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THE ERDŐS-KO-RADO THEOREM FOR THE DERANGEMENT GRAPH OF THE PROJECTIVE GENERAL LINEAR GROUP ACTING ON THE PROJECTIVE SPACE

PABLO SPIGA

Abstract. In this paper we prove an Erdős-Ko-Rado-type theorem for intersecting sets of permutations. We show that an intersecting set of maximal size in the projective general linear group $\mathrm{PGL}_{n+1}(q)$, in its natural action on the points of the n-dimensional projective space, is either a coset of the stabiliser of a point or a coset of the stabiliser of a hyperplane. This gives a positive solution to [15, Conjecture 2].

Keywords derangement graph, independent set, Erdős-Ko-Rado theorem, projective general linear group

1. Introduction

The Erdős-Ko-Rado theorem [5] is one of the most fundamental theorems of extremal combinatorics; this result determines the cardinality and also describes the structure of a set of maximal size of intersecting k-subsets from $\{1, \ldots, n\}$. This theorem consists of two parts: the first part determines the maximal size of intersecting k-subsets; the second part classifies the sets attaining this maximum. There are various applications of the Erdős-Ko-Rado theorem, for example to qualitatively independent sets, problems in finite geometry and in statistics. The importance of the Erdős-Ko-Rado theorem is mainly due to its ubiquity: there are many different proofs and extensions of this theorem and we refer the reader to [3, 7] for a full account. In fact, the book [7] is dedicated primarily to various incarnations of the Erdős-Ko-Rado result.

In this paper, we are concerned with an extension, due to Erdős, of the Erdős-Ko-Rado theorem to permutation groups. Let G be a permutation group on Ω. A subset S of G is said to be intersecting if, for every $g, h \in S$, the permutation gh^{-1} fixes some point of Ω. This definition is very natural: writing $\Omega = \{\omega_1, \omega_2, \ldots, \omega_n\}$ and identifying the permutations g and h with the n-tuples $(\omega_1^g, \omega_2^g, \ldots, \omega_n^g)$ and $(\omega_1^h, \omega_2^h, \ldots, \omega_n^h)$, g and h are intersecting if their corresponding tuples coincide in some coordinate, that is, $\omega_i^g = \omega_i^h$, for some $i \in \{1, \ldots, n\}$. As with the Erdős-Ko-Rado theorem, in this context we are interested in finding the cardinality of an intersecting set of maximal size and classifying the sets that attain this bound.

As in most Erdős-Ko-Rado-type of results, this problem can be formulated in a graph-theoretic terminology. We denote by Γ_G the *derangement graph* of G, the vertices of this graph are the elements of G and the edges are the pairs $\{g, h\}$ such that $g^{-1}h$ fixes no point. (A fixed-point-free permutation is sometimes referred to as a derangement, this term goes back to the 1708 teasing question of Montmort at the end of his book [18, page 185] concerning the "jeu du treize".) An intersecting set of G is simply an independent set of Γ_G.

The finite symmetric group $\mathrm{Sym}(n)$ of degree n is possibly the most interesting permutation group for combinatorialists. Here, the natural extension of the Erdős-Ko-Rado theorem for $\mathrm{Sym}(n)$ was independently proved by Cameron and Ku in [1], and by Larose and Malvenuto in [13]. These papers, using different methods, showed that every intersecting set of $\mathrm{Sym}(n)$ has cardinality at most $(n-1)!$. They both further showed that the only intersecting sets meeting this bound are the cosets of the stabiliser of a point. (Actually, the results in [13] are slightly more general.) The same result was also proved by Godsil and Meagher in [6] using the character theory of $\mathrm{Sym}(n)$. The Godsil-Meagher proof is of paramount importance in this area; in fact, the proof is entirely algebraic and flexible, in the sense that it can be used as a "template" and it can be applied to any permutation group where the character theory is either sufficiently easy or sufficiently well-understood (the symmetric group is a perfect candidate for meeting both requirements). Recently, this method was used in [17] to prove a weak form of the Erdős-Ko-Rado theorem of every 2-transitive group; to the best of our knowledge this is the only result concerning the Erdős-Ko-Rado theorem for groups covering simultaneously a large class of permutation groups.

Currently, it is not clear for which families of permutation groups the complete analogue of the Erdős-Ko-Rado theorem holds.

In this paper we are interested in the q-analogue of the Erdős-Ko-Rado theorem proved in [1, 13]:

Theorem 1.1. *The independent sets of maximal size in the derangement graph of $\mathrm{PGL}_{n+1}(q)$ acting on the points of the projective space $\mathrm{PG}^n(q)$ are the cosets of the stabiliser of a point and the cosets of the stabiliser of a hyperplane.*

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This result was first conjectured in [15, Conjecture 2] and hence Theorem 1.1 gives a positive solution to this question. Prior to this paper, the only evidence towards the veracity of [15, Conjecture 2] was given in [15] settling the case $n = 1$, and in [16] settling the case $n = 2$.

It is said that "the work of the righteous is done by others" and, to some extent, we feel that this citation fits our proof of Theorem 1.1. Indeed, our proof uses all the ideas in [15–17], specifically we combine the character-theoretic arguments developed in [15] with the combinatorial arguments used in [16]. The role of [17] is slightly more marginal. It is remarkable that by combining both methods our proof here works only when $n \geq 3$, hinting to the fact that possibly the cases $n \in \{1, 2\}$ show peculiar behaviour and had to be treated separately. Actually, we could adapt our proof to include the case $n = 2$, but this would make our arguments unnecessarily long, and we opted for a shorter and more unified proof for $n \geq 3$ (for clarity, in the course of our argument we point out where our proof breaks when $n = 1$ and when $n = 2$).

1.1. **Comments on Theorem** 1.1 **and structure of the paper.**

We thank Alex Zalesski for some discussions on this paper and for proving [19, Theorem 1.1] for us. Our approach in proving Theorem 1.1 (and hence settling [15, Conjecture 2]) was adapting the Godsil-Meagher proof for $\mathrm{Sym}(n)$ to $\mathrm{PGL}_{n+1}(q)$, and Theorem 1.1 in [19] is the natural q-analogue of one of the tools used in [6].

Recently, Long, Plaza, Sin and Xiang [14] have proved the Erdős-Ko-Rado theorem for the derangement graph of $\mathrm{PSL}_2(q)$ in its action of on the projective line: this result was conjectured in [15]. In their work, these authors have developed some new ideas and they posed a beautiful conjecture [14, Section 6] concerning the stability of extremal intersecting families in $\mathrm{PSL}_2(q)$, in the same spirit as the results proved by Ellis [4] on the stability of the extremal intersecting families in $\mathrm{Sym}(n)$. Now that [15, Conjecture 2] is proved, we remark the relevance and importance of the work in [14] for a possible strengthening of Theorem 1.1, and we encourage investigations on the stability of extremal intersecting families for general projective linear groups.

The structure of this paper is straightforward. In Section 2, we set some notation that we use throughout the whole paper. In Section 3 we prove Theorem 1.1, assuming the veracity of two facts: Propositions 3.1 and 3.2. The rest of the paper is dedicated in proving Propositions 3.1 and 3.2. In Section 4, we gather some information on the characters of $\mathrm{PGL}_{n+1}(q)$, here the result of Zalesski [19] is fundamental. We prove Proposition 3.1 in Section 5 and we prove Proposition 3.2 in Section 6.

2. Notation and preliminary comments

We let q be a power of a prime, we let $n \geq 3$ be an integer, we denote by $\mathtt{GF}(q)$ a field of size q, and by $\mathtt{GF}(q)^{n+1}$ the $(n+1)$-dimensional vector space over $\mathtt{GF}(q)$ of row vectors with basis $(\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_{n+1})$. We denote by $\mathrm{PG}^n(q)$ the desarguesian projective space over $\mathtt{GF}(q)^{n+1}$, by $\mathcal{P}$ its set of points, by $\mathcal{L}$ its set of lines and by $\mathcal{H}$ its set of hyperplanes. Recall that $\mathrm{PG}^n(q)$ is the collection of all subspaces of $\mathtt{GF}(q)^{n+1}$ with incidence relation given by the usual set inclusion. Thus $\mathcal{P}$, $\mathcal{L}$ and $\mathcal{H}$ consist of the 1-, 2- and n-dimensional subspaces of $\mathtt{GF}(q)^{n+1}$, respectively. In what follows, for denoting the points we will use Greek letters, and for the lines roman letters.

Given two distinct points α and α', we denote by $\alpha \vee \alpha'$ the line spanned by α and α', that is, the line of $\mathrm{PG}^n(q)$ containing both α and α'. Moreover, for simplicity, given two distinct lines ℓ and ℓ' with $\ell \cap \ell' \neq \emptyset$, we denote by $\ell \wedge \ell'$ the point of the intersection of ℓ and ℓ'.

For not making the notation too cumbersome to use, we denote the projective general linear group $\mathrm{PGL}_{n+1}(q)$ simply by G; however, sometimes we break this rule when we want to emphasise some property of $\mathrm{PGL}_{n+1}(q)$. The subgroup of G that fixes the point $\alpha \in \mathcal{P}$ is denoted by G_α; similarly, given $\ell \in \mathcal{L}$ and $\pi \in \mathcal{H}$, we denote by G_ℓ and G_π the setwise stabiliser of ℓ and π in G.

As usual, $\mathbb{C}[G]$ is the group algebra of G over the complex numbers $\mathbb{C}$. For the first part of our argument, we only need the vector space structure on $\mathbb{C}[G]$: a basis for $\mathbb{C}[G]$ is indexed by the group elements $g \in G$. Given a subset S of G, we denote by $\mathbf{1}_S \in \mathbb{C}[G]$ the characteristic vector of S, that is, $(\mathbf{1}_S)_g = 1$ if $g \in S$, and $(\mathbf{1}_S)_g = 0$ otherwise. Observe that when S is a subgroup of G, we may consider $\mathbf{1}_S$ as the class function of S mapping each element of S to 1, that is, we view $\mathbf{1}_S$ as the principal character of S.

There is a natural duality [2] between the points and the hyperplanes of $\mathrm{PG}^n(q)$ and this duality is preserve by $G = \mathrm{PGL}_{n+1}(q)$. Hence, for each $g \in G$, the number of elements of $\mathcal{P}$ fixed by g coincides with the number of elements of $\mathcal{H}$ fixed by g. In particular, we have the equality

$$\sum_{\alpha \in \mathcal{P}} \mathbf{1}_{G_\alpha} = \sum_{\pi \in \mathcal{H}} \mathbf{1}_{G_\pi}. \tag{2.1}$$

3. Proof of Theorem 1.1

Let $\mathcal{P}^2 := \mathcal{P} \times \mathcal{P}$ be the set of ordered pairs of projective points. Let A be the $\{0,1\}$-matrix where the rows are indexed by the elements of G, the columns are indexed by the elements of $\mathcal{P}^2$ and

$$A_{g,(\alpha,\beta)} = \begin{cases} 1 & \text{if } \alpha^g = \beta, \\ 0 & \text{otherwise.} \end{cases}$$

In particular, A has $|G|$ rows and $|\mathcal{P}|^2$ columns. Of course, some ordering must be chosen for the rows and columns. We fix a particular ordering of the rows of A so that the first rows are labelled by the derangements $\mathcal{D}$ of G, and the remaining rows are labelled by the elements of $G \setminus \mathcal{D}$. With this ordering, we get that A is the following block matrix

$$A = \begin{pmatrix} M \\ B \end{pmatrix}.$$

In particular, the rows of M are labelled by elements of $\mathcal{D}$ and the columns of M are labelled by the elements of $\mathcal{P}^2$. Since the columns of A have coordinates indexed by the elements of G, we can view each column of A as an element of $\mathbb{C}[G]$. Similarly, the rows of A can be viewed as characteristic vectors in a suitable vector space, which we now introduce.

Define V to be the $\mathbb{C}$-vector space whose basis consists of all $e_{\alpha\beta}$, where $(\alpha,\beta) \in \mathcal{P}^2$. Given two subsets X and Y of $\mathcal{P}$, we write

$$e_{XY} := \sum_{(\alpha,\beta)\in X\times Y} e_{\alpha\beta}. \tag{3.1}$$

When X has cardinality one, say $X = \{\alpha\}$, we write simply $e_{\alpha Y}$, a similar comment applies when Y has cardinality one.

We now state two pivotal properties of the matrices A and M; we postpone the technical proofs of the two properties to Sections 5 and 6.

Proposition 3.1. *If S is an independent set of maximal size of Γ_G, then $\mathbf{1}_S$ is a linear combination of the columns of A. Moreover, given $\bar\alpha \in \mathcal{P}$, the subspace*

$$\langle e_{\bar\alpha\mathcal{P}} - e_{\alpha\mathcal{P}}, e_{\mathcal{P}\bar\alpha} - e_{\mathcal{P}\alpha} \mid \alpha \in \mathcal{P}\rangle$$

of V is the right kernel of A.

Proposition 3.2. *Given $\bar\alpha \in \mathcal{P}$ and $\bar\pi \in \mathcal{H}$, the subspace*

$$\langle e_{\alpha\alpha}, e_{\bar\alpha\mathcal{P}} - e_{\alpha\mathcal{P}}, e_{\mathcal{P}\bar\alpha} - e_{\mathcal{P}\alpha}, e_{\bar\pi\bar\pi} - e_{\pi\pi} \mid \alpha \in \mathcal{P}, \pi \in \mathcal{H}\rangle$$

of V is the right kernel of M.

Before proving Theorem 1.1, using these two yet unproven propositions, we need to show an elementary fact about $\mathrm{PGL}_{n+1}(q)$.

