Research Article

Conchita Martínez-Pérez, Francesco Matucci and Brita E. A. Nucinkis*

Cohomological finiteness conditions and centralisers in generalisations of Thompson's group V

Abstract: We consider generalisations of Thompson's group V, denoted by $V_r(\Sigma)$, which also include the groups of Higman, Stein and Brin. We show that, under some mild hypotheses, $V_r(\Sigma)$ is the full automorphism group of a Cantor algebra. Under some further minor restrictions, we prove that these groups are of type F_∞ and that this implies that also centralisers of finite subgroups are of type F_∞.

Keywords: Cohomological finiteness conditions, Thompson groups

MSC 2010: 20J05

DOI: 10.1515/forum-2014-0176
Received October 6, 2014; revised February 3, 2015

Communicated by: Frederick R. Cohen

1 Introduction

Thompson's group V is defined as a homeomorphism group of the Cantor set. The group V has many interesting generalisations such as the Higman–Thompson groups $V_{n,r}$ ([10]), Stein's generalisations [14] and Brin's higher-dimensional Thompson groups sV ([3]). All these groups contain any finite group, contain free abelian groups of infinite rank, are finitely presented and of type FP_∞ (see work by several authors in [4, 7, 9, 11, 14]). The first and third authors together with Kochloukova [11, 13] further generalised these groups, denoted by $V_r(\Sigma)$ or $G_r(\Sigma)$, as automorphism groups of certain Cantor algebras. We shall use the notation $V_r(\Sigma)$ in this paper. We show in Theorem 2.5 that they are the full automorphism groups of these algebras.

Fluch, Marschler, Witzel and Zaremsky [7] used Morse-theoretic methods to prove that Brin's groups sV are of type F_∞. By adapting their methods, we show in Theorem 3.1 that under some restrictions on the Cantor algebra, which still comprehend all families mentioned above, $V_r(\Sigma)$ is of type F_∞. We also give some constructions of further examples.

Bleak, Bowman, Gordon, Graham, Hughes, Matucci and Sapir [2] and the first and third authors [13] showed independently that centralisers of finite subgroups Q in $V_{n,r}$ and $V_r(\Sigma)$ can be described as extensions

$$K \rightarrowtail C_{V_r(\Sigma)}(Q) \twoheadrightarrow V_{r_1}(\Sigma) \times \cdots \times V_{r_t}(\Sigma),$$

where K is locally finite and $r_1, \dots, r_t$ are integers uniquely determined by Q. It was conjectured in [13] that these centralisers are of type F_∞ if the groups $V_r(\Sigma)$ are. In Section 4, we expand the description of the centralisers given in [2, 13], which allows us to prove that the conjecture holds true. This also implies that any of the generalised $V_r(\Sigma)$ which are of type F_∞ admit a classifying space for proper actions that is

Conchita Martínez-Pérez: Departamento de Matemáticas, Universidad de Zaragoza, 50009 Zaragoza, Spain, e-mail: conmar@unizar.es
Francesco Matucci: Département de Mathématiques, Bâtiment 425, Faculté des Sciences d'Orsay, Université Paris-Sud 11, 91405 Orsay Cedex, France, e-mail: francesco.matucci@math.u-psud.fr
***Corresponding author: Brita E. A. Nucinkis:** Department of Mathematics, Royal Holloway, University of London, Egham, Surrey TW20 0EX, UK, e-mail: brita.nucinkis@rhul.ac.uk

Brought to you by | Royal Holloway, University of London
Authenticated
Download Date | 10/21/15 12:54 PM

a mapping telescope of cocompact classifying spaces for smaller families of finite subgroups. In other words, these groups are of type quasi-$\underline{F}_\infty$. For definitions and background, the reader is referred to [13].

We conclude with a description of normalisers of finite subgroups in Section 5. These turn up in computations of the source of the rationalised Farrell–Jones assembly map, where one needs to compute not only centralisers, but also the Weyl groups $W_G(Q) = N_G(Q)/C_G(Q)$. For more details, see [12], or [8] for an example where these are computed for Thompson's group T.

2 Background on generalised Thompson groups

2.1 Cantor algebras

We shall follow the notation of [13, Section 2] and begin by defining the Cantor algebras $U_r(\Sigma)$. Consider a finite set of colours $S = \{1, \dots, s\}$ and associate to each $i \in S$ an integer $n_i > 1$, called the arity of the colour i. Let U be a set on which, for all $i \in S$, the following operations are defined: an n_i-ary operation $\lambda_i : U^{n_i} \to U$, and n_i 1-ary operations $\alpha_i^1, \dots, \alpha_i^{n_i}$ with $\alpha_i^j : U \to U$. Denote $\Omega = \{\lambda_i, \alpha_i^j\}_{i,j}$ and call U an Ω-algebra. For more details, see [5] and [11]. We write these operations on the right. We also consider, for each $i \in S$ and $v \in U$, the map $\alpha_i : U \to U^{n_i}$ given by $v\alpha_i := (v\alpha_i^1, v\alpha_i^2, \dots, v\alpha_i^{n_i})$. The maps α_i are called descending operations, or expansions, and the maps λ_i are called ascending operations, or contractions. Any word in the descending operations is called a descending word.

A *morphism* between Ω-algebras is a map commuting with all operations in Ω. Let $\mathfrak{B}_0$ be the category of all Ω-algebras for some Ω. An object $U_0(X) \in \mathfrak{B}_0$ is a *free object* in $\mathfrak{B}_0$ with X as a *free basis* if for any $S \in \mathfrak{B}_0$ any mapping $\theta : X \to S$ can be extended in a unique way to a morphism $U_0(X) \to S$.

