LIFTING A PRESCRIBED GROUP OF AUTOMORPHISMS OF GRAPHS

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ABSTRACT. In this paper we are interested in lifting a prescribed group of automorphisms of a finite graph via regular covering projections. Let Γ be a finite graph and let $\mathrm{Aut}(\Gamma)$ be the automorphism group of Γ. It is well known that we can always find a finite graph $\tilde{\Gamma}$ and a regular covering projection $\wp \colon \tilde{\Gamma} \to \Gamma$ such that $\mathrm{Aut}(\Gamma)$ lifts along $\wp$. However, for constructing peculiar examples and in applications it is often important, given a subgroup G of $\mathrm{Aut}(\Gamma)$, to find $\wp$ along which G lifts but no further automorphism of Γ does, or even that $\mathrm{Aut}(\tilde{\Gamma})$ is the lift of G. In this paper, we address these problems.

1. INTRODUCTION

Covering projections of graphs and lifting automorphisms along them is a classical tool in algebraic graph theory that goes back to Djoković and his proof of the infinitude of cubic 5-arc-transitive graphs [5]. Moreover, several theoretical aspects of lifting graph automorphisms along covering projections, together with their remarkable applications, are considered in a number of papers, for example in [9, 18, 20, 30], to name a few of the most notable ones.

One of many applications of lifting automorphisms along covering projections is the construction of graphs with a prescribed type of symmetry. For example, covering techniques are used to find new peculiar examples of semisymmetric graphs (see [19, 21, 31, 33]), half-arc-transitive graphs (see [10, 13, 26]), and arc-regular graphs (see [7, 14, 15]), to name a few.

In these papers, a typical strategy is to start with a graph Γ and a group $G \leq \mathrm{Aut}(\Gamma)$ with a prescribed type of action on Γ (such as edge-transitive, vertex-transitive, s-arc-transitive for $s \geq 1$, locally primitive, etc.) and then trying to find regular covering projections $\wp : \tilde{\Gamma} \to \Gamma$ along which G lifts. In many applications, it is desirable for the lift to have the following two additional properties:

(1) G is the *largest* subgroup of $\mathrm{Aut}(\Gamma)$ that lifts along $\wp$;
(2) *every* automorphism of $\tilde{\Gamma}$ projects along $\wp$.

If both these requirements are fulfilled, then $\mathrm{Aut}(\tilde{\Gamma})$ is precisely the lift of G.

The problem of finding regular covering projections satisfying (1) has been addressed in an ad hoc way for some fixed pairs (Γ, G) (see, for example, [21, 31, 33])

and determining conditions under which a covering projection satisfies (2) was considered by several authors in the very specific context of canonical double covers (see [29, 32]). There have been some attempts to determine covering projections satisfying simultaneously (1) and (2), but again only for a small number of very specific pairs (Γ, G) (see, for example, [9, 17, 28]).

The aim of this paper is to address the problem of existence of a regular covering projection satisfying (1) and (2) for arbitrary pairs (Γ, G).

In Theorem 6 we prove that if $\mathrm{Aut}(\Gamma)$ acts faithfully on the integral cycle space $\mathrm{H}_1(\Gamma; \mathbb{Z})$, then a regular covering projection onto Γ fulfilling (1) always exists; see Section 2 for notation and terminology. The condition of $\mathrm{Aut}(\Gamma)$ acting faithfully on $\mathrm{H}_1(\Gamma; \mathbb{Z})$ is very mild: in the most interesting cases $\mathrm{Aut}(\Gamma)$ does act faithfully on $\mathrm{H}_1(\Gamma; \mathbb{Z})$; see Lemma 7 and Corollary 8. Moreover, there are examples where $\mathrm{Aut}(\Gamma)$ does not act faithfully on $\mathrm{H}_1(\Gamma; \mathbb{Z})$ and where a regular covering projection as in (1) does not exist: the easiest example is when Γ is a cycle and G is the transitive cyclic subgroup of $\mathrm{Aut}(\Gamma)$.

In Theorem 9 we prove that if $\mathrm{Aut}(\Gamma)$ acts faithfully on $\mathrm{H}_1(\Gamma; \mathbb{Z})$ and (Γ, G) satisfies an additional condition, then there exists a regular covering projection onto Γ satisfying (1) and (2). The extra condition on (Γ, G) is slightly technical and requires some notation and terminology, thus we refer the reader to Section 4 for its definition and details. Here we simply observe that the condition is satisfied by many interesting classes: for instance, when G acts transitively on the 2-arcs of Γ, or when G acts transitively on the arcs of Γ and the valency of Γ is prime.

In Conjecture 11, we conjecture that the additional requirements we put on Γ and G in Theorem 9 are not needed, that is, a regular covering projection satisfying (1) and (2) exists whenever $\mathrm{Aut}(\Gamma)$ acts faithfully on the integral cycle space of Γ.

Both Theorems 6 and 9 do apply to graphs that are not necessarily simple and we refer to Subsection 2.1 for the precise definition of graph in our paper.