Lemma 3.3. *For every $\alpha \in \mathcal{P}$ and for every $\pi \in \mathcal{H}$, there exists $g \in G$ with g fixing only the element α of $\mathcal{P}$ and only the element π of $\mathcal{H}$.*

Proof. Observe that G acts transitively on the sets $\{(\alpha,\pi) \in \mathcal{P}\times\mathcal{H} \mid \alpha \in \pi\}$ and $\{(\alpha,\pi) \in \mathcal{P}\times\mathcal{H} \mid \alpha \notin \pi\}$. In particular, replacing α and π by α^h and π^h for some $h \in G$, we may assume that either $\alpha = \langle\varepsilon_1\rangle$ and $\pi = \langle\varepsilon_1,\dots,\varepsilon_n\rangle$, or $\alpha = \langle\varepsilon_1\rangle$ and $\pi = \langle\varepsilon_2,\dots,\varepsilon_{n+1}\rangle$. In the first case, the unipotent regular element

$$g = \begin{bmatrix} 1 & & & & \\ 1 & 1 & & & \\ & 1 & 1 & & \\ & & \ddots & \ddots & \\ & & & 1 & 1 \end{bmatrix} \in G$$

fixes only the element α of $\mathcal{P}$ and only the element π of $\mathcal{H}$. In the second case, let $C \in \mathrm{GL}_n(q)$ be a Singer cycle, that is, an element of $\mathrm{GL}_n(q)$ of order $q^n - 1$. Then

$$\begin{bmatrix} 1 & \\ & C \end{bmatrix} \in G$$

fixes only the element α of $\mathcal{P}$ and only the element π of $\mathcal{H}$. $\square$

Proof of Theorem 1.1. Let S be an independent set of maximal size of Γ_G. We aim to prove that S is a coset of the stabiliser of a point or of a hyperplane of $\mathrm{PG}^n(q)$. Up to multiplication of S by a suitable element of G, we may assume that the identity of G is in S. In particular, we have to prove that S is the stabiliser of a point or of a hyperplane of $\mathrm{PG}^n(q)$.

From Proposition 3.1, the characteristic vector $\mathbf{1}_S$ of S is a linear combination of the columns of A, and hence, for some vector $x \in V$, we have

$$\mathbf{1}_S = Ax = \begin{pmatrix} M \\ B \end{pmatrix} x = \begin{pmatrix} Mx \\ Bx \end{pmatrix}.$$

As the identity of G is in S, there are no derangements in S. Hence, by our choice of the ordering of the rows of A, we get

$$\mathbf{1}_S = \begin{pmatrix} 0 \\ t \end{pmatrix}$$

and thus $Mx = 0$ and $Bx = t$. Hence x lies in the right kernel of M. Therefore, by Proposition 3.2, given $\bar\alpha \in \mathcal{P}$ and $\bar\pi \in \mathcal{H}$, we have

$$x = \sum_{\alpha\in\mathcal{P}} c_\alpha e_{\alpha\alpha} + \sum_{\alpha\in\mathcal{P}\setminus\{\bar\alpha\}} c_\alpha^1(e_{\bar\alpha\mathcal{P}} - e_{\alpha\mathcal{P}}) + \sum_{\alpha\in\mathcal{P}\setminus\{\bar\alpha\}} c_\alpha^2(e_{\mathcal{P}\bar\alpha} - e_{\mathcal{P}\alpha}) + \sum_{\pi\in\mathcal{H}\setminus\{\bar\pi\}} c_\pi(e_{\bar\pi\bar\pi} - e_{\pi\pi}),$$

for some scalars $c_\alpha, c_\alpha^1, c_\alpha^2, c_\pi \in \mathbb{C}$.

From Proposition 3.1, the vectors $e_{\bar\alpha\mathcal{P}} - e_{\alpha\mathcal{P}}$ and $e_{\mathcal{P}\bar\alpha} - e_{\mathcal{P}\alpha}$ are in the right kernel of A, and hence $B(e_{\bar\alpha\mathcal{P}} - e_{\alpha\mathcal{P}}) = B(e_{\mathcal{P}\bar\alpha} - e_{\mathcal{P}\alpha}) = 0$. In particular,

$$t = Bx = B\left(\sum_{\alpha\in\mathcal{P}} c_\alpha e_{\alpha\alpha} + \sum_{\pi\in\mathcal{H}\setminus\{\bar\pi\}} c_\pi(e_{\bar\pi\bar\pi} - e_{\pi\pi})\right);$$

and hence we may assume that $c_\alpha^1 = c_\alpha^2 = 0$, for every $\alpha \in \mathcal{P}$.

For $\alpha \in \mathcal{P}$, we have

$$Be_{\alpha\alpha} = \mathbf{1}_{G_\alpha}.$$

Moreover, for $\pi \in \mathcal{H}$ and g and element of G that is not a derangement, we have

$$(Be_{\pi\pi})_g = \sum_{\alpha,\beta\in\pi} B_{g,(\alpha,\beta)} = \sum_{\alpha\in\pi,\alpha^g\in\pi} 1 = |\{\alpha\in\pi \mid \alpha^g\in\pi\}| = |\pi\wedge\pi^{g^{-1}}| = \begin{cases} \frac{q^n-1}{q-1} & \text{if } g \in G_\pi, \\ \frac{q^{n-1}-1}{q-1} & \text{if } g \notin G_\pi. \end{cases} \tag{3.2}$$

For the last equality observe that $\pi \wedge \pi^{g^{-1}}$ has co-dimension 2 in $\mathtt{GF}(q)^{n+1}$ when $\pi \neq \pi^g$. Thus

$$Be_{\pi\pi} = q^{n-1}\mathbf{1}_{G_\pi} + \frac{q^{n-1}-1}{q-1}\mathbf{1}_G,$$

and it follows from (3.2) that

$$B(e_{\bar\pi\bar\pi} - e_{\pi\pi}) = q^{n-1}(\mathbf{1}_{G_{\bar\pi}} - \mathbf{1}_{G_\pi}).$$

Putting these two facts together, we get

$$t = Bx = \sum_{\alpha\in\mathcal{P}} c_\alpha \mathbf{1}_{G_\alpha} + q^{n-1}\sum_{\pi\in\mathcal{H}\setminus\{\bar\pi\}} c_\pi(\mathbf{1}_{G_{\bar\pi}} - \mathbf{1}_{G_\pi}). \tag{3.3}$$

As the identity of G is in S, the coordinate of t corresponding to the identity of G is 1 and hence, using the formula in (3.3), we get

$$\sum_{\alpha\in\mathcal{P}} c_\alpha = 1. \tag{3.4}$$

Applying Lemma 3.3 to $\alpha \in \mathcal{P}$ and to the hyperplane $\bar\pi$, we get $g \in G$ that fixes only the point α and only the hyperplane $\bar\pi$. As t is a $\{0,1\}$-vector, the coordinate of t corresponding to g is either 0 or 1; by taking the coordinate corresponding to g on the right-hand side of (3.3), we get

$$c_\alpha + q^{n-1}\sum_{\pi\in\mathcal{H}\setminus\{\bar\pi\}} c_\pi \in \{0,1\}, \tag{3.5}$$

for every $\alpha \in \mathcal{P}$. Applying this argument to $\alpha \in \mathcal{P}$ and to $\pi \in \mathcal{H}\setminus\{\bar\pi\}$, we obtain

$$c_\alpha - q^{n-1}c_\pi \in \{0,1\}, \tag{3.6}$$

for every $\alpha \in \mathcal{P}$ and for every $\pi \in \mathcal{H}\setminus\{\bar\pi\}$.

Write $c := \sum_{\pi\in\mathcal{H}\setminus\{\bar\pi\}} c_\pi$. From (3.5), we have $c_\alpha = -q^{n-1}c$ or $c_\alpha = -q^{n-1}c+1$, for every $\alpha\in\mathcal{P}$. Define the sets

$$\mathcal{P}_{-q^{n-1}c} := \{\alpha\in\mathcal{P} \mid c_\alpha = -q^{n-1}c\},\quad \mathcal{P}_{-q^{n-1}c+1} := \{\alpha\in\mathcal{P} \mid c_\alpha = -q^{n-1}c+1\}.$$

Case 1: Suppose that both $\mathcal{P}_{-q^{n-1}c}$ and $\mathcal{P}_{-q^{n-1}c+1}$ are non-empty.

Let $\alpha\in\mathcal{P}_{-q^{n-1}c+1}$ and let $\beta\in\mathcal{P}_{-q^{n-1}c}$. Applying (3.6) to α and β, we get

$$-q^{n-1}c+1-q^{n-1}c_\pi\in\{0,1\},\quad -q^{n-1}c-q^{n-1}c_\pi\in\{0,1\},$$

for each $\pi\in\mathcal{H}\setminus\{\bar\pi\}$. If $-q^{n-1}c+1-q^{n-1}c_\pi=0$, for some $\pi\in\mathcal{H}\setminus\{\bar\pi\}$, then $-q^{n-1}c-q^{n-1}c_\pi=-1\notin\{0,1\}$, but this is a contradiction. Thus $-q^{n-1}c+1-q^{n-1}c_\pi=1$ and hence $c_\pi=-c$, for every $\pi\in\mathcal{H}\setminus\{\bar\pi\}$. Since $c=\sum_{\pi\in\mathcal{H}\setminus\{\bar\pi\}} c_\pi$, we obtain

$$c=\sum_{\pi\in\mathcal{H}\setminus\{\bar\pi\}}(-c) = -c(|\mathcal{H}|-1).$$

This implies $c=0$, and hence $c_\pi=0$, for each $\pi\in\mathcal{H}\setminus\{\bar\pi\}$. This shows that $t=\sum_{\alpha\in\mathcal{P}} c_\alpha \mathbf{1}_{G_\alpha}$. Now, (3.5) gives $c_\alpha\in\{0,1\}$, for every $\alpha\in\mathcal{P}$, and hence (3.4) implies that there exists a unique $\alpha'\in\mathcal{P}$ with $c_{\alpha'}=1$ and all other scalars are zero. Thus $t=\mathbf{1}_{G_{\alpha'}}$ and S is the stabiliser of the point α'. ■

Case 2: Suppose that $\mathcal{P}_{-q^{n-1}+1}=\emptyset$.

Thus $\mathcal{P}=\mathcal{P}_{-q^{n-1}c}$ and $c_\alpha=-q^{n-1}c$, for each $\alpha\in\mathcal{P}$. In particular, (3.4) gives $-q^{n-1}c|\mathcal{P}|=1$, that is,

$$c=-\frac{q^{1-n}}{|\mathcal{P}|}. \tag{3.7}$$

Moreover, (3.6) gives $-q^{n-1}c-q^{n-1}c_\pi\in\{0,1\}$, that is,

$$c_\pi\in\{-c,-c-q^{1-n}\},$$

for each $\pi\in\mathcal{H}\setminus\{\bar\pi\}$. Let a be the number of hyperplanes $\pi\in\mathcal{H}\setminus\{\bar\pi\}$ with $c_\pi=-c$ and let b be the number of hyperplanes $\pi\in\mathcal{H}\setminus\{\bar\pi\}$ with $c_\pi=-c-q^{1-n}$. Thus $a+b=|\mathcal{H}|-1$. Moreover, by the definition of c, we have

$$a(-c)+b(-c-q^{1-n})=c.$$

Putting these together, we have $-c(|\mathcal{H}|-1)-bq^{1-n}=c$, which implies $b=-cq^{n-1}|\mathcal{H}|=1$ by (3.7). In particular, there exists a unique $\pi'\in\mathcal{H}\setminus\{\bar\pi\}$ with $c_{\pi'}=-c-q^{1-n}$ and all other hyperplanes $\pi\in\mathcal{H}\setminus\{\bar\pi\}$ have $c_\pi=-c$. Therefore, using (2.1) and (3.7), we get

$$\begin{aligned} t &= -cq^{n-1}\sum_{\alpha\in\mathcal{P}}\mathbf{1}_{G_\alpha} + q^{n-1}(-c)\sum_{\pi\in\mathcal{H}\setminus\{\bar\pi\}}(\mathbf{1}_{G_{\bar\pi}}-\mathbf{1}_{G_\pi}) + q^{n-1}\cdot(-q^{1-n})(\mathbf{1}_{G_{\bar\pi}}-\mathbf{1}_{G_{\pi'}})\\ &= -cq^{n-1}\left(\sum_{\alpha\in\mathcal{P}}\mathbf{1}_{G_\alpha}-\sum_{\pi\in\mathcal{H}}\mathbf{1}_{G_\pi}\right) - q^{n-1}c\sum_{\pi\in\mathcal{H}}\mathbf{1}_{G_{\bar\pi}} - (\mathbf{1}_{G_{\bar\pi}}-\mathbf{1}_{G_{\pi'}})\\ &= -cq^{n-1}|\mathcal{H}|\mathbf{1}_{G_{\bar\pi}} - (\mathbf{1}_{G_{\bar\pi}}-\mathbf{1}_{G_{\pi'}}) = \mathbf{1}_{G_{\bar\pi}} - (\mathbf{1}_{G_{\bar\pi}}-\mathbf{1}_{G_{\pi'}}) = \mathbf{1}_{G_{\pi'}}. \end{aligned}$$

In this case, S is the stabiliser of the hyperplane π'. ■

Case 3: Suppose that $\mathcal{P}_{-q^{n-1}c}=\emptyset$.

Thus $\mathcal{P}=\mathcal{P}_{-q^{n-1}c+1}$ and $c_\alpha=-cq^{n-1}+1$, for each $\alpha\in\mathcal{P}$. In particular, (3.4) gives $(-q^{n-1}c+1)|\mathcal{P}|=1$, that is,

$$c=q^{1-n}\frac{|\mathcal{P}|-1}{|\mathcal{P}|}. \tag{3.8}$$

Furthermore, (3.6) gives $-q^{n-1}c+1-q^{n-1}c_\pi\in\{0,1\}$, that is,

$$c_\pi\in\{-c,-c+q^{1-n}\},$$

for each $\pi\in\mathcal{H}\setminus\{\bar\pi\}$. Let a be the number of hyperplanes $\pi\in\mathcal{H}\setminus\{\bar\pi\}$ with $c_\pi=-c+q^{1-n}$ and let b be the number of hyperplanes $\pi\in\mathcal{H}\setminus\{\bar\pi\}$ with $c_\pi=-c$. As in Case 2, we have the equations

$$a+b=|\mathcal{H}|-1,\quad a(-c+q^{1-n})+b(-c)=c,$$

and hence $-c(|\mathcal{H}|-1)+aq^{1-n}=c$. Thus, by (3.8), $a=cq^{n-1}|\mathcal{H}|=|\mathcal{H}|-1$ and $b=0$. Therefore, using (2.1) and (3.8), we get

$$\begin{aligned} t &= (-cq^{n-1}+1)\sum_{\alpha\in\mathcal{P}}\mathbf{1}_{G_\alpha}+q^{n-1}(-c+q^{1-n})\sum_{\pi\in\mathcal{H}\setminus\{\bar\pi\}}(\mathbf{1}_{G_{\bar\pi}}-\mathbf{1}_{G_\pi})\\ &= (-cq^{n-1}+1)\left(\sum_{\alpha\in\mathcal{P}}\mathbf{1}_{G_\alpha}-\sum_{\pi\in\mathcal{H}}\mathbf{1}_{G_\pi}\right)+(-cq^{n-1}+1)\sum_{\pi\in\mathcal{H}}\mathbf{1}_{G_{\bar\pi}}\\ &= (-cq^{n-1}+1)|\mathcal{P}|\mathbf{1}_{G_{\bar\pi}}=\mathbf{1}_{G_{\bar\pi}}. \end{aligned}$$

In this case, S is the stabiliser of the hyperplane $\bar\pi$. □

4. Permutation characters

We let $\mathrm{Irr}_{\mathbb{C}}(G)$ be the set of the irreducible complex characters of G and we denote by $\langle\cdot,\cdot\rangle_G$ the Hermitian form on the $\mathbb{C}$-space of the class functions of G. Given a subgroup H of G and a character η of H, we denote by η^G the induced character.