For every set X, there is an Ω-algebra, free on X, called the Ω-*word algebra on* X and denoted by $W_\Omega(X)$ (see [11, Definition 2.1]). Let $B \subset W_\Omega(X)$, $b \in B$ and let i be a colour of arity n_i. The set

$$(B \setminus \{b\}) \cup \{b\alpha_i^1, \dots, b\alpha_i^{n_i}\}$$

is called a simple expansion of B. Analogously, if $b_1, \dots, b_{n_i} \subseteq B$ are pairwise distinct, then

$$(B \setminus \{b_1, \dots, b_{n_i}\}) \cup \{(b_1, \dots, b_{n_i})\lambda_i\}$$

is a simple contraction of B. A chain of simple expansions (contractions) is an expansion (contraction). A subset $A \subseteq W_\Omega(X)$ is called *admissible* if it can be obtained from the set X by finitely many expansions or contractions.

We shall now define the notion of a Cantor algebra. Fix a finite set X and consider the variety of Ω-algebras satisfying a certain set of identities as follows.

Definition 2.1 ([13, Section 2]). We denote by $\Sigma = \Sigma_1 \cup \Sigma_2$ the following set of laws in the alphabet X.

(i) A set of laws Σ_1 given by

$$u\alpha_i\lambda_i = u, \quad (u_1, \dots, u_{n_i})\lambda_i\alpha_i = (u_1, \dots, u_{n_i})$$

for every $u \in W_\Omega(X)$, $i \in S$ and n_i-tuple $(u_1, \dots, u_{n_i}) \in W_\Omega(X)^{n_i}$.

(ii) A second set of laws

$$\Sigma_2 = \bigcup_{1 \le i < i' \le s} \Sigma_2^{i,i'},$$

where each $\Sigma_2^{i,i'}$ is either empty or consists of the following laws. Consider first an i and fix a map

$$f : \{1, \dots, n_i\} \to \{1, \dots, s\}.$$

For each $1 \le j \le n_i$, we see $\alpha_i^j\alpha_{f(j)}$ as a set of length 2 sequences of descending operations and let $\Lambda_i = \bigcup_{j=1}^{n_i} \alpha_i^j\alpha_{f(j)}$. Do the same for i' (with a corresponding map f') to get $\Lambda_{i'}$. We need to assume that f, f' are chosen so that $|\Lambda_i| = |\Lambda_{i'}|$ and fix a bijection $\phi : \Lambda_i \to \Lambda_{i'}$. Then, $\Sigma_2^{i,i'}$ is the set of laws

$$uv = u\phi(v), \quad v \in \Lambda_i,\ u \in W_\Omega(X).$$

Brought to you by | Royal Holloway, University of London
Authenticated
Download Date | 10/21/15 12:54 PM

Factor out of $W_\Omega(X)$ the fully invariant congruence $\mathfrak{q}$ generated by Σ to obtain an Ω-algebra $W_\Omega(X)/\mathfrak{q}$ satisfying the identities in Σ. The algebra $W_\Omega(X)/\mathfrak{q} = U_r(\Sigma)$, where $r = |X|$, is called a *Cantor algebra*.

As in [11], we say that Σ is *valid* if for any admissible $Y \subseteq W_\Omega(X)$ we have $|Y| = |\overline{Y}|$, where $\overline{Y}$ is the image of Y under the epimorphism $W_\Omega(X) \twoheadrightarrow U_r(\Sigma)$. In particular, this implies that $U_r(\Sigma)$ is a free object on X in the class of those Ω-algebras which satisfy the identities Σ above. In other words, this implies that X is a basis. If the set Σ used to define $U_r(\Sigma)$ is valid, we also say that $U_r(\Sigma)$ is valid. As done for $W_\Omega(X)$, we say that a subset $A \subset U_r(\Sigma)$ is *admissible* if it can be obtained by a finite number of expansions or contractions from $\overline{X}$, where expansions and contractions mean the same as before. We shall, from now on, not distinguish between X and $\overline{X}$. If A can be obtained from a subset B by expansions only, we will say that A is an expansion or a descendant of B and we will write $B \leq A$. If A can be obtained from B by applying a single descending operation, i.e., if

$$A = (B \setminus \{b\}) \cup \{b\alpha_i^1, \dots, b\alpha_i^{n_i}\}$$

for some colour i of arity n_i, then we will say that A is a simple expansion of B.

Remark 2.2. Let B be a basis in a valid $U_r(\Sigma)$ and let $A \leq B$. The fact that A is also a basis implies that for any element $b \in B$ there is a single $A(b) \in A$ such that $A(b)w = b$ for some descending word w. In this case, we say that $A(b)$ is a prefix of b.

Definition 2.3 ([13, Definition 2.12]). Let $U_r(\Sigma)$ be a valid Cantor algebra. Then, $V_r(\Sigma)$ denotes the group of all Ω-algebra automorphisms of $U_r(\Sigma)$ which are induced by a map $V \to W$, where V and W are admissible subsets of the same cardinality.

Throughout this paper, we shall denote group actions on the left.

Remark 2.4. For any basis $A \geq X$ and any $g \in V_r(\Sigma)$, there is some B with $A \leq B, gB$. To see this, take B such that $A, g^{-1}A \leq B$, which exists by [13, Lemma 2.8].

We now explore the relation between admissible subsets and bases. We say that $U_r(\Sigma)$ is *bounded* (see [13, Definition 2.7]) if for all admissible subsets Y and Z such that there is some admissible $A \leq Y, Z$, there is a unique least upper bound of Y and Z. By a unique least upper bound we mean an admissible subset T such that $Y \leq T$ and $Z \leq T$, and whenever there is an admissible set S also satisfying $Y \leq S$ and $Z \leq S$, then $T \leq S$.

