S}^N \cap \mathbb{R}^{N+1}_+$ and $\mathbb{S}^{N-1} := \partial \mathbb{S}^N_+$;
- $S_r^+ := \{r\theta \colon \theta \in \mathbb{S}^N_+\}$ denotes a positive half-sphere with arbitrary radius $r > 0$;
- $\mathrm{d}S$ and $\mathrm{d}S'$ denote the volume element in N and $N-1$ dimensional spheres, respectively;
- for $t \in \mathbb{R}$, $t^+ := \max\{0,t\}$ and $t^- := \max\{0,-t\}$.

2. Preliminaries

In this section we clarify some details about the spaces involved in our exposition and their relation with the fractional Laplace operator, we recall basic known facts and we prove some introductory results.

2.1. Preliminaries on Fractional Sobolev Spaces and weighted spaces in $\mathbb{R}^{N+1}_+$.

Let us consider the homogenous Sobolev space $\mathcal{D}^{s,2}(\mathbb{R}^N)$ defined in Section 1. Thanks to the Hardy-Littlewood-Sobolev inequality

$$S \|u\|^2_{L^{2^*_s}(\mathbb{R}^N)} \le \|u\|^2_{\mathcal{D}^{s,2}(\mathbb{R}^N)}, \tag{2.1}$$

that holds for all functions $u \in \mathcal{D}^{s,2}(\mathbb{R}^N)$, we have that $\mathcal{D}^{s,2}(\mathbb{R}^N)$ is continuously embedded in $L^{2^*_s}(\mathbb{R}^N)$, where $2^*_s := \frac{2N}{N-2s}$ is the critical Sobolev exponent. Combining (1.7) and (1.8) with (1.4) and (2.1), we obtain, respectively

$$\kappa_s \gamma_H \int_{\mathbb{R}^N} \frac{|\mathrm{Tr}\, U|^2}{|x|^{2s}} \,\mathrm{d}x \le \int_{\mathbb{R}^{N+1}_+} t^{1-2s} |\nabla U|^2 \,\mathrm{d}t\,\mathrm{d}x \quad \text{for all } U \in \mathcal{D}^{1,2}(\mathbb{R}^{N+1}_+; t^{1-2s}) \tag{2.2}$$

and

$$\kappa_s S \|\mathrm{Tr}\, U\|^2_{L^{2^*_s}(\mathbb{R}^N)} \le \int_{\mathbb{R}^{N+1}_+} t^{1-2s} |\nabla U|^2 \,\mathrm{d}t\,\mathrm{d}x \quad \text{for all } U \in \mathcal{D}^{1,2}(\mathbb{R}^{N+1}_+; t^{1-2s}). \tag{2.3}$$

Moreover, just as a consequence of (1.7) and (1.8), we have that

$$\kappa_s \|\mathrm{Tr}\, U\|^2_{\mathcal{D}^{s,2}(\mathbb{R}^N)} \le \int_{\mathbb{R}^{N+1}_+} t^{1-2s} |\nabla U|^2 \,\mathrm{d}t\,\mathrm{d}x \quad \text{for all } U \in \mathcal{D}^{1,2}(\mathbb{R}^{N+1}_+; t^{1-2s}). \tag{2.4}$$

Now we state a result providing a compact trace embedding, which will be useful in the following.

Lemma 2.1. *Let $p \in L^{N/2s}(\mathbb{R}^N)$. If $(U_n)_n \subseteq \mathcal{D}^{1,2}(\mathbb{R}^{N+1}_+; t^{1-2s})$ and $U \in \mathcal{D}^{1,2}(\mathbb{R}^{N+1}_+; t^{1-2s})$ are such that $U_n \rightharpoonup U$ weakly in $\mathcal{D}^{1,2}(\mathbb{R}^{N+1}_+; t^{1-2s})$ as $n \to \infty$, then $\int_{\mathbb{R}^N} p|\mathrm{Tr}\, U_n|^2 \,\mathrm{d}x \to \int_{\mathbb{R}^N} p|\mathrm{Tr}\, U|^2 \,\mathrm{d}x$ as $n \to \infty$. In particular, if $p > 0$ a.e. in $\mathbb{R}^N$, the trace operator*

$$\mathcal{D}^{1,2}(\mathbb{R}^{N+1}_+; t^{1-2s}) \hookrightarrow L^2(\mathbb{R}^N; p\,\mathrm{d}x)$$

is compact, where $L^2(\mathbb{R}^N; p\,\mathrm{d}x) := \left\{u \in L^1_{\mathrm{loc}}(\mathbb{R}^N) \colon \int_{\mathbb{R}^N} p\,|u|^2 \,\mathrm{d}x < \infty\right\}$.

Proof. Let $(U_n)_n \subseteq \mathcal{D}^{1,2}(\mathbb{R}^{N+1}_+;t^{1-2s})$ and $U \in \mathcal{D}^{1,2}(\mathbb{R}^{N+1}_+;t^{1-2s})$ be such $U_n \rightharpoonup U$ weakly in $\mathcal{D}^{1,2}(\mathbb{R}^{N+1}_+;t^{1-2s})$ as $n \to \infty$. Hence, in view of continuity of the trace operator (1.5) and classical compactness results for fractional Sobolev spaces (see e.g. [12, Theorem 7.1]), we have that $\operatorname{Tr} U_n \to \operatorname{Tr} U$ in $L^2_{\mathrm{loc}}(\mathbb{R}^N)$ and a.e. in $\mathbb{R}^N$. Furthermore, by continuity of the trace operator (1.5) and (2.3), we have that, for every $\omega \subset \mathbb{R}^N$ measurable,

$$\int_\omega |p||\operatorname{Tr}(U_n - U)|^2 \,\mathrm{d}x \le C\|p\|_{L^{N/(2s)}(\omega)}\|U_n - U\|^2_{\mathcal{D}^{1,2}(\mathbb{R}^{N+1}_+;t^{1-2s})}$$

for some positive constant $C > 0$ independent of ω and n. Therefore, by Vitali's Convergence Theorem we can conclude that $\lim_{n\to\infty}\int_{\mathbb{R}^N}|p||\operatorname{Tr}(U_n - U)|^2\,\mathrm{d}x = 0$, from which the conclusion follows. □

We finally introduce a class of weighted Lebesgue and Sobolev spaces, on bounded open Lipschitz sets $\omega \subseteq \mathbb{R}^{N+1}_+$ in the upper half-space. Namely, we define

$$L^2(\omega;t^{1-2s}) := \left\{U\colon \omega \to \mathbb{R} \text{ measurable: such that } \int_\omega t^{1-2s}\,|U|^2\,\mathrm{d}t\,\mathrm{d}x < \infty\right\}$$

and the weighted Sobolev space

$$H^1(\omega;t^{1-2s}) := \left\{U \in L^2(\omega;t^{1-2s})\colon \nabla U \in L^2(\omega;t^{1-2s})\right\}.$$

From the fact that the weight t^{1-2s} belongs to the second Muckenhoupt class (see, for instance, [14, 13]) and thanks to well known weighted inequalities, one can prove that the embedding $H^1(\omega;t^{1-2s}) \hookrightarrow L^2(\omega;t^{1-2s})$ is compact, see for details [20, Proposition 7.1] and [30]. In addition, in the particular case of $\omega = B_r^+$ one can prove that the trace operators

$$H^1(B_r^+;t^{1-2s}) \hookrightarrow L^2(B_r'), \quad H^1(B_r^+;t^{1-2s}) \hookrightarrow L^2(S_r^+;t^{1-2s}),$$

are well defined and compact, where

$$L^2(S_r^+;t^{1-2s}) := \left\{\psi\colon S_r^+ \to \mathbb{R} \text{ measurable} : \int_{S_r^+} t^{1-2s}\,|\psi|^2\,\mathrm{d}S < \infty\right\}.$$

2.2. **The Angular Eigenvalue Problem.** Let us consider, for any $\lambda \in \mathbb{R}$, the problem

$$\begin{cases} -\operatorname{div}_{\mathbb{S}^N}(\theta_1^{1-2s}\nabla_{\mathbb{S}^N}\psi) = \mu\theta_1^{1-2s}\psi, & \text{in } \mathbb{S}^N_+, \\ -\lim\limits_{\theta_1\to 0^+}\theta_1^{1-2s}\nabla_{\mathbb{S}^N}\psi\cdot\boldsymbol{e}_1 = \kappa_s\lambda\psi, & \text{on } \mathbb{S}^{N-1}, \end{cases} \tag{2.5}$$

where $\boldsymbol{e}_1 = (1,0,\dots,0) \in \mathbb{R}^{N+1}_+$ and $\nabla_{\mathbb{S}^N}$ denotes the gradient on the unit N-dimensional sphere $\mathbb{S}^N$. In order to give a variational formulation of (2.5) we introduce the following Sobolev space

$$H^1(\mathbb{S}^N_+;\theta_1^{1-2s}) := \left\{\psi \in L^2(\mathbb{S}^N_+;\theta_1^{1-2s})\colon \int_{\mathbb{S}^+_N}\theta_1^{1-2s}|\nabla_{\mathbb{S}^N}\psi|^2\,dS < +\infty\right\}.$$

We say that $\psi \in H^1(\mathbb{S}^N_+;\theta_1^{1-2s})$ and $\mu \in \mathbb{R}$ weakly solve (2.5) if

$$\int_{\mathbb{S}^N_+}\theta_1^{1-2s}\nabla_{\mathbb{S}^N}\psi(\theta)\cdot\nabla_{\mathbb{S}^N}\varphi(\theta)\,\mathrm{d}S = \mu\int_{\mathbb{S}^N_+}\theta_1^{1-2s}\psi(\theta)\varphi(\theta)\,\mathrm{d}S + \kappa_s\lambda\int_{\mathbb{S}^{N-1}}\psi(0,\theta')\varphi(0,\theta')\,\mathrm{d}S'$$

for all $\varphi \in H^1(\mathbb{S}^N_+;\theta_1^{1-2s})$. By standard spectral arguments, if $\lambda < \gamma_H$, there exists a diverging sequence of real eigenvalues of problem (2.5)

$$\mu_1(\lambda) \le \mu_2(\lambda) \le \cdots \le \mu_n(\lambda) \le \cdots$$

Moreover, each eigenvalue has finite multiplicity (which is counted in the enumeration above) and $\mu_1(\lambda) > -\left(\frac{N-2s}{2}\right)^2$ (see [16, Lemma 2.2]). For every $n \ge 1$ we choose an eigenfunction $\psi_n \in H^1(\mathbb{S}^N_+;\theta_1^{1-2s}) \setminus \{0\}$, corresponding to $\mu_n(\lambda)$, such that $\int_{\mathbb{S}^N_+}\theta_1^{1-2s}\,|\psi_n|^2\,\mathrm{d}S = 1$. In addition, we choose the family $\{\psi_n\}_n$ in such a way that it is an orthonormal basis of $L^2(\mathbb{S}^N_+;\theta_1^{1-2s})$. We refer to [16] for further details.

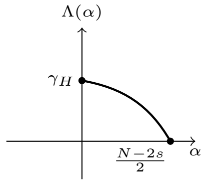

FIGURE 1. The function Λ

In [16] the following implicit characterization of $\mu_1(\lambda)$ is given. For any $\alpha \in \left(0, \frac{N-2s}{2}\right)$ we define

$$\Lambda(\alpha) := 2^{2s} \frac{\Gamma\left(\frac{N+2s+2\alpha}{4}\right)\Gamma\left(\frac{N+2s-2\alpha}{4}\right)}{\Gamma\left(\frac{N-2s+2\alpha}{4}\right)\Gamma\left(\frac{N-2s-2\alpha}{4}\right)}. \tag{2.6}$$

It is known (see e.g. [25] and [16, Proposition 2.3]) that the map $\alpha \mapsto \Lambda(\alpha)$ is continuous and monotone decreasing. Moreover

$$\lim_{\alpha\to 0^+} \Lambda(\alpha) = \gamma_H, \qquad \lim_{\alpha\to \frac{N-2s}{2}} \Lambda(\alpha) = 0, \tag{2.7}$$

see Figure 1. Furthermore, in [16, Proposition 2.3] it is proved that, for every $\alpha \in \left(0, \frac{N-2s}{2}\right)$,

$$\mu_1(\Lambda(\alpha)) = \alpha^2 - \left(\frac{N-2s}{2}\right)^2. \tag{2.8}$$

In particular, for every $\lambda \in (0, \gamma_H)$ there exists one and only one $\alpha \in \left(0, \frac{N-2s}{2}\right)$ such that $\Lambda(\alpha) = \lambda$ and hence $\mu_1(\lambda) = \alpha^2 - \left(\frac{N-2s}{2}\right)^2 < 0$.

We recall the following result from [15].

Lemma 2.2 ([15, Lemma 4.1])**.** *For every $\alpha \in \left(0, \frac{N-2s}{2}\right)$ there exists $\Upsilon_\alpha : \overline{\mathbb{R}^{N+1}_+} \setminus \{0\} \to \mathbb{R}$ such that Υ_α is locally Hölder continuous in $\overline{\mathbb{R}^{N+1}_+} \setminus \{0\}$, $\Upsilon_\alpha > 0$ in $\overline{\mathbb{R}^{N+1}_+} \setminus \{0\}$, and*

$$\begin{cases} -\operatorname{div}(t^{1-2s}\nabla \Upsilon_\alpha) = 0, & \text{in } \mathbb{R}^{N+1}_+, \\ \Upsilon_\alpha(0,x) = |x|^{-\frac{N-2s}{2}+\alpha}, & \text{on } \mathbb{R}^N, \\ -\lim_{t\to 0} t^{1-2s}\dfrac{\partial \Upsilon_\alpha}{\partial t} = \kappa_s \Lambda(\alpha)\, |x|^{-2s}\, \Upsilon_\alpha, & \text{on } \mathbb{R}^N, \end{cases} \tag{2.9}$$

in a weak sense. Moreover, $\Upsilon_\alpha \in H^1(B_R^+; t^{1-2s})$ for every $R > 0$.

The first eigenvalue $\mu_1(\lambda)$ satisfies the properties described in the following lemma.

Lemma 2.3. *Let $\lambda < \gamma_H$. Then the first eigenvalue of problem* (2.5) *can be characterized as*

$$\mu_1(\lambda) = \inf_{\substack{\psi \in H^1(\mathbb{S}^N_+;\theta_1^{1-2s}) \\ \psi \not\equiv 0}} \frac{\displaystyle\int_{\mathbb{S}^N_+} \theta_1^{1-2s}\, |\nabla_{\mathbb{S}^N}\psi|^2 \,\mathrm{d}S - \kappa_s\lambda \int_{\mathbb{S}^{N-1}} |\psi|^2 \,\mathrm{d}S'}{\displaystyle\int_{\mathbb{S}^N_+} \theta_1^{1-2s}\, |\psi|^2 \,\mathrm{d}S}$$

and the above infimum is attained by $\psi_1 \in H^1(\mathbb{S}^N_+; \theta_1^{1-2s})$, which weakly solves (2.5) *for $\mu = \mu_1(\lambda)$. Moreover*

(1) $\mu_1(\lambda)$ is simple, i.e. if ψ attains $\mu_1(\lambda)$ then $\psi = \delta\psi_1$ for some $\delta \in \mathbb{R}$;
(2) either $\psi_1 > 0$ or $\psi_1 < 0$ in $\mathbb{S}^N_+$;
(3) if $\lambda > 0$ and $\psi_1 > 0$, then the trace of ψ_1 on $\mathbb{S}^{N-1}$ is positive and constant;
(4) if $\lambda = 0$ then ψ_1 is constant in $\mathbb{S}^N_+$.

Proof. The proof of the fact that $\mu_1(\lambda)$ is reached is classical, as well as the proofs of points (1) and (2), see for instance [34, Section 8.3.3].

In order to prove (3), let us first observe that, if $\lambda \in (0, \gamma_H)$, there exists one and only one $\alpha \in \big(0, \frac{N-2s}{2}\big)$ such that $\Lambda(\alpha) = \lambda$. For this α let $\Upsilon_\alpha > 0$ be the solution of (2.9). Thanks to [16, Theorem 4.1], it is possible to describe the behaviour of Υ_α near the origin: in particular, since $\Upsilon_\alpha > 0$, we have that there exists $C > 0$ such that

$$\tau^{-a_{\Lambda(\alpha)}} \Upsilon_\alpha(0, \tau\theta') \to C\psi_1(0, \theta') \qquad \text{in } C^{1,\beta}(\mathbb{S}^{N-1}) \quad \text{as } \tau \to 0^+,$$

where

$$a_{\Lambda(\alpha)} = -\frac{N-2s}{2} + \sqrt{\left(\frac{N-2s}{2}\right)^2 + \mu_1(\Lambda(\alpha))}$$

Thanks to (2.8) we have that $a_{\Lambda(\alpha)} = -\frac{N-2s}{2} + \alpha$; then $\tau^{-a_{\Lambda(\alpha)}} \Upsilon_\alpha(0, \tau\theta') \equiv 1$ and so $\psi_1(0, \theta')$ is positive and constant in $\mathbb{S}^{N-1}$.

Finally, if $\lambda = 0$ then $\mu_1(0) = 0$ is clearly attained by every constant function. □

We note that, in view of well known regularity results (see Proposition A.1 in the Appendix), $\psi_1 \in C^{0,\beta}(\overline{\mathbb{S}^N_+})$ for some $\beta \in (0,1)$. Hereafter, we choose the first eigenfunction ψ_1 of problem (2.5) to be strictly positive in $\mathbb{S}^N_+$. With this choice of ψ_1, we also have that, in view of the Hopf type principle proved in [8, Proposition 4.11] (see Proposition A.2),

$$\min_{\overline{\mathbb{S}^N_+}} \psi_1 > 0. \tag{2.10}$$

2.3. **Asymptotic Estimates of Solutions.** In this section, we describe the asymptotic behaviour of solutions to equations of the type $-\operatorname{div}(t^{1-2s}\nabla\Phi) = 0$, with singular potentials appearing in the Neumann-type boundary conditions, either on positive half-balls B_r^+ or on their complement in $\mathbb{R}^{N+1}_+$.