4.1. **Natural actions.** We turn our attention to the permutation character π of the action of $G=\mathrm{PGL}_{n+1}(q)$ on the projective points $\mathcal{P}$ of $\mathrm{PG}^n(q)$. Since G acts 2-transitively on $\mathcal{P}$, we have

$$\pi=\mathbf{1}_G+\chi_0, \tag{4.1}$$

where χ_0 is an irreducible complex character of G of degree $|\mathcal{P}|-1=(q^{n+1}-q)/(q-1)$.

We let $\mathcal{P}^{(2)}:=\{(\alpha,\beta)\in\mathcal{P}^2\mid\alpha\neq\beta\}$ be the set of distinct pairs of points from $\mathcal{P}$. Let $\pi^{(2)}$ be the permutation character of the action of $G=\mathrm{PGL}_{n+1}(q)$ on $\mathcal{P}^{(2)}$. We are interested in decomposing $\pi^{(2)}$ as a sum of irreducible complex characters of G. (This could also be inferred by the work of Green [8], but we prefer a more elementary approach.) To this end, observe that since G is transitive on $\mathcal{P}^{(2)}$, from Frobenius reciprocity, we have

$$\langle\pi^{(2)},\pi^{(2)}\rangle_G=\langle(\pi^{(2)})^2,\mathbf{1}_G\rangle_G=\langle(\pi^{(2)})^2{}_{|G_{\alpha\beta}},\mathbf{1}_{G_{\alpha\beta}}\rangle_{G_{\alpha\beta}}, \tag{4.2}$$

where $(\alpha,\beta)\in\mathcal{P}^{(2)}$ and $\mathbf{1}_{G_{\alpha\beta}}$ is the principal character of $G_{\alpha\beta}$. From the orbit counting lemma and (4.2), we deduce that $\langle\pi^{(2)},\pi^{(2)}\rangle$ equals the number of orbits of $G_{\alpha\beta}$ on $\mathcal{P}^{(2)}$. An easy computation, using the geometry of $\mathrm{PG}^n(q)$ and the action of $G=\mathrm{PGL}_{n+1}(q)$ on $\mathrm{PG}^n(q)$, shows that

$$\langle\pi^{(2)},\pi^{(2)}\rangle_G=\begin{cases}q+14 & \text{when } n\geq 3,\\ q+13 & \text{when } n=2,\\ q+4 & \text{when } n=1.\end{cases} \tag{4.3}$$

Indeed, the orbits of $G_{\alpha\beta}$ on $\mathcal{P}^{(2)}$ are (compare also with Figure 1):

`Case 1:` $\{(\alpha,\beta)\}$,
`Case 2:` $\{(\alpha,\delta)\mid\delta\in\alpha\vee\beta\setminus\{\alpha,\beta\}\}$,
`Case 3:` $\{(\alpha,\delta)\mid\delta\notin\alpha\vee\beta\}$ (this arises when $n\geq 2$),
`Case 4:` $\{(\beta,\alpha)\}$,
`Case 5:` $\{(\beta,\delta)\mid\delta\in\alpha\vee\beta\setminus\{\alpha,\beta\}\}$,
`Case 6:` $\{(\beta,\delta)\mid\delta\notin\alpha\vee\beta\}$ (this arises when $n\geq 2$),
`Case 7:` $\{(\gamma,\alpha)\mid\gamma\in\alpha\vee\beta\setminus\{\alpha,\beta\}\}$,
`Case 8:` $\{(\gamma,\beta)\mid\gamma\in\alpha\vee\beta\setminus\{\alpha,\beta\}\}$,
`Case 9:` fix $\gamma\in\alpha\vee\beta\setminus\{\alpha,\beta\}$, for each $\delta\in\alpha\vee\beta\setminus\{\alpha,\beta,\gamma\}$, $\{(\gamma^x,\delta^x)\mid x\in G_{\alpha\beta}\}$ (there are $q-2$ orbits of this type, in particular, when $q=2$, there are none),
`Case 10:` $\{(\gamma,\delta)\mid\gamma\in\alpha\vee\beta\setminus\{\alpha,\beta\},\delta\notin\alpha\vee\beta\}$ (this arises when $n\geq 2$),
`Case 11:` $\{(\gamma,\alpha)\mid\gamma\notin\alpha\vee\beta\}$ (this arises when $n\geq 2$),
`Case 12:` $\{(\gamma,\beta)\mid\gamma\notin\alpha\vee\beta\}$ (this arises when $n\geq 2$),
`Case 13:` $\{(\gamma,\delta)\mid\gamma\notin\alpha\vee\beta,\delta\in\alpha\vee\beta\setminus\{\alpha,\beta\}\}$ (this arises when $n\geq 2$),
`Case 14:` $\{(\gamma,\delta)\mid\gamma\notin\alpha\vee\beta,\delta\in\alpha\vee\gamma\setminus\{\alpha,\gamma\}\}$ (this arises when $n\geq 2$),
`Case 15:` $\{(\gamma,\delta)\mid\gamma\notin\alpha\vee\beta,\delta\in\beta\vee\gamma\setminus\{\beta,\gamma\}\}$ (this arises when $n\geq 2$),
`Case 16:` $\{(\gamma,\delta)\mid\gamma\notin\alpha\vee\beta,\delta\in\alpha\vee\beta\vee\gamma\setminus\{(\alpha\vee\beta)\cup(\alpha\vee\gamma)\cup(\beta\vee\gamma)\}\}$ (this arises when $n\geq 2$),
`Case 17:` $\{(\gamma,\delta)\mid\gamma\notin\alpha\vee\beta,\delta\notin\alpha\vee\beta\vee\gamma\}$ (this arises when $n\geq 3$).

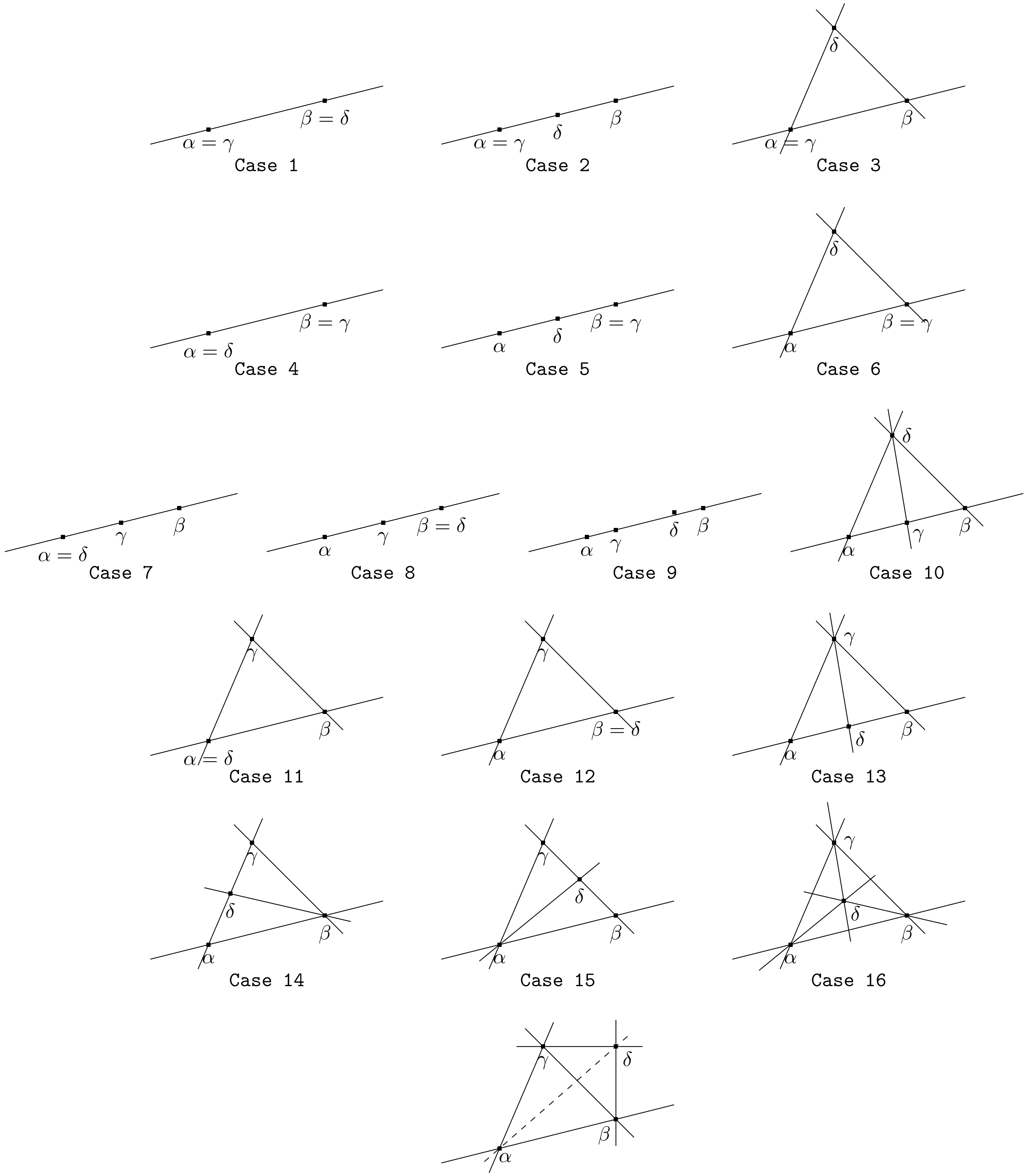

FIGURE 1. Configurations for the orbits of $G_{\alpha\beta}$ on ordered pairs of distinct points

Let α and β be two distinct points of $\mathcal{P}$, let $H := \{g \in G \mid (\alpha \vee \beta)^g = \alpha \vee \beta\}$ be the setwise stabiliser of the line $\alpha \vee \beta$, let $K := G_{\alpha\beta}$ and let

$$\psi := \mathbf{1}_K^H.$$

Then ψ is the permutation character of the action of H on the right cosets of K. By definition of H and K, the permutation group induced by this action is permutation isomorphic to $\mathrm{PGL}_2(q)$ in its natural action on the ordered pairs of distinct points of the projective line $\mathrm{PG}^1(q)$. Therefore, using the character table of $\mathrm{PGL}_2(q)$ (see [12]), we see that, when q is odd, ψ decomposes as

$$\psi = \mathbf{1}_H + (2\psi_{1,q} + \psi_{2,q}) + (\psi_{1,q-1} + \cdots + \psi_{\frac{q-1}{2},q-1}) + (\psi_{1,q+1} + \cdots + \psi_{\frac{q-3}{2},q+1})$$

and, when q is even, ψ decomposes as

$$\psi = \mathbf{1}_H + 2\psi_{1,q} + (\psi_{1,q-1} + \cdots + \psi_{\frac{q}{2},q-1}) + (\psi_{1,q+1} + \cdots + \psi_{\frac{q}{2}-1,q+1}).$$

Here, $(\psi_{i,j})_{i,j}$ denote irreducible complex characters of H having the pointwise stabiliser of the line $\alpha \vee \beta$ in their kernel and having degree j. Observe that this is consistent with (4.3) when $n = 1$, because regardless of whether q is odd or even, we have $\langle \psi, \psi \rangle_H = q + 4$. To include both cases in our argument, we write

$$\psi := \mathbf{1}_H + 2\eta + \psi_1 + \cdots + \psi_{q-1}, \tag{4.4}$$

where η is the only irreducible component of ψ having multiplicity 2, $\psi_1, \ldots, \psi_{q-1}$ are distinct irreducible characters and $\eta(1) = q$.

From (4.4), we have

$$\pi^{(2)} = \mathbf{1}_H^G + 2\eta^G + \psi_1^G + \cdots + \psi_{q-1}^G. \tag{4.5}$$

Recall $n \geq 3$. As $\mathbf{1}_H^G$ is the permutation character of the action of $\mathrm{PGL}_{n+1}(q)$ on the projective lines of $\mathrm{PG}^n(q)$ and since this action has rank three (when $n \geq 3$), we have

$$\mathbf{1}_H^G = \mathbf{1}_G + \chi + \chi',$$

for some $\chi, \chi' \in \mathrm{Irr}_{\mathbb{C}}(G)$. Elementary considerations yield that the degrees of the constituents of $\mathbf{1}_H^G$ are

$$1, \frac{q^{n+1} - q}{q - 1}, \frac{(q^{n+1} - 1)(q^n - q)}{(q - 1)(q^2 - 1)}.$$

Replacing χ by χ' if necessary, we may assume that χ has degree $(q^{n+1} - q)/(q - 1)$. Observe that G_α has two orbits on lines and hence $\langle (\mathbf{1}_H^G)_{|G_\alpha}, \mathbf{1}_{G_\alpha} \rangle_{G_\alpha} = 2$. Therefore, from Frobenius reciprocity, $\langle \mathbf{1}_H^G, \pi \rangle_G = 2$ and we deduce $\chi = \chi_0$ from (4.1). Thus

$$\mathbf{1}_H^G = \mathbf{1}_G + \chi_0 + \chi'.$$

Now, for every $i \in \{1, \ldots, q - 1\}$, using again the character table of $\mathrm{PGL}_2(q)$, we deduce

$$\langle \psi_i^G, \chi_0 \rangle_G = \langle \psi_i, (\chi_0)_{|H} \rangle_H = 0. \tag{4.6}$$

Since G acts transitively on $\mathcal{P}^{(2)}$, we have

$$\langle \pi^{(2)}, \mathbf{1}_G \rangle_G = 1. \tag{4.7}$$

From Frobenius reciprocity, $\langle \pi^{(2)}, \pi \rangle_G = \langle \mathbf{1}_{G_{\alpha\beta}}, \pi_{|G_{\alpha\beta}} \rangle_{G_{\alpha\beta}}$. Since $G_{\alpha\beta}$ has four orbits on $\mathcal{P}$ (namely, $\{\alpha\}$, $\{\beta\}$, $\alpha \vee \beta \setminus \{\alpha, \beta\}$ and $\mathcal{P} \setminus \alpha \vee \beta$), we get

$$\langle \pi^{(2)}, \pi \rangle_G = 4. \tag{4.8}$$

Now, from (4.1), (4.7) and (4.8), we deduce

$$\langle \pi^{(2)}, \chi_0 \rangle_G = 3. \tag{4.9}$$

From (4.5), (4.6) and (4.9), we deduce that χ_0 is a constituent of η^G and hence $\eta^G = \chi_0 + \chi''$ for some $\chi'' \in \mathrm{Irr}_{\mathbb{C}}(G)$. Since η has degree q, η^G has degree $q|G/H| = q|\mathcal{L}| = (q^{n+1} - 1)(q^{n+1} - q)/((q - 1)(q^2 - 1))$. Since χ_0 has degree $(q^{n+1} - q)/(q - 1)$, we get

$$\chi''(1) = \eta^G(1) - \chi_0(1) = q^3 \frac{(q^n - 1)(q^{n-1} - 1)}{(q - 1)(q^2 - 1)}.$$

Moreover,

$$\pi^{(2)} = \mathbf{1}_G + 3\chi_0 + \chi' + 2\chi'' + \psi_1^G + \cdots + \psi_{q-1}^G.$$

Since $\langle \pi^{(2)}, \pi^{(2)} \rangle_G = q + 16$ from (4.3) and since $1^2 + 3^2 + 1^2 + 2^2 + (q + 1) \cdot 1^2 = q + 16$, we deduce that $\mathbf{1}_G, \chi_0, \chi', \chi'', \psi_1^G, \ldots, \psi_{q-1}^G$ are distinct irreducible complex characters of G and are the constituents of G.