We conclude this introductory section giving some applications. Theorem 9 and Corollary 10 reprove, in a unified way, a number of results that have been proved in the past using methods specific to the families of graphs under consideration. For example, one of the consequences of Corollary 10 is a solution to three problems posed by Djoković and Miller [6, Problems 2, 3, and 4] about the existence of finite cubic arc-transitive graphs with a prescribed type of the full automorphism group, which were first solved in [8] by an ad hoc construction. Furthermore, assuming the correctness of Conjecture 11 one can answer a 2001 question of Marušič and Nedela [22, Problem 7.7] about the existence of tetravalent half-arc-transitive graphs of any given possible type, and a number of similar other problems, such as the one of existence of graphs of every possible arc-type; see [11, 12].

2. BACKGROUND MATERIAL AND NOTATION

In this section, we give a brief overview of the definitions and results pertaining to the theory of covering projections of graphs; see [18] and [20].

2.1. **Graph, fundamental group, and integral cycle space.** A *graph* is an ordered 4-tuple $\Gamma = (D, V; \mathrm{beg}, \mathrm{inv})$ where D and V are disjoint non-empty finite sets of *darts* and *vertices*, respectively, $\mathrm{beg} : D \to V$ is a mapping which assigns to each dart x its *tail* $\mathrm{beg}(x)$, and $\mathrm{inv} : D \to D$ is an involutory permutation which interchanges every dart x with its *inverse dart*, denoted by x^{-1}. The vertex $\mathrm{beg}(x^{-1})$ is then called the *head* of the dart x.

An *edge* underlying a dart x is a pair $\{x, x^{-1}\}$ of mutually inverse darts, and $\{\mathrm{beg}(x), \mathrm{beg}(x^{-1})\}$ is the set of *endvertices* of that edge. An edge $\{x, x^{-1}\}$ with $\mathrm{beg}(x) \neq \mathrm{beg}(x^{-1})$ is called a *link*. If $\mathrm{beg}(x) = \mathrm{beg}(x^{-1})$, then the edge $\{x, x^{-1}\}$ is a *loop* if $x \neq x^{-1}$, and is a *semiedge* if $x = x^{-1}$. Two edges with the same set of endvertices are called *parallel*. A graph without loops, semiedges, and parallel edges is *simple* and the usual terminology about simple graphs applies in this case.

The neighbourhood of a vertex v of Γ, denoted $\Gamma(v)$, is the set of darts having v as its tail, and the cardinality of $\Gamma(v)$ is called the *valency* of v.

A *walk* from a vertex v to a vertex u in Γ is a sequence of darts such that v is the tail of the first dart, u is the head of the last dart, and the head of each dart in the walk coincides with the tail of the next dart in the walk. When $v = u$, the walk is said to be *closed*, whereas the walk is *reduced* provided no two consecutive darts are inverse of each other. Note that if x is a dart and $\{x, x^{-1}\}$ is a loop or a semiedge, then (x) is a reduced closed walk. The empty sequence of darts is considered a walk and is called the trivial walk.

For a vertex b of Γ one can define the *fundamental group at b*, denoted $\pi(\Gamma, b)$, as the set of all closed reduced walks starting and ending in b, with the operation being the concatenation (with the deletion of consecutive pairs of mutually inverse darts, if necessary). Note that $\pi(\Gamma, b)$ is a free product of infinite cyclic groups and cyclic groups of order 2, the latter arising from walks consisting of a single semiedge.

The abelianisation $\pi(\Gamma, b)/[\pi(\Gamma, b), \pi(\Gamma, b)]$ of $\pi(\Gamma, b)$, viewed as a $\mathbb{Z}$-module, is called the *first homology group* or the *integral cycle space* and denoted $\mathrm{H}_1(\Gamma; \mathbb{Z})$. When Γ is connected, $\mathrm{H}_1(\Gamma; \mathbb{Z})$ is independent of the choice of the vertex b.

2.2. Graph morphism, regular covering projection, universal covering.

Let $\tilde{\Gamma} = (\tilde{D}, \tilde{V}; \mathrm{beg}_{\tilde{\Gamma}}, \mathrm{inv}_{\tilde{\Gamma}})$ and $\Gamma = (D, V; \mathrm{beg}_\Gamma, \mathrm{inv}_\Gamma)$ be two graphs. A *morphism of graphs*, $f: \tilde{\Gamma} \to \Gamma$, is a function $f: \tilde{V} \cup \tilde{D} \to V \cup D$ such that $f(\tilde{V}) \subseteq V$, $f(\tilde{D}) \subseteq D$, $f \circ \mathrm{beg}_{\tilde{\Gamma}} = \mathrm{beg}_\Gamma \circ f$, and $f \circ \mathrm{inv}_{\tilde{\Gamma}} = \mathrm{inv}_\Gamma \circ f$. A graph morphism is an *epimorphism* (*automorphism*) if it is a surjection (bijection, respectively). A graph epimorphism $\wp: \tilde{\Gamma} \to \Gamma$ is called a *covering projection* provided that it maps the neighbourhood $\tilde{\Gamma}(\tilde{v})$ bijectively onto the neighbourhood $\Gamma(\wp(\tilde{v}))$ for every $\tilde{v} \in \tilde{V}$. (Note that if Γ is connected the surjectivity of $\wp$ follows from the local bijectivity.)

Let $\wp: \tilde{\Gamma} \to \Gamma$ be a covering projection of connected graphs, let $g \in \mathrm{Aut}(\Gamma)$, and let $\tilde{g} \in \mathrm{Aut}(\tilde{\Gamma})$ be such that $\wp(x^{\tilde{g}}) = \wp(x)^g$ for every vertex and for every dart x of $\tilde{\Gamma}$. Then we say that g *lifts* along $\wp$ and that $\tilde{g}$ is a *lift* of g. Similarly, we say that $\tilde{g}$ *projects* along $\wp$ and that g is a *projection* of $\tilde{g}$ along $\wp$. The set of all automorphisms of Γ that lift along $\wp$ is called the *maximal group that lifts along $\wp$*. If G is a subgroup of the maximal group that lifts, then the set $\tilde{G}$ of all lifts of elements of G forms a subgroup of $\mathrm{Aut}(\tilde{\Gamma})$ and is called *the lift of G*. The lift of the maximal group that lifts along $\wp$ is the *maximal group that projects along $\wp$*.

The lift of the identity group is called the *group of covering transformations of $\wp$* and denoted $\mathrm{CT}(\wp)$. Whenever the covering graph $\tilde{\Gamma}$ is connected, the group $\mathrm{CT}(\wp)$ acts semiregularly on each fibre, and if it is transitive (and thus regular) on each fibre, then we say that the covering projection $\wp$ is *regular*.

Regular covering projections can also be defined in terms of *graph quotients*. Let $\tilde{\Gamma}$ be a graph, let $N \leq \mathrm{Aut}(\tilde{\Gamma})$ with the stabiliser N_x being trivial for every vertex of $\tilde{\Gamma}$, and let $\tilde{\Gamma}/N$ be the graph whose vertices and darts are N-orbits of vertices

and darts of $\tilde{\Gamma}$ and with the functions $\mathrm{inv}_{\tilde{\Gamma}/N}$ and $\mathrm{beg}_{\tilde{\Gamma}/N}$ mapping a dart x^N of $\tilde{\Gamma}/N$ to the N-orbit of $\mathrm{inv}_{\tilde{\Gamma}}(x)$ and $\mathrm{beg}_{\tilde{\Gamma}}(x)$, respectively (see [20, Section 2.1]). The *quotient projection* $\wp_N \colon \tilde{\Gamma} \to \tilde{\Gamma}/N$, mapping each vertex or dart of $\tilde{\Gamma}$ to its N-orbit, is a regular covering projection and N is the group $\mathrm{CT}(\wp_N)$ of covering transformations of $\wp_N$. Every regular covering projection arises in this way.

Lemma 1 ([20, Sections 2.2 and 3]). *If $\wp \colon \tilde{\Gamma} \to \Gamma$ is a regular covering projection, then the maximal group that projects along $\wp$ equals the normaliser of $N = \mathrm{CT}(\wp)$ in $\mathrm{Aut}(\tilde{\Gamma})$. Moreover, Γ is isomorphic to the quotient graph $\tilde{\Gamma}/N$, the quotient projection $\tilde{\Gamma} \to \tilde{\Gamma}/N$ is a covering projection isomorphic to $\wp$, and for a group $G \leq \mathrm{Aut}(\Gamma)$ and its lift $\tilde{G}$, we have $G \cong \tilde{G}/N$.*

2.3. **Splitting of covering projections.** Let $\wp \colon \tilde{\Gamma} \to \Gamma$ be a regular covering projection between connected graphs with covering transformation group N, let G be a subgroup of $\mathrm{Aut}(\Gamma)$ that lifts along $\wp$ to $\tilde{G}$, and let K be a normal subgroup of N. Then one can consider the quotient projection $\wp_K \colon \tilde{\Gamma} \to \tilde{\Gamma}/K$ and define $\wp_{N/K} \colon \tilde{\Gamma}/K \to \Gamma$ by $\wp_{N/K}(x^K) = \wp(x)$ for every vertex and for every dart x of $\tilde{\Gamma}$. Then the following lemma holds.

Lemma 2. *The covering projection $\wp_{N/K}$ is regular with covering transformation group N/K and $\wp = \wp_{N/K} \circ \wp_K$. Moreover, if K is normalised by $\tilde{G}$, then G lifts along $\wp_{N/K}$ and its lift is $\tilde{G}/K$.*

If $\mathcal{T}$ is an infinite tree and $\wp \colon \mathcal{T} \to \Gamma$ is a regular covering projection, then we say that $\wp$ is *universal*. It is well known that for every finite graph there is, up to equivalence of covering projections, a unique universal covering projection and that it has the following property.

Lemma 3. *If Γ is a finite connected graph and $\wp \colon \mathcal{T} \to \Gamma$ is the universal covering projection, then $\mathrm{Aut}(\Gamma)$ lifts along $\wp$ and $\mathrm{CT}(\wp)$ is isomorphic to the fundamental group $\pi(\Gamma, b)$ for some (every) vertex b of Γ. Moreover, $\mathrm{CT}(\wp)_x = 1$ for every vertex and for every edge x of $\mathcal{T}$.*

2.4. **Graphs with no semiedges.** In what follows we mention some facts concerning graphs without semiedges relevant for our proofs.

First, if Γ has no semiedges, then the fundamental group $\pi(\Gamma, b)$ is a free group. Consequently, the first homology group $\mathrm{H}_1(\Gamma; \mathbb{Z})$ is isomorphic to $\mathbb{Z}^{m_\Gamma}$ where m_Γ is the *Betti number of* Γ, that is, the number of cotree edges relative to a fixed spanning tree of Γ. In particular m_Γ equals the number of edges minus the number of vertices and plus one. Given a prime number p, we let $\mathbb{Z}_p := \mathbb{Z}/p\mathbb{Z}$ be the finite field of order p. The module $\mathrm{H}_1(\Gamma; \mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{Z}_p \cong \mathbb{Z}_p^{m_\Gamma}$ will be denoted by $\mathrm{H}_1(\Gamma; \mathbb{Z}_p)$.

Second, if Γ has no semiedges and N is a subgroup of $\mathrm{Aut}(\Gamma)$ with $N_x = 1$ for every vertex as well as for every edge x of Γ, then the quotient Γ/N has no semiedges.

Finally, if a graph Γ is simple, then every covering graph of Γ is again simple. Hence the results of this paper apply to the important class of simple graphs.

3. RESULTS AND PROOFS

Given a group X and a subgroup Y, we denote by $[X, Y]$ the commutator subgroup defined by $[X, Y] := \langle x^{-1}y^{-1}xy \mid x \in X, y \in Y \rangle$, by $\mathbf{N}_X(Y)$ the normaliser of Y in X and we write $X^p := \langle x^p \mid x \in X \rangle$. By $\mathbf{Z}(X)$, we denote the centre of X.

Recall that a $\mathbb{Z}_p X$-module V is regular if V is isomorphic to the group algebra $\mathbb{Z}_p X$ (seen as a $\mathbb{Z}_p X$-module). This means that $\dim_{\mathbb{Z}_p}(V) = |X|$ and that V has a $\mathbb{Z}_p$-basis $(v_x \mid x \in X)$ such that the action of X on $(v_x \mid x \in X)$ is permutation isomorphic to the action of X on itself by right multiplication. In other words, $v_x y = v_{xy}$, for each $x, y \in X$. We start with a technical lemma.

Lemma 4. *Let p be an odd prime, let Γ be a connected graph, and let H be a subgroup of $\mathrm{Aut}(\Gamma)$ acting faithfully on $\mathrm{H}_1(\Gamma; \mathbb{Z})$. Then H acts faithfully on $\mathrm{H}_1(\Gamma; \mathbb{Z}_p)$.*

Proof. A simple closed walk in Γ is a walk where no vertex is repeated.

Suppose that $h \in H$ acts trivially on $\mathrm{H}_1(\Gamma; \mathbb{Z}_p)$. Thus, for every $W \in \mathrm{H}_1(\Gamma; \mathbb{Z})$, $W^h = W + pW'$, for some $W' \in \mathrm{H}_1(\Gamma; \mathbb{Z})$. In particular, for every simple closed walk $C \in \mathrm{H}_1(\Gamma; \mathbb{Z})$, we have $C^h = C + pC'$ for some $C' \in \mathrm{H}_1(\Gamma; \mathbb{Z})$. Therefore, $C^h - C$ is a p-multiple of the element $C' \in \mathrm{H}_1(\Gamma; \mathbb{Z})$. Clearly, this happens only when $C^h = -C$ and $p = 2$, or $C^h = C$. Since we are assuming p odd, we have $C^h = C$ for every simple closed walk C of Γ. Since these elements generate $\mathrm{H}_1(\Gamma; \mathbb{Z})$, we obtain $h = 1$. □

Theorem 5. *Let p be an odd prime, let $\mathcal{T}$ be an infinite tree, let $\mathcal{G} \leq \mathrm{Aut}(\mathcal{T})$, let $\mathcal{N}$ be a non-identity normal subgroup of $\mathcal{G}$ of finite index such that $\mathcal{N}_x = 1$ for every vertex and for every edge x of $\mathcal{T}$, and let $\mathcal{H} = \mathbf{N}_{\mathrm{Aut}(\mathcal{T})}(\mathcal{N})$. If $\mathcal{H}/\mathcal{N}$ acts faithfully on $\mathrm{H}_1(\mathcal{T}/\mathcal{N}; \mathbb{Z})$, then there exists a normal subgroup $\mathcal{P}$ of $\mathcal{N}$ of finite index such that $\mathbf{N}_{\mathcal{H}}(\mathcal{P}) = \mathcal{G}$ and that $\mathcal{N}/\mathcal{P}$ is p-group.*

Proof. The idea for the proof of this theorem is inspired by a surprisingly unrelated problem solved by Bryant and Kovács in [3]. We closely follow [3] and we use some of the observations therein. This is the second time that this paper on Lie algebras has proved useful in the context of group actions on graphs; see for instance [24] for another application.

As $\mathcal{N} \neq 1$ and $\mathcal{N}_v = 1$ for every $v \in \mathrm{V}(\mathcal{T})$, from the Bass-Serre theory, we deduce that $\mathcal{N}$ is a non-identity free group; see [4, Proposition 4.5]. Following [3], we construct a filtration of the free group $\mathcal{N}$. Define $\mathcal{N}_1 := \mathcal{N}$ and, for $i \in \mathbb{N} \setminus \{0\}$, $\mathcal{N}_{i+1} := \mathcal{N}_i^p[\mathcal{N}_i, \mathcal{N}]$. By construction, $\mathcal{N}_{i+1}$ is the smallest normal subgroup of $\mathcal{N}$ contained in $\mathcal{N}_i$ such that $\mathcal{N}_i/\mathcal{N}_{i+1}$ has exponent p and is central in $\mathcal{N}/\mathcal{N}_{i+1}$, that is, $\mathcal{N}_i/\mathcal{N}_{i+1} \leq \mathbf{Z}(\mathcal{N}/\mathcal{N}_{i+1})$. Moreover, $\mathcal{N}_{i+1}$ is normal in $\mathcal{G}$ because so is $\mathcal{N}$.

Write $G := \mathcal{G}/\mathcal{N}_1$ and $H := \mathcal{H}/\mathcal{N}_1$. As $\mathcal{G} \leq \mathcal{H}$, we have $G \leq H$. Given $i \in \mathbb{N} \setminus \{0\}$, define $V_i := \mathcal{N}_i/\mathcal{N}_{i+1}$. As $\mathcal{N}_i$ is centralised by $\mathcal{N} = \mathcal{N}_1$ modulo $\mathcal{N}_{i+1}$, the action of $\mathcal{H}$ by conjugation on $\mathcal{N}_i/\mathcal{N}_{i+1} = V_i$ defines a group homomorphism $H = \mathcal{H}/\mathcal{N}_1 \rightarrow \mathrm{Aut}(V_i)$, that is, H acts as a linear group on the $\mathbb{Z}_p$-vector space V_i and hence V_i is a $\mathbb{Z}_p H$-module. The inclusion $G \leq H$ allows us to regard, via the restriction mapping, V_i also as $\mathbb{Z}_p G$-modules.

We now require a few facts from [4] and from [3]. From [4, Theorem 9.2], the $\mathbb{Z}H$-module $\mathrm{H}_1(\mathcal{T}/\mathcal{N}; \mathbb{Z})$ is isomorphic to the $\mathbb{Z}H$-module $\mathcal{N}_1/[\mathcal{N}_1, \mathcal{N}_1]$. Therefore the $\mathbb{Z}_p H$-module $\mathrm{H}_1(\mathcal{T}/\mathcal{N}; \mathbb{Z}_p) = \mathrm{H}_1(\mathcal{T}/\mathcal{N}; \mathbb{Z}) \otimes \mathbb{Z}_p$ is isomorphic to the $\mathbb{Z}_p H$-module $\mathcal{N}/[\mathcal{N}, \mathcal{N}] \otimes \mathbb{Z}_p \cong \mathcal{N}/[\mathcal{N}, \mathcal{N}]\mathcal{N}^p = \mathcal{N}_1/\mathcal{N}_2 = V_1$. Now, the hypothesis in the statement of the theorem and Lemma 4 allow us to conclude that H acts faithfully on $V_1 = \mathcal{N}_1/\mathcal{N}_2$ and hence we can view H as a subgroup of $\mathrm{Aut}(\mathcal{N}_1/\mathcal{N}_2) = \mathrm{Aut}(V_1)$. This fact will allow us to apply directly the results from [3]. Let $\Sigma = \mathrm{Aut}(V_1)$. From [3, page 416] it follows that the action of Σ on V_1 induces an action on V_i and, moreover, the embedding of H in Σ is compatible with the action of H defined on V_i above.

From [3, Theorems 2 and 3], we deduce that there exists a positive integer i such that the $\mathbb{Z}_p\Sigma$-module V_i contains a regular submodule. Let R be a regular $\mathbb{Z}_p\Sigma$-module contained in V_i and let $(r_\sigma \mid \sigma \in \Sigma)$ be a $\mathbb{Z}_p$-basis of R with $r_\sigma\delta = r_{\sigma\delta}$ for every $\sigma, \delta \in \Sigma$. Let $P := \langle r_\sigma \mid \sigma \in G \rangle$ and observe that P is a regular $\mathbb{Z}_pG$-module.

Since $V_i = \mathcal{N}_i/\mathcal{N}_{i+1}$, we may write $P = \mathcal{P}/\mathcal{N}_{i+1}$ for some subgroup $\mathcal{P}$ of $\mathcal{N}_i$ containing $\mathcal{N}_{i+1}$. Observe that $\mathcal{N}/\mathcal{P}$ is a p-group because $\mathcal{N}/\mathcal{P}$ is a quotient of the p-group $\mathcal{N}_1/\mathcal{N}_{i+1}$. Moreover, the index of $\mathcal{P}$ in $\mathcal{N}$ is finite because $\mathcal{N}_{i+1}$ has finite index in $\mathcal{N}_1 = \mathcal{N}$. (Each $\mathbb{Z}_p$-vector space V_i is finite dimensional because $\mathcal{N}_1$ is finitely generated.)

Let $x \in \mathbf{N}_\mathcal{H}(\mathcal{P}) = \mathbf{N}_{\mathrm{Aut}(\mathcal{T})}(\mathcal{N}) \cap \mathbf{N}_{\mathrm{Aut}(\mathcal{T})}(\mathcal{P})$. Since x normalises $\mathcal{N}$, x acts by conjugation as a linear transformation of the vector spaces $\mathcal{N}_1/\mathcal{N}_2 = V_1$ and $\mathcal{N}_i/\mathcal{N}_{i+1} = V_i$. Denote by $\tau \in \mathrm{Aut}(V_1) = \Sigma$ the linear transformation of V_1 induced by the conjugation by x. Now, τ fixes setwise R because R is a $\mathbb{Z}_p\Sigma$-submodule of V_i. Since x normalises $\mathcal{P}$, τ fixes setwise $\mathcal{P}/\mathcal{N}_{i+1} = P$. Since $r_1 \in P$, we see that $r_1\tau = r_\tau \in P = \langle r_\sigma \mid \sigma \in G \rangle$ and hence $\tau \in G$. Let $y \in \mathcal{G}$ be an element projecting to τ. Now, xy^{-1} projects to the identity element of $\Sigma = \mathrm{Aut}(V_1)$. Therefore xy^{-1} centralises $V_1 = \mathcal{N}_1/\mathcal{N}_2$. Observe that xy^{-1} lies in $\mathcal{H}$ because so does x and y. By hypothesis, $\mathcal{H}/\mathcal{N}$ acts faithfully on $\mathrm{H}_1(\mathcal{T}/\mathcal{N}; \mathbb{Z}_p) = V_1$. Therefore $xy^{-1} \in \mathcal{N}$. Since $\mathcal{N} \le \mathcal{G}$ and $y \in \mathcal{G}$, we obtain $x \in \mathcal{G}$. We have thus shown $\mathbf{N}_\mathcal{H}(\mathcal{P}) \le \mathcal{G}$; the inclusion $\mathcal{G} \le \mathbf{N}_\mathcal{H}(\mathcal{P})$ is obvious. $\square$

Theorem 6. *Let p be an odd prime, let Γ be a finite connected graph without semiedges such that the induced action of $\mathrm{Aut}(\Gamma)$ on $\mathrm{H}_1(\Gamma; \mathbb{Z})$ is faithful, and let $G \le \mathrm{Aut}(\Gamma)$. Then there exists a regular covering projection $\wp\colon \tilde{\Gamma} \to \Gamma$ with $\tilde{\Gamma}$ finite, such that the maximal group that lifts along $\wp$ is G and that the group of covering transformations of $\wp$ is a p-group.*

Proof. Let $\mu\colon \mathcal{T} \to \Gamma$ be the universal covering projection; see Section 2. Then $\mathcal{T}$ is an infinite tree and in view of Lemma 3, G lifts along μ to a group $\mathcal{G} \le \mathrm{Aut}(\mathcal{T})$. Let $\mathcal{N} = \mathrm{CT}(\mu)$. Then $\mathcal{N} \ne 1$, $\mathcal{N}_x = 1$ for every vertex and for every edge x of $\mathcal{T}$, $\mathcal{T}/\mathcal{N} \cong \Gamma$, and we may identify Γ with $\mathcal{T}/\mathcal{N}$ in such a way that $\mathcal{G}/\mathcal{N} = G$ and that the quotient projection $\wp_\mathcal{N}\colon \mathcal{T} \to \mathcal{T}/\mathcal{N}$ is μ.

Let $\mathcal{H} = \mathbf{N}_{\mathrm{Aut}(\mathcal{T})}(\mathcal{N})$. By Lemma 1, $\mathcal{H}$ is the largest group that projects along μ and thus, since $\mathrm{Aut}(\Gamma)$ lifts along μ, $\mathcal{H}$ is the lift of $\mathrm{Aut}(\Gamma)$.

Since $\mathrm{Aut}(\Gamma)$ acts faithfully on $\mathrm{H}_1(\Gamma; \mathbb{Z})$, by Theorem 5, there exists a normal subgroup $\mathcal{P}$ of $\mathcal{N}$ of finite index such that $\mathbf{N}_\mathcal{H}(\mathcal{P}) = \mathcal{G}$ and $\mathcal{N}/\mathcal{P}$ is a p-group.

Let $\tilde{\Gamma} = \mathcal{T}/\mathcal{P}$ and $\tilde{G} = \mathcal{G}/\mathcal{P} \le \mathrm{Aut}(\tilde{\Gamma})$. In view of Lemma 1, the quotient projection $\wp_\mathcal{P}\colon \mathcal{T} \to \tilde{\Gamma}$ is a covering projection and there exists a regular covering projection $\wp\colon \tilde{\Gamma} \to \Gamma$ such that $\mu = \wp \circ \wp_\mathcal{P}$. Moreover, since $\mathcal{G}$ normalises $\mathcal{P}$, the group G lifts along $\wp$ and its lift is $\tilde{G}$.