Lemma 2.4. *Let $R_0 > 0$, $\lambda < \gamma_H$ and let $\Phi \in H^1(B_{R_0}^+; t^{1-2s})$, $\Phi \geq 0$ a.e. in $B_{R_0}^+$, $\Phi \not\equiv 0$, be a weak solution of the following problem*

$$\begin{cases} -\operatorname{div}(t^{1-2s}\nabla\Phi) = 0, & \text{in } B_{R_0}^+, \\ -\lim\limits_{t\to 0} t^{1-2s}\dfrac{\partial\Phi}{\partial t} = \kappa_s(\lambda\,|x|^{-2s} + q)\Phi, & \text{on } B'_{R_0}, \end{cases}$$

where $q \in C^1(B'_{R_0} \setminus \{0\})$ is such that

$$|q(x)| + |x \cdot \nabla q(x)| = O(|x|^{-2s+\varepsilon}) \qquad \text{as } |x| \to 0,$$

for some $\varepsilon > 0$. Then there exist $C_1 > 0$ and $R \leq R_0$ such that

$$\frac{1}{C_1}\,|z|^{a_\lambda} \leq \Phi(z) \leq C_1\,|z|^{a_\lambda} \qquad \text{for all } z \in B_R^+, \tag{2.11}$$

where

$$a_\lambda = -\frac{N-2s}{2} + \sqrt{\left(\frac{N-2s}{2}\right)^2 + \mu_1(\lambda)}. \tag{2.12}$$

Furthermore, if $0 \leq \lambda < \gamma_H$, then there exists $C_2 > 0$ such that

$$\lim_{|x|\to 0} |x|^{-a_\lambda}\,\Phi(0, x) = C_2. \tag{2.13}$$

Proof. Since $\Phi \geq 0$ a.e., $\Phi \not\equiv 0$, from [16, Theorem 4.1] we know that there exists $C > 0$ such that

$$\tau^{-a_\lambda}\Phi(\tau\theta) \to C\psi_1(\theta) \qquad \text{in } C^{0,\beta}(\overline{\mathbb{S}^N_+}) \quad \text{as } \tau \to 0. \tag{2.14}$$

Estimate (2.11) follows from the above convergence and (2.10). Convergence (2.13) follows from (2.14) and statements (3–4) of Lemma 2.3. □

Lemma 2.5. *Let $R_0 > 0$, $\lambda < \gamma_H$ and let $\Phi \in \mathcal{D}^{1,2}(\mathbb{R}^{N+1}_+; t^{1-2s})$, $\Phi \geq 0$ a.e. in $\mathbb{R}^{N+1}_+$, $\Phi \not\equiv 0$, be a weak solution of the following problem*

$$\begin{cases} -\operatorname{div}(t^{1-2s}\nabla\Phi) = 0, & \text{in } \mathbb{R}^{N+1}_+ \setminus B^+_{R_0}, \\ -\lim\limits_{t\to 0} t^{1-2s}\dfrac{\partial\Phi}{\partial t} = \kappa_s(\lambda\,|x|^{-2s} + q)\Phi, & \text{on } \mathbb{R}^N \setminus B'_{R_0}, \end{cases}$$

where $q \in C^1(\mathbb{R}^N \setminus B'_{R_0})$ is such that

$$|q(x)| + |x \cdot \nabla q(x)| = O(|x|^{-2s-\varepsilon}) \qquad \text{as } |x| \to +\infty,$$

for some $\varepsilon > 0$. Then there exist $C_3 > 0$ and $R \geq R_0$ such that

$$\frac{1}{C_3}\,|z|^{-(N-2s)-a_\lambda} \leq \Phi(z) \leq C_3\,|z|^{-(N-2s)-a_\lambda} \qquad \text{for all } z \in \mathbb{R}^{N+1}_+ \setminus B^+_R.$$

Furthermore, if $0 \leq \lambda < \gamma_H$, then there exists $C_4 > 0$ such that

$$\lim_{|x|\to\infty} |x|^{N-2s+a_\lambda}\,\Phi(0,x) = C_4.$$

Proof. The proof follows by considering the equation solved by the Kelvin transform of Φ

$$\tilde{\Phi}(z) := |z|^{-(N-2s)}\,\Phi\left(\frac{z}{|z|^2}\right) \tag{2.15}$$

(see [18, Proposition 2.6]) and applying Lemma 2.4. □

Lemma 2.6. *Let $R_0 > 0$ and let $\Phi \in H^1(B^+_{R_0}; t^{1-2s})$, $\Phi \geq 0$ a.e. in $\mathbb{R}^{N+1}_+$, $\Phi \not\equiv 0$, be a weak solution of the following problem*

$$\begin{cases} -\operatorname{div}(t^{1-2s}\nabla\Phi) = 0, & \text{in } B^+_{R_0}, \\ -\lim\limits_{t\to 0} t^{1-2s}\dfrac{\partial\Phi}{\partial t} = \kappa_s q\Phi, & \text{on } B'_{R_0}, \end{cases}$$

where $q \in C^1(B'_{R_0} \setminus \{0\})$ is such that

$$|q(x)| + |x \cdot \nabla q(x)| = O(|x|^{-2s+\varepsilon}) \qquad \text{as } |x| \to 0,$$

for some $\varepsilon > 0$. Then there exists $C_5 > 0$ such that

$$\lim_{|z|\to 0} \Phi(z) = \lim_{|x|\to 0} \Phi(0,x) = C_5.$$

Proof. The thesis is a direct consequence of the regularity result of [27, Proposition 2.4] (see Proposition A.1 in the Appendix) combined with the Hopf type principle in [8, Proposition 4.11] (see Proposition A.2). It can be also derived as a particular case of [16, Theorem 4.1] with $\lambda = 0$, taking into account that, for $\lambda = 0$, ψ_1 is a positive constant on $\mathbb{S}^N_+$, as observed in Lemma 2.3. □

Lemma 2.7. *Let $R_0 > 0$ and let $\Phi \in \mathcal{D}^{1,2}(\mathbb{R}^{N+1}_+; t^{1-2s})$, $\Phi \geq 0$ a.e. in $\mathbb{R}^{N+1}_+$, $\Phi \not\equiv 0$, be a weak solution of the following problem*

$$\begin{cases} -\operatorname{div}(t^{1-2s}\nabla\Phi) = 0, & \text{in } \mathbb{R}^{N+1}_+ \setminus B^+_{R_0}, \\ -\lim\limits_{t\to 0} t^{1-2s}\dfrac{\partial\Phi}{\partial t} = \kappa_s q\Phi, & \text{on } \mathbb{R}^N \setminus B'_{R_0}, \end{cases}$$

where $q \in C^1(\mathbb{R}^N \setminus B'_{R_0})$ is such that

$$|q(x)| + |x \cdot \nabla q(x)| = O(|x|^{-2s-\varepsilon}) \qquad \text{as } |x| \to +\infty,$$

for some $\varepsilon > 0$. Then there exists $C_6 > 0$ such that

$$\lim_{|z|\to\infty} |z|^{N-2s}\,\Phi(z) = \lim_{|x|\to\infty} |x|^{N-2s}\,\Phi(0,x) = C_6$$

Proof. The proof follows by considering the equation solved by the Kelvin transform of Φ given in (2.15) and applying Lemma 2.6. □

3. Proof of Theorem 1.5

Proof of Theorem 1.5. First, assume $\sum_{i=1}^k \lambda_i^+ < \gamma_H$. By Hardy inequality (1.4) we deduce that

$$Q_{\lambda_1,\dots,\lambda_k,a_1,\dots,a_k}(u) \geq \left(1 - \frac{\sum_{i=1}^k \lambda_i^+}{\gamma_H}\right) \|u\|^2_{\mathcal{D}^{s,2}(\mathbb{R}^N)} \quad \text{for all } u \in \mathcal{D}^{s,2}(\mathbb{R}^N),$$

thus implying that $Q_{\lambda_1,\dots,\lambda_k,a_1,\dots,a_k}$ is positive definite.

Now we assume that $\sum_{i=1}^k \lambda_i^+ > \gamma_H$. By optimality of the constant γ_H in Hardy inequality, it follows that there exists $\varphi \in C_c^\infty(\mathbb{R}^N)$ such that

$$\|\varphi\|^2_{\mathcal{D}^{s,2}(\mathbb{R}^N)} - \sum_{i=1}^k \lambda_i^+ \int_{\mathbb{R}^N} \frac{|\varphi|^2}{|x|^{2s}}\, \mathrm{d}x < 0. \tag{3.1}$$

Let $\varphi_\rho(x) := \rho^{-\frac{N-2s}{2}} \varphi(x/\rho)$. Then, taking into account Lemma 8.1, we have that

$$\begin{aligned}\|\varphi_\rho\|^2_{\mathcal{D}^{s,2}(\mathbb{R}^N)} - \sum_{i=1}^k \lambda_i^+ \int_{\mathbb{R}^N} \frac{|\varphi_\rho|^2}{|x-a_i|^{2s}}\, \mathrm{d}x &= \|\varphi\|^2_{\mathcal{D}^{s,2}(\mathbb{R}^N)} - \sum_{i=1}^k \lambda_i^+ \int_{\mathbb{R}^N} \frac{|\varphi|^2}{|x-a_i/\rho|^{2s}}\, \mathrm{d}x \\ &\to \|\varphi\|^2_{\mathcal{D}^{s,2}(\mathbb{R}^N)} - \sum_{i=1}^k \lambda_i^+ \int_{\mathbb{R}^N} \frac{|\varphi|^2}{|x|^{2s}}\, \mathrm{d}x < 0,\end{aligned}$$

as $\rho \to +\infty$. Therefore, there exists $\tilde{\rho}$ such that

$$\|\psi\|^2_{\mathcal{D}^{s,2}(\mathbb{R}^N)} - \sum_{i=1}^k \lambda_i^+ \int_{\mathbb{R}^N} \frac{|\psi|^2}{|x-a_i|^{2s}}\, \mathrm{d}x < 0, \tag{3.2}$$

where $\psi := \varphi_{\tilde{\rho}}$. Let $R > 0$ be such that $\operatorname{supp} \psi \subset B'_R$. Then

$$\int_{\mathbb{R}^N} \frac{|\psi|^2}{|x-a|^{2s}} \leq \frac{1}{(|a|-R)^{2s}} \int_{B'_R} |\psi|^2\, \mathrm{d}x \tag{3.3}$$

for $|a|$ sufficiently large. Hence, from (3.2) and (3.3) it follows that

$$\begin{aligned}Q_{\lambda_1,\dots,\lambda_k,a_1,\dots,a_k}(\psi) &= \|\psi\|^2_{\mathcal{D}^{s,2}(\mathbb{R}^N)} - \sum_{i=1}^k \lambda_i^+ \int_{\mathbb{R}^N} \frac{|\psi|^2}{|x-a_i|^{2s}}\, \mathrm{d}x + \sum_{i=1}^k \lambda_i^- \int_{\mathbb{R}^N} \frac{|\psi|^2}{|x-a_i|^{2s}}\, \mathrm{d}x \\ &\leq \|\psi\|^2_{\mathcal{D}^{s,2}(\mathbb{R}^N)} - \sum_{i=1}^k \lambda_i^+ \int_{\mathbb{R}^N} \frac{|\psi|^2}{|x-a_i|^{2s}}\, \mathrm{d}x + \sum_{i=1}^k \lambda_i^- \frac{1}{(|a_i|-R)^{2s}} \int_{B'_R} |\psi|^2\, \mathrm{d}x < 0\end{aligned}$$

if the poles a_i's, corresponding to negative λ_i's, are sufficiently far from the origin. The proof is thereby complete. □

Remark 3.1. We observe that, in the case of two poles (i.e. $k = 2$), Theorem 1.5 implies the sufficiency of condition (1.15) for the existence of a configuration of poles that makes the quadratic form $Q_{\lambda_1,\dots,\lambda_k,a_1,\dots,a_k}$ positive definite. Indeed, if $k = 2$ condition (1.15) directly implies (1.16).

4. A positivity criterion in the class Θ

In this section, we provide the proof of Lemma 1.2, that is a criterion for establishing positivity of Schrödinger operators with potentials in Θ, in relation with existence of positive supersolutions, in the spirit of Allegretto-Piepenbrink theory.

We first prove the equivalent formulation of the infimum in (1.11) stated in Lemma 1.1.

Proof of Lemma 1.1. Let's fix $\tilde{u} \in \mathcal{D}^{s,2}(\mathbb{R}^N) \setminus \{0\}$ and let's call $\tilde{U} \in \mathcal{D}^{1,2}(\mathbb{R}^{N+1}_+; t^{1-2s})$ its extension. Since

$$\kappa_s \|\tilde{u}\|^2_{\mathcal{D}^{s,2}(\mathbb{R}^N)} = \int_{\mathbb{R}^{N+1}_+} t^{1-2s} |\nabla \tilde{U}|^2\, \mathrm{d}t\, \mathrm{d}x,$$

where κ_s is defined in (1.9), then

$$\frac{\|\tilde{u}\|^2_{\mathcal{D}^{s,2}(\mathbb{R}^N)} - \int_{\mathbb{R}^N} V \tilde{u}^2 \, \mathrm{d}x}{\|\tilde{u}\|^2_{\mathcal{D}^{s,2}(\mathbb{R}^N)}} = \frac{\int_{\mathbb{R}^{N+1}_+} t^{1-2s} |\nabla \tilde{U}|^2 \, \mathrm{d}t \, \mathrm{d}x - \kappa_s \int_{\mathbb{R}^N} V |\operatorname{Tr} \tilde{U}|^2 \, \mathrm{d}x}{\int_{\mathbb{R}^{N+1}_+} t^{1-2s} |\nabla \tilde{U}|^2 \, \mathrm{d}t \, \mathrm{d}x}.$$

Therefore

$$\frac{\|\tilde{u}\|^2_{\mathcal{D}^{s,2}(\mathbb{R}^N)} - \int_{\mathbb{R}^N} V |\tilde{u}|^2 \, \mathrm{d}x}{\|\tilde{u}\|^2_{\mathcal{D}^{s,2}(\mathbb{R}^N)}} \geq \inf_{\substack{U \in \mathcal{D}^{1,2}(\mathbb{R}^{N+1}_+; t^{1-2s}) \\ U \not\equiv 0}} \frac{\int_{\mathbb{R}^{N+1}_+} t^{1-2s} |\nabla U|^2 \, \mathrm{d}t \, \mathrm{d}x - \kappa_s \int_{\mathbb{R}^N} V |\operatorname{Tr} U|^2 \, \mathrm{d}x}{\int_{\mathbb{R}^{N+1}_+} t^{1-2s} |\nabla U|^2 \, \mathrm{d}t \, \mathrm{d}x}$$

and then we can pass to the inf also on the left-hand quotient.

On the other hand, from (2.4), we have that, for any $U \in \mathcal{D}^{1,2}(\mathbb{R}^{N+1}_+; t^{1-2s}) \setminus \{0\}$

$$\frac{\int_{\mathbb{R}^{N+1}_+} t^{1-2s} |\nabla U|^2 \, \mathrm{d}t \, \mathrm{d}x - \kappa_s \int_{\mathbb{R}^N} V |\operatorname{Tr} U|^2 \, \mathrm{d}x}{\int_{\mathbb{R}^{N+1}_+} t^{1-2s} |\nabla U|^2 \, \mathrm{d}t \, \mathrm{d}x} \geq 1 - \frac{\int_{\mathbb{R}^N} V |u|^2 \, \mathrm{d}x}{\|u\|^2_{\mathcal{D}^{s,2}(\mathbb{R}^N)}}$$

where $u = \operatorname{Tr} U$. Taking the infimum to both sides concludes the proof. □

Now we are able to provide the proof of the positivity criterion.

Proof of Lemma 1.2. Let us first prove (I). Let $U \in C_c^\infty(\overline{\mathbb{R}^{N+1}_+} \setminus \{(0,a_1), \dots, (0,a_k)\}) \setminus \{0\}$, $U \not\equiv 0$ on $\mathbb{R}^N$. Note that $U^2/\Phi \in \mathcal{D}^{1,2}(\mathbb{R}^{N+1}_+; t^{1-2s})$ and $U^2/\Phi \geq 0$, hence we can choose U^2/Φ in (1.13) as a test function. Easy computations yield

$$\int_{\mathbb{R}^{N+1}_+} t^{1-2s} |\nabla U|^2 \, \mathrm{d}t \, \mathrm{d}x - \kappa_s \int_{\mathbb{R}^N} V |\operatorname{Tr} U|^2 \, \mathrm{d}x \geq \varepsilon \kappa_s \int_{\mathbb{R}^N} \tilde{V} |\operatorname{Tr} U|^2 \, \mathrm{d}x$$

which, taking into account the hypothesis on $\tilde{V}$, implies that

$$\kappa_s \int_{\mathbb{R}^N} V |\operatorname{Tr} U|^2 \, \mathrm{d}x \leq \frac{1}{1+\varepsilon} \int_{\mathbb{R}^{N+1}_+} t^{1-2s} |\nabla U|^2 \, \mathrm{d}t \, \mathrm{d}x.$$

Hence

$$\frac{\int_{\mathbb{R}^{N+1}_+} t^{1-2s} |\nabla U|^2 \, \mathrm{d}t \, \mathrm{d}x - \kappa_s \int_{\mathbb{R}^N} V |\operatorname{Tr} U|^2 \, \mathrm{d}x}{\int_{\mathbb{R}^{N+1}_+} t^{1-2s} |\nabla U|^2 \, \mathrm{d}t \, \mathrm{d}x} \geq \frac{\varepsilon}{1+\varepsilon}. \tag{4.1}$$

Therefore (I) follows by density of $C_c^\infty(\overline{\mathbb{R}^{N+1}_+} \setminus \{(0,a_1), \dots, (0,a_k)\})$ in $\mathcal{D}^{1,2}(\mathbb{R}^{N+1}_+; t^{1-2s})$ (see Lemma A.4).

Now we prove (II). First of all we notice that, thanks to Hölder's inequality, (2.2) and (2.3)

$$\begin{aligned} \kappa_s \int_{\mathbb{R}^N} \tilde{V} |\operatorname{Tr} U|^2 \, \mathrm{d}x &\leq \kappa_s \int_{\mathbb{R}^N} |V| |\operatorname{Tr} U|^2 \, \mathrm{d}x \\ &\leq \left[\frac{1}{\gamma_H} \left(\sum_{i=1}^k |\lambda_i| + |\lambda_\infty| \right) + S^{-1} \|W\|_{L^{N/2s}(\mathbb{R}^N)} \right] \int_{\mathbb{R}^{N+1}_+} t^{1-2s} |\nabla U|^2 \, \mathrm{d}t \, \mathrm{d}x \end{aligned}$$

for all $U \in \mathcal{D}^{1,2}(\mathbb{R}^{N+1}_+; t^{1-2s})$. If

$$0 < \varepsilon < \frac{\mu(V)}{2} \left[\frac{1}{\gamma_H} \left(\sum_{i=1}^k |\lambda_i| + |\lambda_\infty| \right) + S^{-1} \|W\|_{L^{N/2s}(\mathbb{R}^N)} \right]^{-1}, \tag{4.2}$$

then

$$\int_{\mathbb{R}^{N+1}_+} t^{1-2s} |\nabla U|^2 \, \mathrm{d}t \, \mathrm{d}x - \kappa_s \int_{\mathbb{R}^N} (V + \varepsilon \tilde{V}) |\operatorname{Tr} U|^2 \, \mathrm{d}x \geq \frac{\mu(V)}{2} \int_{\mathbb{R}^{N+1}_+} t^{1-2s} |\nabla U|^2 \, \mathrm{d}t \, \mathrm{d}x \tag{4.3}$$

for all $U \in \mathcal{D}^{1,2}(\mathbb{R}^{N+1}_+; t^{1-2s})$. Hence, for any fixed $p \in L^{N/2s}(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$, Hölder continuous and positive, the infimum

$$m_p = \inf_{\substack{U \in \mathcal{D}^{1,2}(\mathbb{R}^{N+1}_+; t^{1-2s}) \\ \operatorname{Tr} U \not\equiv 0}} \frac{\displaystyle\int_{\mathbb{R}^{N+1}_+} t^{1-2s} |\nabla U|^2 \, \mathrm{d}t \, \mathrm{d}x - \kappa_s \int_{\mathbb{R}^N} (V + \varepsilon \tilde{V}) |\operatorname{Tr} U|^2 \, \mathrm{d}x}{\displaystyle\int_{\mathbb{R}^N} p \, |\operatorname{Tr} U|^2 \, \mathrm{d}x}$$

is nonnegative. Also m_p is achieved by some function $\Phi \in \mathcal{D}^{1,2}(\mathbb{R}^{N+1}_+; t^{1-2s}) \setminus \{0\}$, that (by evenness) can be chosen to be nonnegative: indeed, thanks to Hardy inequality (2.2) and (4.3) it's easy to prove that the map

$$\mathcal{D}^{1,2}(\mathbb{R}^{N+1}_+; t^{1-2s}) \ni U \mapsto \int_{\mathbb{R}^{N+1}_+} t^{1-2s} |\nabla U|^2 \, \mathrm{d}t \, \mathrm{d}x - \kappa_s \int_{\mathbb{R}^N} (V + \varepsilon \tilde{V}) |\operatorname{Tr} U|^2 \, \mathrm{d}x$$

is weakly lower semicontinuous (since its square root is an equivalent norm in $\mathcal{D}^{1,2}(\mathbb{R}^{N+1}_+; t^{1-2s})$), while Lemma 2.1 yields the compactness of the trace map from $\mathcal{D}^{1,2}(\mathbb{R}^{N+1}_+; t^{1-2s})$ into $L^2(\mathbb{R}^N, p \, \mathrm{d}x)$. Moreover Φ satisfies in a weak sense

$$\begin{cases} -\operatorname{div}(t^{1-2s} \nabla \Phi) = 0, & \text{in } \mathbb{R}^{N+1}_+, \\ -\lim\limits_{t \to 0^+} t^{1-2s} \dfrac{\partial \Phi}{\partial t} = \kappa_s (V + \varepsilon \tilde{V}) \operatorname{Tr} \Phi + m_p p \operatorname{Tr} \Phi, & \text{in } \mathbb{R}^N, \end{cases} \tag{4.4}$$

i.e.