We sum up in the following lemma what we have shown so far.

Lemma 4.1. *Let π be the permutation character of the action of G on $\mathcal{P}$, let $\pi^{(2)}$ be the permutation character of the action of G on $\mathcal{P}^{(2)}$ and let π_ℓ be the permutation character of the action of G on $\mathcal{L}$. Then, π, $\pi^{(2)}$ and π_ℓ decompose as the sum of irreducible complex characters as follows:*

$$\begin{aligned}\pi &= \mathbf{1}_G + \chi_0,\\ \pi^{(2)} &= \mathbf{1}_G + 3\chi_0 + \chi' + 2\chi'' + \psi_1^G + \psi_2^G + \cdots + \psi_{q-1}^G,\\ \pi_\ell &= \mathbf{1}_G + \chi_0 + \chi',\end{aligned}$$

where $\mathbf{1}_G, \chi_0, \chi', \chi'', \psi_1^G, \psi_2^G, \ldots, \psi_{q-1}^G$ are distinct irreducible complex characters. The characters $\psi_1^G, \psi_2^G, \ldots, \psi_{q-1}^G$ are induced from irreducible complex characters of the setwise stabiliser of a line. Moreover, χ'' has degree $q^3((q^n - 1)(q^{n-1}-1))/((q-1)(q^2-1))$.

4.2. **Permutation character induced from a Singer cycle.** We let c be an element of order $\frac{q^{n+1}-1}{q-1} = |\mathcal{P}|$ in G, that is,

(4.10) $\qquad$ c is a Singer cycle of the projective general linear group $\mathrm{PGL}_{n+1}(q)$ and $C := \langle c\rangle$.

The proof of the following lemma is rather short, but it uses a deep result of Zalesski on $\mathbf{1}_C^G$. (The author wishes to express his gratitude to Alex Zalesski for proving [19, Theorem 1.1], our argument for proving Theorem 1.1 heavily depends on [19] via Lemma 4.2.)

Lemma 4.2. *Suppose $(n, q) \neq (2, 2)$. Let $\eta \in \mathrm{Irr}_{\mathbb{C}}(G)$ with $\langle \mathbf{1}_C^G - \mathbf{1}_G, \eta\rangle_G = 0$. Then either $\eta(1) = 1$ or $\eta = \chi_0$ (recall that χ_0 is the non-principal constituent of the permutation character of G in its action on the points of $\mathrm{PG}^n(q)$).*

Proof. From [19, Theorem 1.1], we see that, when $(n, q) \neq (2, 2)$, if $\langle \mathbf{1}_C^G - \mathbf{1}_G, \eta\rangle_G = 0$, then $\eta(1) = 1$ or $\eta(1) = (q^{n+1}-q)/(q-1)$. In particular, to conclude the proof of this lemma, we need to consider the case that $\eta(1) = (q^{n+1}-q)/(q-1)$ and we need to show that $\eta = \chi_0$. Note that all irreducible characters of G of degree $(q^{n+1}-q)/(q-1)$ differ by a multiple which is a linear character, see [19, page 524] or [9, Table II]. Therefore, $\eta = \xi\chi_0$, where ξ is a linear character of G.

Suppose $\xi \neq \mathbf{1}_G$. As $\mathbf{1}_G$ is not a constituent of the irreducible character $\xi\chi_0$, we have

$$\begin{aligned}\langle \mathbf{1}_C^G - \mathbf{1}_G, \xi\chi_0\rangle_G &= \langle \mathbf{1}_C^G, \xi\chi_0\rangle_G = \langle \mathbf{1}_C, (\xi\chi_0)_{|C}\rangle_C = \frac{1}{|C|}\sum_{x\in C}\xi(x)\chi_0(x) = \frac{1}{|C|}\left(\frac{q^{n+1}-q}{q-1} - \sum_{x\in C\setminus\{1\}}\xi(x)\right)\\ &= \frac{1}{|C|}\left(\frac{q^{n+1}-q}{q-1} + 1 - \sum_{x\in C}\xi(x)\right) = \frac{1}{|C|}\left(\frac{q^{n+1}-1}{q-1} - \langle \xi, \mathbf{1}_C\rangle_C\right) = \frac{1}{|C|}\frac{q^{n+1}-1}{q-1} \neq 0.\end{aligned}$$

□

Proposition 4.5 is yet another technical result concerning characters that we need to prove. This result seems out of context, but it will be useful for computing the eigenvalues of a certain matrix. Let $(\alpha, \beta) \in \mathcal{P}^{(2)}$. We are interested in

$$\frac{\eta(1)}{|G|}\sum_{g\in G_{\alpha\beta}C}\eta(g^{-1}),$$

where $\eta \in \mathrm{Irr}_{\mathbb{C}}(G)$. Here, the results in [11, Section 4] are relevant. For the reader's benefit, we report some results from [11] tailored to our application.

Lemma 4.3. *Let χ be a character of G and let H be a subgroup of G with $\langle \mathbf{1}_{|H}, \mathbf{1}_H\rangle_H = 1$. Let $\mathcal{X}$ be a representation affording χ such that $\mathcal{X}$ restricted to H has the form:*

$$\chi(h) = \begin{pmatrix}1 & 0\\ 0 & \mathcal{Y}(h)\end{pmatrix},$$

for every $h \in H$, where $\mathcal{Y}$ is a representation of H. Let $x \in G$ and suppose that the entry in row 1 and column 1 of $\mathcal{X}(x)$ is $a \in \mathbb{C}$. Then

$$\sum_{h\in H}\chi(xh) = |H|a.$$

Proof. This is Corollary 4.2 in [11]. □

Lemma 4.4. *Let χ be a character of G, let H be a subgroup of G with $\langle \mathbf{1}_{|H}, \mathbf{1}_H\rangle_H = 0$ and let $x \in G$. Then*

$$\sum_{h\in H}\chi(xh) = 0.$$

Proof. This follows from the proof of Lemma 4.1 and the proof of Corollary 4.2 in [11]. □

Proposition 4.5. *Let η be a constituent of $\pi^{(2)}$ having multiplicity 1 with $\eta \neq \chi'$ (see Lemma 4.1 for the definition of χ'). Then*

$$\frac{\eta(1)}{|G|}\sum_{g\in G_{\alpha\beta}C}\eta(g^{-1})\neq 0.$$

Proof. Fix $(\alpha,\beta)\in\mathcal{P}^{(2)}$. Observe that $\langle\pi^{(2)},\eta\rangle_G=\langle\mathbf{1}_{G_{\alpha\beta}}^G,\eta\rangle_G$ and hence $\langle\eta_{|G_{\alpha\beta}},\mathbf{1}_{G_{\alpha\beta}}\rangle_{G_{\alpha\beta}}=1$ from Frobenius reciprocity. Moreover,

$$\frac{\eta(1)}{|G|}\sum_{g\in G_{\alpha\beta}C}\eta(g^{-1})=\frac{\eta(1)}{|G|}\sum_{x\in C}\left(\sum_{h\in G_{\alpha\beta}}\eta(xh)\right). \tag{4.11}$$

Therefore we are in the position to apply Lemma 4.3 with $\chi=\eta$ and $H=G_{\alpha\beta}$. Before doing so, observe that the statement of the lemma is obvious when $\eta=\mathbf{1}_G$. Now, we assume $\eta\neq\mathbf{1}_G$ and we let $\mathcal{X}$ be a representation affording η.

We now use Lemma 4.1. Suppose $\eta=\psi_i^G$, for some $i\in\{1,\ldots,q-1\}$. It is clear from the definition of induced character applied to ψ_i^G and to its representation $\mathcal{X}$ that the $(1,1)$-entry of the matrix $\mathcal{X}(x)$ is zero, unless $x=1$. Therefore, from (4.11),

$$\frac{\psi_i^G(1)}{|G|}\sum_{g\in G_{\alpha\beta}C}\psi_i^G(g^{-1})=\frac{\psi_i^G(1)}{|G|}\sum_{h\in G_{\alpha\beta}}\psi_i^G(h)=\frac{\psi_i^G(1)|G_{\alpha\beta}|}{|G|}\langle(\psi_i^G)_{|G_{\alpha\beta}},\mathbf{1}_{G_{\alpha\beta}}\rangle_{G_{\alpha\beta}}=\frac{\psi_i^G(1)|G_{\alpha\beta}|}{|G|}\langle\psi_i^G,\pi^{(2)}\rangle_G\neq 0,$$

where in the last equality we have used, as usual, Frobenius reciprocity. □

We were not able to prove Proposition 4.5 when $\eta=\chi'$, although we have strong computational evidence that the result holds also in this case. We pay this deficiency by having an ad-hoc argument for χ' in Section 6.

5. Proof of Proposition 3.1

We start with an elementary observation.

Lemma 5.1. *Let S be an intersecting set of maximal size of G. Then $|S|=|G|/|\mathcal{P}|$.*

Proof. Let C be as in (4.10). Then C is a clique and S is an independent set of Γ_S. From the clique-coclique bound, $|G|\geq|S||C|=|S||\mathcal{P}|$ and hence $|S|\leq|G|/|\mathcal{P}|$. Since a point-stabiliser is an independent set of cardinality $|G|/|\mathcal{P}|$, the proof of the lemma follows. □

It is elementary to see that A has rank $(|\mathcal{P}|-1)^2+1$, for instance, this can be shown exactly as in [15, Proposition 3.2]. Here we present a slightly longer and difficult proof, however this detour has the advantage of proving also the stronger statement in Proposition 3.1.

Let J be the subspace of $\mathbb{C}[G]$ spanned by the characteristic vectors $\mathbf{1}_S$ of the independent sets S of maximal size of Γ_G, and let Z be the subspace of $\mathbb{C}[G]$ spanned by the columns of A.

As each column of A is the characteristic vector of a coset of the stabiliser of a point, and hence the characteristic vector of an independent set of maximal size by Lemma 5.1, Z is a subspace of J, that is, $Z\leq J$.

We now use the algebra structure of $\mathbb{C}[G]$. We claim that

$$J \text{ and } Z \text{ are ideals of } \mathbb{C}[G].$$

Indeed, for every independent set S of maximal size of Γ_G and for every $g\in G$, we have $\mathbf{1}_Sg=\mathbf{1}_{Sg}$ and $g\mathbf{1}_S=\mathbf{1}_{gS}$. Since Sg and gS are both independent sets of maximal size of Γ_G, we deduce $\mathbf{1}_Sg, g\mathbf{1}_S\in J$. Therefore, J is an ideal of $\mathbb{C}[G]$. A similar argument yields that Z is also an ideal of $\mathbb{C}[G]$.

We now recall some basic facts on group algebras, see [10]. The minimal ideals $(I_\eta)_{\eta\in\mathrm{Irr}_{\mathbb{C}}(G)}$ of $\mathbb{C}[G]$ are indexed by the irreducible complex characters $\mathrm{Irr}_{\mathbb{C}}(G)$ and we have the direct sum decomposition

$$\mathbb{C}[G]=\bigoplus_{\eta\in\mathrm{Irr}_{\mathbb{C}}(G)}I_\eta. \tag{5.1}$$

For each $\eta\in\mathrm{Irr}_{\mathbb{C}}(G)$, we have $\dim_{\mathbb{C}}(I_\eta)=\eta(1)^2$ and the minimal ideal I_η contains the principal idempotent element

$$e_\eta:=\frac{\eta(1)}{|G|}\sum_{g\in G}\eta(g^{-1})g. \tag{5.2}$$

In particular, for each $\eta, \eta' \in \mathrm{Irr}_{\mathbb{C}}(G)$, we have

$$e_\eta e_{\eta'} = \begin{cases} e_\eta & \text{when } \eta = \eta', \\ 0 & \text{when } \eta \neq \eta'. \end{cases}$$

Since $\mathbb{C}[G]$ is a semi-simple algebra,

$$J = \bigoplus_{\eta \in \mathcal{J}} I_\eta, \tag{5.3}$$

for some subset $\mathcal{J}$ of $\mathrm{Irr}_{\mathbb{C}}(G)$. Similarly,

$$Z = \bigoplus_{\eta \in \mathcal{Z}} I_\eta,$$

for some subset $\mathcal{Z}$ of $\mathrm{Irr}_{\mathbb{C}}(G)$.