Theorem 2.5. *Let $U_r(\Sigma)$ be a valid and bounded Cantor algebra. Then, $V_r(\Sigma)$ is the full group of Ω-algebra automorphisms of $U_r(\Sigma)$.*

Proof. Any Ω-algebra automorphism of $U_r(\Sigma)$ is induced by a bijective map between two bases V and W with the same cardinality. Thus, from the definition of $V_r(\Sigma)$, we need to show that, under our hypotheses, a subset of $U_r(\Sigma)$ is admissible if and only if it is a basis.

Since every admissible subset is a basis of $U_r(\Sigma)$, see [11, Lemma 2.5], we only need to show that any basis of $U_r(\Sigma)$ is admissible. Let $Y = \{y_1, \dots, y_n\}$ be an arbitrary basis. Since X is a basis, it generates all of $U_r(\Sigma)$. Hence, for each $y_i \in Y$, there exists some admissible subset T_i of $U_r(\Sigma)$ containing y_i. Now, let Z be a common upper bound of the T_i, $i = 1, \dots, n$. This exists by [13, Lemma 2.8] using the argument of [11, Proposition 3.4]. The set Z is an admissible subset containing a set $\widehat{Y}$ whose elements are obtained by performing finitely many descending operations in Y. Denote by $\widehat{Y}_i$ the subsets of $\widehat{Y}$ given by $\{y_i\} \leq \widehat{Y}_i$ and $\widehat{Y} = \bigcup \widehat{Y}_i$. Since Y and Z are bases and $Y \leq Z$, then Remark 2.2 implies that $\widehat{Y}_i \cap \widehat{Y}_j = \varnothing$ for $i \neq j$. By Remark 2.6, since $\widehat{Y}$ is admissible, it is a basis. Remark 2.6 also implies that Z is a basis. It follows from the definition of a free basis, see, e.g., [11, p. 3], that no proper subset of a basis is a basis. Hence, $\widehat{Y} = Z$ is admissible, thus Y is admissible as well. □

Remark 2.6. Any set obtained from a basis by performing expansions or contractions is also a basis. Furthermore, the cardinality m of every admissible subset satisfies $m \equiv r \bmod d$ for $d := \gcd\{n_i - 1 : i = 1, \dots, s\}$. In particular, any basis with m elements can be transformed into one of r elements. Hence, $U_r(\Sigma) = U_m(\Sigma)$ and we may assume that $r \leq d$.

Brought to you by | Royal Holloway, University of London
Authenticated
Download Date | 10/21/15 12:54 PM

2.2 Brin-like groups

In this subsection, we give some examples of the groups $V_r(\Sigma)$, which generalise both Brin's groups sV ([3]) and Stein's groups $V(l, A, P)$ ([14]). Furthermore, these groups satisfy the conditions of Definition 2.14 below and we show in Section 3 that they are of type F_∞.

Example 2.7. (i) We begin by recalling the definition of the Brin algebra [11, Section 2], [13, Example 2.4]. Consider the set of s colours $S = \{1, \dots, s\}$, all of which have arity 2, together with the relations $\Sigma := \Sigma_1 \cup \Sigma_2$ with

$$\Sigma_2 := \{\alpha_i^l \alpha_j^t = \alpha_j^t \alpha_i^l : 1 \le i \ne j \le s,\ l, t = 1, 2\}.$$

Then, $V_r(\Sigma) = sV$ is Brin's group.

(ii) Furthermore, one can also consider s colours, all of arity $n_i = n \in \mathbb{N}$, for all $1 \le i \le s$. Let

$$\Sigma_2 := \{\alpha_i^l \alpha_j^t = \alpha_j^t \alpha_i^l : 1 \le i \ne j \le s,\ 1 \le l, t \le n\}.$$

Here, $V_r(\Sigma) = sV_n$ is Brin's group of arity n. It was shown in [13, Example 2.9] that in this case $U_r(\Sigma)$ is valid and bounded.

(iii) We can also mix arities. Consider s colours, each of arity $n_i \in \mathbb{N}$, $i = 1, \dots, s$, together with $\Sigma := \Sigma_1 \cup \Sigma_2$, where

$$\Sigma_2 := \{\alpha_i^l \alpha_j^t = \alpha_j^t \alpha_i^l : 1 \le i \ne j \le s,\ 1 \le l \le n_i,\ 1 \le t \le n_j\}.$$

We denote these mixed-arity Brin groups by $V_r(\Sigma) = V_{\{n_1\},\dots,\{n_s\}}$. The same argument as in [11, Lemma 3.2] yields that the Cantor algebra $U_r(\Sigma)$ in this case is also valid and bounded.

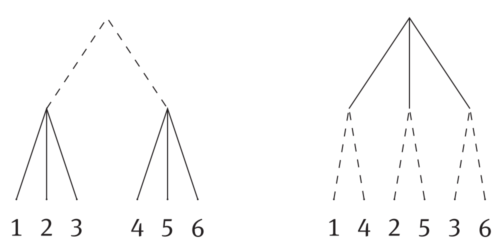

Figure 1. Visualising the identities in Σ_2 for $V_{\{2\},\{3\}}$.