$$\int_{\mathbb{R}^{N+1}_+} t^{1-2s} \nabla \Phi \cdot \nabla W \, \mathrm{d}t \, \mathrm{d}x = \kappa_s \int_{\mathbb{R}^N} (V + \varepsilon \tilde{V}) \operatorname{Tr} \Phi \operatorname{Tr} W \, \mathrm{d}x + m_p \int_{\mathbb{R}^N} p \operatorname{Tr} \Phi \operatorname{Tr} W \, \mathrm{d}x$$

for all $W \in \mathcal{D}^{1,2}(\mathbb{R}^{N+1}_+; t^{1-2s})$. From [27, Proposition 2.6] (see also Proposition A.1 in the Appendix) we have that Φ is locally Hölder continuous in $\overline{\mathbb{R}^{N+1}_+} \setminus \{(0, a_1), \ldots, (0, a_k)\}$; in particular $\Phi \in C^0\left(\overline{\mathbb{R}^{N+1}_+} \setminus \{(0, a_1), \ldots, (0, a_k)\}\right)$. Moreover, the classical Strong Maximum Principle implies that $\Phi > 0$ in $\mathbb{R}^{N+1}_+$; then, in the case when $V, \tilde{V}$ are locally Hölder continuous in $\mathbb{R}^N \setminus \{a_1, \ldots, a_k\}$, the Hopf type principle proved in [8, Proposition 4.11] (which is recalled in the Proposition A.2 of the Appendix) ensures that $\Phi(0, x) > 0$ for all $x \in \mathbb{R}^N \setminus \{a_1, \ldots, a_k,\}$; we observe that assumption (A.1) of Proposition A.2 is satisfied thanks to [8, Lemma 4.5], see Lemma A.3. ☐

5. Upper and lower bounds for $\mu(V)$

In this section we prove bounds from above and from below (in Lemma 5.1 and 5.2, respectively) for the quantity $\mu(V)$.

Lemma 5.1. *For any* $V(x) = \sum_{i=1}^k \frac{\lambda_i \chi_{B'(a_i, r_i)}(x)}{|x - a_i|^{2s}} + \frac{\lambda_\infty \chi_{\mathbb{R}^N \setminus B'_R}(x)}{|x|^{2s}} + W(x) \in \Theta$, *there holds:*

(i) $\mu(V) \leq 1$;
(ii) *if* $\max_{i=1,\ldots,k,\infty} \lambda_i > 0$, *then* $\mu(V) \leq 1 - \frac{1}{\gamma_H} \max_{i=1,\ldots,k,\infty} \lambda_i$.

Proof. Let us fix $u \in C^\infty_c(\mathbb{R}^N)$, $u \not\equiv 0$ and $P \in \mathbb{R}^N \setminus \{a_1, \ldots, a_k\}$. For every $\rho > 0$ we define $u_\rho(x) := \rho^{-\frac{(N-2s)}{2}} u(\frac{x-P}{\rho})$ and we notice that, by scaling properties,

$$\|u_\rho\|_{\mathcal{D}^{s,2}(\mathbb{R}^N)} = \|u\|_{\mathcal{D}^{s,2}(\mathbb{R}^N)} \quad \text{and} \quad \|u_\rho\|_{L^{2^*_s}(\mathbb{R}^N)} = \|u\|_{L^{2^*_s}(\mathbb{R}^N)}. \tag{5.1}$$

Moreover, since $\operatorname{supp}(u_\rho) = P + \rho \operatorname{supp}(u)$, we have that $a_1, \ldots, a_k \notin \operatorname{supp}(u_\rho)$ for $\rho >$ sufficiently small, hence

$$V \in L^{N/2s}(\operatorname{supp}(u_\rho)). \tag{5.2}$$

Therefore, from the definition of $\mu(V)$, thanks also to (5.1), (5.2), Hölder inequality, and (2.1), we deduce that

$$\mu(V) \leq 1 - \frac{\int_{\mathbb{R}^N} V |u_\rho|^2 \, \mathrm{d}x}{\|u_\rho\|^2_{\mathcal{D}^{s,2}(\mathbb{R}^N)}} \leq 1 + \frac{\int_{\mathrm{supp}(u_\rho)} |V| \, |u_\rho|^2 \, \mathrm{d}x}{\|u\|^2_{\mathcal{D}^{s,2}(\mathbb{R}^N)}}$$

$$\leq 1 + \frac{\|V\|_{L^{N/2s}(\mathrm{supp}(u_\rho))} \|u\|^2_{L^{2^*_s}(\mathbb{R}^N)}}{\|u\|^2_{\mathcal{D}^{s,2}(\mathbb{R}^N)}} \leq 1 + S^{-1} \|V\|_{L^{N/2s}(\mathrm{supp}(u_\rho))} = 1 + o(1),$$

as $\rho \to 0^+$. By density we may conclude the first part of the proof.

Now let us assume $\max_{i=1,\dots,k,\infty} \lambda_i > 0$ and let us first consider the case $\max_{i=1,\dots,k,\infty} \lambda_i = \lambda_j$ for a certain $j = 1, \dots, k$. From optimality of the best constant in Hardy inequality (1.4) and from the density of $C_c^\infty(\mathbb{R}^N \setminus \{a_1, \dots, a_k, 0\})$ in $\mathcal{D}^{s,2}(\mathbb{R}^N)$ (see Lemma A.4), we have that, for any $\varepsilon > 0$, there exists $\varphi \in C_c^\infty(\mathbb{R}^N \setminus \{a_1, \dots, a_k, 0\})$ such that

$$\|\varphi\|^2_{\mathcal{D}^{s,2}(\mathbb{R}^N)} < (\gamma_H + \varepsilon) \int_{\mathbb{R}^N} \frac{|\varphi|^2}{|x|^{2s}} \, \mathrm{d}x. \tag{5.3}$$

Now, for any $\rho > 0$ we define $\varphi_\rho(x) := \rho^{-\frac{(N-2s)}{2}} \varphi(\frac{x - a_j}{\rho})$. From the definition of $\mu(V)$ and from (5.1) we deduce that

$$\mu(V) \leq 1 - \frac{\int_{\mathbb{R}^N} V |\varphi_\rho|^2 \, \mathrm{d}x}{\|\varphi\|^2_{\mathcal{D}^{s,2}(\mathbb{R}^N)}}. \tag{5.4}$$

On the other hand, we can split the numerator as

$$\int_{\mathbb{R}^N} V |\varphi_\rho|^2 \, \mathrm{d}x = \lambda_j \int_{B'(a_j, r_j)} |x - a_j|^{-2s} |\varphi_\rho|^2 \, \mathrm{d}x + \sum_{i \neq j} \lambda_i \int_{B'(a_i, r_i)} |x - a_i|^{-2s} |\varphi_\rho|^2 \, \mathrm{d}x$$

$$+ \lambda_\infty \int_{\mathbb{R}^N \setminus B'_R} |x|^{-2s} |\varphi_\rho|^2 \, \mathrm{d}x + \int_{\mathbb{R}^N} W |\varphi_\rho|^2 \, \mathrm{d}x.$$

From Hölder inequality and (5.1) we have that

$$\left| \int_{\mathbb{R}^N} W \varphi_\rho^2 \, \mathrm{d}x \right| \leq \|W\|_{L^{N/2s}(\mathrm{supp}(\varphi_\rho))} \|\varphi\|^2_{L^{2^*_s}} \to 0, \quad \text{as } \rho \to 0^+, \tag{5.5}$$

while, just by a change of variable

$$\int_{B'(a_j, r_j)} |x - a_j|^{-2s} |\varphi_\rho|^2 \, \mathrm{d}x = \int_{B'_{r_j/\rho}} |x|^{-2s} |\varphi|^2 \, \mathrm{d}x \to \int_{\mathbb{R}^N} |x|^{-2s} |\varphi|^2 \, \mathrm{d}x, \tag{5.6}$$

as $\rho \to 0^+$. Moreover $\mathrm{supp}(\varphi_\rho) = a_j + \rho \, \mathrm{supp}(\varphi)$, and therefore, thanks to (5.5) and (5.6), we have that, as $\rho \to 0^+$,

$$\int_{\mathbb{R}^N} V |\varphi_\rho|^2 \, \mathrm{d}x = \lambda_j \int_{\mathbb{R}^N} |x|^{-2s} |\varphi|^2 \, \mathrm{d}x + o(1). \tag{5.7}$$

Hence, combining (5.4) with (5.7) and (5.3), we obtain that

$$\mu(V) \leq 1 - \lambda_j (\gamma_H + \varepsilon)^{-1},$$

for all $\varepsilon > 0$, which implies that $\mu(V) \leq 1 - \lambda_j / \gamma_H$. Finally, let us assume $\max_{i=1,\dots,k,\infty} \lambda_i = \lambda_\infty$. Letting $\varphi_\rho(x) := \rho^{-\frac{(N-2s)}{2}} \varphi(x/\rho)$, we observe that $\varphi_\rho \to 0$ uniformly, as $\rho \to +\infty$. So, arguing as before, one can similarly obtain that $\mu(V) \leq 1 - \frac{\lambda_\infty}{\gamma_H}$. The proof is thereby complete. □

The following result provides the positivity in the case of potentials with subcritical masses supported in sufficiently small neighbourhoods of the poles. In the following we fix two cut-off functions $\zeta, \tilde{\zeta} : \mathbb{R}^N \to \mathbb{R}$ such that $\zeta, \tilde{\zeta} \in C^\infty(\mathbb{R}^N)$, $0 \leq \zeta(x) \leq 1$, $0 \leq \tilde{\zeta}(x) \leq 1$, and

$$\zeta(x) = 1 \quad \text{for } |x| \leq \frac{1}{2}, \quad \zeta(x) = 0 \quad \text{for } |x| \geq 1,$$

$$\tilde{\zeta}(x) = 0 \quad \text{for } |x| \leq 1, \quad \tilde{\zeta}(x) = 1 \quad \text{for } |x| \geq 2.$$

Lemma 5.2. *Let $\{a_1, a_2, \dots, a_k\} \subset B'_R$, $a_i \neq a_j$ for $i \neq j$, and $\lambda_1, \lambda_2, \dots, \lambda_k, \lambda_\infty \in \mathbb{R}$ be such that $m := \max_{i=1,\dots,k,\infty} \lambda_i < \gamma_H$. For any $0 < h < 1 - \frac{m}{\gamma_H}$, there exists $\delta = \delta(h) > 0$ such that*

$$\mu\left(\sum_{i=1}^{k} \frac{\lambda_i \zeta(\frac{x-a_i}{\delta})}{|x-a_i|^{2s}} + \frac{\lambda_\infty \tilde{\zeta}(\frac{x}{R})}{|x|^{2s}}\right) \geq \begin{cases} 1 - \frac{m}{\gamma_H} - h, & \text{if } m > 0 \\ 1, & \text{if } m \leq 0. \end{cases}$$

Proof. Let us assume that $m > 0$, otherwise the statement is trivial. First, let us fix $0 < \varepsilon < \frac{\gamma_H}{m} - 1$, so that

$$\tilde{\lambda}_i := \lambda_i + \varepsilon \lambda_i^+ < \gamma_H \quad \text{for all } i = 1, \dots, k, \infty.$$

In order to prove the statement, it is sufficient to find $\delta = \delta(\varepsilon) > 0$ and $\Phi \in \mathcal{D}^{1,2}(\mathbb{R}^{N+1}_+; t^{1-2s})$ such that $\Phi \in C^0(\overline{\mathbb{R}^{N+1}_+} \setminus \{(0, a_1/\delta), \dots, (0, a_k/\delta)\})$, $\Phi > 0$ in $\overline{\mathbb{R}^{N+1}_+} \setminus \{(0, a_1/\delta), \dots, (0, a_k/\delta)\}$, and

$$\int_{\mathbb{R}^{N+1}_+} t^{1-2s} \nabla\Phi \cdot \nabla U \, \mathrm{d}t \, \mathrm{d}x - \sum_{i=1}^{k} \kappa_s \int_{\mathbb{R}^N} V_i \operatorname{Tr}\Phi \operatorname{Tr} U \, \mathrm{d}x - \kappa_s \int_{\mathbb{R}^N} V_\infty \operatorname{Tr}\Phi \operatorname{Tr} U \, \mathrm{d}x \geq 0 \tag{5.8}$$

for all $U \in \mathcal{D}^{1,2}(\mathbb{R}^{N+1}_+; t^{1-2s})$, $U \geq 0$ a.e., where

$$V_i(x) = \frac{\tilde{\lambda}_i \zeta(x - \frac{a_i}{\delta})}{|x - a_i/\delta|^{2s}}, \qquad V_\infty(x) = \frac{\tilde{\lambda}_\infty \tilde{\zeta}(\frac{\delta}{R} x)}{|x|^{2s}}.$$

Indeed, thanks to scaling properties in (5.8) and to Lemma 1.2, (5.8) implies that

$$\mu\left(\sum_{i=1}^{k} \frac{\lambda_i \zeta(\frac{x-a_i}{\delta})}{|x-a_i|^{2s}} + \frac{\lambda_\infty \tilde{\zeta}(\frac{x}{R})}{|x|^{2s}}\right) \geq \frac{\varepsilon}{1+\varepsilon},$$

so that, letting $h := 1 - \frac{m}{\gamma_H} - \frac{\varepsilon}{\varepsilon+1}$, we obtain the result. Hence, we seek for some Φ positive and continuous in $\overline{\mathbb{R}^{N+1}_+} \setminus \{(0, a_1/\delta), \dots, (0, a_k/\delta)\}$ satisfying (5.8). Let us set, for some $0 < \tau < 1$,

$$p_i(x) := p\left(x - \frac{a_i}{\delta}\right) \quad \text{for } i = 1, \dots, k, \quad p_\infty(x) = \left(\frac{\delta}{R}\right)^{2s} p\left(\frac{\delta x}{R}\right),$$

where $p(x) = \frac{1}{|x|^{2s-\tau}(1+|x|^2)^\tau}$. We observe that $p_i, p_\infty \in L^{N/2s}(\mathbb{R}^N)$. Therefore, thanks to Lemma 2.1, the weighted eigenvalue

$$\mu_i = \inf_{\substack{\Phi \in \mathcal{D}^{1,2}(\mathbb{R}^{N+1}_+; t^{1-2s}) \\ \operatorname{Tr}\Phi \not\equiv 0}} \frac{\int_{\mathbb{R}^{N+1}_+} t^{1-2s} |\nabla\Phi|^2 \, \mathrm{d}t \, \mathrm{d}x - \kappa_s \int_{\mathbb{R}^N} V_i |\operatorname{Tr}\Phi|^2 \, \mathrm{d}x}{\int_{\mathbb{R}^N} p_i |\operatorname{Tr}\Phi|^2 \, \mathrm{d}x}$$

is positive and attained by some nontrivial, nonnegative function $\Phi_i \in \mathcal{D}^{1,2}(\mathbb{R}^{N+1}_+; t^{1-2s})$ that weakly solves

$$\begin{cases} -\operatorname{div}(t^{1-2s} \nabla\Phi_i) = 0, & \text{in } \mathbb{R}^{N+1}_+, \\ -\lim_{t\to 0} t^{1-2s} \dfrac{\partial\Phi_i}{\partial t} = (\kappa_s V_i + \mu_i p_i) \operatorname{Tr}\Phi_i, & \text{on } \mathbb{R}^N. \end{cases}$$

From the classical Strong Maximum Principle we deduce that $\Phi_i > 0$ in $\mathbb{R}^{N+1}_+$, while Proposition A.1 yields that Φ_i is locally Hölder continuous in $\overline{\mathbb{R}^{N+1}_+} \setminus \{(0, a_i/\delta)\}$. Moreover, from the Hopf type lemma in Proposition A.2 (whose assumption (A.1) is satisfied thanks to Lemma A.3 outside $\{a_i/\delta\}$) we deduce that $\operatorname{Tr}\Phi_i > 0$ in $\mathbb{R}^N \setminus \{a_i/\delta\}$. Similarly

$$\mu_\infty = \inf_{\substack{\Phi \in \mathcal{D}^{1,2}(\mathbb{R}^{N+1}_+; t^{1-2s}) \\ \operatorname{Tr}\Phi \not\equiv 0}} \frac{\int_{\mathbb{R}^{N+1}_+} t^{1-2s} |\nabla\Phi|^2 \, \mathrm{d}t \, \mathrm{d}x - \kappa_s \int_{\mathbb{R}^N} V_\infty |\operatorname{Tr}\Phi|^2 \, \mathrm{d}x}{\int_{\mathbb{R}^N} p_\infty |\operatorname{Tr}\Phi|^2 \, \mathrm{d}x}$$

is positive and reached by some nontrivial, nonnegative function $\Phi_\infty \in \mathcal{D}^{1,2}(\mathbb{R}^{N+1}_+; t^{1-2s})$ such that Φ_∞ is locally Hölder continuous in $\overline{\mathbb{R}^{N+1}_+}$ and $\Phi_\infty > 0$ in $\overline{\mathbb{R}^{N+1}_+}$. Moreover, Φ_∞ weakly solves

$$\begin{cases} -\operatorname{div}(t^{1-2s}\nabla\Phi_\infty) = 0, & \text{in } \mathbb{R}^{N+1}_+, \\ -\lim_{t\to 0} t^{1-2s}\dfrac{\partial \Phi_\infty}{\partial t} = (\kappa_s V_\infty + \mu_\infty p_\infty)\operatorname{Tr}\Phi_\infty, & \text{on } \mathbb{R}^N. \end{cases}$$

Lemmas 2.4–2.7 (and continuity of the Φ_i's outside the poles) imply that there exists $C_0 > 0$ (independent of δ) such that

$$\frac{1}{C_0}\left|x - \frac{a_i}{\delta}\right|^{a_{\tilde\lambda_i}} \le \operatorname{Tr}\Phi_i \le C_0\left|x - \frac{a_i}{\delta}\right|^{a_{\tilde\lambda_i}}, \qquad \text{in } B'(a_i/\delta, 1), \tag{5.9}$$

$$\frac{1}{C_0}\left|x - \frac{a_i}{\delta}\right|^{-(N-2s)} \le \operatorname{Tr}\Phi_i \le C_0\left|x - \frac{a_i}{\delta}\right|^{-(N-2s)}, \qquad \text{in } \mathbb{R}^N \setminus B'(a_i/\delta, 1), \tag{5.10}$$

$$\frac{1}{C_0}\left|\frac{\delta x}{R}\right|^{-(N-2s)-a_{\tilde\lambda_\infty}} \le \operatorname{Tr}\Phi_\infty \le C_0\left|\frac{\delta x}{R}\right|^{-(N-2s)-a_{\tilde\lambda_\infty}}, \qquad \text{in } \mathbb{R}^N \setminus B'_{R/\delta}, \tag{5.11}$$

$$\frac{1}{C_0} \le \operatorname{Tr}\Phi_\infty \le C_0, \qquad \text{in } B'_{R/\delta}. \tag{5.12}$$

Let $\Phi := \sum_{i=1}^k \Phi_i + \eta\Phi_\infty$, with $0 < \eta < \inf\{\frac{\mu_i}{4C_0^2\tilde\lambda_i} : i = 1,\dots,k,\ \tilde\lambda_i > 0\}$. Therefore,

$$\begin{aligned} &\int_{\mathbb{R}^{N+1}_+} t^{1-2s}\nabla\Phi\cdot\nabla U \,\mathrm{d}t\,\mathrm{d}x - \sum_{i=1}^k \int_{\mathbb{R}^N} V_i \operatorname{Tr}\Phi\operatorname{Tr}U\,\mathrm{d}x - \int_{\mathbb{R}^N} V_\infty\operatorname{Tr}\Phi\operatorname{Tr}U\,\mathrm{d}x \\ &= \int_{\mathbb{R}^N}\left[\sum_{i=1}^k\Big(\mu_i p_i - V_\infty - \sum_{j\neq i} V_j\Big)\operatorname{Tr}\Phi_i + \eta\Big(\mu_\infty p_\infty - \sum_{i=1}^k V_i\Big)\operatorname{Tr}\Phi_\infty\right]\operatorname{Tr}U\,\mathrm{d}x \\ &=: \int_{\mathbb{R}^N} g(x)\operatorname{Tr}U(x)\,\mathrm{d}x \end{aligned} \tag{5.13}$$

for all $U \in \mathcal{D}^{1,2}(\mathbb{R}^{N+1}_+; t^{1-2s})$. Hereafter, let us assume $U \ge 0$ a.e. in $\mathbb{R}^{N+1}_+$. We will split the integral into three parts and prove that each of these is nonnegative. First, let us consider $x \in B'_{R/\delta} \setminus \big(\cup_{i=1}^k B'(a_i/\delta, 1)\big)$: here $V_i = V_\infty = 0$ and we have that

$$g(x) = \sum_{i=1}^k \mu_i p_i \operatorname{Tr}\Phi_i + \eta\mu_\infty p_\infty\operatorname{Tr}\Phi_\infty \ge 0.$$

Now let us take $n \in \{1,\dots,k\}$ and $x \in B'(a_n/\delta, 1)$, where $V_\infty = V_i = 0$ for $i \neq n$. Then

$$\begin{aligned} g(x) &= \sum_{i=1}^k \mu_i p_i\operatorname{Tr}\Phi_i - V_n\sum_{i\neq n}\operatorname{Tr}\Phi_i + \eta(\mu_\infty p_\infty - V_n)\operatorname{Tr}\Phi_\infty \\ &\ge \mu_n p_n \operatorname{Tr}\Phi_n - V_n\Big(\sum_{\substack{i=1\\ i\neq n}}^k \operatorname{Tr}\Phi_i + \eta\operatorname{Tr}\Phi_\infty\Big). \end{aligned}$$

If $\tilde\lambda_n \le 0$ this is clearly nonnegative; so let us assume $\tilde\lambda_n > 0$. Thanks to (5.9), (5.10) and (5.12) we can estimate this quantity from below by

$$\left|x - \frac{a_n}{\delta}\right|^{-2s}\left[\frac{\mu_n}{2^\tau C_0}\left|x - \frac{a_n}{\delta}\right|^{\tau + a_{\tilde\lambda_n}} - \tilde\lambda_n C_0\Big(\sum_{i\neq n}\left|x - \frac{a_i}{\delta}\right|^{-(N-2s)} + \eta\Big)\right]. \tag{5.14}$$

We observe that $|x - a_n/\delta|^{\tau + a_{\tilde\lambda_n}} \ge 1$, since $\tilde\lambda_n > 0$ implies that $a_{\tilde\lambda_n} < 0$ and we can choose $\tau < -a_{\tilde\lambda_n}$. Moreover it's not hard to prove that, for $i \neq n$,

$$\left|x - \frac{a_i}{\delta}\right|^{-(N-2s)} \le \left(\frac{2}{|a_n - a_i|}\right)^{N-2s}\delta^{N-2s} < \frac{\eta}{k-1},$$

for $\delta > 0$ sufficiently small. Thanks to this and to the choice of η we have that the expression in (5.14) and then $g(x)$ is nonnegative in $B'(a_n/\delta, 1)$. Finally, if $x \in \mathbb{R}^N \setminus B'_{R/\delta}$, then the function g in (5.13) becomes

$$\sum_{i=1}^{k} (\mu_i p_i - V_\infty) \operatorname{Tr} \Phi_i + \eta \mu_\infty p_\infty \operatorname{Tr} \Phi_\infty. \tag{5.15}$$

Again, if $\tilde{\lambda}_\infty \leq 0$ this quantity is nonnegative. If $\tilde{\lambda}_\infty > 0$, thanks to (5.10) and (5.11), we have that the function in (5.15) is greater than or equal to

$$|x|^{-2s} \left[- C_0 \tilde{\lambda}_\infty \sum_{i=1}^{k} \left| x - \frac{a_i}{\delta} \right|^{-(N-2s)} + \frac{\eta \mu_\infty}{2^\tau C_0} \left| \frac{\delta x}{R} \right|^{-(N-2s) - a_{\tilde{\lambda}_\infty} + \tau} \right]. \tag{5.16}$$

Now, one can easily see that

$$\left| x - \frac{a_i}{\delta} \right| \geq \left(1 - \frac{a}{R} \right) |x| \qquad \text{for all } x \in \mathbb{R}^N \setminus B'_{R/\delta}, \quad \text{where } a = \max_{j=1,\dots,k} |a_j| ,$$

so that we can estimate (5.16) from below obtaining that, for all $x \in \mathbb{R}^N \setminus B'_{R/\delta}$,

$$\begin{aligned} g(x) &\geq |x|^{-N} \left[- C_0 \tilde{\lambda}_\infty k \left(1 - \frac{a}{R} \right)^{-(N-2s)} + \frac{\eta \mu_\infty}{2^\tau C_0} \left| \frac{\delta}{R} \right|^{-(N-2s) - a_{\tilde{\lambda}_\infty} + \tau} |x|^{-a_{\tilde{\lambda}_\infty} + \tau} \right] \\ &\geq |x|^{-N} \left[- C_0 \tilde{\lambda}_\infty k \left(1 - \frac{a}{R} \right)^{-(N-2s)} + \frac{\eta \mu_\infty}{2^\tau C_0} \left| \frac{\delta}{R} \right|^{-(N-2s)} \right] \geq 0 \end{aligned}$$

for $\delta > 0$ sufficiently small, since $a_{\tilde{\lambda}_\infty} < 0$ if $\tilde{\lambda}_\infty > 0$. The proof is thereby complete. $\square$

6. Perturbation at infinity and at poles

In this section, we investigate the persistence of the positivity when the mass is increased at infinity (Theorem 6.3) and at poles (Theorem 6.4).

In order to make use of Lemmas 2.4–2.7, we may need to restrict the class Θ to some more regular potentials and to have a control on their growth at infinity.

For any $\delta > 0$, we define

$$\mathcal{P}_\infty^\delta := \Big\{ f \colon \mathbb{R}^N \to \mathbb{R} \colon f \in C^1(\mathbb{R}^N \setminus B'_{R_\infty}) \text{ for some } R_\infty > 0 \\ \text{and } |f(x)| + |x \cdot \nabla f(x)| = O(|x|^{-2s-\delta}) \text{ as } |x| \to +\infty \Big\}. \tag{6.1}$$

Moreover, in order to prove some intermediary, technical lemmas based on the positivity criterion Lemma 1.2, the need for even more regular potentials occasionally arises. So, let us introduce the class

$$\Theta^* := \left\{ V \in \Theta \colon V \in C^1(\mathbb{R}^N \setminus \{a_1, \dots, a_k\}) \right\}. \tag{6.2}$$

Then, we will recover the full generality of the class Θ, thanks to an approximation procedure, which is based on the following lemma.

Lemma 6.1. *Let $V_1, V_2 \in \Theta$ be such that $V_1 - V_2 \in L^{N/2s}(\mathbb{R}^N)$. Then*

$$|\mu(V_2) - \mu(V_1)| \leq S^{-1} \| V_2 - V_1 \|_{L^{N/2s}(\mathbb{R}^N)} ,$$

where $S > 0$ is the best constant in the Sobolev embedding (2.1).

Proof. From the definition of $\mu(V_2)$, Hölder inequality and (2.3), for all $U \in \mathcal{D}^{1,2}(\mathbb{R}^{N+1}_+; t^{1-2s})$ we have that

$$\int_{\mathbb{R}^{N+1}_+} t^{1-2s} |\nabla U|^2 \, dt \, dx - \kappa_s \int_{\mathbb{R}^N} V_1 |\mathrm{Tr}\, U|^2 \, dx$$
$$\geq \left(\mu(V_2) - S^{-1} \|V_2 - V_1\|_{L^{N/2s}(\mathbb{R}^N)}\right) \int_{\mathbb{R}^{N+1}_+} t^{1-2s} |\nabla U|^2 \, dt \, dx, \quad (6.3)$$

which implies that

$$\mu(V_1) \geq \mu(V_2) - S^{-1} \|V_2 - V_1\|_{L^{N/2s}(\mathbb{R}^N)} .$$

Analogously one can prove that $\mu(V_2) \geq \mu(V_1) - S^{-1} \|V_2 - V_1\|_{L^{N/2s}(\mathbb{R}^N)}$, thus concluding the proof. □

Lemma 6.2. *Let $V \in \mathcal{H}$, $a_1, \dots, a_k \in \mathbb{R}^N$, and $R > 0$ be such that*

$$V \in C^1(\mathbb{R}^N \setminus \{a_1, \dots, a_k\}) \quad \text{and} \quad V(x) = \frac{\lambda_\infty}{|x|^{2s}} + W(x) \quad \text{in } \mathbb{R}^N \setminus B'_R,$$

where $\lambda_\infty < \gamma_H$ and $W \in \mathcal{P}^\delta_\infty \cap L^\infty(\mathbb{R}^N)$ for some $\delta > 0$. Assume that $\mu(V) > 0$ and let $\nu_\infty \in \mathbb{R}$ be such that $\lambda_\infty + \nu_\infty < \gamma_H$. Then there exist $\tilde{R} > R$ and $\Phi \in \mathcal{D}^{1,2}(\mathbb{R}^{N+1}_+; t^{1-2s})$ such that Φ is locally Hölder continuous in $\overline{\mathbb{R}^{N+1}_+} \setminus \{(0, a_1), \dots, (0, a_k)\}$, $\Phi > 0$ in $\overline{\mathbb{R}^{N+1}_+} \setminus \{(0, a_1), \dots, (0, a_k)\}$, and

$$\int_{\mathbb{R}^{N+1}_+} t^{1-2s} \nabla \Phi \cdot \nabla U \, dt \, dx - \kappa_s \int_{\mathbb{R}^N} \left[V + \frac{\nu_\infty}{|x|^{2s}} \chi_{\mathbb{R}^N \setminus B'_{\tilde{R}}}\right] \mathrm{Tr}\, \Phi \, \mathrm{Tr}\, U \, dx \geq 0,$$

for all $U \in \mathcal{D}^{1,2}(\mathbb{R}^{N+1}_+; t^{1-2s})$ with $U \geq 0$ a.e.

Proof. By (2.7) we can fix $\varepsilon \in \left(0, \frac{N-2s}{2}\right)$ such that

$$\Lambda(\varepsilon) - \lambda_\infty > 0 \quad \text{and} \quad \Lambda(\varepsilon) - \lambda_\infty - \nu_\infty > 0. \quad (6.4)$$

Since $W \in \mathcal{P}^\delta_\infty \cap L^\infty(\mathbb{R}^N)$, there exists $C_0 > 0$ such that

$$W(x) \leq \frac{C_0}{|x|^{2s+\delta}} \qquad \text{in } \mathbb{R}^N. \quad (6.5)$$

Let $R_0 \geq \max\left\{R, \frac{1}{2}\left[\frac{C_0}{\Lambda(\varepsilon) - \lambda_\infty}\right]^{1/\delta}\right\}$, so that

$$\Lambda(\varepsilon) - \lambda_\infty - C_0 (2R_0)^{-\delta} \geq 0. \quad (6.6)$$

From Lemma 2.2 there exists a positive, locally Hölder continuous function $\Upsilon_\varepsilon : \overline{\mathbb{R}^{N+1}_+} \setminus \{0\} \to \mathbb{R}$ such that $\Upsilon_\varepsilon \in \bigcap_{r>0} H^1(B_r^+; t^{1-2s})$ and

$$\begin{cases} -\mathrm{div}(t^{1-2s} \nabla \Upsilon_\varepsilon), = 0 & \text{in } \mathbb{R}^{N+1}_+, \\ \Upsilon_\varepsilon(0, x) = |x|^{-\frac{N-2s}{2} + \varepsilon}, & \text{on } \mathbb{R}^N, \\ -\lim\limits_{t \to 0^+} t^{1-2s} \dfrac{\partial \Upsilon_\varepsilon}{\partial t}, = \kappa_s \Lambda(\varepsilon) |x|^{-2s} \, \mathrm{Tr}\, \Upsilon_\varepsilon & \text{on } \mathbb{R}^N, \end{cases} \quad (6.7)$$

in a weak sense. Direct calculations (see e.g. [18, Proposition 2.6]) yield that the Kelvin transform

$$\tilde{\Upsilon}_\varepsilon(z) = |z|^{-(N-2s)} \, \Upsilon_\varepsilon(z / |z|^2)$$

of Υ_ε weakly satisfies

$$\begin{cases} -\mathrm{div}(t^{1-2s} \nabla \tilde{\Upsilon}_\varepsilon) = 0, & \text{in } \mathbb{R}^{N+1}_+ \setminus \{0\}, \\ \tilde{\Upsilon}_\varepsilon(0, x) = |x|^{\frac{2s-N}{2} - \varepsilon}, & \text{on } \mathbb{R}^N \setminus \{0\}, \\ -\lim\limits_{t \to 0^+} t^{1-2s} \dfrac{\partial \tilde{\Upsilon}_\varepsilon}{\partial t} = \kappa_s \Lambda(\varepsilon) |x|^{-2s} \, \mathrm{Tr}\, \tilde{\Upsilon}_\varepsilon, & \text{on } \mathbb{R}^N \setminus \{0\}, \end{cases} \quad (6.8)$$

$\tilde\Upsilon_\varepsilon > 0$ in $\overline{\mathbb{R}^{N+1}_+} \setminus \{0\}$ and $\tilde\Upsilon_\varepsilon$ is locally Hölder continuous in $\overline{\mathbb{R}^{N+1}_+} \setminus \{0\}$. Moreover we have that

$$\int_{\mathbb{R}^{N+1}_+ \setminus B_r^+} t^{1-2s} |\nabla \tilde\Upsilon_\varepsilon|^2 \, dt \, dx + \int_{\mathbb{R}^{N+1}_+ \setminus B_r^+} t^{1-2s} \frac{|\tilde\Upsilon_\varepsilon|^2}{|x|^2 + t^2} \, dt \, dx < +\infty \quad \text{for all } r > 0. \tag{6.9}$$

Let $\eta \in C^\infty(\overline{\mathbb{R}^{N+1}_+})$ be a cut-off function such that η is radial, i.e. $\eta(z) = \eta(|z|)$, $|\nabla \eta| \le \frac{2}{R_0}$ in $\overline{\mathbb{R}^{N+1}_+}$,

$$\eta(z) := \begin{cases} 0, & \text{in } B_{R_0}^+ \cup B'_{R_0} \\ 1, & \text{in } \left(\mathbb{R}^{N+1}_+ \setminus B^+_{2R_0}\right) \cup \left(\mathbb{R}^N \setminus B'_{2R_0}\right), \end{cases}$$

and $\eta > 0$ in $\overline{\mathbb{R}^{N+1}_+} \setminus \overline{B^+_{R_0}}$. We point out that

$$\frac{\partial \eta}{\partial t}(0,x) = 0 \quad \text{and} \quad \frac{1}{t}\left|\frac{\partial \eta}{\partial t}(t,x)\right| = O(1) \quad \text{as } t \to 0 \text{ (uniformly in } x).$$

We let $\Phi_1 := \eta \tilde\Upsilon_\varepsilon$. By its construction, Φ_1 is continuous on the whole $\overline{\mathbb{R}^{N+1}_+}$ and $\Phi_1 > 0$ in $\overline{\mathbb{R}^{N+1}_+} \setminus \overline{B^+_{R_0}}$, whereas (6.9) implies that $\Phi_1 \in \mathcal{D}^{1,2}(\mathbb{R}^{N+1}_+; t^{1-2s})$. Moreover direct computations yield that Φ_1 weakly solves

$$\begin{cases} -\mathrm{div}(t^{1-2s} \nabla \Phi_1) = t^{1-2s} F_1, & \text{in } \mathbb{R}^{N+1}_+, \\ -\lim\limits_{t \to 0^+} t^{1-2s} \dfrac{\partial \Phi_1}{\partial t} = \kappa_s \Lambda(\varepsilon) |x|^{-2s} \operatorname{Tr} \Phi_1, & \text{on } \mathbb{R}^N, \end{cases} \tag{6.10}$$

where

$$F_1 := (2s-1) \frac{1}{t} \frac{\partial \eta}{\partial t} \tilde\Upsilon_\varepsilon - 2 \nabla \tilde\Upsilon_\varepsilon \cdot \nabla \eta - \tilde\Upsilon_\varepsilon \Delta \eta.$$

We observe that $F_1 \in C^\infty(\mathbb{R}^{N+1}_+)$ and $\mathrm{supp}(F_1) \subset \overline{B^+_{2R_0} \setminus B^+_{R_0}}$. Given

$$f_1(x) := \kappa_s \Lambda(\varepsilon) \, |x|^{-2s} \, \chi_{B'_{2R_0} \setminus B'_{R_0}} \Phi_1(0,x),$$

we can choose a smooth, compactly supported function $f_2 \colon \mathbb{R}^N \to \mathbb{R}$ such that

$$f_1 + f_2 \ge 0 \quad \text{in } \mathbb{R}^N, \qquad H := f_2 + \left[W + \lambda_\infty \, |x|^{-2s} \right] \chi_{B'_{2R_0} \setminus B'_{R_0}} \operatorname{Tr} \Phi_1 \ge 0 \quad \text{in } \mathbb{R}^N. \tag{6.11}$$

We also choose another smooth, positive, compactly supported function $F_2 \colon \overline{\mathbb{R}^{N+1}_+} \to \mathbb{R}$ such that $F_1 + F_2 \ge 0$ in $\overline{\mathbb{R}^{N+1}_+}$. Since $\mu(V) > 0$ and $H \in L^{\frac{2N}{N+2s}}(\mathbb{R}^N)$, by Lax-Milgram Lemma there exists $\Phi_2 \in \mathcal{D}^{1,2}(\mathbb{R}^{N+1}_+; t^{1-2s})$ such that

$$\begin{cases} -\mathrm{div}(t^{1-2s} \nabla \Phi_2) = t^{1-2s} F_2, & \text{in } \mathbb{R}^{N+1}_+, \\ -\lim\limits_{t \to 0^+} t^{1-2s} \dfrac{\partial \Phi_2}{\partial t} = \kappa_s \left[V \operatorname{Tr} \Phi_2 + H \right], & \text{on } \mathbb{R}^N, \end{cases} \tag{6.12}$$

holds in a weak sense. From Proposition A.1 we know that Φ_2 is locally Hölder continuous in $\overline{\mathbb{R}^{N+1}_+} \setminus \{(0,a_1), \dots, (0,a_k)\}$.

In order to prove that Φ_2 is strictly positive in $\overline{\mathbb{R}^{N+1}_+} \setminus \{(0,a_1), \dots, (0,a_k)\}$, we compare it with the unique weak solution $\Phi_3 \in \mathcal{D}^{1,2}(\mathbb{R}^{N+1}_+; t^{1-2s})$ to the problem

$$\begin{cases} -\mathrm{div}(t^{1-2s} \nabla \Phi_3) = 0, & \text{in } \mathbb{R}^{N+1}_+, \\ -\lim\limits_{t \to 0^+} t^{1-2s} \dfrac{\partial \Phi_3}{\partial t} = \kappa_s \left[V \operatorname{Tr} \Phi_3 + H \right], & \text{on } \mathbb{R}^N, \end{cases} \tag{6.13}$$

whose existence is again ensured by the Lax-Milgram Lemma. The difference $\widetilde\Phi = \Phi_2 - \Phi_3$ belongs to $\mathcal{D}^{1,2}(\mathbb{R}^{N+1}_+; t^{1-2s})$ and weakly solves

$$\begin{cases} -\mathrm{div}(t^{1-2s} \nabla \widetilde\Phi) = t^{1-2s} F_2, & \text{in } \mathbb{R}^{N+1}_+, \\ -\lim\limits_{t \to 0^+} t^{1-2s} \dfrac{\partial \widetilde\Phi}{\partial t} = \kappa_s V \operatorname{Tr} \widetilde\Phi, & \text{on } \mathbb{R}^N. \end{cases}$$

By directly testing the above equation with $-\widetilde{\Phi}^-$, since $\mu(V) > 0$ we obtain that $\widetilde{\Phi} \geq 0$ in $\mathbb{R}^{N+1}_+$, i.e. $\Phi_2 \geq \Phi_3$. Furthermore, testing the equation for Φ_3 with $-\Phi_3^-$, we also obtain that $\Phi_3 \geq 0$ in $\mathbb{R}^{N+1}_+$. The classical Strong Maximum Principle, combined with Proposition A.2 (whose assumption (A.1) for (6.13) is satisfied thanks to the assumption $V \in C^1(\mathbb{R}^N \setminus \{a_1, \dots, a_k\})$ and Lemma A.3), yields $\Phi_3 > 0$ in $\overline{\mathbb{R}^{N+1}_+} \setminus \{(0, a_1), \dots, (0, a_k)\}$ and hence

$$\Phi_2 > 0 \quad \text{in} \quad \overline{\mathbb{R}^{N+1}_+} \setminus \{(0, a_1), \dots, (0, a_k)\}.$$

Finally, from Lemma 2.5 and from the continuity of Φ_2, there exists $C_1 > 0$ such that

$$\frac{1}{C_1} |x|^{-(N-2s)-a_{\lambda_\infty}} \leq \Phi_2(0, x) \leq C_1 |x|^{-(N-2s)-a_{\lambda_\infty}} \quad \text{in } \mathbb{R}^N \setminus B'_{2R_0}. \tag{6.14}$$

Now we set $\Phi = \Phi_1 + \Phi_2$. We immediately observe that $\Phi \in \mathcal{D}^{1,2}(\mathbb{R}^{N+1}_+; t^{1-2s})$ is locally Hölder continuous and strictly positive in $\overline{\mathbb{R}^{N+1}_+} \setminus \{(0, a_1), \dots, (0, a_k)\}$. We claim that, for $\tilde{R} > 0$ sufficiently large,

$$\int_{\mathbb{R}^{N+1}_+} t^{1-2s} \nabla\Phi \cdot \nabla U \, dt \, dx - \kappa_s \int_{\mathbb{R}^N} \left[V + \frac{\nu_\infty}{|x|^{2s}} \chi_{\mathbb{R}^N \setminus B'_{\tilde{R}}} \right] \operatorname{Tr}\Phi \operatorname{Tr} U \, dx \geq 0, \tag{6.15}$$

for all $U \in \mathcal{D}^{1,2}(\mathbb{R}^{N+1}_+; t^{1-2s})$ with $U \geq 0$ a.e.

The function Φ weakly satisfies

$$\begin{cases} -\operatorname{div}(t^{1-2s}\nabla\Phi) = t^{1-2s}(F_1 + F_2), & \text{in } \mathbb{R}^{N+1}_+, \\ -\lim\limits_{t\to 0^+} t^{1-2s} \dfrac{\partial\Phi}{\partial t} = \kappa_s \left[\Lambda(\varepsilon)\, |x|^{-2s} \operatorname{Tr}\Phi_1 + V \operatorname{Tr}\Phi_2 + H \right], & \text{on } \mathbb{R}^N. \end{cases} \tag{6.16}$$

Hence, if $U \in \mathcal{D}^{1,2}(\mathbb{R}^{N+1}_+; t^{1-2s})$, $U \geq 0$ a.e.,

$$\begin{aligned} &\int_{\mathbb{R}^{N+1}_+} t^{1-2s} \nabla\Phi \cdot \nabla U \, dt \, dx - \kappa_s \int_{\mathbb{R}^N} \left[V + \frac{\nu_\infty}{|x|^{2s}} \chi_{\mathbb{R}^N \setminus B'_{\tilde{R}}} \right] \operatorname{Tr}\Phi \operatorname{Tr} U \, dx \\ &\geq \int_{\mathbb{R}^N} \Bigg[\kappa_s \frac{\Lambda(\varepsilon) - \lambda_\infty - \nu_\infty}{|x|^{2s}} \chi_{\mathbb{R}^N \setminus B'_{\tilde{R}}} \operatorname{Tr}\Phi_1 + \kappa_s \frac{\Lambda(\varepsilon) - \lambda_\infty - |x|^{2s} W}{|x|^{2s}} \chi_{B'_{\tilde{R}} \setminus B'_{2R_0}} \operatorname{Tr}\Phi_1 \\ &- \kappa_s \left(W \operatorname{Tr}\Phi_1 + \frac{\nu_\infty}{|x|^{2s}} \operatorname{Tr}\Phi_2 \right) \chi_{\mathbb{R}^N \setminus B'_{\tilde{R}}} + f_1 + f_2 \Bigg] \operatorname{Tr} U \, dx =: \int_{\mathbb{R}^N} F(x) \operatorname{Tr} U(x) \, dx. \end{aligned}$$

If $x \in B'_{2R_0}$, then $F(x) = f_1(x) + f_2(x) \geq 0$. If $x \in B'_{\tilde{R}} \setminus B'_{2R_0}$, then from (6.11), (6.5) and (6.6)

$$F(x) \geq \kappa_s(\Lambda(\varepsilon) - \lambda_\infty - |x|^{2s} W)\, |x|^{-2s} \operatorname{Tr}\Phi_1 \geq \kappa_s(\Lambda(\varepsilon) - \lambda_\infty - C_0(2R_0)^{-\delta})\, |x|^{-2s} \operatorname{Tr}\Phi_1 \geq 0.$$

Finally, if $x \in \mathbb{R}^N \setminus B'_{\tilde{R}}$, then from the definition of Φ_1, (6.11), (6.14) and (6.5) we have that

$$F(x) \geq \kappa_s(\Lambda(\varepsilon) - \lambda_\infty - \nu_\infty)\, |x|^{-\frac{N+2s}{2} - \varepsilon} - \kappa_s C_0\, |x|^{-\frac{N+2s}{2} - \varepsilon - \delta} - \kappa_s C_1 \nu_\infty\, |x|^{-N - a_{\lambda_\infty}}.$$

Since the function $\lambda \mapsto \mu_1(\lambda)$ is strictly decreasing and $\lambda_\infty < \Lambda(\varepsilon)$, from (2.8) it follows that $\mu_1(\lambda_\infty) > \varepsilon^2 - \left(\frac{N-2s}{2}\right)^2$ which yields $-N - a_{\lambda_\infty} < -\frac{N+2s}{2} - \varepsilon$. Hence, if $\tilde{R}$ is sufficiently large, $F(x) \geq 0$ for all $x \in \mathbb{R}^N \setminus B'_{\tilde{R}}$. This concludes the proof. □

Combining Lemma 6.2 with the positivity criterion Lemma 1.2 and an approximation procedure based on Lemma 6.1, we prove the persistence of the positivity under perturbations at infinity for potentials in the class Θ.

Theorem 6.3. *Let*

$$V(x) = \sum_{i=1}^{k} \frac{\lambda_i \chi_{B'(a_i, r_i)}(x)}{|x - a_i|^{2s}} + \frac{\lambda_\infty \chi_{\mathbb{R}^N \setminus B'_R}(x)}{|x|^{2s}} + W(x) \in \Theta.$$

Assume $\mu(V) > 0$ *and let* $\nu_\infty \in \mathbb{R}$ *be such that* $\lambda_\infty + \nu_\infty < \gamma_H$. *Then there exists* $\tilde{R} > R$ *such that*

$$\mu\left(V + \frac{\nu_\infty}{|x|^{2s}}\chi_{\mathbb{R}^N\setminus B'_{\tilde{R}}}\right) > 0.$$

Proof. Since $V \in \Theta$ and $\mu(V) > 0$, arguing as in (4.3) we have that, for ε chosen sufficiently small as in (4.2), $\mu(V+\varepsilon V) > \frac{\mu(V)}{2} > 0$. Moreover we can choose ε such that $\lambda_\infty+\nu_\infty+\varepsilon(\lambda_\infty+\nu_\infty) < \gamma_H$ and $\lambda_i + \varepsilon\lambda_i < \gamma_H$ for all $i = 1, \dots, k, \infty$. Let $\sigma = \sigma(\varepsilon)$ be such that

$$0 < \sigma < \min\{S\varepsilon, S\mu(V)/2\}. \tag{6.17}$$

By density of $C_c^\infty(\mathbb{R}^N)$ in $L^{\frac{N}{2s}}(\mathbb{R}^N)$ there exists

$$\hat{V}(x) = \sum_{i=1}^{k} \frac{\lambda_i\chi_{B'(a_i,r_i)}(x)}{|x-a_i|^{2s}} + \frac{\lambda_\infty\chi_{\mathbb{R}^N\setminus B'_R}(x)}{|x|^{2s}} + \hat{W}(x) \in \Theta^*$$

such that $\hat{W} \in \mathcal{P}_\infty^\delta$ for some $\delta > 0$ and

$$\|\hat{V} - V\|_{L^{N/2s}(\mathbb{R}^N)} < \frac{\sigma}{1+\varepsilon}. \tag{6.18}$$

Then from Lemma 6.1, taking into account (6.17) and (6.18), we have that

$$\mu(\hat{V} + \varepsilon\hat{V}) \geq \mu(V+\varepsilon V) - (1+\varepsilon)S^{-1}\|\hat{V} - V\|_{L^{N/2s}(\mathbb{R}^N)} > 0.$$

Now, thanks to Lemma 6.2, there exists $\tilde{R} > R$ and a function $\Phi \in \mathcal{D}^{1,2}(\mathbb{R}^{N+1}_+; t^{1-2s})$ such that Φ is strictly positive and locally Hölder continuous in $\overline{\mathbb{R}^{N+1}_+} \setminus \{(0,a_1), \dots, (0,a_k)\}$ and

$$\int_{\mathbb{R}^{N+1}_+} t^{1-2s}\nabla\Phi\cdot\nabla U \,\mathrm{d}t\,\mathrm{d}x - \kappa_s\int_{\mathbb{R}^N}\left[\hat{V} + \varepsilon\hat{V} + \frac{\nu_\infty + \varepsilon\nu_\infty}{|x|^{2s}}\chi_{\mathbb{R}^N\setminus B'_{\tilde{R}}}\right]\operatorname{Tr}\Phi\operatorname{Tr}U\,\mathrm{d}x \geq 0,$$

for all $U \in \mathcal{D}^{1,2}(\mathbb{R}^{N+1}_+; t^{1-2s})$ with $U \geq 0$ a.e. Therefore Lemma 1.2 yields

$$\mu\left(\hat{V} + \frac{\nu_\infty}{|x|^{2s}}\chi_{\mathbb{R}^N\setminus B'_{\tilde{R}}}\right) \geq \frac{\varepsilon}{1+\varepsilon}.$$

Finally, thanks to Lemma 6.1, (6.17) and (6.18), we have the estimate

$$\mu\left(V + \frac{\nu_\infty}{|x|^{2s}}\chi_{\mathbb{R}^N\setminus B'_{\tilde{R}}}\right) \geq \mu\left(\hat{V} + \frac{\nu_\infty}{|x|^{2s}}\chi_{\mathbb{R}^N\setminus B'_{\tilde{R}}}\right) - S^{-1}\|\hat{V} - V\|_{L^{N/2s}(\mathbb{R}^N)} > 0$$

which yields the conclusion. □

Swapping the singularity at a pole for a singularity at infinity through the Kelvin transform, we obtain the analog of Theorem 6.3 when perturbing the mass of a pole.

Theorem 6.4. *Let*

$$V(x) = \sum_{i=1}^{k} \frac{\lambda_i\chi_{B'(a_i,r_i)}(x)}{|x-a_i|^{2s}} + \frac{\lambda_\infty\chi_{\mathbb{R}^N\setminus B'_R}(x)}{|x|^{2s}} + W(x) \in \Theta.$$

Assume $\mu(V) > 0$ *and let* $i_0 \in \{1, \dots, k\}$ *and* $\nu \in \mathbb{R}$ *be such that* $\lambda_{i_0} + \nu < \gamma_H$. *Then there exists* $\delta \in (0, r_{i_0})$ *such that*

$$\mu\left(V + \frac{\nu}{|x-a_{i_0}|^{2s}}\chi_{B'(a_{i_0},\delta)}\right) > 0.$$

Before proving Theorem 6.4, it is convenient to make the following remark.

Remark 6.5. (i) By the invariance by translation of the norm $\|\cdot\|_{\mathcal{D}^{s,2}(\mathbb{R}^N)}$, we have that, if $V \in \mathcal{H}$, then, for any $a \in \mathbb{R}^N$, the translated potential $V_a := V(\cdot + a)$ belongs to $\mathcal{H}$ and $\mu(V_a) = \mu(V)$.

(ii) If $V \in \mathcal{H}$ and $V_K(x) := |x|^{-4s}V\big(\frac{x}{|x|^2}\big)$, then $V_K \in \mathcal{H}$ and $\mu(V_K) = \mu(V)$. To prove this statement, we observe that, by the change of variables $y = \frac{x}{|x|^2}$,

$$\int_{\mathbb{R}^N} |V_K(x)|^2 u^2(x)\,\mathrm{d}x = \int_{\mathbb{R}^N} |V(y)|^2 (\mathcal{K}u)^2(y)\,\mathrm{d}y \quad \text{for any } u \in \mathcal{D}^{s,2}(\mathbb{R}^N),$$

where $(\mathcal{K}u)(x) := |x|^{2s-N}u\big(\frac{x}{|x|^2}\big)$ is the Kelvin transform of u. The claim then follows from the fact that $\mathcal{K}$ is an isometry on $\mathcal{D}^{s,2}(\mathbb{R}^N)$ (see [18, Lemma 2.2]).

Proof of Theorem 6.4. Let $V_1(x) := V(x + a_{i_0})$. We have that

$$V_1(x) = \frac{\lambda_{i_0}\chi_{B'_{r_{i_0}}}(x)}{|x|^{2s}} + \sum_{i \neq i_0} \frac{\lambda_i \chi_{B'(a_i - a_{i_0}, r_i)}(x)}{|x - (a_i - a_{i_0})|^{2s}} + \frac{\lambda_\infty \chi_{\mathbb{R}^N \setminus B'_R}(x)}{|x|^{2s}} + W_1(x) \in \Theta$$

and, in view of Remark 6.5 (i), $\mu(V_1) = \mu(V) > 0$. Then we can choose some ε sufficiently small so that $\mu(V_1 + \varepsilon V_1) > \frac{\mu(V)}{2} > 0$ (see (4.3)) and $\lambda_{i_0} + \nu + \varepsilon(\lambda_{i_0} + \nu) < \gamma_H$, $\lambda_i + \varepsilon\lambda_i < \gamma_H$ for all $i = 1, \dots, k, \infty$. Let $\sigma = \sigma(\varepsilon) \in (0, \min\{S\varepsilon, S\mu(V)/2\})$. By density of $C^\infty_c(\mathbb{R}^N \setminus \{a_1, \dots, a_k\})$ in $L^{\frac{N}{2s}}(\mathbb{R}^N)$ there exists

$$V_2(x) = \frac{\lambda_{i_0}\chi_{B'_{r_{i_0}}}(x)}{|x|^{2s}} + \sum_{i \neq i_0} \frac{\lambda_i \chi_{B'(a_i - a_{i_0}, r_i)}(x)}{|x - (a_i - a_{i_0})|^{2s}} + \frac{\lambda_\infty \chi_{\mathbb{R}^N \setminus B'_R}(x)}{|x|^{2s}} + W_2(x) \in \Theta^*$$

such that $W_2 \in L^{N/2s}(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ vanishes in a neighbourhood of any pole and in a neighbourhood of ∞ and

$$\|V_2 - V_1\|_{L^{N/2s}(\mathbb{R}^N)} < \frac{\sigma}{1+\varepsilon}.$$

Let $V_3(x) := |x|^{-4s}V_2\big(\frac{x}{|x|^2}\big)$. Then

$$V_3 \in C^1\left(\mathbb{R}^N \setminus \left\{0, \tfrac{a_i - a_{i_0}}{|a_i - a_{i_0}|^2}\right\}_{i \neq i_0}\right)$$

and there exists $r > 0$ such that

$$V_3(x) = \frac{\lambda_{i_0}}{|x|^{2s}} \quad \text{in } \mathbb{R}^N \setminus B'_r.$$

Moreover, from Remark 6.5 (ii) and Lemma 6.1 it follows that $V_3 \in \mathcal{H}$ and

$$\begin{aligned}\mu(V_3 + \varepsilon V_3) &= \mu(V_2 + \varepsilon V_2)\\ &\geq \mu(V_1 + \varepsilon V_1) - S^{-1}(1+\varepsilon)\|V_1 - V_2\|_{L^{N/2s}(\mathbb{R}^N)} > \frac{\mu(V)}{2} - S^{-1}\sigma > 0.\end{aligned}$$

From Lemma 6.2 we deduce that there exists $\tilde{R} > r$ and a function $\Phi \in \mathcal{D}^{1,2}(\mathbb{R}^{N+1}_+; t^{1-2s})$ such that Φ is strictly positive and locally Hölder continuous in $\overline{\mathbb{R}^{N+1}_+} \setminus \left\{0, \frac{a_i - a_{i_0}}{|a_i - a_{i_0}|^2}\right\}_{i \neq i_0}$ and

$$\int_{\mathbb{R}^{N+1}_+} t^{1-2s}\nabla\Phi \cdot \nabla U \,\mathrm{d}t\,\mathrm{d}x - \kappa_s \int_{\mathbb{R}^N} \left[V_3 + \varepsilon V_3 + \frac{\nu + \varepsilon\nu}{|x|^{2s}}\chi_{\mathbb{R}^N \setminus B'_{\tilde{R}}}\right] \operatorname{Tr}\Phi \operatorname{Tr} U \,\mathrm{d}x \geq 0,$$

for all $U \in \mathcal{D}^{1,2}(\mathbb{R}^{N+1}_+; t^{1-2s})$ with $U \geq 0$ a.e. Therefore Lemma 1.2 yields

$$\mu\left(V_3 + \frac{\nu}{|x|^{2s}}\chi_{\mathbb{R}^N \setminus B'_{\tilde{R}}}\right) \geq \frac{\varepsilon}{1+\varepsilon}.$$

From Remark 6.5 (ii) we have that $\mu\left(V_3+\frac{\nu}{|x|^{2s}}\chi_{\mathbb{R}^N\setminus B'_{\tilde R}}\right)=\mu\left(V_2+\frac{\nu}{|x|^{2s}}\chi_{B'_{1/\tilde R}}\right)\geq\frac{\varepsilon}{1+\varepsilon}$. Hence, letting $\delta=1/\tilde R$, from Remark 6.5 (i) and Lemma 6.1 we deduce that

$$\begin{aligned}\mu\left(V+\frac{\nu}{|x-a_{i_0}|^{2s}}\chi_{B'(a_{i_0},\delta)}\right)&=\mu\left(V_1+\frac{\nu}{|x|^{2s}}\chi_{B'_\delta}\right)\\&\geq\mu\left(V_2+\frac{\nu}{|x|^{2s}}\chi_{B'_\delta}\right)-S^{-1}\|V_1-V_2\|_{L^{N/2s}(\mathbb{R}^N)}\geq\frac{\varepsilon}{1+\varepsilon}-\frac{S^{-1}\sigma}{1+\varepsilon}>0\end{aligned}$$

which yields the conclusion. □

Corollary 6.6. *Let*

$$V(x)=\sum_{i=1}^k\frac{\lambda_i\chi_{B'(a_i,r_i)}(x)}{|x-a_i|^{2s}}+\frac{\lambda_\infty\chi_{\mathbb{R}^N\setminus B'_R}(x)}{|x|^{2s}}+W(x)\in\Theta$$

be such that $\mu(V)>0$. *Then there exists*

$$\widetilde V(x)=\sum_{i=1}^k\frac{\tilde\lambda_i\chi_{B'(a_i,\tilde r_i)}(x)}{|x-a_i|^{2s}}+\frac{\lambda_\infty\chi_{\mathbb{R}^N\setminus B'_R}(x)}{|x|^{2s}}+\widetilde W(x)\in\Theta \tag{6.19}$$

such that

$$\widetilde V-V\in C^\infty(\mathbb{R}^N\setminus\{a_1,\dots,a_k\}),\quad \widetilde V\geq V,\quad \mu(\widetilde V)>0,\quad \textit{and}\quad \tilde\lambda_i>0 \textit{ for all } i=1,\dots,k.$$

Proof. For every $i=1,\dots,k$, let ν_i be such that $\nu_i>0$ and $\lambda_i+\nu_i\in(0,\gamma_H)$. From Theorem 6.4 we have that, for every $i=1,\dots,k$, there exists δ_i such that, letting

$$\widehat V=V+\sum_{i=1}^k\frac{\nu_i}{|x-a_i|^{2s}}\chi_{B'(a_i,\delta_i)},$$

$\mu(\widehat V)>0$. Let us consider a cut-off function $\zeta:\mathbb{R}^N\to\mathbb{R}$ such that $\zeta\in C^\infty(\mathbb{R}^N)$, $0\leq\zeta(x)\leq1$, $\zeta(x)=1$ for $|x|\leq\frac12$, and $\zeta(x)=0$ for $|x|\geq1$. Let

$$\widetilde V(x)=V+\sum_{i=1}^k\frac{\nu_i}{|x-a_i|^{2s}}\zeta\left(\frac{x-a_i}{\delta_i}\right).$$

Then $\widetilde V-V\in C^\infty(\mathbb{R}^N\setminus\{a_1,\dots,a_k\})$ and $\widetilde V\geq V$. Moreover $\widetilde V$ is of the form (6.19) with $\tilde\lambda_i=\lambda_i+\nu_i>0$ and, in view of (1.11) and the fact that $\widetilde V\leq\widehat V$, $\mu(\widetilde V)\geq\mu(\widehat V)>0$. The proof is thereby complete. □

7. Localization of Binding

This section is devoted to the proof of Theorem 1.3, which is the main tool needed in order to prove our main result. Indeed this tool ensures, inside the class Θ, that the sum of two positive operators is positive, provided one of them is translated sufficiently far.

For any $\delta>0$ and $a_1,\dots,a_k\in\mathbb{R}^N$, we define

$$\mathcal{P}^\delta_{a_1,\dots,a_k}:=\mathcal{P}^\delta_\infty\cap\left(\bigcap_{j=1}^k\mathcal{P}^\delta_{a_j}\right) \tag{7.1}$$

where $\mathcal{P}^\delta_\infty$ is defined in (6.1) and, for all $j=1,\dots,k$,

$$\mathcal{P}^\delta_{a_j}=\Big\{f:\mathbb{R}^N\to\mathbb{R}:f\in C^1(B'(a_j,R_j)\setminus\{a_j\})\text{ for some }R_j>0\\ \text{and }|f(x)|+|(x-a_j)\cdot\nabla f(x)|=O(|x-a_j|^{-2s+\delta})\text{ as }x\to a_j\Big\}.$$

Lemma 7.1. *Let*

$$V_1(x) = \sum_{i=1}^{k_1} \frac{\lambda_i^1 \chi_{B'(a_i^1, r_i^1)}(x)}{|x - a_i^1|^{2s}} + \frac{\lambda_\infty^1 \chi_{\mathbb{R}^N \setminus B'_{R_1}}(x)}{|x|^{2s}} + W_1(x) \in \Theta^*,$$

$$V_2(x) = \sum_{i=1}^{k_2} \frac{\lambda_i^2 \chi_{B'(a_i^2, r_i^2)}(x)}{|x - a_i^2|^{2s}} + \frac{\lambda_\infty^2 \chi_{\mathbb{R}^N \setminus B'_{R_2}}(x)}{|x|^{2s}} + W_2(x) \in \Theta^*,$$

with $W_1 \in \mathcal{P}^\delta_{a_1^1,\dots,a_{k_1}^1}$, $W_2 \in \mathcal{P}^\delta_{a_1^2,\dots,a_{k_2}^2}$ *for some* $\delta > 0$. *If* $\mu(V_1), \mu(V_2) > 0$ *and* $\lambda_\infty^1 + \lambda_\infty^2 < \gamma_H$, *then there exists* $R > 0$ *such that for every* $y \in \mathbb{R}^N \setminus \overline{B'_R}$ *there exists* $\Phi_y \in \mathcal{D}^{1,2}(\mathbb{R}^{N+1}_+; t^{1-2s})$ *such that* Φ_y *is strictly positive and locally Hölder continuous in* $\overline{\mathbb{R}^{N+1}_+} \setminus \{(0, a_i^1), (0, a_i^2 + y)\}_{i=1,\dots,k_j, j=1,2}$ *and*

$$\int_{\mathbb{R}^{N+1}_+} t^{1-2s} \nabla \Phi_y \cdot \nabla U \,\mathrm{d}t\,\mathrm{d}x \geq \kappa_s \int_{\mathbb{R}^N} (V_1(x) + V_2(x - y)) \operatorname{Tr} \Phi_y \operatorname{Tr} U \,\mathrm{d}x \tag{7.2}$$

for all $U \in \mathcal{D}^{1,2}(\mathbb{R}^{N+1}_+; t^{1-2s})$, *with* $U \geq 0$ *a.e.*

Proof. First of all we observe that it is not restrictive to assume that $\lambda_i^j > 0$ for all $i = 1, \dots, k_j$, $j = 1, 2$. Indeed, letting V_1, V_2 as in the assumptions, from Corollary 6.6 there exist $\widetilde{V}_1, \widetilde{V}_2 \in \Theta^*$ with positive masses at poles such that $\widetilde{V}_j \geq V_j$ and $\mu(\widetilde{V}_j) > 0$ for $j = 1, 2$. If the theorem is true under the further assumption of positivity of masses at poles, we conclude that, for every y with $|y|$ sufficiently large, there exists $\Phi_y \in \mathcal{D}^{1,2}(\mathbb{R}^{N+1}_+; t^{1-2s})$ strictly positive and locally Hölder continuous in $\overline{\mathbb{R}^{N+1}_+} \setminus \{(0, a_i^1), (0, a_i^2 + y)\}_{i=1,\dots,k_j, j=1,2}$ such that (7.2) holds with $\widetilde{V}_1(x) + \widetilde{V}_2(x - y)$ in the right hand side integral instead of $V_1(x) + V_2(x - y)$. Since $\widetilde{V}_1(x) + \widetilde{V}_2(x - y) \geq V_1(x) + V_2(x - y)$ we obtain (7.2). Then we can assume that $\lambda_i^j > 0$ for all $i = 1, \dots, k_j$, $j = 1, 2$, without loss of generality.

Let $\varepsilon \in (0, \gamma_H)$ be such that $\lambda_\infty^1 + \lambda_\infty^2 < \gamma_H - \varepsilon$, $\lambda_\infty^1 < \gamma_H - \varepsilon$, and $\lambda_\infty^2 < \gamma_H - \varepsilon$ and let $\Lambda := \gamma_H - \varepsilon$. Let us set

$$\nu_\infty^1 := \Lambda - \lambda_\infty^1, \quad \nu_\infty^2 := \Lambda - \lambda_\infty^2,$$

so that $\nu_\infty^1, \nu_\infty^2 > 0$. Let $0 < \eta < 1$ be such that

$$\lambda_\infty^2 < \nu_\infty^1 (1 - 2\eta) \quad \text{and} \quad \lambda_\infty^1 < \nu_\infty^2 (1 - 2\eta). \tag{7.3}$$

Let us choose $\bar{R} > 0$ large enough so that

$$\cup_{i=1}^{k_j} B'(a_i^j, r_i^j) \subset B'_{\bar{R}} \quad \text{for } j = 1, 2.$$

We observe that, by Theorem 6.3, there exists $\tilde{R}_j > 0$ such that

$$\mu \left(V_j + \frac{\nu_\infty^j}{|x|^{2s}} \chi_{\mathbb{R}^N \setminus B'_{\tilde{R}_j}} \right) > 0.$$

Since $\lambda_i^j > 0$ implies that $a_{\lambda_i^j} < 0$, we can fix some $\sigma > 0$ such that $\sigma < 2s$ and $\sigma < -a_{\lambda_i^j}$ for all $i = 1, \dots, k_j$, $j = 1, 2$. Let us consider, for $j = 1, 2$, $p_j \in C^\infty(\mathbb{R}^N \setminus \{a_1^j, \dots, a_{k_j}^j\}) \cap \mathcal{P}^\sigma_{a_1^j,\dots,a_{k_j}^j}$ such that $p_j(x) > 0$ for all $x \in \mathbb{R}^N$ and

$$p_j(x) \geq \frac{1}{|x - a_i^j|^{2s-\sigma}} \text{ if } x \in B'(a_i^j, r_i^j), \quad p_j(x) \geq 1 \text{ if } x \in B'_{\bar{R}} \setminus \cup_{i=1}^{k_j} B(a_i^j, r_i^j). \tag{7.4}$$

Since $p_j \in L^{\frac{N}{2s}}(\mathbb{R}^N)$ satisfies the hypotheses of Lemma 2.1 the infimum

$$\mu_j = \inf_{\substack{U \in \mathcal{D}^{1,2}(\mathbb{R}^{N+1}_+; t^{1-2s}) \\ \operatorname{Tr} U \not\equiv 0}} \frac{\int_{\mathbb{R}^{N+1}_+} t^{1-2s} |\nabla U|^2 \,\mathrm{d}t\,\mathrm{d}x - \kappa_s \int_{\mathbb{R}^N} \left[V_j + \frac{\nu_\infty^j}{|x|^{2s}} \chi_{\mathbb{R}^N \setminus B'_{\tilde{R}_j}} \right] |\operatorname{Tr} U|^2 \,\mathrm{d}x}{\int_{\mathbb{R}^N} p_j |\operatorname{Tr} U|^2 \,\mathrm{d}x} > 0$$

is achieved by some nonnegative $\Psi_j \in \mathcal{D}^{1,2}(\mathbb{R}^{N+1}_+; t^{1-2s})$, for $j = 1, 2$. In addition, Ψ_j weakly solves

$$\begin{cases} -\mathrm{div}(t^{1-2s}\nabla\Psi_j) = 0, & \text{in } \mathbb{R}^{N+1}_+, \\ -\lim\limits_{t\to0^+} t^{1-2s}\dfrac{\partial\Psi_j}{\partial t} = \kappa_s \left[V_j + \dfrac{\nu^j_\infty}{|x|^{2s}}\chi_{\mathbb{R}^N\setminus B'_{\tilde{R}_j}} + \mu_j p_j\right] \mathrm{Tr}\,\Psi_j, & \text{on } \mathbb{R}^N. \end{cases} \tag{7.5}$$

From Proposition A.1 we know that Ψ_j is locally Hölder continuous in $\overline{\mathbb{R}^{N+1}_+}\setminus\{(0,a^j_1),\dots,(0,a^j_{k_j})\}$).

In order to prove that Ψ_j is strictly positive in $\overline{\mathbb{R}^{N+1}_+}\setminus\{(0,a^j_1),\dots,(0,a^j_{k_j})\}$, we compare it with the unique weak solution $\widetilde{\Psi}_j \in \mathcal{D}^{1,2}(\mathbb{R}^{N+1}_+; t^{1-2s})$ to the problem

$$\begin{cases} -\mathrm{div}(t^{1-2s}\nabla\widetilde{\Psi}_j) = 0, & \text{in } \mathbb{R}^{N+1}_+, \\ -\lim\limits_{t\to0^+} t^{1-2s}\dfrac{\partial\widetilde{\Psi}_j}{\partial t} = \kappa_s V_j\,\mathrm{Tr}\,\widetilde{\Psi}_j + \kappa_s\mu_j p_j\,\mathrm{Tr}\,\Psi_j, & \text{on } \mathbb{R}^N, \end{cases} \tag{7.6}$$

whose existence directly follows from the Lax-Milgram Lemma. The difference $\widetilde{\Phi}_j = \Psi_j - \widetilde{\Psi}_j$ belongs to $\mathcal{D}^{1,2}(\mathbb{R}^{N+1}_+; t^{1-2s})$ and weakly solves

$$\begin{cases} -\mathrm{div}(t^{1-2s}\nabla\widetilde{\Phi}_j) = 0, & \text{in } \mathbb{R}^{N+1}_+, \\ -\lim\limits_{t\to0^+} t^{1-2s}\dfrac{\partial\widetilde{\Phi}_j}{\partial t} = \kappa_s V_j\,\mathrm{Tr}\,\widetilde{\Phi}_j + \kappa_s\dfrac{\nu^j_\infty}{|x|^{2s}}\chi_{\mathbb{R}^N\setminus B'_{\tilde{R}_j}}\,\mathrm{Tr}\,\Psi_j, & \text{on } \mathbb{R}^N. \end{cases}$$

By testing the above equation with $-\widetilde{\Phi}_j^-$ and recalling that $\mu(V_j) > 0$, we obtain that $\widetilde{\Phi}_j \geq 0$ in $\mathbb{R}^{N+1}_+$ and hence $\Psi_j \geq \widetilde{\Psi}_j$. Moreover, testing (7.6) with $-\widetilde{\Psi}_j^-$, we also obtain that $\widetilde{\Psi}_j \geq 0$ in $\mathbb{R}^{N+1}_+$. From the classical Strong Maximum Principle and Proposition A.2 (whose assumption (A.1) for (7.6) is satisfied thanks to Lemma A.3 and the assumption $V_j \in \Theta^*$) it follows that $\widetilde{\Psi}_j > 0$ in $\overline{\mathbb{R}^{N+1}_+}\setminus\{(0,a^j_1),\dots,(0,a^j_{k_j})\}$ and hence

$$\Psi_j > 0 \quad \text{in} \quad \overline{\mathbb{R}^{N+1}_+}\setminus\{(0,a^j_1),\dots,(0,a^j_{k_j})\}.$$

Lemma 2.5 yields

$$\lim_{|x|\to\infty} \Psi_j(0,x)|x|^{N-2s+a_\Lambda} = \ell_j > 0,$$

for some $\ell_j > 0$ (see (2.12) for the notation a_Λ). Hence, the function $\Phi_j(t,x) := \frac{\Psi_j(t,x)}{\ell_j}$ satisfies (7.5) and $\Phi_j(0,x) \sim |x|^{-(N-2s+a_\Lambda)}$ for $|x| \to \infty$. Therefore, there exists $\rho > \max\{\tilde{R}_1, \tilde{R}_2, \bar{R}\}$ such that

$$(1-\eta^2)|x|^{-(N-2s+a_\Lambda)} \leq \Phi_j(0,x) \leq (1+\eta)|x|^{-(N-2s+a_\Lambda)} \tag{7.7}$$

and

$$|W_1(x)| \leq \frac{\eta\nu^2_\infty}{|x|^{2s}}, \qquad |W_2(x)| \leq \frac{\eta\nu^1_\infty}{|x|^{2s}} \tag{7.8}$$

for all $x \in \mathbb{R}^N \setminus B'_\rho$. Also, form Lemma 2.4 we know that there exists $C > 0$ such that

$$\frac{1}{C}|x-a^j_i|^{a_{\lambda^j_i}} \leq \Phi_j(0,x) \leq C|x-a^j_i|^{a_{\lambda^j_i}} \quad \text{in } B'(a^j_i, r^j_i), \tag{7.9}$$

for $i = 1, \dots k_j$, $j = 1, 2$. For any $y \in \mathbb{R}^N$, we define

$$\Phi_y(t,x) := \nu^2_\infty\Phi_1(t,x) + \nu^1_\infty\Phi_2(t,x-y) \in \mathcal{D}^{1,2}(\mathbb{R}^{N+1}_+; t^{1-2s}).$$

Then

$$\int_{\mathbb{R}^{N+1}_+} t^{1-2s}\nabla\Phi_y\cdot\nabla U\,\mathrm{d}t\,\mathrm{d}x - \kappa_s\int_{\mathbb{R}^N}(V_1(x)+V_2(x-y))\,\mathrm{Tr}\,\Phi_y\,\mathrm{Tr}\,U\,\mathrm{d}x = \int_{\mathbb{R}^N} g_y(x)\,\mathrm{Tr}\,U\,\mathrm{d}x$$

for all $U \in \mathcal{D}^{1,2}(\mathbb{R}^{N+1}_+; t^{1-2s})$, where

$$g_y(x) := \kappa_s \bigg[\mu_1 \nu^2_\infty p_1(x) \Phi_1(0,x) + \frac{\nu^1_\infty \nu^2_\infty}{|x|^{2s}} \chi_{\mathbb{R}^N \setminus B'_{\tilde{R}_1}}(x) \Phi_1(0,x) + \mu_2 \nu^1_\infty p_2(x-y) \Phi_2(0,x-y)$$

$$+ \frac{\nu^1_\infty \nu^2_\infty}{|x-y|^{2s}} \chi_{\mathbb{R}^N \setminus B'(y,\tilde{R}_2)}(x) \Phi_2(0,x-y) - \nu^1_\infty V_1(x) \Phi_2(0,x-y) - \nu^2_\infty V_2(x-y) \Phi_1(0,x) \bigg].$$

Therefore, to conclude the proof it is enough to show that $g_y \geq 0$ a.e. in $\mathbb{R}^N$.

From (7.3), (7.7) and (7.8), it follows that in $\mathbb{R}^N \setminus \big(B'_\rho \cup B'(y,\rho)\big)$

$$\begin{aligned}
g_y(x) &\geq \kappa_s \bigg[\frac{\nu^1_\infty \nu^2_\infty}{|x|^{2s}} \Phi_1(0,x) + \frac{\nu^1_\infty \nu^2_\infty}{|x-y|^{2s}} \Phi_2(0,x-y) \\
&\quad - \nu^1_\infty \left(\frac{\lambda^1_\infty}{|x|^{2s}} + W_1(x) \right) \Phi_2(0,x-y) - \nu^2_\infty \left(\frac{\lambda^2_\infty}{|x-y|^{2s}} + W_2(x-y) \right) \Phi_1(0,x) \bigg] \\
&> \kappa_s \nu^1_\infty \nu^2_\infty (1-\eta^2) \Big[|x|^{-(N+a_\Lambda)} + |x-y|^{-(N+a_\Lambda)} \\
&\quad - |x|^{-(N-2s+a_\Lambda)} |x-y|^{-2s} - |x|^{-2s} |x-y|^{-(N-2s+a_\Lambda)} \Big] \\
&= \kappa_s \nu^1_\infty \nu^2_\infty (1-\eta^2) \left(\frac{1}{|x|^{N-2s+a_\Lambda}} - \frac{1}{|x-y|^{N-2s+a_\Lambda}} \right) \left(\frac{1}{|x|^{2s}} - \frac{1}{|x-y|^{2s}} \right) \geq 0.
\end{aligned}$$

For $|y| > R > 2\rho$, we have $B'_\rho \cap B'(y,\rho) = \emptyset$. From (7.4), (7.7), (7.8), (7.9) and the choice of σ we have that, in $B(a^1_i, r^1_i)$,

$$\begin{aligned}
g_y(x) &\geq \kappa_s \bigg[\mu_1 \nu^2_\infty p_1(x) \Phi_1(0,x) + \frac{\nu^1_\infty \nu^2_\infty}{|x-y|^{2s}} \Phi_2(0,x-y) \\
&\qquad - \nu^1_\infty V_1(x) \Phi_2(0,x-y) - \nu^2_\infty V_2(x-y) \Phi_1(0,x) \bigg] \\
&\geq \kappa_s |x-a^1_i|^{a_{\lambda^1_i} - 2s + \sigma} \bigg[\frac{\mu_1 \nu^2_\infty}{C} \\
&\qquad - \nu^1_\infty (1+\eta) |x-y|^{-(N-2s+a_\Lambda)} |x-a^1_i|^{-a_{\lambda^1_i} - \sigma} (\lambda^1_i + \|W_1\|_{L^\infty(\mathbb{R}^N)} |x-a^1_i|^{2s}) \\
&\qquad - \nu^2_\infty \nu^1_\infty (1-\eta) C |x-y|^{-2s} |x-a^1_i|^{2s-\sigma} \bigg] \\
&\geq \kappa_s |x-a^1_i|^{a_{\lambda^1_i} - 2s + \sigma} \left[\frac{\mu_1 \nu^2_\infty}{C} + o(1) \right],
\end{aligned}$$

as $|y| \to \infty$. Now let $x \in B'_\rho \setminus \Big(\cup_{i=1}^{k_1} B'(a^1_i, r^1_i)\Big)$: since Φ_1 is positive and continuous we have $\tilde{C}^{-1} > \Phi_1(0,x) > \tilde{C}$, for some $\tilde{C} > 0$, and so, thanks to (7.4), (7.7) and (7.8), there holds

$$g_y(x) \geq \kappa_s \mu_1 \nu^2_\infty \tilde{C} + o(1),$$

as $|y| \to \infty$. One can similarly prove that, for $|y|$ sufficiently large, $g_y(x) \geq 0$ in $B'(y,\rho)$ as well. The proof is thereby complete. □

Proof of Theorem 1.3. First, let

$$0 < \varepsilon < \min \left\{ 2S\mu(V_j), \frac{\mu(V_j)}{2} \left[\frac{1}{\gamma_H} \left(\sum_{i=1}^{k} |\lambda^j_i| + |\lambda^j_\infty| \right) + S^{-1} \|W\|_{L^{N/2s}(\mathbb{R}^N)} \right]^{-1} \right\} \tag{7.10}$$

for $j = 1,2$, such that, in addition, $\lambda^1_\infty + \lambda^2_\infty + \varepsilon(\lambda^1_\infty + \lambda^2_\infty) < \gamma_H$ and $\lambda^j_i + \varepsilon \lambda^j_i < \gamma_H$ for all $i = 1, \dots, k_j, \infty$. Similarly to (4.3), one can prove that $\mu(V_j + \varepsilon V_j) > \frac{\mu(V_j)}{2} > 0$ for $j = 1,2$. Moreover, let $\sigma = \sigma(\varepsilon)$ be such that

$$0 < \sigma < \min \left\{ \frac{S\mu(V_1)}{2}, \frac{S\mu(V_2)}{2}, \frac{S\varepsilon}{2} \right\}. \tag{7.11}$$

Let, for $j = 1, 2$,

$$\hat{V}_j(x) = \sum_{i=1}^{k_j} \frac{\lambda_i^j \chi_{B'(a_i^j, r_i^j)}(x)}{|x - a_i^j|^{2s}} + \frac{\lambda_\infty^j \chi_{\mathbb{R}^N \setminus B'_{R_j}}(x)}{|x|^{2s}} + \hat{W}_j(x) \in \Theta^*,$$

be such that $\hat{W}_j \in \mathcal{P}^\delta_{a_1^j, \dots, a_{k_j}^j}$ for some $\delta > 0$ and

$$\|\hat{V}_j - V_j\|_{L^{N/2s}(\mathbb{R}^N)} < \frac{\sigma}{1+\varepsilon}. \tag{7.12}$$

From Lemma 6.1, (7.11) and (7.12) we deduce that

$$\mu(\hat{V}_j + \varepsilon \hat{V}_j) \ge \mu(V_j + \varepsilon V_j) - (1+\varepsilon) S^{-1} \|\hat{V}_j - V_j\|_{L^{N/2s}(\mathbb{R}^N)} > 0.$$

Hence we infer from Lemma 7.1 that there exists $R > 0$ such that, for all $y \in \mathbb{R}^N \setminus \overline{B'_R}$, there exists $\Phi_y \in \mathcal{D}^{1,2}(\mathbb{R}^{N+1}_+; t^{1-2s})$ such that Φ_y is strictly positive and locally Hölder continuous in $\overline{\mathbb{R}^{N+1}_+} \setminus \{(0, a_i^1), (0, a_i^2 + y)\}_{i=1,\dots,k_j, j=1,2}$ and

$$\int_{\mathbb{R}^{N+1}_+} t^{1-2s} \nabla \Phi_y \cdot \nabla U \, \mathrm{d}t \, \mathrm{d}x - \kappa_s \int_{\mathbb{R}^N} \left[\hat{V}_1(x) + \varepsilon \hat{V}_1(x) + \hat{V}_2(x-y) + \varepsilon \hat{V}_2(x-y) \right] \operatorname{Tr} \Phi_y \operatorname{Tr} U \, \mathrm{d}x \ge 0$$

for all $U \in \mathcal{D}^{1,2}(\mathbb{R}^{N+1}_+; t^{1-2s})$, with $U \ge 0$ a.e. Therefore, thanks to the positivity criterion (Lemma 1.2), we know that

$$\mu(\hat{V}_1(\cdot) + \hat{V}_2(\cdot - y)) \ge \frac{\varepsilon}{\varepsilon + 1}.$$

Combining Lemma 6.1 with (7.11) and (7.12), we finally deduce that

$$\begin{aligned} \mu(V_1(\cdot) + V_2(\cdot - y)) &\ge \mu(\hat{V}_1(\cdot) + \hat{V}_2(\cdot - y)) \\ &\quad - S^{-1} \|V_1 - \hat{V}_1\|_{L^{N/2s}(\mathbb{R}^N)} - S^{-1} \|V_2(\cdot - y) - \hat{V}_2(\cdot - y)\|_{L^{N/2s}(\mathbb{R}^N)} > 0, \end{aligned}$$

thus completing the proof. □

8. Proof of Theorem 1.4

In order to prove Theorem 1.4, we first need the following lemma, concerning the left-hand side in Hardy inequality (1.4).

Lemma 8.1. *We have that*

$$\lim_{|\xi| \to 0} \int_{\mathbb{R}^N} \frac{|u(x)|^2}{|x+\xi|^{2s}} \, \mathrm{d}x = \int_{\mathbb{R}^N} \frac{|u(x)|^2}{|x|^{2s}} \, \mathrm{d}x \quad \text{and} \quad \lim_{|\xi| \to +\infty} \int_{\mathbb{R}^N} \frac{|u(x)|^2}{|x+\xi|^{2s}} \, \mathrm{d}x = 0$$

for any $u \in \mathcal{D}^{s,2}(\mathbb{R}^N)$.

Proof. The proof easily follows from density of $C_c^\infty(\mathbb{R}^N \setminus \{0\})$ in $\mathcal{D}^{s,2}(\mathbb{R}^N)$ (see Lemma A.4), the Dominated Convergence Theorem and the fractional Hardy inequality (1.4). □

We are now able to prove Theorem 1.4.

Proof of Theorem 1.4. First we prove that condition (1.15) is sufficient for the existence of at least one configuration of poles $a_1, \dots, a_k$ such that the quadratic form associated to $\mathcal{L}_{\lambda_1, \dots, \lambda_k, a_1, \dots, a_k}$ is positive definite. In order to do this, we argue by induction on the number of poles k. For any k we assume the masses to be sorted in increasing order $\lambda_1 \le \cdots \le \lambda_k$. If $k = 2$ the claim is proved in Remark 3.1. Suppose now the claim is proved for $k - 1$. If $\lambda_k \le 0$ the proof is trivial, so let us assume $\lambda_k > 0$: since (1.15) holds, it is true also for $\lambda_1, \dots, \lambda_{k-1}$, hence there exists a configuration of poles $a_1, \dots, a_{k-1}$ such that $Q_{\lambda_1, \dots, \lambda_{k-1}, a_1, \dots, a_{k-1}}$ is positive definite. If we let

$$V_1(x) = \sum_{i=1}^{k-1} \frac{\lambda_i}{|x - a_1|^{2s}} \quad \text{and} \quad V_2(x) = \frac{\lambda_k}{|x|^{2s}},$$

we have that $V_1, V_2 \in \Theta$ satisfy the assumptions of Theorem 1.3. Therefore there exists $a_k \in \mathbb{R}^N$ such that

$$\mathcal{L}_{\lambda_1,\dots,\lambda_k,a_1,\dots,a_k} = (-\Delta)^s - (V_1 + V_2(\cdot - a_k))$$

is positive definite. This concludes the first part.

We now prove the necessity of condition (1.15). Let $\varepsilon > 0$ be such that

$$\|u\|^2_{\mathcal{D}^{s,2}(\mathbb{R}^N)} - \sum_{i=1}^k \lambda_i \int_{\mathbb{R}^N} \frac{|u(x)|^2}{|x-a_i|^{2s}}\,\mathrm{d}x \geq \varepsilon \|u\|^2_{\mathcal{D}^{s,2}(\mathbb{R}^N)} \tag{8.1}$$

for all $u \in \mathcal{D}^{s,2}(\mathbb{R}^N)$ and let $\delta \in (0, \varepsilon\gamma_H)$. Assume by contradiction that $\lambda_j \geq \gamma_H$ for some $j \in \{1,\dots,k\}$. By optimality of γ_H in Hardy inequality (1.4) and by density of $C_c^\infty(\mathbb{R}^N)$ in $\mathcal{D}^{s,2}(\mathbb{R}^N)$, we have that there exists $\varphi \in C_c^\infty(\mathbb{R}^N)$ such that

$$\|\varphi\|^2_{\mathcal{D}^{s,2}(\mathbb{R}^N)} - \lambda_j \int_{\mathbb{R}^N} \frac{|\varphi(x)|^2}{|x|^{2s}}\,\mathrm{d}x < \delta \int_{\mathbb{R}^N} \frac{|\varphi(x)|^2}{|x|^{2s}}\,\mathrm{d}x. \tag{8.2}$$

If we let $\varphi_\rho := \rho^{-\frac{N-2s}{2}}\varphi(x/\rho)$, we have that

$$\begin{aligned} Q_{\lambda_1,\dots,\lambda_k,a_1,\dots,a_k}(\varphi_\rho(\cdot - a_j)) &= \|\varphi\|^2_{\mathcal{D}^{s,2}(\mathbb{R}^N)} - \lambda_j \int_{\mathbb{R}^N} \frac{|\varphi(x)|^2}{|x|^{2s}}\,\mathrm{d}x - \sum_{i\neq j} \lambda_i \int_{\mathbb{R}^N} \frac{|\varphi(x)|^2}{\left|x - \frac{a_i - a_j}{\rho}\right|^{2s}}\,\mathrm{d}x \\ &\to \|\varphi\|^2_{\mathcal{D}^{s,2}(\mathbb{R}^N)} - \lambda_j \int_{\mathbb{R}^N} \frac{|\varphi(x)|^2}{|x|^{2s}}\,\mathrm{d}x \qquad \text{as } \rho \to 0^+, \end{aligned} \tag{8.3}$$

in view of Lemma 8.1. Combining (8.1), (8.2), (8.3) and Hardy inequality (1.4) we obtain

$$\varepsilon \|\varphi\|^2_{\mathcal{D}^{s,2}(\mathbb{R}^N)} \leq \|\varphi\|^2_{\mathcal{D}^{s,2}(\mathbb{R}^N)} - \lambda_j \int_{\mathbb{R}^N} \frac{|\varphi(x)|^2}{|x|^{2s}}\,\mathrm{d}x < \delta \int_{\mathbb{R}^N} \frac{|\varphi(x)|^2}{|x|^{2s}}\,\mathrm{d}x \leq \frac{\delta}{\gamma_H} \|\varphi\|^2_{\mathcal{D}^{s,2}(\mathbb{R}^N)},$$

which is a contradiction, because of the choice of δ.

Now suppose that $K := \sum_{i=1}^k \lambda_i \geq \gamma_H$. Arguing analogously, there exists $\varphi \in C_c^\infty(\mathbb{R}^N)$ such that

$$\|\varphi\|^2_{\mathcal{D}^{s,2}(\mathbb{R}^N)} - K \int_{\mathbb{R}^N} \frac{|\varphi(x)|^2}{|x|^{2s}}\,\mathrm{d}x < \delta \int_{\mathbb{R}^N} \frac{|\varphi(x)|^2}{|x|^{2s}}\,\mathrm{d}x.$$

The function $\varphi_\rho(x) := \rho^{-\frac{N-2s}{2}}\varphi(x/\rho)$ satisfies

$$\begin{aligned} Q_{\lambda_1,\dots,\lambda_k,a_1,\dots,a_k}(\varphi_\rho) &= \|\varphi\|^2_{\mathcal{D}^{s,2}(\mathbb{R}^N)} - \sum_{i=1}^k \lambda_i \int_{\mathbb{R}^N} \frac{|\varphi(x)|^2}{|x - a_i/\rho|^{2s}}\,\mathrm{d}x \\ &\to \|\varphi\|^2_{\mathcal{D}^{s,2}(\mathbb{R}^N)} - K \int_{\mathbb{R}^N} \frac{|\varphi(x)|^2}{|x|^{2s}}\,\mathrm{d}x \qquad \text{as } \rho \to +\infty, \end{aligned}$$

thanks to Lemma 8.1. With the same argument as above, we again reach a contradiction. □

9. Proof of Proposition 1.6

Finally, in this section we present the proof of Proposition 1.6, that is independent of the previous results from the point of view of the technical approach.

Proof of Proposition 1.6. First, let us denote $\bar\lambda = \max\{0, \lambda_1, \dots, \lambda_k, \lambda_\infty\}$. By hypothesis there exists $\alpha \in \left(0, 1 - \frac{\bar\lambda}{\gamma_H}\right)$ such that $\mu(V) \leq 1 - \frac{\bar\lambda}{\gamma_H} - \alpha$. From Lemma 5.2 we know that there exists $\delta > 0$ such that, denoting by

$$\bar V = \sum_{i=1}^k \frac{\lambda_i \zeta(\frac{x - a_i}{\delta})}{|x - a_i|^{2s}} + \frac{\lambda_\infty \tilde\zeta(\frac{x}{R})}{|x|^{2s}},$$

with $\zeta, \tilde{\zeta}$ being as in Lemma 5.2, we have that

$$\mu(\bar{V}) \geq 1 - \frac{\bar{\lambda}}{\gamma_H} - \frac{\alpha}{2}. \tag{9.1}$$

if $\bar{\lambda} > 0$ and $\mu(\bar{V}) \geq 1$ if $\bar{\lambda} = 0$. We can write $V = \bar{V} + \bar{W}$ for some $\bar{W} \in L^{N/2s}(\mathbb{R}^N)$. Now let $\{U_n\}_n \subseteq \mathcal{D}^{1,2}(\mathbb{R}^{N+1}_+; t^{1-2s})$ be a minimizing sequence for $\mu(V)$, i.e.

$$\int_{\mathbb{R}^{N+1}_+} t^{1-2s} |\nabla U_n|^2 \, \mathrm{d}t \, \mathrm{d}x - \kappa_s \int_{\mathbb{R}^N} V \left|\mathrm{Tr}\, U_n\right|^2 \, \mathrm{d}x = \mu(V) + o(1), \quad \text{as } n \to \infty \tag{9.2}$$

and $\int_{\mathbb{R}^{N+1}_+} t^{1-2s} |\nabla U_n|^2 \, \mathrm{d}t \, \mathrm{d}x = 1$. Since $\{U_n\}_n$ is bounded in $\mathcal{D}^{1,2}(\mathbb{R}^{N+1}_+; t^{1-2s})$, there exists $U \in \mathcal{D}^{1,2}(\mathbb{R}^{N+1}_+; t^{1-2s})$ such that, up to a subsequence (still denoted by $\{U_n\}_n$),

$$U_n \rightharpoonup U \quad \text{weakly in } \mathcal{D}^{1,2}(\mathbb{R}^{N+1}_+; t^{1-2s}) \quad \text{and} \quad U_n \to U \quad \text{a.e. in } \mathbb{R}^{N+1}_+, \tag{9.3}$$

as $n \to \infty$. There holds

$$\begin{aligned} \mu(\bar{V}) &\leq \int_{\mathbb{R}^N} t^{1-2s} |\nabla U_n|^2 \, \mathrm{d}t \, \mathrm{d}x - \kappa_s \int_{\mathbb{R}^N} V \left|\mathrm{Tr}\, U_n\right|^2 \, \mathrm{d}x + \kappa_s \int_{\mathbb{R}^N} \bar{W} \left|\mathrm{Tr}\, U_n\right|^2 \, \mathrm{d}x \\ &= \mu(V) + \kappa_s \int_{\mathbb{R}^N} \bar{W} \left|\mathrm{Tr}\, U_n\right|^2 \, \mathrm{d}x + o(1), \quad \text{as } n \to \infty. \end{aligned}$$

Hence, from (9.3), (9.1), the choice of α, and Lemma 2.1 we deduce that (if $\bar{\lambda} > 0$)

$$1 - \frac{1}{\gamma_H}\bar{\lambda} - \frac{\alpha}{2} \leq \mu(V) + \kappa_s \int_{\mathbb{R}^N} \bar{W} \left|\mathrm{Tr}\, U\right|^2 \, \mathrm{d}x \leq 1 - \frac{\bar{\lambda}}{\gamma_H} - \alpha + \kappa_s \int_{\mathbb{R}^N} \bar{W} \left|\mathrm{Tr}\, U\right|^2 \, \mathrm{d}x,$$

and so $\kappa_s \int_{\mathbb{R}^N} \bar{W} \left|\mathrm{Tr}\, U\right|^2 \, \mathrm{d}x \geq \frac{\alpha}{2} > 0$, which implies that $U \not\equiv 0$. The same conclusion easily follows in the case $\bar{\lambda} = 0$. From the weak convergence $U_n \rightharpoonup U$ in $\mathcal{D}^{1,2}(\mathbb{R}^{N+1}_+; t^{1-2s})$, the continuity of the trace map $\mathrm{Tr} : \mathcal{D}^{1,2}(\mathbb{R}^{N+1}_+; t^{1-2s}) \to L^{N/2s}(\mathbb{R}^N)$ and the definition of $\mu(V)$, we have that

$$\begin{aligned} \mu(V) &\leq \frac{\displaystyle\int_{\mathbb{R}^{N+1}_+} t^{1-2s} |\nabla U|^2 \, \mathrm{d}t \, \mathrm{d}x - \kappa_s \int_{\mathbb{R}^N} V \left|\mathrm{Tr}\, U\right|^2 \, \mathrm{d}x}{\displaystyle\int_{\mathbb{R}^{N+1}_+} t^{1-2s} |\nabla U|^2 \, \mathrm{d}t \, \mathrm{d}x} \\ &= \frac{\mu(V) - \left[\displaystyle\int_{\mathbb{R}^{N+1}_+} t^{1-2s} |\nabla (U_n - U)|^2 \, \mathrm{d}t \, \mathrm{d}x - \kappa_s \int_{\mathbb{R}^N} V \left|\mathrm{Tr}\, U_n - \mathrm{Tr}\, U\right|^2 \, \mathrm{d}x\right] + o(1)}{\displaystyle\int_{\mathbb{R}^{N+1}_+} t^{1-2s} |\nabla U_n|^2 \, \mathrm{d}t \, \mathrm{d}x - \int_{\mathbb{R}^{N+1}_+} t^{1-2s} |\nabla (U_n - U)|^2 \, \mathrm{d}t \, \mathrm{d}x + o(1)} \\ &\leq \mu(V) \, \frac{1 - \displaystyle\int_{\mathbb{R}^{N+1}_+} t^{1-2s} |\nabla (U_n - U)|^2 \, \mathrm{d}t \, \mathrm{d}x + o(1)}{1 - \displaystyle\int_{\mathbb{R}^{N+1}_+} t^{1-2s} |\nabla (U_n - U)|^2 \, \mathrm{d}t \, \mathrm{d}x + o(1)} = \mu(V) + \frac{o(1)}{\displaystyle\int_{\mathbb{R}^{N+1}_+} t^{1-2s} |\nabla U|^2 \, \mathrm{d}t \, \mathrm{d}x + o(1)}. \end{aligned}$$

Letting $n \to \infty$ yields the fact that $\mu(V)$ is attained by U and this concludes the proof. □

Appendix A.

In this appendix, we recall some known results about properties of solutions to equations on the extended, positive half-space.

We start by recalling a regularity result.

Proposition A.1 ([17] Proposition 3, [27] Proposition 2.6)**.** *Let $a, b \in L^p(B_1')$, for some $p > \frac{N}{2s}$ and $c, d \in L^q(B_1^+; t^{1-2s})$, for some $q > \frac{N+2-2s}{2}$. Let $w \in H^1(B_1^+; t^{1-2s})$ be a weak solution of*

$$\begin{cases} -\mathrm{div}(t^{1-2s} \nabla w) + t^{1-2s} c(z) w = t^{1-2s} d(z), & \text{in } B_1^+, \\ -\lim_{t \to 0^+} t^{1-2s} \frac{\partial w}{\partial t} = a(x) w + b(x), & \text{on } B_1'. \end{cases}$$

Then $w \in C^{0,\beta}(\overline{B^+_{1/2}})$ *and in addition*

$$\|w\|_{C^{0,\beta}(\overline{B^+_{1/2}})} \le C\left(\|w\|_{L^2(B_1^+)} + \|b\|_{L^p(B_1')} + \|d\|_{L^q(B_1^+;t^{1-2s})}\right),$$

with $C, \beta > 0$ *depending only on* $N, s, \|a\|_{L^p(B_1')}, \|c\|_{L^q(B_1^+;t^{1-2s})}$.

Now we recall, from [8], an Hopf-type Lemma.

Proposition A.2 ([8] Proposition 4.11)**.** *Let* $\Phi \in C^0(B_R^+ \cup B_R') \cap H^1(B_R^+; t^{1-2s})$ *satisfy*

$$\begin{cases} -\operatorname{div}(t^{1-2s}\nabla\Phi) \ge 0, & \text{in } B_R^+, \\ \Phi > 0, & \text{in } B_R^+, \\ \Phi(0,0) = 0. \end{cases}$$

Then

$$-\limsup_{t\to 0^+} t^{1-2s}\frac{\Phi(t,0)}{t} < 0.$$

In addition, if

$$t^{1-2s}\frac{\partial\Phi}{\partial t} \in C^0(B_R^+ \cup B_R'), \tag{A.1}$$

then

$$-\left(t^{1-2s}\frac{\partial\Phi}{\partial t}\right)(0,0) < 0.$$

In several points of the present paper we used the following result from [8] to verify the validity of assumption (A.1) needed to apply Proposition A.2.

Lemma A.3 ([8] Lemma 4.5)**.** *Let* $s \in (0,1)$ *and* $R > 0$. *Let* $\varphi \in C^{0,\sigma}(B'_{2R})$ *for some* $\sigma \in (0,1)$ *and* $\Phi \in L^\infty(B^+_{2R}) \cap H^1(B^+_{2R}; t^{1-2s})$ *be a weak solution to*

$$\begin{cases} -\operatorname{div}(t^{1-2s}\nabla\Phi) = 0, & \text{in } B^+_{2R}, \\ -\lim_{t\to 0^+} t^{1-2s}\frac{\partial\Phi}{\partial t} = \varphi(x), & \text{on } B'_{2R}. \end{cases}$$

Then there exists $\beta \in (0,1)$ *depending only on* N, s, σ *such that*

$$\Phi \in C^{0,\beta}(\overline{B_R^+}) \quad \text{and} \quad t^{1-2s}\frac{\partial\Phi}{\partial t} \in C^{0,\beta}(\overline{B_R^+}).$$

Finally, we prove a density result: the idea behind is that removing a point does not impair the definition of $\mathcal{D}^{1,2}(\mathbb{R}^{N+1}_+; t^{1-2s})$ and $\mathcal{D}^{s,2}(\mathbb{R}^N)$; in other words, a point in $\mathbb{R}^N$ has null fractional s-capacity if $N > 2s$, see also [1, Example 2.5].

Lemma A.4. *Let* $z_0 \in \overline{\mathbb{R}^{N+1}_+}$, $N > 2s$. *Then* $C_c^\infty(\overline{\mathbb{R}^{N+1}_+} \setminus \{z_0\})$ *is dense in* $\mathcal{D}^{1,2}(\mathbb{R}^{N+1}_+; t^{1-2s})$. *As a consequence, if* $x_0 \in \mathbb{R}^N$, *then* $C_c^\infty(\mathbb{R}^N \setminus \{x_0\})$ *is dense in* $\mathcal{D}^{s,2}(\mathbb{R}^N)$.

Proof. Assume $z_0 \in \partial\overline{\mathbb{R}^{N+1}_+} = \mathbb{R}^N$ (the proof is completely analogous if $z_0 \in \mathbb{R}^{N+1}_+$). Moreover, without loss of generality, we can assume $z_0 = 0$. Let $U \in C_c^\infty(\overline{\mathbb{R}^{N+1}_+})$ and let $\xi_n \in C^\infty(\overline{\mathbb{R}^{N+1}_+})$ be a cut-off function such that

$$\xi_n(z) = \begin{cases} 1, & \text{if } z \in \overline{\mathbb{R}^{N+1}_+ \setminus B^+_{2/n}}, \\ 0, & \text{if } z \in \overline{B^+_{1/n}}, \end{cases}$$

$$\xi_n \quad \text{is radial, i.e. } \xi_n(z) = \xi_n(|z|), \qquad |\xi_n| \le 1, \quad |\nabla\xi_n| \le 2n.$$

Trivially $\xi_n U \in C_c^\infty(\overline{\mathbb{R}^{N+1}_+} \setminus \{0\})$. We claim that $\xi_n U \to U$ in $\mathcal{D}^{1,2}(\mathbb{R}^{N+1}_+; t^{1-2s})$. Indeed, thanks to Dominated Convergence Theorem,

$$\int_{\mathbb{R}^{N+1}_+} t^{1-2s} \left|\nabla((\xi_n - 1)U)\right|^2 \, \mathrm{d}t \, \mathrm{d}x$$

$$\leq 2 \int_{\mathbb{R}^{N+1}_+} t^{1-2s} \left|\xi_n - 1\right|^2 \left|\nabla U\right|^2 \, \mathrm{d}t \, \mathrm{d}x + 2 \int_{\mathbb{R}^{N+1}_+} t^{1-2s} \left|U\right|^2 \left|\nabla \xi_n\right|^2 \, \mathrm{d}t \, \mathrm{d}x$$

$$\leq o(1) + Cn^2 \int_{B^+_{2/n} \setminus B^+_{1/n}} t^{1-2s} \, \mathrm{d}t \, \mathrm{d}x.$$

Moreover

$$n^2 \int_{B^+_{2/n} \setminus B^+_{1/n}} t^{1-2s} \, \mathrm{d}t \, \mathrm{d}x = O(n^{2s-N}),$$

which concludes the proof of the claim, in view of the assumption $N > 2s$ and the density of $C_c^\infty(\overline{\mathbb{R}^{N+1}_+})$ in $\mathcal{D}^{1,2}(\mathbb{R}^{N+1}_+; t^{1-2s})$.

For what concerns the second statement, as before, without loss of generality, we can assume $x_0 = 0$. Let $u \in \mathcal{D}^{s,2}(\mathbb{R}^N)$ and let $U \in \mathcal{D}^{1,2}(\mathbb{R}^{N+1}_+; t^{1-2s})$ be its extension. By the density of $C_c^\infty(\overline{\mathbb{R}^{N+1}_+} \setminus \{0\})$ in $\mathcal{D}^{1,2}(\mathbb{R}^{N+1}_+; t^{1-2s})$ just proved, there exists a sequence $\{U_n\} \subset C_c^\infty(\overline{\mathbb{R}^{N+1}_+} \setminus \{0\})$ such that $U_n \to U$ in $\mathcal{D}^{1,2}(\mathbb{R}^{N+1}_+; t^{1-2s})$. Then $\mathrm{Tr}(U_n) \in C_c^\infty(\mathbb{R}^N \setminus \{0\})$ and $\mathrm{Tr}(U_n) \to \mathrm{Tr}\, U = u$ in $\mathcal{D}^{s,2}(\mathbb{R}^N)$, thanks to the continuity of the trace map $\mathrm{Tr}\colon \mathcal{D}^{1,2}(\mathbb{R}^{N+1}_+; t^{1-2s}) \to \mathcal{D}^{s,2}(\mathbb{R}^N)$. □

Acknowledgments. V. Felli is partially supported by the PRIN2015 grant "Variational methods, with applications to problems in mathematical physics and geometry". D. Mukherjee's research is supported by the Czech Science Foundation, project GJ19–14413Y. V. Felli and R. Ognibene are partially supported by the INDAM-GNAMPA 2018 grant "Formula di monotonia e applicazioni: problemi frazionari e stabilità spettrale rispetto a perturbazioni del dominio". This work was started while D. Mukherjee was visiting the University of Milano - Bicocca supported by INDAM-GNAMPA.

References

[1] Abatangelo, L., Felli, V., and Noris, B. On simple eigenvalues of the fractional Laplacian under removal of small fractional capacity sets. *Communications in Contemporary Mathematics*, to appear.

[2] Agmon, S. On positivity and decay of solutions of second order elliptic equations on Riemannian manifolds. In *Methods of functional analysis and theory of elliptic equations (Naples, 1982)*. Liguori, Naples, 1983, pp. 19–52.

[3] Agmon, S. Bounds on exponential decay of eigenfunctions of Schrödinger operators. In *Schrödinger operators (Como, 1984)*, vol. 1159 of *Lecture Notes in Math.* Springer, Berlin, 1985, pp. 1–38.

[4] Allegretto, W. On the equivalence of two types of oscillation for elliptic operators. *Pacific J. Math. 55* (1974), 319–328.

[5] Berchio, E., Ganguly, D., and Grillo, G. Improved multipolar Poincaré-Hardy inequalities on Cartan-Hadamard manifolds. *Annali di Matematica Pura ed Applicata* (2019).

[6] Bosi, R., Dolbeault, J., and Esteban, M. J. Estimates for the optimal constants in multipolar Hardy inequalities for Schrödinger and Dirac operators. *Commun. Pure Appl. Anal. 7*, 3 (2008), 533–562.

[7] Brändle, C., Colorado, E., de Pablo, A., and Sánchez, U. A concave-convex elliptic problem involving the fractional Laplacian. *Proc. Roy. Soc. Edinburgh Sect. A 143*, 1 (2013), 39–71.

[8] Cabré, X., and Sire, Y. Nonlinear equations for fractional Laplacians, I: Regularity, maximum principles, and Hamiltonian estimates. *Ann. Inst. H. Poincaré Anal. Non Linéaire 31*, 1 (2014), 23–53.

[9] Caffarelli, L., and Silvestre, L. An extension problem related to the fractional Laplacian. *Comm. Partial Differential Equations 32*, 7-9 (2007), 1245–1260.

[10] Cazacu, C., and Zuazua, E. Improved multipolar Hardy inequalities. In *Studies in phase space analysis with applications to PDEs*, vol. 84 of *Progr. Nonlinear Differential Equations Appl.* Birkhäuser/Springer, New York, 2013, pp. 35–52.

[11] de Almeida, M. F., Ferreira, L. C. F., and Precioso, J. C. On the heat equation with nonlinearity and singular anisotropic potential on the boundary. *Potential Anal. 46*, 3 (2017), 589–608.

[12] Di Nezza, E., Palatucci, G., and Valdinoci, E. Hitchhiker's guide to the fractional Sobolev spaces. *Bull. Sci. Math. 136*, 5 (2012), 521–573.

[13] Fabes, E. B., Garofalo, N., and Lin, F.-H. A partial answer to a conjecture of B. Simon concerning unique continuation. *J. Funct. Anal. 88*, 1 (1990), 194–210.
[14] Fabes, E. B., Kenig, C. E., and Serapioni, R. P. The local regularity of solutions of degenerate elliptic equations. *Comm. Partial Differential Equations 7*, 1 (1982), 77–116.
[15] Fall, M. M. Semilinear elliptic equations for the fractional Laplacian with Hardy potential. *Nonlinear Analysis* (2018).
[16] Fall, M. M., and Felli, V. Unique continuation property and local asymptotics of solutions to fractional elliptic equations. *Comm. Partial Differential Equations 39*, 2 (2014), 354–397.
[17] Fall, M. M., and Felli, V. Unique continuation properties for relativistic Schrödinger operators with a singular potential. *Discrete Contin. Dyn. Syst. 35*, 12 (2015), 5827–5867.
[18] Fall, M. M., and Weth, T. Nonexistence results for a class of fractional elliptic boundary value problems. *J. Funct. Anal. 263*, 8 (2012), 2205–2227.
[19] Faraci, F., Farkas, C., and Kristály, A. Multipolar Hardy inequalities on Riemannian manifolds. *ESAIM Control Optim. Calc. Var. 24*, 2 (2018), 551–567.
[20] Felli, V., and Ferrero, A. Unique continuation principles for a higher order fractional Laplace equation. *Preprint 2018,* arXiv:1809.09496.
[21] Felli, V., Marchini, E. M., and Terracini, S. On Schrödinger operators with multipolar inverse-square potentials. *J. Funct. Anal. 250*, 2 (2007), 265–316.
[22] Felli, V., and Terracini, S. Elliptic equations with multi-singular inverse-square potentials and critical nonlinearity. *Comm. Partial Differential Equations 31*, 1-3 (2006), 469–495.
[23] Ferreira, L. C. F., and Castañeda Centurión, N. F. A Fourier analysis approach to elliptic equations with critical potentials and nonlinear derivative terms. *Milan J. Math. 85*, 2 (2017), 187–213.
[24] Ferreira, L. C. F., and Mesquita, C. A. A. S. Existence and symmetries for elliptic equations with multipolar potentials and polyharmonic operators. *Indiana Univ. Math. J. 62*, 6 (2013), 1955–1982.
[25] Frank, R. L., Lieb, E. H., and Seiringer, R. Hardy-Lieb-Thirring inequalities for fractional Schrödinger operators. *J. Amer. Math. Soc. 21*, 4 (2008), 925–950.
[26] Herbst, I. W. Spectral theory of the operator $(p^2 + m^2)^{1/2} - Ze^2/r$. *Comm. Math. Phys. 53*, 3 (1977), 285–294.
[27] Jin, T., Li, Y., and Xiong, J. On a fractional Nirenberg problem, part I: blow up analysis and compactness of solutions. *J. Eur. Math. Soc. (JEMS) 16*, 6 (2014), 1111–1171.
[28] Moroz, V., and Van Schaftingen, J. Nonlocal Hardy type inequalities with optimal constants and remainder terms. *Ann. Univ. Buchar. Math. Ser. 3(LXI)*, 2 (2012), 187–200.
[29] Moss, W. F., and Piepenbrink, J. Positive solutions of elliptic equations. *Pacific J. Math. 75*, 1 (1978), 219–226.
[30] Opic, B., and Kufner, A. *Hardy-type inequalities*, vol. 219 of *Pitman Research Notes in Mathematics Series.* Longman Scientific & Technical, Harlow, 1990.
[31] Piepenbrink, J. Nonoscillatory elliptic equations. *J. Differential Equations 15* (1974), 541–550.
[32] Pinchover, Y. On the localization of binding for Schrödinger operators and its extension to elliptic operators. *J. Anal. Math. 66* (1995), 57–83.
[33] Pinchover, Y., and Psaradakis, G. On positive solutions of the (p, A)-Laplacian with potential in Morrey space. *Anal. PDE 9*, 6 (2016), 1317–1358.
[34] Salsa, S. *Partial differential equations in action*, third ed., vol. 99 of *Unitext.* Springer, [Cham], 2016.
[35] Simon, B. Brownian motion, L^p properties of Schrödinger operators and the localization of binding. *J. Funct. Anal. 35*, 2 (1980), 215–229.

Veronica Felli
Dipartimento di Matematica e Applicazioni, Università degli Studi di Milano-Bicocca,
Via Cozzi 55, 20125 Milano, Italy.

E-mail address: veronica.felli@unimib.it

Debangana Mukherjee
Department of Mathematics and Statistics, Masaryk University,
Kotlářská 267/2, 611 37 Brno, Czech Republic.

E-mail address: mukherjeed@math.muni.cz

Roberto Ognibene
Dipartimento di Matematica e Applicazioni, Università degli Studi di Milano-Bicocca,
Via Cozzi 55, 20125 Milano, Italy.

E-mail address: roberto.ognibene@unimib.it