Let $M \le \mathrm{Aut}(\Gamma)$ be the maximal group that lifts along $\wp$ and let $\tilde{M} \le \mathrm{Aut}(\tilde{\Gamma})$ be its lift. Clearly, $G \le M$ and thus $\tilde{G} \le \tilde{M}$. Since $\mathcal{T}$ is a tree, $\wp_\mathcal{P}$ is a universal covering projection, and in view of Lemma 3, $\tilde{M}$ lifts along $\wp_\mathcal{P}$ to some $\mathcal{M} \le \mathrm{Aut}(\mathcal{T})$. But then $\mathcal{M}$ is the lift of M along $\mu = \wp \circ \wp_\mathcal{P}$, and thus $\mathcal{M} \le \mathbf{N}_{\mathrm{Aut}(\mathcal{T})}(\mathcal{N}) = \mathcal{H}$. On the other hand, $\mathcal{M}$ normalises $\mathcal{P}$ and so $\mathcal{M} \le \mathbf{N}_\mathcal{H}(\mathcal{P}) = \mathcal{G}$. But then $\tilde{M} = \mathcal{M}/\mathcal{P} \le \mathcal{G}/\mathcal{P} = \tilde{G}$, and hence $\tilde{M} = \tilde{G}$. Therefore, $M = \tilde{M}/(\mathcal{N}/\mathcal{P}) = \tilde{G}/(\mathcal{N}/\mathcal{P}) = G$, and thus G is the maximal group that lifts along $\wp$, as required. $\square$

Let us now discuss the condition of G acting faithfully on $\mathrm{H}_1(\Gamma; \mathbb{Z})$.

Lemma 7. *If Γ is a simple 3-edge-connected graph, then* $\mathrm{Aut}(\Gamma)$ *acts faithfully on* $\mathrm{H}_1(\Gamma; \mathbb{Z})$.

Proof. Suppose to the contrary that the action of $\mathrm{Aut}(\Gamma)$ on $\mathrm{H}_1(\Gamma; \mathbb{Z})$ is not faithful. Then there exists an automorphism g fixing every element of $\mathrm{H}_1(\Gamma; \mathbb{Z})$ and a vertex v such that $v^g \neq v$. Let u, w, and z be three neighbours of v. By 3-edge-connectivity of Γ it follows that there is a cycle C_1 through the 2-path uvw and a cycle C_2 through the 2-path uvz. Now fix an orientation of C_1 and C_2 in such a way that u is a predecessor of v in both C_1 and C_2. Consider C_1 and C_2 as elements of $\mathrm{H}_1(\Gamma; \mathbb{Z})$. By assumption, g preserves C_1 and C_2 together with their orientation. In particular, the vertex v^g lies on C_1 and on C_2. For $i \in \{1, 2\}$, let P_i be the path from v^g to u following the cycle C_i in the positive direction with respect to the chosen orientation. Note that, since g preserves the orientation, u^g belongs to neither P_1 nor P_2. Now consider the closed walk C obtained by concatenating P_1 with the reverse of P_2 and consider it as an element of $\mathrm{H}_1(\Gamma; \mathbb{Z})$. By assumption, C is fixed by g. However, the vertex u belongs to C, while u^g does not, a contradiction. $\square$

Since in connected vertex-transitive graphs the edge connectivity equals the valency (see for instance [16, Lemma 3.3.3]), we get the following corollary.

Corollary 8. *If Γ is a simple connected vertex-transitive graph of valency at least 3, then* $\mathrm{Aut}(\Gamma)$ *acts faithfully on* $\mathrm{H}_1(\Gamma; \mathbb{Z})$.

4. A COROLLARY

In this section we strengthen Theorem 6 in the case of certain arc-transitive graphs. Recall that an *s-arc* of a graph Γ is an ordered $(s+1)$-tuple of vertices of Γ such that any two consecutive vertices are adjacent and any three consecutive vertices are pairwise distinct. Thus a 1-arc is simply an ordered pair of adjacent vertices and is also called an *arc*. If a group $G \leq \mathrm{Aut}(\Gamma)$ acts transitively on the set of s-arcs of Γ, we say that Γ is *(G, s)-arc-transitive* (or simply, *G-arc-transitive* if $s = 1$). The symbol G can be dropped from this notation if $G = \mathrm{Aut}(\Gamma)$.

Given a graph Γ, $G \leq \mathrm{Aut}(\Gamma)$ and a vertex v of Γ, we let $G_v^{\Gamma(v)}$ denote the permutation group induced by the action of the vertex-stabiliser G_v on the neighbourhood $\Gamma(v)$ of the vertex v. A finite transitive permutation group L is *graph-restrictive* provided there exists a constant $c = c(L)$ such that whenever Γ is a connected G-arc-transitive graph with $G_v^{\Gamma(v)}$ being permutationally isomorphic to L, the order of the stabiliser G_v is at most $c(L)$. This notion was introduced and studied in [23] and is relevant in the context of the Weiss conjecture: using this terminology, Weiss conjecture states that every primitive group is graph-restrictive.

We will call a transitive permutation group L acting on a set Ω *strongly graph-restrictive* if every group T with $L \leq T \leq \mathrm{Sym}(\Omega)$ is graph-restrictive. Examples of strongly graph-restrictive permutation groups are provided by certain classes of primitive groups. As the culmination of work of Weiss and Trofimov, every 2-transitive group is graph-restrictive and, as an overgroup of a 2-transitive group is still 2-transitive, we deduce that 2-transitive groups are strongly graph-restrictive. The proof of the Weiss conjecture for 2-transitive groups is scattered over many papers and hence this result is somewhat part of folklore; see [23, Section 6] and the references therein for an overview of the argument. Other examples of strongly graph-restrictive groups are provided by primitive groups with abelian socle, that

is, primitive groups of affine-type. Recently in [27] it was proved that Weiss conjecture does hold for this class of primitive groups. In many interesting cases, every overgroup of a primitive group of affine-type is either affine or 2-transitive and hence these groups are strongly graph-restrictive. (For instance, most affine groups whose point stabilisers are primitive linear groups satisfy this property.) Now, it would take us too far astray to describe the primitive groups of affine-type where each overgroup is either affine or 2-transitive, thus we simply refer to [1, 2] or [25] for a thorough analysis of the inclusions among primitive groups. Here we simply observe that every primitive group of prime degree is strongly graph-restrictive. In conclusion, if the Weiss conjecture proved to be true, then every primitive group is strongly graph-restrictive.

For a strongly graph-restrictive group L, we let $c_*(L)$ denote the maximum of all constants $c(T)$ with $L \leq T \leq \mathrm{Sym}(\Omega)$.

Theorem 9. *Let Γ be a finite connected G-arc-transitive graph without semiedges of valency at least 3 such that $G_v^{\Gamma(v)}$ is strongly graph-restrictive. Then there exists a regular covering projection $\wp \colon \tilde{\Gamma} \to \Gamma$ with $\tilde{\Gamma}$ finite, such that the maximal group that lifts along $\wp$ is G and every automorphism of $\tilde{\Gamma}$ projects along $\wp$.*

Proof. Let n be the order of Γ and let p be a prime with $p > n c_*(G_v^{\Gamma(v)})$. By Corollary 8, $\mathrm{Aut}(\Gamma)$ acts faithfully on $\mathrm{H}_1(\Gamma; \mathbb{Z}_p)$, and then by Theorem 6, there exists a regular covering projection $\wp \colon \tilde{\Gamma} \to \Gamma$ with $\tilde{\Gamma}$ finite, such that the maximal group that lifts along $\wp$ is G and that the group of covering transformations of $\wp$ is a p-group.

Let $\tilde{A}$ be the automorphism group of $\tilde{\Gamma}$, let $\tilde{G}$ be the lift of G along $\wp$, and let $N = \mathrm{CT}(\wp)$. From Lemma 1, $\tilde{G}/N \cong G$ and $\tilde{G} = \mathbf{N}_{\tilde{A}}(N)$. Let $\tilde{v}$ be a vertex of $\tilde{\Gamma}$, let $v = \wp(\tilde{v})$, and set $c = c_*(G_v^{\Gamma(v)})$. Since $G_v^{\Gamma(v)} \cong \tilde{G}_{\tilde{v}}^{\tilde{\Gamma}(\tilde{v})}$ is strongly graph-restrictive and $\tilde{G}_{\tilde{v}}^{\tilde{\Gamma}(\tilde{v})} \leq \tilde{A}_{\tilde{v}}^{\tilde{\Gamma}(\tilde{v})}$, we have $|\tilde{A}_{\tilde{v}}| \leq c$. Since $\tilde{G}$ is transitive on the vertices of $\tilde{\Gamma}$, we have $\tilde{A} = \tilde{A}_{\tilde{v}} \tilde{G}$ and hence $[\tilde{A} : \tilde{G}] = [\tilde{A}_{\tilde{v}} : \tilde{G}_{\tilde{v}}] \leq |\tilde{A}_{\tilde{v}}| \leq c < p$. Moreover, $[\tilde{G} : N] = |G| = n|G_v| \leq nc < p$. Therefore, $[\tilde{A} : N] = [\tilde{A} : \tilde{G}][\tilde{G} : N]$ is not divisible by p and hence N is a Sylow p-subgroup of $\tilde{A}$.

By the Sylow theorem, the number of Sylow p-subgroups of $\tilde{A}$ is $[\tilde{A} : \mathbf{N}_{\tilde{A}}(N)] = [\tilde{A} : \tilde{G}] < p$ and is congruent to 1 modulo p. Therefore, $\tilde{A} = \mathbf{N}_{\tilde{A}}(N)$ and hence $\tilde{A} = \tilde{G}$. □

Corollary 10. *Let Γ be a finite connected G-arc-transitive graph of valency at least 3. If G acts transitively on the 2-arcs of Γ or if the valency of Γ is prime, then there exists a regular covering projection $\wp \colon \tilde{\Gamma} \to \Gamma$ with $\tilde{\Gamma}$ finite, such that the maximal group that lifts along $\wp$ is G and every automorphism of $\tilde{\Gamma}$ projects along $\wp$.*

We conclude by daring to conjecture that the requirement of $G_v^{\Gamma(v)}$ being strongly graph-restrictive in Theorem 9 is not necessary.

Conjecture 11. *Let Γ be a finite connected graph such that the induced action of $\mathrm{Aut}(\Gamma)$ on $\mathrm{H}_1(\Gamma; \mathbb{Z})$ is faithful, and let $G \leq \mathrm{Aut}(\Gamma)$. Then there exists a regular covering projection $\wp \colon \tilde{\Gamma} \to \Gamma$ with $\tilde{\Gamma}$ finite, such that the maximal group that lifts along $\wp$ is G and such that $\mathrm{Aut}(\tilde{\Gamma})$ projects along $\wp$.*

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