Example 2.8. We now recall the laws Σ_2 for Stein's groups [14]. Let $P \subseteq \mathbb{Q}_{>0}$ be a finitely generated multiplicative group. Consider a basis of P of the form $\{n_1, \dots, n_s\}$ with all $n_i \ge 1$ integers, $i = 1, \dots, s$. Consider s colours of arities $\{n_1, \dots, n_s\}$ and let $\Sigma = \Sigma_1 \cup \Sigma_2$ with Σ_2 the set of identities given by the order-preserving identification

$$\begin{aligned}&\{\alpha_i^1 \alpha_j^1, \dots, \alpha_i^1 \alpha_j^{n_j}, \alpha_i^2 \alpha_j^1, \dots, \alpha_i^2 \alpha_j^{n_j}, \dots, \alpha_i^{n_i} \alpha_j^1, \dots, \alpha_i^{n_i} \alpha_j^{n_j}\}\\ &\quad = \{\alpha_j^1 \alpha_i^1, \dots, \alpha_j^1 \alpha_i^{n_i}, \alpha_j^2 \alpha_i^1, \dots, \alpha_j^2 \alpha_i^{n_i}, \dots, \alpha_j^{n_j} \alpha_i^1, \dots, \alpha_j^{n_j} \alpha_i^{n_i}\},\end{aligned}$$

where $i \ne j$ and $i, j \in \{1, \dots, s\}$.

The resulting Brown–Stein algebra $U_r(\Sigma)$ is valid and bounded, see, e.g., [13, Lemma 2.11]. We denote the resulting groups by $V_r(\Sigma) = V_{\{n_1,\dots,n_s\}}$.

Definition 2.9. Let S be a set of s colours together with arities n_i for each $i = 1, \dots, s$. Suppose S can be partitioned into m disjoint subsets S_k such that for each k, the set $\{n_i : i \in S_k\}$ is a basis for a finitely generated multiplicative group $P_k \subseteq \mathbb{Q}_{>0}$.

Consider Ω-algebras on s colours with arities as above and the set of identities $\Sigma = \Sigma_1 \cup \Sigma_2$, where $\Sigma_2 = \Sigma_{2_1} \cup \Sigma_{2_2}$ is given as follows.

Brought to you by | Royal Holloway, University of London
Authenticated
Download Date | 10/21/15 12:54 PM

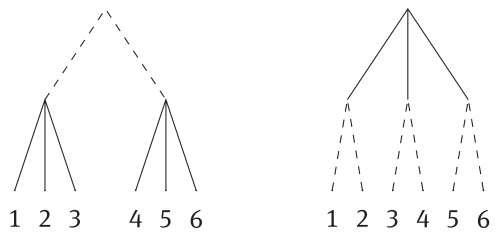

Figure 2. Visualising the identities in Σ_2 for $V_{\{2,3\}}$.

- Σ_{2_1} is given by the following order-preserving identifications (as in the Brown–Stein algebra in Example 2.8). For each $k \le m$, we have
$$\{\alpha_i^1\alpha_j^1, \dots, \alpha_i^1\alpha_j^{n_j}, \alpha_i^2\alpha_j^1, \dots, \alpha_i^2\alpha_j^{n_j}, \dots, \alpha_i^{n_i}\alpha_j^1, \dots, \alpha_i^{n_i}\alpha_j^{n_j}\}$$
$$= \{\alpha_j^1\alpha_i^1, \dots, \alpha_j^1\alpha_i^{n_i}, \alpha_j^2\alpha_i^1, \dots, \alpha_j^2\alpha_i^{n_i}, \dots, \alpha_j^{n_j}\alpha_i^1, \dots, \alpha_j^{n_j}\alpha_i^{n_i}\},$$
where $i \neq j$ and $i, j \in S_k$.
- Σ_{2_2} is given by Brin-like identifications (as in Example 2.7). For all $i \in S_k$ and $j \in S_l$ such that $S_k \cap S_l = \varnothing$, $k \neq l$, $k, l \le m$, we have
$$\Sigma_{2_2} := \{\alpha_i^l\alpha_j^t = \alpha_j^t\alpha_i^l : 1 \le l \le n_i,\ 1 \le t \le n_j\}.$$

We call the resulting Cantor algebra $U_r(\Sigma)$ Brin-like and denote the generalised Higman–Thompson group by $V_r(\Sigma) = V_{\{n_i:i\in S_1\},\dots,\{n_i:i\in S_m\}}$.

Example 2.10. From Definition 2.9, we notice the following examples.
(i) If $m = s$, we have the Brin groups as in Example 2.7 (iii).
(ii) If $m = 1$, we have Stein groups as in Example 2.8.
(iii) Suppose that we have $\{n_i : i \in S_k\} = \{n_i : i \in S_l\}$ for each $l, k \le m$. Then, the resulting group can be viewed as a higher-dimensional Stein group $mV_{\{n_i:i\in S_m\}}$.

Question 2.11. Suppose $m \notin \{1, s\}$. What are the conditions on the arities for the groups $V_{\{n_i:i\in S_1\},\dots,\{n_i:i\in S_m\}}$ not be isomorphic to any of the known generalised Thompson groups such as the Higman–Thompson groups, Stein's groups or Brin's groups? More generally, when are two of these groups non-isomorphic? See [6] for some special cases.

Remark 2.12. We can view these groups as bijections of m-dimensional cuboids in the m-dimensional Cartesian product of the Cantor set, similarly to the description given for sV, the Brin–Thompson groups. In each direction, we get subdivisions of the Cantor set as in the Stein–Brown groups given by Σ_{2_1}.

Lemma 2.13. *The Brin-like Cantor algebras are valid and bounded.*

Proof. Using the description of Remark 2.12, we can apply the same argument as in [11, Lemma 3.2]. □

All groups defined in this subsection satisfy the following condition on the relations in Σ, and hence satisfy the conditions needed in Section 3.

