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Aspects of Compactifications and Dualities in Superconformal Theories

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Abstract. This thesis focuses on various non-perturbative aspects of supersymmetric gauge theories in dimensions 2,3 and 4 and several constructions that relate properties of superconformal quantum field theories among different dimensionalities. Various techniques have been applied the most prominent of them being dimensional reduction and compactifications with decorations of the internal manifolds.

The thesis deals with two main topics; the first is the study of compactifications of 4d superconformal field theories placed on a Riemann surface with a particular choice of background data along the internal dimensions, the choice of which is imposed by the requirement that supersymmetry is unbroken by the curved geometry. An extensive analysis of this construction is carried out at the formal level for any genus both for minimal and extended supersymmetry. The results obtained provide a systematic classification of all 2d theories that can be constructed via this technique. We then apply the results to the study of several specific 4d models. By restricting to a special class of theories endowed with a toric structure on their moduli space we are able to show a direct connection between the toric geometry and the explicit form of the 2d central charge and anomalies.

The second topic is the study of circle compactifications of 4d dualities. We consider the reduction of Seiberg duality and its generalizations to SQCD with symplectic gauge group and adjoint plus fundamental matter fields. A remarkable property of these 4d theories, called E_7 surprise, carries over to 3d and it is shown to be responsible for the appearance in the infrared theory of a pattern of duality and global symmetry enhancement. We conjecture the existence of such IR fixed points and support our claim of the 3d dualities by providing explicit checks of the 3d partition functions computed via supersymmetric localization. Finally, we obtain similar results for theories with power-law superpotentials for the antisymmetric tensor field as well as confining theories with 6 fundamentals.

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Declaration

I declare that the thesis has been composed by myself and that the work has not be submitted for any other degree or professional qualification. I confirm that the work submitted is my own, except where work which has formed part of jointly-authored publications has been included. I confirm that appropriate credit has been given within this thesis where reference has been made to the work of others.

The work presented in this thesis was previously published in the following research articles:

1) A. Amariti, L. Cassia and S. Penati, *Surveying 4d SCFTs twisted on Riemann surfaces*, *JHEP* **06** (2017) 056 [1703.08201].
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3) A. Amariti and L. Cassia, *$USp(2N_c)$ $SQCD_3$ with antisymmetric: dualities and symmetry enhancements*, 1809.03796.

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CHAPTER 1

Introduction

Superconformal field theories (SCFTs) in low dimensions are the testing ground for the observation of many phenomena in quantum systems. In two dimensions they play a central role in the worldsheet description of string theory and in higher dimensions they describe the worldvolume theories on branes in type II string theory and M-theory. With the advent of the AdS/CFT correspondence it was also realized that they are of fundamental importance in describing the dual supergravity solutions in the bulk of the AdS geometry. With respect to their position in the landscape of all quantum field theories it is clear that they constitute a very special subclass because of the rich structure they exhibit due to the interplay of supersymmetry and conformal invariance. Nevertheless one can hope to obtain very general non-perturbative results on the much larger class of non-conformal and non-supersymmetric quantum field theories (QFTs) by first taking advantage of the exact computational tools available in this context. One example of this paradigm is the recent development of non-supersymmetric 3d IR dualities which in many cases are obtained by studying the parallel example of supersymmetric duality for which the partition function can be computed exactly via localization.

The derivation of such results is still a difficult task in general and finding new insight into the nature of (super)conformal theories is far from straightforward. A classification of superconformal theories is not known in general, especially in lower dimensions where less constraints are imposed by supersymmetry. It is therefore an open problem to find new ways to explore the space of such theories and find possible relations between the known ones.

A classical approach to the problem is the engineering of Lagrangian models with manifest supersymmetry which possess an IR fixed point along their RG flow. While Lagrangian constructions usually yield more tractable theories at the analytical level, they are hard to find in the first place and it is sometimes conjectured that they form a negligible set inside the space of all possible theories. For this reason it is interesting to study alternative constructions. One example that we are going to explore in this thesis consists in constructing novel supersymmetric models by compactifications of known ones. Compactification reduces the number of dimensions and in general breaks supersymmetry but with the appropriate choice of decorations of the internal manifold one can still end up with theories where much of the symmetry is preserved and many computations can be performed by reduction of the information about the 4d parent. This strategy gives rise to a way to make contact between theories in different dimensions, which seem otherwise completely independent of each other and to manifest wildly different phenomena.

Another aspect we already mentioned is the possible existence of many dual descriptions for the same theory. It is in fact observed that some theories admit two or more Lagrangian formulations in which the fundamental degrees of freedom are different across the dual phases. A prominent example of this phenomenon is Seiberg duality which itself can be viewed as a generalization of electro-magnetic duality. With the help of supersymmetric localization then one can analytically verify the equivalence of the different descriptions by showing the identity between the corresponding partition functions (or indices).

In this dissertation we consider the question of reduction of such dualities from 4d to 3d and study the way the lower dimensional dynamics must be modified in order to preserve the duality. The main example in this case will be that of supersymmetric QCD (SQCD) with symplectic gauge groups and various types of matter representations.

In the following sections we sketch the main ideas contained in this thesis.

1.1. Topological twist

SCFTs in 2d can be obtained by reducing 4d SCFTs on compact 2d manifolds. The first step of the compactification consists in putting the theory on a curved background of the form $\mathbb{R}^{1,1} \times \Sigma$ where Σ is a compact, closed Riemann surface of arbitrary genus g. The spin connection ω in general has non-trivial holonomy on Σ and this means that fields that are coupled to it (i.e. vectors, spinors etc.) might transform non-trivially under supersymmetry transformations. In particular, the supersymmetry variation of the gravitino field, i.e., the (odd) 1-form connection associated to the Q-supercharges, contains the covariant derivative of the supersymmetry parameters. These parameters are sections of the spinor bundle and as such they couple to ω. If supersymmetry with respect to those parameters is to be preserved then they must be covariantly constant so that the gravitino variation vanishes. Spinors with this property are said to be Killing and usually only exist on manifold with reduced holonomy, which is not the case for Σ. However, as suggested in [**1**] (see also [**2, 3**]), if spinors are also charged under other global symmetries then one can introduce a twist of the spinor bundle which reduces the structure group and allows us to find non-zero covariantly constant sections. Let us see what this means in detail: when a background connection A for a global symmetry, i.e., R-symmetry, is turned on, the spinor supercharges become sections of the tensor product of the spinor bundle and some vector bundle associated to that connection. Holonomies with respect to A and ω are generically non-trivial however there might be a common subgroup of both the spin and R-symmetry groups such that there are some sections that transform in opposite ways under it. If this is the case, the structure group of the product bundle can be restricted to that subgroup and the bundle decomposes into the direct sum of vector bundles associated to smaller irreducible representations of that subgroup. If the trivial representation appears in the decomposition, the associated sub-bundle admits non-zero flat sections, i.e., (charged) Killing spinors.

For example, in the case of a Riemann surface, the structure group of the spinor bundle is the spin group $Spin(2) \cong U(1)$, hence it is a complex line bundle $\mathcal{L}_{\pm}^{\text{spin}}$ with abelian connection ω associated to a 1-dimensional representation of charge ± 1 (according to chirality). If we assume the existence of a R-symmetry $U(1)_R$ such that supercharges also have charge ± 1 under it, then there is a second line bundle $\mathcal{L}_{\pm}^{R}$ such that:

$$Q \in \Gamma(\mathcal{L}_{+}^{\text{spin}} \otimes \mathcal{L}_{+}^{R}) \qquad \bar{Q} \in \Gamma(\mathcal{L}_{-}^{\text{spin}} \otimes \mathcal{L}_{-}^{R}) \tag{1.1}$$

The tensor product bundles have structure group $Spin(2) \times U(1)_R$ and connection $\omega + A$. If we restrict to the subgroup generated by elements of the form $(\mathrm{e}^{\mathrm{i}\theta}, \mathrm{e}^{-\mathrm{i}\theta})$ and we choose $A = -\omega$, then the product bundle becomes topologically trivial and flat. All holonomies identically vanish and covariantly constant spinors can be found.

More generally, one can turn on fluxes for any global symmetry, also the flavor ones. In this case preserving supersymmetry also requires to set to zero the associated gaugino variations. These backgrounds are most naturally modeled by non-dynamical (conformal) supergravity multiplets for the appropriate amount of supersymmetry. These are the main tools that we use in Chapter 2 to study the reduction of 4d theories on Σ. While the cases of $\mathcal{N} = 1, 2$ are well known in the literature we provide a complete classification of the solutions to the topologically twisted Killing spinor equations also for the $\mathcal{N} = 3, 4$ cases, for which new solutions are found. The Lagrangian of the resulting theories are not known in general and, even though this procedure does not allow to extract the matter content of the 2d theory, useful information on its IR behavior is given by the 2d global anomalies that can be obtained in terms of the 4d ones and of the background fluxes [**4**]. They provide consistency checks and impose several constraints on the behavior of RG flows and the existence of IR fixed points.

A well studied anomaly in four dimensions is the coefficient of the Euler density appearing in the conformal anomaly. This quantity is referred to as the central charge a and it satisfies an analogue of the 2d c-theorem of [**5**], i.e., it is decreasing along the RG flow from UV to IR [**6, 7**]. When considering $\mathcal{N} = 1$ superconformal field theories, the central charge a, non-perturbatively obtained in [**8**], is maximized by the exact R-current of the superconformal algebra [**9**]. The exact R-current turns out to be a linear combination of the UV R-current and the currents associated to the other global symmetries of the theory. By maximizing the central charge the mixing coefficients can be exactly determined. A peculiar feature of a-maximization in four dimensions is the absence of mixing of baryonic currents in the resulting expression for the exact R-current [**10**]. In two dimensions the conformal anomaly is given by the central charge $c = c_r - c_l$. In the case of 2d $\mathcal{N} = (0, 2)$ SCFTs the corresponding right-moving central charge c_r is extremized by the exact 2d R-current that turns out to be a linear combination of a trial R-current and the currents corresponding to the other abelian global symmetries. The program of constructing $\mathcal{N} = (0, 2)$ 2d SCFTs from 4d became an intense field of research[1] after such an extremization principle was derived in [**4**].

When considering 4d theories with an AdS_5 holographic dual description the topological twist can be reproduced at the gravitational level by turning on properly quantized fluxes for the (abelian) gauge symmetries in the bulk [**23**]. This triggers a RG flow across dimensions that, when restricting to the supergravity approximation, connects the original AdS_5 description to a warped $\mathrm{AdS}_3 \times \Sigma$ geometry. Alternatively, one can consider the full 10d geometry. Solving the BPS equations in this case should lead to a warped product $\mathrm{AdS}_3 \times \mathcal{M}_7$, where the general properties of the seven manifold $\mathcal{M}_7$ were originally discussed in [**24, 25**]. This approach was taken in [**18**] for the infinite class of twisted Y^{pq} quiver gauge theories of [**26, 27, 28**]. A comparison between the structure of the exact IR R-current in the 4d $\mathcal{N} = 1$ and in the 2d $\mathcal{N} = (0, 2)$ theories reveals that baryonic symmetries that do not mix with the R-current in 4d, do mix non-trivially with the 2d R-current.

In this thesis we consider a more general class of 2d $\mathcal{N} = (0, 2)$ SCFTs which is the one obtained by a partial topological twist of 4d $\mathcal{N} = 1$ toric quiver gauge theories that describe a stack of N D3 branes probing the tip of a toric Calabi-Yau threefold CY_3 over a 5d Sasaki-Einstein (SE) base X_5 with $U(1)^3$ isometry (see [**29, 30**] and references therein). The trial central charge of the 4d toric theory can be determined either geometrically, in terms of the geometrical data

[1] See [**11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22**] for related work.

of the associated toric diagram [**31, 32**] (see (3.12)), or alternatively, through the holographic correspondence that provides a in terms of the X_5 volume parametrized by the Reeb vector of the dual supergravity solution [**33, 34, 35, 36, 37, 10, 38, 39, 40, 41, 42**] (see (3.14)). The holographic dictionary translates a-maximization into the minimization of the X_5 volume [**35, 10**].

While for some specific examples (see [**43, 18**]) the correspondence between the 2d central charge c_r and the volume of the 7-manifold $\mathcal{M}_7$ is known, it is still an open problem to find a general formula analogous to the one in 4d. A possible obstruction in finding a volume formula dual to c-extremization arises from the non-trivial mixing of baryonic currents in the 2d exact R-current. In fact, as a consequence, a putative volume formula for c_r should probably involve symmetries that are not necessarily isometries of the seven manifold, so making the generalization of the results in [**35**] to these cases not straightforward. In Chapter 3 we tackle the problem of finding geometric and holographic prescriptions for computing the central charge c_r for 2d SCFTs corresponding to twisted compactification of toric theories.

1.2. Duality reduction

Another fascinating field of research, attracting the interest of both the high energy and the condensed matter communities, consists of the 3d analogue of 2d bosonization. This phenomenon can be more generally thought of as a limiting case of a broad web of non-supersymmetric 3d dualities (see for example [**44, 45, 46, 47, 48**] for an incomplete list of references). These 3d dualities share many common properties with their supersymmetric counterparts, and some attempts to derive them from the supersymmetric case appeared in [**49, 50, 51, 52, 53**]. This provides one of the main motivations for further investigations on the supersymmetric side of 3d dualities.

So far most of the non-supersymmetric dualities discussed in the literature refer to gauge theories with fundamental matter fields. Recently dualities involving QCD_3 with two-index tensor matter fields appeared in [**54, 55, 56**]. In the supersymmetric case, models with two-index tensor matter fields played a relevant role in the generalizations of 4d Seiberg duality, starting from the original example of [**57**]. Furthermore, 4d theories with fundamental and adjoint matter fields have been recently used as a perturbative description of 4d $\mathcal{N} = 2$ non-Lagrangian SCFTs [**58, 59**]. In general 3d dualities involving two-index tensor matter fields have been derived by a circle reduction of the 4d cases [**60, 61, 62**] by following the prescription of [**63**]. There it was observed that if one simply puts the theory on $\mathbb{R}^3 \times \mathbb{S}^1$ and takes the small radius limit then the dimensional reduction does not give rise to a 3d duality. In order to correctly reduce the 4d duality one has to modify the limiting procedure as follows. First one needs to find an effective 3d description of the 4d duality on $\mathbb{R}^3 \times \mathbb{S}^1$ which can be thought of as a new 3d IR duality, UV completed by the 4d physics. Due to the presence of non-trivial holonomies of the gauge connection along the circle, the effective theory develops a compact Coulomb branch parametrized by periodic scalars. Non-trivial contribution coming from 4d instanton configurations also appear, giving rise to KK monopole superpotentials that modify the 3d dynamics. Finally the 3d limit can be taken by real mass and Higgs flows. By applying this procedure 4d Seiberg duality (and its generalizations) reduces to 3d Aharony duality (and its generalizations).

A different 3d limit was recently considered in [**64**] for the reduction of $USp(2N_c)$ $SQCD_4$. This led the authors to discover new interesting families of 3d dualities with non-trivial monopole superpotentials [2]. The generalization of these new dualities to the cases with tensorial matter fields is the subject of Chapter 4.

[2]See also [**65, 66, 67, 59, 68**] for other applications of monopole superpotentials to 3d $\mathcal{N} = 2$ theories.

CHAPTER 2

4d SCFTs on a Riemann Surface

2.1. Overview

In this chapter we engineer the partial topological twist in the natural setup of conformal supergravity and systematically study the twisted compactification on Riemann surfaces of 4d SCFTs with different amount of supersymmetry. In this unified framework we investigate the cases of $\mathcal{N} = 1,2,3,4$ conformal supergravity corresponding to 4d geometries of the form $\mathbb{R}^{1,1} \times \Sigma$ where Σ is a genus g Riemann surface. We study what are the conditions necessary to preserve different amounts of supersymmetry in 2d by solving the Killing spinor equations arising from setting to zero the variations of the gravitino and of the auxiliary fermions in the Weyl multiplet (Sections 2.2, 2.3, 2.4 and 2.5). When possible, i.e., in cases with $\mathcal{N} = 1,2$ supersymmetry, we also turn on vector multiplets associated to global flavor symmetries. In this case an additional constraining equation for Killing spinors arises from setting to zero the variation of the corresponding gaugino.

All possible solutions to the twisted Killing spinor equations are listed in Tables 3, 5, 8 and 11 for $\mathcal{N} = 1,2,3$ and 4, respectively. We observe that for $\mathcal{N} = 1,2$ the presence of global gauged non-R symmetries can in general decrease the number of supersymmetries, but never below $\mathcal{N} = (0,2)$ or $(2,0)$. For $\mathcal{N} = 3,4$ theories, where flavor symmetries are absent, we discuss a different approach to flavor symmetry twists. This consists in a first twist along an abelian subgroup of $SU(3)_R \times U(1)_R$ (or $SU(4)_R$) which partially breaks conformal supersymmetry allowing for vector multiplets associated to non-R global symmetries to be introduced. A further twist along such symmetries corresponds to $\mathcal{N} = 1$ or $\mathcal{N} = 2$ gaugings and preserves half of the supercharges. In Section 2.6 we provide further details on the vanishing of the supersymmetry variation for the auxiliary fermions in the $\mathcal{N} = 3,4$ cases.

In Section 2.7 we derive the 't Hooft anomaly coefficients of the 2d theories and, in the case of $\mathcal{N} = (0,2)$ supersymmetry, we apply c-extremization to obtain the central charges expressed in terms of the background fluxes and the 4d anomalies. The explicit expression for the exact 2d R-current is also obtained as a linear combination of the 4d R-current and global flavor symmetries.

2.2. Topological twist in $\mathcal{N} = 1$ conformal supergravity

We begin by considering a $\mathcal{N} = 1$ superconformal theory on the four dimensional spin manifold $M = \mathbb{R}^{1,1} \times \Sigma$, where Σ is a Riemann surface of genus g and scalar curvature $\kappa\eta_\Sigma$. Twisted compactification of this class of theories has been already discussed in [**69, 17, 18**]. Here we review the procedure in a $\mathcal{N} = 1$ superconformal gravity setup to fix the general scheme that we will use in the $\mathcal{N}$-extended cases.

We call (x^0, x^1) the coordinates on $\mathbb{R}^{1,1}$ and (x^2, x^3) those on Σ. The spin connection ω_μ on Σ then has non-vanishing curvature:

$$\frac{1}{2\pi}\int_\Sigma R(\omega) = 2 - 2g \tag{2.1}$$

where $R(\omega) = \mathrm{d}\omega$ is the curvature 2-form of the tangent bundle of Σ. For later convenience we define:

$$\kappa = \mathrm{sgn}(2-2g) \quad \text{and} \quad \eta_\Sigma = \begin{cases} |2-2g| & \kappa \neq 0 \\ 1 & \kappa = 0 \end{cases} \tag{2.2}$$

so that we can write:

$$R(\omega) = \kappa\,\Omega \tag{2.3}$$

with $\Omega = \eta_\Sigma\,\mathrm{dvol}_\Sigma$ the normalized volume form.

In general, compactification on Σ breaks supersymmetry completely, since on arbitrarily curved manifolds there are no covariantly constant Killing spinors. Along the lines of [**70**], in order to put a 4d theory on a curved manifold and preserve some supersymmetry we couple the theory to a conformal supergravity background that reproduces the desired spacetime geometry. The whole superconformal group is gauged and the corresponding gauge fields are organized into the Weyl multiplet as follows (we use notations and conventions of [**71**]) Here P_a,K_a are vector

Table 1. Generators and gauge fields of $\mathcal{N}=1$ conformal supergravity.

generator	P_a	M_{ab}	Δ	K_a	T_R	Q	S
field	e_μ^a	ω_μ^{ab}	b_μ	f_μ^a	A_μ	ψ_μ	ϕ_μ

generators of translations and special conformal transformations, M_{ab} and Δ are generators of Lorentz rotations and dilatations, Q and S are the spinorial supercharges. The $U(1)_R$ R-symmetry generator T_R assigns charge -1 to the positive chirality supercharges Q_α and S_α and charge $+1$ to their conjugates $\bar{Q}_{\dot\alpha} = (Q_\alpha)^\dagger$ and $\bar{S}_{\dot\alpha} = (S_\alpha)^\dagger$. When the R-symmetry generator acts on the supercharges we will often write $T_R = -\gamma_5$ with $\gamma_5 = \mathrm{i}\gamma_{0123}$.

The supersymmetry transformation laws of the independent gauge fields read

$$\delta e_\mu^a = \frac{1}{2}\bar{\varepsilon}\gamma^a\psi_\mu \tag{2.4}$$

$$\delta b_\mu = \frac{1}{2}\bar{\varepsilon}\phi_\mu - \frac{1}{2}\bar{\eta}\psi_\mu \tag{2.5}$$

$$\delta A_\mu = \frac{1}{2}\mathrm{i}\bar{\varepsilon}\gamma_5\phi_\mu + \frac{1}{2}\mathrm{i}\bar{\eta}\gamma_5\psi_\mu \tag{2.6}$$

$$\delta\psi_\mu = \mathcal{D}_\mu\varepsilon - e_\mu^a\gamma_a\eta \tag{2.7}$$

where ε, η are the Majorana spinors associated to Q and S transformations, respectively. The covariant derivative is defined as $\mathcal{D}_\mu\varepsilon \equiv (\partial_\mu + \frac{1}{2}b_\mu + \frac{1}{4}\omega_\mu^{ab}\gamma_{ab} - \mathrm{i}A_\mu T_R)\varepsilon$.

Since we are only interested in theories on curved manifolds with rigid supersymmetry, we fix the Weyl multiplet to be a collection of background fields describing the geometry of spacetime. In order to preserve Lorentz invariance on $\mathbb{R}^{1,1}$ we set all the spinor fields to zero and assign possibly non-vanishing components to bosonic forms only in the (x^2, x^3) directions. As follows from (2.7), in general this choice breaks superconformal invariance, however some Q-supersymmetry can survive if the geometry admits non-trivial solutions of $\delta\psi_\mu = 0$. In the following we consider backgrounds such that $\eta = 0$ [1]. so that supersymmetry is preserved if there exist covariantly constant spinor fields, i.e., solutions to the equation $\mathcal{D}_\mu\varepsilon = 0$. This equation may have non-trivial solutions if we turn on a non-zero background

[1] To begin with one could solve the equation $\delta\psi_\mu = 0$ for non-vanishing η, by setting $\eta = \frac{1}{4}\not{\mathcal{D}}\varepsilon$ [**72**]. The solution $\eta = 0, \mathcal{D}_\mu\varepsilon = 0$ is compatible with this condition.

also for the R-symmetry gauge connection A_μ [**1**] such that the two contributions coming from A_μ and ω_μ^{ab} in the covariant derivative cancel each other.

More precisely, focusing on constant solutions, we first apply the exterior derivative to $\delta\psi_\mu$, so that the Killing spinor equation $\mathcal{D}_\mu\varepsilon = 0$ is traded with

$$2\partial_{[\mu}\delta\psi_{\nu]} = \left[\frac{1}{2}R_{\mu\nu}(\omega^{23})\gamma_{23} - \mathrm{i}R_{\mu\nu}(A)\gamma_5\right]\varepsilon = 0 \tag{2.8}$$

where $R_{\mu\nu}(\omega^{23})$ and $R_{\mu\nu}(A)$ are the curvatures of the connections ω_μ^{23} and A_μ, respectively. Given the particular form of the curvature $R_{\mu\nu}(\omega) = \kappa\,\Omega_{\mu\nu}$, we choose A_μ such that its curvature is also proportional to the normalized volume form $\Omega_{\mu\nu}$

$$R_{\mu\nu}(A) = -a\,\Omega_{\mu\nu} \tag{2.9}$$

where the parameter a is constrained by the Dirac quantization condition

$$\frac{1}{2\pi}\int_\Sigma R(A) = -a\int_\Sigma \frac{\Omega}{2\pi} = -a\eta_\Sigma \in \mathbb{Z} \tag{2.10}$$

Substituting (2.9) in (2.8), we then obtain

$$\left[\frac{\kappa}{2}\mathrm{i}\gamma_{23} - a\gamma_5\right]\varepsilon = 0 \tag{2.11}$$

We postpone the search and classification of non-vanishing solutions to Section 2.2.2.

2.2.1. Twisting with flavors. We now consider the case in which the original 4d theory also admits a global abelian non-R symmetry that can be either flavor or baryonic symmetry. With an abuse of notation, we call it $U(1)_{\text{flavor}}$.

This symmetry can be weakly gauged by turning on a background connection [2]. However, in order to preserve the original superconformal symmetry one has to turn on a whole abelian $\mathcal{N} = 1$ superconformal gauge multiplet (B_μ, λ, Y) whose field content consists of the gauge vector potential B_μ, the gaugino λ and the auxiliary scalar Y, all in the adjoint representation of the flavor symmetry. The corresponding supersymmetry transformations are

$$\begin{aligned} \delta B_\mu &= -\frac{1}{2}\bar{\varepsilon}\gamma_\mu\lambda \\ \delta\lambda &= \left[\frac{1}{4}\gamma^{ab}R_{ab}(B) + \frac{1}{2}Y\mathrm{i}\gamma_5\right]\varepsilon \\ \delta Y &= \frac{1}{2}\mathrm{i}\bar{\varepsilon}\gamma_5\gamma^\mu\mathcal{D}_\mu\lambda \end{aligned} \tag{2.12}$$

where $R_{\mu\nu}(B)$ is the curvature 2-form of the gauge connection B_μ and the covariant derivative on spinors is defined as in eq. (2.7).

Similarly to the case of the R-symmetry background in (2.9), we can choose a $U(1)_{\text{flavor}}$ connection with curvature:

$$R_{\mu\nu}(B) = b\,\Omega_{\mu\nu}, \qquad b\nu \in \mathbb{Z} \tag{2.13}$$

together with vanishing background gaugino. In order to preserve some supersymmetry we have to require

$$\delta\lambda = \left[\frac{b}{2}|e|\Omega_{23}\gamma_{23} + \frac{1}{2}Y\mathrm{i}\gamma_5\right]\varepsilon = 0 \tag{2.14}$$

where $|e| = e^{22}e^{33} - e^{23}e^{32}$ is the vielbein determinant on Σ.

[2]Similar discussions appeared in [**73, 11, 13**].

Writing $\gamma_5 = \mathrm{i}\gamma_{23}\gamma_{01}$ in the previous equation allows to factor out a gamma matrix γ_{23}. Therefore, setting $Y = \pm b|e|\Omega_{23}$ we finally obtain the condition

$$(1 \mp \gamma_{01})\,\varepsilon = 0 \tag{2.15}$$

We then see that in principle, turning on a background for an abelian non-R global symmetry, introduces additional constraints on the supersymmetry generators.

More generally, we can consider 4d theories with rank-n flavor symmetry group, i.e., with n generators T_i in the Cartan subalgebra. In this case we can gauge one vector multiplet $(B^i_\mu, \lambda^i, Y^i)$ for each Cartan generator. If the corresponding auxiliary scalars are fixed by the same equation $Y_i = +b_i|e|\Omega_{23}$ (or $Y_i = -b_i|e|\Omega_{23}$) we are led to the same constraints (2.15).

2.2.2. Classification of the solutions. We are now ready to discuss the most general solutions of the two supersymmetry preserving conditions

$$\left[\frac{\kappa}{2}\mathrm{i}\gamma_{23} - a\gamma_5\right]\varepsilon = 0 \qquad , \qquad b_i\,(1 \mp \gamma_{01})\,\varepsilon = 0 \tag{2.16}$$

where the constant a signals the presence of a non-trivial $U(1)_R$ background, eq. (2.9), while b_i are associated to B^i_μ connections for $U(1)_{\text{flavor}}$ symmetries, eq. (2.13). We note that the second equation is nothing but a 2d (anti)chirality condition.

In order to find solutions to these equations, we write the Majorana spinor ε in terms of its Weyl components, $\varepsilon = (\epsilon_\alpha, \bar\epsilon^{\dot\alpha})$, and with no loss of generality we restrict the discussion to the positive chiral spinor ϵ_α transforming in the $\mathbf{2}$ of $SL(2,\mathbb{C})$.

On the product manifold $\mathbb{R}^{1,1} \times \Sigma$ the original Lorentz group of 4d Minkowski is reduced as $Spin(3,1) \to Spin(1,1) \times Spin(2)_\Sigma$, and consequently the spinorial representation of ϵ_α also splits as

$$\mathbf{2} \to [\mathbf{1}_{1,1} \oplus \mathbf{1}_{-1,-1}] \tag{2.17}$$

Here the representations on the right hand side are labelled by the eigenvalues of the hermitian generators γ_{01} and $\mathrm{i}\gamma_{23}$ of $Spin(1,1)$ and $Spin(2)_\Sigma$, respectively. The generator γ_{01} corresponds also to the chirality operator on $\mathbb{R}^{1,1}$, hence we refer to $\mathbf{1}_{1,1}$ and $\mathbf{1}_{-1,-1}$ as the 2d positive (left) and negative (right) chirality representations respectively, and denote the corresponding spinors as ϵ_+ and ϵ_-.

TABLE 2. Supersymmetry generators and their charges under $Spin(1,1)$, $Spin(2)_\Sigma$ and R-symmetry. Since the $U(1)_R$ generator can be written as $\gamma_5 = (\gamma_{01})(\mathrm{i}\gamma_{23})$, it follows that $\epsilon_\pm$ are automatically irreducible representations of the R-symmetry group corresponding to charge 1.

supersymmetry	chirality	representation	γ_{01}	$\mathrm{i}\gamma_{23}$	γ_5	$\delta\psi_\mu = 0$
ϵ_+	L	$\mathbf{1}_{+1,+1}$	+1	+1	+1	$a - \kappa/2 = 0$
ϵ_-	R	$\mathbf{1}_{-1,-1}$	−1	−1	+1	$a + \kappa/2 = 0$

As summarized in Table 2, for $\kappa \neq 0$ solutions to the first eq. in (2.16) correspond to ϵ_+ for $a = \frac{\kappa}{2}$ and ϵ_- for $a = -\frac{\kappa}{2}$. The second equation in (2.16) does not restrict the Killing spinors any further, since we can always choose b_i such that (2.15) projects on the same chirality as that of the Killing spinor. Therefore, independently of the presence of gauged flavor symmetries, the resulting 2d theory is $\mathcal{N} = (2,0)$ for $a = \frac{\kappa}{2}$ and $\mathcal{N} = (0,2)$ for $a = -\frac{\kappa}{2}$. These solutions are compatible with the quantization condition $a\eta_\Sigma \in \mathbb{Z}$, being $\kappa\eta_\Sigma$ an even number.

In the special case of compactification on a torus, $\kappa = 0$, when no flavor symmetry is gauged ($b_i = 0$) there is no need for twisting. In fact, setting A_μ to zero, the Killing spinor equation reduces to $\partial_\mu \epsilon = 0$ and is automatically satisfied for every constant section ϵ. Therefore, supersymmetry is not broken and the resulting 2d theory is $\mathcal{N} = (2,2)$ with R-symmetry $U(1)_{\text{left}} \times U(1)_{\text{right}}$ generated by the two combinations $T_\pm = \frac{1}{2} T_R \pm M_{23}$, where M_{23} is the Lorentz generator on Σ. Supersymmetry can be reduced by gauging some flavor symmetry. In this case, in fact, the second equation in (2.16) constrains the supercharges to be of definite chirality and reduces supersymmetry to $\mathcal{N} = (2,0)$ for $Y_i = +b_i|e|\Omega_{23}$ or $\mathcal{N} = (0,2)$ for $Y_i = -b_i|e|\Omega_{23}$.

The complete picture of topological twisted reduction of $\mathcal{N} = 1$ SCFTs is summarized in Table 3, where the resulting 2d theories are classified in terms of the surviving amount of supersymmetry.

TABLE 3. Classification of topologically twisted 4d $\mathcal{N} = 1$ SCFTs on Riemann surfaces of curvature $\kappa = \pm 1, 0$ in terms of the surviving amount of supersymmetry in 2d. We include the possibility of a twist along the flavor symmetries, with flux b.

$\kappa \neq 0$	$a = \frac{\kappa}{2}$	$a = -\frac{\kappa}{2}$	$\kappa = 0$	$a = 0$
$b = 0$	$\mathcal{N} = (2,0)$	$\mathcal{N} = (0,2)$	$b = 0$	$\mathcal{N} = (2,2)$
$b \neq 0$	$\mathcal{N} = (2,0)$	$\mathcal{N} = (0,2)$	$b \neq 0$	$\mathcal{N} = (2,0)$ or $(0,2)$

2.3. Topological twist in $\mathcal{N} = 2$ conformal supergravity

We now consider a $\mathcal{N} = 2$ SCFT with R-symmetry group $SU(2)_R \times U(1)_R$. The Lie algebra of $SU(2)_R$ is spanned by anti-hermitian matrices $i\sigma_A$, where $\sigma_{A=1,2,3}$ are the three Pauli matrices.

The four-dimensional chiral supercharges $Q_{\alpha I}$ are in the $(\mathbf{2}, \bar{\mathbf{2}})_{-1}$ representation of the group $Spin(3,1) \times SU(2)_R \times U(1)_R$, while their complex conjugates $\bar{Q}^I_{\dot\alpha} = (Q_{\alpha I})^\dagger$ transform in the $(\bar{\mathbf{2}}, \mathbf{2})_{+1}$ representation. In particular, the $U(1)_R$ generator T_R acts on the supercharges as $-\gamma_5$.

The $\mathcal{N} = 2$ superconformal algebra contains a $\mathcal{N} = 1$ subalgebra with R-symmetry group $U(1)_R^{\mathcal{N}=1}$ generated by the combination

$$T_R^{\mathcal{N}=1} = \frac{2}{3}\sigma_3 + \frac{1}{3} T_R \tag{2.18}$$

Twisted compactifications of $\mathcal{N} = 2$ SCFTs have been already considered in [**74, 75, 17**]. Here we give a systematic derivation within the superconformal gravity setup.

Analogously to the $\mathcal{N} = 1$ case, a $\mathcal{N} = 2$ SCFT can be consistently defined on a curved manifold $M = \mathbb{R}^{1,1} \times \Sigma$, by first coupling it to the extended $\mathcal{N} = 2$ superconformal gravity and then gauge fixing the background Weyl multiplet as to reproduce the desired geometry with possibly non-trivial fluxes turned on in order to preserve some supersymmetry.

We recall that the $\mathcal{N} = 2$ Weyl multiplet contains the gauge fields of the conformal group $e^a_\mu, f^a_\mu, b_\mu, \omega^{ab}_\mu$, the superconnections $\psi^I_\mu, \phi_{\mu I}$ associated to supersymmetries Q_I and S^I, the connections A_μ and V^A_μ for the R-symmetry groups $U(1)_R$ and $SU(2)_R$ and the auxiliary fields T^-_{ab}, D (bosonic) and χ^I (fermionic), needed to close the algebra off-shell.

Under supersymmetry transformations the fermionic fields of the gravity multiplet transform as

$$\delta\psi^I_\mu = \left[\partial_\mu + \frac{1}{2}b_\mu + \frac{1}{4}\omega^{ab}_\mu\gamma_{ab} - A_\mu \mathrm{i}\gamma_5\right]\varepsilon^I - V^A_\mu(\mathrm{i}\sigma_A)^I{}_J\varepsilon^J - \frac{1}{16}\gamma^{ab}T^-_{ab}\varepsilon^{IJ}\gamma_\mu\varepsilon_J \tag{2.19}$$

$$\delta\chi^I = \frac{1}{2}D\varepsilon^I - \frac{1}{6}\gamma^{ab}\left[\frac{1}{4}\not{D}T^-_{ab}\varepsilon^{IJ}\varepsilon_J - R_{ab}(A)\mathrm{i}\gamma_5\varepsilon^I - R_{ab}(V^A)(\mathrm{i}\sigma_A)^I{}_J\varepsilon^J\right] \tag{2.20}$$

In order to preserve Lorentz invariance on $\mathbb{R}^{1,1}$ the background fermions must be set to zero. This choice automatically sets to zero the Q-supersymmetry variation of all bosonic fields, which can then be chosen such that the Q-variation of the fermions vanish as well.

From (2.19) and (2.20) we deduce that we can safely set the background fields b_μ and T^-_{ab} to zero and simplify these expressions to

$$\delta\psi^I_\mu = \left[\partial_\mu + \frac{1}{4}\omega^{ab}_\mu\gamma_{ab} - A_\mu\mathrm{i}\gamma_5\right]\varepsilon^I - V^A_\mu(\mathrm{i}\sigma_A)^I{}_J\varepsilon^J \equiv 0 \tag{2.21}$$

$$\delta\chi^I = \frac{1}{2}D\varepsilon^I + \frac{1}{6}\gamma^{ab}\left[R_{ab}(A)\mathrm{i}\gamma_5\varepsilon^I + R_{ab}(V^A)(\mathrm{i}\sigma_A)^I{}_J\varepsilon^J\right] \equiv 0 \tag{2.22}$$

The remaining background connections A_μ and V^A_μ can then be used to perform partial topological twist as we now describe.

Turning on a background flux for V^A_μ breaks explicitly the $SU(2)_R$ invariance of the theory down to a $U(1)$ subgroup of it. Without loss of generality we choose this subgroup to be the one generated by $\mathrm{i}\sigma_3$. Namely, we parametrize the R-symmetry gauging as follows

$$R_{\mu\nu}(A) = -a_1\Omega_{\mu\nu}, \quad R_{\mu\nu}(V^{A=1,2}) = 0, \quad R_{\mu\nu}(V^3) = -a_2\Omega_{\mu\nu} \tag{2.23}$$

where the parameters $a_{i=1,2}$, are constrained by the quantization condition $a_i\nu \in \mathbb{Z}$, and $\Omega_{\mu\nu}$ is the normalized volume form of Σ. This choice is actually equivalent to gauging the 1-parameter subgroup of $SU(2)_R \times U(1)_R$ generated by $a_1T_R + a_2\sigma_3$.

Looking for constant spinor solutions of (2.21) and (2.22) we can apply the exterior covariant derivative to $\delta\psi_\mu$ thus turning the Killing spinor equation into an equation for the curvatures. Substituting the background (2.23) we find

$$\begin{aligned}2\partial_{[\mu}\delta\psi^I_{\nu]} &= \left[\frac{1}{2}R_{\mu\nu}(\omega^{23})\gamma_{23} - R_{\mu\nu}(A)\mathrm{i}\gamma_5\right]\varepsilon^I - R_{\mu\nu}(V^3)(\mathrm{i}\sigma_3)^I{}_J\varepsilon^J \\ &= \mathrm{i}\Omega_{\mu\nu}\left[-\frac{\kappa}{2}\mathrm{i}\gamma_{23}\delta^I_J + a_1\gamma_5\delta^I_J + a_2(\sigma_3)^I{}_J\right]\varepsilon^J = 0\end{aligned} \tag{2.24}$$

$$\delta\chi^I = \frac{1}{2}\left[D - \frac{\kappa}{6}|e|\Omega_{23}\right]\varepsilon^I = 0 \tag{2.25}$$

where (2.25) is obtained by substituting (2.24) in (2.22) and therefore it is only valid on the components of ϵ^I that are actual solutions of the Killing spinor equation.

The χ^I variation can be set to zero by fixing the auxiliary field as $D = \frac{\kappa}{6}|e|\Omega_{23}$. We are then left with a single defining equation for Killing spinors.

2.3.1. Twisting with flavors. Before solving the Killing spinor equation (2.24) we generalize the discussion to the case of 4d SCFTs admitting some global abelian non-R symmetry $U(1)_{\text{flavor}}$. Weakly gauging this symmetry implies turning on a non-vanishing background $\mathcal{N} = 2$ vector multiplet $(B_\mu, X, \lambda^I, Y^A)$. Such a multiplet contains one gauge field B_μ with curvature $R_{\mu\nu}(B)$, one complex scalar X, two gaugini λ^I forming a $SU(2)$ doublet, and one auxiliary field Y^A transforming in the adjoint of the R-symmetry group. Setting the fermions $\lambda^I = 0$, the supersymmetry variations of the bosonic components of the multiplet are identically vanishing, and they can be chosen to satisfy

$$\delta\lambda^I = \left[\frac{1}{4}R_{ab}(B)\gamma^{ab}\delta^I_J + Y^A(\mathrm{i}\sigma_A)^I_J\right]\varepsilon^J \equiv 0 \tag{2.26}$$

Gauging the global symmetry along Σ with $R_{\mu\nu}(B) = b\,\Omega_{\mu\nu}$, and setting for instance $Y^{1,2}=0$, $Y^3 = -\frac{b}{2}$ for the positive chirality component of ε^J we obtain

$$\frac{b}{2}\left[\gamma^{23}\delta^I_J - (\mathrm{i}\sigma_3)^I_J\right]\epsilon^J = 0 \quad \Rightarrow \quad \begin{cases} (\gamma_{01}+1)\epsilon^1 = 0 \\ (\gamma_{01}-1)\epsilon^2 = 0 \end{cases} \tag{2.27}$$

where we have used $\mathrm{i}\gamma_{23} = \gamma_{01}\gamma_5$ and $\gamma_5\epsilon^J = \epsilon^J$.

The previous condition is equivalent to requiring that the two components of the ϵ^I doublet have opposite chirality. Setting $Y^3 = \frac{b}{2}$ would simply interchange the conditions on ϵ^1 and ϵ^2.

Another possibility to perform the flavor twist would be via a two-step procedure. We first gauge a $\mathcal{N}=1$ vector multiplet that breaks explicitly $\mathcal{N}=2$ supersymmetry even before coupling the theory to a curved background. We then identify the $\mathcal{N}=1$ subsector of the $\mathcal{N}=2$ theory which is compatible with this gauging, and apply the twist as in Section 2.2. Observe that we could engineer such a reduction also in the absence of flavor symmetries. In that case we should first perform a R-symmetry twist that preserves four supercharges. This twist would break R-symmetry and leave an unbroken $U(1)$ that could be treated as flavor symmetry useful for further twisting.

2.3.2. Classification of the solutions. In order to find solutions to eq. (2.24) we observe that the selected background breaks $Spin(3,1)\times SU(2)_R \to Spin(1,1)\times Spin(2)_\Sigma \times U(1)_{\sigma_3}$, and correspondingly the positive chirality components ϵ^I_α in the $(\mathbf{2},\mathbf{2})$ representation as

$$\epsilon^I_\alpha \to \epsilon^1_+ \oplus \epsilon^1_- \oplus \epsilon^2_+ \oplus \epsilon^2_- \tag{2.28}$$

where on the r.h.s. $\pm$ indices denote the 2d chirality of the reduced spinors

$$\gamma_{01}\epsilon^I_\pm = \pm\epsilon^I_\pm, \qquad \mathrm{i}\gamma_{23}\epsilon^I_\pm = \pm\epsilon^I_\pm \tag{2.29}$$

We can find solutions to (2.24) by appropriately choosing the values of the twisting parameters a_i as summarized in Table 4. A further constraint comes from eq. (2.27)

TABLE 4. Supersymmetry equations for $\mathcal{N}=2$ theories. The supersymmetries in the left column are preserved when the twisting parameters a_i satisfy the corresponding equations in the column on the right.

supersymmetry	$\delta\psi^I_\mu = 0$
ϵ^1_+	$a_1 + a_2 - \kappa/2 = 0$
ϵ^1_-	$a_1 + a_2 + \kappa/2 = 0$
ϵ^2_+	$a_1 - a_2 - \kappa/2 = 0$
ϵ^2_-	$a_1 - a_2 + \kappa/2 = 0$

when a global non-R symmetry is also gauged.
We discuss in detail the solutions for $\kappa \neq 0$ and $\kappa = 0$, separately.

$\kappa \neq 0$. For the case of non-zero curvature, we give a prototype of twist for each fixed amount of supersymmetry preserved in 2d. All the other choices are related by a trivial change of basis of the symmetries or a different choice of sign for the auxiliary fields.

• For $a_1 = -\frac{\kappa}{2}$ and $a_2 = 0$ the preserved Killing spinors are $\epsilon^1_- \oplus \epsilon^2_-$ which form a $SU(2)_R$ doublet. The 4d R-symmetry is left unbroken and the 2d theory is a chiral $\mathcal{N} = (0,4)$ theory. If we add a flux for an external vector B_μ, then equations (2.27) imply that only one of the two components of the doublet can be preserved according to the particular choice of the auxiliary field Y^A in the vector multiplet, hence supersymmetry is necessarily broken to $\mathcal{N} = (0,2)$.

• For $a_1 = 0$ and $a_2 = -\frac{\kappa}{2}$ the preserved supersymmetries are $\epsilon^1_- \oplus \epsilon^2_+$. R-symmetry is broken to $U(1)^2$ with generators $T_\pm \equiv \frac{1}{2}T_R \pm M_{23}$ and the preserved supersymmetry in two dimensions is $\mathcal{N} = (2,2)$. The global symmetry generated by the background along $T \equiv M_{23} + \frac{1}{2}\sigma_3$ becomes a flavor symmetry in two dimensions since, by definition, the preserved supercharges transform trivially under it. In this case, gauging a global non-R symmetry with the corresponding connection B_μ together with the choice of auxiliary $Y^3 = -\frac{b}{2}$, does not constrain the Killing spinors any further (see eq. (2.27)) and the 2d theory maintains $\mathcal{N} = (2,2)$ supersymmetry.

• For $a_1 + a_2 = -\frac{\kappa}{2}$ the only preserved supersymmetry is ϵ^1_-, hence the theory is $\mathcal{N} = (0,2)$ with $U(1)$ R-symmetry. In this case there are two new abelian flavor symmetries that were not present in the original 4d theory, generated by the two combinations

$$T_1 \equiv \frac{1}{2}T_R + M_{23} \quad \text{and} \quad T_2 \equiv \frac{1}{2}\left(T_R - \sigma_3\right) \tag{2.30}$$

Turning on a flavor flux B_μ does not constrain this solution any further.

$\kappa = 0$. In the case of compactification on a torus we have two possible solutions.

• The trivial solution corresponds to $a_1 = a_2 = 0$, and $D = 0$ in (2.25). This is the case where there is no twist, since the dimensional reduction on flat space preserves all supersymmetry. The compactified theory flows to $\mathcal{N} = (4,4)$ in 2d with global symmetry $SU(2)\times U(1)^2$ where the two abelian groups are generated by the combinations $T_\pm \equiv \frac{1}{2}T_R \pm M_{23}$. Both sectors $(4,0)$ and $(0,4)$ provide a four dimensional real representation of the $SU(2)$ R-symmetry group. This however poses a puzzle because the only possible superconformal algebra compatible with $(4,4)$ supersymmetry and this $SU(2)$ action on the supercharges is the small $\mathcal{N} = (4,4)$ superconformal algebra which only admits a $SU(2)$ R-symmetry. The remaining abelian factors $U(1)\times U(1)$, while acting effectively on the supercharges, are not compatible with any known superconformal algebra. A possible resolution of the issue is to regard them as global symmetries coming from outer automorphisms of the algebra.

• Another possible choice of supersymmetry preserving background on the torus corresponds to $a_1 + a_2 = 0$ with both fluxes different from zero. Solutions of (2.24) are then spinors $\epsilon^1_+ \oplus \epsilon^1_-$ that transform trivially with respect to the background symmetry

$$T \equiv \frac{1}{2}\left(T_R - \sigma_3\right) \tag{2.31}$$

The theory flows to $\mathcal{N} = (2,2)$ in 2d with $U(1)^2$ R-symmetry given by

$$T_\pm \equiv \frac{1}{2}T_R \pm M_{23} \tag{2.32}$$

Turning on a background for an external global symmetry, $R_{\mu\nu}(B) = b\,\Omega_{\mu\nu}$, together with the auxiliary $Y^3 = -\frac{b}{2}$ further breaks supersymmetry to ϵ^1_-, as can be seen from (2.27). In this case, the theory is $\mathcal{N} = (0,2)$ with $U(1)$ R-symmetry T_R

and two flavor symmetries which correspond precisely to the T background (2.31) and the left R-symmetry T_+ (under which the right sector is invariant). Alternatively, choosing $Y^3 = +\frac{b}{2}$, the theory flows to $\mathcal{N} = (2,0)$ with two flavor symmetries T and T_-.

The results of this section are summarized in the Table 5.

TABLE 5. Classification of topologically twisted 4d $\mathcal{N} = 2$ SCFTs on Riemann surfaces of curvature $\kappa = \pm 1, 0$ in terms of the surviving amount of supersymmetry in 2d. We include the possibility of a twist along the flavor symmetries, with flux b.

$\kappa = 0$	$a_1 = a_2 = 0$	$a_1 + a_2 = 0$
$b = 0$	$\mathcal{N} = (4,4)$	$\mathcal{N} = (2,2)$
$b \neq 0$	$\mathcal{N} = (2,2)$	$\mathcal{N} = (0,2)$ or $(2,0)$

$\kappa \neq 0$	$a_1 = -\frac{\kappa}{2}, a_2 = 0$	$a_1 = 0, a_2 = -\frac{\kappa}{2}$	$a_1 + a_2 = -\frac{\kappa}{2}$
$b = 0$	$\mathcal{N} = (0,4)$	$\mathcal{N} = (2,2)$	$\mathcal{N} = (0,2)$
$b \neq 0$	$\mathcal{N} = (0,2)$	$\mathcal{N} = (2,2)$	$\mathcal{N} = (0,2)$

2.4. Topological twist in $\mathcal{N} = 3$ conformal supergravity

It has been recently claimed [**76, 77, 78, 79**] that 4d $\mathcal{N} = 3$ SCFTs with no enhancement to $\mathcal{N} = 4$ can exist at strong coupling. These theories have $SU(3)_R \times U(1)_R$ R-symmetry and their matter content coincides with the one of 4d $\mathcal{N} = 4$ SYM. As a consequence there are no non-R global symmetries.

Considering a $\mathcal{N} = 3$ SCFT compactified on $M = \mathbb{R}^{1,1} \times \Sigma$, a partial topological twist can be performed on Σ using an abelian subgroup of the R-symmetry group. In this section we study all possible solutions of the Killing spinor equations for such a twist, classifying all different configurations of preserved supercharges in two dimensions in terms of the different choices of the fluxes for the R-symmetry group.

As discussed above, the most natural framework where twisting a $\mathcal{N} = 3$ SCFT on a curved manifold is $\mathcal{N} = 3$ conformal supergravity [**80, 81, 82**], whose Weyl multiplet and the corresponding non-linear supersymmetry transformations have been recently derived in [**83**].

TABLE 6. Field content of the Weyl multiplet in $\mathcal{N} = 3$ conformal supergravity.

field	e_μ^a	b_μ	A_μ	V_μ^A	E_I	T_{ab}^I	D_J^I	ψ_μ^I	Λ	χ_{IJ}	ζ^I
$SU(3)_R \times U(1)_R$	$\mathbf{1}_0$	$\mathbf{1}_0$	$\mathbf{1}_0$	$\mathbf{8}_0$	$\bar{\mathbf{3}}_2$	$\mathbf{3}_{-2}$	$\mathbf{8}_0$	$\mathbf{3}_1$	$\mathbf{1}_3$	$\mathbf{6}_1$	$\mathbf{3}_1$
# of real d.o.f.	5	0	3	24	6	18	8	24	4	24	12

The $\mathcal{N} = 3$ Weyl multiplet in four dimensions is given in Table 6. In particular, A_μ and V_μ^A, $A = 1, \cdots, 8$ are the gauge fields associated to the R-symmetry $U(1)_R$ and $SU(3)_R$ transformations, respectively.

The R-symmetry group $SU(3)_R$ is generated by antihermitian matrices $(\mathrm{i}\lambda_A)$, with $A = 1, .., 8$. We choose a basis in which the $SU(3)$ can be embedded into the top left 3×3 block of $SU(4)$, so that the first 8 generators of $SU(4)$ reduce straightforwardly to the generators of $SU(3)$. The $U(1)_R$ group is obtained by mixing the $U(1)$ from the decomposition of $SU(4)_R$ into $SU(3)_R \times U(1)$ and the chiral $U(1)$ symmetry that enhances the superalgebra $PSU(2,2|4)$ to $U(2,2|4)$ [**80, 84**]. We observe that these two $U(1)$ groups act proportionally to each other on the components of the $\mathcal{N} = 4$ Weyl multiplet that survive in the projection to the $\mathcal{N} = 3$ Weyl multiplet.

As in the previous cases, we are interested in preserving supersymmetry while coupling the SCFT to a curved background describing the geometry of the manifold M. We choose a background Weyl multiplet where, together with the fermions, all the bosonic fields are set to zero except for e_μ^a, A_μ, V_μ^A and D^I_J. Consequently, the conditions for the fermion variations to vanish read [**83**]

$$\delta\psi^I_\mu = \left[\partial_\mu + \frac{1}{4}\omega_\mu^{ab}\gamma_{ab} - A_\mu \mathrm{i}\gamma_5\right]\varepsilon^I - V_\mu^A(\mathrm{i}\lambda_A)^I_J\varepsilon^J = 0 \tag{2.33}$$

$$\delta\chi_{IJ} = -\frac{1}{2}\varepsilon_{KL(I}D^K_{J)}\varepsilon^L - \frac{1}{4}\varepsilon_{KL(I}\gamma^{ab}R_{ab}(V^A)(\mathrm{i}\lambda_A)^K_{J)}\varepsilon^L = 0 \tag{2.34}$$

$$\delta\zeta^I = \frac{1}{4}D^I_K\varepsilon^K - \frac{1}{24}\gamma^{ab}R_{ab}(V^A)(\mathrm{i}\lambda_A)^I_K\varepsilon^K + \frac{1}{3}\gamma^{ab}R_{ab}(A)\mathrm{i}\gamma_5\varepsilon^I = 0 \tag{2.35}$$

$$\delta\Lambda = 0 \tag{2.36}$$

These provide the set of constraints that select the surviving Killing spinors in two dimensions. In order to find non-trivial solutions, we choose the R-symmetry V_μ^A and A_μ background fields such that

$$R_{\mu\nu}(V^3) = -a_1\Omega_{\mu\nu}\ , \quad R_{\mu\nu}(V^8) = -\sqrt{3}a_2\Omega_{\mu\nu}\ , \quad R_{\mu\nu}(V^A) = 0 \quad \text{for} \quad A \neq 3,8$$
$$R_{\mu\nu}(A) = -a_3\Omega_{\mu\nu} \tag{2.37}$$

and subject to appropriate quantization conditions (see the remark at the end of Section 2.5). The non-trivial Killing spinor equations then reduce to

$$\begin{aligned} 2\partial_{[\mu}\delta\psi^I_{\nu]} &= \frac{1}{2}R_{\mu\nu}(\omega^{23})\gamma_{23}\varepsilon^I - R_{\mu\nu}(A)\mathrm{i}\gamma_5\varepsilon^I - \left[R_{\mu\nu}(V^3)(\mathrm{i}\lambda_3)^I_J + R_{\mu\nu}(V^8)(\mathrm{i}\lambda_8)^I_J\right]\varepsilon^J \\ &= \mathrm{i}\Omega_{\mu\nu}\left[-\frac{\kappa}{2}\mathrm{i}\gamma_{23}\delta^I_J + a_1(\lambda_3)^I_J + a_2\sqrt{3}(\lambda_8)^I_J + a_3\gamma_5\delta^I_J\right]\varepsilon^J = 0 \end{aligned} \tag{2.38}$$

together with the two auxiliary conditions (2.34, 2.35).

2.4.1. Classification of the solutions. In order to find non-trivial solutions to equation (2.38) we restrict the discussion to the positive chirality components of the ε^I spinors. We observe that under the breaking $Spin(3,1) \times SU(3)_R \times U(1)_R \to Spin(1,1) \times Spin(2)_\Sigma \times U(1)_{\lambda_3} \times U(1)_{\lambda_8} \times U(1)_R$ realized by the chosen geometry, the original 4d chiral parameters ϵ^I_α, $I = 1,2,3$, split as

$$\epsilon^I_\alpha \to \epsilon^1_+ \oplus \epsilon^1_- \oplus \epsilon^2_+ \oplus \epsilon^2_- \oplus \epsilon^3_+ \oplus \epsilon^3_- \tag{2.39}$$

where ± still indicate the 2d chirality as defined in (2.29). The spinors are charged under $U(1)_{\lambda_3} \times U(1)_{\lambda_8} \times U(1)_R$ according to:

	$U(1)_{\lambda_3}$	$U(1)_{\lambda_8}$	$U(1)_R$
$\epsilon^1_\pm$	1	$\frac{1}{\sqrt{3}}$	1
$\epsilon^2_\pm$	−1	$\frac{1}{\sqrt{3}}$	1
$\epsilon^3_\pm$	0	$-\frac{2}{\sqrt{3}}$	1

(2.40)

Supersymmetry preserving equations are then given in Table 7. Once the equation $\delta\psi^I_\mu = 0$ has been solved for a particular set of a_i parameters, equations (2.34, 2.35) need to be satisfied. We defer to Section 2.6 the discussion of the existence of

TABLE 7. Supersymmetry equations for $\mathcal{N}=3$ theories.

supersymmetry	$\delta\psi^I_\mu = 0$
$\epsilon^1_\pm$	$a_1 + a_2 + a_3 \mp \kappa/2 = 0$
$\epsilon^2_\pm$	$-a_1 + a_2 + a_3 \mp \kappa/2 = 0$
$\epsilon^3_\pm$	$-2a_2 + a_3 \mp \kappa/2 = 0$

solutions to $\delta\chi_{IJ} = 0$ and $\delta\zeta^I = 0$. There we show that solutions always exist if we appropriately choose the background value of the auxiliary field D^I_J.

In Table 8 we list all possible solutions to the equations in Table 7 together with the corresponding preserved supersymmetries and the remaining 2d R-symmetry. We focus on the cases with mostly right supersymmetry and for each possibility we pick just one choice of fluxes. All the other possibilities can be obtained through a change of basis for the $SU(3)_R$ generators. In all the $\kappa \neq 0$ cases a $U(1)$ flavor

TABLE 8. Classification of topologically twisted 4d $\mathcal{N}=3$ SCFTs in terms of the surviving amount of supersymmetry in 2d. In the last column we indicate the subgroup of 4d R-symmetry that is compatible with the twisted compactification.

$\kappa = 0$	fluxes	supersymmetries	R-symmetry
$(6,6)$	$a_1 = 0, a_2 = 0, a_3 = 0$	$\epsilon^1_\pm \oplus \epsilon^2_\pm \oplus \epsilon^3_\pm$	$SU(3) \times U(1)$
$(4,4)$	$a_1 = 0, a_2 + a_3 = 0$	$\epsilon^1_\pm \oplus \epsilon^2_\pm$	$SU(2) \times U(1)$
$(2,2)$	$a_1 + a_2 + a_3 = 0$	$\epsilon^1_\pm$	$U(1)$

$\kappa \neq 0$	fluxes	supersymmetries	R-symmetry
$(2,4)$	$a_1 = 0, a_2 = -\frac{\kappa}{3}, a_3 = -\frac{\kappa}{6}$	$\epsilon^3_+ \oplus \epsilon^1_- \oplus \epsilon^2_-$	$SU(2) \times U(1)$
$(0,6)$	$a_1 = 0, a_2 = 0, a_3 = -\frac{\kappa}{2}$	$\epsilon^1_- \oplus \epsilon^2_- \oplus \epsilon^3_-$	$SU(3) \times U(1)$
$(2,2)$	$a_1 = -\frac{\kappa}{2}, a_2 + a_3 = 0$	$\epsilon^2_+ \oplus \epsilon^1_-$	$U(1) \times U(1)$
$(0,4)$	$a_1 = 0, a_2 + a_3 = -\frac{\kappa}{2}$	$\epsilon^1_- \oplus \epsilon^2_-$	$SU(2) \times U(1)$
$(0,2)$	$a_1 + a_2 + a_3 = -\frac{\kappa}{2}$	ϵ^1_-	$U(1)$

symmetry survives in two dimensions, being it associated to the diagonal generator $(\frac{\kappa}{2}\mathrm{i}\gamma_{23} - T)$, where $T = a_1\lambda_3 + a_2\sqrt{3}\lambda_8 + a_3\gamma_5$, under which, by definition, the surviving Killing spinors are neutral. However, in the $\mathcal{N}=(2,4)$ case, one extra $U(1)$ symmetry emerges from the topological twist, which is generated by T itself (or any linear combination of T with the flavor symmetry generator). Although under T the supercharges are charged, this symmetry cannot be a R-symmetry of the low energy SCFT. It might be that this symmetry is not a symmetry of the 2d theory, or that it appears as an outer automorphism of the 2d superalgebra. However, in order to get more insight on it one should know the actual SCFT algebra that emerges from the twisted reduction and the relation of T with the rest of the superalgebra generators. A similar interpretation can be given to the global symmetries found in the $(6,6)$, $(4,4)$ and $(0,6)$, $(0,4)$ solutions where the corresponding superconformal algebras are not know or are not compatible with the global symmetries resulting from our analysis. Another possibility for the $(4,4)$ and $(0,4)$ cases is that in the IR there is an enhancement of symmetry from $SU(2)\times U(1)$ to $SO(4)$ which would then be compatible with a large $\mathcal{N}=4$ 2d superconformal algebra.

From Table 8 we note that, while for $\kappa \neq 0$ we can reduce supersymmetry in two dimensions to $\mathcal{N}=(0,2)$, in the case of the torus the minimum amount of supersymmetry that we obtain by partial topological twist is $\mathcal{N}=(2,2)$. This is a consequence of the fact that in the $\mathcal{N}=3$ case there are no flavor symmetries that can be weakly gauged in order to further reduce supersymmetry.

However, also in the $\kappa=0$ case we can reduce supersymmetry to $\mathcal{N}=(0,2)$ by a two-step procedure similar to the one already discussed in Section 2.3 for $\mathcal{N}=2$ theories without flavor symmetries. This works as follows. First we perform a R-symmetry twist that preserves either four or eight supercharges. This twist breaks R-symmetry as well, leaving some flavor symmetries with the associated vector multiplets. The second step of this reduction is performed by introducing a ($\mathcal{N}=1$ or $\mathcal{N}=2$) background for the vector multiplet that preserves only half of the supercharges. For example, if we use this procedure in the case of $a_1+a_2+a_3=0$ we preserve in the first step a 4d $\mathcal{N}=1$ subalgebra of the original $\mathcal{N}=3$. The left-over R-symmetry is just $U(1)$, while the residual $SU(2)\times U(1)$ from the original $SU(3)_R\times U(1)_R$ survives as flavor symmetry. In the second step we can gauge an abelian subgroup of this flavor symmetry. The corresponding gaugino background then breaks supersymmetry to $\mathcal{N}=(2,0)$ or $\mathcal{N}=(0,2)$ as we can see from (2.14).

2.5. Topological twist in $\mathcal{N}=4$ conformal supergravity

This case has been extensively discussed in the literature [**2, 3, 23, 4, 11**]. For completeness, here we briefly review the main results in the language of conformal supergravity.

The supercharges are in the antifundamental representation of the $SU(4)_R$ R-symmetry group. The generators are traceless hermitian matrices λ_A, $A=1,...,15$. We choose a basis in which the Cartan subalgebra is spanned by

$$\lambda_3 = \mathrm{diag}(1,-1,0,0) \tag{2.41}$$

$$\lambda_8 = \frac{1}{\sqrt{3}}\mathrm{diag}(1,1,-2,0) \tag{2.42}$$

$$\lambda_{15} = \frac{1}{\sqrt{6}}\mathrm{diag}(1,1,1,-3) \tag{2.43}$$

The Weyl multiplet of the $\mathcal{N}=4$ conformal supergravity contains the gauge fields e_μ^a, b_μ V_μ^A and ψ_μ^I, the bosonic auxiliary fields C, E_{IJ}, T_{ab}^{IJ} D_{KL}^{IJ} and the fermionic auxiliaries Λ_I, χ_K^{IJ}. In Table 9 we list the corresponding $SU(4)_R$ representations. For a complete description of $\mathcal{N}=4$ supergravity we refer to [**80, 81**]. As

TABLE 9. Field content of the Weyl multiplet in $\mathcal{N}=4$ conformal supergravity.

field	e_μ^a	b_μ	V_μ^A	C	E_{IJ}	T_{ab}^{IJ}	D_{KL}^{IJ}	ψ_μ^I	Λ_I	χ_K^{IJ}
$SU(4)_R$	**1**	**1**	**15**	**1**	$\bar{\mathbf{10}}$	**6**	**20**	**4**	$\bar{\mathbf{4}}$	**20**
# of real d.o.f.	5	0	45	2	20	36	20	32	16	80

in the previous cases, we define the theory on the curved manifold[3] $M = \mathbb{R}^{1,1} \times \Sigma$, by freezing the Weyl multiplet to contain as only non-vanishing components the vielbein, a R-symmetry background V_μ^A and an auxiliary field D^{IJ}_{KL}. Supersymmetry is (partially) preserved if there exist spinor parameters ε^I_α satisfying

$$0 = \delta\psi^I_\mu = \partial_\mu \varepsilon^I + \frac{1}{4}\omega^{ab}_\mu \gamma_{ab}\varepsilon^I - V^A_\mu (\mathrm{i}\lambda_A)^I{}_J \varepsilon^J \tag{2.44}$$

$$\begin{aligned} 0 = \delta\chi^{IJ}_K = \frac{1}{2} D^{IJ}_{KL}\varepsilon^L - \frac{1}{2}\gamma^{ab} R_{ab}(V^A)(\mathrm{i}\lambda_A)^{[I}_K \varepsilon^{J]} \\ -\frac{1}{6}\gamma^{ab}\delta^{[I}_K R_{ab}(V^A)(\mathrm{i}\lambda_A)^{J]}_L \varepsilon^L \end{aligned} \tag{2.45}$$

while $\delta\Lambda_I$ is identically zero in the selected background. In order to find non-trivial solutions we choose the R-symmetry gauge field such that

$$R_{\mu\nu}(V^3) = -a_1\Omega_{\mu\nu}, \quad R_{\mu\nu}(V^8) = -\sqrt{3}a_2\Omega_{\mu\nu}, \quad R_{\mu\nu}(V^{15}) = -\sqrt{6}a_3\Omega_{\mu\nu}, \tag{2.46}$$

$$R_{\mu\nu}(V^A) = 0 \quad \text{for} \quad A \neq 3, 8, 15 \tag{2.47}$$

subject to appropriate quantization conditions (see the remark at the end of this section). Equations (2.44) and (2.45) then reduce to

$$\begin{aligned} 2\partial_{[\mu}\delta\psi^I_{\nu]} &= \frac{1}{2}R_{\mu\nu}(\omega^{23})\gamma_{23}\varepsilon^I - R_{\mu\nu}(V^A)(\mathrm{i}\lambda_A)^I_J\varepsilon^J \\ &= \mathrm{i}\Omega_{\mu\nu}\left[-\frac{\kappa}{2}\mathrm{i}\gamma_{23}\delta^I_J + a_1(\lambda_3)^I_J + a_2\sqrt{3}(\lambda_8)^I_J + a_3\sqrt{6}(\lambda_{15})^I_J\right]\varepsilon^J \\ &= 0 \end{aligned} \tag{2.48}$$

2.5.1. Classification of the solutions. The selected background induces the breaking $Spin(3,1) \times SU(4)_R \to Spin(1,1) \times Spin(2)_\Sigma \times U(1)_{\lambda_3} \times U(1)_{\lambda_8} \times U(1)_{\lambda_{15}}$ under which the chiral supersymmetry parameters split as

$$\epsilon^I_\alpha \to \epsilon^1_+ \oplus \epsilon^1_- \oplus \epsilon^2_+ \oplus \epsilon^2_- \oplus \epsilon^3_+ \oplus \epsilon^3_- \oplus \epsilon^4_+ \oplus \epsilon^4_- \tag{2.49}$$

where, once again, the ± indices indicate chirality as defined in (2.29). The spinors are charged under $U(1)_{\lambda_3} \times U(1)_{\lambda_8} \times U(1)_{\lambda_{15}}$ according to:

	$U(1)_{\lambda_3}$	$U(1)_{\lambda_8}$	$U(1)_{\lambda_{15}}$
$\epsilon^1_\pm$	1	$\frac{1}{\sqrt{3}}$	$\frac{1}{\sqrt{6}}$
$\epsilon^2_\pm$	-1	$\frac{1}{\sqrt{3}}$	$\frac{1}{\sqrt{6}}$
$\epsilon^3_\pm$	0	$-\frac{2}{\sqrt{3}}$	$\frac{1}{\sqrt{6}}$
$\epsilon^4_\pm$	0	0	$-\frac{3}{\sqrt{6}}$

(2.50)

Therefore, equation (2.48) translates into the set of supersymmetry preserving equations listed in Table 10. For any set of a_i parameters satisfying one of the

TABLE 10. Supersymmetry equations for $\mathcal{N} = 4$ theories.

supersymmetry	$\delta\psi^I_\mu = 0$
$\epsilon^1_\pm$	$a_1 + a_2 + a_3 \mp \kappa/2 = 0$
$\epsilon^2_\pm$	$-a_1 + a_2 + a_3 \mp \kappa/2 = 0$
$\epsilon^3_\pm$	$-2a_2 + a_3 \mp \kappa/2 = 0$
$\epsilon^4_\pm$	$-3a_3 \mp \kappa/2 = 0$

[3]Four dimensional $\mathcal{N} = 4$ superconformal theories on curved backgrounds have been considered in [**85**].

conditions in the previous table, equation (2.45) can be satisfied by a suitable choice of the background auxiliary fields D^{IJ}_{KL} without further constraining the ϵ^I parameters.

In Table 11 we list explicit solutions for the a_i parameters and the corresponding 2d surviving supersymmetry with its R-symmetry group. We focus on the cases with mostly right-handed supersymmetry and for each possibility we pick just one particular configuration of fluxes. Similarly to what happens in the $\mathcal{N} = 3$ case,

TABLE 11. Classification of topologically twisted 4d $\mathcal{N} = 4$ SYM on Riemann surfaces of curvature $\kappa = \pm 1, 0$ in terms of the surviving amount of supersymmetry in 2d. In the last column we indicate the subgroup of 4d R-symmetry that is compatible with the twisted compactification.

$\kappa = 0$	fluxes	supersymmetries	R-symmetry
$(8,8)$	$a_1 = 0, a_2 = 0, a_3 = 0$	$\epsilon^1_\pm \oplus \epsilon^2_\pm \oplus \epsilon^3_\pm \oplus \epsilon^4_\pm$	$SU(4)$
$(4,4)$	$a_1 = 0, a_2 + a_3 = 0$	$\epsilon^1_\pm \oplus \epsilon^2_\pm$	$SU(2) \times U(1)$
$(2,2)$	$a_1 + a_2 + a_3 = 0$	$\epsilon^1_\pm$	$U(1)$

$\kappa \neq 0$	fluxes	supersymmetries	R-symmetry
$(4,4)$	$a_1 = 0, a_2 = -\frac{\kappa}{3}, a_3 = -\frac{\kappa}{6}$	$\epsilon^3_+ \oplus \epsilon^4_+ \oplus \epsilon^1_- \oplus \epsilon^2_-$	$SU(2) \times SU(2)$
$(0,6)$	$a_1 = 0, a_2 = 0, a_3 = -\frac{\kappa}{2}$	$\epsilon^1_- \oplus \epsilon^2_- \oplus \epsilon^3_-$	$SU(3) \times U(1)$
$(2,2)$	$a_1 = -\frac{\kappa}{2}, a_2 + a_3 = 0$	$\epsilon^2_+ \oplus \epsilon^1_-$	$U(1) \times U(1)$
$(0,4)$	$a_1 = 0, a_2 + a_3 = -\frac{\kappa}{2}$	$\epsilon^1_- \oplus \epsilon^2_-$	$SU(2) \times U(1)$
$(0,2)$	$a_1 + a_2 + a_3 = -\frac{\kappa}{2}$	ϵ^1_-	$U(1)$

for the $\mathcal{N} = (4,4)$ solution with $\kappa \neq 0$ one extra $U(1)$ symmetry generated by $T = a_1\lambda_3 + a_2\sqrt{3}\lambda_8 + a_3\sqrt{6}\lambda_{15}$ emerges from the topological twist. Although T acts non-trivially on the supercharges, this cannot be a R-symmetry of the low energy SCFT, but it could be identified as an outer automorphism of the 2d superconformal algebra.

We conclude this analysis by observing that, as in the case of $\mathcal{N} = 3$ theories, although there are no flavor symmetries, we can further reduce supersymmetry by performing a two step reduction. The first step consists of turning on an R-symmetry twist, breaking supersymmetry to $\mathcal{N} = 2$ or $\mathcal{N} = 1$. The second step consists of introducing a background $\mathcal{N} = 2$ or $\mathcal{N} = 1$ vector multiplet for the left-over non-R *flavor* symmetry, such that only half of the supercharges are preserved.

Remark: In the $\mathcal{N} = 3, 4$ cases the background quantization conditions $a_i\eta_\Sigma \in \mathbb{Z}$ used for $\mathcal{N} = 1, 2$ are too restrictive, but fortunately they can be partially relaxed. For example, if we look at the $\mathcal{N} = (4,4)$, $\kappa \neq 0$ case in Table 11 the solutions $a_2 = -\kappa/3$ and $a_3 = -\kappa/6$ would be incompatible with such a quantization condition and consequently the R-symmetry bundle would be ill-defined. However, in this case the quantization condition that one has to actually impose is that the combination $T \equiv a_2\sqrt{3}\lambda_8 + a_3\sqrt{6}\lambda_{15}$ (i.e., the background symmetry that has been gauged by the twist) assigns integer charges to every field/representation of the theory. Substituting the explicit values of a_2 and a_3 we can see that the background symmetry T corresponds precisely to the $U(1)$ R-symmetry of the $\mathcal{N}=(4,4)$ theory

$$T = -\frac{\kappa}{2}\left[\frac{2}{3}(\sqrt{3}\lambda_8) + \frac{1}{3}(\sqrt{6}\lambda_{15})\right] = -\frac{\kappa}{2}\mathrm{diag}(1,1,-1,-1) \tag{2.51}$$

The quantization condition then becomes $\frac{\kappa}{2}\eta_\Sigma \in \mathbb{Z}$, which is satisfied for any choice of genus g. A similar analysis applies to the other cases, leading to the same conclusion.

2.6. Supersymmetry variations of the auxiliary fields

In this section we show that it is always possible to choose a background compatible with the topological twist and supersymmetry such that the variations of the auxiliary fermionic fields in the Weyl multiplet are identically zero.

The case of the $\mathcal{N}=1$ supergravity is trivial as the Weyl multiplet already does not contain any auxiliary fermions. For $\mathcal{N}=2$ this condition is non-trivial (see (2.25)) but a solution can be always found by setting the bosonic auxiliary field $D = \frac{\kappa}{6}|e|\Omega_{23}$. The cases of $\mathcal{N}=3,4$ are more involved and we treat them more carefully in this section.

Note to the reader: in this section we do not assume Einstein summation notation for repeated R-symmetry indices.

We begin by considering the $\mathcal{N}=4$ case. Since we gauge the background R-symmetry along a subgroup of the Cartan of $SU(4)$, the curvature $R(V)^I_J \equiv R(V^A)(\mathrm{i}\lambda_A)^I_J$ is diagonal in the adjoint indices (I,J). As a consequence, the Killing spinor equations (2.48) split into a set of four decoupled equations for ε^I, $I=1,\dots,4$. Non-trivial ε^I solutions correspond to the preserved supersymmetries, whereas the remaining spinor components do not satisfy the Killing equation and must be set to zero. Having this in mind, we now discuss the condition $\delta\chi^{IJ}_K = 0$, where the variation is generated only by the preserved supercharges (i.e., the Killing spinors). Three possible cases can arise.

- If $K \neq I,J$ from (2.45) we immediately find:

$$\delta\chi^{IJ}_K = \frac{1}{2}\sum_L D^{IJ}_{KL}\varepsilon^L = 0 \tag{2.52}$$

  which can be immediately solved by setting the corresponding components D^{IJ}_{KL} to zero.
- The second case corresponds to $K = I \neq J$ with non-vanishing ε^I and ε^J. Restricting as usual to the positive chirality transformation, the χ^{IJ}_I variation reads:

$$\delta\chi^{IJ}_I = \frac{1}{2}D^{IJ}_{IJ}\epsilon^J - \frac{1}{4}\gamma^{ab}R_{ab}(V)^I_I\epsilon^J - \frac{1}{12}\gamma^{ab}R_{ab}(V)^J_J\epsilon^J = 0 \tag{2.53}$$

  where we chose D^{IJ}_{IL} to be diagonal in the J,L indices. After the compactification the ϵ^J spinors decompose as $\mathrm{i}\gamma_{23}$ eigenvectors and we write $\mathrm{i}\gamma_{23}\epsilon^J = s_J\epsilon^J$ with eigenvalue $s_J = \pm 1$ according to the 2d chirality of the spinor. Using equation (2.48):

$$\frac{1}{2}R_{\mu\nu}(\omega^{23})\gamma_{23}\epsilon^J = R_{\mu\nu}(V)^J_J\epsilon^J \quad \text{with} \quad R_{23}(\omega^{23}) = \kappa\Omega_{23} \tag{2.54}$$

  we eventually find:

$$\delta\chi^{IJ}_I = \left[\frac{1}{2}D^{IJ}_{IJ} + \frac{\kappa}{4}|e|\Omega_{23}\left(s_Is_J + \frac{1}{3}\right)\right]\varepsilon^J = 0 \tag{2.55}$$

If $\kappa = 0$ this equation is easily satisfied by $D^{IJ}_{IJ} = 0$. If $\kappa \neq 0$, from Table 11 it turns out that for each given solution ϵ^J only one chirality is present and (2.55) can be always satisfied by an appropriate choice of the D^{IJ}_{IJ} components.

- Finally, if ε^J is Killing but ε^I is not, then then χ^{IJ}_I variations do not vanish in general. However it is possible to show with a case by case analysis that these components always decouple from the representation of the 2d superalgebra hence they are not relevant for the counting of the supersymmetries.

For the $\mathcal{N} = 3$ case, solutions to (2.34, 2.35) can be derived from the general $\mathcal{N} = 4$ solution by recalling that the fermionic auxiliary components of the $\mathcal{N} = 3$ Weyl multiplet can be obtained from the $\mathcal{N} = 4$ ones according to the following decomposition [**83**]:

$$\chi_{(KL)} + \sum_M \varepsilon_{KLM}\zeta^M \equiv \sum_{IJ} \frac{1}{2}\varepsilon_{LIJ4}\chi^{IJ}_K \tag{2.56}$$

$$D^M_N \equiv \sum_{IJKL} \frac{1}{4}\varepsilon^{MKL4}\varepsilon_{NIJ4}D^{IJ}_{KL} \tag{2.57}$$

Therefore, exploiting the previous results, we conclude that also in the $\mathcal{N} = 3$ case it is always possible to choose a non-vanishing D^M_N background that sets the supersymmetry variations $\delta\chi_{IJ}$ and $\delta\zeta^I$ to zero.

2.7. 't Hooft anomalies and 2d central charges

In this section we focus on the special case of two dimensional $\mathcal{N} = (0,2)$ theories obtained by twisted compactification of $\mathcal{N}$-extended supersymmetric theories in four dimensions, as described in the previous sections. In particular, we determine a general expression for the 2d global anomalies and central charges.

Generalizing the prescription developed in [**18**] for $\mathcal{N} = 1$ SCFTs, we begin with the 4d anomaly polynomial I_6 for the $U(1)$ global symmetries, including the abelian symmetry coupled to the twisting supergravity background, and integrate it along the Σ directions. The resulting expression is a 4-form that can be identified with the anomaly polynomial I_4 of the 2d theory. From this expression we can then infer the 2d anomalies as functions of the 4d anomalies and of the background fluxes.

In this procedure we have to take into account that, even if the R-symmetry we start with is the exact R-symmetry in 4d, along the dimensional flow the $U(1)_R$ can mix with other abelian flavor symmetries. The exact 2d central charge is then reconstructed by extremizing a trial central charge as a function of the mixing coefficients [**4**]. Because of this potential mixing, in the reduction procedure we can start with any trial $U(1)$ R-symmetry T_R in four dimensions, as different choices will simply shift the mixing parameters of the 2d theory without affecting the final result of the extremization procedure. We remark here that a necessary condition for c-extremization to be well defined is that there are no accidental continuous global symmetries in the IR. In this section we will assume that this is the case.

2.7.1. Computation of the anomalies. We consider a generic SCFT in four dimensions with an arbitrary amount of supersymmetry that flows to a $\mathcal{N} = (0,2)$ theory in two dimensions. As it turns out to be clear from our discussion in Section 2.2, in the $\mathcal{N} = 1$ case the 4d trial T_R generator can be identified with the original $U(1)$ R-symmetry generator of the $\mathcal{N} = 1$ algebra. We call t_R the corresponding abelian generator in the reduced $\mathcal{N} = (0,2)$ theory. In general the two $U(1)$ symmetries will have different matrix forms but they can still be identified up to a mixing with the abelian flavor symmetries:

$$T_R \to t_R + \sum_{i=1}^{n} \xi_i t_i \tag{2.58}$$

where t_i are the generators of the symmetries $U(1)_i$ in the 2d representation, while ξ_i are the mixing coefficients. The relation (2.58) represents the most general trial 2d R-current, involving abelian currents that do not necessarily mix with the R-current in the 4d SCFT, as is the case for baryonic symmetries in toric quiver gauge theories [**86, 10**].

Our discussion can be applied also to the case of extended supersymmetry. In that case we can identify the generator t_R with the four dimensional R-current of the $\mathcal{N} = 1$ subalgebra. When reducing to 2d $\mathcal{N} = (0,2)$ all the other abelian global currents have to be treated as flavor symmetries that can potentially mix with the 2d R-symmetry. In the rest of this section we restrict to the case of 4d $\mathcal{N} = 1$ SCFT. The case of extended supersymmetry can be analyzed similarly by formulating the theory in $\mathcal{N} = 1$ language.

In order to compute the anomaly polynomial I_6, which encodes all the global and gravitational anomalies of the twisted theory[4], we first couple each global symmetry to a background connection in the two directions orthogonal to the Riemann surface. The topological twist introduces additional background components for $U(1)_R$ and $U(1)_i$ also along the Σ directions. Following the notations of Appendix B.3 we denote x_R the first Chern class of the R-symmetry bundle and x_i the classes associated to the gauging of the abelian $U(1)_i$ flavor symmetries. Then we can write:

$$x_R = x_R^{\text{2d}} + x_R^{\Sigma} \qquad \text{and} \qquad x_i = x_i^{\text{2d}} + x_i^{\Sigma} \tag{2.59}$$

where the components in the direction of Σ are defined by (2.9) and (2.13) as:

$$x_R^{\Sigma} = -a\left[\frac{\Omega}{2\pi}\right] \qquad \text{and} \qquad x_i^{\Sigma} = b_i\left[\frac{\Omega}{2\pi}\right] \tag{2.60}$$

so that the total Chern class of the global symmetry bundle E (see Appendix B.3 for the definition) restricted to the Riemann surface Σ is:

$$c_1(E)\Big|_{\Sigma} = \mathsf{Tr}[\gamma_5 T_R]x_R^{\Sigma} + \sum_i \mathsf{Tr}[\gamma_5 T_i]x_i^{\Sigma} = \mathsf{Tr}[\gamma_5(-aT_R + \textstyle\sum_i b_i T_i)]\left[\frac{\Omega}{2\pi}\right] \tag{2.61}$$

where T_R and T_i are the 4d generators and γ_5 is the 4d chirality operator. Here the twisting parameter a is fixed by the Killing spinor equation (2.11) to the value $-\frac{\kappa}{2}$. We can then interpret the combination $T \equiv \frac{\kappa}{2}T_R + \sum_i b_i T_i$ to be the abelian symmetry which generates the topological twist on Σ.

According to formula (B.23), the anomaly polynomial is given by the 6-form:

$$\begin{aligned} I_6 \;\; = \;\; & \frac{1}{6}\mathsf{Tr}[\gamma_5 T_R^3]x_R^3 + \frac{1}{2}\sum_i \mathsf{Tr}[\gamma_5 T_R^2 T_j]x_R^2 x_i \\ & + \frac{1}{2}\sum_{ij} \mathsf{Tr}[\gamma_5 T_R T_i T_j]x_R x_i x_j + \frac{1}{6}\sum_{ijk}\mathsf{Tr}[\gamma_5 T_i T_j T_k]x_i x_j x_k \\ & - \frac{1}{24}p_1 \mathsf{Tr}[\gamma_5 T_R]x_R - \frac{1}{24}p_1 \sum_i \mathsf{Tr}[\gamma_5 T_i]x_i \end{aligned} \tag{2.62}$$

[4]The gauge theory is assumed to be free of local gauge anomalies, i.e., anomalies for symmetries coupled to dynamical gauge vectors.

where $\mathsf{Tr}[\gamma_5 T_{i_1} \cdots T_{i_\ell}] \equiv k_{i_1 \ldots i_\ell}$ are the degree-ℓ 't Hooft anomaly coefficients of the 4d theory and p_1 is the first Pontryagin class of the gravitational background.

Having compactified the theory on Σ it is natural to identify the anomaly polynomial of the corresponding two-dimensional theory with the expression obtained by integrating I_6 on the Riemann surface (see Appendix B.4 for a discussion on this point). The result of the integration is:

$$\int_\Sigma I_6 = \eta_\Sigma \left[\frac{\mathsf{Tr}[\gamma_5 T_R^2 T]}{2} x_R^2 + \sum_i \mathsf{Tr}[\gamma_5 T_R T_i T] x_R x_i + \sum_{ij} \frac{\mathsf{Tr}[\gamma_5 T_i T_j T]}{2} x_i x_j - \frac{k}{24} p_1 \right] \tag{2.63}$$

which can be compared to the general formula for the anomaly polynomial in 2d:

$$I_4 = \frac{k_{RR}}{2} x_R^2 + \sum_i k_{Ri} x_R x_i + \sum_{ij} \frac{k_{ij}}{2} x_i x_j - \frac{k}{24} p_1 \tag{2.64}$$

leading to the following identities

$$\begin{aligned} k_{RR} &= \eta_\Sigma \mathsf{Tr}[\gamma_5 T_R^2 T] \\ k_{Ri} &= \eta_\Sigma \mathsf{Tr}[\gamma_5 T_R T_i T] \\ k_{ij} &= \eta_\Sigma \mathsf{Tr}[\gamma_5 T_i T_j T] \\ k &= \eta_\Sigma \mathsf{Tr}[\gamma_5 T] \end{aligned} \tag{2.65}$$

where η_Σ is defined as in (2.2). We note that (2.65) relates 4d 't Hooft anomaly coefficients on the right hand side with 2d anomaly coefficients, $k_{AB} \equiv \mathsf{Tr}[\gamma_3^{2\mathrm{d}} t_A t_B]$, on the left hand side.

2.7.2. c-extremization. Assuming that the dimensional flow leads to a 2d fixed point with both supersymmetry and conformal invariance, then the superconformal algebra introduces very precise relations between the central charges of the theory and the global anomalies. More specifically, one can show that for $(0,2)$ theories the central charges must obey:

$$c_r = 3k_{RR} \tag{2.66}$$

$$c_l = c_r - k \tag{2.67}$$

where c_l, c_r are the left/right central charges, $k_{RR} = \mathsf{Tr}[\gamma_3^{2\mathrm{d}} t_R t_R]$ is the quadratic anomaly of the $U(1)_R$ R-symmetry and $k = \mathsf{Tr}[\gamma_3^{2\mathrm{d}}]$ is the gravitational anomaly. In order to match our formulas with those in the literature here we define the 2d chirality operator as $\gamma_3^{2d} = -\gamma_{01}$ (see Appendix A). In the following we will focus mostly on the right-moving central charge as the left-moving one can be straightforwardly derived from c_r once k is known.

In order to compute c_r we first need to determine the exact spectrum of charges of the 2d R-symmetry at the fixed point. As shown in [**4**], this can be obtained by allowing $U(1)_R$ to mix with all the non-anomalous abelian symmetries of the 2d theory and then extremizing the central charge c_r with respect to the mixing parameters. Therefore, reinterpreting equation (2.65) in a two-dimensional language, requires substituting the generator T_R with (2.58). Explicitly, we find:

$$k_{RR}^{\mathrm{trial}} = \eta_\Sigma \left[\xi_i \xi_j \left(\frac{\kappa}{2} k_{ijR} + b_k k_{ijk} \right) + 2\xi_i \left(\frac{\kappa}{2} k_{RiR} + b_j k_{Rij} \right) + \left(\frac{\kappa}{2} k_{RRR} + b_i k_{RRi} \right) \right] \tag{2.68}$$

$$k_{Ri}^{\mathrm{trial}} = \eta_\Sigma \left[\left(\frac{\kappa}{2} k_{ijR} + b_k k_{ijk} \right) \xi_j + \left(\frac{\kappa}{2} k_{RiR} + b_j k_{Rij} \right) \right] \tag{2.69}$$

$$k_{ij} = \eta_\Sigma \left(\frac{\kappa}{2} k_{ijR} + b_k k_{ijk} \right) \tag{2.70}$$

$$k = \eta_\Sigma \left(\frac{\kappa}{2} k_R + b_i k_i \right) \tag{2.71}$$

The mixing parameters ξ_i are now determined by extremizing the trial central charge c_r^{trial}:

$$0 = \frac{\partial c_r^{\text{trial}}}{\partial \xi_i} = 6 k_{Ri}^{\text{trial}} \tag{2.72}$$

which implies:

$$k_{ij}\xi_j + \eta_\Sigma \left(\frac{\kappa}{2} k_{RRi} + b_j k_{Rij} \right) = 0 \tag{2.73}$$

Equation (2.73) can be solved by inverting the matrix k_{ij}, provided that it has non-vanishing determinant. The expression for the extremized central charge is finally given by:

$$c_r = -3\eta_\Sigma^2 \left(\frac{\kappa}{2} k_{RRi} + b_k k_{Rki} \right) k_{ij}^{-1} \left(\frac{\kappa}{2} k_{RRj} + b_l k_{Rlj} \right) + 3\eta_\Sigma \left(\frac{\kappa}{2} k_{RRR} + b_m k_{RRm} \right) \tag{2.74}$$

in terms of the anomaly coefficients of the original four dimensional SCFT.

Solutions to (2.73) can then be plugged in (2.58) to yield the exact R-current and the R-charges of the fields at the superconformal fixed point.

2.8. Conclusions

In this chapter we obtained a complete classifications of solutions of Killing spinor equations for 4d theories partially twisted on a Riemann surface. The methods of conformal supergravity were employed in order to ensure the vanishing of the variations of the background fermionic fields. The solutions were classified by the amount of supersymmetry preserved by the twist and by the global symmetries compatible with the compactification. Further analysis of the 't Hooft anomalies of the 2d theory was performed in order to establish a relation with the anomalies of the 4d theory and to compute, when possible, the central charge via the technique of c-extremization. While providing a unified treatment for the Killing spinor equations in the presence of topological twist, we also show the first examples of twisted compactifications of $\mathcal{N} = 3$ theories giving rise to rather exotic 2d $\mathcal{N} = (2,4)$ and $\mathcal{N} = (0,6)$ solutions. Moreover we find a previously unknown $\mathcal{N} = (0,6)$ twist of $\mathcal{N} = 4$ SYM (see Table 11).

Here we collect a few observation regarding our analysis. Firstly we observe that a generic feature of these type of compactifications is that all symmetries involved in the twisting procedure commute with the complex structure of the 4-dimensional spinor supercharges and for this reason in 2d (where the spinors are real) the supercharges always come in pairs. Holomorphy is preserved and as a result supersymmetry is of the form $\mathcal{N} = (2p, 2q)$ with the R-symmetry being a unitary subgroup of $U(p) \times U(q)$.

Whenever the 2d theory admits an IR fixed point after the dimensional flow one can obtain the superconformal $(0,2)$ R-current and central charges by studying the reduced 't Hooft anomalies and applying the c-extremization procedure. The formula for the central charge in this case has been given in Section 2.7 for the most general type of abelian twist. We remark that when such a superconformal fixed point exists, equation (2.73) determines the mixing coefficients of the flavor symmetries. However these coefficients may differ from the ones appearing in the 4d exact R-current obtained by a-maximization. There are abelian currents that do not mix in 4d but their mixing in 2d is in general not excluded. This was first observed in [**18**] for the case of the baryonic symmetries of the infinite family of Y^{pq} models of [**28**].

In the next chapter we will show that this is indeed a generic phenomenon by applying our results to several classes of 4d quiver SCFTs with a gravitational dual. These models are described by D3 branes probing the conical singularity of some Calabi-Yau three-fold and are characterized by the presence of two $U(1)$ mesonic flavor symmetries and many $U(1)$ baryonic symmetries. These baryonic symmetries arise as the non-anomalous combinations of the $U(1)$ gauge factors of the quiver, which in the IR become free and decouple. While in the Y^{pq} case there is just one such baryonic symmetry, in other cases one can have a richer structure. The formalism developed in this section is therefore necessary for extending the discussion to such families.

We conclude with a comment on the reduction of non-Lagrangian theories. As described in this chapter the computation of the 2d 't Hooft anomalies relies on our ability to compute the anomaly coefficients of the 4d theory. In the case of non-Lagrangian theories it would appear that one cannot obtain this information by explicitly computing traces of matrix generators, however in some cases there other methods to compute the anomaly polynomial, e.g. from the geometry of the holographic dual theory for example. For the $\mathcal{N} = 3$ theories that we study in Section 2.4 this is not necessary as in this case the theories are obtained by deforming $\mathcal{N} = 4$ SYM to a strong coupling point in the moduli space. Because of this the interactions become non-Lagrangian but the field content is still known exactly from the representation theory of the superalgebra.

CHAPTER 3

Twisted Compactification of Toric Gauge Theories

3.1. Overview

In the previous chapter we have been able to study and classify the possible 2d theories obtained by partial topological twist and compactification by using the formalism of conformal supergravity. The analysis we performed so far is universal in the sense that it does not depend on the particular details of the theory in consideration except for number of supersymmetries. The prescription to obtain the 't Hooft anomalies of the reduced theory was also derived in a manned that can be generically applied to any 4d SCFT.

In this chapter we focus on the application of our results to the study of specific models. In particular we concentrate on the problem of finding a geometric prescription to compute the central charge in a specific set of quiver gauge theories whose moduli space is a toric variety. Because of this peculiar property we are able to make use of the tools of toric geometry and obtain exact results regarding the central charges of the 2d theories. The general formula that expresses c_r at large N in terms of the toric data of the 4d parent theory is computed. Having assigned the generator $\mathcal{T}$ of the topological twist in four dimensions, our prescription is based on the non-trivial identification of the abelian fluxes on the internal manifold with the $\mathcal{T}$-charges of the Perfect Matching (PM) variables associated to the toric diagram of the 4d theory, and the mixing parameters in the 2d trial central charge with the PM R-charges. Assuming the validity of these identifications, the resulting formula for the trial central charge turns out to be very compact. Precisely, it reads

$$c_r = \frac{3\eta_\Sigma N^2}{2} |\det(V_I, V_J, V_K)| n_{\pi_I} \Delta_{\pi_J} \Delta_{\pi_K} \tag{3.1}$$

where the PM R-charges Δ_{π_J} and their charges n_{π_I} under the symmetry $\mathcal{T}$ satisfy the constraints (3.9) and (3.23), respectively, as a remnant of the 4d conformal symmetry. The exact central charge is obtained by extremizing this expression as a function of Δ_{π_I}.

Result (3.1) represents the field theory dual of the holographic formula obtained by studying the $\text{AdS}_5 \to \text{AdS}_3$ flow in gauged supergravity, in the presence of a generic number of vector multiplets [**11, 12, 20, 15**].

We provide several checks of our proposal by applying prescription (3.1) to 2d SCFTs for which the field content and the corresponding charges are known as functions of the mixing parameters and the fluxes on Σ. In all the cases the central charge coincides with the large N expression computed using the prescription in [**4**]. Moreover, we extend our prescription to the case of twisted compactification of 4d toric theories with singular horizons.

As already mentioned, in four dimensions the exact R-symmetry of toric quiver gauge theories is a mixture of the $U(1)^3$ symmetries of X_5, whereas the baryonic ones, corresponding to the non-anomalous combination of the $U(1) \subset U(N)$ gauge groups, decouple in the IR and do not play any role. When flowing to two dimensions, baryonic symmetries can mix with the exact R-current as explicitly shown in the particular case of $X_5 = \mathsf{Y}^{pq}$ for which there is a single baryonic symmetry [**18**].

By studying several examples of increasing complexity, we show that this picture is general and holds for models with a larger amount of baryonic symmetries.

The chapter is organized as follows. In Section 3.2 we review some basic aspects of toric quiver gauge theories. In Section 3.3 we derive the expression of c_r in terms of the toric data of the 4d theory. We first study cases with smooth horizons, correctly reproducing the behavior of c_r as a function of the R-charges. We then confirm the validity of this formula by studying several examples of increasing complexity. In Section 3.4 we consider the case of non-smooth horizons, and describe the prescription for obtaining c_r in these cases. In Section 3.5 we study the compactification of del Pezzo gauge theories, dP_2 and dP_3, with two and three non-anomalous baryonic symmetries, respectively, showing their mixing in the exact 2d R-current. Finally, we study a case with a generic number of baryonic symmetries, by showing the mechanism in necklace quivers, denoted as L^{pqp} theories.

3.2. Review of toric quiver gauge theories

We start our discussion by reviewing the main aspects of toric quiver gauge theories and their twisted compactification on Riemann surfaces. For exhaustive reviews on toric gauge theories we refer the reader to [**29, 30**].

Toric quiver gauge theories [**87**] describe the near horizon limit of a stack of N D3 branes probing the tip of a CY$_3$ cone over a 5d SE X_5, characterized by a $U(1)^3$ action on the metric. The dual $\mathcal{N} = 1$ SCFTs are described by quiver gauge theories whose nodes carry $U(N)$ gauge factors and are connected by oriented arrows, representing bifundamental matter fields.

In order to exemplify the discussion we consider the explicit case of a gauge theory living on a stack of N D3 branes probing the first del Pezzo singularity, dP_1. It has four gauge groups and the corresponding quiver is represented in Figure 1. The superpotential:

$$W = -\epsilon_{\alpha\beta} X_{12} X_{23}^{(\alpha)} X_{34}^{(3)} X_{41}^{(\beta)} + \epsilon_{\alpha\beta} X_{23}^{(\alpha)} X_{34}^{(\beta)} X_{42} + \epsilon_{\alpha\beta} X_{13} X_{34}^{(\alpha)} X_{41}^{(\beta)} \tag{3.2}$$

is subject to the toric condition, which requires that each field appears in exactly two terms having opposite signs. This model has a $SU(2) \times U(1)$ flavor symmetry that, together with the $U(1)_R$ R-symmetry, builds up the isometry group of dP_1. In general, there are also baryonic symmetries associated to the non-trivial second cohomology group of X_5. These symmetries can be obtained from the $U(1) \subset U(N)$ gauge factors. They are IR free and at low energies decouple from the dynamics, becoming global symmetries. In quivers with a chiral-like matter content as the ones considered here, some of these $U(1)$'s are anomalous. The non-anomalous abelian factors correspond to the aforementioned baryonic symmetries. For the specific example of dP_1, to begin with there are four $U(1)_i \subset U(N)_i$ global symmetries of baryonic type with $T_{i=1,\dots,4}$ generators. Two combinations are anomalous and one decouples. We are then left with just a single non-anomalous baryonic symmetry that can be for example identified with the combination $2T_1 - T_2 + T_3$.

When flowing to the IR fixed point abelian flavor symmetries can mix with the R-current to form the exact R-symmetry, whereas the baryonic symmetries do not mix, as discussed in [**86, 10**]. This is a general feature of this family of 4d SCFTs.

For a quiver theory with n_G gauge groups the mixing coefficients of global symmetries into the exact R-symmetries are obtained by maximizing the central

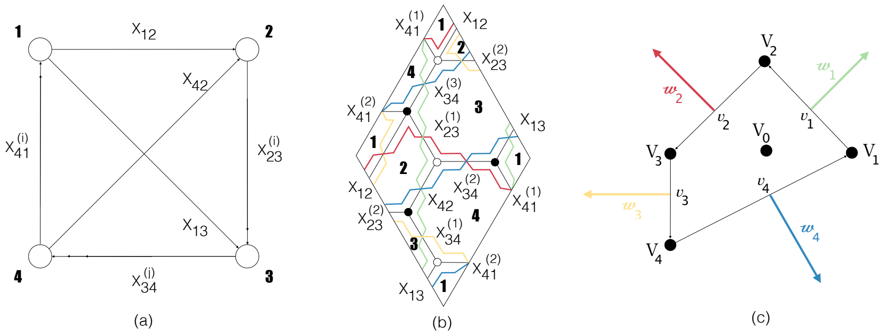


FIGURE 1. Quiver, dimer and toric diagram of the dP_1 model. In quiver (a) the number of arrows on the straight lines indicates the number of fields connecting two nodes. In figure (c) primitive normal vectors w_I of the toric diagram are also indicated and the different colors clarify their relation with the zig-zag paths in the dimer, figure (b). This is useful for reading the R-charges of the fields in terms of the charges of the zig-zag paths or of the perfect matchings.

charge [**9**]

$$\begin{aligned} a_{FT} &= \frac{3}{32}(3\,\mathsf{Tr}R^3 - \mathsf{Tr}R) \\ &= \frac{3}{32}\left[2n_G(N^2-1) + \sum_{i=1}^{n_F} \dim(\rho_i)(3(R_i-1)^3 - (R_i-1))\right] \end{aligned} \tag{3.3}$$

where the first term is the contribution of the gaugini, n_F it the total amount of matter multiplets, $\dim(\rho_i)$ is the dimension of the corresponding representation and R_i the R-charge of the scalar component of the i-th multiplet. For matter multiplets in the bifundamental and/or adjoint representations, at large N the central charge is further simplified by the constraint $\mathsf{Tr}R = 0$ and we read:

$$a_{FT} = \frac{9}{32}N^2\left(n_G + \sum_{i=1}^{n_F}(R_i-1)^3\right) + \mathcal{O}(1) \tag{3.4}$$

For toric gauge theories the R_i charges can be determined directly from the geometric data of the singularity [**33, 34, 35, 36, 37, 10, 38, 39, 31, 32, 40, 41, 42**], as we now review.

First of all, we recollect how to construct the toric diagram corresponding to a given quiver gauge theory. One embeds the quiver diagram in a two dimensional torus. The resulting planar diagram can be dualized by inverting the role of faces and nodes, thus obtaining a bipartite diagram, called dimer, where faces correspond to gauge groups, edges to chiral fields and nodes to superpotential interactions. For the dP_1 model these diagrams are represented in Figure 1 (a) and (b). The toric condition of the superpotential translates into a bipartite structure of the dimer. From the dimer one can construct the so called perfect matchings, i.e., collections of edges X_{ij} (chiral fields) characterized by the property that each node is connected to one and only one edge of the PM. One can then introduce a new set of formal variables π_I associated to each PM. These variables are defined by the relations:

$$X_{ij} \equiv \prod_I (\pi_I)^{M_I(X_{ij})} \tag{3.5}$$

where the product is taken over all the PMs and

$$M_I(X_{ij}) = \begin{cases} 0 & \text{if } X_{ij} \text{ does not belong to the set of } \pi_I \\ 1 & \text{if } X_{ij} \text{ belongs to the set of } \pi_I \end{cases} \tag{3.6}$$

The π_I's provide a convenient set of variables that can be used to parametrize the abelian moduli space of the quiver gauge theory. The advantage of using the PMs variables π_I instead of the more natural set of scalar components of the chiral fields X_{ij} comes from the fact that using definition (3.5) the F-term equations are trivially satisfied. This is a consequence of the fact that, in this basis, each term in the superpotential becomes equal to $\pm \prod_I \pi_I$, with I ranging over all PMs. To each PM we can associate a signed intersection number, ±1 or 0, with respect to a basis of 1-cycles of the first homology group of the 2-torus. The signs can be inferred from the bipartite structure of the dimer. For each PM these two intersection numbers are the first two coordinates of 3d vectors V_I inside a 3d integer lattice and they define a convex integral polygon, named toric diagram. The Calabi-Yau condition fixes the third coordinate of the "primitive vectors" V_I to be equal to 1.

In the dP_1 case, the toric diagram is given by the 3d vectors V_I that are associated to the PM's as follows:

PM	primitive vector
$\pi_1 = \{X_{34}^{(3)}, X_{42}, X_{13}\}$	$V_1 = (1,0,1)$
$\pi_2 = \{X_{23}^{(1)}, X_{34}^{(1)}, X_{41}^{(1)}\}$	$V_2 = (0,1,1)$
$\pi_3 = \{X_{12}, X_{34}^{(1)}, X_{34}^{(2)}\}$	$V_3 = (-1,0,1)$
$\pi_4 = \{X_{23}^{(2)}, X_{34}^{(2)}, X_{41}^{(2)}\}$	$V_4 = (-1,-1,1)$
$\pi_5 = \{X_{12}, X_{13}, X_{42}\}$	$V_0 = (0,0,1)$
$\pi_6 = \{X_{13}, X_{23}^{(1)}, X_{23}^{(2)}\}$	$V_0 = (0,0,1)$
$\pi_7 = \{X_{34}^{(1)}, X_{34}^{(2)}, X_{34}^{(3)}\}$	$V_0 = (0,0,1)$
$\pi_8 = \{X_{41}^{(1)}, X_{41}^{(2)}, X_{42}\}$	$V_0 = (0,0,1)$

(3.7)

and the corresponding toric diagram is drawn in Figure 1 (c).

It is also useful to introduce the notion of zig-zag paths. Given the set of primitive vectors V_I, one can define primitive normal vectors w_I, orthogonal to the edges of the toric diagram, $v_I \equiv (V_{I+1} - V_I)$, $I = 1,..,4$ with $V_5 \equiv V_1$ (see Figure 1 (c)). These vectors are in 1-to-1 correspondence with a set of paths, made out of edges of the dimer, called zig-zag paths and represented in Figure 1 (b). They are oriented closed loops on the dimer which turn maximally left (right) at the black (white) nodes. The zig-zag paths correspond to differences of consecutive PM's lying at the corners of the toric diagram (and, if present, on the perimeter) and are associated to the $U(1)$ global symmetries of the superpotential.

Conversely, given a particular toric diagram with d external points, it is possible to identify the main features of the corresponding quiver gauge theory as follows:

- The number of $U(N)$ gauge groups describing the quiver is given by twice the area of the toric diagram.
- The matter content of the theory (type of bifundamental fields and their degeneracy) can be inferred from the edges v_I of the toric diagram [**88**], up to Seiberg duality, or equivalently toric phases, corresponding to Yang-Baxter transformations on the zig zag paths [**89**]. In its minimal toric phase, a set Φ_{IJ} of bifundamental fields X_{ij} is assigned to each pair $(I,J)_{I,J=1,..,d}$ with degeneracy $|\det(v_I, v_J)|$. See [**38**] for a geometrical interpretation of this fact in terms of zig-zag paths.

- The corresponding spectrum of charges $R_{IJ} \equiv R[\Phi_{IJ}]$ is determined by assigning a R-charge Δ_{π_I} to each PM on the boundary of the toric diagram and using the prescription [**38**]:

$$R_{IJ} = \begin{cases} \sum_{K=I+1}^{J} \Delta_{\pi_K} & I < J \\ 2 - \sum_{K=J+1}^{I} \Delta_{\pi_K} & I > J \end{cases} \tag{3.8}$$

where the charges Δ_{π_I} are subject to the constraint [1]:

$$\sum_{I=1}^{d} \Delta_{\pi_I} = 2 \tag{3.9}$$

to ensure that each superpotential term has R-charge equal to 2.
- The number d of external vertices of the toric diagram also determines the total number of non-anomalous $U(1)$ global symmetries of the gauge theory, which are identified as one R-symmetry, two flavor symmetries and $(d-3)$ baryonic symmetries. Analogously to (3.9), the condition for the superpotential to be neutral with respect to any non-R symmetry translates into:

$$\sum_{I=1}^{d} Q^{\mathbf{J}}_{\pi_I} = 0 \qquad \mathbf{J} = 1, \dots, d-1 \tag{3.10}$$

where $Q^{\mathbf{J}}_{\pi_I}$ is the charge of the I-th PM with respect to the $\mathbf{J}$-th symmetry.
- From the geometric data one can also identify the anomalies of the theory. The crucial observation is that the areas of the triangles of the toric diagram are related to the coefficients of the 't Hooft anomalies of the field theory as [**31, 32**]:

$$\mathsf{Tr}_{4\mathrm{d}}(\mathcal{T}_I \mathcal{T}_J \mathcal{T}_K) = \frac{N^2}{2} |\det(V_I, V_J, V_K)| \tag{3.11}$$

where $\mathcal{T}_I$ are global symmetry generators[2] and the trace $\mathsf{Tr}_{4\mathrm{d}}$ is taken over the 4d fermions with the insertion of the 4d chirality operator. The central charge can be written as [**32**]:

$$a_{\mathrm{geom}} \equiv \frac{9}{64} N^2 \, |\det(V_I, V_J, V_K)| \, \Delta_{\pi_I} \, \Delta_{\pi_J} \, \Delta_{\pi_K} \tag{3.12}$$

This expression is equivalent to (3.4) once we take into account the mapping between the two sets of charges R_i and Δ_{π_I} in (3.8).

In the case of the dP_1 model, using definition (3.5) we find the following map between the sets of fields Φ_{IJ} obtained from the toric diagram and the fields given by the quiver description:

(I,J)	$\lvert\det(v_I, v_J)\rvert$	Φ_{IJ}	R_{IJ}
$(4,1)$	3	$\{X_{13}, X_{34}^{(3)}, X_{42}\}$	Δ_{π_1}
$(1,2)$	2	$\{X_{23}^{(1)}, X_{41}^{(1)}\}$	Δ_{π_2}
$(2,3)$	1	$\{X_{12}\}$	Δ_{π_3}
$(3,4)$	2	$\{X_{23}^{(2)}, X_{41}^{(2)}\}$	Δ_{π_4}
$(1,3)$	1	$\{X_{34}^{(1)}\}$	$\Delta_{\pi_2} + \Delta_{\pi_3}$
$(2,4)$	1	$\{X_{34}^{(2)}\}$	$\Delta_{\pi_3} + \Delta_{\pi_4}$

(3.13)

[1] A geometrical interpretation of (3.9) can be given in terms of the *isoradial embedding* of [**89**].

[2] Here we use calligraphic symbols to indicate that the generators are written in the basis associated to the PM variables.

where Δ_{π_I} satisfy (3.9) and all internal PMs are assigned zero charge under all $U(1)$ symmetries.

The example we have considered has a smooth horizon where all the external points of the toric diagram correspond to corners. In this case the prescription for assigning R-charges to the bifundamental fields is unambiguously given in (3.8). In the case of singular horizons there are also points on the perimeter of the toric diagram which do not correspond to any corner. These points have a degeneracy (given by a binomial coefficient), as they correspond to more than one PM. The assignment of the R-charges in terms of the external PM's may then become ambiguous. According to the prescription in [**38, 39**], at these points one sets to zero the R-charges of the PM's that do not determine any zig-zag path, being then left with an unambiguous assignment of Δ_{π_I} charges.

The holographic correspondence provides the following relation between the central charge and the X_5 volume [**33**]:

$$a_{\text{holo}} \equiv \frac{N^2\pi^3}{4\,\text{vol}(X_5(\mathsf{b}))} \tag{3.14}$$

where the volume is parameterized in terms of the components of the Reeb vector b, a constant norm Killing vector that commutes with the isometries of X_5. It follows that the a-maximization prescription that determines the exact R-current in field theory corresponds to the volume minimization in the gravity dual. When the cone over X_5 is toric, the central charge can be directly obtained from the toric geometry. In fact, the X_5 volume can be expressed as [**35**]:

$$\text{vol}(X_5) = \frac{\pi}{6}\sum_{I=1}^{d}\text{vol}(\Sigma_I) \tag{3.15}$$

where d represents the number of vertices and $\text{vol}(\Sigma_I)$ corresponds to the volume of a 3-cycle Σ_I, on which D3 branes, corresponding to dibaryons, are wrapped [**90**]. Holographic data also determines the charges Δ_{π_I} which can be parameterized in terms of the components of the Reeb vector b. Using the explicit parameterization [**34**]:

$$\Delta_{\pi_I}(\mathsf{b}) = \frac{\pi}{3}\frac{\text{vol}(\Sigma_I(\mathsf{b}))}{\text{vol}(X_5(\mathsf{b}))} \tag{3.16}$$

it is straightforward to show the equivalence between a_{geom} in (3.12) and a_{holo} in (3.14).

3.2.1. Partial topological twist and compactification. Following the prescription that we have developed in Chapter 2 for topologically twisted compactifications of 4d theories on Riemann surfaces, here we proceed to compactify the toric models previously described.

The most general twist is performed along the generator:

$$T = \kappa T_R + \sum_{\mathbf{I}=1}^{n_A} b_{\mathbf{I}} T_{\mathbf{I}} \tag{3.17}$$

where κ is the normalized curvature defined in (2.2). Here n_A refers to the number of abelian $T_{\mathbf{I}}$ generators of non-R global symmetries (both flavor and baryonic ones) and $b_{\mathbf{I}}$ are the corresponding background fluxes. For convenience we have defined all the abelian symmetry generators to act with half-integer charges in the same way that the generator of the rotations in the tangent space to Σ acts with half-integer charges. This way we can rescale all the background fluxes and get rid of the factors of 1/2 that appeared in the formulas of the previous chapter.

For generic choices of the fluxes we can obtain 2d $\mathcal{N} = (0,2)$ supersymmetry with a $U(1)_R$ R-symmetry generated by a combination of the 4d generator T_R together with the other generators $T_{\mathbf{I}}$:

$$R = T_R + \sum_{\mathbf{I}=1}^{n_A} \epsilon_{\mathbf{I}} T_{\mathbf{I}} \tag{3.18}$$

where $\epsilon_{\mathbf{I}}$ are the mixing coefficients and $T_R, T_{\mathbf{I}}$ are meant to act on fields reorganized in 2d representations.

At the IR fixed point the 2d central charge c_r is extremized as in (2.74) and the mixing coefficients are given by (2.73):

$$\epsilon^*_{\mathbf{I}} = -\eta_\Sigma k^{-1}_{\mathbf{IJ}} \left(k_{R\mathbf{JK}} b_{\mathbf{K}} + \kappa k_{RR\mathbf{J}} \right) \tag{3.19}$$

where:

$$k_{\mathbf{IJ}} = \eta_\Sigma \left(\kappa k_{R\mathbf{IJ}} + b_{\mathbf{K}} k_{\mathbf{KIJ}} \right) \tag{3.20}$$

From (3.19) we observe that coefficients $\epsilon^*_{\mathbf{I}}$ are generically non-vanishing for any choice of the $b_{\mathbf{I}}$ fluxes. In particular, this is true for the coefficients associated to baryonic symmetries, which then do mix with the exact R-current in two dimensions, even if they do not in the original 4d theory. This pattern has been already observed in [**18**] for the Y^{pq} family. In Section 3.5 we will study toric quiver gauge theories with a larger amount of baryonic symmetries, confirming that they generically mix with the 2d exact R-current after the twisted compactification.

Starting from a 4d toric theory with n_G $U(N)$ gauge groups and n_F massless chiral fermions, to each 2d field surviving the compactification on Σ we can associate a T-charge n_i and a R-charge R_i according to (see eqs. (3.17) and (3.18))

$$n_i = \kappa\, r_i + Q^{\mathbf{J}}_i b_{\mathbf{J}}, \quad R_i = r_i + Q^{\mathbf{J}}_i \epsilon_{\mathbf{J}} \qquad i = 1, \dots, n_F \tag{3.21}$$

where r_i is the R-charge respect to the 4d R-current and $Q^{\mathbf{J}}_i$ is the charge matrix of the fermions respect to the global $U(1)$ non-R symmetries, inherited from the 4d parent fields. Therefore, applying prescription (2.74) we find that at large N the central charge before extremization is given by

$$c_r = 3N^2 \eta_\Sigma \left(\kappa\, n_G + \sum_{i=1}^{n_F} (n_i - \kappa)(R_i - 1)^2 \right) + \mathcal{O}(1) \tag{3.22}$$

This formula is general and applies to any 2d SCFT obtained from compactification of a 4d quiver gauge theory on a Riemann surface with curvature κ. Through R_i it depends parametrically on the mixing coefficients $\epsilon_{\mathbf{I}}$ that need to be determined by the 2d extremization procedure.

As reviewed in the previous section, in the case of 4d toric quiver theories we can parametrize the $U(1)$ charges in terms of PM variables Δ_{π_I} and $Q^{\mathbf{J}}_{\pi_I}$. When twisting, we can also assign to PMs a further n_{π_I} charge with respect to the twisting T symmetry (3.17) as (in this case $n_A = d - 1$)

$$n_{\pi_I} = \kappa \Delta_{\pi_I} + b_{\mathbf{J}} Q^{\mathbf{J}}_{\pi_I} \qquad \text{with} \qquad \sum_{I=1}^{d} n_{\pi_I} = 2\kappa \tag{3.23}$$

where the constraint on n_{π_I} follows from (3.9) and (3.10).

The r_i and $Q^{\mathbf{J}}_i$ charge assignments in two dimensions, eq. (3.21), need necessarily to respect the original constraints arising from the condition of superconformal invariance for the 4d superpotential. In particular, given the superpotential

$W = \sum_\alpha W_\alpha$, these constraints imply that for each superpotential term W_α the conditions $\sum_{i\in W_\alpha} r_i = 2$ and $\sum_{i\in W_\alpha} Q_i^{\mathbf{J}} = 0$ hold. Consequently, from (3.21) we read

$$\sum_{i\in W_\alpha} R_i = 2\,, \qquad \sum_{i\in W_\alpha} n_i = 2\kappa \tag{3.24}$$

Now, we can think of the dimensional flow from the original 4d theory to the resulting 2d one as being accompanied by the set of toric data $(\Delta_{\pi_I}, Q^{\mathbf{J}}_{\pi_I}, n_{\pi_I})$ that parametrize the $U(1)$ charges in 4d and, consequently, that can still be used to parametrize the corresponding charges in two dimensions. Using this parametrization reinterpreted as charge parametrization for 2d fields, constraints (3.24) are traded with (3.9), (3.10) and (3.23).

3.3. Central charges from toric geometry

For the class of 2d SCFTs obtained from the topologically twisted reduction of toric quiver gauge theories, we now provide a general prescription for determining the central charge c_r directly in terms of the geometry of the toric diagram associated to the original 4d parent theory. This is the main result of the chapter, which we are going to check in the successive sections for a number of explicit examples.

3.3.1. Reading the 2d central charge from the toric diagram. To this end, we consider a toric gauge theory twisted along the abelian generator:

$$T = \sum_{I=1}^{d} a_I \mathcal{T}_I \qquad \text{with} \qquad \sum_{I=1}^{d} a_I = 2\kappa \tag{3.25}$$

where I runs over the d external points of the toric diagram. To be consistent with the conventions used so far, the abelian $\mathcal{T}_I$ generators are chosen so that they assign charge one to the superpotential of the 4d theory. It is always possible to construct such a set of generators by combining the generators of the 4d trial R-current, the two flavor symmetries and the $(d-3)$ non-anomalous baryonic symmetries that appear in (3.17). The new fluxes are subject to the constraint in (3.25) in order to ensure $\mathcal{N} = (0,2)$ supersymmetry in 2d. They need to be further constrained in such a way that each flux $b_{\mathbf{I}}$ in (3.17) is properly quantized.

Accordingly, the 2d trial R-symmetry can be written as

$$R = \sum_{I=1}^{d} \epsilon_I \mathcal{T}_I \qquad \text{with} \qquad \sum_{I=1}^{d} \epsilon_I = 2 \tag{3.26}$$

where the constraint follows from the requirement for R to be a canonical normalized R-current.

The 2d central charge c_r expressed in terms of the 4d anomaly coefficients $\mathsf{Tr}_{4\text{d}}(\mathcal{T}_I\mathcal{T}_J\mathcal{T}_K)$, the a_I fluxes and the mixing parameters ϵ_I becomes (see eq. (2.74))

$$c_r = 3\,\eta_\Sigma \mathsf{Tr}_{4\text{d}}(TR^2) = 3\,\eta_\Sigma \mathsf{Tr}_{4\text{d}}(\mathcal{T}_I\mathcal{T}_J\mathcal{T}_K)\, a_I\epsilon_J\epsilon_K \tag{3.27}$$

In the case of toric theories the anomaly coefficients are given by (3.11) in terms of the areas of the triangles of the toric diagram. Therefore, the 2d central charge can be rewritten as

$$c_r = \frac{3\eta_\Sigma N^2}{2} |\det(V_I, V_J, V_K)| a_I \epsilon_J \epsilon_K \tag{3.28}$$

In order to complete the map between the 2d field theory and the 4d geometric data we need to find a prescription for parametrizing the a_I fluxes and the mixing parameters ϵ_I in terms of the PM's associated to the external vertices of the toric diagram. To this end, we observe that the constraints satisfied by a_I and ϵ_I, eqs.

(3.25, 3.26), are the same as the constraints satisfied by Δ_{π_I}, eq. (3.9) and n_{π_I}, eq. (3.23), and we are naturally led to identify $\epsilon_I \equiv \Delta_{\pi_I}$ and $a_I \equiv n_{\pi_I}$. Therefore, in the large N limit the central charge c_r for the 2d SCFT obtained from a 4d toric quiver gauge theory topologically twisted on a 2d Riemann surface can be expressed entirely in terms of the toric data by the formula

$$c_r = \frac{3\eta_\Sigma N^2}{2} |\det(V_I, V_J, V_K)| n_{\pi_I} \Delta_{\pi_J} \Delta_{\pi_K} \tag{3.29}$$

with Δ_{π_J} and n_{π_I} satisfying constraint (3.9) and (3.23). The exact central charge for the 2d SCFT is then obtained by extremizing (3.29) as a function of Δ_{π_I}.

We note that equation (3.29) gives also the left central charge c_l. In fact, in the large N limit the gravitational anomaly $k = c_r - c_l$ is vanishing, being a linear combination of $\mathsf{Tr}_{4\text{d}} R$ and $\mathsf{Tr}_{4\text{d}} T_{\mathbf{I}}$ that for toric theories are subleading in N [**91**] (whereas the traces of the baryonic symmetries vanish also at finite N [**10**]).

Our proposal (3.29) requires some direct check on explicit examples that we report below. However, a holographical confirmation can be already found in the analysis of the $\text{AdS}_5 \to \text{AdS}_3$ flow engineered in gauged supergravity [**23**]. In this case we need to consider a consistent truncation of $\text{AdS}_5 \times X_5$, a 5d theory with a gravity multiplet, n_V vector multiplets and n_H hypermultiplets. The graviphoton plays the role of the R-symmetry current, while the n_V vector multiplets correspond to the non-R global currents of the holographic dual field theory that remain as massless vector multiplets in a given truncation. In general $n_V \leq n_A$. The hypermultiplets impose constraints that correspond to the vanishing of the R-current anomaly in the dual field theory [**37**]. When flowing to AdS_3 and using the Brown-Henneaux formula [**92**] in this setup, it was observed [**12, 11, 20**] that c_r can be expressed in terms of R-charges $\hat{r}^I$ and fluxes $\hat{a}^J$ as

$$c_r = \frac{2\pi^3 N^2 \eta_\Sigma}{3\text{vol}(X_5)} C_{IJK} \hat{a}^I \hat{r}^J \hat{r}^K \tag{3.30}$$

where the constraints $\sum \hat{r}^I = 2$ and $\sum \hat{a}^I = 2\kappa$ need to be imposed. In this formula C_{IJK} are the Chern-Simons coefficients of the dual supergravity, the R-charges $\hat{r}^I$ are obtained from the sections of the special geometry corresponding to the (constrained) scalars in the vector multiplets, and the prepotentials of $\mathcal{N} = 2$ AdS_5 gauged supergravity. The constants $\hat{a}^I$ are the coefficients of the volume forms in the reduction of the 5d vector multiplets to 3d.

On the other hand, the C_{IJK} coefficients are the holographic duals of the cubic 't Hooft anomaly coefficients, which for toric quiver gauge theories correspond to the areas of the triangles in the toric diagrams, eq. (3.11). Therefore

$$C_{IJK} = \frac{N^2}{2} |\det(V_I, V_J, V_K)| \tag{3.31}$$

If we naturally identify the R-charges $\hat{r}^I$ with the Δ_{π_I} charges assigned to the PM's, and similarly the $\hat{a}_I$ fluxes with the set of n_I fluxes (they satisfy the same constraints $\sum \Delta_{\pi_I} = 2$ and $\sum n_{\pi_I} = 2\kappa$) we obtain our proposal (3.29).

In the remaining part of this section we test formula (3.29) on examples of increasing complexity. As a warm-up we consider the cases of $X_5 = \mathbb{S}^5$ corresponding to $\mathcal{N} = 4$ SYM and $X_5 = T^{1,1}$ corresponding to the conifold [**93**]. Then we move to two more complicated cases, namely the second and third del Pezzo surfaces. We conclude the analysis by considering infinite families of quiver gauge theories associated to the Y^{pq} [**26, 27, 28**], L^{pqr} [**94, 95**] and X^{pq} [**96**] geometries.

The strategy is the following. For each 4d model we use the general formula (3.22) to compute the central charge of the corresponding 2d SCFT obtained after twisted compactification. Then, we determine the parametrization of the R-charges and fluxes in terms of the toric data according to our prescription in Section 3.3.1. Finally, we check that using this parametrization in (3.22) we obtain the central charge as given by (3.29).

3.3.2. $\mathcal{N} = 4$ **SYM.** The first example that we consider corresponds to the case of $X_5 = \mathbb{S}^5$. In this case the dual gauge theory is $\mathcal{N} = 4$ SYM and its twisted compactification on a Riemann surface has been discussed in [**2, 3, 4**]. The 4d field theory can be studied as a toric quiver gauge theory in $\mathcal{N} = 1$ language. In this formulation the global symmetry corresponds to the $U(1)^3$ abelian subgroup of $SO(6)_R$. The quiver has a single node with three adjoint superfields Φ_i and superpotential:

$$W = \Phi_1[\Phi_2, \Phi_3] \tag{3.32}$$

The dimer, the zig-zag paths and the toric diagram are shown in Figure 2.

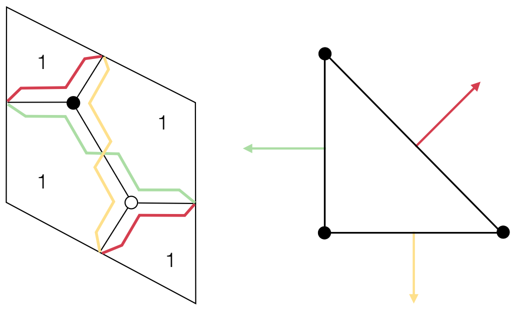


FIGURE 2. Dimer, zig-zag paths and toric diagram of $\mathbb{C}^3 \cong \mathrm{Cone}(\mathbb{S}^5)$.

By reducing this theory on Σ the topological twist is performed along the $U(1)^3$ subgroup of the $SO(6)_R$. This corresponds to turning on three fluxes, one for each $U(1)$ factor, constraining their sum to be equal to the curvature κ. From the general expression (3.22) we can read the 2d central charge at large N:

$$c_r = 3N^2\eta_\Sigma\left(\kappa + \sum_{i=1}^{3}(n_{\Phi_i} - \kappa)(R_{\Phi_i} - 1)^2\right) \tag{3.33}$$

where R_{Φ_i} are the R-charges and n_{Φ_i} the associated fluxes of the three adjoint fields. These variables are constrained by the relations $R_{\Phi_1} + R_{\Phi_2} + R_{\Phi_3} = 2$ and $n_{\Phi_1} + n_{\Phi_2} + n_{\Phi_3} = 2\kappa$.

Alternatively, we can compute the 2d central charge from (3.29) and find:

$$c_r = 3N^2\eta_\Sigma\left(n_{\pi_1}\Delta_{\pi_2}\Delta_{\pi_3} + n_{\pi_2}\Delta_{\pi_3}\Delta_{\pi_1} + n_{\pi_3}\Delta_{\pi_1}\Delta_{\pi_2}\right) \tag{3.34}$$

In order to check this result against (3.33) we need to express R-charges and fluxes in terms of the ones of the PM's. This can be done with the prescription discussed in Section 3.2. The three zig-zag paths in Figure 2 are the three possible combinations of two adjoints, $\Phi_i\Phi_j$. It follows that each adjoint field corresponds to the intersection of two primitive normal vectors w_I of the toric diagram. Furthermore in this case each external PM corresponds to one of the adjoint fields. Therefore the charge and the flux assigned to each field correspond to the charge and the flux assigned to each external PM:

$$\begin{array}{lll} R_{\Phi_1} = \Delta_{\pi_1} & R_{\Phi_2} = \Delta_{\pi_2} & R_{\Phi_3} = \Delta_{\pi_3} \\ n_{\Phi_1} = n_{\pi_1} & n_{\Phi_2} = n_{\pi_2} & n_{\Phi_3} = n_{\pi_3} \end{array} \tag{3.35}$$

By substituting this parameterization in (3.33) we can easily prove that in this case the central charge is equivalent to (3.34) if constraints (3.9) and (3.23) are imposed.

3.3.3. The conifold. As a second example we study the case of the conifold, corresponding to $X_5 = T^{1,1}$. The model consists of a $SU(N) \times SU(N)$ gauge theory with two pairs of bifundamental a_i and anti-bifundamental b_i fields connecting the gauge groups and interacting through the superpotential

$$W = \epsilon_{ij}\epsilon_{lk} a_i b_l a_j b_k \tag{3.36}$$

The dimer, the zig-zag paths and the toric diagram are shown in Figure 3. In

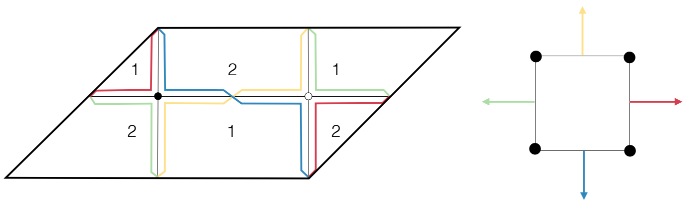


FIGURE 3. Dimer, zig-zag paths and toric diagram of $T^{1,1}$

this case the flavor symmetry is $SU(2)^2$ and one baryonic $U(1)$ symmetry is also present. The R-charges of the four fields, R_{a_i} and R_{b_i} are constrained by $R_{a_1} + R_{a_2} + R_{b_1} + R_{b_2} = 2$.

When twisting the theory on Σ we introduce T-fluxes defined in (3.21). In this case they are n_{a_1}, n_{a_2}, n_{b_1} and n_{b_2}, constrained by $n_{a_1} + n_{a_2} + n_{b_1} + n_{b_2} = 2\kappa$.

The 2d central charge can be written at large N, using eq. (3.22)

$$c_r = 3N^2 \eta_\Sigma \left[2\kappa + \sum_{i=1}^{2} \left((n_{a_i} - \kappa)(R_{a_i} - 1)^2 + (n_{b_i} - \kappa)(R_{b_i} - 1)^2 \right) \right] \tag{3.37}$$

This formula can be reproduced from the geometry of the toric diagram using prescription (3.29). To prove it, we start by ordering the vectors V_I in the toric diagram as

$$V_1 = (0,0,1), \quad V_2 = (1,0,1), \quad V_3 = (1,1,1) \quad V_4 = (0,1,1) \tag{3.38}$$

The four zig-zag paths in Figure 3 are the four possible combinations of two bifundamentals, $a_i b_j$. It follows that each bifundamental field corresponds to the intersection of two consecutive primitive normal vectors of the toric diagram. Furthermore in this case each external PM corresponds to one of the bifundamental fields. Again the charge and the flux assigned to each bifundamental field correspond to the charge and the flux assigned to each external PM

$$\begin{array}{llll} R_{a_1} = \Delta_{\pi_1} & R_{b_1} = \Delta_{\pi_2} & R_{a_2} = \Delta_{\pi_3} & R_{b_2} = \Delta_{\pi_4} \\ n_{a_1} = n_{\pi_1} & n_{b_1} = n_{\pi_2} & n_{a_2} = n_{\pi_3} & n_{b_2} = n_{\pi_4} \end{array} \tag{3.39}$$

By substituting parameterization (3.39) in (3.37) we can check directly that the central charge c_r coincides with the one obtained from (3.29), under the conditions

$$\sum_{I=1}^{4} \Delta_{\pi_I} = 2, \qquad \sum_{I=1}^{4} n_{\pi_I} = 2\kappa \tag{3.40}$$

3.3.4. dP_2 theory. We now consider the quiver gauge theory living on a stack of D3 branes probing the tip of the complex cone over dP_2 (see [**18**] for a discussion of the universal twist of dP_k theories). There are two Seiberg dual realizations of such a theory. Here we focus on the case with the minimal number of fields. This phase is usually referred to as the first phase and denoted as $\mathsf{dP}_2^{(I)}$. It is a quiver gauge theory (see figure 4) with five $SU(N)$ gauge groups and superpotential

$$\begin{aligned} W &= X_{13}X_{34}X_{41} - Y_{12}X_{24}X_{41} + X_{12}X_{24}X_{45}Y_{51} - X_{13}X_{35}Y_{51} \\ &\quad + Y_{12}X_{23}X_{35}X_{51} - X_{12}X_{23}X_{34}X_{45}X_{51}. \end{aligned} \tag{3.41}$$

The model has five non-anomalous abelian global symmetries. There are a $U(1)_R$ symmetry and two $U(1)$ flavor symmetries corresponding to the $U(1)^3$ isometry of the SE geometry. There are also five baryonic currents: Two of them are non-anomalous, two are anomalous and one is redundant.

We perform the c_r calculation from the geometry and we show the validity of formula (3.29) by matching the geometric result with the one obtained from the field theory analysis.

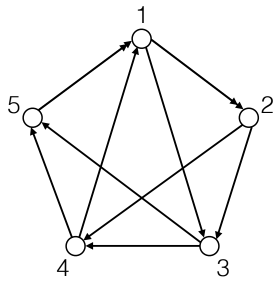


FIGURE 4. Quiver of the $\mathsf{dP}_2^{(I)}$ model

The dimer, the zig-zag paths and the toric diagram are shown in Figure 5. The

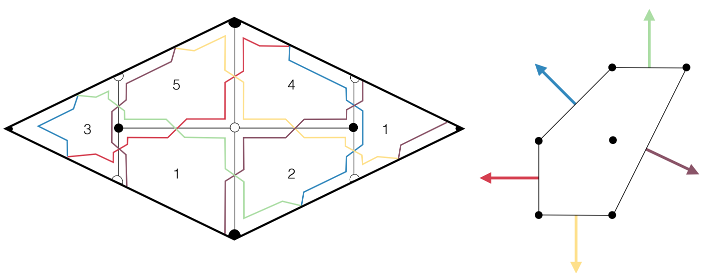


FIGURE 5. Dimer, zig-zag paths and toric diagram of dP_2.

toric diagram is identified by the lattice points

$$\begin{gathered} V_1 = (1,1,1) \quad V_2 = (0,1,1) \quad V_3 = (-1,0,1) \\ V_4 = (-1,-1,1) \quad V_5 = (0,-1,1) \end{gathered} \tag{3.42}$$

The R-charges and the fluxes of the fields can be parameterized in terms of the Δ_{π_I} charges and the n_{π_I} fluxes as

	R_{ϕ_i}	n_{ϕ_i}
$\phi_1 = X_{13}$	$\Delta_{\pi_4} + \Delta_{\pi_5}$	$n_{\pi_4} + n_{\pi_5}$
$\phi_2 = X_{24}$	Δ_{π_5}	n_{π_5}
$\phi_3 = X_{51}$	Δ_{π_5}	n_{π_5}
$\phi_4 = X_{23}$	Δ_{π_2}	n_{π_2}
$\phi_5 = X_{41}$	$\Delta_{\pi_1} + \Delta_{\pi_2}$	$n_{\pi_1} + n_{\pi_2}$
$\phi_6 = Y_{51}$	$\Delta_{\pi_2} + \Delta_{\pi_3}$	$n_{\pi_2} + n_{\pi_3}$
$\phi_7 = Y_{12}$	$\Delta_{\pi_3} + \Delta_{\pi_4}$	$n_{\pi_3} + n_{\pi_4}$
$\phi_8 = X_{45}$	Δ_{π_4}	n_{π_4}
$\phi_9 = X_{12}$	Δ_{π_1}	n_{π_1}
$\phi_{10} = X_{35}$	Δ_{π_1}	n_{π_1}
$\phi_{11} = X_{34}$	Δ_{π_3}	n_{π_3}

(3.43)

subject to the constraints $\sum_{I=1}^5 \Delta_{\pi_I} = 2$ and $\sum_{I=1}^5 n_{\pi_I} = 2\kappa$. This parameterization satisfies the constraints $\sum_{a\in W} R_{\phi_a} = 2$ and $\sum_{a\in W} n_{\phi_a} = 2\kappa$. In this case there are 5 gauge groups and the central charge is obtained from the formula

$$c_r = 3N^2\eta_\Sigma \left(5\kappa + \sum_{i=1}^{11}(n_{\phi_i} - \kappa)(R_{\phi_i} - 1)^2\right) \tag{3.44}$$

By substituting parameterization (3.43) in (3.44) we can see show that (3.44) is equivalent to (3.29) once the constraints (3.9) and (3.23) are imposed.

3.3.5. dP_3 theory. Here we consider the quiver gauge theory living on a stack of D3 branes probing the tip of the complex cone over dP_3. There are four Seiberg dual realizations of such a theory, and we focus on the case with the minimal number of fields, usually called the first phase and denoted as $\mathsf{dP}_3^{(I)}$. The quiver is

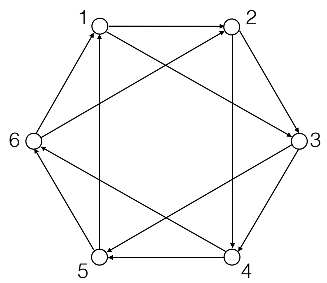


FIGURE 6. Quiver of the $\mathsf{dP}_3^{(I)}$ model

represented in Figure 6, and it has six gauge groups. The superpotential is

$$\begin{aligned} W &= X_{12}X_{24}X_{45}X_{51} - X_{24}X_{46}X_{62} + X_{23}X_{35}X_{56}X_{62} - X_{35}X_{51}X_{13} \\ &\quad + X_{34}X_{46}X_{61}X_{13} - X_{12}X_{23}X_{34}X_{45}X_{56}X_{61}. \end{aligned} \tag{3.45}$$

The model possesses six non-anomalous abelian global symmetries. There are a $U(1)_R$ symmetry and two $U(1)$ flavor symmetries corresponding to the $U(1)^3$ isometry of the SE geometry. There are also six baryonic currents: Three are non-anomalous, two are anomalous and one is redundant. The dimer, the zig-zag paths and the toric diagram are shown in Figure 7. Again we can perform the

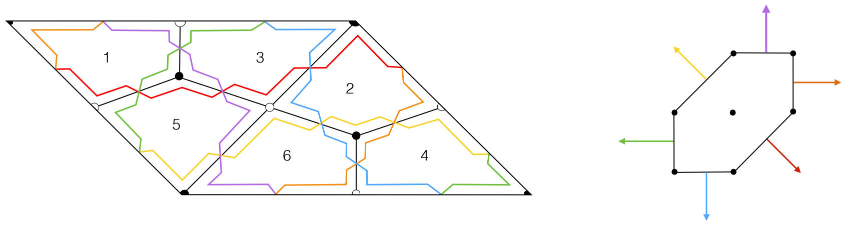


FIGURE 7. Dimer, zig-zag paths and toric diagram of dP_3

calculation from the geometry, showing the validity of formula (3.29). The toric diagram is identified by the lattice points

$$\begin{array}{lll} V_1 = (1,1,1) & V_2 = (0,1,1) & V_3 = (-1,0,1) \\ V_4 = (-1,-1,1) & V_5 = (0,-1,1) & V_6 = (1,0,1) \end{array} \quad (3.46)$$

The R-charges and the fluxes of the fields can be parameterized in terms of the Δ_{π_I} charges and of the n_{π_I} fluxes as

	R_{ϕ_i}	n_{ϕ_i}
$\phi_1 = X_{12}$	Δ_{π_6}	n_{π_6}
$\phi_2 = X_{13}$	$\Delta_{\pi_2} + \Delta_{\pi_3}$	$n_{\pi_2} + n_{\pi_3}$
$\phi_3 = X_{23}$	Δ_{π_5}	n_{π_5}
$\phi_4 = X_{24}$	$\Delta_{\pi_1} + \Delta_{\pi_2}$	$n_{\pi_1} + n_{\pi_2}$
$\phi_5 = X_{34}$	Δ_{π_4}	n_{π_4}
$\phi_6 = X_{35}$	$\Delta_{\pi_1} + \Delta_{\pi_6}$	$n_{\pi_1} + n_{\pi_6}$
$\phi_7 = X_{45}$	Δ_{π_3}	n_{π_3}
$\phi_8 = X_{46}$	$\Delta_{\pi_5} + \Delta_{\pi_6}$	$n_{\pi_5} + n_{\pi_6}$
$\phi_9 = X_{56}$	Δ_{π_2}	n_{π_2}
$\phi_{10} = X_{51}$	$\Delta_{\pi_4} + \Delta_{\pi_5}$	$n_{\pi_4} + n_{\pi_5}$
$\phi_{11} = X_{61}$	Δ_{π_1}	n_{π_1}
$\phi_{12} = X_{62}$	$\Delta_{\pi_3} + \Delta_{\pi_4}$	$n_{\pi_3} + n_{\pi_4}$

(3.47)

with the constraints $\sum_{I=1}^{6} \Delta_{\pi_I} = 2$ and $\sum_{I=1}^{6} n_{\pi_I} = 2\kappa$. This parameterization satisfies the constraints $\sum_{a\in W} R_{\phi_a} = 2$ and $\sum_{a\in W} n_{\phi_a} = 2\kappa$. The central charge is obtained from the formula

$$c_r = 3N^2\eta_\Sigma \left(6\kappa + \sum_{i=1}^{12} (n_{\phi_i} - \kappa)(R_{\phi_i} - 1)^2\right) \quad (3.48)$$

By substituting parameterization (3.47) in (3.48) we can easily see that (3.48) is equivalent to (3.29) provided the constraints (3.9) and (3.23) are imposed.

3.3.6. Y^{pq} theories. We can prove the validity of (3.29) also for infinite families of quiver gauge theories. The first family that we consider is $X_5 = \mathsf{Y}^{pq}$. These models has been derived in [**28**]. They are quiver gauge theories with $2p$ gauge groups and bifundamental matter. For generic values of p and q the models have a $SU(2) \times U(1)$ flavor symmetry and one non-anomalous baryonic $U(1)$ symmetry. At the 2d fixed point this baryonic symmetry generically mixes with the R-current.

The general prescription to obtain the exact 2d central charge after twisted compactification has been given in [**18**] and detailed explicitly there for some cases of particular interest. Knowing the field content of these theories as summarized in

Table (3.51), at large N we can use the general formula (3.22) to write

$$c_r = 3N^2\eta_\Sigma \left(2p\kappa + \sum_{i=1}^{6} d_i(n_{\phi_i} - \kappa)(R_{\phi_i} - 1)^2\right) \tag{3.49}$$

We now show how to reproduce this expression from our geometric formulation (3.29).

For generic values of p and q the toric diagram has four external corners. There are also internal lattice points, associated to the anomalous baryonic symmetries, that do not play any role in our analysis. The corners of the toric diagram are associated to the vectors

$$V_1 = (0,0,1), \quad V_2 = (1,0,1), \quad V_3 = (0,p,1) \quad V_4 = (-1,p-q,1) \tag{3.50}$$

The parameterization of the R-charges and fluxes for the various fields in terms of the toric data can be read from the following table

	multiplicity	R_{ϕ_i}	n_{ϕ_i}
$\phi_1 = Y$	$p+q$	Δ_{π_1}	n_{π_1}
$\phi_2 = U_1$	p	Δ_{π_2}	n_{π_2}
$\phi_3 = Z$	$p-q$	Δ_{π_3}	n_{π_3}
$\phi_4 = U_2$	p	Δ_{π_4}	n_{π_4}
$\phi_5 = V_1$	q	$\Delta_{\pi_2} + \Delta_{\pi_3}$	$n_{\pi_2} + n_{\pi_3}$
$\phi_6 = V_2$	q	$\Delta_{\pi_3} + \Delta_{\pi_4}$	$n_{\pi_3} + n_{\pi_4}$

(3.51)

The charges are subject to constraints (3.9). This parameterization satisfies the constraints $\sum_{a\in W} R_{\phi_a} = 2$ and $\sum_{a\in W} n_{\phi_a} = 2\kappa$ at each node of the dimer.

It is now easy to check that substituting these expressions for the R-charges and the fluxes in (3.49) and taking into account constraints (3.9) and (3.23) we reproduce exactly what we would obtain from (3.29).

3.3.7. L^{pqr} theories. We now consider a second infinite family, corresponding to $X_5 = \mathsf{L}^{pqr}$, for $p \neq r$ (the degenerate case $p = r$ will be treated in Section 3.4). These models have been derived in [**97, 98, 90**]. They can be described in terms of a necklace quiver, i.e. a set of $p+q$ $SU(N)$ gauge groups such that each node is connected to its nearest neighbors by a bifundamental and an anti-bifundamental fields. In general there may be also additional adjoint chiral multiplets, depending on the value of p and q and on the Seiberg dual phase that we are considering.

The central charge at large N can be easily obtained from (3.22) taking into account the field content of these theories in their the minimal phase, as summarized in Table (3.54):

$$c_r = 3N^2\eta_\Sigma \left((p+q)\kappa + \sum_{i=1}^{6} d_i(n_{\phi_i} - \kappa)(R_{\phi_i} - 1)^2\right) \tag{3.52}$$

To check the equivalence with the geometric prescrition (3.29) we first assign the external corners of the toric diagrams to the following vectors:

$$V_1 = (0,0,1), \quad V_2 = (1,0,1), \quad V_3 = (P,s,1) \quad V_4 = (-k,q,1) \tag{3.53}$$

where $r - Ps - kq = 0$ and $p + q = r + s$ and $p \le r \le q \le s$. R-charges and fluxes parametrized in terms of the PM's are:

	multiplicity	R_{ϕ_i}	n_{ϕ_i}
$\phi_1 = Y$	q	Δ_{π_1}	n_{π_1}
$\phi_2 = W_2$	s	Δ_{π_2}	n_{π_2}
$\phi_3 = Z$	p	Δ_{π_3}	n_{π_3}
$\phi_4 = X_1$	r	Δ_{π_4}	n_{π_4}
$\phi_5 = W_1$	$q - s$	$\Delta_{\pi_2} + \Delta_{\pi_3}$	$n_{\pi_2} + n_{\pi_3}$
$\phi_6 = X_1$	$q - r$	$\Delta_{\pi_3} + \Delta_{\pi_4}$	$n_{\pi_3} + n_{\pi_4}$

(3.54)

with the constraints $\sum_{I=1}^{4} \Delta_{\pi_I} = 2$ and $\sum_{I=1}^{4} n_{\pi_I} = 2\kappa$. This parameterization satisfies the constraints $\sum_{a\in W} R_{\phi_a} = 2$ and $\sum_{a\in W} n_{\phi_a} = 2\kappa$.

Substituting this parameterization in (3.52) we directly obtain an expression equivalent to (3.29), once constraints (3.9) and (3.23) are taken into account.

3.3.8. X^{pq} theories. Finally we consider the infinite family of models corresponding to $X_5 = \mathsf{X}^{pq}$. They have been constructed in [**96**]. In this case there are $2p + 1$ gauge groups and taking into account the spectrum of fields and their multiplicities as given in Table (3.56), the 2d central charge as read from (3.22) is

$$c_r = 3N^2 \eta_\Sigma \left((2p+1)\kappa + \sum_{i=1}^{10} d_i (n_{\phi_i} - \kappa)(R_{\phi_i} - 1)^2 \right) \tag{3.55}$$

To check it against the geometric calculation (3.22), we first label the external corners of the toric diagrams as (we take $p > q$)

$$V_1 = (1, p, 1), \quad V_2 = (0, p - q + 1, 1), \quad V_3 = (0, p - q, 1) \quad V_4 = (1, 0, 1) \quad V_5 = (2, 0, 1)$$

The R-charges and fluxes parametrization in terms of the PM's is given by

	multiplicity	R_{ϕ_i}	n_{ϕ_i}
ϕ_1	$p + q - 1$	Δ_{π_1}	n_{π_1}
ϕ_2	1	Δ_{π_2}	n_{π_2}
ϕ_3	1	Δ_{π_3}	n_{π_3}
ϕ_4	$p - q$	Δ_{π_4}	n_{π_4}
ϕ_5	p	Δ_{π_5}	n_{π_5}
ϕ_6	$p - 1$	$\Delta_{\pi_2} + \Delta_{\pi_3}$	$n_{\pi_2} + n_{\pi_3}$
ϕ_7	1	$\Delta_{\pi_3} + \Delta_{\pi_4}$	$n_{\pi_3} + n_{\pi_4}$
ϕ_8	$q - 1$	$\Delta_{\pi_2} + \Delta_{\pi_3} + \Delta_{\pi_4}$	$n_{\pi_2} + n_{\pi_3} + n_{\pi_4}$
ϕ_9	1	$\Delta_{\pi_1} + \Delta_{\pi_2}$	$n_{\pi_1} + n_{\pi_2}$
ϕ_{10}	q	$\Delta_{\pi_4} + \Delta_{\pi_5}$	$n_{\pi_4} + n_{\pi_5}$

(3.56)

with the constraints $\sum_{I=1}^{5} \Delta_{\pi_I} = 2$ and $\sum_{I=1}^{5} n_{\pi_I} = 2\kappa$. Once again, this parameterization satisfies the constraints $\sum_{a\in W} R_{\phi_a} = 2$ and $\sum_{a\in W} n_{\phi_a} = 2\kappa$.

Using this parameterization it is easy to check that result (3.55) is equivalent to (3.29), once constraints (3.9) and (3.23) are imposed.

3.4. Singular horizons and lattice points lying on the perimeter

In this section we discuss the case of toric diagrams with some external lattice points that are not corners but lie along the perimeter. These diagrams are associated to theories with non-smooth horizons, usually arising from the action of an orbifold.

In this case, as discussed in [**10**], the geometric procedure to extract the central charge a from the toric diagram needs some modification. The reason is that the lattice points lying on the perimeter are associated to a multiple number of PM's. Therefore, this requires a change in the prescription for assigning R-charges to the fields in terms of the charges of the PM's.

The prescription that we propose follows the one described in [**38**] and it works as follows. First divide the PM's in two sets, the ones associated to corners of the toric diagram and the degenerate ones lying on the perimeter, namely π^c and π^p respectively. Then we associate a R-charge $\Delta_{\pi_I^c}$ to the PM's at the corners, as done before. For the PM's on the perimeter, observing that at each point on the perimeter only one of the degenerate PM's enters the definition of the zig-zag paths, we assign a non-zero charge $\Delta_{\pi_I^p}$ to this PM and set the charge of all the other PM's associated to the same I-th lattice point to zero. With this modification of charge assignments we can then parameterize the R-charges R_i and the fluxes n_i unambiguously as described in Section 3.3.

We have checked in a large set of examples that by applying this prescription the 2d central charge computed from the field theory analysis, eq. (3.22), matches with the one computed using formula (3.29). In the following we report the explicit check for a couple of examples in the L^{pqp} class.

3.4.1. L^{222} model. For this particular representative of the L^{pqr} family the quiver diagram, the dimer with the zig-zag paths and the toric diagram are depicted in Figure 8. The superpotential of this model is:

$$W = X_{12}X_{23}X_{32}X_{21} - X_{23}X_{34}X_{43}X_{32} + X_{34}X_{41}X_{14}X_{43} - X_{41}X_{12}X_{21}X_{14} \quad (3.57)$$

The central charge can be obtained from formula (3.22) once we take into account

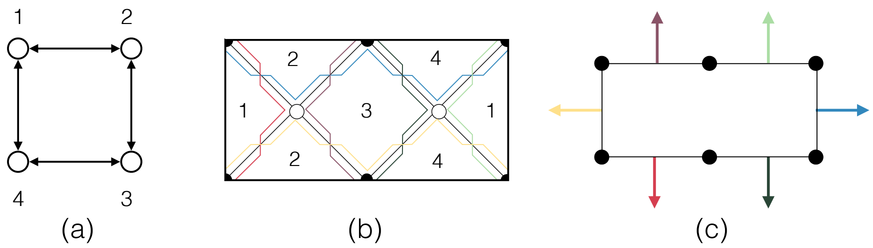


FIGURE 8. Quiver, Dimer with zig zag paths and toric diagram of L^{222}

the specific field content of the theory that can be read from the quiver diagram or in Table (3.61). We obtain

$$c_r = 3N^2\eta_\Sigma\left(4\kappa + \sum_{i=1}^{8}(n_{\phi_i} - \kappa)(R_{\phi_i} - 1)^2\right) \quad (3.58)$$

In order to match this expression with (3.29) we first observe that the PM's are related to the lattice points as follows

PM	Lattice point
$\pi_1 = \{X_{12}, X_{34}\}$	$V_1 = (0,0,1)$
$\pi_2 = \{X_{21}, X_{34}\}$	$V_2 = (1,0,1)$
$\pi_3 = \{X_{12}, X_{43}\}$	$V_2 = (1,0,1)$
$\pi_4 = \{X_{21}, X_{43}\}$	$V_3 = (2,0,1)$
$\pi_5 = \{X_{32}, X_{14}\}$	$V_4 = (2,1,1)$
$\pi_6 = \{X_{32}, X_{41}\}$	$V_5 = (1,1,1)$
$\pi_7 = \{X_{23}, X_{14}\}$	$V_5 = (1,1,1)$
$\pi_8 = \{X_{23}, X_{41}\}$	$V_6 = (0,1,1)$

(3.59)

The two points on the perimeter, identified as V_2 and V_5, are degenerate since they correspond to two different PM's. According to our prescription in Section 3.3.1, we set $\Delta_{\pi_3} = \Delta_{\pi_7} = 0$ and $n_{\pi_3} = n_{\pi_7} = 0$. The other non-vanishing charges and fluxes are constrained by the relations

$$\begin{aligned} \Delta_{\pi_1} + \Delta_{\pi_2} + \Delta_{\pi_4} + \Delta_{\pi_5} + \Delta_{\pi_6} + \Delta_{\pi_8} &= 2 \\ n_{\pi_1} + n_{\pi_2} + n_{\pi_4} + n_{\pi_5} + n_{\pi_6} + n_{\pi_8} &= 2\kappa \end{aligned} \tag{3.60}$$

From here we can read the charges and the fluxes of every single field

	R_{ϕ_i}	n_{ϕ_i}
$\phi_1 = X_{12}$	Δ_{π_1}	n_{π_1}
$\phi_2 = X_{21}$	$\Delta_{\pi_4} + \Delta_{\pi_2}$	$n_{\pi_4} + n_{\pi_2}$
$\phi_3 = X_{23}$	Δ_{π_8}	n_{π_8}
$\phi_4 = X_{32}$	$\Delta_{\pi_5} + \Delta_{\pi_6}$	$n_{\pi_5} + n_{\pi_6}$
$\phi_5 = X_{34}$	$\Delta_{\pi_1} + \Delta_{\pi_2}$	$n_{\pi_1} + n_{\pi_2}$
$\phi_6 = X_{43}$	Δ_{π_4}	n_{π_4}
$\phi_7 = X_{41}$	$\Delta_{\pi_6} + \Delta_{\pi_8}$	$n_{\pi_6} + n_{\pi_8}$
$\phi_8 = X_{14}$	Δ_{π_5}	n_{π_5}

(3.61)

This parameterization satisfies the constraints $\sum_{a\in W} R_{\phi_a} = 2$ and $\sum_{a\in W} n_{\phi_a} = 2\kappa$.

By substituting this parametrization in (3.58) we can easily prove that it is equivalent to (3.29) once constraints (3.60) are imposed.

3.4.2. L^{131} model. As a second example, we consider the L^{131} model associated to the quiver, dimer and toric diagram drawn in Figure 9. In this case the superpotential reads

$$\begin{aligned} W \;=\; & X_{12}X_{21}X_{14}X_{41} - X_{12}X_{22}X_{21} + X_{32}X_{22}X_{23} \\ & - X_{23}X_{33}X_{32} + X_{43}X_{33}X_{34} - X_{14}X_{43}X_{34}X_{41} \end{aligned} \tag{3.62}$$

Given the particular field content, the central charge computed from (3.22) reads

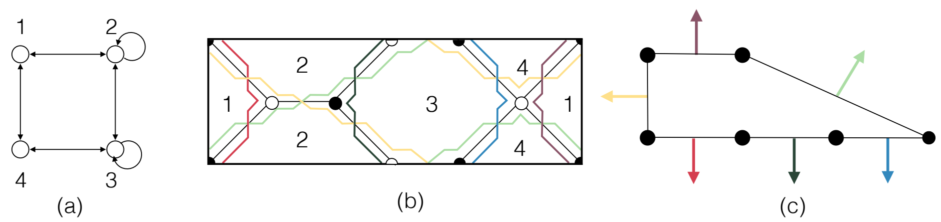


FIGURE 9. Dimer, zig-zag paths and toric diagram of L^{131}

$$c_r = 3N^2\eta_\Sigma\left(4\kappa + \sum_{i=1}^{10}(n_{\phi_i} - \kappa)(R_{\phi_i} - 1)^2\right) \tag{3.63}$$

In this case the PM's are related to the lattice points as follows

PM	Lattice point
$\pi_1 = \{X_{12}, X_{23}, X_{34}\}$	$V_1 = (0,0,1)$
$\pi_2 = \{X_{21}, X_{23}, X_{34}\}$	$V_2 = (1,0,1)$
$\pi_3 = \{X_{12}, X_{32}, X_{34}\}$	$V_2 = (1,0,1)$
$\pi_4 = \{X_{12}, X_{23}, X_{43}\}$	$V_2 = (1,0,1)$
$\pi_5 = \{X_{21}, X_{32}, X_{34}\}$	$V_3 = (2,0,1)$
$\pi_6 = \{X_{21}, X_{23}, X_{43}\}$	$V_3 = (2,0,1)$
$\pi_7 = \{X_{12}, X_{32}, X_{43}\}$	$V_3 = (2,0,1)$
$\pi_8 = \{X_{21}, X_{32}, X_{43}\}$	$V_4 = (3,0,1)$
$\pi_9 = \{X_{14}, X_{22}, X_{33}\}$	$V_5 = (1,1,1)$
$\pi_{10} = \{X_{41}, X_{22}, X_{33}\}$	$V_6 = (0,1,1)$

(3.64)

There are still two perimeter points, this time with degeneracy three. We set $\Delta_{\pi_2} = \Delta_{\pi_3} = \Delta_{\pi_5} = \Delta_{\pi_6} = 0$ and correspondingly, $n_{\pi_2} = n_{\pi_3} = n_{\pi_5} = n_{\pi_6} = 0$. The remaining charges and fluxes satisfy:

$$\begin{aligned} \Delta_{\pi_1} + \Delta_{\pi_4} + \Delta_{\pi_7} + \Delta_{\pi_8} + \Delta_{\pi_9} + \Delta_{\pi_{10}} &= 2 \\ n_{\pi_1} + n_{\pi_4} + n_{\pi_7} + n_{\pi_6} + n_{\pi_9} + n_{\pi_{10}} &= 2\kappa \end{aligned} \tag{3.65}$$

The R-charges R_{ϕ_i} and the fluxes n_{ϕ_i} of the fields can be expressed in terms of the charges Δ_{π_I} and the fluxes n_{π_I} of the PM's as

	R_{ϕ_i}	n_{ϕ_i}
$\phi_1 = X_{12}$	$\Delta_{\pi_1} + \Delta_{\pi_4} + \Delta_{\pi_7}$	$n_{\pi_1} + n_{\pi_4} + n_{\pi_7}$
$\phi_2 = X_{21}$	Δ_{π_8}	n_{π_8}
$\phi_3 = X_{22}$	$\Delta_{\pi_9} + \Delta_{\pi_{10}}$	$n_{\pi_9} + n_{\pi_{10}}$
$\phi_4 = X_{23}$	$\Delta_{\pi_1} + \Delta_{\pi_4}$	$n_{\pi_1} + n_{\pi_4}$
$\phi_5 = X_{32}$	$\Delta_{\pi_7} + \Delta_{\pi_8}$	$n_{\pi_7} + n_{\pi_8}$
$\phi_6 = X_{33}$	$\Delta_{\pi_9} + \Delta_{\pi_{10}}$	$n_{\pi_9} + n_{\pi_{10}}$
$\phi_7 = X_{34}$	Δ_{π_1}	n_{π_1}
$\phi_8 = X_{43}$	$\Delta_{\pi_4} + \Delta_{\pi_7} + \Delta_{\pi_8}$	$n_{\pi_4} + n_{\pi_7} + n_{\pi_8}$
$\phi_9 = X_{41}$	$\Delta_{\pi_{10}}$	$n_{\pi_{10}}$
$\phi_{10} = X_{14}$	Δ_{π_9}	n_{π_9}

(3.66)

This parameterization satisfies the constraints:

$$\sum_{a\in W} R_{\phi_a} = 2 \quad \text{and} \quad \sum_{a\in W} n_{\phi_a} = 2\kappa \tag{3.67}$$

It is now easy to substitute this parameterization in (3.63) and check that the resulting expression is equivalent to (3.29) once we take into account constraints (3.65).

3.5. Mixing of the baryonic symmetries

In this section, by studying the twisted compactification of some of the 4d $\mathcal{N} = 1$ toric quiver gauge theories discussed above, we provide further evidence that both flavor and baryonic symmetries mix with the R-current at the 2d fixed point. We compute the central charge with the formalism reviewed in Section 3.2.1 showing its positivity for many choices of curvature and fluxes.

3.5.1. dP_2 theory. We begin by identifying the global currents of the dP_2 model. There are a UV R-current R_0, two flavor currents $F_{1,2}$ and two non-anomalous baryonic currents $B_{1,2}$. Having the model five gauge groups, to begin with we have five classically conserved baryonic currents, associated to the decoupling of the gauge abelian factors $U(1)_i \subset U(N)_i$. As usual one of such currents is redundant. Among the other global baryonic $U(1)$'s some of the combinations can be anomalous at quantum level. After the identification of the two non-anomalous baryonic currents the charges of the fields respect to all the global currents are

	Y_{51}	X_{51}	X_{23}	X_{35}	X_{41}	X_{34}	X_{13}	X_{24}	X_{45}	X_{12}	Y_{12}
R_0	2	0	2	0	2	0	0	0	0	0	0
F_1	−2	1	−3	1	−2	1	1	1	0	1	1
F_2	1	1	−1	1	0	2	−2	1	−3	1	−1
B_1	−1	−1	−1	1	0	0	0	−1	1	1	1
B_2	1	1	−1	0	−1	2	−1	1	−2	0	0

(3.68)

Any linear combination

$$R_{\text{trial}} = R_0 + \epsilon_1 T_{F_1} + \epsilon_2 T_{F_2} + \eta_1 T_{B_1} + \eta_2 T_{B_2} \tag{3.69}$$

is still an R-current. Such an ambiguity is fixed by maximizing the central charge with respect to the mixing parameters ϵ_i and η_i [**9**]

$$\frac{\partial a}{\partial \epsilon_i} = \frac{3}{32}[9\mathsf{Tr}(R_{\text{trial}}^2 F_i) - \mathsf{Tr}(F_i)] = 0 \tag{3.70}$$

$$\frac{\partial a}{\partial \eta_i} = \frac{3}{32}[9\mathsf{Tr}(R_{\text{trial}}^2 B_i) - \mathsf{Tr}(B_i)] = 0. \tag{3.71}$$

By using the relations $\mathsf{Tr}(B_iB_jB_k) = 0 = \mathsf{Tr}(B_i)$ equations (3.71) reduce to a linear system in the η_i variables. Substituting the solution back into (3.70) we are left with two free mixing parameters. Therefore, one can always linearly combine the global symmetries in such a way that at the fixed point $\eta_1 = \eta_2 = 0$. This signals the fact that the baryonic symmetries do not mix with the 4d exact R-current.

Solving the remaining equations we obtain:

$$\epsilon_1 = \frac{1}{8}\left(\sqrt{33} - 1\right), \quad \epsilon_2 = \frac{1}{16}\left(3\sqrt{33} - 19\right) \tag{3.72}$$

We can proceed by twisting the theory on Σ. The partial topological twist is performed along the generator:

$$T = \kappa T_R + b_1 T_{F_1} + b_2 T_{F_2} + b_3 T_{B_1} + b_4 T_{B_2} \tag{3.73}$$

and the central charge c_r of the 2d theory can be obtained from (2.74).

The general results are rather involved, so we restrict to some simple choices of fluxes $b_{\mathbf{I}}$ for the case $\kappa \neq 0$. We have the following cases:

- $b_1 \neq 0$

$$\begin{aligned}
\epsilon_1 &= \frac{7b_1^4 + 85b_1^3 - 1800b_1^2 - 4500b_1 + 60000}{13b_1^4 + 310b_1^3 - 6375b_1^2 + 7000b_1 + 100000}\\
\epsilon_2 &= \frac{11b_1^4 + 110b_1^3 - 225b_1^2 - 3250b_1 - 10000}{13b_1^4 + 310b_1^3 - 6375b_1^2 + 7000b_1 + 100000}\\
\eta_1 &= -\frac{3b_1^4 - 20b_1^3 - 525b_1^2 + 2000b_1}{13b_1^4 + 310b_1^3 - 6375b_1^2 + 7000b_1 + 100000}\\
\eta_2 &= -\frac{18b_1^4 - 10b_1^3 - 600b_1^2 - 500b_1}{13b_1^4 + 310b_1^3 - 6375b_1^2 + 7000b_1 + 100000}
\end{aligned} \tag{3.74}$$

- $b_2 \neq 0$

$$\begin{aligned}
\epsilon_1 &= -\frac{8b_2^4 - 270b_2^3 + 2600b_2^2 - 1250b_2 - 45000}{180b_2^3 - 3825b_2^2 + 4500b_2 + 75000} \\
\epsilon_2 &= -\frac{8b_2^3 - 175b_2^2 + 525b_2 + 500}{12b_2^3 - 255b_2^2 + 300b_2 + 5000} \\
\eta_1 &= -\frac{4b_2^4 - 135b_2^3 + 850b_2^2 - 1000b_2}{180b_2^3 - 3825b_2^2 + 4500b_2 + 75000} \\
\eta_2 &= -\frac{8b_2^4 - 270b_2^3 + 2000b_2^2 + 250b_2}{180b_2^3 - 3825b_2^2 + 4500b_2 + 75000}
\end{aligned} \tag{3.75}$$

- $b_3 \neq 0$

$$\epsilon_1 = \frac{3}{5}, \quad \epsilon_2 = -\frac{1}{10}, \quad \eta_1 = -\frac{b_3}{10}, \quad \eta_2 = 0 \tag{3.76}$$

- $b_4 \neq 0$

$$\begin{aligned}
\epsilon_1 &= \frac{95b_4^3 - 900b_4^2 + 45000}{3b_4^4 - 1950b_4^2 + 1500b_4 + 75000} \\
\epsilon_2 &= -\frac{4b_4^4 + 5b_4^3 - 1000b_4^2 + 1000b_4 + 5000}{2b_4^4 - 1300b_4^2 + 1000b_4 + 50000} \\
\eta_2 &= -\frac{12b_4^4 + 175b_4^3 + 750b_4^2 - 1500b_4}{6b_4^4 - 3900b_4^2 + 3000b_4 + 150000} \\
\eta_2 &= \frac{6b_4^4 + 80b_4^3 - 1800b_4^2 - 6000b_4}{3b_4^4 - 1950b_4^2 + 1500b_4 + 75000}
\end{aligned} \tag{3.77}$$

It is interesting to observe that in each case all the mixing parameters are non-vanishing, showing the generic fact that the baryonic symmetries have a non-trivial mixing with the R-current in 2d.

We conclude by showing in Figure 10, 11 and 12 the central charge for different values of the discrete fluxes for dP_2 compactified on $\Sigma = \mathbb{T}^2$, $\Sigma = \mathbb{S}^2$ and $\Sigma = \mathbb{H}^2$, respectively. The scale of colors represents the value of the central charge in units of N^2.

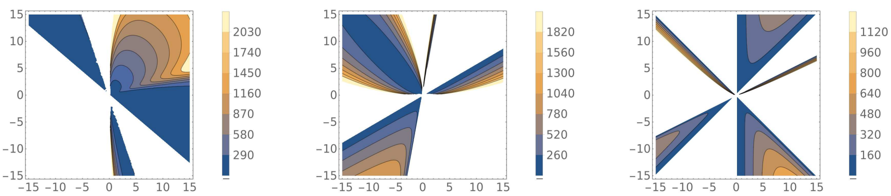


FIGURE 10. Central charge of dP_2 on $\Sigma = \mathbb{T}^2$ for different values of the integer fluxes. We plot the regions of fluxes b_i in which the central charge assumes a positive value. In the first case we have fixed $b_1 = x$, $b_2 = y$ and $b_3 = b_4 = 0$. In the second case we have fixed $b_3 = x$ and $b_4 = y$ and $b_1 = b_2 = 0$. In the third case we have fixed $b_2 = x$, $b_3 = y$ and $b_1 = b_4 = 0$.

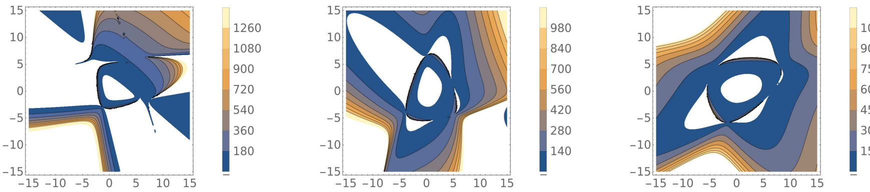


FIGURE 11. Central charge of dP_2 on $\Sigma = \mathbb{S}^2$ for different values of the integer fluxes. We plot the regions of fluxes b_i in which the central charge assumes a positive value. In the first case we have fixed $b_1 = x$, $b_2 = y$ and $b_3 = b_4 = 0$. In the second case we have fixed $b_3 = x$ and $b_4 = y$ and $b_1 = b_2 = 0$. In the third case we have fixed $b_2 = x$, $b_3 = y$ and $b_1 = b_4 = 0$.

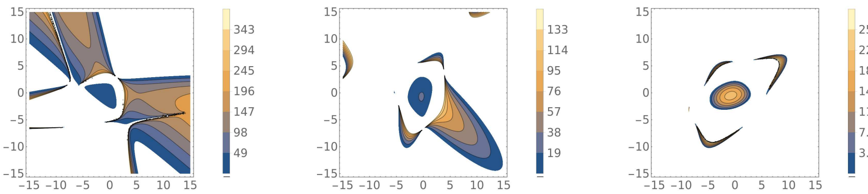


FIGURE 12. Central charge of dP_2 on $\Sigma = \mathbb{H}^2$ for different values of the integer fluxes. We plot the regions of fluxes b_i in which the central charge assumes a positive value. In the first case we have fixed $b_1 = x$, $b_2 = y$ and $b_3 = b_4 = 0$. In the second case we have fixed $b_3 = x$ and $b_4 = y$ and $b_1 = b_2 = 0$. In the third case we have fixed $b_2 = x$, $b_3 = y$ and $b_1 = b_4 = 0$.

3.5.2. dP_3 theory. We now consider the quiver gauge theory living on a stack of D3 branes probing the tip of the complex cone over dP_3. The global currents of the model are a UV R-current R_0, two flavor currents $F_{1,2}$ and three non-anomalous baryonic currents $B_{1,2,3}$.

We can identify the baryonic currents as follows. The model has six gauge groups and classically there are six conserved baryonic currents. Two of them are anomalous and one is redundant. One is left with three non-anomalous baryonic symmetries. We can choose the charges of the fields with respect of the global symmetries as

	X_{12}	X_{13}	X_{23}	X_{24}	X_{34}	X_{35}	X_{45}	X_{46}	X_{56}	X_{51}	X_{61}	X_{62}
R_0	2	0	0	0	0	2	0	2	0	0	0	0
F_1	−1	1	0	−1	1	−2	1	−1	0	1	−1	2
F_2	3	0	−1	−4	1	0	1	2	−1	0	−3	2
B_1	2	1	−1	−1	0	0	0	1	1	−1	−2	0
B_2	1	0	−1	−1	0	1	1	0	−1	−1	0	1
B_3	−1	−1	0	1	1	0	−1	−1	0	1	1	0

(3.78)

Any linear combination:

$$R_{\text{trial}} = R_0 + \epsilon_1 T_{F_1} + \epsilon_2 T_{F_2} + \eta_1 T_{B_1} + \eta_2 T_{B_2} + \eta_3 T_{B_3} \tag{3.79}$$

is still an R-current. The mixing coefficients in (3.79) are fixed by maximizing the central charge, and in this case we find:

$$\epsilon_1 = \frac{2}{3}, \quad \epsilon_2 = -\frac{1}{3}, \quad \eta_i = 0 \tag{3.80}$$

Again we chose a parameterization such that at the superconformal fixed point the contribution of the baryonic symmetries vanishes. The partial topological twist on Σ is performed along the generator:

$$T = \kappa T_R + b_1 T_{F_1} + b_2 T_{F_2} + b_3 T_{B_1} + b_4 T_{B_2} + b_5 T_{B_3} \tag{3.81}$$

and the central charge c_r is obtained from (2.74).

The final expressions are rather complicated and we do not learn much in writing them out explicitly for the most general choice of fluxes. In the following we consider only the case $\kappa \neq 0$ and we only fix one non-vanishing flux for each choice. We have the following cases:

- $b_1 \neq 0$

$$\begin{aligned}
\epsilon_1 &= \frac{16979 b_1 - 6982 b_1^2 \kappa - 740 b_1^3 + 171360\kappa}{52998 b_1 - 4416 b_1^2 \kappa - 1104 b_1^3 + 235116\kappa} \\
\epsilon_2 &= -\frac{8165 b_1 - 844 b_1^2 \kappa - 188 b_1^3 + 42840\kappa}{26499 b_1 - 2208 b_1^2 \kappa - 552 b_1^3 + 117558\kappa} \\
\eta_1 &= -\frac{105 b_1 + 16 b_1^2 \kappa + 4 b_1^3 + 630\kappa}{8833 b_1 - 736 b_1^2 \kappa - 184 b_1^3 + 39186\kappa} \\
\eta_2 &= \frac{920 b_1 - 1948 b_1^2 \kappa - 464 b_1^3 - 630\kappa}{26499 b_1 - 2208 b_1^2 \kappa - 552 b_1^3 + 117558\kappa} \\
\eta_3 &= \frac{605 b_1 - 1996 b_1^2 \kappa - 476 b_1^3 - 2520\kappa}{52998 b_1 - 4416 b_1^2 \kappa - 1104 b_1^3 + 235116\kappa}
\end{aligned} \tag{3.82}$$

- $b_2 \neq 0$

$$\begin{aligned}
\epsilon_1 &= -\frac{4\left(5640 b_2 - 41298 b_2^2 \kappa + 840 b_2^3 + 737 b_2^4 \kappa - 52 b_2^5 + 514080\kappa\right)}{3\left(43536 b_2 + 116452 b_2^2 \kappa - 1952 b_2^3 - 2053 b_2^4 \kappa + 104 b_2^5 - 940464\kappa\right)} \\
\epsilon_2 &= \frac{199452 b_2 - 177054 b_2^2 \kappa + 1767 b_2^3 + 4138 b_2^4 \kappa - 338 b_2^5 + 1028160\kappa}{130608 b_2 + 349356 b_2^2 \kappa - 5856 b_2^3 - 6159 b_2^4 \kappa + 312 b_2^5 - 2821392\kappa} \\
\eta_1 &= \frac{51420 b_2 + 15446 b_2^2 \kappa - 8649 b_2^3 - 1289 b_2^4 \kappa + 156 b_2^5 + 15120\kappa}{43536 b_2 + 116452 b_2^2 \kappa - 1952 b_2^3 - 2053 b_2^4 \kappa + 104 b_2^5 + 940464\kappa} \\
\eta_2 &= \frac{960 b_2 + 66888 b_2^2 \kappa - 39906 b_2^3 - 4054 b_2^4 \kappa + 728 b_2^5 + 15120\kappa}{130608 b_2 + 349356 b_2^2 \kappa - 5856 b_2^3 - 6159 b_2^4 \kappa + 312 b_2^5 - 2821392\kappa} \\
\eta_3 &= \frac{79500 b_2 + 63198 b_2^2 \kappa - 30447 b_2^3 - 4835 b_2^4 \kappa + 364 b_2^5 + 30240\kappa}{3\left(43536 b_2 + 116452 b_2^2 \kappa - 1952 b_2^3 - 2053 b_2^4 \kappa + 104 b_2^5 - 940464\kappa\right)}
\end{aligned} \tag{3.83}$$

- $b_3 \neq 0$

$$
\begin{aligned}
\epsilon_1 &= -\frac{15660b_3\kappa - 19524b_3^2 + 1857b_3^3\kappa - 41b_3^4 + 2056320}{3\left(25344b_3\kappa + 27384b_3^2 - 41b_3^4 - 940464\right)} \\
\epsilon_2 &= \frac{18180b_3\kappa - 21588b_3^2 + 3729b_3^3\kappa + 41b_3^4 + 1028160}{3\left(25344b_3\kappa + 27384b_3^2 - 41b_3^4 - 940464\right)} \\
\eta_1 &= \frac{107484b_3\kappa - 3564b_3^2 - 3245b_3^3\kappa + 2b_3^4 + 15120}{25344b_3\kappa + 27384b_3^2 - 41b_3^4 - 940464} \\
\eta_2 &= -\frac{2\left(11340b_3\kappa + 366b_3^2 + 1749b_3^3\kappa - 164b_3^4 - 7560\right)}{3\left(25344b_3\kappa + 27384b_3^2 - 41b_3^4 - 940464\right)} \\
\eta_3 &= \frac{4\left(21690b_3\kappa + 246b_3^2 - 357b_3^3\kappa + 44b_3^4 + 7560\right)}{3\left(25344b_3\kappa + 27384b_3^2 - 41b_3^4 - 940464\right)}
\end{aligned}
\tag{3.84}
$$

- $b_4 \neq 0$

$$
\begin{aligned}
\epsilon_1 &= -\frac{3450b_4\kappa - 15095b_4^2 + 1028160}{27288b_4\kappa + 25014b_4^2 - 1410696} \\
\epsilon_2 &= -\frac{3450b_4\kappa + 7645b_4^2 - 514080}{27288b_4\kappa + 25014b_4^2 - 1410696} \\
\eta_1 &= \frac{3330b_4\kappa - 65b_4^2 + 7560}{9096b_4\kappa + 8338b_4^2 - 470232} \\
\eta_2 &= \frac{62178b_4 - 1894b_4^2\kappa - 1137b_4^3 + 3780\kappa}{13644b_4\kappa^2 + 12507b_4^2\kappa - 705348\kappa} \\
\eta_3 &= \frac{7410b_4\kappa + 565b_4^2 + 7560}{13644b_4\kappa + 12507b_4^2 - 705348}
\end{aligned}
\tag{3.85}
$$

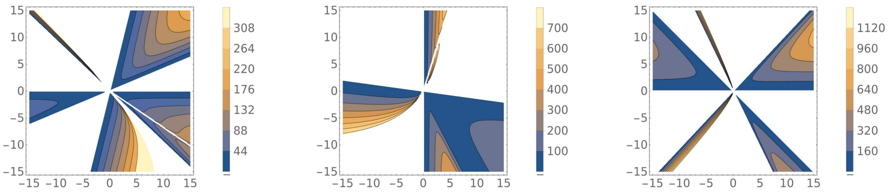
- $b_5 \neq 0$

$$
\begin{aligned}
\epsilon_1 &= -\frac{3450b_5\kappa + 15095b_5^2 - 1028160}{27288b_5\kappa - 25014b_5^2 + 1410696} \\
\epsilon_2 &= \frac{3450b_5\kappa + 3725b_5^2 - 257040}{13644b_5\kappa - 12507b_5^2 + 705348} \\
\eta_1 &= -\frac{60b_5\kappa - 65b_5^2 + 3780}{4548b_5\kappa - 4169b_5^2 + 235116} \\
\eta_2 &= -\frac{7590b_5\kappa - 760b_5^2 + 3780}{13644b_5\kappa - 12507b_5^2 + 705348} \\
\eta_3 &= \frac{3593b_5^2\kappa - 2274b_5^3 + 114366b_5 + 15120\kappa}{25014b_5^2\kappa - 27288b_5 - 1410696\kappa}
\end{aligned}
\tag{3.86}
$$

As in the dP_2 case we observe that the mixing parameters η_i, that were vanishing in the 4d case, are non-zero in two dimensions.

We conclude by showing in Figure 13, 14 and 15 the central charge for different values of the discrete fluxes for dP_2 compactified on $\Sigma = \mathbb{T}^2$, $\Sigma = \mathbb{S}^2$ and $\Sigma = \mathbb{H}^2$, respectively.

3.5.3. L^{pqp} theories. As a last example we consider models with a higher number of baryonic symmetries, i.e. L^{pqp} models [**97, 98, 90**]. In order to have a comprehensive discussion we pick up a particular (Seiberg dual) phase, that can be easily visualized by the description of the system in terms of D4 and NS branes in type IIA string theory (the other phases are obtained by exchanging the NS



$b_1 = x,\ b_2 = y$
$b_3 = b_4 = b_5 = 0$

$b_3 = x,\ b_4 = y$
$b_1 = b_2 = b_5 = 0$

$b_2 = x,\ b_3 = y$
$b_1 = b_4 = b_5 = 0$

FIGURE 13. Central charge of dP_3 on $\Sigma = \mathbb{T}^2$ for different values of the integer fluxes. We plot the regions of fluxes b_i in which the central charge assumes a positive value. The variable x is plotted on the horizontal axis while y on the vertical one.

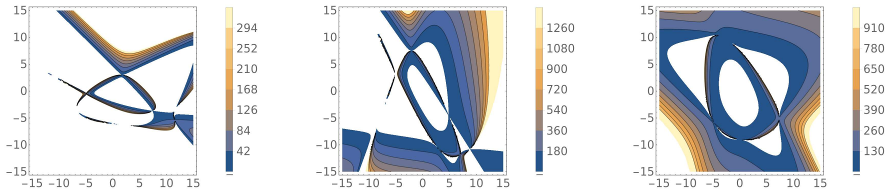


FIGURE 14. Central charge of dP_3 on $\Sigma = \mathbb{S}^2$ for different values of the integer fluxes. We plot the regions of fluxes b_i in which the central charge assumes a positive value. In the first case we have fixed $b_1 = x$, $b_2 = y$ and $b_3 = b_4 = b_5 0$. In the second case we have fixed $b_3 = x$ and $b_4 = y$ and $b_1 = b_2 = b_5 0$. In the third case we have fixed $b_2 = x$, $b_3 = y$ and $b_1 = b_4 = b_5 = 0$.

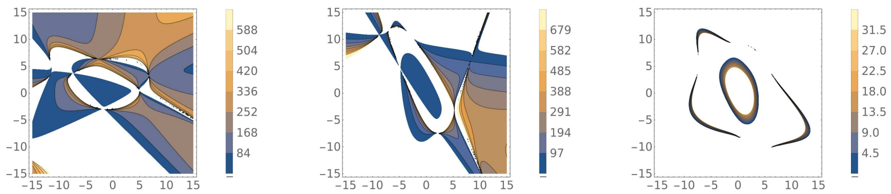


FIGURE 15. Central charge of dP_3 on $\Sigma = \mathbb{H}^2$ for different values of the integer fluxes. We plot the regions of fluxes b_i in which the central charge assumes a positive value. In the first case we have fixed $b_1 = x$, $b_2 = y$ and $b_3 = b_4 = b_5 0$. In the second case we have fixed $b_3 = x$ and $b_4 = y$ and $b_1 = b_2 = b_5 0$. In the third case we have fixed $b_2 = x$, $b_3 = y$ and $b_1 = b_4 = b_5 = 0$.

branes). We consider a stack of N D4 branes extended along x_{0123} and wrapping the compact direction x_6. Then we consider two sets of p NS and q NS' branes. The NS branes are extended along x_{012345} and the NS' along x_{012389}. We order the NS branes and then the NS' branes clockwise along x_6.

Each gauge group is associated to a segment of N D4 branes on x_6, suspended between two consecutive NS branes. The resulting field theory is a $U(N)$ necklace quiver gauge theory with different types of nodes. By counting clockwise on x_6 we have

- a set of $p-1$ nodes with an adjoint of type C;
- a node without any adjoint;
- a set of $q-1$ nodes with an adjoint of type $\tilde{C}$;
- a node without any adjoint.

The $2p$ bifundamental matter fields crossing a NS brane are of type A or B depending on their orientation and the second set of $2q$ bifundamental fields, crossing a NS' brane, are of type $\tilde{A}$ and $\tilde{B}$. We can visualize the situation in the quiver of Figure 16 for the case or $p = 2$ and $q = 4$.

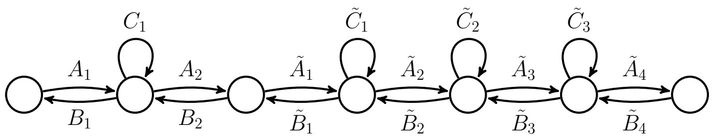


FIGURE 16. Quiver of the L^{242} model. The first and the last node are identified.

These theories are characterized by one R-current, two flavor currents and $p+q$ baryonic currents [**97, 98, 90**]. One baryonic current is redundant, being the quiver necklace. The vector-like nature of the field content ensures that the other $p+q-1$ currents are all conserved at quantum level. The charge assignment of flavor and R-currents is summarized in the following table[3]

field	mult	F_1	F_2	R_0
A	p	0	1	0
B	p	-1	0	0
$\tilde{A}$	q	1	0	1
$\tilde{B}$	q	0	-1	1
C	$p-1$	1	-1	2
$\tilde{C}$	$q-1$	-1	1	0

(3.87)

The baryonic currents are associated to the $U(1)_i \subset U(N)_i$ gauge factors and the charges are read from the representation of each field under the gauge groups. Fundamental fields of $SU(N)_i$ have charge +1 and anti-fundamental fields of $SU(N)_i$ have charge -1 under the baryonic $U(1)_i$.

The 4d R-charge mixes with the global symmetries through the combination

$$R = R_0 + \epsilon_1 F_1 + \epsilon_2 T_{F_2} + \sum_{i=1}^{p+q-1} \eta_i T_{B_i} \tag{3.88}$$

with mixing parameters

$$\epsilon_1 = \frac{\sqrt{p^2 - pq + q^2} - 2p + q}{3(p-q)}, \quad \epsilon_2 = \frac{p}{\sqrt{p^2 - pq + q^2} + 2p - q}, \quad \eta_i = 0 \tag{3.89}$$

determined by the a-maximization.

When the theory is partially topologically twisted on Σ along the generator

$$T = \kappa T_R + b_1 T_{F_1} + b_2 T_{F_2} + \sum_{i=1}^{p+q-1} b_{i+2} B_i \tag{3.90}$$

the central charge c_r is obtained from (2.74) and extremized respect to the ϵ_I parameters. The final formulas are too involved and we do not report them here. The

[3]We refer to R_0 as a trial R-charge obtained after the maximization on the baryonic charges, as described after (3.71).

mixing parameters are non-vanishing for generic choices of the curvature and of the fluxes $b_{\mathbf{I}}$. This signals the fact that the baryonic symmetries mix with the 2d exact R-current.

In the following we show some numerical results for the 2d central charge of the L^{222} and the L^{131} gauge theories. In both cases there are four gauge groups and three non-anomalous baryonic symmetries and we observe that the baryonic symmetries mix for generic values of the with the R-current at the 2d fixed point.

In Figure 17 and 18 we represent the central charge for different values of the discrete fluxes for L^{222} compactified on $\Sigma = \mathbb{H}^2$ and $\Sigma = \mathbb{S}^2$, respectively.

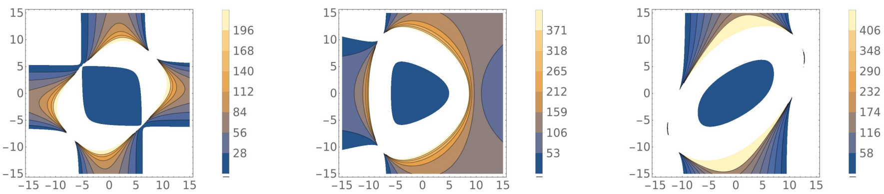


FIGURE 17. Central charge of L^{222} on $\Sigma = \mathbb{H}^2$ for different values of the integer fluxes. We plot the regions of fluxes b_i in which the central charge assumes a positive value. In the first case we have fixed $b_1 = x$, $b_2 = y$ and $b_3 = b_4 = b_5 = 0$. In the second case we have fixed $b_3 = x$ and $b_4 = y$ and $b_1 = b_2 = b_5 = 0$. In the third case we have fixed $b_2 = x$, $b_4 = y$ and $b_1 = b_3 = b_5 = 0$.

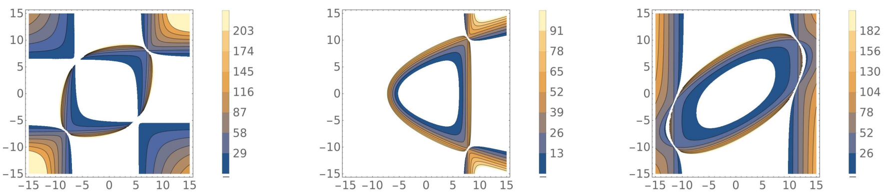


FIGURE 18. Central charge of L^{222} on $\Sigma = \mathbb{S}^2$ for different values of the integer fluxes. We plot the regions of fluxes b_i in which the central charge assumes a positive value. In the first case we have fixed $b_1 = x$, $b_2 = y$ and $b_3 = b_4 = b_5 = 0$. In the second case we have fixed $b_3 = x$ and $b_4 = y$ and $b_1 = b_2 = b_5 = 0$. In the third case we have fixed $b_2 = x$, $b_4 = y$ and $b_1 = b_3 = b_5 = 0$.

In the case of the torus reduction ($\kappa = 0$) the formulae are simpler and we can provide the analytical expression for c_r extremized with respect to the mixing parameters, in terms of the b_i fluxes:

$$c_r(\mathsf{L}^{222})_{\mathbb{T}^2} = \frac{6(b_3^2+b_1b_3+b_2b_3-2b_4b_3+2b_4^2+b_5^2+2b_1b_2-(b_1+b_2+2b_4)b_5)(b_3^2+b_5^2+b_2(b_5-b_3)+b_1(2b_2-b_3+b_5))}{(b_1-b_2)(b_4^2-b_3b_4+b_1b_3+b_2b_3-(b_1+b_2+b_4)b_5)} \quad (3.91)$$

The parameters ϵ_i^* are generically non-vanishing for both the flavor and the baryonic global symmetries.

For the case of L^{131} we represent the central charge for different values of the discrete fluxes on $\Sigma = \mathbb{H}^2$ in Figure 19 and on $\Sigma = \mathbb{S}^2$ in Figure 20.

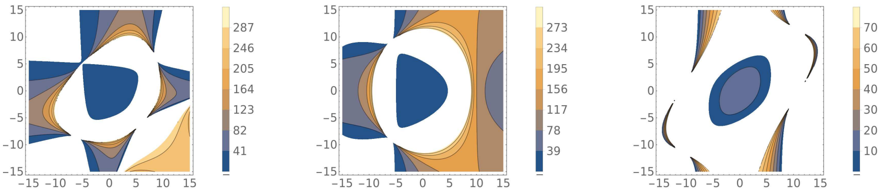


FIGURE 19. Central charge of L^{131} on $\Sigma = \mathbb{H}^2$ for different values of the integer fluxes. We plot the regions of fluxes b_i in which the central charge assumes a positive value. In the first case we have fixed $b_1 = x$, $b_2 = y$ and $b_3 = b_4 = b_5 = 0$. In the second case we have fixed $b_3 = x$ and $b_4 = y$ and $b_1 = b_2 = b_5 = 0$. In the third case we have fixed $b_2 = x$, $b_4 = y$ and $b_1 = b_3 = b_5 = 0$.

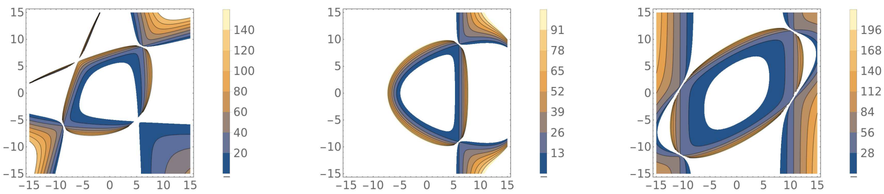


FIGURE 20. Central charge of L^{131} on $\Sigma = \mathbb{S}^2$ for different values of the integer fluxes. We plot the regions of fluxes b_i in which the central charge assumes a positive value. In the first case we have fixed $b_1 = x$, $b_2 = y$ and $b_3 = b_4 = b_5 = 0$. In the second case we have fixed $b_3 = x$ and $b_4 = y$ and $b_1 = b_2 = b_5 = 0$. In the third case we have fixed $b_2 = x$, $b_4 = y$ and $b_1 = b_3 = b_5 = 0$.

As in the previous case, the formulae are simpler for the case of the torus reduction ($\kappa = 0$) and we can provide the analytical expression for the extremized central charge c_r:

$$c_r(\mathsf{L}^{131})_{\mathbb{T}^2} = 6\left(\frac{(b_1^2-3b_2b_1+b_2^2-b_3^2+(b_1+b_2)b_3)^2}{(b_1-b_2)(b_1^2+b_2^2+b_4^2+b_5^2+b_3b_1-b_2b_1+b_2b_3-b_3b_4-b_4b_5)} - \frac{b_1^2+4b_2b_1-2b_3b_1-b_2^2+2b_3^2-2b_2b_3}{b_1-b_2}\right) \tag{3.92}$$

Also here we observe that the parameters ϵ_i^* are generically non-vanishing for both the flavor and the baryonic global symmetries.

3.6. Conclusions

In this chapter we have studied c-extremization for 2d $\mathcal{N} = (0,2)$ SCFTs obtained by applying the the twisted compactification described in Chapter 2 to infinite families of 4d quiver gauge theories holographically dual to D3 branes probing the tip of CY_3 cones over a base 5-manifold X_5 admitting a $U(1)^3$ toric action. In such cases we have been able to develop a simple geometric formulation for the 2d trial central charge c_r, which in the large N limit turns out to be expressible in terms of combinatorial information associated to the toric diagram of the 4d parent theory (see (3.29)). Our result represents the field theory dual of the holographic formula found in [**11, 12, 20**].

This formulation borrows many ideas and constructions developed in the 4d parent theory, in which it has been demonstrated that the conformal anomaly a is proportional to the inverse of the volume of the SE manifold X_5. The geometric analogy with the 4d formulation that we have discussed may be helpful in understanding the possible relation between c_r and the volume of the 7-manifold [**24, 25**] in the conjectured $\mathrm{AdS}_3 \times \mathcal{M}_7$ correspondence (or 8-manifolds in M-theory). Progress on the subject has been recently made in [**99**].

CHAPTER 4

Circle Reduction of 4d Dualities

4.1. Overview

Up to now we have been concerned with compactifications of 4d theories to 2d and correspondingly we have been able to discuss the role of background fluxes in the reduction of (conformal)supersymmetry as well as anomalies and central charges. In this chapter we turn our attention to a different aspect of compactification which is the reduction of supersymmetric dualities. The compactifications we will consider are of the type of circle compactifications to 3d effective field theories. The geometry of the internal manifold does not allow in this case to perform the twists previously discussed however it has been shown that other non-perturbative effects do contribute to the dynamics of the 3d theory. One particular example is the appearance of KK monopoles coming form the reduction of 4d instanton configurations wrapping around the compact dimension. As we will show, such contributions turn out to be fundamental in establishing the duality at the level of the 3d effective theory.

The theories we consider are $USp(2N_c)$ SQCD_4 with one antisymmetric matter field. The 4d dualities involving this gauge and matter content have been originally discussed in [**100**], in the presence of a power law superpotential for the antisymmetric field. The generalization of Seiberg duality for such models was obtained and many tests have been performed. The dynamics of these theories in the absence of the tree level superpotential for the antisymmetric field has been discussed originally in [**101**], in the presence of six fundamentals. It was shown that this theory confines in the IR without chiral symmetry breaking. More recently the analysis has been extended to $USp(2N_c)$ gauge theories with eight fundamentals, an antisymmetric and a set of extra singlets. Making use of exact mathematical identities for $\mathcal{N} = 1$ superconformal indices the authors of [**102**] constructed a large number of magnetic duals of this theory, all of which are related by the action of a reflection group of the type of the E_7-root system Weyl group. In [**65**] it was then shown that in the rank-1 case these dualities can become a self-duality of a single theory that exhibits an IR global symmetry enhancement to the full E_7 algebra and in [**103**] this result was generalized to arbitrary rank also showing that the symmetry actually enhances to $E_7 \times U(1)$.

Motivated by this discussion we study the dimensional reduction of these 4d models, finding large classes of new relations and dualities. In Section 4.2 we study the reduction of $USp(2N_c)$ theories with $2N_f$ fundamentals Q, an antisymmetric A and superpotential $W = \mathsf{Tr}\, A^{k+1}$. We generalize the structure of RG flows and 3d dualities already worked out for the case without antisymmetric matter. The main results obtained there are highlighted in red in Figure 1. In Section 4.3 we study the reduction of the 72 dual phases involving $USp(2N_c)$ gauge groups, eight fundamentals and an antisymmetric. We show that the Weyl group of E_7 is still at work, preserving the 4d dualities. Moreover we show that by real mass flow we can generalize the result of [**65**] where the Weyl group of D_6 relates two classes of dual theories. On one hand one has $USp(2N_c)$ theories with six fundamentals and one antisymmetric. On the other hand there are $U(N_c)$ gauge theories with four pairs of fundamentals and anti-fundamentals and an adjoint. By further real mass flow a large web involving USp/U dualities can be constructed. We have summarized this construction in Figure 2. It is interesting to observe that a generalization of the $SU(3)$ global symmetry enhancements, discussed recently in [**104, 105, 106**] for $U(1)$ with two fundamentals, is obtained here for $U(N_c)$ with two fundamentals and one adjoint. In Section 4.4 we study the dimensional reduction of the confining $USp(2N_c)$ gauge theory with one antisymmetric and six fundamentals. We obtain new 3d confining theories with both $USp(2N_c)$ with fundamentals and one antisymmetric and $U(N_c)$ gauge group with fundamentals, antifundamentals and one adjoint.

4.2. $2N_f$ fundamentals, one antisymmetric A and $W = \mathrm{Tr}\, A^{k+1}$

In this section we study the reduction to three dimensions of 4d theories with $USp(2N_c)$ gauge group, $2N_f$ fundamental fields Q and one antisymmetric field A, with superpotential

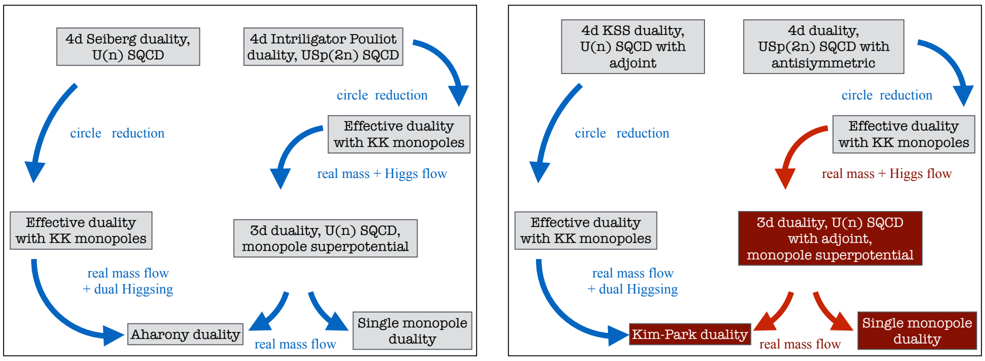
$$W = \mathrm{Tr}\, A^{k+1}. \tag{4.1}$$

This theory has a Seiberg-like dual description [**100**], corresponding to a $USp(2(k(N_f - 2) - N_c))$ gauge theory, with $2N_f$ dual fundamentals q and one antisymmetric a. In this case the superpotential of the dual theory is

$$W = \mathrm{Tr}\, a^{k+1} + \sum_{j=0}^{k-1} \mathrm{Tr}\, M_{k-j-1} q a^j q\,, \tag{4.2}$$

where the generalized mesons M_j are identified with the gauge invariant combinations QA^jQ and the contractions with the symplectic forms are left implicit. While the reduction on $\mathbb{S}^1$ for this model has already been discussed in the literature [**61, 62**], here we will study some further flows, constructing dualities involving 3d $U(N_c)$ gauge theories with fundamental flavor, adjoint matter and monopole superpotentials. These flows are triggered by large real masses and large expectation values for the real scalars in the 3d $\mathcal{N} = 2$ vector multiplet (a.k.a. Higgs flows). When $k = 1$ they reduce to the flows studied in [**64**] for $USp(2N_c)$ SQCD. In Figure 1 we present the various steps of the dimensional reduction, the real mass and Higgs flows studied in the literature for the cases without and with matter fields in tensorial representations. We highlight in red the flows and the models that we are going to study in this section in order to complete the classification. Observe that here we are not considering dualities with CS terms, as the ones discussed in [**107, 108, 109, 110**].

4.2.1. The 3d duality with a monopole superpotential. The reduction to three dimensions of this model has already been discussed in the literature in [**61, 62**]. This consists of the stepwise procedure introduced in [**63**] and reviewed in Appendix C.1. Observe that in this case the matter content allows the generation of the KK superpotential even in the absence of superpotential interactions (see Appendix C.2.7 for details). This generates an effective duality on $\mathbb{S}^1$. This effective duality can be deformed into a 3d duality between $USp(2N_c)$ gauge theories as discussed in [**61, 62**], by a real mass flow. There is another interesting flow that can be performed, consisting of a mixing between a real mass and Higgs flow, of the type introduced in [**65**] and thoroughly investigated in [**66, 64**]. Such a flow can be triggered by splitting the real masses μ_a of the $2N_f$ fundamentals into



(a) (b)

FIGURE 1. In figure (a) we represented two different strategies, appeared in the literature, that starting from 4d dualities led to Aharony duality. In the first case one starts from Seiberg duality [**111**], reduces on the circle obtaining a new effective duality [**63**] and then, by a real mass (and a Higgs flow in the dual phase), one recovers Aharony duality [**112**]. In the second case one starts from the duality of [**113**], reduces on a circle and then flows to a new duality [**64**], in the presence of monopole superpotentials. It is then possible to either flow to the duality with a single term in the monopole superpotential or to the conventional Aharony duality by a real mass flow. In figure (b) we consider the same type of reductions for theories with two-index tensor matter. The first reduction connects the 4d duality of [**57**] to the 3d duality of [**114**] and it has been studied in [**60**]. In the second case we observe that the 3d duality of [**114**] can be recovered also starting from the 4d duality with $USp(2N_c)$ gauge group, with fundamentals and an antisymmetric [**100**]. This duality has been already reduced on a circle in [**61, 62**], while the other steps have not been performed yet in the literature. We highlight in red these new steps, as they are the subject of the current analysis.

N_f components m_a and N_f components n_a and considering the following shift of parameters in the 3d partition function:

$$\mu_a \to m_a + s, \qquad \mu_{a+N_f} \to n_a + s,, \qquad a = 1, \dots, N_f \tag{4.3}$$

with $s \to \infty$ [1]. At the same time one needs to consider the Higgs flow $\sigma_i \to \sigma_i + s$, with $i = 1, \dots, n$. This leads to a 3d duality between a $U(N_c)$ theory with N_f fundamentals Q, N_f anti-fundamentals $\widetilde{Q}$ and one adjoint X with superpotential

$$W = \mathrm{Tr}\, X^{k+1} + T_0 + \widetilde{T}_0 \tag{4.4}$$

and a $U(k(N_f - 2) - N_c)$ theory with N_f fundamentals q, N_f anti-fundamentals $\widetilde{q}$ and one adjoint x, with superpotential

$$W = \mathrm{Tr}\, x^{k+1} + \sum_{j=0}^{k-1} \mathrm{Tr}\, M_{k-j-1} q x^j \widetilde{q} + t_0 + \widetilde{t}_0 \tag{4.5}$$

[1]This is the procedure that is needed to trigger the flow at the level of the partition function on the squashed 3-sphere. On flat space the situation is slightly different, in fact it is sufficient to turn on a non-zero real mass for the fundamental flavors which then, being massive, decouple in the infrared.

where the generalized mesons are $M_j = QX^j\widetilde{Q}$. Turning on the linear monopole superpotentials in (4.4) and in (4.5) breaks the topological and the axial symmetry and it fixes the R-charges of the fundamentals. Furthermore this is consistent with the duality map: imposing in the electric theory the monopole and the anti-monopole R-charge as $\Delta_{T_0} = \Delta_{\widetilde{T}_0} = 2$ fixes the R-charge of the fundamentals as

$$\Delta_Q = \Delta_{\widetilde{Q}} = \frac{N_f - 2 - \Delta_X(N_c - 1)}{N_f} \tag{4.6}$$

In the magnetic theory a similar computation gives

$$\Delta_q = \Delta_{\widetilde{q}} = \frac{2 - N_f + \Delta_x(N_c + N_f - 1)}{N_f} \tag{4.7}$$

The duality map $\Delta_q = \Delta_X - \Delta_Q$ is satisfied by (4.6) and (4.7).

We conclude this discussion with a remark on the linear monopole superpotentials in (4.4) and in (4.5). The generation of such a superpotential for a $U(N_c)$ gauge theory with adjoint matter may appear incorrect, because the adjoint field adds two further zero modes to the ones carried by the gaugino. Nevertheless we claimed that the linear superpotential for the bare monopole and anti-monopole can be generated. This is motivated by the nature of the UV completion of our theory: indeed the $U(N_c)$ theory that we are describing so far is UV-completed by a $USp(2N_c)$ gauge theory with an antisymmetric field. By performing the counting of the zero modes in this setup (see Appendix C.2) one can observe that the antisymmetric field does not lead to any further zero mode in the monopole backgrounds that we are considering, and this signals the presence of the linear monopole superpotentials. In other words the linear monopoles are inserted in the UV $USp(2N_c)$ theory, where they are perfectly consistent with the zero mode counting, and they modify the Coulomb branch of the IR $U(N_c)$, treated as an effective theory.

4.2.2. Flowing to Kim-Park duality. Starting from the duality above we can flow to the duality of [**114**] as follows. First we consider a $U(N_c)$ gauge theory with $N_f + 2$ flavors and one adjoint with the superpotential (4.4). The dual theory has $U(kN_f - N_c)$ gauge group, $N_f + 2$ flavors, one adjoint and the superpotential coincides with (4.5). Then we trigger the flow by shifting the real masses as

$$\begin{aligned} m_{N_f+1} &\to m_{N_f+1} + s, \quad m_{N_f+2} \to m_{N_f+2} - s, \\ n_{N_f+1} &\to n_{N_f+1} - s, \quad\ n_{N_f+2} \to n_{N_f+2} + s, \end{aligned} \tag{4.8}$$

with $s \to \infty$. In the electric theory we obtain a 3d theory with $U(N_c)$ gauge group, N_f flavors, an adjoint and superpotential $W = \operatorname{Tr} X^{k+1}$. In the dual theory the situation is more involved. Indeed in this case some of the components of the original mesons remain massless even if they are not associated to the massless mesons of the electric theory. These are the $(N_f + 1)$-th and the $(N_f + 2)$-th diagonal components of M_j. These fields are light gauge singlets in the magnetic theory and their quantum numbers are compatible with the following superpotential interactions

$$W = t_0 \sum_{j=0}^{k-1} (M_j)_{N_f+1,N_f+1} \operatorname{Tr} x^j + \widetilde{t}_0 \sum_{j=0}^{k-1} (M_j)_{N_f+2,N_f+2} \operatorname{Tr} x^j \tag{4.9}$$

where t_0 and $\widetilde{t}_0$ are the bare monopole and anti-monopole of the $U(kN_f - N_c)$ theory. The dressed monopoles of the dual theory can be defined as $t_j = t_0 \operatorname{Tr} x^j$ and $\widetilde{t}_j = \widetilde{t}_0 \operatorname{Tr} x^j$. On the other hand the singlets $(M_j)_{N_f+1,N_f+1}$ and $(M_j)_{N_f+2,N_f+2}$ can be identified with the dressed monopoles of the electric theory, i.e. $T_j = T_0 \operatorname{Tr} X^j$

and $\widetilde{T}_j = \widetilde{T}_0 \operatorname{Tr} X^j$. The final form of the superpotential of the dual theory is then

$$W = \operatorname{Tr} x^{k+1} + \sum_{j=0}^{k-1} \operatorname{Tr} M_{k-j-1} q x^j \widetilde{q} + \sum_{j=0}^{k-1} (t_j T^{k-1-j} + \widetilde{t}_j \widetilde{T}^{k-1-j}) \tag{4.10}$$

reproducing the dual superpotential of the Kim-Park duality.

4.2.3. Duality with a single monopole superpotential. This case can be studied starting with $N_f + 1$ flavors and monopole superpotential (4.4). The dual theory in this case has rank $U(k(N_f - 1) - N_c)$. We consider the real mass flow

$$m_{N_f+1} = \eta + s, \quad n_{N_f+1} = \eta - s \tag{4.11}$$

In the electric theory this real mass flow reduces the number of flavors, and removes the contribution of $\widetilde{T}_0$ to the superpotential (4.4). This gives a $U(N_c)$ electric theory with N_f flavors and superpotential

$$W = \operatorname{Tr} X^{k+1} + T_0 \tag{4.12}$$

Observe that at the level of global currents this superpotential restores a combination of the axial $U(1)_A$ and the topological $U(1)_J$ symmetries.

On the dual side the real mass deformation (4.11) reduces the number of flavors, while it leaves the rank of the dual group invariant. The deformation (4.11) removes the contribution of $\widetilde{t}_0$ to the monopole superpotential and it reduces the number of singlets, from $k(N_f + 1)^2$ to $k(N_f^2 + 1)$. The first kN_f^2 components correspond to the generalized mesons of the electric theory, $M_j = QX^j\widetilde{Q}$. The last k components correspond components of the generalized mesons that remain light despite (4.11). We refer to these k components as S_j, with $j = 0, \ldots, k-1$. Their quantum numbers are compatible with the superpotential

$$W = \operatorname{Tr} x^{k+1} + \sum_{j=0}^{k-1} \operatorname{Tr} M_j q x^j \widetilde{q} + t_0 + \sum_{j=0}^{k-1} S_j \widetilde{t}_j \tag{4.13}$$

where $\widetilde{t}_j = \widetilde{t}_0 \operatorname{Tr} X^j$ are the dressed anti-monopoles of the magnetic theory. It is then natural to identify S_j with the dressed monopoles of the electric theory.

4.2.4. Partition functions. The sequence of reductions and dualities discussed above can be studied at the level of localization, where the partition function is obtained as a certain kind of hyperbolic hypergeometric integral which depends on the complex variables $\vec{\mu} \in \mathbb{C}^{N_f}$ being the mass parameters associated to the fundamental fields, and $\tau \in \mathbb{C}$ being the mass parameter of the antisymmetric[2]. The 4d/3d reduction can therefore be analyzed by reducing the identity between the superconformal indices relating the 4d duality (see Appendix C.3 for details). This provides a relation between the 3d partition functions leading to the integral identity:

$$\mathcal{Z}_{USp(2N_c)}(\vec{\mu};\tau) = \prod_{j=0}^{k-1} \prod_{1 \le a < b \le 2N_f} \Gamma_h(j\tau + \mu_a + \mu_b)\, \mathcal{Z}_{USp(2\widetilde{N}_c)}(\vec{\widetilde{\mu}};\tau) \tag{4.14}$$

We remark that the identity (4.14) only holds when the parameters of the electric theory satisfy the *balancing condition*:

$$\sum_{a=1}^{2N_f} \mu_a = 2\left(\omega(N_f - 2) - (N_c - 1)\tau\right) \tag{4.15}$$

[2]Physically speaking, the real part of the parameters $\vec{\mu}$ and τ represents the real masses, i.e., the fugacities associated to the Cartan of the flavor symmetry, while their imaginary part correspond to the R-charges (up to a constant factor depending on the squashing).

which is to be regarded as a constraint on the global charges of the matter fields, coming from the presence of a non-trivial monopole superpotential (4.12). Similarly, the parameter τ is fixed by the superpotential to the value $\tau = \frac{\omega}{k+1}$, and indeed the antisymmetric field in this case is not charged under any non-R global symmetry because of the superpotential. The parameter τ is purely imaginary and corresponds to the R-charge of the antisymmetric field. The rank of the dual group is $\widetilde{N}_c = k(N_f - 2) - N_c$. The parameters $\widetilde{\mu}_a$ are related to the electric ones by the duality map $\widetilde{\mu}_a = \tau - \mu_a$.

Starting from the relation (4.14) we can shift the scalars $\sigma_i \to \sigma_i + s$ and consider the real mass flow $\mu_a = m_a + s$, $\mu_{a+N_f} = n_a - s$ for $1 \le a \le N_f$. This does not affect the balancing condition (4.15) but it leads to the relation

$$\mathcal{Z}_{U(N_c)}(\vec{m};\vec{n};\tau) = \prod_{j=0}^{k-1}\prod_{a,b=1}^{N_f}\Gamma_h(j\tau + m_a + n_b)\,\mathcal{Z}_{U(\widetilde{N}_c)}(\vec{\widetilde{m}};\vec{\widetilde{n}};\tau) \tag{4.16}$$

where $\widetilde{m}_a = \tau - m_a$, $\widetilde{n}_a = \tau - n_a$. Relation (4.16) represents the equivalence between the partition functions of the models discussed in 4.2.1. The presence of the monopole superpotential is encoded in the constraint (4.15). Observe that the presence of this constraint breaks the otherwise non-anomalous axial symmetry. The breaking of $U(1)_J$ is encoded in the absence of an FI.

We can also reproduce the flow to the Kim-Park duality on the partition function. The final relation was originally proved in [**115**] by considering the reduction of KSS duality on $\mathbb{S}^1$. The flow from the effective duality on $\mathbb{S}^1$ to the Kim-Park duality required a Higgsing in the dual phase. This Higgsing led to a product of gauge groups. One of them represented the dual gauge group while the other needed to be dualized to set of singlet, corresponding to the electric monopoles acting as singlets in the dual phase. On the partition function this dualization was possible because of an exact relation at the level of the 3d partition function, discussed in [**116**]. Interestingly this relation played a relevant role recently in [**117**], in the reduction of AD theories to 3d.

Here we arrive at the same final identity proven in [**115**] by following a different strategy. First we consider the monopole duality with $N_f + 2$ flavors. Then, on the electric side we shift the flavors as discussed in (4.8). The final identity is

$$\begin{aligned}\mathcal{Z}_{U(N_c)}(\vec{m};\vec{n};\tau;\Lambda) = \prod_{j=0}^{k-1}\Gamma_h\left(\pm\frac{\Lambda}{2} + \omega N_f + (j - N_c + 1)\tau - \sum_{a=1}^{N_f}\frac{m_a + n_a}{2}\right)\times \\ \times\prod_{a,b=1}^{N_f}\Gamma_h(j\tau + m_a + n_b)\mathcal{Z}_{U(\widetilde{N}_c)}(\vec{\widetilde{m}};\vec{\widetilde{n}};\tau;-\Lambda)\end{aligned} \tag{4.17}$$

Observe that the balancing condition becomes

$$\sum_{a=1}^{N_f}(m_a + n_a) + 2(m_{N_f+1} + m_{N_f+2}) = 2(\omega N_f - (N_c - 1)\tau) \tag{4.18}$$

The parameters m_{N_f+1} and m_{N_f+2} are free, signaling the absence of a balancing condition on the mass parameters m_a and n_a. On the physical side the combinations $m_{N_f+1} + m_{N_f+2}$ represents the presence of an axial symmetry while the parameter $\Lambda = 2(m_{N_f+1} - m_{N_f+2})$ represents the Fayet-Iliopoulos (FI) term. It indeed appears as $e^{i\pi\Lambda\sum\sigma_i}$ on the LHS of (4.17) and with an opposite side on the RHS. The first term on the RHS of (4.17) represents the contribution of the dressed monopoles of the electric theory acting as singlets on the dual side.

We conclude the analysis by considering the real mass flow studied in 4.2.3, leading to the theory with a single monopole superpotential. In this case we consider the electric theory with $N_f + 1$ flavors and deform it as in (4.11). The balancing condition becomes

$$2\tau(N_c - 1) + \sum_{a=1}^{N_f}(m_a + n_a) + 2\eta = 2\omega(N_f - 1) \tag{4.19}$$

The presence of η in (4.19) signals the fact that an extra abelian symmetry has been generated by the real mass flow. To understand the origin of this symmetry one can look at the final identity relating the partition functions of the electric and of the magnetic theory. In this case case we have

$$\begin{aligned}\mathcal{Z}_{U(N_c)}(\vec{m};\vec{n};\tau;\omega-\eta) = \mathrm{e}^{\frac{\mathrm{i}\pi}{2}k\sum_{a=1}^{N_f}(m_a^2-n_a^2)}\prod_{a,b=1}^{N_f}\Gamma_h(j\tau+m_a+n_b)\times \\ \times\prod_{j=0}^{k-1}\Gamma_h\left(\eta+\omega(N_f-1)+\tau(j-N_c+1)-\frac{1}{2}\sum_{a=1}^{N_f}(m_a+n_a)\right)\times \\ \times\mathcal{Z}_{U(\widetilde{N}_c)}(\vec{\widetilde{m}};\vec{\widetilde{n}};\tau;\tau-\omega-\eta)\end{aligned} \tag{4.20}$$

where $\widetilde{m}_a = \tau - m_a$, $\widetilde{n}_a = \tau - n_a$ and $\widetilde{N}_c = k(N_f - 1) - N_c$. The result matches the expectations from the field theory analysis. Indeed one can read the charges of the singlets from the partition function and check that they coincide with those obtained for the singlets S_j in the superpotential (4.13).

4.3. Eight fundamentals and E_7 symmetry

In this section we re-consider the gauge and field content discussed above, i.e., supersymmetric gauge theories with a symplectic gauge group, fundamentals Q (here we restrict to $2N_f = 8$) and one anti-symmetric tensor matter field A. However, here we have models without a power law superpotential for the field A. These theories have been analyzed in [**102, 65, 103**] where it has been shown that these models present an IR enhancement of the global symmetry to $E_7 \times U(1)$. In the following we will study the reduction of these models to 3d, showing the appearance of monopole superpotentials and constructing an intricate web of dualities. These new dualities generalize the 3d $SU(2)/U(1)$ duality discovered in [**65**] to a $USp(2N_c)/U(N_c)$ duality. We also emphasize the key role played by monopole superpotentials in the analysis and check many of the claims by testing them with the three sphere partition function.

4.3.1. The 4d theory. The 4d theories have been largely discussed in [**102, 103**] and here we will just briefly review some of the main aspects of these models necessary for our analysis. One can divide the 4d theories into 4 classes, depending on the global symmetry and on the presence of singlets that can be added without further breaking of the flavor symmetry. The first two classes, (**A**) and (**B**) have a classically unbroken $SU(8)$ global symmetry, while the other two classes, (**C**) and (**D**) have a smaller $SU(4) \times SU(4)$ classical global symmetry. In the following we discuss some of the salient features of these theories.

(**A**) The theory has a global $SU(8) \times U(1) \times U(1)_R$ symmetry group and the fields transform under the gauge and global symmetries as in the following table:

	$USp(2N_c)$	$SU(8)$	$U(1)'$	$U(1)_R$
Q	$2N_c$	8	$\frac{1-N_c}{4}$	R_Q
A	$N_c(2N_c-1)-1$	1	1	R_A

The anomaly freedom of the R-symmetry imposes the constraint

$$(N_c+1)+(N_c-1)(R_A-1)+4(R_Q-1)=0 \tag{4.21}$$

(**B**) This is a Seiberg dual phase, the global symmetry group visible in the lagrangian is maximal, i.e. $SU(8)\times U(1)\times U(1)_R$. This dual theory has N_c mesons in the two index antisymmetric representatiton of the non-abelian $SU(8)$. These 28 dimensional mesons act as singlets in the dual phase and they can be expressed in terms of the matter fields of the electric phase as

$$M_{rs}^{(j)}=Q_rQ_sA^{N_c-1-j} \tag{4.22}$$

with $1\leq r<s\leq 8$ and $j=0,\dots,N_c-1$. The eight dual fundamentals q and the dual antisymmetric a interact with the meson through the superpotential

$$W_B=\sum_{j=0}^{N_c-1}\mathsf{Tr}\, M^{(j)}qqa^j \tag{4.23}$$

(**C**) In this case the global $SU(8)$ symmetry is explicitly broken to $SU(4)\times SU(4)\times U(1)_B$, where the subscript B in the abelian factor indicates that this symmetry acts like a baryonic symmetry giving an opposite charge to the fundamentals of the two $SU(4)$ factors. There are N_c mesons $M^{(j)}$ in the $4\times\overline{4}$ representation of the non-abelian symmetry group and there is a superpotential

$$W_C=\sum_{j=0}^{N_c-1}\mathsf{Tr}\, M^{(j)}qpa^j \tag{4.24}$$

where the four anti-fundamentals q refer to the first $SU(4)$ factor and the four fundamentals p refer to the second $SU(4)$ factor. Up to permutations there are $\frac{1}{2}\binom{8}{4}=35$ inequivalent theories, having the same field content in terms of gauge group and charged matter. All these models are claimed to be dual to the ones presented in (**A**) and (**B**).

(**D**) There is a second family of theories with an $SU(4)^2$ manifest global symmetry group. This theory has two types of mesons, each one in the antisymmetric representation of one of the two $SU(4)$ factors. Referring to these mesons as $M^{(j)}$ and $N^{(j)}$ the superpotential becomes

$$W_D=\sum_{j=0}^{N_c-1}\mathsf{Tr}\left(M^{(j)}qqa^j+N^{(j)}ppa^j\right) \tag{4.25}$$

Also in this case there are 35 inequivalent theories and they are claimed to be dual to the ones discussed in (**A**), (**B**) and (**C**).

The duality among these 72 models has been claimed in [**102**], by use of the integral identities of [**118**] between their superconformal indices. These identities correspond to the action of the Weyl group of E_7 on the chemical potentials associated to the global symmetries. One can then imagine that the set of 72 dual theories forms an orbit for the action of the Weyl group of E_7 with stabilizer the parabolic subgroup $S_8\cong W(A_7)$ corresponding to the manifest global symmetry of the lagrangian which acts by permutation of the fundamental fields. The size of the orbit is then given by the ratio of the orders of the two groups, which is precisely the number 72. In the case of even N_c it has been also observed [**103**] that all the models can be deformed in such a way that one deals with 72 self dual phases. In such cases the (self)-duality group enhances the $SU(8) \times U(1)$ global symmetry to $E_7 \times U(1)$.

4.3.2. Reduction to 3d. The models described above can be reduced to 3d by a circle compactification. The prescription of [**63**], reviewed in Appendix C.1, is necessary in order to preserve the duality among the different 72 phases. We reduce the spectrum and the interactions of each phase and then add the KK monopole superpotential (see C.2.7 for details). The presence of the KK monopole superpotential imposes further constraints on the 3d real masses of the matter fields. In this case the constraint is:

$$(N_c + 1) + (N_c - 1)(\Delta_A - 1) + 4(\Delta_Q - 1) = 0 \tag{4.26}$$

where Δ_Q and Δ_A are the 3d R-charges of the fundamentals and of the antisymmetric respectively. Observe that the constraint (4.26) is equivalent to the one imposed in 4d by the anomaly freedom of the R-current. The KK monopole superpotential constrains the global symmetries as well, preventing the generation of possible axial symmetries.

This procedure preserves the duality among the 72 $USp(2N_c)$ theories with eight fundamentals and an antisymmetric. This can indeed be thought of as a duality between 3d effective theories. This claim can be tested by reducing the identities between the superconformal indices to identities between the three sphere partition functions. The final identities already appeared in the literature in [**116**] (see [**119**] for a seminal work). Here we translate in a physical language many of the results of [**116**], deriving an interesting set of new 3d $\mathcal{N} = 2$ dualities. The starting point consists of considering the exact mathematical identity

$$\begin{aligned} \mathcal{Z}_{USp(2N_c)}(\vec{\mu};\tau) = & \prod_{j=0}^{N_c-1} \prod_{1\le r<s\le 4} \Gamma_h(j\tau + \mu_r + \mu_s) \times \\ & \times \prod_{5\le r<s\le 8} \Gamma_h(j\tau + \mu_r + \mu_s)\ \mathcal{Z}_{USp(2N_c)}(\vec{\widetilde{\mu}};\tau) \end{aligned} \tag{4.27}$$

where we defined $\widetilde{\mu} = \{\mu_r + \zeta, \mu_{r+4} - \zeta\}$ for $r = 1, \ldots, 4$ and

$$2\zeta = \sum_{r=5}^{8} \mu_r - 2\omega + (N_c - 1)\tau = -\sum_{r=1}^{4} \mu_r + 2\omega - (N_c - 1)\tau \tag{4.28}$$

This identity holds provided the constraint

$$2(N_c - 1)\tau + \sum_{r=1}^{8} \mu_r = 4\omega \tag{4.29}$$

is imposed on the mass parameters μ_r and τ. The relation (4.27) together with the invariance of the integral under permutations of the eight μ_r variables provides the invariance under the action of $W(E_7)$ [**119, 116**]. Observe that (4.27) can be viewed as a master relation and that all possible other dualities can be proved by alternating (4.27) and permutation.

For example the duality between the two models with a manifest $SU(8)$ global symmetry follows from (4.27). It is obtained by alternating the transformation (4.27) and three permutations. More precisely one first applies (4.27) to the μ_i ordered as above. Then one permutes the $\widetilde{\mu}$ variables exchanging $\widetilde{\mu}_3$ and $\widetilde{\mu}_4$ with $\widetilde{\mu}_5$ and $\widetilde{\mu}_6$ and apply (4.27) again. The last permutation corresponds to exchange $\widetilde{\widetilde{\mu}}_3$ and $\widetilde{\widetilde{\mu}}_4$ with $\widetilde{\widetilde{\mu}}_7$ and $\widetilde{\widetilde{\mu}}_8$ and apply (4.27) for the third time. Observe that each time we apply the transformation (4.27) we generate $12N_c$ new mesons, corresponding to N_c times the two antisymmetric representation of each $SU(4)$ global symmetry group. However the duality with the manifest $SU(8)$ global symmetry has N_c mesons in the antisymmetric representation of $SU(8)$ corresponding to $28N_c$ components. One can observe explicitly that the extra $8N_c$ components are pairwise massive and eliminate them on the integral identity by iterating the relation (C.53). The final relation that one obtains is:

$$\mathcal{Z}_{USp(2N_c)}(\vec{\mu};\tau) = \prod_{j=0}^{N_c-1} \prod_{r<s} \Gamma_h(j\tau + \mu_r + \mu_s)\, \mathcal{Z}_{USp(2N_c)}(\vec{\widetilde{\mu}};\tau) \tag{4.30}$$

with $\widetilde{\mu}_r = \omega - \frac{N_c-1}{2}\tau - \mu_r$

It is interesting to observe that this last duality reduces to one of the cases discussed in Section 4.2 if we add to the superpotential the deformation $W = A^{N_c+1}$. This superpotential deformation corresponds on the dual side to the contribution $W = a^{N_c+1}$ for the dual antisymmetric field. This deformation breaks the $U(1)'$ symmetry and it forces $\tau = \frac{2\omega}{N_c+1}$. It corresponds to the reduction of the duality of [**120**] studied in [**62**]. Indeed in this case the dual mode must have $USp(2(k(N_f-2)-N_c))$ gauge symmetry, where here $k = N_c$ and $N_f = 4$. The identity (4.30) corresponds to the one obtained in [**62**] if the actual value of τ is inserted in the balancing condition.

We conclude by observing that the global symmetry of the integrals can enhance to $W(E_7)$ if N_c is even, similarly to the 4d case. First one adds the superpotential:

$$\Delta W = \sum_{j=0}^{\frac{N_c}{2}-1} \mathrm{Tr}\, M^{(j)} Q Q A^j + \sum_{j=2}^{N_c} \beta_j \,\mathrm{Tr}\, A^j \tag{4.31}$$

Then one observes that all the theories are self dual if this deformation is provided. At the integral level this signals the fact the we must have an integral identity in which each phase is just a re-parametrization of the real masses, without further uncharged matter fields distinguishing the different phases. This corresponds to an enhanced symmetry and not to a duality. This can be proven on the partition function by showing that the generator of the Weyl reflection that does not correspond to a permutation is just a re-parametrization of the masses. The further generator is the one generating the identity (4.27). The addition of the superpotential (4.31) corresponds to multiplying the identity (4.27) by the terms

$$\prod_{r<s} \prod_{j=0}^{\frac{N_c}{2}-1} \Gamma_h(2\omega - j\tau - \mu_r - \mu_s) \times \prod_{j=2}^{N_c} \Gamma_h(2\omega - j\tau) \tag{4.32}$$

On the LHS some of the terms simplify against the contributions of the mesons in (4.27). The mesonic contributions that do not simplify correspond to the term

$$\begin{aligned}\prod_{j=\frac{N_c}{2}}^{N_c-1} \Bigg[\prod_{1\le r<s\le 4} \Gamma_h(2\omega - j\tau - \mu_r - \mu_s) \prod_{5\le r<s\le 8} \Gamma_h(2\omega - j\tau - \mu_r - \mu_s) \Bigg] \times \\ \times \prod_{j=0}^{\frac{N_c}{2}-1} \prod_{r=1}^{4} \prod_{s=5}^{8} \Gamma_h(2\omega - j\tau - \mu_r - \mu_s)\end{aligned} \tag{4.33}$$

We can substitute in this expression the real masses $\widetilde{\mu}_r = \mu_r - \zeta$ for $r = 1,\dots,4$ and $\widetilde{\mu}_r = \mu_r + \zeta$ for $r = 5,\dots,8$ and when necessary plug in the condition (4.28). The final result is that in the dual theory we remain with the same pre-factor added to the LHS of (4.27) in terms of the $\widetilde{\mu}$ masses.

$$\prod_{r<s} \prod_{j=0}^{\frac{N_c}{2}-1} \Gamma_h(2\omega - j\tau - \widetilde{\mu}_r - \widetilde{\mu}_s) \times \prod_{j=2}^{N_c} \Gamma_h(2\omega - j\tau) \tag{4.34}$$

Recalling that the identity (4.27) together with the permutations of μ generates the group $W(E_7)$, proves that this is a symmetry of the integrals and supports the claim that the model with the superpotential (4.31) enhances the global symmetry group to E_7.

4.3.3. Real mass flows: $USp(2N_c)$ models and the action of D_6. The next step consists of removing the KK monopole superpotential to obtain conventional 3d $\mathcal{N} = 2$ models. This is done by integrating out some flavors, i.e. by assigning a large real mass to them. When this procedure is done consistently on a pair of dual theories such a duality can be preserved in the 3d limit [**63**]. In the case discussed here we have a set of dual phases connected by a larger symmetry group than the one expected from the action. The concepts of global symmetry and of duality are here strongly connected, and depending on the details of the model the action of a duality group can become the action of a global symmetry group. The group underlining this web of theories is the Weyl group of E_7.

We have just reviewed its action on the complex combinations μ, representing the real masses and the R-charges of the matter fields. By a real mass flow the $W(E_7)$ symmetry group is generically broken to a subgroup. This subgroup is associated to the action of the global symmetry group of the IR theory. In the following we study an explicit realization of such a mechanism by assigning an opposite large mass to two fundamentals in the $USp(2N_c)$ theory with one antisymmetric and eight fundamentals. This assignment must be done consistently with the duality map (4.28). For example we can assign the large masses as $\mu_7 = M + \xi_7$ and $\mu_8 = -M + \xi_8$, with $M > 0$. The parameters $\xi_{7,8}$ can be eliminated after using the original constraints between the real masses (4.29), signaling the fact that we will be left with a set of unconstrained real masses at the end of the flow. This is consistent with the generation of an extra symmetry, constrained before by the presence of the KK monopole superpotential. This is similar to the generation of the axial symmetry in the reduction of 4d Seiberg duality to 3d Aharony duality. This signals the fact that the monopole superpotential vanishes as well.

Let us consider the effect of such a mass deformation in one of the dual phases introduced above. We proceed as follows: we pick up a pair of dual models, treating them as a representative of the duality. We discuss the real mass flow for this pair of dual models and then extract the necessary informations to reconstruct the full duality symmetry. We first study the following reduction between a pair of dual models:

- On the electric side we consider 3d $\mathcal{N} = 2$ $USp(2N_c)$ gauge theory with an anti-symmetric and eight fundamental quarks. In this case the electric theory, after the real mass flow, becomes $USp(2N_c)$ with six fundamentals and one antisymmetric. The real mass parameters are unconstrained.
- On the magnetic side we consider 3d $\mathcal{N} = 2$ $USp(2N_c)$ gauge theory with an anti-symmetric, eight fundamental quarks and superpotential W_D in (4.25). The situation in this dual theory is more interesting. The mesons $M^{(j)}$ are light and survive in the low energy spectrum. The other mesons that survive are the components $N_{56}^{(j)}$ and $N_{78}^{(j)}$. While the first set corresponds to the N_c generalized mesons of the $SU(2) \subset SU(4)$ original flavor symmetry, the second set is associated to N_c new singlets, that we denote as T_j. By looking at the charges of these singlets, they correspond to the dressed electric monopoles, combinations of the bare monopole T_0 of the electric theory with powers of the antisymmetric field X, i.e.,

$T_j = T_0 \mathrm{Tr}\, A^j$. This is consistent with a superpotential of the form

$$W_{T_j} = \sum_{j=1}^{N_c} T_{N_c-j} \mathrm{Tr}\, p_5 p_6 a^{j-1} \tag{4.35}$$

The interaction with the other meson $N_{56}^{(j)}$ vanishes, because the fields p_7 and p_8 are massive in the dual phase. Nevertheless the charges of this meson are consistent [3] with the interaction

$$W_{N_{56}} = \sum_{j=1}^{N_c} N_{56}^{(j)} t_{j-1} \tag{4.36}$$

where t_j represents the dressed monopole of the magnetic theory $t_j = t_0 \mathrm{Tr}\, a^j$. The final structure of the superpotential of the dual theory is

$$W = \sum_{j=1}^{N_c} \left(\mathrm{Tr}\, M^{(j)} qqa^{j-1} + T_{N_c-j} \mathrm{Tr}\, p_5 p_6 a^{j-1} + N_{56}^{(j)} t_{j-1} \right) \tag{4.37}$$

One can also follow this real mass flow on the partition function. The duality is preserved if the divergent terms coincide in the relation (4.27) after the infinite shifts are performed. The real mass flow is performed by using the relation (C.54). The divergent term in the electric partition function is a phase $e^{\frac{i\pi}{\omega_1 \omega_2}\phi_e}$ with

$$\phi_e = 2N_c \left(2M + \xi_7 - \xi_8\right) \left(\xi_7 + \xi_8 - 2\omega\right) \tag{4.38}$$

In the dual model there are two phases contributing to the divergent term. The first comes from the mesons and the second one from the dual fundamentals, $\phi_m = \phi_{\text{mes.}} + \phi_{\text{fund.}}$. They are

$$\begin{aligned} \phi_{\text{mes.}} &= 2N_c \left(2M + \xi_7 - \xi_8\right) \left(\mu_5 + \mu_6 + (N_c - 1)\tau + \xi_7 + \xi_8 - 2\omega\right) \\ \phi_{\text{fund.}} &= -2N_c \left(2M + \xi_7 - \xi_8\right) \left(\mu_5 + \mu_6 + (N_c - 1)\tau\right) \end{aligned} \tag{4.39}$$

One can check that $\phi_e = \phi_m$ leading to the equality

$$\begin{aligned} \mathcal{Z}_{USp(2N_c)_0}(\vec{\mu};\tau) = \prod_{j=0}^{N_c-1} \prod_{1\le r<s\le 4} \Gamma_h(j\tau + \mu_r + \mu_s)\Gamma_h(j\tau + \mu_5 + \mu_6) \times \\ \times \Gamma_h\left(4\omega - (2N_c - 2 + j)\tau - \sum_{r=1}^{6} \mu_r\right) \mathcal{Z}_{USp(2N_c)_0}(\vec{\widetilde{\mu}};\tau) \end{aligned} \tag{4.40}$$

where $\widetilde{\mu}_r = \mu_r + \zeta$ for $r = 1, \ldots, 4$ and $\widetilde{\mu}_r = \mu_r - \zeta$ for $r = 5, 6$ and $2\zeta = 2\omega - \sum_{r=1}^{4} \mu_r - (N_c-1)\tau$. Observe that this relation corresponds to **Theorem 5.6.11** of [**116**] after applying the identity $\Gamma_h(2\omega - x)\Gamma_h(x) = 1$ on the last term in (4.40). As explained there, the integrals have an $W(D_6)$ symmetry, generated by the combined action of the permutation of the parameters μ_r and by the transformation (4.40).

One can also study the real mass flow triggered by $\mu_7 = M + \xi_7$ and $\mu_8 = -M + \xi_8$ on the dual model that preserves the whole $SU(8)$ flavor symmetry, identified by the superpotential W_B in (4.23). In this case some of the components of the mesons of the dual theory are massive, while there are N_c singlets, associated to the combination $Q_7 Q_8 A^j$, massless in the dual theory, that do not give rise to any generalized meson in the dual theory. They correspond to the dressed monopole operators of the electric theory, $T_0 \,\mathrm{Tr}\, A^j$, acting as singlet in the dual phase. By looking at the charges of these singlets under the global symmetry one can observe that there

[3] Observe that further checks are necessary to prove the existence of such an interaction.

is a superpotential interaction compatible with the presence of these fields. The interaction is

$$W = \sum_{j=0}^{N_c-1} T_j \, t_{N_c-1-j} \tag{4.41}$$

where $t_j = t_0 \operatorname{Tr} a^j$ are the dressed monopoles of the dual theory. We can perform some checks of this duality.

- As a first consistency check we observe that if the antisymmetric acquires a mass term and it is integrated out, the superpotential (4.41) corresponds to the one expected for the Aharony duality for $USp(2N_c)$ with six fundamentals and $N_c = 1$.
- Another check of the duality just stated consists of studying the real mass flow on the partition function. Proceeding as above we arrive at the relation:

$$\begin{aligned} \mathcal{Z}_{USp(2N_c)}(\vec{\mu};\tau) = \prod_{j=0}^{N_c-1} \prod_{1\le r<s\le 6} \Gamma_h(j\tau + \mu_r + \mu_s) \times \\ \times \Gamma_h\left(4\omega - (2N_c - 2 + j)\tau - \sum_{r=1}^{6} \mu_r\right) \mathcal{Z}_{USp(2N_c)}(\vec{\widetilde{\mu}};\tau) \end{aligned} \tag{4.42}$$

  with $\widetilde{\mu}_r = \omega - \frac{n-1}{2}\tau - \mu_r$, for $r = 1, \ldots, 6$. From this relation we can read the real mass $\hat{m}_{\text{ele}}^{(j)}$ of the j-th electric dressed monopole $\hat{m}_{\text{ele}}^{(j)} = 4\omega - (2N_c - 2 + j)\tau - \sum_{r=1}^{6} \mu_r$. The j-th magnetic dressed monopole has real mass $\hat{m}_{\text{mag}}^{(j)} = 4\omega - (2N_c - 2 + j)\tau - \sum_{r=1}^{6} \widetilde{\mu}_r$. It follows that $\hat{m}_{\text{ele}}^{(j)} + \hat{m}_{\text{mag}}^{(N_c-1-j)} = 2\omega$, corresponding to the constraint imposed by the superpotential (4.41).
- As a last check we can show that also in this case the duality reduces to the one studied in [**62**] if we add a superpotential $W = \operatorname{Tr} A^{k+1}$ to the antisymmetric field in the electric side and an analogous one on the dual side. In this case the dual theory is expected to have $USp(2(k(N_f - 1) - N_c))$ gauge symmetry, where $N_f = 3$ and $k = N_c$. The identity (4.42) coincides with the one derived in [**62**] after the actual value of τ is inserted in the balancing condition.

We can modify this duality to a self duality if N_c is even. On the field theory side this can be done by flipping half of the singlets T_j and $M^{(j)}$. On the partition function this is done equivalently by multiplying both sides of the identity (4.40) by

$$\prod_{j=0}^{\frac{N_c}{2}-1} \frac{1}{\prod_{1\le r<s\le 6} \Gamma_h(j\tau + \mu_r + \mu_s) \Gamma_h\left(4\omega - (2N_c - 2 + j)\tau - \sum_{r=1}^{6} \mu_r\right)} \tag{4.43}$$

and then using $\Gamma_h(x)\Gamma_h(2\omega - x) = 1$ together with the balancing condition. By proceeding in a similar fashion one can work out the explicit matter content and superpotentials of the other possible phases. We can also count the number of dual phases: there are $|W(D_6)|/|W(A_5)| = 32$ dual phases. This corresponds to the calculation performed in [**65**]. We will further comment on this number in Section 4.3.5, where we will explain its algebraic origin and study further real mass flows, constructing a full duality web.

4.3.4. Higgs flow and new $U(N_c)/USp(2N_c)$ dualities. The action of the Weyl group of D_6 on the real mass parameters can be also obtained by engineering a different flow on the original duality. This is essentially the same type of flow studied in [**65**], that led the authors to conjecture an $SU(2)/U(1)$ duality. Similarly here we will claim the existence of an $USp(2N_c)/U(N_c)$ duality. In this case the real mass flow has to be supplemented by an Higgs flow, that has indeed the effect of breaking the $USp(2N_c)$ gauge symmetry to $U(N_c)$. In order to study such a flow here we first order the masses μ_i as $(m_1, m_2, m_3, n_4, n_1, n_2, n_3, m_4)$ and then we consider the infinite shifts $m_r + M$ and $n_r - M$ for $r = 1, \ldots, 4$ and $M > 0$. The change in the labels of the masses is done only to match with the notations of [**116**].

The Higgs flow is triggered by assigning a vev to the scalar σ in the vector multiplet. This is equivalent to consider the shift $\sigma_i \to \sigma_i + M$. This shift breaks $USp(2N_c)$ with eight fundamentals and an antisymmetric into $U(N_c)$ with four flavors and one adjoint. The mass parameters are still constrained by the relation

$$2(N_c - 1)\tau + \sum_{r=1}^{4}(m_r + n_r) = 4\omega \tag{4.44}$$

signaling the presence of a superpotential

$$W = T_0 + \widetilde{T}_0 \tag{4.45}$$

This is the type of flow from symplectic to unitary gauge groups studied in [**65, 64**], and indeed the superpotential (4.45) is the generalization of the one obtained in the case without antisymmetric matter. As already observed in Section 4.2.1 this superpotential is generated because the UV completion of this model is a $USp(2N_c)$ gauge theory with an antisymmetric matter field. Such a matter content does not induce further zero modes allowing the generation of (4.45).

We then study the R charges and the global charges of the monopoles to infer the constraints on the real masses induced by the superpotential (4.45). For the R-charges we have

$$\Delta_{T_0} = \Delta_{\widetilde{T}_0} = 2(1 - \Delta_Q) + 2(1 - \Delta_{\widetilde{Q}}) - \Delta_X(N_c - 1) \tag{4.46}$$

Similar relations can be written down for the other global charges. By imposing $\Delta_{T_0} = \Delta_{\widetilde{T}_0} = 2$ (and $Q^{(i)}_{T_0} = Q^{(i)}_{\widetilde{T}_0} = 0$ for the other global symmetries) we observe that the constraint (4.44) is recovered . This analysis can be performed in all of the dual phases, leading to the same constraint (4.44). It has been shown in [**116**] that also in this case $W(D_6)$ is a symmetry of the three sphere partition function. We can also count the number of dual phases: there are $|W(D_6)|/|W(A_3)|^2 = \frac{2^5 6!}{(4!)^2} = 40$ dual phases; again this corresponds to the calculation performed in [**65**].

We now have two different theories in which $W(D_6)$ acts as a symmetry on the mass parameters. It is tempting to conjecture a duality among such theories. Such a duality can be proven at the level of the partition function.

4.3.4.1. *$USp(2N_c)/U(N_c)$ dualities from the partition function.* Let us apply the Higgs and real mass flow just discussed on the LHS of (4.27), performing the large M limit on the various terms in both sides of the duality enconded in (4.27). We can simplify the calculation as explained in [**64**] by using the symmetries of the integrand. This is necessary also to obtain the correct dimension of the Weyl group when flowing from $USp(2N_c)$ to $U(N_c)$.

Then we need to compare the divergent terms and only if they coincide we can read the equivalence between the finite parts. We will now just compute their phases by using (C.54). In the electric theory there are three sources of divergences, the contribution from the fundamental quarks, the contribution from the antisymmetric field and the contribution from the gauge sector. After using the balancing condition (4.29) the contribution $\sum_{i=1}^{N_c} \sigma_i$ vanishes. It signals the absence of an FI in the final result, as expected. This is consistent with the presence of the superpotential (4.45).

The phase is

$$\phi_e = -4MN_c(2\omega + \tau(N_c - 1)) - 2N_c\omega \sum_{i=1}^{4}(m_r - n_r) + N_c \sum_{r=1}^{4}(m_r^2 - n_r^2) \tag{4.47}$$

Next we need to study the phase corresponding to the divergent pre-factor in the partition function of the dual theory. In this case there is no Higgsing taking place and we have just two sources generating the large shift in M, the dual quarks and the mesons. The dual quarks generating the shift are the one parameterized by $\nu_4+\zeta$ and $\mu_4-\zeta$. Their contribution to the phase together with the contribution of the massive mesons cancels (4.47) after imposing the balancing condition (4.29).

Summarizing: all the phases cancel and one remains with the identity of **Theorem 5.6.15** in [**116**] with the same constraints on the masses and the same duality map. The identity is

$$\begin{aligned}\mathcal{Z}_{U(N_c)_0}(\vec{m};\vec{n};\tau) = \prod_{j=0}^{N_c-1}\prod_{r=1}^{3}\Gamma_h(j\tau + m_r + n_4, j\tau + m_4 + n_r)\times \\ \times \mathcal{Z}_{USp(2N_c)_0}(\vec{\mu}_\zeta;\tau)\end{aligned} \tag{4.48}$$

where

$$\mu_\zeta = \{m_1+\zeta, m_2+\zeta, m_3+\zeta, n_1-\zeta, n_2-\zeta, n_3-\zeta\} \tag{4.49}$$

There is a constraint

$$\begin{aligned} 2\zeta &= m_4 + n_1 + n_2 + n_3 - 2\omega + (N_c-1)\tau \\ &= 2\omega - (N_c-1)\tau - n_4 - m_1 - m_2 - m_3 \end{aligned} \tag{4.50}$$

This shows that one can obtain the duality between the $U(N_c)$ and the $USp(2N_c)$ theory discussed above flowing from the $W(E_7)$ invariant case on the partition function. The electric theory has superpotential (4.45), coming from the Higgs flow from $USp(2N_c)$ to $U(N_c)$.

In the $USp(2N_c)$ dual theory the monopole superpotential is set to zero because of the real mass flow and we are dealing with a set of unconstrained real masses. The correct constraints are imposed by the dual superpotential, involving the gauge singlets, identified with some of the mesons of the $U(n)$ theory. The superpotential of the $USp(2N_c)$ dual theory [4] corresponds to the one discussed in [**103**] and it is

$$W = \sum_{j=1}^{N_c} \mathsf{Tr}\left(M^{(j)}qqA^{j-1} + N^{(j)}ppA^{j-1}\right) \tag{4.51}$$

where M and N are gauge singlets, q and p are the quarks of the two $SU(3)$ global symmetries, and A is in the antisymmetric of $USp(2N_c)$. The real masses of the quarks q correspond to the first three entries in (4.49) while the ones of the quarks p to the last three entries. The duality then identifies the singlets $M^{(j)}$ and $N^{(j)}$ with the mesons $Q_r\widetilde{Q}_4X^{N_c-j}$ and $Q_4\widetilde{Q}_rX^{N_c-j}$ respectively, where $r=1,2,3$ and $j=1,\ldots,N_c$. This imposes the constraints on ζ discussed in (4.50), breaking a baryonic-like symmetry. It is interesting to observe that while the monopole superpotential of the electric theory (4.45) breaks the topological symmetry, the superpotential (4.51) of the dual theory, involving the mesonic operators, breaks the baryonic symmetry. This behavior is reminiscent of mirror symmetry and may play a crucial role in a deeper understanding of this duality.

4.3.5. The general scheme. In this section we study the general web of dualities and enhancement of the Weyl group symmetry that can be derived from the

[4] Observe that this superpotential has already appeared in the literature in [**121**]. The duality discussed here can indeed be reduced to the one conjectured in [**121**] through a real mass flow.

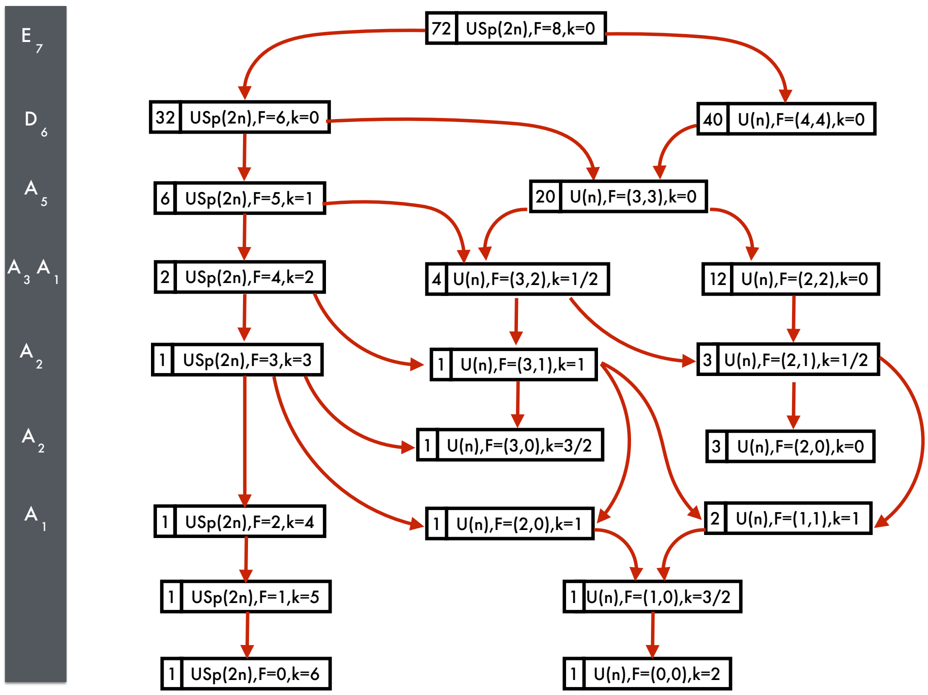


FIGURE 2. Weyl group symmetry enhancements obtained from $USp(2N_c)$ with eight fundamentals and antisymmetric defined on $\mathbb{R}^3 \times \mathbb{S}^1$. The rectangles represent the partition function of sets of 3d $\mathcal{N} = 2$ models, with gauge group $USp(2N_c)_{2k}$ or $U(N_c)_{-k}$, where k refers to the CS level. In the symplectic case there is one antisymmetric matter field A and F fundamentals, while in the unitary case there is one adjoint and a pair $F = (F_1, F_2)$ of fundamentals and anti-fundamentals.. The numbers appearing in a square on the left of each box correspond to the degeneration of integrands with the same gauge and charged matter content (up to spacetime parity and charge conjugation). They can in principle differ by the presence of hyperbolic gamma functions corresponding to extra singlets, i.e. that do not appear in the integrands. These degenerations are obtained by modding the enhanced Weyl group symmetry of the integrands, defined in the grey column on the left of the picture, by the manifest Weyl group symmetry, specified by the value(s) of F. The red arrows in the figure specify the RG flow connecting the UV to the IR models in the diagram.

reduction of 4d $\mathcal{N} = 1$ $USp(2N_c)$ with eight fundamentals and one antisymmetric. This web is obtained generalizing the real mass and Higgs flows that led to the dual models with the $W(D_6)$ enhancement. In order to do that let's discuss formally the flow from the case with $W(E_7)$ enhancement to the case with $W(D_6)$ enhancement.

In that case we reduced the manifest global symmetry by assigning real masses. The partition function had a reduced discrete symmetry group, corresponding to a

$W(D_6)$ subgroup of $W(E_7)$. There are different ways to construct such subgroups, corresponding to new dualities, transforming for example $USp(2N_c)$ into $U(N_c)$.

The discussion was made mathematically precise in [**116**] by showing that the symmetry of the new integrals is a subgroup of $W(E_7)$. This subgroup acts as a discrete symmetry on the IR partition functions and it is obtained by letting the variables μ of the E_7 case go to infinity in the direction of a vector in eight dimensional Euclidean space. This vector corresponds to $\vec{b_1} = (0,0,0,0,0,0,1,-1)$ in the first case and to $\vec{b_2} = (1,1,1,1,-1,-1,-1,-1)$ in the second case (up to normalizations). Observe that the vectors $\vec{b_i}$ are defined modulo permutations. These permutations explore the degenerations of the 32 $USp(2N_c)$ models with six fundamentals and of the 40 $U(N_c)$ models with four flavors. The vectors orthogonal to $\vec{b_i}$ form two different embeddings of the root system of D_6. The discrete symmetry of the IR partition functions corresponds to the reflections in these roots. It is possible to transform the system defined by $\vec{b_1}$ to the system defined by $\vec{b_2}$ by acting with the "broken" elements of $W(E_7)$: this is the mathematical interpretation of the $USp(2N_c)/U(N_c)$ duality that we discussed above at physical level. The classification scheme has been completely carried out in [**116**] and here we report the results, translating them in a physical language.

In order to classify the other possible Weyl group symmetry enhancements we need to iterate the flow, by further real mass and Higgs flows. In this way we can construct models with $USp(2N_c)_{2k}$ and $U(N_c)_k$ gauge groups, antisymmetric matter and a lower amount of fundamentals. These flows preserve the dualities and this translates in a possible enhancement of the Weyl group symmetry for some of the IR theories. For example starting from the $W(D_6)$ case one can flow to a case with enhancement of the Weyl group symmetry to $W(A_5)$. In terms of the gauge group and of the charged matter content we have two possibilities (counted up to parity transformations).

- $USp(2N_c)_1$ with five fundamentals and an antisymmetric. The Weyl group symmetry in this case enhances to $W(A_5)$. Accordingly, there are $\frac{|W(A_5)|}{|W(A_4)|} = 6$ dual models.
- $U(N_c)_0$ with three fundamental flavors and an adjoint. Also in this case the Weyl group symmetry enhances to $W(A_5)$ and there are $\frac{|W(A_5)|}{|W(A_2)|^2} = 20$ models.

In Figure 2 we reported the full structure of the RG flow, by iterating the procedure. At each level we specify the degeneration of the models, and we can observe that is it always consistent with the ratio of the orders of the enhanced Weyl group and that of the Weyl group of the naive global symmetry of the classical action.

From the figure we can extract some physical consequences.

Each row represents a set of models with the same three sphere partition function. This is the first step to claim a duality between these models. The equivalences among the various partition functions hold if the correct duality maps, the CS contact terms and the balancing conditions are specified. These constraints can be obtained by studying the flows from the UV models. We refer the interested reader to [**116**] where these results have been computed and listed.

Moreover each box in the figure is associated to a degeneration of integrals, and consequently of models. The manifest global symmetry in each of these models can enhance to a larger symmetry group. This can happen by consistently deforming the superpotentials, in the various dual phases, transforming the duality into a self duality. In this way the Weyl group symmetry enhancement of the integrands becomes a discrete symmetry enhancement of the full partition functions. This is a necessary condition for the global symmetry enhancement. More refined analyses (for example the analysis of the operators counted by the superconformal index, or the study of the Hilbert series) are then necessary in order to see if the Weyl group symmetry enhancement can be promoted to an enhancement of global symmetry to the full group/algebra.

We conclude with an observation about the $U(N_c)_0$ model with two fundamentals and an adjoint. In this case one can observe an enhancement of the Weyl group symmetry to $W(A_2)$. In the limit of $N_c = 1$ this model coincides with the model discussed in [**104, 105, 106**], where the global symmetry has been indeed conjectured to enhance to $SU(3)$. Here we observe [5], at the level of the partition function, that the model is also dual to a $U(1)_{-3/2}$ gauge theory with three fields at charge 1 (see also [**122**]). This last theory has a non-trivial monopole superpotential and it corresponds to a self dual case of the duality studied in [**64**] with $U(N_c)_{k/2}$ gauge groups, N_f fundamentals and $N_f - k$ anti-fundamentals.

Finally, we also observe that the enhancement of $U(N_c)$ with two pairs of fundamental flavors to $A_3 \times A_1$ corresponds to the $SO(6)$ enhancement discussed in [**106**].

4.4. Six fundamentals and confining theories

In this section we study the dimensional reduction of a $USp(2N_c)$ gauge theory with an antisymmetric and six fundamentals. This model can be obtained from the one with eight fundamentals by a superpotential mass deformation. It has been observed in [**102**] that in this case the superconformal index supports an enhancement of the global symmetry to E_6 [6]. Here we will not comment on the enhancement of the global symmetry for this case. We will rather study the consequences of this mass deformation in the dual model with the maximal amount of global symmetry, which becomes a confined WZ model. The 4d theory was studied in [**101**], and it was indeed shown that, in the IR, this theory confines without breaking the chiral symmetry. The confined degrees of freedom are expressed in terms of gauge invariant combination of the matter fields. They correspond to the gauge invariant operators

$$S_k \propto \mathrm{Tr}\, A^{k+2},\; M_k \propto Q A^k Q, \quad k = 0, 1, \ldots, N_c - 1 \tag{4.52}$$

There is also a superpotential interaction in the confined description, with a number of terms rapidly growing with N_c. For $N_c = 2, 3, 4$ these superpotentials have been given in [**101**].

This theory can be reduced to 3d, in both the confining and in the confined phase. In the first case one has a 3d effective $USp(2N_c)$ gauge theory with an antisymmetric and six fundamentals. This theory has also a KK monopole superpotential, $W = \eta Y$. The confined theory on the other hand has the same fields and interaction of the 4d parent. This is consistent with the results of [**124, 125**], where the reduction of $U(N_c)$ confining gauge theories was discussed in details.

The 3d duality obtained by this reduction has a well studied mathematical counterpart in the analysis of hyperbolic integrals. Indeed the matching between the partition function relating the two theories was already proven by [**126, 127, 128**]. The explicit relation is

$$\mathcal{Z}_{USp(2N_c)}(\vec{\mu};\tau) = \prod_{j=2}^{N_c} \Gamma_h(j\tau) \prod_{j=0}^{N_c-1} \prod_{1\le a<b\le 6} \Gamma_h(j\tau + \mu_a + \mu_b) \tag{4.53}$$

[5] Furthermore in [**122**] it has been shown that this model is dual to an $SU(3)_{5/2}$ gauge theory with a manifest $SU(3)$ global symmetry.

[6] See also [**123**] for related discussions.

with the relation among the mass parameters

$$2(N_c - 1)\tau + \sum_{a=1}^{6} \mu_a = 2\omega \tag{4.54}$$

This relation, that corresponds to impose the anomaly free constraints on the 4d R-current, signals the presence of the KK monopole superpotential in the confining theory. In the confined case it is consistent with the superpotentials of [**101**].

For concreteness let us analyze the dynamics of the $USp(4)$ case. Higher ranks can be studied in a similar fashion. First we need to obtain the R-charges at the fixed point. This calculation is performed by extremizing the three sphere partition function. We obtain $\Delta_\tau \simeq 0.184$ and $\Delta_{1,\dots,6} \simeq 0.272$. The R-charges of the singlets are $\Delta_{S_0} = 0.369$, $\Delta_{M_0} = 0.544$, and $\Delta_{M_1} = 0.728$. This shows that the singlet S_0 hit the bound of unitarity, $\Delta < \frac{1}{2}$. The WZ superpotential in this case is

$$W = S_0 M_0^3 + M_0 M_1^2 \tag{4.55}$$

and it implies that the first term is irrelevant and flows to zero in the IR. One can take care of the presence such an accidental symmetry also in the electric theory by adding a new singlet, β_2, with a superpotential interaction $\sim \beta_2 \,\mathsf{Tr} A^2$. A similar argument applies for higher ranks, and this modifies the electric superpotential, that eventually becomes

$$W = \sum_{j=2}^{N_c} \beta_j \,\mathsf{Tr} A^j + \eta Y \tag{4.56}$$

On the other hand the dual superpotential is cubic and corresponds to

$$W = \sum_{ijk} M_i M_j M_k \delta_{i+j+k,2(N_c-1)} \tag{4.57}$$

where M^3 corresponds to the contraction of the $SU(6)$ indices with an ε tensor. This procedure can be easily carried on from the point of view of the partition function. Indeed one can just divide both members of (4.53) by the contributions of the S_k singlets and then use the relation $\Gamma_h(x)\Gamma_h(2\omega - x) = 1$. The final equality is

$$\mathcal{Z}_{USp(2N_c)}(\vec{\mu};\tau) \prod_{j=2}^{N_c} \Gamma_h(2\omega - j\tau) = \prod_{j=0}^{N_c-1} \prod_{1\le a<b\le 6} \Gamma_h(j\tau + \mu_a + \mu_b) \tag{4.58}$$

We can summarize the duality as follows

- In the electric side we have a $USp(2N_c)$ gauge theory with six fundamentals, one antisymmetric and $N_c - 1$ singlets β_j. The superpotential is given in (4.56).
- In the magnetic side we have a WZ model, with singlets $M_j = QA^jQ$, $j = 0, \dots, N_c - 1$, interacting through the $SU(6)$ invariant superpotential (4.57).

The field content of the electric and of the magnetic theory is:

	$USp(2N_c)$	$SU(6)$	$U(1)_R$	$U(1)'$
Q	$2N_c$	6	Δ	$\frac{1-N_c}{3}$
A	$N_c(2N_c-1)$	1	$\frac{1-3\Delta}{N_c-1}$	1
β_j	1	1	$2-j\frac{1-3\Delta}{N_c-1}$	$-j$
M_j	1	15	$2\Delta+j\frac{1-3\Delta}{N_c-1}$	$\frac{2(1-N_c)}{3}+j$

It is possible to study a 3d conventional limit by real mass and Higgs flow. We consider these two cases separately in the following sections.

4.4.1. Real mass flow. In this case we assign a large mass to two fundamentals reducing the theory to $USp(2N_c)$ with four fundamentals and an antisymmetric. In such cases the meson M_{ij}, in the antisymmetric representation of $SU(6)$ splits into an antisymmetric meson of $SU(4)$ and a monopole. For example, if we consider the case $N_c=2$, the superpotential of the confined phase, before turning on the real mass, is

$$W=S_2\varepsilon_{ijklmn}M_0^{ij}M_0^{kl}M_0^{mn}+\frac{1}{3}\varepsilon_{ijklmn}M_0^{ij}M_1^{kl}M_1^{mn} \tag{4.59}$$

In the IR we have

$$W=S_2Y_0\varepsilon_{ijkl}M_0^{ij}M_0^{kl}+\frac{1}{3}Y_0\varepsilon_{ijkl}M_1^{ij}M_1^{kl}+\frac{1}{3}Y_1\varepsilon_{ijkl}M_0^{ij}M_1^{kl} \tag{4.60}$$

where the fields $Y_i=Y\,\mathsf{Tr}\,A^i$ corresponds to the dressed monopoles of the confining theory acting as singlets in the confined phase. It can be indeed checked that the charges of Y_i obtained from the superpotential (4.60) correspond to the ones obtained by using the quantum formula for the monopole charge (C.47) in the electric theory. This claim can be corroborated by studying the partition function. Indeed in this case we arrive at the identity

$$\begin{aligned}\mathcal{Z}_{USp(2N_c)}(\vec{\mu};\tau) &= \prod_{j=0}^{N_c-1}\Gamma_h\Big(2\omega-(2N_c-2-j)\tau-\sum_{a=1}^{4}\mu_a\Big)\\ &\times \prod_{j=2}^{N_c}\Gamma_h(j\tau)\prod_{1\le a<b\le 4}\Gamma_h(j\tau+\mu_a+\mu_b)\end{aligned} \tag{4.61}$$

where the mass parameters are unconstrained. The first term in the RHS of (4.61) corresponds to the contribution of the dressed monopoles Y_i, for $i=0,\dots,N_c-1$.

Again we can study the dynamics of the models and observe that the superpotential terms involving the singlet S_k are irrelevant. Proceeding as above we end up with the following duality:

- In the electric side we have a $USp(2N_c)$ gauge theory with four fundamentals, one antisymmetric and N_c-1 singlets β_j. The superpotential is given by

$$W=\sum_{j=2}^{N_c}\beta_j\,\mathsf{Tr}A^j \tag{4.62}$$

- In the magnetic side we have a WZ model, with singlets $M_j=QA^jQ$ and Y_i, with $j=0,\dots,N_c-1$, interacting through the superpotential

$$W=\sum_{i,j,k}Y_iM_jM_k\delta_{i+j+k,2(N_c-1)} \tag{4.63}$$

The field content of the electric and of the magnetic theory is:

	$USp(2N_c)$	$SU(4)$	$U(1)_R$	$U(1)'$	$U(1)_A$
Q	$2N_c$	4	Δ	0	1
A	$N_c(2N_c-1)$	1	Δ_A	1	0
β_j	1	1	$2-j\Delta_A$	$-j$	0
M_j	1	6	$2\Delta+j\Delta_A$	j	2
Y_j	1	1	$2-4\Delta+(j-2(N_c-1))\Delta_A$	$j-2(N_c-1)$	-4

4.4.2. Higgs flow. A second interesting 3d limit can be taken by shifting the scalars σ_i by a large real quantity s. If this flow is supported by a real mass flow $\mu_a \to m_a + s$ and $\mu_{a+3} \to n_a - s$ for $a = 1,2,3$, the final theory has $U(N_c)$ gauge group, three pairs of fundamental and antifundamental quarks and a monopole superpotential

$$W = T_0 + \widetilde{T}_0 \tag{4.64}$$

This theory is dual to a set of singlets interacting through a superpotential. For example if $N_c = 2$ we have

$$W = S_2 \varepsilon_{ijk}\varepsilon_{lmn} M_0^{il} M_0^{jm} M_0^{kn} + \frac{1}{3}\varepsilon_{ijk}\varepsilon_{lmn} M_0^{il} M_1^{jm} M_1^{kn} \tag{4.65}$$

where the mesons $M_j^{ab} = Q_a X^j \widetilde{Q}_b$ are in the bifundamental representation of the $SU(3)$ non-abelian flavor symmetry group. Here X represents the adjoint matter field and $j = 0,\dots,N_c-1$. In this case there are no monopoles of the electric theory acting as singlets in the magnetic dual. Indeed all the massless components of the original (antisymmetric) mesons become components of the new (bifundamental) meson in the theory with the reduced flavor. It is possible to reproduce this behavior on the partition function. Indeed this duality corresponds to an exact identity obtained in [**116**]. The identity is

$$\mathcal{Z}_{U(n)_0}(\vec{m};\vec{n};\tau) = \prod_{j=2}^{N_c}\Gamma_h(j\tau)\prod_{j=0}^{N_c-1}\prod_{a,b=1}^{4}\Gamma_h(j\tau+m_a+n_b) \tag{4.66}$$

where the parameters satisfy the relation

$$2(N_c-1)\tau + \sum_{a=1}^{3}(m_a+n_a) = 2\omega \tag{4.67}$$

Observe that the symmetries of the integrals in (4.61) and (4.66) are consistent with the D_5 enhancement [**129**]. It would be interesting to further study this enhancement along the lines of section 4.3.

The IR dynamics of this model can be obtained after decoupling the irrelevant superpotential terms in the dual phase. We end up with the following duality:

- In the electric side we have a $U(N_c)$ gauge theory with three pairs of fundamentals Q and anti-fundamentals $\widetilde{Q}$, one adjoint X and N_c-1 singlets β_j. The superpotential is given by

$$W = \sum_{j=2}^{N_c}\beta_j\,\mathrm{Tr}X^j + T_0 + \widetilde{T}_0 \tag{4.68}$$

- In the magnetic side we have a WZ model, with singlets $M_j = QX^j\widetilde{Q}$ interacting through a cubic $SU(3)_L \times SU(3)_R$ invariant superpotential.

The field content of the electric and of the magnetic theory is:

	$U(N_c)$	$SU(3)_R$	$SU(3)_L$	$U(1)_R$	$U(1)'$
Q	N_c	3	1	Δ	$\frac{1}{3}(N_c-1)$
$\widetilde{Q}$	$\overline{N_c}$	1	3	Δ	$\frac{1}{3}(N_c-1)$
X	N_c^2-1	1	1	$\frac{1-3\Delta}{N-1}$	1
β_j	1	1	1	$2-j\frac{1-3\Delta}{N-1}$	$-j$
M_j	1	3	3	$2\Delta+j\frac{1-3\Delta}{N-1}$	$\frac{2}{3}(N_c-1)+j$

4.5. Conclusions

In this chapter we have considered 4d theories with antisymmetric matter fields and we have performed their reduction to 3d, finding interesting new relations and dualities. We summarize here our main results.

In Section 4.2 we completed the picture in Figure 1, showing that the general aspects of the reduction of $USp(2N_c)/U(N_c)$ dualities with fundamental matter persist when adding anti-symmetric/adjoint matter with a power law superpotential. We have provided arguments from field theory and localization to confirm our claims.

In Section 4.3 we obtained a family of 3d effective $USp(2N_c)$ theory with eight fundamentals and one anti-symmetric matter field in which the action of the E_7 Weyl group is manifest on the real masses, leading to the generalization of the dualities of [**102**] to 3d. Furthermore we constructed a whole web of $USp(2N_c)/U(N_c)$ dualities, generalizing the $SU(2)/U(1)$ results of [**65**] to higher ranks and to lower symmetry groups, as summarized in Figure 2.

In Section 4.4 we reduced 4d confining $USp(2N_c)$ theories with six fundamentals and an antisymmetric matter field. Also in this case we obtained new relations for both $USp(2N_c)$ and for $U(N_c)$ theories.

CHAPTER 5

Discussion and Future Directions

In this thesis we have been concerned with various non-perturbative aspects of supersymmetric gauge theories in dimensions 2,3 and 4. More specifically we have studied twisted compactifications of 4d SCFTs by methods of conformal supergravity, 't Hooft anomaly computations and toric geometry as well as aspects of reduction of supersymmetric dualities. In this chapter we discuss some open questions and possible implications of our results for future lines of research.

In Chapter 2 we have provided a classification of 2d supersymmetric field theories obtained by partial topological twist and compactification along a Riemann surface of 4d theories with minimal or extended supersymmetry. We have shown that several degrees of supersymmetry can be obtained in 2d by appropriately tuning the parameters of R-symmetry and flavor symmetry twists.

A first generalization of the program of constructing 2d SCFTs from four dimensions consists of decorating the Riemann surfaces with codimension 2 defects, i.e., punctures. A possible way to study such a problem consists of exploiting the doubling trick discussed in [**130, 131**]. In this case one can gain information on the effective number of 2d chiral fermions by gluing a Riemann surface with a copy of itself (with the opposite orientation), thus obtaining a closed surface. This direction of research is particularly stimulating because, following the analogy with the six dimensional case, one could hope for an almost-lagrangian description of the 2d solutions in terms of more tractable blocks similar to the trinions of [**132**].

Another straightforward generalization is the case of partial topological twist in other dimensions, as was considered for instance in [**133**]. In particular it is interesting the case where the compactification manifold is larger than just a Riemann surface as this would allow for non-abelian holonomy and therefore a richer variety of topological twists. Especially the case of flows between even and odd dimensions proves to be a non-trivial generalization because in this case one cannot study the reduction by following the behavior of the anomaly polynomial as this quantity is not defined in odd dimensional field theories.

From the gravity side of the story one could study the problem from the AdS dual setup along the lines of [**18**], reconstructing the central charge from the gravitational perspective. The solution in this case should correspond to D1 branes probing a type IIB warped $\mathrm{AdS}_3 \times_\omega \mathcal{M}_7$ geometry, where $\mathcal{M}_7$ is a $U(1)$ bundle over a 6d Kahler-Einstein manifold. It should be possible to formulate the central charge and its extremization in terms of the volumes of $\mathcal{M}_7$, in the spirit of [**35**].

It would also be interesting to study models arising from the compactification of 6d theories, such as class S theories [**132**] or theories with lower supersymmetry, as the S_k models [**134**] or the models of [**135**]. Finally, the analysis of $\mathcal{N} = 3$ theories is an interesting novel development of the subject, and it would be especially fruitful to consider given that the central charges a and c can be computed as in [**78**]. The analysis of the gravitational dual mechanism of the topological twist in this case can be performed by studying the consistent truncation of [**84**] in gauged supergravity. The twisted compactification of those theories would require to further truncate the $\mathcal{N} = 6$ theory to a $\mathcal{N} = 2$ subsector once the fluxes are turned on. In such a setup it might be possible to compare the field theory and the supergravity results.

In the analysis of Chapter 3 we have shown that the 2d central charge, expressed in terms of the mixing parameters, can be reformulated in the language of the toric geometry underlining the moduli space of the 4d theory. Nevertheless we did not give a general discussion on the extremization of this function. This point certainly deserves a separate and deeper analysis. Indeed, the existence of an extremum is not guaranteed, as discussed in [**4**]. The main obstructions are due to the absence of a normalizable vacuum of the 2d CFT and to the presence of accidental symmetries at the IR fixed point. The study of this problem would be simplified by the knowledge of the spectrum and the interactions of the 2d models. Progresses in these directions have been made in [**73, 13, 17**]. On the geometric side, the dual of c-extremization was recently studied in [**99**]. It would also be interesting to see if some of the tools developed in 4d (e.g. the zonotope discussed in [**40**]) can be fruitfully applied to the analysis of the extremization properties of the 2d central charge.

We would also like to mention that infinite families of 2d SCFTs have been recently obtained by exploiting the role of the toric geometry [**136, 137**]. These so called brane brick models are expected to describe the worldvolume theory of stacks of D1 branes probing the tip of toric CY_4 cones in type IIB. It has been shown that in such cases the techniques of toric geometry can be used to obtain the elliptic genus [**138**]. It would be interesting to further explore the role of toric geometry in these 2d SCFTs and look for possible connections, if any, with our results.

In Chapter 4 we reduced 4d $USp(2N_c)$ theories with antisymmetric matter fields to 3d. There are some general lessons that we can extract from our analysis there. One of them regards the structure of the monopole superpotentials in the presence of unitary gauge groups and adjoint matter. If such a theory is considered as UV complete then the zero modes counting does not allow the generation of any monopole superpotential, because each adjoint field carries two additional zero modes. Nevertheless, as we widely discussed there, we claim the existence of linear monopole/anti-monopole superpotentials. Such terms correspond to monopoles of the UV completion, i.e., $USp(2N_c)$ models with anti-symmetric matter, where the zero mode counting allows the generation of the monopole superpotentials. It would be interesting to further investigate this phenomenon, finding other examples of its possible application and discussing its relation to the index theorems that in general prevent the generation of the superpotentials that we have constructed. The problem of finding an UV completion should be analyzed together with the study of accidental symmetries. This requires the minimization of the partition function [**139**] similarly to the analysis of [**140, 141, 142**].

Another result that requires further investigation is the generalization of the flows considered in Figure 1 to the case of Brodie duality [**143**], involving two adjoints and a non-trivial superpotential. Recently, this duality has been reduced to 3d in [**144**]. The main difference that emerged from the analysis is the presence of superpotentials involving monopoles with charge 2. It would be interesting to study how this behavior modifies our analysis and if a structure similar to the one in Figure 1 does appear for this case as well. Moreover, one might also consider the possibility to investigate the existence of 3d dualities between orthogonal and unitary gauge theories as well as whether one can find 4d analogs of these dualities.

We conclude with a comment on global symmetry enhancements. In this thesis we have observed the fact that, thanks to localization, it is possible to prove that the action of the Weyl group is larger that the one expected from the classical global symmetry. This is in itself an indication of the possibility of a global symmetry enhancement but further investigations are usually necessary for a complete understanding. One is nevertheless led to think that this mechanism can be used as a general guideline for recognizing the existence of this sort of enhancements.

APPENDIX A

Spinors and Supersymmetry in 2,3 and 4 dimensions

In this appendix we review some general notions about Clifford algebras, spinors and supersymmetry.

A.1. Clifford algebras and spinors

Let us start by defining the notion of real Clifford algebra associated to an inner product space. Let V be a real vector space of dimension d endowed with a non-degenerate quadratic form $Q(\cdot) : V \to \mathbb{R}$. An inner product $\langle \cdot, \cdot \rangle$ can be defined via the polarization formula for Q:

$$\langle x, y \rangle = \frac{1}{2} \left(Q(x+y) - Q(x) - Q(y) \right) \tag{A.1}$$

for $x, y \in V$. We allow for indefinite signature (s,t) where s is the number of positive eigenvalues and t is the number of negative ones, so that $d = s + t$. The tensor algebra of V is defined as the $\mathbb{N}$-graded vector space:

$$\mathbb{T}(V) = \bigoplus_{n=0}^{\infty} V^{\otimes n} = \mathbb{R} \oplus V \oplus (V \otimes V) \oplus \dots \tag{A.2}$$

where the grading is given by the number of factors in the products. The structure of graded associative algebra is given by the product $\otimes$ [1].

The real Clifford algebra over (V, Q) is defined as the quotient:

$$Cl(s,t) = \mathbb{T}(V)/I_Q \tag{A.3}$$

by the proper ideal I_Q generated by elements of the form:

$$x \otimes x + Q(x)1 \tag{A.4}$$

for $x \in V$. The $\mathbb{N}$-grading of $\mathbb{T}(V)$ is not preserved by the quotient because the ideal is not generated by homogeneous elements, however, since I_Q is homogeneous mod 2, there is a $\mathbb{Z}_2$-grading on $Cl(s,t)$ compatible with the product of the algebra. As a vector space, $Cl(s,t)$ is isomorphic to the exterior algebra $\Lambda^\bullet V$ and we can decompose it according to the degree mod 2 as:

$$Cl(s,t) = Cl(s,t)_0 \oplus Cl(s,t)_1 \tag{A.5}$$

where we call $Cl(s,t)_0$ the even subalgebra.

We denote the Clifford product of two elements x, y in V as:

$$xy = x \cdot y + x \wedge y \tag{A.6}$$

where $x \cdot y$ and $x \wedge y$ are the symmetric and antisymmetric parts, respectively. One can check that by definition, $x \cdot y = -\langle x, y \rangle 1$ is a scalar while $x \wedge y$ is a bivector, i.e., an element of degree 2 in the algebra.

[1] Because we are working with real vector spaces, the tensor products are also taken w.r.t. the base field $\mathbb{R}$.

Given vectors $a, x \in V$ we can compute the reflection of x in the hyperplane orthogonal to a by use of the Clifford product as:

$$\begin{aligned} P_a(x) &= x - 2\frac{a \cdot x}{a \cdot a} a \\ &= x - \frac{ax + xa}{a \cdot a} a \\ &= x - \frac{axa}{a \cdot a} - x \\ &= -a^{-1} x a \end{aligned} \tag{A.7}$$

By Dieudonné's theorem all rotations in V are generated by compositions of an even number of reflections, therefore given two vectors $a, b \in V$ the rotation of x in the plane generated by a and b is:

$$R_{ab}(x) = P_b(P_a(x)) = (ab)^{-1} x (ab) \tag{A.8}$$

If we write $B = \frac{a \wedge b}{|a||b|\sin(\theta)}$ where θ is the angle between vectors a and b, then $B^2 = -1$ and we can use the power series formula for the exponential to write:

$$R_{ab}(x) = \mathrm{e}^{-\theta B}\, x\, \mathrm{e}^{\theta B} \tag{A.9}$$

One can then prove that this provides a linear representation of the double cover of the rotation group, i.e., the spin group $Spin(s,t) \to SO(s,t)$. Moreover, since its generators are the bivectors[2], there is an isomorphism between the even subalgebra and the universal enveloping algebra of the spin group Lie algebra. This implies that all irreducible representations of the spin group are also irreducible representations of $Cl(s,t)_0$. We then define *spinors* to be the smallest non-trivial irreducible representation(s) of this algebra.

In order to classify all possible representations of $Spin(s,t)$ we first need to classify all the Clifford algebras $Cl(s,t)$ and their even subalgebras. This can be done by noting the following recursion relations[3]:

$$Cl(s,t) = Cl(1,1) \otimes_{\mathbb{R}} Cl(s-1, t-1) \tag{A.10}$$

$$Cl(s,0) = Cl(2,0) \otimes_{\mathbb{R}} Cl(0, s-2) \tag{A.11}$$

$$Cl(0,t) = Cl(0,2) \otimes_{\mathbb{R}} Cl(t-2, 0) \tag{A.12}$$

together with the initial isomorphisms:

$$\begin{array}{ll} Cl(0,1) \cong \mathbb{R}(1) \oplus \mathbb{R}(1) & Cl(1,0) \cong \mathbb{C}(1) \\ Cl(1,1) \cong Cl(0,2) \cong \mathbb{R}(2) & Cl(2,0) \cong \mathbb{H}(1) \end{array} \tag{A.13}$$

where by $\mathbb{A}(k)$ we mean the algebra of k-by-k matrices with entries in the division algebra $\mathbb{A} = \mathbb{R}, \mathbb{C}, \mathbb{H}$. The even subalgebras can also be obtained by using the isomorphisms:

$$Cl(s,t)_0 \cong Cl(s-1,t) \quad \text{and} \quad Cl(s,t)_0 \cong Cl(t,s)_0 \tag{A.14}$$

All the possible cases are summarized in Table 1.

Finally we introduce the notion of pseudoscalar element of a Clifford algebra. Suppose we pick an orthonormal basis of V given by vectors γ_μ such that the inner product $\langle \cdot, \cdot \rangle$ becomes diagonal in that basis:

$$\gamma_\mu \cdot \gamma_\nu = -\langle \gamma_\mu, \gamma_\nu \rangle 1 = -\eta_{\mu\nu} 1 \tag{A.15}$$

[2]The Lie bracket of the Lie algebra is given by the commutator of bivectors in the Clifford algebra.

[3]See [**145, 146**] for proofs of these statements.

TABLE 1. Clifford algebras and spinors. In even dimensions there are two inequivalent choices of spinor representations, one for each eigenvalue of the pseudoscalar.

$s-t \mod 8$	$Cl(s,t)_0$	spinors	k
0	$\mathbb{R}(k)\oplus\mathbb{R}(k)$	$\mathbb{R}^k_+$ or $\mathbb{R}^k_-$	$2^{\frac{d-2}{2}}$
1, 7	$\mathbb{R}(k)$	$\mathbb{R}^k$	$2^{\frac{d-1}{2}}$
2, 6	$\mathbb{C}(k)$	$\mathbb{C}^k$ or $\bar{\mathbb{C}}^k$	$2^{\frac{d-2}{2}}$
3, 5	$\mathbb{H}(k)$	$\mathbb{H}^k$	$2^{\frac{d-3}{2}}$
4	$\mathbb{H}(k)\oplus\mathbb{H}(k)$	$\mathbb{H}^k_+$ or $\mathbb{H}^k_-$	$2^{\frac{d-4}{2}}$

The *pseudoscalar* of the algebra is defined as the (ordered) product of all the basis vectors:

$$\gamma_* = \gamma_1\gamma_2\dots\gamma_d \tag{A.16}$$

and up to a scalar coefficient it is the only element of degree d in the algebra. By definition γ_* satisfies:

$$\gamma_*^2 = (-1)^{\frac{d(d-1)}{2}+s} \tag{A.17}$$

In odd dimensions γ_* is in the center of $Cl(s,t)$, hence it acts as multiplication by a scalar in all irreducible representations.

In even dimensions γ_* anti-commutes with the generators of degree 1, hence it is in the center of $Cl(s,t)_0$ and it acts as multiplication by scalar on the spinor representations. For $s-t = 0 \mod 4$, $\gamma_*^2 = 1$ and the two spinor representations correspond to the eigenvalues ± 1. For $s-t = 2 \mod 4$, $\gamma_*^2 = -1$ and the two spinor representations correspond to the eigenvalues $\pm\mathrm{i}$ (they are complex conjugate representations and γ_* is the complex structure).

By defining $\gamma_{d+1} = \mathrm{i}^{\frac{d(d-1)}{2}+s}\gamma_*$ (so that $\gamma_{d+1}^2 = +1$), we can call positive chirality spinors those that are in the representation of eigenvalue $+1$ and negative chirality spinors those in the representation of eigenvalue -1.

Now that we have defined spinor representations we can also define spinor fields as follows. Consider a spacetime manifold X with pseudo-Riemannian signature (s,t) and apply the previous construction to $V = T_pX$ for every point p in an open subset $U \subset X$. This defines a vector bundle $\mathcal{S} \to U$ in which each fiber is a spinor representation of the Lorentz group. $\mathcal{S}$ is called the *spinor bundle* and it can be considered as the vector bundle associated to the lift of the structure group of the tangent bundle from *SO* to its double cover *Spin*. In general there might be topological obstructions to defining $\mathcal{S}$ globally on X, in fact one can show that this can only be done if the second Stiefel-Whitney characteristic class of TX vanishes. When this is the case, X is said to be a *spin manifold* and the sections of $\mathcal{S}$ are the spinor fields.

A.2. Supersymmetry in low dimensions

A superalgebra is a $\mathbb{Z}_2$-graded vector space $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$ together with a graded Lie bracket:

$$[x,y] = -(-1)^{\deg x \deg y}[y,x] \tag{A.18}$$

which satisfies the super-Jacobi identity:

$$(-1)^{\deg x \deg z}[x,[y,z]] + (-1)^{\deg y \deg x}[y,[z,x]] + (-1)^{\deg z \deg y}[z,[x,y]] = 0 \tag{A.19}$$

Elements of $\mathfrak{g}_0$ are called the even generators and they form a bosonic subalgebra, while elements of $\mathfrak{g}_1$ are called the odd or fermionic generators and they form an irreducible module for $\mathfrak{g}_0$, i.e., $\mathfrak{g}_0$ acts on $\mathfrak{g}_1$ by derivations.

In applications to physics one usually requires that the even part contains the Poincaré algebra and that the odd part is a spinor module for it, so that the odd generators transform in the spinor representation of the Lorentz group. These generators are then called the supercharges Q_α and they anti-commute to give the generators of the translations P_μ:

$$[Q_\alpha, Q_\beta] = \Gamma^\mu_{\alpha\beta} P_\mu + \dots \tag{A.20}$$

where $\Gamma^\mu_{\alpha\beta}$ are structure constant of the algebra. Observe that this is only possible if the symmetric product of two spinors representations decomposes into irreducible representations of the Lorentz group, one of which is the vector representation. For $(s,t) = (d-1,1)$ this is always the case, but other signatures also have this property. Generators for all the other representations that appear in the decomposition can be introduced and correspond to the so called central charges Z, which we are not going to discuss.

If the odd part $\mathfrak{g}_1$ contains multiple copies of the basic spinor representation, then one says that the algebra is an extended superalgebra. For example, if there are $\mathcal{N}$ copies of the odd generators, then the supercharges Q^I_α carry an extra index $I = 1,\dots,\mathcal{N}$ labeling each copy. In this case we can think of them as being in the tensor product of two representations, the spinor one and a $\mathcal{N}$-dimensional representation of the so called R-symmetry algebra. By definition this is a subalgebra of $Der(\mathfrak{g}_1)$ which commutes with $\mathfrak{g}_0$. In general it only acts as an outer derivation but it can be made part of the actual superalgebra by taking the semidirect product of the two. Let us remark here that even when $\mathcal{N} = 1$, the R-symmetry might be non-trivial; this is the case for example when the spinor representation is a vector space over some field (division algebra) $\mathbb{A}$ such that $Der(\mathbb{A})$ is non-empty, e.g., for $\mathbb{A} = \mathbb{C}$ there is always a $U(1)$ R-symmetry while for $\mathbb{A} = \mathbb{H}$ there is a $USp(2)$ R-symmetry.

Finally, if $\mathfrak{g}_0$ contains the algebra of the conformal group $SO(s+1,t+1)$ and $\mathfrak{g}_1$ transforms as a spinor representation under it, then the superalgebra is a *superconformal algebra*. By restricting to the Lorentz subgroup $SO(s,t)$, the supercharges decompose as the direct sum of the usual supercharges Q and the conformal supercharges S [4]. While superalgebras exist for any value of d, superconformal algebras only exist for $d \leq 6$. They are also special in that the R-symmetry is always a an inner derivation of the algebra, which means that the R-symmetry generators appear in the superbracket of two supercharges.

Similarly to how Einstein gravity is the Cartan geometry of the inclusion of the spin group in the Poincaré group, i.e., the gauge theory for the Lorentz and translation symmetries of a pseudo-Riemannian manifold and its tangent bundle, *supergravity* (conformal supergravity) is the super-Cartan geometry for the inclusion of the spin group in a supersymmetric extension of the Poincaré group (conformal group).

For a detailed introduction to supergravity and conformal supergravity we refer the reader to the standard references [**71, 147, 148**]. For a treatment of the subject in terms of Cartan geometry see [**149**].

[4]One can show this by using the relations in (A.10) and (A.13).

A.2.1. Two dimensions. For $d = 2$ and signature $(s,t) = (1,1)$ the even Clifford algebra $Cl(1,1)_0$ is $\mathbb{R} \oplus \mathbb{R}$, therefore the spinors are real vectors of length 1. Moreover we see that they are chiral and the chirality is given by the eigenvalue of $\gamma_3 = -\gamma_* = -\gamma_0\gamma_1$. The full Clifford algebra is $Cl(1,1) \cong \mathbb{R}(2)$ and we can find a basis of generators given by the real matrices:

$$\gamma_0 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \quad \gamma_1 = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \tag{A.21}$$

The double cover of $SO(1,1)$ is $Spin(1,1) \cong GL(1,\mathbb{R})$.

Because spinors are chiral and real, supersymmetry in two dimensions can be generated by an arbitrary number of supercharges of both chiralities. One then denotes the amount of supersymmetry as $\mathcal{N} = (p,q)$ and the R-symmetry can be as large as $O(p) \times O(q)$.

A.2.2. Three dimensions. For $d = 3$ and signature $(s,t) = (2,1)$ the even Clifford algebra $Cl(2,1)_0$ is $\mathbb{R}(2)$, therefore the spinors are real vectors of length 2 with no chirality. The full Clifford algebra is $Cl(2,1) \cong \mathbb{C}(2)$ and we can find a basis of generators given by the complex matrices:

$$\gamma_0 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \quad \gamma_1 = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \quad \gamma_2 = \begin{bmatrix} \mathrm{i} & 0 \\ 0 & -\mathrm{i} \end{bmatrix} \tag{A.22}$$

The double cover of $SO(2,1)$ is $Spin(2,1) \cong SL(2,\mathbb{R})$.

Because spinors are real but not chiral, supersymmetry in three dimensions can be generated by an arbitrary (even) number of real supercharges. One then denotes the amount of supersymmetry as $\mathcal{N} = n$ and the R-symmetry can be as large as $O(n)$.

Under dimensional reduction to $d = 2$, a 3d spinor decomposes as the direct sum of two 2d spinors of opposite chirality:

$$\begin{array}{ccc} 3d & & 2d \\ \mathbb{R}^2 & \to & \mathbb{R}_+ \oplus \mathbb{R}_- \end{array} \tag{A.23}$$

A.2.3. Four dimensions. For $d = 4$ and signature $(s,t) = (3,1)$ the even Clifford algebra $Cl(3,1)_0$ is $\mathbb{C}(2)$, therefore the spinors are complex vectors of length 2. Moreover we see that they are chiral and the chirality is given by the eigenvalue of $\gamma_5 = \mathrm{i}\gamma_* = \mathrm{i}\gamma_0\gamma_1\gamma_2\gamma_3$. The full Clifford algebra is $Cl(3,1) \cong \mathbb{H}(2)$. By taking the tensor product with $\mathbb{C}$ we can represent the generators by the complex 4-by-4 matrices[5]:

$$\begin{gathered} \gamma_0 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \qquad \gamma_1 = \begin{bmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix} \\ \gamma_2 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{bmatrix} \qquad \gamma_3 = \begin{bmatrix} 0 & 0 & 0 & -\mathrm{i} \\ 0 & 0 & \mathrm{i} & 0 \\ 0 & \mathrm{i} & 0 & 0 \\ -\mathrm{i} & 0 & 0 & 0 \end{bmatrix} \end{gathered} \tag{A.24}$$

The double cover of $SO(3,1)$ is $Spin(3,1) \cong SL(2,\mathbb{C})$.

[5]Here we use the isomorphism of algebras $\mathbb{H}(k) \otimes_{\mathbb{R}} \mathbb{C} \cong \mathbb{C}(2k)$.

Because spinors are chiral and complex, in four dimensions equation (A.20) makes sense only if there is an equal number of positive and negative chirality supercharges, which must also be complex conjugate of each other. Therefore supersymmetry is non-chiral and it is denoted as $\mathcal{N} = n$ for a total of $4n$ real supercharges. The R-symmetry in this case can be as large as $U(n)$.

Under dimensional reduction to $d = 3$, a 4d chiral spinor decomposes as the direct sum of two 3d spinors:

$$\begin{array}{ccccc} 4d & & 3d & & 2d \\ \mathbb{C}^2 & \to & \mathbb{R}^2 \oplus \mathrm{i}\mathbb{R}^2 & \to & (\mathbb{R}_+ \oplus \mathbb{R}_-) \oplus \mathrm{i}(\mathbb{R}_+ \oplus \mathbb{R}_-) \end{array} \tag{A.25}$$

By further reducing to 2d one can re-assemble the 2d spinors as:

$$\begin{array}{c} \mathbb{R}_+ \oplus \mathrm{i}\mathbb{R}_+ \cong \mathbb{R}_+ \otimes_{\mathbb{R}} \mathbb{C} \\ \mathbb{R}_- \oplus \mathrm{i}\mathbb{R}_- \cong \mathbb{R}_- \otimes_{\mathbb{R}} \bar{\mathbb{C}} \end{array} \tag{A.26}$$

so that they can be seen as the tensor product of spinors of $Spin(1,1)$ and spinors of $Spin(2) \cong U(1)$.

APPENDIX B

The Anomaly Polynomial

In this appendix we review the general formalism of the anomaly polynomial that has been used in section 2.7. We refer the reader to the seminal paper [**150**] for the original construction. For a detailed review see also [**151**] and references therein.

B.1. Gauge anomalies

We consider a gauge theory on a $2l$-dimensional manifold X which for simplicity we assume to be a sphere $\mathbb{S}^{2l}$. A gauge theory with gauge group G is specified by a principal G-bundle on X with connection A and some associated vector bundle whose sections are to be regarded as the matter fields of the theory. In what follows we will consider fermionic matter fields ψ transforming as spinors under the Lorentz group and as vectors in some irreducible representation $\mathcal{R}$ under the gauge group G. If we define $\mathcal{S}_\pm$ the chiral spinor bundles over X and E the vector bundle associated to the representation $\mathcal{R}$ then the fields ψ are sections of the tensor product of the two:

$$\psi_\pm \in \Gamma(\mathcal{S}_\pm \otimes E) \tag{B.1}$$

Let us choose for definiteness a left-handed Weyl fermion ψ_+ whose partition function in the background of the gauge connection A is given by:

$$\mathcal{Z}(A) = \mathrm{e}^{-W(A)} = \int \mathrm{d}\psi \mathrm{d}\bar{\psi}\ \exp\left(-\int_X \mathrm{d}x\ \bar{\psi}\mathrm{i}\nabla_+\psi\right) = \det(\mathrm{i}\nabla_+) \tag{B.2}$$

where $\nabla_+ = \gamma^\mu(\partial_\mu + \omega_\mu + A_\mu)$ is the covariant derivative on $\mathcal{S}_+ \otimes E$. The partition function can therefore be computed as the regularized product of the eigenvalues of the operator ∇_+. However this operator maps positive chirality spinors into negative chirality spinors hence it does not make sense to ask what are its eigenvalues. In order to compute its determinant we introduce an auxiliary fermion of opposite chirality ψ_- not coupled to the gauge background and consider the Dirac operator $\hat{D}$ acting on the Dirac fermion ψ:

$$\mathrm{i}\hat{D} = \begin{bmatrix} 0 & \mathrm{i}\partial_- \\ \mathrm{i}\nabla_+ & 0 \end{bmatrix} \qquad \psi = \begin{bmatrix} \psi_+ \\ \psi_- \end{bmatrix} \tag{B.3}$$

The new Dirac operator can be diagonalized however one finds that its spectrum is not gauge invariant and the determinant is invariant only in modulus:

$$|\det(\mathrm{i}\hat{D})|^2 = \det(\mathrm{i}\hat{D}(\mathrm{i}\hat{D})^\dagger) = \det(\mathrm{i}\partial_-\mathrm{i}\partial_+)\ \det(\mathrm{i}\nabla_+\mathrm{i}\nabla_-) \tag{B.4}$$

which means that if we perform a gauge transformation g on A the partition function picks up a non-trivial anomalous phase $\delta_g W(A) = \mathrm{i}w(g, A)$. If this happens we say that the theory has a quantum gauge anomaly which manifests itself as a failure in the conservation of the corresponding gauge current:

$$\mathcal{D}_\mu \langle J^\mu \rangle = \mathcal{D}_\mu \frac{\delta W}{\delta A_\mu} \neq 0 \tag{B.5}$$

We give now a topological description of the anomalous phase of the determinant in (B.2). Let us define $\mathcal{A}$ the space of all possible gauge connections A_μ and

$\mathcal{G}$ the group of all (finite) gauge transformations of the gauge bundle E. Then $\mathcal{A}$ is an infinite dimensional affine space on which the infinite dimensional group $\mathcal{G}$ acts as:

$$g \cdot A \equiv A^g = g^{-1}(A + \mathrm{d})g \qquad \text{for } g \in \mathcal{G},\ A \in \mathcal{A} \tag{B.6}$$

and Feynman's path integral prescription says that we have to integrate over $\mathcal{A}$ the partition function of the system in order to make the gauge field dynamical and properly quantize the gauge theory. If the fermionic partition function $\mathcal{Z}(A)$ is constant along the orbits of $\mathcal{G}$ then one should be able to split the integral over $\mathcal{A}$ first by integrating along the orbit and then over the quotient $\mathcal{A}/\mathcal{G}$:

$$\int_{\mathcal{A}} \mathcal{Z}(A) = \int_{\mathcal{A}/\mathcal{G}} \int_{\mathcal{G}} \mathcal{Z}(A) = \mathrm{vol}(\mathcal{G}) \int_{\mathcal{A}/\mathcal{G}} \mathcal{Z}([A]) \tag{B.7}$$

where $[A]$ is the gauge equivalence class of A in $\mathcal{A}/\mathcal{G}$ and we assumed that the action of the group of gauge transformations is free, i.e., there are no reducible connections so that the orbits are all isomorphic to $\mathcal{G}$. We remark that $\mathcal{A}$ is a principal $\mathcal{G}$-bundle over the space of orbits and that the function $\mathcal{Z}$ being constant along the fiber means that it can be "pushed down" to a function $\mathcal{Z}_\downarrow$ on the base of the fibration:

$$\begin{array}{ccccc} \mathcal{G} & \longrightarrow & \mathcal{A} & \xrightarrow{\ \mathcal{Z}\ } & \mathbb{C} \\ & & \downarrow & \nearrow_{\mathcal{Z}_\downarrow} & \\ & & \mathcal{A}/\mathcal{G} & & \end{array} \tag{B.8}$$

As we have seen before however, the phase of $\mathcal{Z}$ has non-trivial variation $w(g, A)$ as we move along the fiber hence it will not define a function on the quotient but rather a section of a complex line bundle $\mathcal{L}$ over it, the determinant line bundle. This is the rank-1 vector bundle on $\mathcal{A}/\mathcal{G}$ associated to the principal bundle $\mathcal{A}$ via the cocycle $w : \mathcal{A} \times \mathcal{G} \to U(1)$. The cocycle condition on w reads:

$$w(hg, A) = w(h, g \cdot A) + w(g, A) \tag{B.9}$$

and it expresses the compability with the group action of $\mathcal{G}$ [1].

When the function $\mathcal{Z}$ is constant along the orbits the cocycle w is trivial as well as the line bundle $\mathcal{L}$ so that its sections are just complex functions over the base and it make sense to compute their integral. When the cocycle is non-trivial one cannot find covariantly constant global sections of $\mathcal{L}$ and it does not make sense to integrate over the base. We can therefore say that the gauge anomaly is precisely encoded by the non-triviality of the determinant bundle $\mathcal{L}$, i.e., the non-vanishing of the cocycle w whose homotopy class in $H^2(\mathcal{A}/\mathcal{G}, \mathbb{Z})$ corresponds to the first Chern class of $\mathcal{L}$. The precise form of the cocycle can be computed via the index of a Dirac operator as we will review in the next section.

B.2. The Atiyah-Singer index theorem

In order to compute w we restrict to an embedded 2-sphere inside of $\mathcal{A}/\mathcal{G}$ and consider the pullback of $\mathcal{L}$ to this $\mathbb{S}^2$. By doing so the pullback of w, i.e., the first Chern class, becomes a class in $H^2(\mathbb{S}^2, \mathbb{Z}) \cong \mathbb{Z}$ hence we can canonically identify it

[1] In more abstract terms w is a functor from the action groupoid $\mathcal{A}//\mathcal{G}$ to the delooping of $U(1)$ obtained by composition of the cocycle μ classifying the principal fibration and the complex character χ specifying the representation of $\mathcal{G}$ on the fibers of $\mathcal{L}$:

$$\mathcal{A}//\mathcal{G} \xrightarrow{\ \mu\ } \mathbf{B}\mathcal{G} \xrightarrow{\ \chi\ } \mathbf{B}U(1), \qquad w = \chi \circ \mu$$

with an integer number.

If we locally trivialize the bundle by choosing two patches being the disks covering the two hemispheres and intersecting on the equator, the first Chern class of the bundle is given by the winding number of the transition function $\lambda : \mathbb{S}^1 \to U(1)$ that identifies the fibers at the intersection of the patches. By definition of the pullback bundle on $\mathbb{S}^2$, the map λ is given by:

$$\lambda(\theta) = \exp(\mathrm{i}w(g(\theta), A)) \tag{B.10}$$

for $g : \mathbb{S}^1 \to \mathcal{G}$ the transition function of the pullback of the principal bundle $\mathcal{A} \to \mathcal{A}/\mathcal{G}$, so that:

$$c_1([\mathbb{S}^2]) = \frac{1}{2\pi\mathrm{i}} \int_{\mathbb{S}^1} \lambda^{-1} \mathrm{d}\lambda = \frac{1}{2\pi} \int_0^{2\pi} \frac{\partial w(g(\theta), A)}{\partial \theta} \mathrm{d}\theta \tag{B.11}$$

In [**150**] it was shown that the winding number of λ can be computed as the index of an elliptic Dirac operator $\mathrm{i}\nabla_{2l+2}$ over $\mathbb{S}^2 \times \mathbb{S}^{2l}$:

$$c_1([\mathbb{S}^2]) = \mathrm{ind}(\mathrm{i}\nabla_{2l+2}) = \int_{\mathbb{S}^2 \times \mathbb{S}^{2l}} \mathrm{ch}_{l+1}(\hat{E}) \tag{B.12}$$

where $\mathrm{ch}(\hat{E})$ is the Chern character of the extension of the bundle E to $\mathbb{S}^2 \times \mathbb{S}^{2l}$ defined by using $g : \mathbb{S}^1 \times \mathbb{S}^{2l} \to G$ as transition function [2]. Then $\mathrm{i}\nabla_{2l+2}$ is the corresponding Dirac operator in $2l+2$ dimensions and the equality in (B.12) between its analytical and topological index is the content of the Atiyah-Singer theorem.

More generally, in [**152, 153**] it was shown that a curvature and connection on the bundle $\mathcal{L} \to \mathcal{A}/\mathcal{G}$ can be defined canonically in terms of the curvature and connection on a generic spacetime X. One needs to consider not just the product of X with an $\mathbb{S}^2$ inside of $\mathcal{A}/\mathcal{G}$ but rather the total space of the fibration:

$$\begin{array}{ccc} X & \longrightarrow & Z \\ & & \downarrow \\ & & \mathcal{A}/\mathcal{G} \end{array} \tag{B.13}$$

where $Z = \mathcal{A} \times X/\mathcal{G}$. The bundle E can be canonically extended to the total space Z and the push-forward of its Chern character to the base $\mathcal{A}/\mathcal{G}$ is the curvature characteristic class of $\mathcal{L}$:

$$c_1(\mathcal{L}) = \left[\int_X \hat{A}(T_{\mathrm{vert}} Z) \mathrm{ch}(\hat{E}) \right]_{(2)} \tag{B.14}$$

where the A-roof genus has also been introduced to keep track of the contribution of the curvature of the tangent bundle of X or more precisely, the vertical part of TZ. For convenience we write the first few terms in the power expansion of $\hat{A}$ in terms of the Pontryagin classes p_i:

$$\hat{A} = 1 - \frac{1}{24} p_1 + \frac{1}{5760} [7p_1^2 - 4p_2] + \dots \tag{B.15}$$

The integrand in (B.14) is an invariant polynomial of degree $l+1$ in the curvatures of $\hat{E}$ and $T_{\mathrm{vert}} Z$, and it is called the anomaly polynomial I_{2l+2}:

$$I_{2l+2} = \left[\hat{A}(T_{\mathrm{vert}} Z) \mathrm{ch}(\hat{E}) \right]_{(2l+2)} \tag{B.16}$$

[2]Here we used the isomorphism $\mathrm{Hom}(\mathbb{S}^1, \mathcal{G}) \cong \mathrm{Hom}(\mathbb{S}^1 \times X, G)$ for $\mathcal{G} \cong \mathrm{Hom}(X, G)$. If we restrict to the group of pointed gauge transformations, the product $\mathbb{S}^1 \times \mathbb{S}^{2l}$ becomes the (reduced) suspension $\mathbb{S}^1 \wedge \mathbb{S}^{2l} \cong \mathbb{S}^{2l+1}$ so that $[g] \in \pi_{2l+1}(G)$.

B.3. Anomaly polynomial for abelian symmetries

We consider here the special case in which the structure group G is abelian. This is the case of study of section 2.7.

Let us start by considering $G = U(1)$. In this case all the irreducible representations are 1-dimensional and every vector bundle E can be decomposed as a Whitney sum of line bundles $E = \mathcal{L}^{(1)} \oplus \cdots \oplus \mathcal{L}^{(n)}$, where n is the rank of E. If we call $F \in \Omega^2(X)$ the curvature of the associated charge-1 bundle, then the curvature of E can be written as:

$$\Omega^{(E)} = T \otimes F \quad \text{with} \quad T \equiv \text{diag}(q^{(1)}, \dots, q^{(n)}) \tag{B.17}$$

where $q^{(r)} \in \mathbb{Z}$ are the charges with which the structure group acts on each eigenbundle $\mathcal{L}^{(r)}$. We write $x = [F/2\pi] \in H^2(X, \mathbb{Z})$ for the Chern class of the principal bundle so that:

$$c_1(\mathcal{L}^{(r)}) = q^{(r)} x \tag{B.18}$$

$$c_1(E) = \mathsf{Tr}[T] x \tag{B.19}$$

By the additive property of the Chern character we then find:

$$\text{ch}(E) = \mathsf{Tr}\ \mathrm{e}^{\Omega^{(E)}/2\pi} = \sum_{k=0}^{\infty} \frac{\mathsf{Tr}[T^k]}{k!} x^k \tag{B.20}$$

More generally we can consider a family of n particles charged under m abelian symmetries $G = \prod_{i=1}^m U(1)_i$. In this case the gauge bundle is:

$$E = \bigoplus_{r=1}^{n} \mathcal{L}^{(r)} \qquad \text{with} \qquad \mathcal{L}^{(r)} = \mathcal{L}_1^{(r)} \otimes \cdots \otimes \mathcal{L}_m^{(r)} \tag{B.21}$$

where each $\mathcal{L}^{(r)}$ is a tensor product representation for the group G, labeled by the set of charges $(q_1^{(r)}, \dots, q_m^{(r)})$. If as before we define x_i to be the Chern class of the principal $U(1)$-bundle associated to the i-th symmetry, then we can write:

$$\begin{aligned} \text{ch}_k(E) &= \frac{1}{k!}\left[c_1(\mathcal{L}^{(1)})^k + \cdots + c_1(\mathcal{L}^{(n)})^k\right] \\ &= \frac{1}{k!}\sum_{r=1}^{n}\left(\sum_{i=1}^{m} q_i^{(r)} x_i\right)^k \\ &= \frac{1}{k!}\sum_{i_1\cdots i_k}^{m}\left(\sum_{r=1}^{n} q_{i_1}^{(r)}\cdots q_{i_k}^{(r)}\right) x_{i_1}\cdots x_{i_k} \\ &= \frac{1}{k!}\sum_{i_1\cdots i_k}^{m} \mathsf{Tr}\left[T_{i_1}\cdots T_{i_k}\right] x_{i_1}\cdots x_{i_k} \end{aligned} \tag{B.22}$$

where $T_i = \text{diag}(q_i^{(1)}, \dots, q_i^{(n)})$ is the matrix generator of the i-th symmetry.

Using formula (B.14) we can compute the anomaly polynomial in 4 dimensions as:

$$\begin{aligned} I_6 &= \text{ch}_3(\hat{E}) - \frac{1}{24} p_1(T_{\text{vert}} Z)\text{ch}_1(\hat{E}) \\ &= \frac{1}{3!}\sum_{ijk} \mathsf{Tr}[T_i T_j T_k] x_i x_j x_k - \frac{1}{24} p_1(T_{\text{vert}} Z) \sum_i \mathsf{Tr}[T_i] x_i \end{aligned} \tag{B.23}$$

where by abuse of notation we write x_i for both the classes on X and those on Z. The coefficients $\mathsf{Tr}[T_i T_j T_k]$ and $\mathsf{Tr}[T_i]$ are called the 't Hooft anomaly coefficients.

Similarly, in 2 dimensions the anomaly polynomial is:

$$\begin{aligned} I_4 &= \mathrm{ch}_2(\hat{E}) - \frac{1}{24} p_1(T_{\mathrm{vert}} Z)\mathrm{ch}_0(\hat{E}) \\ &= \frac{1}{2!}\sum_{ij} \mathsf{Tr}[T_i T_j] x_i x_j - \frac{n}{24} p_1(T_{\mathrm{vert}} Z) \end{aligned} \tag{B.24}$$

with 't Hooft anomaly coefficients $\mathsf{Tr}[T_i T_j]$ and $n = \mathrm{rank} E$ the gravitational anomaly coefficient.

Observe that for a Weyl fermion of negative chirality the anomaly polynomial gets a minus sign in front. This means that for a theory with many fermions of both chiralities one should compute the 't Hooft anomaly coefficients by inserting the chirality operator γ_{d+1} inside of the traces so that contributions from opposite chirality fermions are counted with the appropriate signs.

B.4. Dimensional reduction of the anomaly polynomial

Let us call $\Omega^{(\mathcal{L})}$ the curvature 2-form of the determinant line bundle $\mathcal{L} \to \mathcal{A}/\mathcal{G}$, $\Omega^{(T_{\mathrm{vert}} Z)}$ the curvature of $T_{\mathrm{vert}} Z$ in (B.13) and $\Omega^{(\hat{E})}$ the curvature of $\hat{E}$, so that:

$$\Omega^{(\mathcal{L})} = \left[2\pi \mathrm{i} \int_X \hat{A}(\Omega^{(T_{\mathrm{vert}} Z)}) \mathrm{ch}(\Omega^{(\hat{E})}) \right]_{(2)} \tag{B.25}$$

If we assume that X is itself a (possibly trivial) fibration:

$$\begin{array}{ccc} \Sigma & \longrightarrow & X \\ & & \downarrow \\ & & X' \end{array} \tag{B.26}$$

of a Riemann surface over a $2l-2$ dimensional manifold X', then we know that the anomalies of the theory on X and those of the theory on X' (after compactification of the Riemann surface Σ) will coincide because the compactification only discards massive KK modes that do not contribute to the anomaly. We therefore expect that the curvature $\Omega^{(\mathcal{L})}$ is unmodified by the compactification. However we can ask: what is the anomaly polynomial of the compactified gauge theory on X'?

In order to see this we observe that the fiber integration on X in (B.25) can be performed in two steps. First one integrates out Σ and obtains the differential form:

$$I_{2l} = \int_\Sigma I_{2l+2} \tag{B.27}$$

defined over the sub-bundle Z' of Z whose fiber is just X':

$$\begin{array}{ccccc} \Sigma & \longrightarrow & X & \longrightarrow & Z \\ & & \downarrow & & \downarrow \\ & & X' & \longrightarrow & Z' \\ & & & & \downarrow \\ & & & & \mathcal{A}/\mathcal{G} \end{array} \qquad \begin{array}{c} I_{2l+2} \\ \downarrow {\scriptstyle \int_\Sigma} \\ I_{2l} \\ \downarrow {\scriptstyle \int_{X'}} \\ \Omega^{(\mathcal{L})} \end{array} \quad \left(I_{2l+2} \xrightarrow{\int_X} \Omega^{(\mathcal{L})} \right) \tag{B.28}$$

The form I_{2l} then should be interpreted as the anomaly polynomial of the gauge theory on X' because, by construction, it reduces to $\Omega^{(\mathcal{L})}$ when it is integrated over the reduced spacetime X':

$$\Omega^{(\mathcal{L})} = \int_{X'} I_{2l} = \int_{X'} \int_\Sigma I_{2l+2} = \int_X I_{2l+2} \tag{B.29}$$

APPENDIX C

Aspects of 3d $\mathcal{N} = 2$ Theories

In this appendix we collect some relevant results on 3d $\mathcal{N} = 2$ theories. We focus on the reduction of 4d dualities to 3d dualities and its realization via the technique of localization.

C.1. 4d/3d reduction and KK monopole

Preserving a 4d supersymmetric duality in 3d can be done by compactifying the dual phases on a finite size circle. The procedure consists of dimensionally reduce the field content and to add the effective 3d dynamics due to the finite size. This lifts possible 4d anomalous symmetries that can potentially become non-anomalous in 3d. Such symmetries are indeed broken by the presence of superpotential terms involving the KK monopoles. The KK monopoles contribute to the effective superpotential if in the spectrum there are only two fermionic zero modes, coming from the gaugino in the adjoint representation, while the matter fields do not carry further fermionic zero modes [1]. The counting of these zero modes follows from an application of the index theorem. Essentially the circle compactification splits the 4d instanton in a set of BPS monopoles, counted by the Callias theorem, and in one KK monopole. The total amount of zero modes for these configurations corresponds to the number of zero modes of the original 4d instanton, obtained from the Atiyah-Singer index theorem. The difference between the two indices counts the number of zero modes in the KK monopole. A more direct result follows from [**154**], where an index theorem on $\mathbb{R}^3 \times \mathbb{S}^1$ was derived (see also [**155**]). In this case the counting of the zero modes in the KK monopole background from each matter fields associated to the affine root. For example the presence of fundamental matter fields does not modify the number of zero modes and the KK monopole is generated. Here we have been interested in $USp(2N_c)$ gauge groups with antisymmetric matter. The index of [**154**] has been computed for this representation in [**156**], and one can see that also in this case the KK monopole superpotential is generated, because no further fermionic zero modes associated to the affine root are present.

C.2. Counting of zero modes

In this section we review the counting of zero modes of fermions in a monopole/instanton background. Our derivation will closely follow that of [**157**] and [**154**].

C.2.1. Lie algebra conventions. We start by collecting a few well known facts about root systems of Lie algebras which are useful in the counting of zero modes. For a standard reference on the subject see [**158**].

Let G be a connected, simply connected and semisimple Lie group with Lie algebra $\mathfrak{g}$. The rank r of G is the dimension of any of its maximal torus subgroups or equivalently the dimension of the corresponding Cartan subalgebra. Having

[1]Actually this condition can be made milder in the presence of potential interactions involving the fermions carrying the extra zero modes.

chosen a particular such Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}$ we pick a basis of commuting generators $\{h_i\}_{i=1}^r \subset \mathfrak{h}$ that satisfy:

$$\mathsf{Tr}_{\mathcal{R}}[h_i h_j] = T(\mathcal{R})\delta_{ij} \tag{C.1}$$

for any irreducible representation $\mathcal{R}$ of $\mathfrak{g}$.

We denote the roots of the algebra as $\Delta = \{\alpha_i\}_{i=1}^{\dim(G)-r} \subset \mathfrak{h}^*$, the simple roots as $\{\beta_i\}_{i=1}^r \subset \Delta$ and the inverse of any root α in the root system Δ as:

$$\alpha^\vee = \tfrac{2\alpha}{\alpha\cdot\alpha} \in \mathfrak{h}^* \qquad \alpha\cdot\alpha \equiv \textstyle\sum_{i=1}^r \alpha(h_i)\alpha(h_i) \tag{C.2}$$

We define the co-roots H_i to be the duals of the inverse simple roots $\beta_j^\vee$, i.e., those elements of the Cartan algebra that satisfy the relation:

$$w(H_i) = \beta_i^\vee \cdot w \tag{C.3}$$

for any weight $w \in \mathfrak{h}^*$ of the algebra.

A particular choice of simple roots defines an associated fundamental Weyl chamber corresponding to the convex subset $\{v \in \mathfrak{h} \,|\, \beta_i(v) > 0, \forall i = 1, \dots, r\} \subset \mathfrak{h}$. Moreover one can split the root system Δ into two components:

$$\Delta = \Delta^+ \cup \Delta^- \tag{C.4}$$

where Δ^+ (Δ^-) are the positive (negative) roots, i.e., those that are positive (negative) integer combinations of the simple roots. The Weyl vector ρ is then defined as the half-sum of all the positive roots:

$$\rho = \frac{1}{2}\sum_{\alpha\in\Delta^+} \alpha \tag{C.5}$$

Given a root system Δ and a choice of simple roots $\{\beta_i\}$ one can define a partial order on Δ^+ as follows. For any positive root $\alpha = \sum_{i=1}^r m_i\beta_i$, define the degree (or level) of α as:

$$\deg(\alpha) = \sum_{i=1}^r m_i \in \mathbb{Z} \tag{C.6}$$

then the degree map endowes Δ^+ with the structure of a partially ordered set, the root poset. The highest root is the root with the highest degree and it is unique with respect to this property. It is customary to write the highest root and its inverse as:

$$\theta = \sum_{i=1}^r k_i\beta_i \quad \text{and} \quad \theta^\vee = \sum_{i=1}^r k_i^\vee \beta_i^\vee \tag{C.7}$$

where k_i are called the Kac labels of the algebra and $k_i^\vee$ are the Dynkin numbers. We will refer to the lowest root $-\theta$ as the affine root and define its dual co-root H_0 as:

$$H_0 = -\sum_{i=1}^r k_i^\vee H_i \tag{C.8}$$

so that $w(H_0) = -\theta^\vee \cdot w$ for every weight w.

The weights of G form a lattice in $\mathfrak{h}^*$ generated by the fundamental weights $\{\lambda_i\}$ defined by the relation:

$$\beta_i^\vee \cdot \lambda_j = \lambda_j(H_i) = \delta_{ij} \tag{C.9}$$

By definition then the weight lattice and the co-root lattice are integral dual to each other. It is a well known result that the sum of all fundamental weights coincides with the Weyl vector:

$$\rho = \sum_{i=1}^r \lambda_i \tag{C.10}$$

which then implies that $\rho(H_i) = \beta_i^\vee \cdot \rho = 1$ for every co-root H_i.

Finally, we recall the useful formula:

$$\Lambda \cdot (\Lambda + 2\rho) = C_2(\mathcal{R}) \tag{C.11}$$

where Λ is the highest weight of the representation $\mathcal{R}$ and $C_2(\mathcal{R})$ is the value of the quadratic Casimir element in that representation.

C.2.2. Callias index theorem. Consider a Euclidean theory on $\mathbb{R}^3 \times \mathbb{S}^1$ with a massless Dirac fermion ψ in the representation $\mathcal{R}$ of the gauge group G. Coordinates are chosen as $\{x^i\}_{i=1,2,3}$ on $\mathbb{R}^3$ and x^4 on $\mathbb{S}^1$. We look for solutions of the Dirac equation for ψ, i.e., zero eigenfunctions of the Dirac operator:

$$D\psi = \gamma^\mu(\partial_\mu + A_\mu)\psi = \begin{bmatrix} 0 & -\nabla^\dagger \\ \nabla & 0 \end{bmatrix} \begin{bmatrix} \psi^+ \\ \psi^- \end{bmatrix} = 0 \tag{C.12}$$

where ∇ and $\nabla^\dagger$ are Fredholm operators acting on spinors of definite chirality and, on an anti-selfdual background, satisfy:

$$\nabla^\dagger \nabla = -D_\mu D^\mu + 2\gamma^m B_m \quad \text{and} \quad \nabla \nabla^\dagger = -D_\mu D^\mu \tag{C.13}$$

where $B_m = \frac{1}{2}\varepsilon_{mlk} F_{lk} = F_{4m}$ is the magnetic field on $\mathbb{R}^3$.

What we are interested in computing is the difference in the number of zero modes of ∇ and those of $\nabla^\dagger$. This quantity is a topological invariant and is called the index of ∇:

$$I_{\mathcal{R}} \equiv \operatorname{ind}(\nabla) = \dim\ker(\nabla) - \dim\ker(\nabla^\dagger) \tag{C.14}$$

Using the fact that $\ker(\nabla^\dagger\nabla) = \ker(\nabla)$ and $\ker(\nabla\nabla^\dagger) = \ker(\nabla^\dagger)$, the index can be conveniently computed by the formula:

$$I_{\mathcal{R}} = \lim_{M^2\to 0} \mathsf{Tr}_{\mathcal{R}}\left[\frac{M^2}{\nabla^\dagger\nabla + M^2}\right] - \mathsf{Tr}_{\mathcal{R}}\left[\frac{M^2}{\nabla\nabla^\dagger + M^2}\right] \tag{C.15}$$

Observe that the trace in (C.15) is both over the representation $\mathcal{R}$ and over the Hilbert space on which the differential operator $\gamma^\mu\partial_\mu$ acts, i.e., the Hilbert space of sections of the spinor bundle.

In terms of the 4d Dirac operator we can write:

$$I_{\mathcal{R}}(M^2) = \mathsf{Tr}_{\mathcal{R}}\left[\gamma_5 \frac{M^2}{-D^2 + M^2}\right] = M\,\mathsf{Tr}_{\mathcal{R}}\left[\gamma_5 \frac{D+M}{-D^2 + M^2}\right] = M\,\mathsf{Tr}_{\mathcal{R}}\left[\gamma_5 \frac{1}{-D + M}\right] \tag{C.16}$$

Observe that in the previous formula it appears the propagator of the Dirac fermion ψ as:

$$\langle \psi(x_1)\bar{\psi}(x_2)\rangle = \left\langle x_1 \left| \frac{1}{D - M} \right| x_2 \right\rangle \tag{C.17}$$

corresponding to the Euclidean action $-\bar{\psi}(-D + M)\psi$ where the mass M has been introduced as an auxiliary parameter. Hence we can write:

$$I_{\mathcal{R}}(M^2) = -M\,\mathsf{Tr}_{\mathcal{R}}[\gamma_5\langle\psi\bar{\psi}\rangle] = \int_{\mathbb{S}^1} \mathrm{d}x^4 \int_{\mathbb{R}^3} \mathrm{d}^3x\, M\,\mathsf{Tr}_{\mathcal{R}}\langle\bar{\psi}\gamma_5\psi\rangle \tag{C.18}$$

The r.h.s. of (C.18) can be expressed using the Atiyah-Singer index theorem for the abelian anomaly of a 4d theory with a massive Dirac fermion ψ:

$$\partial_\mu J_5^\mu \equiv \partial_\mu(\bar{\psi}\gamma^\mu\gamma_5\psi) = -2M\bar{\psi}\gamma_5\psi - \mathrm{ch}_2(F) \tag{C.19}$$

where the second term on the right of (C.19) is the second Chern character of the bundle associated to the representation $\mathcal{R}$:

$$\mathrm{ch}_2(F) = \frac{1}{8\pi^2}\mathsf{Tr}_{\mathcal{R}}[F \wedge F] \tag{C.20}$$

Substituting (C.19) in (C.18) the index can finally be rewritten as:

$$I_{\mathcal{R}}(M^2) = -\frac{1}{2}\int_{\mathbb{S}^1} \mathrm{d}x^4 \int_{\mathbb{S}^2_\infty} \varepsilon_{ijk}\langle J_5^i\rangle \mathrm{d}x^j \mathrm{d}x^k - \frac{1}{2}\int_{\mathbb{R}^3\times\mathbb{S}^1} \mathrm{ch}_2(F) \tag{C.21}$$

C.2.3. BPS monopole background. The first type of background for which we are interested in counting fermionic zero modes is that of a static 4d monopole solution of 't Hooft-Polyakov. For a more general gauge group G we consider the embedding of the $SU(2)$ solution into the group G as in [**159, 160**]. These are usually referred to as BPS monopole backgrounds.

Because the solution is "static", the fourth component A_4 of the gauge connection behaves effectively as a Higgs field Φ for the connection A_i on $\mathbb{R}^3$. The theorem of Callias then states that the index of (C.14) depends on the topology of the Higgs field by counting the winding number of the map $|\Phi|^{-1}\Phi$ as it goes around the 2-sphere at spatial infinity.

In this set up we have that the Higgs field $\Phi \sim A_4$ is constant in x^4 but varies along $\mathbb{R}^3$. Finiteness of the energy of the solution imposes the following restrictions:

$$|\Phi| \to 1, \quad F \to 0, \quad D\Phi \to 0 \quad \text{for} \quad |x| \to \infty \tag{C.22}$$

so that the connection is asymptotically pure gauge. Because, the Higgs field is covariantly constant on the sphere at spatial infinity, we can write:

$$\Phi(x)|_{\mathbb{S}^2_\infty} = \mathrm{Ad}_{g(x)}\Phi_\infty \tag{C.23}$$

where $\Phi_\infty \equiv \Phi(p)$ is the value of the Higgs field at some fixed reference point $p \in \mathbb{S}^2_\infty$ and $g : \mathbb{S}^2_\infty \to G$. This corresponds to a global trivialization of the adjoint bundle of which Φ is a section and can always be done since the bundle is topologically trivial on $\mathbb{R}^3$.

The field Φ is then equivalent to the pair (Φ_∞, g) and defines a map to the orbit of the VEV Φ_∞ in the Lie algebra $\mathfrak{g}$ under the adjoint action of G. We assume that the VEV breaks the gauge group maximally, i.e., we choose Φ_∞ such that its stabilizer in G is a maximal torus $T \cong U(1)^r \subset G$. This implies that orbit is isomorphic the coset space G/T and:

$$\Phi : \mathbb{S}^2_\infty \to G/T \tag{C.24}$$

The homotopy class of the Higgs field then specifies an element of the second homotopy group of this coset, which for G simple and simply connected is:

$$\pi_2(G/T) \cong \mathbb{Z}^r \tag{C.25}$$

With this choice of VEV the BPS solution can be written explicitly as:

$$A_4|_{\mathbb{S}^2_\infty} = \Phi_\infty \tag{C.26}$$

$$F|_{\mathbb{S}^2_\infty} = \frac{\mathsf{n}}{2}\frac{\varepsilon_{ijk}x^i \mathrm{d}x^j \mathrm{d}x^k}{|x|^3} \quad \text{with} \quad \mathsf{n} \equiv \sum_{i=1}^{r} n_i H_i \tag{C.27}$$

where we have performed a patch-wise gauge transformation to make the Higgs field constant.

The solution is described by the following parameters:

- $\Phi_\infty \in \mathfrak{h}$ parametrizes the choice of asymptotic VEV for the Higgs field A_4. We choose Φ_∞ such that:

$$\beta_i(\Phi_\infty) > 0 \tag{C.28}$$

for any simple root $\beta_i \in \mathfrak{h}^*$. It follows that Φ_∞ is *regular* with respect to the chosen basis of simple roots β_i and lies into the *fundamental Weyl chamber* (other choices are possible but are all related by the action of the Weyl group).
- $n_i \in \mathbb{Z}$ is the magnetic charge[2] of the fundamental BPS monopole associated to the simple co-root $H_i \in \mathfrak{h}$; these integers describe the topology of the Higgs field and formally correspond to elements of $\pi_2(G/T)$.

Substituting (C.26) and (C.27) in (C.21), the first contribution to the index is given by:

$$\int_{\mathbb{S}^2_\infty} \frac{\varepsilon_{ijk} x^i \mathrm{d}x^j \mathrm{d}x^k}{|x|^3} \sum_{p\in\mathbb{Z}} \mathsf{Tr}_{\mathcal{R}} \left[\mathsf{n} \left(\frac{2\pi p}{L} + \Phi_\infty \right) \int_{\mathbb{R}^3} \frac{\mathrm{d}^3 k}{(2\pi)^3} \frac{1}{[k^2 + M^2 + (\frac{2\pi p}{L} + \Phi_\infty)^2]^2} \right] \tag{C.29}$$

where we expanded in Fourier modes along the circle $\mathbb{S}^1$ so that $\int_{\mathbb{S}^1} \mathrm{d}x^4 \to \sum_{p\in\mathbb{Z}}$ and $-\mathrm{i}D_4 \to \frac{2\pi p}{L} + A_4$. Here L is the length of $\mathbb{S}^1$.

In the $M \to 0$ limit the momentum integral can be computed using the following formula:

$$\lim_{M\to 0} \int_{\mathbb{R}^3} \frac{\mathrm{d}^3 k}{(2\pi)^3} [k^2 + M^2 + a^2]^{-2} = 4\pi \lim_{M\to 0} \int_{\mathbb{R}^3} \frac{k^2 \mathrm{d}k}{(2\pi)^3} [k^2 + M^2 + a^2]^{-2} = \frac{1}{8\pi |a|} \tag{C.30}$$

which substituted into (C.29) gives:

$$\frac{1}{2} \mathsf{Tr}_{\mathcal{R}} \left[\mathsf{n} \sum_{p\in\mathbb{Z}} \frac{\frac{2\pi p}{L} + \Phi_\infty}{\left|\frac{2\pi p}{L} + \Phi_\infty\right|} \right] \tag{C.31}$$

The series can now be regularized using the η-invariant:

$$\eta(s) = \sum_{p\in\mathbb{Z}} \frac{\mathrm{sgn}\left(p + \frac{\Phi_\infty L}{2\pi}\right)}{\left|p + \frac{\Phi_\infty L}{2\pi}\right|^s} \tag{C.32}$$

and we obtain:

$$\begin{aligned} I^{\mathrm{BPS}}_{\mathcal{R}}(0) &= \frac{1}{2} \mathsf{Tr}_{\mathcal{R}} [\mathsf{n}\, \eta(0)] - \frac{1}{2} \int_{\mathbb{R}^3 \times \mathbb{S}^1} \mathrm{ch}_2(F) \\ &= \mathsf{Tr}_{\mathcal{R}} \left[\left(-\frac{\Phi_\infty L}{2\pi} + \left\lfloor \frac{\Phi_\infty L}{2\pi} \right\rfloor \right) \mathsf{n} \right] - \frac{1}{2} \int_{\mathbb{R}^3 \times \mathbb{S}^1} \mathrm{ch}_2(F) \end{aligned} \tag{C.33}$$

where $\lfloor x \rfloor = \max\{m \in \mathbb{Z} \,|\, m \le x\}$ is the floor function.

The second contribution to the index can be computed by observing that $H^4(\mathbb{R}^3 \times \mathbb{S}^1, \mathbb{R}) = 0$ so that the Chern character is trivialized by the globally defined

[2]For each fundamental monopole to be properly quantized as an $SU(2)$ BPS solution embedded into the larger gauge group G we need n to be a vector in the co-root lattice of G. In fact, if n is an integer linear combination of co-roots then its eigenvalues are integers in all representations. This can be shown by considering that every weight w is a linear combination of fundamental weights λ_i with integer coefficients, so that, using (C.9) we have $w(\mathsf{n}) \in \mathbb{Z}$.

Chern-Simons form which can then be integrated over the asymptotic boundary:

$$\begin{aligned} -\frac{1}{2}\int_{\mathbb{R}^3\times\mathbb{S}^1}\mathrm{ch}_2(F) &= \frac{1}{8\pi^2}\int_{\mathbb{S}^1}\mathrm{d}x^4\int_{\mathbb{S}^2_\infty}\mathsf{Tr}_{\mathcal{R}}[A_4F_{jk}]\mathrm{d}x^j\mathrm{d}x^k \\ &= \frac{L}{16\pi^2}\mathsf{Tr}_{\mathcal{R}}\left[\Phi_\infty\mathsf{n}\right]\int_{\mathbb{S}^2_\infty}\frac{\varepsilon_{ijk}x^i\mathrm{d}x^j\mathrm{d}x^k}{|x|^3} \\ &= \mathsf{Tr}_{\mathcal{R}}\left[\frac{\Phi_\infty L}{2\pi}\mathsf{n}\right] \end{aligned} \tag{C.34}$$

Combining the two contributions gives the index in the BPS monopole background:

$$I^{\mathrm{BPS}}_{\mathcal{R}} = \mathsf{Tr}_{\mathcal{R}}\left[\left\lfloor\frac{\Phi_\infty L}{2\pi}\right\rfloor\mathsf{n}\right] = \sum_{i=1}^{\dim(\mathcal{R})}\left\lfloor w_i\left(\frac{\Phi_\infty L}{2\pi}\right)\right\rfloor w_i(\mathsf{n}) \tag{C.35}$$

where w_i are the weights of the representation $\mathcal{R}$.

In the small radius limit we have $|w_i\left(\frac{\Phi_\infty L}{2\pi}\right)| \ll 1$ for every weight w_i so that:

$$\left\lfloor w_i\left(\frac{\Phi_\infty L}{2\pi}\right)\right\rfloor = \begin{cases} 0 & \text{for} \quad 0 \le w_i\left(\frac{\Phi_\infty L}{2\pi}\right) < 1 \\ -1 & \text{for} \quad -1 < w_i\left(\frac{\Phi_\infty L}{2\pi}\right) < 0 \end{cases} \tag{C.36}$$

Then we can rewrite (C.35) as a sum over the "negative" weights:

$$I^{\mathrm{BPS}}_{\mathcal{R}} = -\sum_{\{w|w(\Phi_\infty)<0\}} w(\mathsf{n}) = \sum_{i=1}^{\dim(\mathcal{R})}\left[\frac{-1+\mathrm{sgn}(w_i(\Phi_\infty))}{2}\right]w_i(\mathsf{n}) \tag{C.37}$$

and, using the fact that the weights of any representation sum to zero, we get the final formula:

$$I^{\mathrm{BPS}}_{\mathcal{R}} = \sum_{i=1}^{\dim(\mathcal{R})}\frac{1}{2}\mathrm{sgn}(w_i(\Phi_\infty))\,w_i(\mathsf{n}) \tag{C.38}$$

Observe that because of the static nature of the solution, the counting of zero modes on $\mathbb{R}^3\times\mathbb{S}^1$ gives the same result as that of a 3d theory obtained in the limit of vanishing radius for $\mathbb{S}^1$. That might not be the case for KK monopole backgrounds coming from 4d instanton configurations.

C.2.4. KK monopole background. When the theory lives on $\mathbb{R}^3\times\mathbb{S}^1$ there is also a second type of topologically non-trivial background called winding instanton or KK monopole. This type of solution is the compactification of a standard 4d instanton on $\mathbb{R}^4$ and can be obtained from a BPS solution by applying an anti-periodic "gauge transformation"[3] along the x^4 direction [**161, 162**]. In this case the fourth component of the gauge field cannot be taken to be constant along the compact direction and in fact it defines a non-trivial Wilson line that wraps the $\mathbb{S}^1$. A similar computation to the one in (C.2.3) yields the index of the Dirac operator in a KK monopole background as:

$$\begin{aligned} I^{\mathrm{KK}}_{\mathcal{R}} &= \mathsf{Tr}_{\mathcal{R}}\left[\left\lfloor\frac{\Phi_\infty L}{2\pi}\right\rfloor n_0H_0\right] + \frac{1}{2}n_0\mathsf{Tr}_{\mathcal{R}}[H_0H_0] \\ &= \sum_{i=1}^{\dim(\mathcal{R})}\frac{1}{2}\mathrm{sgn}(w_i(\Phi_\infty))w_i(n_0H_0) + 2n_0\frac{T(\mathcal{R})}{\theta\cdot\theta} \end{aligned} \tag{C.39}$$

where H_0 is the "affine" co-root (C.8), $T(\mathcal{R})$ is the Dynkin index of the representation and $n_0 \in \mathbb{Z}$ is the KK monopole charge.

[3]The quotation marks here are due to the fact that because the transformation is not periodic, it does not define a proper gauge transformation. In fact, the transformed solution is not gauge equivalent to the BPS one.

C.2.5. Adjoint representation. Here we show that, for a suitable choice of adjoint scalar VEV, a fermion ψ in the adjoint representation of any gauge group G carries exactly two zero modes for every unit of BPS or KK monopole charge.

The counting of BPS zero modes is given by the index of (C.38) which can be written as:

$$I_{\text{adj}}^{\text{BPS}} = \sum_{i=1}^{\dim(G)} \frac{1}{2}\,\text{sgn}(\alpha_i(\Phi_\infty))\,\alpha_i(\mathsf{n}) \tag{C.40}$$

Because we have chosen the VEV of the Higgs field to lie in the fundamental Weyl chamber (C.28), the sum ranges over the negative roots with a minus sign and over the positive roots with a plus sign and the 1/2 in front takes care of the double counting, therefore we can write:

$$I_{\text{adj}}^{\text{BPS}} = \sum_{\alpha\in\Delta^+} \alpha(\mathsf{n}) = 2\rho(\mathsf{n}) = \sum_{i=1}^{r} 2n_i \tag{C.41}$$

where ρ is the Weyl vector (C.11). The computation of the index for a KK monopole goes as follows:

$$\begin{aligned} I_{\text{adj}}^{\text{KK}} &= -2n_0\rho\cdot\theta^\vee + 2n_0\frac{T(\text{adj})}{\theta\cdot\theta} \\ &= 2n_0\left(\frac{-2\theta\cdot\rho + T(\text{adj})}{\theta\cdot\theta}\right) \\ &= 2n_0\left(1 + \frac{-\theta\cdot(\theta+2\rho) + T(\text{adj})}{\theta\cdot\theta}\right) \end{aligned} \tag{C.42}$$

where $\theta\cdot(\theta+2\rho)$ is the value of the quadratic Casimir element $C_2(\text{adj})$ on the adjoint representation. Using the fact that $C_2(\text{adj}) = T(\text{adj})$ we finally obtain the desired result:

$$I_{\text{adj}}^{\text{KK}} = 2n_0 \tag{C.43}$$

C.2.6. ***USp*** **and** ***U*** **case.** In section 4.3 we consider $USp(2N)$ theories with matter in the fundamental and antisymmetric representations as well as $U(N)$ theories with matter in the fundamental and adjoint. Here we give the result for the counting of zero modes for those groups and those representations.

In the case of a $USp(2N)$ gauge group we have one zero mode contribution coming from the BPS monopole associated to the long simple root:

$$I_{\text{fund.}} = n_N \tag{C.44}$$

and one for every unit of monopole charge associated to the short simple roots coming from the antisymmetric representation:

$$I_{\text{antisymm.}} = \sum_{i=1}^{N-1} 2n_i \tag{C.45}$$

In particular, we observe that in both representations there are no contributions to KK monopole zero modes.

In the case of a $SU(N)$ gauge group we have:

$$I_{\text{fund.}} = n_i \tag{C.46}$$

for i the largest integer such that $w_i(\Phi_\infty) > 0$ where w_i is the i-th weight of the fundamental representation of $SU(N)$. Therefore there is one contribution coming from the i-th BPS monopole and none coming from the KK monopole. Similarly, for the case of a $U(N)$ gauge group there are no fundamental zero mode contributions to the KK monopole [**63**].

C.2.7. Monopole superpotentials. Once a 4d duality is reduced on $\mathbb{S}^1$ we are in the presence of a new 3d effective duality. Such a duality has the field and gauge content of the 4d theory and in addition an extra superpotential involving the KK monopole. These effective dualities can then be transformed into more conventional 3d dualities, by real mass and higgs flow. Actually, richer structure of RG flows have been more recently analyzed in [**66**], leading to families of new 3d dualities with non-trivial monopole superpotentials. For example it has been shown that the 4d duality of [**113**], that relates $USp(2N_c)$ gauge theories with fundamentals, can be reduced in this way to a duality between unitary theories with linear monopoles in the superpotential. Many of the salient features of these reductions can be captured reducing the 4d superconformal index to the 3d partition function. This reduction gives indeed the 3d identities between the 3d dualities obtained from the field theory side. An important aspect of these reductions regards the constraints between the fugacities in the superconformal index. These constraints are necessary in 4d to enforce the constraints imposed by superconformality, i.e. the vanishing of the beta function or equivalently the anomaly freedom of the R-symmetry current. The constraints translate in a constraint on the parameters of the 3d partition functions (the real masses of the associated field theory). These constraints signal the presence of monopole superpotentials and are usually referred to in the mathematical literature as balancing conditions. We encounter such conditions often in our analysis.

Observe that the presence of a monopole in a superpotential fixes the R-charge and the abelian flavor charges of such a superpotential term, $R[W] = 2$ and $F_k[W] = 0$ where the index k runs over the abelian non-R global symmetries. The charge of a monopole operator with magnetic charge n as in (C.27) under any global symmetry can be computed in terms of the charges of the fermions of the theory by a one loop computation. The quantum correction to the monopole charge is obtained at one loop and it is

$$Q_A[\text{monopole}] = -\frac{1}{2}\sum_i Q_A[\psi_i]|w_i(\mathsf{n})| \tag{C.47}$$

where w_i is the weight of the i-th fermion ψ_i under the gauge group and $Q_A[\psi_i]$ is its charge under the abelian symmetry A.

C.3. Squashed three sphere partition function

Here we provide some more formulas used in Chapter 4. The partition function of a 3d $\mathcal{N} = 2$ gauge theory, with gauge group G, on a squashed three sphere is given by the general formula [**163, 139, 164, 165**]

$$\mathcal{Z}_{G;k}(\lambda;\vec{\mu}) = \frac{1}{|W|}\int\prod_{i=1}^{G}\frac{\mathrm{d}\sigma_i}{\sqrt{-\omega_1\omega_2}}\mathrm{e}^{\frac{k\pi\mathrm{i}\sigma_i^2}{\omega_1\omega_2}+\frac{2\pi\mathrm{i}\lambda\sigma_i}{\omega_1\omega_2}}\frac{\prod_I\Gamma_h\left(\omega\Delta_I+\rho_I(\sigma)+\widetilde{\rho}_I(\mu)\right)}{\prod_{\alpha\in G_+}\Gamma_h\left(\pm\alpha(\sigma)\right)}, \tag{C.48}$$

where the hyperbolic gamma function Γ_h (see for example [**116**]) correspond to the contributions of the one loop determinants and are defined as

$$\Gamma_h(x;\omega_1,\omega_2) \equiv \Gamma_h(x) \equiv \mathrm{e}^{\frac{\pi\mathrm{i}}{2\omega_1\omega_2}\left((x-\omega)^2-\frac{\omega_1^2+\omega_2^2}{12}\right)}\prod_{j=0}^{\infty}\frac{1-\mathrm{e}^{\frac{2\pi\mathrm{i}}{\omega_1}(\omega_2-x)}\mathrm{e}^{\frac{2\pi\mathrm{i}\omega_2 j}{\omega_1}}}{1-\mathrm{e}^{-\frac{2\pi\mathrm{i}}{\omega_2}x}\mathrm{e}^{-\frac{2\pi\mathrm{i}\omega_1 j}{\omega_2}}}. \tag{C.49}$$

We denote as b the squashing parameter of the ellipsoid defined by the relation

$$\frac{x_1^2 + x_2^2}{b^2} + \frac{x_3^2 + x_4^2}{1/b^2} = 1 \tag{C.50}$$

and define $\omega_1 = \mathrm{i}b$, $\omega_2 = \mathrm{i}b^{-1}$ and $2\omega \equiv \omega_1 + \omega_2$. In formula (C.48) σ and μ are real quantities, in the Cartan of the gauge and of the flavor symmetry. We denoted with an α the positive roots of the gauge group and with $\rho(\sigma)$ and $\widetilde{\rho}(\mu)$ the weights of the gauge and of the flavor symmetry respectively, necessary to parameterize the one loop contribution of each chiral field. The parameter λ corresponds to a possible FI term, while the R charge of each chiral field is identified by Δ_I. The Gaussian factor in the integrand corresponds to the contribution of the classical action and it is identified with the CS term at level k. Possible CS terms involving the flavor symmetries can be turned on and are associated to the contact terms as discussed in [**166, 167**].

In the Chapter 4 we mainly studied the partition function of $USp(2N_c)$ gauge theories with $2N_f$ fundamentals and one antisymmetric matter field. By calling μ_a ($a = 1, \dots, 2N_f$) the mass parameters of the fundamentals and τ the mass parameter of the antisymmetric, the three sphere partition function becomes

$$\begin{aligned} \mathcal{Z}_{USp(2N_c)}(\vec{\mu};\tau) = \frac{\Gamma_h(\tau)^{N_c-1}}{2^{N_c} N_c! (-\omega_1\omega_2)^{N_c/2}} \int \prod_{i=1}^{N_c} \mathrm{d}\sigma_i \frac{\prod_{a=1}^{2N_f} \Gamma_h(\widetilde{\mu}_a \pm \sigma_i)}{\Gamma_h(\pm 2\sigma_i)} \times \\ \times \prod_{i<j} \frac{\Gamma_h(\tau \pm \sigma_i \pm \sigma_j)}{\Gamma_h(\pm\sigma_i \pm \sigma_j)} \end{aligned} \tag{C.51}$$

We also used the partition function of $U(N_c)$ gauge theories with N_f fundamental flavors and an adjoint. In this case the partition function has the general form In this case we define two mass parameters for the flavor, m_a and n_a ($i = a, \dots, N_f$), for the fundamentals and the anti-fundamentals respectively. We denote as τ the mass parameter of the adjoint. The three sphere partition function in this case is

$$\begin{aligned} \mathcal{Z}_{U(N_c)}(\vec{m};\vec{n};\tau;\Lambda) = \frac{\Gamma_h(\tau)^{N_c-1}}{N_c!(-\omega_1\omega_2)^{N_c/2}} \int \Bigg(\prod_{i=1}^{N_c} \mathrm{d}\sigma_i e^{\pi \mathrm{i} \Lambda \sigma_i} \prod_{a=1}^{N_f} \Gamma_h(m_a + \sigma_i) \times \\ \times \prod_{a=1}^{N_f} \Gamma_h(n_a - \sigma_i) \Bigg) \prod_{1 \le i<j \le N_c} \frac{\Gamma_h(\tau \pm (\sigma_i - \sigma_j))}{\Gamma_h(\pm(\sigma_i - \sigma_j))} \end{aligned} \tag{C.52}$$

where the parameter Λ refers to the FI term. Observe that we did not consider possible constraints among the parameters. Such constraints have to be added in the presence of non-trivial superpotential interactions, like monopole superpotentials.

In the analysis we made use of two relevant formulas relating the hyperbolic gamma function. The first formula

$$\Gamma_h(2\omega - x)\Gamma_h(x) = 1 \tag{C.53}$$

allows to integrate out pairs of fields associated to superpotential mass terms. The second formula

$$\lim_{x \to \pm\infty} \Gamma_h(x) = e^{-\frac{\pi \mathrm{i}}{2} \mathrm{sgn}(x)(x-\omega)^2} \tag{C.54}$$

allows to integrate out fields with a large real mass. The gaussian factor in this formula reproduces the CS terms generated on the field theory side.

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