Lemma 5.2. *$\mathcal{Z} = \{\mathbf{1}, \chi_0\}$ and A has rank $(|\mathcal{P}|-1)^2+1$. In particular, $\langle e_{\bar{\alpha}\mathcal{P}} - e_{\alpha\mathcal{P}}, e_{\mathcal{P}\bar{\alpha}} - e_{\mathcal{P}\alpha} \mid \alpha \in \mathcal{P}\rangle$ is the right kernel of A.*

Proof. Fix $\alpha \in \mathcal{P}$ and $x \in G$. Let $\eta \in \mathrm{Irr}_{\mathbb{C}}(G)$. We have

$$\begin{aligned} e_\eta \cdot \mathbf{1}_{G_\alpha x} &= \frac{\eta(1)}{|G|} \sum_{g \in G} \eta(g^{-1}) \mathbf{1}_{gG_\alpha x} = \frac{\eta(1)}{|G|} \sum_{g \in G} \eta(g^{-1}) \left(\sum_{h \in gG_\alpha x} h \right) \\ &= \frac{\eta(1)}{|G|} \sum_{h \in G} \left(\sum_{g \in hG_\alpha x} \eta(g^{-1}) \right) h = \frac{\eta(1)}{|G|} \sum_{h \in G} \left(\sum_{g \in hxG_\alpha} \eta(g^{-1}) \right) h, \end{aligned} \tag{5.4}$$

where in the last equality we used the fact that η is a class function and hence $\eta((hyx)^{-1}) = \eta(x^{-1}y^{-1}h^{-1}) = \eta(y^{-1}x^{-1}h^{-1}) = \eta((hxy)^{-1})$, for every y.

If $\langle \eta_{|G_\alpha}, \mathbf{1}_{G_\alpha} \rangle_{G_\alpha} = 0$, then from Lemma 4.4 and (5.4), we deduce $e_\eta \cdot \mathbf{1}_{G_\alpha x} = 0$. As $\mathbf{1}_{G_\alpha x}$ is an arbitrary generator of Z, we deduce $\mathcal{Z} \subseteq \{\eta \in \mathrm{Irr}_{\mathbb{C}}(G) \mid \langle \eta_{|G_\alpha}, \mathbf{1}_{G_\alpha}\rangle \neq 0\}$. From Frobenius reciprocity and (4.1), $\mathcal{Z} \subseteq \{\mathbf{1}_G, \chi_0\}$.

When $\eta = \mathbf{1}_G$, we have $e_{\mathbf{1}_G} \cdot \mathbf{1}_{G_\alpha x} = |G_\alpha| e_{\mathbf{1}_G} \neq 0$. Finally, when $\eta = \chi_0$, an application of Lemma 4.3 to (5.4) yields

$$\begin{aligned} e_{\chi_0} \cdot \mathbf{1}_{G_\alpha x} &= \frac{\chi_0(1)}{|G|} \sum_{h \in xG_\alpha} \left(\sum_{g \in x^{-1}hG_\alpha} \chi_0(g^{-1}) \right) h = \frac{|\mathcal{P}|-1}{|G|} \sum_{h \in xG_\alpha} \left(\sum_{g \in G_\alpha} \chi_0(g) \right) h \\ &= \frac{|\mathcal{P}|-1}{|G|} \sum_{h \in xG_\alpha} h = \frac{|\mathcal{P}|-1}{|G|} \mathbf{1}_{xG_\alpha} \neq 0. \end{aligned}$$

Thus $\mathcal{Z} = \{\mathbf{1}_G, \chi_0\}$ and $\dim(Z) = \deg(\mathbf{1}_G)^2 + \deg(\chi_0)^2 = 1 + (|\mathcal{P}|-1)^2$.

Since A has $|\mathcal{P}|^2$ columns, the right kernel of A has dimension $|\mathcal{P}|^2 - ((|\mathcal{P}|-1)^2+1) = 2(|\mathcal{P}|-1)$.

An elementary computation shows that $\langle e_{\bar{\alpha}\mathcal{P}} - e_{\alpha\mathcal{P}}, e_{\mathcal{P}\bar{\alpha}} - e_{\mathcal{P}\alpha} \mid \alpha \in \mathcal{P}\rangle$ is contained in the right kernel of A and has dimension $2(|\mathcal{P}|-1)$; thus the lemma is proven. □

The argument for determining $\mathcal{J}$, and hence J, is similar to the argument in Lemma 5.2.

Proof of Proposition 3.1. From Lemma 5.2 and the discussion above, it remains to show that $Z = I$.

Let I be the ideal of $\mathbb{C}[G]$ generated by $\mathbf{1}_C^G - \mathbf{1}_G$. From [10, Chapter 1], we have

$$I = \bigoplus_{\substack{\eta \in \mathrm{Irr}_{\mathbb{C}}(G) \\ \langle \mathbf{1}_C^G - \mathbf{1}_G, \eta\rangle_G \neq 0}} I_\eta. \tag{5.5}$$

Let S be an independent set of maximal size of G. Then, from Lemma 5.1,

$$\langle \mathbf{1}_S, \mathbf{1}_C^G - \mathbf{1}_G \rangle_G = \langle \mathbf{1}_S, \mathbf{1}_C^G \rangle_G - \langle \mathbf{1}_S, \mathbf{1}_G \rangle_G = \langle (\mathbf{1}_S)_{|C}, \mathbf{1}_C \rangle_C - \frac{|S|}{|G|} = \frac{1}{|C|} - \frac{1}{|C|} = 0. \tag{5.6}$$

In particular, from (5.3), (5.5) and (5.6), we deduce

$$\mathcal{J} \subseteq \{\eta \in \mathrm{Irr}_{\mathbb{C}}(G) \mid \langle 1_C^G - \mathbf{1}, \eta \rangle = 0\}.$$

Now, Lemma 4.2 gives $\mathcal{J} \subseteq \{\chi_0, \xi \mid \xi \in \mathrm{Irr}_{\mathbb{C}}(G), \xi(1) = 1\}$.

Let $\xi \in \mathrm{Irr}_{\mathbb{C}}(G)$ with $\xi \neq \mathbf{1}_G$ and $\xi(1) = 1$, and let S be an independent set of maximal size of G. Let $T := \mathrm{PSL}_{n+1}(q)$, let $Tx_1, \ldots, Tx_d$ be the right cosets of T in G, where $d := \gcd(n, q-1)$, and let $S_i := Tx_i \cap S$ for

$i \in \{1, \ldots, d\}$. Observe that $S_i := (S \cap Tx_i)x_i^{-1}$ are intersecting sets for $T = \mathrm{PSL}_{n+1}(q)$, for every $i \in \{1, \ldots, d\}$. From [17], an intersecting set of maximal size of T has cardinality $|T|/|\mathcal{P}|$. Therefore,

$$|S \cap Tx_i| = |S_i| \leq \frac{|T|}{|\mathcal{P}|} = \frac{|G|}{d|\mathcal{P}|}.$$

As $S = S_1 \cup S_2 \cup \cdots \cup S_d$, we deduce $|S_i| = |S|/d$ for each i, that is, S is equally distributed among the cosets of $\mathrm{PSL}_{n+1}(q)$ in $\mathrm{PGL}_{n+1}(q)$. Since ξ contains T in its kernel, an easy computation yields $\langle \xi, \mathbf{1}_S \rangle_G = 0$. Since this argument is independent on ξ and S, we obtain $\mathcal{J} \subseteq \{\mathbf{1}_G, \chi_0\}$ and hence $J \leq= I_{\mathbf{1}_G} \oplus I_{\chi_0} = Z$. Since $Z \leq J$, we finally have $Z = J$. □

6. Proof of Proposition 3.2

For simplifying some of the notation later, we define

$$u := \frac{|G|}{|\mathcal{P}|(|\mathcal{P}|-1)}, \quad v := \frac{|G|}{|\mathcal{P}|(|\mathcal{P}|-1)(|\mathcal{P}|-q-1)}. \tag{6.1}$$

Recall that from (4.10), c is a Singer cycle of the projective general linear group $\mathrm{PGL}_{n+1}(q)$ and $C = \langle c \rangle$. We let $\mathcal{C}$ denote the $\mathrm{PGL}_{n+1}(q)$-conjugacy class of c.

Since C acts transitively on $\mathcal{P}$, from the Frattini argument, we have

$$G = CG_\alpha = G_\alpha C,$$

for each $\alpha \in \mathcal{P}$. Moreover, since C is abelian, $\mathbf{C}_G(C) = C$ and so $C \cap G_\alpha = \mathbf{C}_G(C) \cap G_\alpha = 1$, that is,

$$\mathcal{C} = \{c^x \mid x \in G_\alpha\}, \quad |\mathcal{C}| = |G_\alpha| = \frac{|G|}{|\mathcal{P}|}. \tag{6.2}$$

Given two distinct points α and β of $\mathrm{PG}^n(q)$, we define

$$\mathcal{C}_{\alpha\beta} := \{g \in \mathcal{C} \mid \alpha^g = \beta\}.$$

Clearly, $\mathcal{C}_{\alpha\beta} \neq \emptyset$. In fact, given $g \in \mathcal{C}$, since G_α is transitive on $\mathcal{P} \setminus \{\alpha\}$, there exists $x \in G_\alpha$ with $(\alpha^g)^x = \beta$, that is, $\alpha^{x^{-1}gx} = \beta$ and $x^{-1}gx \in \mathcal{C}_{\alpha\beta}$. Fix, once and for all, an arbitrary element

$$g_{\alpha\beta} \in \mathcal{C}_{\alpha\beta}$$

and let $x \in G_\alpha$. Then, $g_{\alpha\beta}^x \in \mathcal{C}_{\alpha\beta}$ if and only if $\alpha^{g_{\alpha\beta}^x} = \beta$; as $\alpha^x = \alpha$, this happens only when $\beta^x = \beta$, that is, $x \in G_{\alpha\beta}$. Combining this with (6.1) and (6.2), we have

$$\mathcal{C}_{\alpha\beta} = \{g_{\alpha\beta}^x \mid x \in G_{\alpha\beta}\}, \quad |\mathcal{C}_{\alpha\beta}| = |G_{\alpha\beta}| = \frac{|G|}{|\mathcal{P}|(|\mathcal{P}|-1)} = u. \tag{6.3}$$

6.1. **Two important matrices.** We now introduce two matrices $\mathscr{M}$ and $\mathscr{C}$ of paramount importance for our argument. The rows of $\mathscr{M}$ are index by the elements of $\mathcal{C}$ (or by the elements of G_α in view of (6.2)) and the columns of $\mathscr{M}$ are indexed by the elements of $\mathcal{P}^{(2)}$ and, given $g \in \mathcal{C}$ and $(\alpha, \beta) \in \mathcal{P}^{(2)}$, we have

$$\mathscr{M}_{g,(\alpha,\beta)} := \begin{cases} 1 & \text{if } \alpha^g = \beta, \\ 0 & \text{if } \alpha^g \neq \beta. \end{cases}$$

Then we define

$$\mathscr{C} := \mathscr{M}^T \mathscr{M}.$$

Therefore $\mathscr{C}$ is a symmetric matrix with $|\mathcal{P}^{(2)}| = |\mathcal{P}|(|\mathcal{P}|-1)$ rows. Moreover, given $(\alpha, \beta), (\gamma, \delta) \in \mathcal{P}^{(2)}$, we have

$$\mathscr{C}_{(\alpha,\beta),(\gamma,\delta)} = |\{g \in \mathcal{C} \mid \alpha^g = \beta, \gamma^g = \delta\}| = |\{g \in \mathcal{C}_{\alpha\beta} \mid \gamma^g = \delta\}|. \tag{6.4}$$

In the rest of this section, we are interested in computing the entries of $\mathscr{C}$. Before embarking in this task, some remarks are necessary for simplifying some of the computations. It is clear from the definition that

$$\mathscr{C}_{(\alpha,\beta),(\gamma,\delta)} = \mathscr{C}_{(\gamma,\delta),(\alpha,\beta)}. \tag{6.5}$$

Although, $\mathcal{C}$ is not closed by inversion, that is, c is not necessarily conjugate to c^{-1} in G, we still have

$$\mathscr{C}_{(\alpha,\beta),(\gamma,\delta)} = \mathscr{C}_{(\beta,\alpha),(\delta,\gamma)}. \tag{6.6}$$

This can be shown observing that c is conjugate to c^{-1} via the inverse-transposed automorphism of G. Set $\mathcal{G}_{\alpha\beta\gamma\delta} := \{x \in G_{\alpha\beta} \mid \gamma^{x^{-1}g_{\alpha\beta}x} = \delta\}$. Then, by (6.3), we have

$$\mathcal{C}_{(\alpha,\beta),(\gamma,\delta)} = |\mathcal{G}_{\alpha\beta\gamma\delta}|. \tag{6.7}$$

An easy computation gives $\mathcal{G}_{\alpha\beta\gamma\delta}G_{\alpha\beta\gamma\delta} := \{gx \mid g \in \mathcal{G}_{\alpha\beta\gamma\delta}, x \in G_{\alpha\beta\gamma\delta}\} = \mathcal{G}_{\alpha\beta\gamma\delta}$ and hence

$$\mathcal{G}_{\alpha\beta\gamma\delta} \text{ is a union of left } G_{\alpha\beta\gamma\delta}\text{-cosets.} \tag{6.8}$$

There is one more elementary fact we aim to observe:

$$\text{if } x \in \mathcal{G}_{\alpha\beta\gamma\delta}, \text{ then } xG_{\alpha\beta\gamma\delta} = xG_{\alpha\beta\gamma} \cap \mathcal{G}_{\alpha\beta\gamma\delta} = xG_{\alpha\beta\delta} \cap \mathcal{G}_{\alpha\beta\gamma\delta} = xG_{\alpha\gamma\delta} \cap \mathcal{G}_{\alpha\beta\gamma\delta} = xG_{\beta\gamma\delta} \cap \mathcal{G}_{\alpha\beta\gamma\delta}. \tag{6.9}$$

Proposition 6.1. *Let (α, β) and (γ, δ) be two pairs of distinct points of $\mathrm{PG}^n(q)$, and consider Figure 1. Then we have*

$$\mathscr{C}_{(\alpha,\beta),(\gamma,\delta)} = \begin{cases} u & \text{when } \alpha,\beta,\gamma,\delta \text{ are as in } \texttt{Case 1}, \\ 0 & \text{when } \alpha,\beta,\gamma,\delta \text{ are as in } \texttt{Cases 2, 3, 4, 5, 7, 8, 9, 12}, \\ v & \text{when } \alpha,\beta,\gamma,\delta \text{ are as in } \texttt{Cases 6, 10, 11, 13, 14, 15, 17}, \\ \frac{q-2}{q-1}v & \text{when } \alpha,\beta,\gamma,\delta \text{ are as in } \texttt{Case 16}. \end{cases}$$

Proof. When $\alpha, \beta, \gamma, \delta$ are as in `Case 1`, the proof follows from (6.3).

When $\alpha, \beta, \gamma, \delta$ are as in `Cases 2, 3, 4, 5, 7, 8, 9, 12`, the proof is clear because in each of these cases, every element g of G, with $\alpha^g = \beta$ and $\gamma^g = \delta$, fixes a 1- or a 2-dimensional subspace of $\mathtt{GF}(q)^{n+1}$, but every element of $\mathcal{C}$ acts irreducibly on $\mathtt{GF}(q)^{n+1}$ (observe that we are using $n \geq 2$).

Suppose that $\alpha, \beta, \gamma, \delta$ are as in `Cases 6, 10, 11, 13, 14, 15, 17`. The fact that in all of these cases $\mathscr{C}_{(\alpha,\beta),(\gamma,\delta)}$ equals v is a miracle in our opinion and we do not fully understand the reason of this behaviour. We first consider `Case 6`. Let $G_{(\alpha\vee\beta)} := \{x \in G \mid \xi^x = \xi, \forall \xi \in \alpha \vee \beta\}$. We claim that, for every $x \in G_{\alpha\beta}$, there exists $y \in G_{(\alpha\vee\beta)}$ with $xy \in \mathcal{G}_{\alpha\beta\gamma\delta}$. Then, let $x \in G_{\alpha\beta}$. If $\gamma^{g_{\alpha\beta}} \in \alpha \vee \beta$, then $(\alpha\vee\beta)^{g_{\alpha\beta}} = (\alpha\vee\gamma)^{g_{\alpha\beta}} = \alpha^{g_{\alpha\beta}} \vee \gamma^{g_{\alpha\beta}} = \beta \vee \gamma^{g_{\alpha\beta}} = \alpha \vee \beta$ and hence $g_{\alpha\beta}$ fixes $\alpha \vee \beta$ setwise, a contradiction. Therefore $\gamma^{g_{\alpha\beta}} \notin \alpha \vee \beta$. Since $x \in G_{\alpha\beta}$, x fixes setwise $\alpha \vee \beta$ and hence $\gamma^{x^{-1}g_{\alpha\beta}x} \notin \alpha \vee \beta$. As $G_{(\alpha\vee\beta)}$ is transitive on $\mathcal{P} \setminus \alpha \vee \beta$ and $\delta \notin \alpha \vee \beta$, there exists $y \in G_{(\alpha\vee\beta)}$ with $\alpha^{x^{-1}g_{\alpha\beta}xy} = \delta$. It is now immediate to check that $xy \in \mathcal{G}_{\alpha\beta\gamma\delta}$ and hence our claim is proved.

From the previous claim, there exists a set $x_1, \ldots, x_{q-1}$ of representatives for the cosets of $G_{(\alpha\vee\beta)}$ in $G_{\alpha\beta}$ with $x_1, \ldots, x_{q-1} \in \mathcal{G}_{\alpha\beta\gamma\delta}$. Therefore

$$\mathcal{G}_{\alpha\beta\gamma\delta} = G_{\alpha\beta} \cap \mathcal{G}_{\alpha\beta\gamma\delta} = \left(\bigcup_{i=1}^{q-1} x_i G_{(\alpha\vee\beta)}\right) \cap \mathcal{G}_{\alpha\beta\gamma\delta} = \bigcup_{i=1}^{q-1} \left(x_i G_{(\alpha\vee\beta)} \cap \mathcal{G}_{\alpha\beta\gamma\delta}\right) = \bigcup_{i=1}^{q-1} x_i G_{\alpha\beta\gamma\delta},$$

where in the last equality we used (6.9). Therefore

$$\mathscr{C}_{(\alpha,\beta),(\gamma,\delta)} = |\mathcal{G}_{\alpha\beta\gamma\delta}| = (q-1)|G_{\alpha\beta\gamma\delta}| = (q-1)\frac{|G_{\alpha\beta\delta}|}{[G_{\alpha\beta\delta} : G_{\alpha\beta\gamma\delta}]} = |G_{\alpha\beta\delta}|$$
$$= \frac{|G_{\alpha\beta}|}{[G_{\alpha\beta} : G_{\alpha\beta\delta}]} = \frac{|G_{\alpha\beta}|}{|\mathcal{P}| - (q+1)} = \frac{|G|}{|\mathcal{P}|(|\mathcal{P}|-1)(|\mathcal{P}|-q-1)} = v.$$

When $\alpha, \beta, \gamma, \delta$ are as in `Case 11`, the proof follows from `Case 6` and (6.6).

Suppose that $\alpha, \beta, \gamma, \delta$ are as in `Case 10`. The argument in this case does follow the argument above, but with some slight differences. We claim that, for every $x \in G_{\alpha\beta}$, there exists $y \in G_{\alpha\beta\gamma}$ with $xy \in \mathcal{G}_{\alpha\beta\gamma\delta}$. Then, let $x \in G_{\alpha\beta}$. Arguing as above, $\gamma^{g_{\alpha\beta}} \notin \alpha \vee \beta$, because otherwise $g_{\alpha\beta}$ fixes $\alpha \vee \beta$ setwise, which is a contradiction. Therefore $\gamma^{g_{\alpha\beta}} \notin \alpha \vee \beta$. Since $x \in G_{\alpha\beta}$, x fixes setwise $\alpha \vee \beta$ and hence $\gamma^{x^{-1}g_{\alpha\beta}x} \notin \alpha \vee \beta$. As $G_{\alpha\beta\gamma}$ is transitive on $\mathcal{P} \setminus \alpha \vee \beta$ and $\delta \notin \alpha \vee \beta$, there exists $y \in G_{(\alpha\vee\beta)}$ with $\alpha^{x^{-1}g_{\alpha\beta}xy} = \delta$. It is now immediate to check that $xy \in \mathcal{G}_{\alpha\beta\gamma\delta}$ and hence our claim is proved. The rest of the argument follows verbatim the argument in `Case 6`.

When $\alpha, \beta, \gamma, \delta$ are as in `Cases 13, 14` and `15`, the proof follows from `Case 10` applying (6.5), or (6.6), or (6.5) and (6.6).

It is remarkable (and in our opinion mysterious) that also in `Case 17` $\mathscr{C}_{(\alpha,\beta),(\gamma,\delta)}$ equals v. We denote by $\mathcal{L}$ the set of lines of $\mathrm{PG}^n(q)$ and we define $\mathcal{L}_0 := \{\ell \in \mathcal{L} \mid \ell \wedge \ell^{g_{\alpha\beta}} = \emptyset\}$ and $\mathcal{L}_1 := \{\ell \in \mathcal{L} \mid \ell \wedge \ell^{g_{\alpha\beta}} \neq \emptyset\}$. We claim that

$$|\mathcal{L}_0| = \frac{(q^{n+1}-1)(q^n-q)}{(q-1)(q^2-1)}. \tag{6.10}$$

To this end, consider $\mathcal{S} := \{(\xi, \ell) \in \mathcal{P} \times \mathcal{L} \mid \xi = \ell \wedge \ell^{g_{\alpha\beta}}\}$. Observe that, if $(\xi, \ell) \in \mathcal{S}$, then $\ell = \xi \vee \xi^{g_{\alpha\beta}}$ and hence ℓ is uniquely determined by ξ. This shows $|\mathcal{S}| = |\mathcal{P}|$. On the other hand,

$$|\mathcal{S}| = \sum_{\ell \in \mathcal{L}} |\ell \wedge \ell^{g_{\alpha\beta}}| = |\mathcal{L}_1|.$$

Thus $|\mathcal{L}_0| = |\mathcal{L}| - |\mathcal{L}_1| = |\mathcal{L}| - |\mathcal{P}|$ and (6.10) follows.

Write $\mathcal{L}_{0,\alpha} := \{\ell \in \mathcal{L}_0 \mid \alpha \in \ell\}$. Since each line contains $q+1$ points, an elementary double counting argument on the set $\{(\xi,\ell) \in \mathcal{P}\times\mathcal{L} \mid \xi \in \ell\}$ yields $(q+1)|\mathcal{L}_0| = |\mathcal{P}||\mathcal{L}_{0,\alpha}|$ and hence

$$|\mathcal{L}_{0,\alpha}| = \frac{q^n - q^2}{q-1}. \tag{6.11}$$

An explicit computation with the matrices in $G_{\alpha\beta}$ shows that, for every $\xi,\xi',\varepsilon,\varepsilon' \in \mathcal{P}$ with $(\alpha\vee\xi)\wedge(\beta\vee\varepsilon)=\emptyset$ and $(\alpha\vee\xi')\wedge(\beta\vee\varepsilon')=\emptyset$, there exists $x \in G_{\alpha\beta}$ with $\xi^x = \xi'$ and $\varepsilon^x = \varepsilon'$.

Let $x \in \mathcal{G}_{\alpha\beta\gamma\delta}$. Then $(\alpha\vee\gamma)^{x^{-1}g_{\alpha\beta}x} = \beta\vee\delta$. As $(\alpha\vee\gamma)\wedge(\beta\vee\delta)=\emptyset$, we deduce $((\alpha\vee\gamma)^{x^{-1}})^{g_{\alpha\beta}}\wedge(\alpha\vee\gamma)^{x^{-1}}=\emptyset$, that is, $(\alpha\vee\gamma)^{x^{-1}}$ is a line, let us call it ℓ, in $\mathcal{L}_{0,\alpha}$. From (6.11) we have $(q^n-q^2)/(q-1)$ possibilities for ℓ. Next, $\gamma^{x^{-1}} \in \ell\setminus\{\alpha\}$ and hence we have q possibilities for the point $\gamma^{x^{-1}}$. From the previous paragraph, we deduce

$$|\mathcal{G}_{\alpha\beta\gamma\delta}| = |G_{\alpha\beta\gamma\delta}|\frac{q^n-q^2}{q-1}q = \frac{|G_{\alpha\beta\gamma}|}{|\mathcal{P}|-(q^2+q+1)}\frac{q^{n+1}-q^3}{q-1} = |G_{\alpha\beta\gamma}| = \frac{|G|}{|\mathcal{P}|(|\mathcal{P}|-1)(|\mathcal{P}|-q-1)} = v.$$

Suppose finally that $\alpha,\beta,\gamma,\delta$ are as in `Case 16` and define $\kappa := \mathscr{C}_{(\alpha,\beta),(\gamma,\delta)}$. Clearly, $\mathcal{C}_{\alpha\beta} = \bigcup_{\xi\in\mathcal{P}}\{g_{\alpha\beta}^x \mid x \in \mathcal{G}_{\alpha\beta\gamma\xi}\}$ and hence

$$\begin{aligned}|\mathcal{C}_{\alpha\beta}| = \sum_{\xi\in\mathcal{P}}|\mathcal{G}_{\alpha\beta\gamma\xi}| = &|\mathcal{G}_{\alpha\beta\gamma\alpha}| \\ &+ \sum_{\xi\in\alpha\vee\beta\setminus\{\alpha,\beta\}}|\mathcal{G}_{\alpha\beta\gamma\alpha}| + \sum_{\xi\in\alpha\vee\gamma\setminus\{\alpha,\gamma\}}|\mathcal{G}_{\alpha\beta\gamma\alpha}| + \sum_{\xi\in\beta\vee\gamma\setminus\{\beta,\gamma\}}|\mathcal{G}_{\alpha\beta\gamma\alpha}| \\ &+ \sum_{\xi\notin\alpha\vee\beta\vee\gamma}|\mathcal{G}_{\alpha\beta\gamma\alpha}| \\ &+ \sum_{\xi\in\alpha\vee\beta\vee\gamma\setminus(\alpha\vee\beta\cup\alpha\vee\gamma\cup\beta\vee\gamma)}|\mathcal{G}_{\alpha\beta\gamma\xi}|.\end{aligned}$$

The summands that we have singled out in the first three lines correspond to `Cases 11, 13, 14, 15` and `17`, whereas the summand in the fourth line correspond to the configuration in `Case 16`. Thus

$$u = |\mathcal{C}_{\alpha\beta}| = v\left(1+(q-1)+(q-1)+(q-1)+\left(|\mathcal{P}|-(q^2+q+1)\right)\right)+\kappa(q-1)^2$$

and the result follows by expressing κ using the remaining terms. □

6.2. **Eigenvectors of $\mathscr{C}$.** We determine some eigenvectors of $\mathscr{C}$. Since the rows and the columns of $\mathscr{C}$ are indexed by the elements of $\mathcal{P}^{(2)}$, just for this section, for simplifying some of our equations, we slightly modify the definition in (3.1). Given two subsets X and Y of $\mathcal{P}$, we write

$$e_{XY} := \sum_{\substack{(x,y)\in X\times Y\\ x\neq y}} e_{xy}.$$

Lemma 6.2. *Let $\ell\in\mathcal{L}$ and let $\gamma\in\ell$. Then*

$$\mathscr{C}e_{\gamma\ell} = ue_{\gamma\ell} + qv(e_{(\mathcal{P}\setminus\ell)\mathcal{P}} + e_{(\ell\setminus\{\gamma\})(\mathcal{P}\setminus\ell)}) - v\sum_{\alpha\in\mathcal{P}\setminus\ell}(e_{\alpha(\alpha\vee\ell)} - e_{\alpha(\alpha\vee\gamma)}).$$

Proof. Let α,β be two distinct points of $\mathcal{P}$ and define $r := \alpha\vee\beta$. We have

$$(\mathscr{C}e_{\gamma\ell})_{(\alpha,\beta)} = \left(\sum_{\delta\in\ell\setminus\{\gamma\}}\mathcal{C}e_{\gamma\delta}\right)_{(\alpha,\beta)} = \sum_{\delta\in\ell\setminus\{\gamma\}}\mathscr{C}_{(\alpha,\beta),(\gamma,\delta)}.$$

The value of this sum depends on the geometric configuration of the lines ℓ and r. Geometrically, that is, up to G-conjugation, we have one of the ten cases depicted in Figure 2. Now, from Figure 1, Proposition 6.1 and Figure 2, we deduce

$$(\mathscr{C}e_{\gamma\ell})_{(\alpha,\beta)} = \begin{cases} u & \text{in Case (i)},\\ 0 & \text{in Cases (ii), (iii), (iv)},\\ qv & \text{in Cases (v), (vi), (vii), (x)},\\ (q-1)v & \text{in Cases (viii), (ix)}.\end{cases} \tag{6.12}$$

These computations are straightforward and we do not include them all here, we only discuss `Case (ix)`, that is, ℓ and r meet in a point different from α,β and γ. Let $\delta\in\ell\setminus\{\gamma\}$. If $\delta\in r$, then $\alpha,\beta,\gamma,\delta$ are as in `Case 13` of

Figure 1 and hence $\mathscr{C}_{(\alpha,\beta),(\gamma,\delta)} = v$ by Proposition 6.1. If $\delta \notin r$, then $\alpha, \beta, \gamma, \delta$ are as in `Case 16` of Figure 1 and hence $\mathscr{C}_{(\alpha,\beta),(\gamma,\delta)} = (q-2)v/(q-1)$ by Proposition 6.1. Therefore, when ℓ and r are as in `Case (ix)`, we find

$$(\mathscr{C}e_{\gamma\ell})_{(\alpha,\beta)} = v + (q-1)\frac{q-2}{q-1}v = (q-1)v.$$

All other cases are similar.

The proof now follows easily from (3.1) and (6.12). $\square$

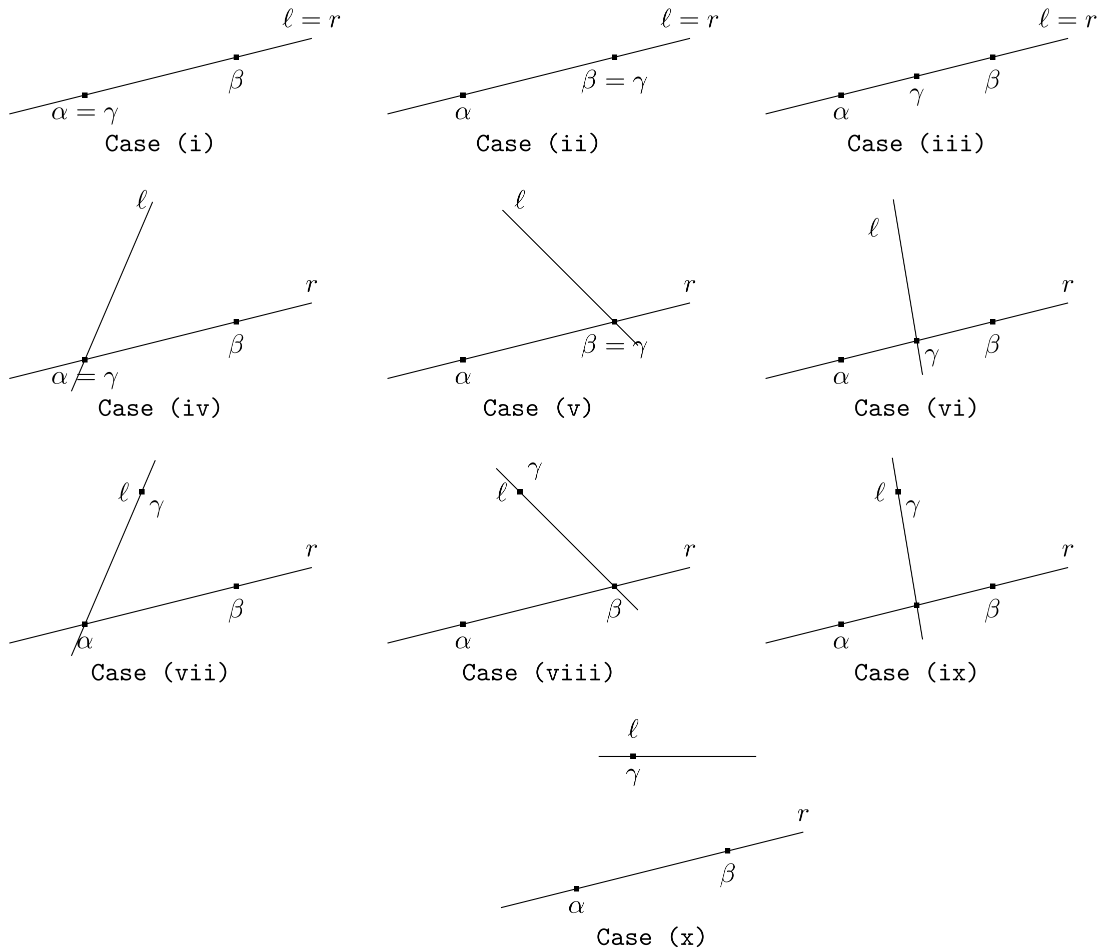

FIGURE 2. Figure for the proof of Lemma 6.2

Corollary 6.3. *Let $\ell \in \mathcal{L}$ and let γ and γ' be two distinct points in ℓ. Then*

$$\mathscr{C}(e_{\gamma\ell} - e_{\gamma'\ell}) = u(e_{\gamma\ell} - e_{\gamma'\ell}) + qv(e_{\gamma'(\mathcal{P}\setminus\ell)} - e_{\gamma(\mathcal{P}\setminus\ell)}) - v\sum_{\alpha\in\mathcal{P}\setminus\ell}(e_{\alpha(\alpha\vee\gamma')} - e_{\alpha(\alpha\vee\gamma)}).$$

Proof. From Proposition 6.1, we have

$$\begin{aligned}\mathscr{C}(e_{\gamma\ell} - e_{\gamma\ell'}) =& u(e_{\gamma\ell} - e_{\gamma'\ell}) \\ &+ qv(e_{(\mathcal{P}\setminus\ell)\mathcal{P}} + e_{(\ell\setminus\{\gamma\})(\mathcal{P}\setminus\ell)} - e_{(\mathcal{P}\setminus\ell)\mathcal{P}} - e_{(\ell\setminus\{\gamma'\})(\mathcal{P}\setminus\ell)}) \\ &- v\left(\sum_{\alpha\in\mathcal{P}\setminus\ell}(e_{\alpha(\alpha\vee\ell)} - e_{\alpha(\alpha\vee\gamma)}) - \sum_{\alpha\in\mathcal{P}\setminus\ell}(e_{\alpha(\alpha\vee\ell)} - e_{\alpha(\alpha\vee\gamma')})\right).\end{aligned}$$

Let us call the summand in the second row A and the summand in the third row B. From (3.1), we immediately have

$$A = qv(e_{(\ell\setminus\{\gamma\})(\mathcal{P}\setminus\ell)} - e_{(\ell\setminus\{\gamma'\})(\mathcal{P}\setminus\ell)}) = qv(e_{\gamma'(\mathcal{P}\setminus\ell)} - e_{\gamma(\mathcal{P}\setminus\ell)}).$$

In B many summands readily cancel:

$$B = - v \sum_{\alpha \in \mathcal{P}\setminus \ell} (e_{\alpha(\alpha\vee\gamma')} - e_{\alpha(\alpha\vee\gamma)}).$$

□

We are now ready to obtain some eigenvectors of $\mathscr{C}$. For each $\alpha\in\mathcal{P}$, $\beta\in\mathcal{P}\setminus\{\alpha\}$ and $\gamma\in\mathcal{P}\setminus\alpha\vee\beta$, define

$$\begin{aligned}
e^1_{\alpha\beta\gamma} &:= e_{\alpha(\alpha\vee\beta)} - e_{\beta(\alpha\vee\beta)} + e_{\beta(\beta\vee\gamma)} - e_{\gamma(\beta\vee\gamma)} + e_{\gamma(\alpha\vee\gamma)} - e_{\alpha(\alpha\vee\gamma)}, \quad W^1 := \langle e^1_{\alpha\beta\gamma} \mid \alpha,\beta,\gamma\rangle,\\
e^2_{\alpha\beta\gamma} &:= e_{(\alpha\vee\beta)\alpha} - e_{(\alpha\vee\beta)\beta} + e_{(\beta\vee\gamma)\beta} - e_{(\beta\vee\gamma)\gamma} + e_{(\alpha\vee\gamma)\gamma} - e_{(\alpha\vee\gamma)\alpha}, \quad W^2 := \langle e^2_{\alpha\beta\gamma} \mid \alpha,\beta,\gamma\rangle.
\end{aligned}$$

Proposition 6.4. *The subspaces W^1 and W^2 are two eigenspaces of the matrix $\mathscr{C}$ for the eigenvalue $u+(q+1)v$ and $W^1\cap W^2 = 0$.*

Proof. We first consider $e^1_{\alpha\beta\gamma}$; we apply Corollary 6.3 three times: to the line $\alpha\vee\beta$ and to the points $\alpha,\beta\in\alpha\vee\beta$, to the line $\alpha\vee\gamma$ and to the points $\alpha,\gamma\in\alpha\vee\gamma$, and to the line $\beta\vee\gamma$ and to the points $\beta,\gamma\in\beta\vee\gamma$. We obtain:

$$\begin{aligned}
\mathscr{C} e^1_{\alpha\beta\gamma} =& u e^1_{\alpha\beta\gamma}\\
&+ qv(e_{\beta(\mathcal{P}\setminus\alpha\vee\beta)} - e_{\alpha(\mathcal{P}\setminus\alpha\vee\beta)} + e_{\gamma(\mathcal{P}\setminus\beta\vee\gamma)} - e_{\beta(\mathcal{P}\setminus\beta\vee\gamma)} + e_{\alpha(\mathcal{P}\setminus\alpha\vee\gamma)} - e_{\gamma(\mathcal{P}\setminus\alpha\vee\gamma)})\\
&- v\left(\sum_{\xi\in\mathcal{P}\setminus\alpha\vee\beta}(e_{\xi(\xi\vee\beta)} - e_{\xi(\xi\vee\alpha)}) + \sum_{\xi\in\mathcal{P}\setminus\beta\vee\gamma}(e_{\xi(\xi\vee\gamma)} - e_{\xi(\xi\vee\beta)}) + \sum_{\xi\in\mathcal{P}\setminus\alpha\vee\gamma}(e_{\xi(\xi\vee\alpha)} - e_{\xi(\xi\vee\gamma)})\right)
\end{aligned}$$

Let us call the summand in the second row A and the summand in the third row B.

From (3.1), we immediately have

$$\begin{aligned}
A =& qv((e_{\beta(\mathcal{P}\setminus\alpha\vee\beta)} - e_{\beta(\mathcal{P}\setminus\beta\vee\gamma)}) + (e_{\alpha(\mathcal{P}\setminus\alpha\vee\gamma)} - e_{\alpha(\mathcal{P}\setminus\alpha\vee\beta)}) + (e_{\gamma(\mathcal{P}\setminus\beta\vee\gamma)} - e_{\gamma(\mathcal{P}\setminus\alpha\vee\gamma)}))\\
=& qv((e_{\beta(\beta\vee\gamma)} - e_{\beta(\alpha\vee\beta)}) + (e_{\alpha(\alpha\vee\beta)} - e_{\alpha(\alpha\vee\gamma)}) + (e_{\gamma(\alpha\vee\gamma)} - e_{\gamma(\beta\vee\gamma)})) = qv e^1_{\alpha\beta\gamma}.
\end{aligned}$$

In B, many summands readily cancel. Indeed, if $\xi\notin(\alpha\vee\beta)\cup(\beta\vee\gamma)\cup(\alpha\vee\gamma)$, then

$$(e_{\xi(\xi\vee\beta)} - e_{\xi(\xi\vee\alpha)}) + (e_{\xi(\xi\vee\gamma)} - e_{\xi(\xi\vee\beta)}) + (e_{\xi(\xi\vee\alpha)} - e_{\xi(\xi\vee\gamma)}) = 0.$$

Therefore, the only summands in B that potentially can give a non-zero contribution arise when $\xi\in(\alpha\vee\beta)\cup(\beta\vee\gamma)\cup(\alpha\vee\gamma)$. Thus, we may write

$$\begin{aligned}
B = - v\Bigg(&\left(\sum_{\xi\in\alpha\vee\gamma\setminus\{\alpha\}}(e_{\xi(\xi\vee\beta)} - e_{\xi(\xi\vee\alpha)}) + \sum_{\xi\in\beta\vee\gamma\setminus\{\beta,\gamma\}}(e_{\xi(\xi\vee\beta)} - e_{\xi(\xi\vee\alpha)})\right)\\
&+\left(\sum_{\xi\in\alpha\vee\beta\setminus\{\beta\}}(e_{\xi(\xi\vee\gamma)} - e_{\xi(\xi\vee\beta)}) + \sum_{\xi\in\alpha\vee\gamma\setminus\{\alpha,\gamma\}}(e_{\xi(\xi\vee\gamma)} - e_{\xi(\xi\vee\beta)})\right)\\
&+\left(\sum_{\xi\in\alpha\vee\beta\setminus\{\alpha\}}(e_{\xi(\xi\vee\alpha)} - e_{\xi(\xi\vee\gamma)}) + \sum_{\xi\in\beta\vee\gamma\setminus\{\beta,\gamma\}}(e_{\xi(\xi\vee\alpha)} - e_{\xi(\xi\vee\gamma)})\right)\Bigg).
\end{aligned}$$

With this new expression for B, we see now that all summands with $\xi\notin\{\alpha,\beta,\gamma\}$ cancel. For instance, when $\xi\in\alpha\vee\beta\setminus\{\alpha,\gamma\}$, the term $e_{\xi(\xi\vee\beta)}$ in the first sum cancels the term $-e_{\xi(\xi\vee\beta)}$ in the fourth sum. Using this observation, we find

$$B = - v((e_{\gamma(\gamma\vee\beta)} - e_{\gamma(\gamma\vee\alpha)}) + (e_{\alpha(\alpha\vee\gamma)} - e_{\alpha(\alpha\vee\beta)}) + (e_{\beta(\beta\vee\alpha)} - e_{\beta(\beta\vee\gamma)})) = v e^1_{\alpha\beta\gamma}.$$

Therefore $\mathscr{C} e^1_{\alpha\beta\gamma} = (u+(q+1)v)e^1_{\alpha\beta\gamma}$ and hence $e^1_{\alpha\beta\gamma}$ is an eigenvector for $\mathscr{C}$ for the eigenvalue $u+(q+1)v$. From (6.6), we deduce that also $e^2_{\alpha\beta\gamma}$ is an eigenvector for $\mathscr{C}$ for the eigenvalue $u+(q+1)v$.

We skip the proof that $W_1\cap W_2 = 0$; in fact, now that we have shown that W_1 and W_2 are eigenspaces of $\mathscr{C}$, the argument follows the ideas in the proof of the analogous statement in [16, Lemma 5.9]. □

6.3. **Proof of Proposition** 3.2. From Proposition 3.1, $\langle e_{\bar{\alpha}\mathcal{P}} - e_{\alpha\mathcal{P}}, e_{\mathcal{P}\bar{\alpha}} - e_{\mathcal{P}\alpha} \mid \alpha \in \mathcal{P}\rangle$ is the right kernel of A and hence is contained in the right kernel of M. If $\alpha \in \mathcal{P}$, then $Me_{\alpha\alpha} = 0$ because no derangement of G fixes α. Therefore $\langle e_{\alpha\alpha} \mid \alpha \in \mathcal{P}\rangle$ is contained in the right kernel of M. Let $\pi \in \mathcal{H}$ and let g be a derangement of G. Then, the definition of M gives

$$(M(e_{\bar{\pi}\bar{\pi}} - e_{\pi\pi}))_g = |\bar{\pi} \wedge \bar{\pi}^{g^{-1}}| - |\pi \wedge \pi^{g^{-1}}| = \frac{q^n-1}{q-1} - \frac{q^n-1}{q-1} = 0.$$

Hence $\langle e_{\bar{\pi}\bar{\pi}} - e_{\pi\pi} \mid \alpha \in \mathcal{P}, \pi \in \mathcal{H}\rangle$ is also contained in the right kernel of M.

We have shown that

$$\langle e_{\alpha\alpha}, e_{\bar{\alpha}\mathcal{P}} - e_{\alpha\mathcal{P}}, e_{\mathcal{P}\bar{\alpha}} - e_{\mathcal{P}\alpha}, e_{\bar{\pi}\bar{\pi}} - e_{\pi\pi} \mid \alpha \in \mathcal{P}, \pi \in \mathcal{H}\rangle$$

is contained in the right kernel of M. Moreover, it is easy to prove that this subspace of $\mathbb{C}\mathcal{P}^2$ has dimension $4|\mathcal{P}|-3$.

By definition, $\mathscr{M}$ is a submatrix of M; namely, $\mathscr{M}$ is the submatrix of M obtained by considering the rows of M labelled by the elements of $\mathcal{C}$ and the columns of M labelled by the elements of $\mathcal{P}^{(2)}$. Therefore, to conclude the proof, it suffices to show that $\mathscr{M}$ has rank at least $|\mathcal{P}|^2 - 4|\mathcal{P}| + 3$. Observe that $\mathscr{M}$ and $\mathscr{C} = \mathscr{M}^T\mathscr{M}$ have the same rank.

Consider the permutation modules $\mathbb{C}\mathcal{P}^{(2)}$ and $\mathbb{C}\mathcal{C}$ for the natural action of G on $\mathcal{P}^{(2)}$ and for the action of G on $\mathcal{C}$ by conjugation, respectively. Moreover, consider the function:

$$\begin{aligned}\mathfrak{m} : \mathbb{C}\mathcal{P}^{(2)} &\longrightarrow \mathbb{C}\mathcal{C}\\ e_{\gamma\delta} &\longmapsto \mathscr{M}e_{\gamma\delta} = \sum_{\substack{g\in\mathcal{C}\\ \gamma^g=\delta}} e_g\end{aligned}$$

As $M_{(g^x,(\alpha^x,\beta^x))} = M_{g,(\alpha,\beta)}$ for every $g \in \mathcal{C}$ and for every $(\alpha,\beta) \in \mathcal{P}^{(2)}$, $\mathfrak{m}$ is a $\mathbb{C}G$-module homomorphism and the rank of $\mathscr{M}$ is the dimension of the image of this mapping.

From (5.1) and (5.2),

$$\mathbb{C}G = \bigoplus_{\eta\in\mathrm{Irr}_{\mathbb{C}}(G)} I_\eta = \bigoplus_{\eta\in\mathrm{Irr}_{\mathbb{C}}(G)} \mathbb{C}G\cdot e_\eta. \tag{6.13}$$

Now, fix $(\alpha,\beta) \in \mathcal{P}^{(2)}$ with $\beta = \alpha^c$, recall the definition of c in (4.10). As $\mathbb{C}\mathcal{P}^{(2)}$ is generated by $e_{\alpha\beta}$ as a $\mathbb{C}G$-module, we have $\mathbb{C}\mathcal{P}^{(2)} = \mathbb{C}G\cdot e_{\alpha\beta}$, and from (6.13) we deduce

$$\mathbb{C}\mathcal{P}^{(2)} = \bigoplus_{\eta\in\mathrm{Irr}_{\mathbb{C}}(G)} \mathbb{C}G\cdot e_\eta e_{\alpha\beta} = \bigoplus_{\substack{\eta\in\mathrm{Irr}_{\mathbb{C}}(G)\\ \langle\eta,\pi^{(2)}\rangle_G\neq 0}} \mathbb{C}G\cdot e_\eta e_{\alpha\beta},$$

where in the last equality we used some standard facts on G-modules, see [10, Chapter 1 and 2]. Analogously, we have

$$\mathrm{Im}(\mathfrak{m}) = \mathfrak{m}(\mathbb{C}\mathcal{P}^{(2)}) = \bigoplus_{\substack{\eta\in\mathrm{Irr}_{\mathbb{C}}(G)\\ \langle\eta,\pi^{(2)}\rangle_G\neq 0}} \mathbb{C}G\cdot \mathfrak{m}(e_\eta e_{\alpha\beta}).$$

From (5.2), we have

$$e_\eta e_{\alpha\beta} = \frac{\eta(1)}{|G|}\sum_{g\in G}\eta(g^{-1})e_{\alpha^g\beta^g}$$

and hence

$$\mathfrak{m}(e_\eta e_{\alpha\beta}) = \frac{\eta(1)}{|G|}\sum_{g\in G}\eta(g^{-1})\mathscr{M}e_{\alpha^g\beta^g} = \frac{\eta(1)}{|G|}\sum_{g\in G}\eta(g^{-1})\sum_{\substack{h\in\mathcal{C}\\ \alpha^{gh}=\beta^g}} e_h = \frac{\eta(1)}{|G|}\sum_{h\in\mathcal{C}}\left(\sum_{\substack{g\in G\\ \alpha^{gh}=\beta^g}}\eta(g^{-1})\right)e_h.$$

Therefore the e_c-coordinate of $\mathfrak{m}(e_\eta e_{\alpha\beta})$ is

$$\frac{\eta(1)}{|G|}\sum_{\substack{g\in G\\ \alpha^{gc}=\beta^g}}\eta(g^{-1}). \tag{6.14}$$

Consider the set $\mathcal{S} := \{g \in G \mid \alpha^{gc} = \beta\}$ indexing the summation in (6.14). Since $\alpha^c = \beta$, we have $1 \in \mathcal{C}$. Since $C = \mathbf{C}_G(c)$ is transitive on $\mathcal{P}$, we have $G = G_\alpha\mathbf{C}_G(c)$. Let $g \in \mathcal{S}$. Then, we may write $g = xy$, for some $x \in G_\alpha$ and some $y \in \mathbf{C}_G(c) = C$. Thus

$$\beta = \alpha^{gc} = \alpha^{xyc} = \alpha^{yc} = \alpha^{cy} = \beta^y$$

and we infer $y \in G_{\alpha\beta}$. This shows that $\mathcal{S} = G_{\alpha\beta}C$ and the e_c-coordinate of the vector $\mathfrak{m}(e_\eta e_{\alpha\beta})$ becomes

$$\frac{\eta(1)}{|G|}\sum_{g\in G_{\alpha\beta}C}\eta(g^{-1}). \tag{6.15}$$

We now refer to Lemma 4.1 for the constituents η of $\pi^{(2)}$ and their multiplicities.

Case $\eta = \chi_0$. From what we have shown at the beginning of this proof, we deduce that $\langle e_{\bar\alpha\mathcal{P}} - e_{\alpha\mathcal{P}}, e_{\mathcal{P}\alpha} - e_{\mathcal{P}\alpha}, e_{\bar\pi\bar\pi - e_{\pi,\pi}} \mid \alpha \in \mathcal{P}, \pi \in \mathcal{H}\rangle$ has dimension $3(|\mathcal{P}|-1) = 3\chi_0(1)$, is contained in the right kernel of $\mathscr{M}$ (and hence in the kernel of $\mathfrak{m}$), and equals $\mathbb{C}G\cdot e_{\chi_0}e_{\alpha\beta}$. Therefore, $\mathbb{C}G\cdot\mathfrak{m}(e_{\chi_0}e_{\alpha\beta}) = 0$. ∎

Case $\eta = \mathbf{1}_G$ or $\eta = \psi_i^G$. As η has multiplicity 1 in $\pi^{(2)}$, the module $\mathbb{C}G\cdot e_\eta e_{\alpha\beta}$ is a simple $\mathbb{C}G$-module. Therefore, from Schur's lemma, either $\mathfrak{m}(e_\eta e_{\alpha\beta}) = 0$ or $\mathfrak{m}$ maps injectively $\mathbb{C}G\cdot e_\eta e_{\alpha\beta}$ into $\mathbb{C}\mathcal{C}$. We deduce from Proposition 4.5 that $\mathfrak{m}(e_\eta e_{\alpha\beta}) \neq 0$ because its e_c-coordinate is non-zero from (6.15). This shows that $\dim_{\mathbb{C}}(\mathbb{C}G\cdot\mathfrak{m}(e_\eta e_{\alpha\beta})) = \eta(1)$. ∎

Case $\eta = \chi'$. Since we were not able to prove Proposition 4.5 when $\eta = \chi'$, we need an ad-hoc argument here. For each line $\ell \in \mathcal{L}$, define

$$\varepsilon_\ell := \sum_{\substack{\gamma,\delta\in\ell\\ \gamma\neq\delta}} e_{\gamma\delta} \in \mathbb{C}\mathcal{P}^{(2)}.$$

Then, define $L := \langle \varepsilon_\ell \mid \ell \in \mathcal{L}\rangle \leq \mathbb{C}\mathcal{P}^{(2)}$. As usual, let $\mathbb{C}\mathcal{L}$ be the permutation module for the action of G on $\mathcal{P}$ with standard bases $(e_\ell)_{\ell\in\mathcal{L}}$ indexed by the elements of $\mathcal{L}$. The mapping

$$\begin{array}{ccc}\mathbb{C}\mathcal{L} & \longrightarrow & \mathbb{C}\mathcal{P}^{(2)}\\ e_\ell & \longmapsto & \varepsilon_\ell\end{array}$$

is an injective $\mathbb{C}G$-module homomorphism and hence $\mathbb{C}\mathcal{L} \cong L$. Therefore, from the decomposition of the permutation character π_ℓ in Lemma 4.1, we deduce $L = \mathbb{C}G\cdot e_{\mathbf{1}_G}e_{\alpha\beta} \oplus \mathbb{C}G\cdot e_{\chi_0}e_{\alpha\beta} \oplus \mathbb{C}G\cdot e_{\chi'}e_{\alpha\beta}$. Thus

$$\mathfrak{m}(L) = \mathbb{C}G\cdot\mathfrak{m}(e_{\mathbf{1}_G}e_{\alpha\beta}) \oplus \mathbb{C}G\cdot\mathfrak{m}(e_{\chi_0}e_{\alpha\beta}) \oplus \mathbb{C}G\cdot\mathfrak{m}(e_{\chi'}e_{\alpha\beta}) = \mathbb{C}G\cdot\mathfrak{m}(e_{\mathbf{1}_G}e_{\alpha\beta}) \oplus \mathbb{C}G\cdot\mathfrak{m}(e_{\chi'}e_{\alpha\beta}).$$

As χ' has multiplicity 1 in $\pi^{(2)}$, the module $\mathbb{C}G\cdot e_{\chi'}e_{\alpha\beta}$ is a simple $\mathbb{C}G$-module. Therefore, from Schur's lemma, either $\mathbb{C}G\cdot e_{\chi'}e_{\alpha\beta}$ is contained in the kernel of $\mathfrak{m}$ or $\mathfrak{m}$ maps injectively $\mathbb{C}G\cdot e_{\chi'}e_{\alpha\beta}$ into $\mathbb{C}\mathcal{C}$. If the first possibility happens, then $\mathfrak{m}(L)$ equals the one-dimensional trivial module $\mathbb{C}G\cdot e_{\mathbf{1}_G}e_{\alpha\beta}$. However this is impossible; in fact, a computation shows that, given $\ell \in \mathcal{L}$, the vector $\mathfrak{m}(\varepsilon_\ell)$ is not a scalar multiple of the all ones vector in $\mathbb{C}\mathcal{C}$. Thus $\mathfrak{m}$ maps injectively $\mathbb{C}G\cdot e_{\chi'}e_{\alpha\beta}$ into $\mathbb{C}\mathcal{C}$, that is, $\dim_{\mathbb{C}}(\mathbb{C}G\cdot\mathfrak{m}(e_{\chi'}e_{\alpha\beta})) = \chi'(1)$. ∎

Case $\eta = \chi''$. Consider the eigenspaces W^1 and W^2 in Proposition 6.4. From their definition, we see that W^1 and W^2 are isomorphic $\mathbb{C}G$-submodules of $\mathbb{C}\mathcal{P}^{(2)}$. From the decomposition of the permutation character $\pi^{(2)}$ in Lemma 4.1, we are forced to have $W^1 \oplus W^2 = \mathbb{C}G\cdot e_{\chi''}e_{\alpha\beta}$ (no other $\mathbb{C}G$-submodule X of $\mathbb{C}\mathcal{P}^{(2)}$ allows a decomposition $X = Y \oplus Z$ with $Y \cong Z$).

Since $W^1 \oplus W^2$ is an eigenspace for $\mathscr{C} = \mathscr{M}^T\mathscr{M}$ for a non-zero eigenvalue, we deduce that $W^1 \oplus W^2$ is mapped injectively by $\mathfrak{m}$ to a submodule of $\mathbb{C}\mathcal{C}$, that is, $\dim_{\mathbb{C}}(\mathbb{C}G\cdot\mathfrak{m}(e_{\chi''}e_{\alpha\beta})) = 2\chi''(1)$. ∎

Summing up,

$$\begin{aligned}\dim_{\mathbb{C}}(\mathrm{Im}(\mathfrak{m})) =& \mathbf{1}_G(1) + \chi'(1) + 2\chi''(1) + \psi_1^G(1) + \cdots + \psi_{q-1}^G(1) = \pi^{(2)} - 3\chi_0(1)\\ =& |\mathcal{P}^{(2)}| - 3(|\mathcal{P}|-1) = |\mathcal{P}|^2 - 4|\mathcal{P}| + 3\end{aligned}$$

and the proof is completed.

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