Definition 2.14. Using the notation of Definition 2.1, suppose that for all $i \neq i'$, $i, i' \in S$, we have $\Sigma_2^{i,i'} \neq \varnothing$, and that $f(j) = i'$ for all $j = 1, \dots, n_i$ and $f'(j') = i$ for all $j' = 1, \dots, n_{i'}$. Then, we say that Σ (or, equivalently, $U_r(\Sigma)$) is *complete*.

Remark 2.15. The Brin-like Cantor algebras are complete.

3 Finiteness conditions

In this section, we prove the following result.

Brought to you by | Royal Holloway, University of London
Authenticated
Download Date | 10/21/15 12:54 PM

Theorem 3.1. *Let Σ be valid, bounded and complete. Then, $V_r(\Sigma)$ is of type* F_∞.

We closely follow [7], where it is shown that Brin's groups sV are of type F_∞. We shall use a different notation, which is more suited to our set-up, and we will explain where the original argument has to be modified in order to get the more general case. Throughout this section, $U_r(\Sigma)$ denotes a valid, bounded and complete Cantor algebra.

Definition 3.2. Let $B \leq A$ be admissible subsets of $U_r(\Sigma)$. We say that the expansion $B \leq A$ is *elementary* if there are no repeated colours in the paths from leaves in B to their descendants in A. Since Σ is complete, this condition is preserved by the relations in Σ. We denote an elementary expansion by $B \preceq A$. We say that the expansion is *very elementary* if all paths have length at most 1. In this case, we write $B \sqsubseteq A$.

Remark 3.3. If $A \preceq B$ is elementary (very elementary) and $A \leq C \leq B$, then $A \preceq C$ and $C \preceq B$ are elementary (very elementary).

Lemma 3.4. *Let Σ be complete, valid and bounded. Then, any admissible basis A has a unique maximal elementary admissible descendant denoted by $\mathcal{E}(A)$.*

Proof. Let $\mathcal{E}(A)$ be the admissible subset of $n_1 \cdots n_s|A|$ elements obtained by applying all descending operations exactly once to every element of A. □

3.1 The Stein subcomplex

Denote by $\mathcal{P}_r$ the poset of admissible bases in $U_r(\Sigma)$. The same argument as in [11, Lemma 3.5 and Remark 3.7] shows that its geometric realisation $|\mathcal{P}_r|$ is contractible and that $V_r(\Sigma)$ acts on $\mathcal{P}_r$ with finite stabilisers. In [11, 13], this poset was denoted by $\mathfrak{A}$, but here we will follow the notation of [7]. This poset is essentially the same as the poset of [7], denoted there by $\mathcal{P}_r$ as well.

We now construct the Stein complex $\mathcal{S}_r(\Sigma)$, which is a subcomplex of $|\mathcal{P}_r|$. The vertices in $\mathcal{S}_r(\Sigma)$ are given by the admissible subsets of $U_r(\Sigma)$. The k-simplices are given by chains of expansions $Y_0 \leq \cdots \leq Y_k$, where $Y_0 \preceq Y_k$ is an elementary expansion.

Lemma 3.5. *Let $A, B \in \mathcal{P}_r$ with $A < B$. There exists a unique $A < B_0 \leq B$ such that $A \prec B_0$ is elementary and for any $A \leq C \leq B$ with $A \preceq C$ elementary, we have $C \preceq B_0$.*

Proof. Let $\mathcal{E}(A)$ be as in the proof of Lemma 3.4. Let $B_0 = \mathrm{glb}(\mathcal{E}(A), B)$, which exists by [11, Lemma 3.14]. If $A \preceq C \leq B$, then $C \leq \mathcal{E}(A)$ and so $C \leq B_0$. □

Lemma 3.6. *For every r and every valid, bounded and complete Σ, the Stein space $\mathcal{S}_r(\Sigma)$ is contractible.*

Proof. By [11, Lemma 3.5], $|\mathcal{P}_r|$ is contractible. Now, use the same argument of [7, Corollary 2.5] to deduce that $\mathcal{S}_r(\Sigma)$ is homotopy equivalent to $|\mathcal{P}_r|$. Essentially, the idea is to use Lemma 3.5 to show that each simplex in $|\mathcal{P}_r|$ can be pushed to a simplex in $\mathcal{S}_r(\Sigma)$. □

Remark 3.7. Notice that the action of $V_r(\Sigma)$ on $\mathcal{P}_r$ induces an action of $V_r(\Sigma)$ on $\mathcal{S}_r(\Sigma)$ with finite stabilisers.

Consider the Morse function $t(A) = |A|$ in $\mathcal{S}_r(\Sigma)$ and filter the complex with respect to t, i.e.,

$$\mathcal{S}_r(\Sigma)^{\leq n} := \text{full subcomplex supported on } \{A \in \mathcal{S}_r(\Sigma) : t(A) \leq n\}.$$

By the same argument as in [11, Lemma 3.7], $\mathcal{S}_r(\Sigma)^{\leq n}$ is finite modulo the action of $V_r(\Sigma)$. Let $\mathcal{S}_r(\Sigma)^{<n}$ be the complex given by the vertex set $\{A \in \mathcal{S}_r(\Sigma) : t(A) < n\}$.

Provided that

$$\text{the connectivity of the pair } (\mathcal{S}_r(\Sigma)^{\leq n}, \mathcal{S}_r(\Sigma)^{<n}) \text{ tends to } \infty \text{ as } n \to \infty, \tag{3.1}$$

Brown's theorem [4, Corollary 3.3] implies that $V_r(\Sigma)$ is of type F_∞, thus proving Theorem 3.1. The rest of this section is devoted to proving (3.1).

Brought to you by | Royal Holloway, University of London
Authenticated
Download Date | 10/21/15 12:54 PM

3.2 Connectivity of descending links

Recall that for any $A \in \mathcal{S}_r(\Sigma)$, the descending link $L(A) := \mathrm{lk}{\downarrow}^t(A)$ with respect to t is defined to be the intersection of the link $\mathrm{lk}(A)$ with $\mathcal{S}_r(\Sigma)^{<n}$, where $t(A) = n$. To show (3.1), we proceed as in [7]. Using Morse theory, the problem is reduced to showing that for A as before, the connectivity of $L(A)$ tends to ∞ when $t(A) = n \to \infty$. Whenever this happens, we will say that $L(A)$ is *n-highly connected*. More generally, assume we have a family of complexes $(X_\alpha)_{\alpha\in\Lambda}$ together with a map $n : \Lambda \to \mathbb{Z}_{>0}$ such that the set $\{n(\alpha)_{\alpha\in\Lambda}\}$ is unbounded. Assume further that whenever $n(\alpha) \to \infty$, the connectivity of the associated complexes X_α tends to ∞. In this case, we will say that the family is n-highly connected.

Note that $L(A)$ is the subcomplex of $\mathcal{S}_r(\Sigma)$ generated by $\{B : B \prec A$ is an elementary expansion$\}$. Following [7], define a height function h for $B \in L(A)$ by

$$h(B) := (c_s, \dots, c_2, b),$$

where $b = |B|$ and c_i, $i = 2, \dots, s$, is the number of elements in A whose length as descendants of their parent in B is i. We order these heights lexicographically. Let $c(B) = (c_s, \dots, c_2)$, which are also ordered lexicographically. Denote by $L_0(A)$ the subcomplex of $\mathcal{S}_r(\Sigma)$ generated by $\{B : B \sqsubset A$ is a very elementary expansion$\}$. Then, for any $B \in L(A)$, $B \in L_0(A)$ if and only if $h(B) = (0, \dots, 0, |B|)$.

Lemma 3.8. *The set of complexes of the form $L_0(A)$ is $t(A)$-highly connected.*

Proof. For any $n \geq 0$, we define a complex denoted by K_n as follows. Start with a set A with n elements. The vertex set of K_n consists of labelled subsets of A where the possible labels are the colours $\{1, \dots, s\}$ and where a subset labelled i has precisely n_i elements. Recall that n_i is the arity of the colour i. A k-simplex $\{\sigma_0, \dots, \sigma_k\}$ in K_n is given by an unordered set of pairwise disjoint σ_j. This complex is isomorphic to the barycentric subdivision of $L_0(A)$ for $n = t(A)$. To prove that K_n is n-highly connected, proceed as in the proof of [4, Lemma 4.20]. □

Now, consider descending links in $L(A)$ with respect to the height function h, i.e., for $B \in L(A)$, let $\mathrm{lk}{\downarrow}^h(B)$ be the subcomplex of $L(A)$ generated by $\{C \in L(A) : h(C) \leq h(B)$ and either $B < C$ or $C > B\}$. Consider the following two cases.

(i) $B \in L(A) \setminus L_0(A)$ and there is at least one element of B that is expanded precisely once to obtain A.

(ii) $B \in L(A) \setminus L_0(A)$ and no element of B is expanded precisely once to obtain A.

The next two lemmas show that in either case $\mathrm{lk}{\downarrow}^h(B)$ is $t(A)$-highly connected.

As in [7], the descending link $\mathrm{lk}{\downarrow}^h(B)$ of some $B \in L(A)$ with respect to h can be viewed as the join of two subcomplexes, the downlink and the uplink. The downlink consists of those elements C such that $C < B$ and $h(C) < h(B)$. Hence, $c(B) = c(C)$. The uplink consists of those C that $B < C$, $h(C) < h(B)$, and therefore $c(B) > c(C)$.

Lemma 3.9. *Let $B \in L(A)$ as in* (i)*. Then,* $\mathrm{lk}{\downarrow}^h(B)$ *is contractible.*

Proof. It suffices to follow the proof of [7, Lemma 3.7]. We briefly sketch this proof using our notation. Let $b \in B$ be an element that is expanded precisely once to obtain A. Given $B \prec A$ and let $b \in B$, which is expanded precisely once to get to A, then there is an M such that $B \preceq M \sqsubset A$ and $b \in M$. The existence of M follows from a variation of Lemma 3.5. Now, for any $C \in \mathrm{lk}{\downarrow}^h(B)$ lying in the uplink, we let $B \prec C_0 \sqsubseteq C$, where C_0 is obtained by performing all expansions in B needed to get C, except the one of b.

One easily checks that $C_0 \leq M$, that C_0 and M lie in $\mathrm{lk}{\downarrow}^h(B)$ and that both C_0 and M lie in the uplink. Hence, $M \geq C_0 \leq C$ provides a contraction of the uplink. As $\mathrm{lk}{\downarrow}^h(B)$ is the join of the downlink and the uplink, we get the result. □

Lemma 3.10. *Let B be as in* (ii)*. Then,* $\mathrm{lk}{\downarrow}^h(B)$ *is $t(A)$-highly connected.*

Proof. As before, we follow the proof of [7, Lemma 3.8] with only minor changes. With our notation, we let k_s be the number of elements in B that are also leaves of A and let k_b be the remaining leaves. Then, one checks that the uplink in $\mathrm{lk}{\downarrow}^h(B)$ is k_b-highly connected and that the downlink is k_s-highly connected. As $t(A) = n \leq k_b n_1 \cdots n_s + k_s$, we get the result. □

Brought to you by | Royal Holloway, University of London
Authenticated
Download Date | 10/21/15 12:54 PM

Finally, using Morse theory as in [7], we deduce that the pair $(L(A), L_0(A))$ is $t(A)$-highly connected. As a result, $L(A)$ is also $t(A)$-highly connected, establishing (3.1) and, hence, Theorem 3.1.

Some time after a preprint of this work was posted, we learned of Thumann's work [15], where he provides a generalised framework of groups defined by operads to apply the techniques introduced in [7]. We believe that automorphism groups of valid, bounded and complete Cantor algebras might be obtained making a suitable choice of cube cutting operads, see [15, Section 3.5.2]. Therefore, Theorem 3.1 could also be seen as a special case of [15, Section 4.7.2].

4 Finiteness conditions for centralisers of finite subgroups

From now on, unless mentioned otherwise, we assume that the Cantor algebra $U_r(\Sigma)$ is valid and bounded.

Definition 4.1. Let L be a finite group. The set of bases in $U_r(\Sigma)$ together with the expansion maps can be viewed as a directed graph. Let $(U_r(\Sigma), L)$ be the following diagram of groups associated to this graph. To each basis A, we associate $\mathrm{Maps}(A, L)$, the set of all maps from A to L. Each simple expansion $A \leq B$ corresponds to the diagonal map $\delta : \mathrm{Maps}(A, L) \to \mathrm{Maps}(B, L)$ with $\delta(f)(a\alpha_i^j) = f(a)$, where $a \in A$ is the expanded element, i.e., $B = (A \setminus \{a\}) \cup \{a\alpha_i^1, \dots, a\alpha_i^{n_i}\}$ for some colour i of arity n_i. To arbitrary expansions, we associate the composition of the corresponding diagonal maps.

Centralisers of finite subgroups in $V_r(\Sigma)$ have been described in [13, Theorem 4.4] and also in [2, Theorem 1.1] for the Higman–Thompson groups $V_{n,r}$. This last description is more explicit and makes use of the action of $V_{n,r}$ on the Cantor set (see Remark 4.3 below).

We will use the following notation, which was used in [13]. Let $Q \leq V_r(\Sigma)$ be a finite subgroup and let t be the number of transitive permutation representations $\varphi_i : Q \to S_{m_i}$ of Q. Here, $1 \leq i \leq t$, m_i is the orbit length and S_{m_i} is the symmetric group of degree m_i. Also, let $L_i = C_{S_{m_i}}(\varphi_i(Q))$.

There is a basis Y setwise fixed by Q and which is of minimal cardinality. The group Q acts on Y by permutations. Thus, there exist integers $0 \leq r_1, \dots, r_t \leq d$ such that $Y = \bigcup_{i=1}^t W_i$ with W_i the union of exactly r_i Q-orbits of type φ_i. See Remark 2.6 for the definition of d.

The next result combines the descriptions in [13, Theorem 4.4] and [2, Theorem 1.1] giving a more detailed description of the centralisers of finite subgroups in $V_r(\Sigma)$.

Theorem 4.2. *Let Q be a finite subgroup of $V_r(\Sigma)$. Then,*

$$C_{V_r(\Sigma)}(Q) = \prod_{i=1}^t G_i,$$

where $G_i = K_i \rtimes V_{r_i}(\Sigma)$ and $K_i = \varinjlim(U_{r_i}(\Sigma), L_i)$. Here, $V_r(\Sigma)$ acts on K_i as follows. Let $g \in V_{r_i}(\Sigma)$ and let A be a basis in $U_{r_i}(\Sigma)$. The action of g on K_i is induced, in the colimit, by the map $\mathrm{Maps}(A, L) \to \mathrm{Maps}(gA, L)$ *obtained contravariantly from*

$$gA \xrightarrow{g^{-1}} A.$$

Proof. The decomposition of $C_{V_r(\Sigma)}(Q)$ into a finite direct product of semi-direct products was shown in [13, Theorem 4.4]. Hence, for the first claim, all that remains to be checked is that $K_i = \varinjlim(U_{r_i}(\Sigma), L_i)$. We use the same notation as in the proof of [13, Theorem 4.4].

Fix $\varphi = \varphi_i$, $l := r_i$, $L := L_i$, $m := m_i$ and $K := K_i = \ker\tau$. Let $x \in K = \ker\tau$, where $\tau : C_{V_r(\Sigma)}(Q) \twoheadrightarrow V_l(\Sigma)$ is the split surjection of the proof of [13, Theorem 4.4]. With Y as above, there is a basis $Y_1 \geq Y$ with $xY_1 = Y_1$ and Y_1 is also Q-invariant. Then, the basis Y_1 decomposes as a union of l Q-orbits (all of them of type φ) and x fixes these orbits setwise. We denote these orbits by $\{C_1, \dots, C_l\}$. In each of the C_j, there is a marked element. Since φ is transitive, this can be used to fix a bijection $C_j \to \{1, \dots, m\}$ corresponding to φ. Then, the action of x on C_j yields a well-defined $l_j \in L$. This means that we may represent x as $(l_j)_{1\leq j\leq l}$. Let A be the basis of $U_l(\Sigma)$ obtained from Y_1 by identifying all elements in the same Q-orbit, i.e., $A = \tau^{\Sigma l}(Y_1)$ with the notation of [13]. Denote $A = \{a_1, \dots, a_l\}$ with a_j coming from C_j. Then, the element x described before can be viewed

Brought to you by | Royal Holloway, University of London
Authenticated
Download Date | 10/21/15 12:54 PM

as the map $x : A \to L$ with $x(a_j) = l_j$. Suppose we chose a different basis Y_2 fixed by x. It is a straightforward check to see that there is a basis Y_3, also fixed by x, such that $Y_1, Y_2 \leq Y_3$, and that this representation is compatible with the associated expansion maps.

To prove the second claim, consider an element $g \in V_l(\Sigma)$ viewed as an element in $C_{V_r(\Sigma)}(Q)$ using the splitting τ above. This means that g maps Q-fixed bases to Q-fixed bases and that g preserves the set of marked elements. Let Y_1, A and $x \in K$ be as above. Then, the basis gY_1 is the union of the Q-orbits $\{gC_1, \dots, gC_l\}$ and $\tau^{\mathfrak{U}}(gY_1) = gA$. Also, for any $c_i \in C_i$, $gxg^{-1}gc_i = gxc_i$, which means that if the action of x on C_i is given by $l_i \in L$, then the action of x^g on gC_i is given also by l_i. Therefore, the map $gA \to L$, which represents x^g, is the composition of the maps $g^{-1} : gA \to A$ and the map $A \to L$, which represents x. □

Remark 4.3. In [2], where the ordinary Higman–Thompson group $V_r(\Sigma) = V_{n,r}$ is considered, the subgroups K_i are described as $\mathrm{Map}^0(\mathfrak{C}, L)$, where $\mathfrak{C}$ denotes the Cantor set and Map^0 the set of continuous maps. Here, the Cantor set is viewed as the set of right infinite words in the descending operations.

It is a straightforward check to see that both descriptions are equivalent in this case. In fact, $x : A \to L$ corresponds to the element in $\mathrm{Map}^0(\mathfrak{C}, L)$ mapping each $\varsigma \in \mathfrak{C}$ to $x(a)$ for the only $a \in A$ which is a prefix of ς. Similarly, one can describe K_i when $V_{r_i}(\Sigma) = sV$ is a Brin group, using the fact that these groups act on $\mathfrak{C}^s$, see [6].

We shall now show that for each i, the action of $V_{r_i}(\Sigma)$ on K_i^n has finitely many orbits for any n.

Notation 4.4. Any element of $U_r(\Sigma)$ which is obtained from the elements in X by applying descending operations only is called a *leaf*. We denote by $\mathcal{L}$ the set of leaves. Observe that $\mathcal{L}$ depends on X. Note also that for any leaf l, there is some basis $A \geq X$ with $l \in A$. Let $l \in \mathcal{L}$, define

$$l(\mathcal{L}) := \{b \in \mathcal{L} : lw = bw' \text{ for descending words } w, w'\}$$

and for a set of leaves $B \subseteq \mathcal{L}$, put also

$$B(\mathcal{L}) = \bigcup_{b \in B} b(\mathcal{L}).$$

Let

$$\Omega := \{B(\mathcal{L}) : B \subset \mathcal{L} \text{ finite}\} \cup \{\varnothing\}.$$

We also denote

$$\Omega^n := \underbrace{\Omega \times \cdots \times \Omega}_{n\text{-times}} = \{(\omega_1, \dots, \omega_n) : \omega_i \in \Omega\}, \quad \Omega_c^n := \left\{(\omega_1, \dots, \omega_n) \in \Omega^n : \bigcup_{i=1}^{n} \omega_i = \mathcal{L}\right\}.$$

Note that the Ω here has no connection to the Ω of the Ω-algebra used in Section 2.1.

Lemma 4.5. *The following statements hold.*
(i) *Let $B \geq A \geq X$ be bases, $B_1 \subseteq B$. Let $A_1 := \{a \in A : a$ is a prefix of an element in $B_1\}$. Then, $A_1(\mathcal{L}) = B_1(\mathcal{L})$.*
(ii) *Let $A \geq X$ be a basis. Then, $A(\mathcal{L}) = \mathcal{L}$.*
(iii) *For any $(\omega_1, \dots, \omega_n) \in \Omega^n$, there is some basis A with $X \leq A$ and some $A_i \subseteq A$, $1 \leq i \leq n$, such that $\omega_i = A_i(\mathcal{L})$.*
(iv) *Let $A \geq X$ be a basis, $A_1, A_2 \subseteq A$ and $\omega_i = A_i(\mathcal{L})$ for $i = 1, 2$. Then, $\omega_1 = \omega_2$ if and only if $A_1 = A_2$.*
(v) *Let $A, B \geq X$ be two bases and $\omega \in \Omega$ be such that for some $A_1 \subseteq A$, $B_1 \subseteq B$, we have $\omega = A_1(\mathcal{L}) = B_1(\mathcal{L})$. Then, $|A_1| \equiv |B_1| \bmod d$ and $|A_1| = 0$ if and only if $|B_1| = 0$.*
(vi) *Let $A, B \geq X$ be two bases and $A_1, A_2 \subseteq A$, $B_1, B_2 \subseteq B$ with $A_1(\mathcal{L}) = B_1(\mathcal{L})$ and $A_2(\mathcal{L}) = B_2(\mathcal{L})$. Then, $A_1 \cap A_2 = \varnothing$ if and only if $B_1 \cap B_2 = \varnothing$.*

Proof. It suffices to prove (i) in the case when B is obtained by a simple expansion from A. Moreover, we may assume that $A_1 = \{a\}$ and $B_1 = \{a\alpha_i^1, \dots, a\alpha_i^{n_i}\}$ for some colour i of arity n_i. