Contributions to the qualitative theory of scattering

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Abstract—

Whenever a problem is difficult from the topological, algebraic or functional analytical points of view, a "qualitative theory" arises. The example of choice is the qualitative theory of dynamical systems, founded by Henri Poincaré in 1881 - 1882. In the context of electromagnetics and optics there exist comprehensive accounts on group theoretical methods [1] and on symmetries [2,3]. This work aims at contributing results in two directions, limited to fixed wavenumber k.

1 - *Symmetries and ε-false symmetries in scalar potential scattering at fixed wavenumber.* By letting n $(= 0, 1, \dots)$ denote the ordinal of a term in the Born sequence, one has:

$$u^{(n+1)}_{\mathrm{sc},\gamma}[\vec{x};\vec{\alpha},k,q_\gamma] - u^{(n+1)}_{\mathrm{sc},0}[\vec{x};\vec{\alpha},k,q_0] = \int_{B_a[\mathbf{O}]} \Phi[\vec{x}-\vec{y}]\left(\mathrm{e}^{\mathrm{i}k\vec{\alpha}\cdot\vec{y}} + u^{(n)}_{\mathrm{sc},\gamma}[\vec{y};\vec{\alpha},k,q_\gamma]\right) q_\gamma[\vec{y}]\mathrm{d}^3y -$$
$$- \int_{B_a[\mathbf{O}]} \Phi[\vec{x}-\vec{y}]\left(\mathrm{e}^{\mathrm{i}k\vec{\alpha}\cdot\vec{y}} + u^{(n)}_{\mathrm{sc},0}[\vec{y};\vec{\alpha},k,q_0]\right) q_0[\vec{y}]\mathrm{d}^3y \ , \tag{1}$$

with $u^{(0)}_{\mathrm{sc},0}[\vec{\cdot}\,;\vec{\cdot},k,\cdot] = u^{(0)}_{\mathrm{sc},\gamma}[\vec{\cdot}\,;\vec{\cdot},k,\cdot] = 0$. Eq. (1) compares scalar waves at location $\vec{x} \in \mathbb{R}^3$ resulting from the scattering of the unit-amplitude incident plane wave $\mathrm{e}^{\mathrm{i}k\vec{\alpha}\cdot\vec{y}}$ by two different potentials. The reference potential, $q_0[.]$, is a real-valued function of sufficient regularity and supported in the sphere $B_a[\mathbf{O}]$ of radius a (> 0) and centered at the origin. The "transformed" potential, $q_\gamma[.]$, is the representation of $q_0[.]$ in a coordinate frame rotated by the matrix $\mathbf{D}_\gamma \in$ SO(3) with respect to the original one. $\Phi[.-.]$ is the fundamental solution of the scalar Helmholtz equation of wave number k (> 0). Eq. 1 can be shown to have a non-local counterpart.

Applications. From Eq. (1) and its non local counterpart, one derives the following. $(A1)$ Lipschitz estimates for scattered waves with respect to $q_\gamma[.]$ and $q_0[.]$. $(A2)$ If $\varepsilon > 0$ and $\mathcal{U}$ is a suitable normed space, a class of transformations $\mathbf{C}_{\delta[\epsilon]} \notin$ SO(3), although related to $\mathbf{D}_\gamma$, which yield scattered waves $u_{\mathrm{sc},\delta}$ asymptotically complying with

$$||u^{(n+1)}_{\mathrm{sc},\delta} - u^{(n+1)}_{\mathrm{sc},0}||_{\mathcal{U}} < \varepsilon \ .$$

Experimentally, such $u_{\mathrm{sc},\delta}$ is likely to be mistaken for $u_{\mathrm{sc},0}$ by the detecting system. The transformations $\mathbf{C}_{\delta[\epsilon]}$ thus produce *ε-false symmetries.*

2 - *Approximate propagators in scalar diffraction at fixed wavenumber and related symmetries.* The second part of the presentation will focus on group properties and related symmetries of the Helmholtz equation and of its approximations. Aperture diffraction in the halfspace of a scalar wave at fixed k is described by a propagator. A suitable asymptotic approximation yields the Fresnel ($\equiv$ paraxial) propagator. Further approximation leads to the Fourier propagator. The related group properties and information contents of both approximation stages will be compared by taking some available results [4-8] into account.

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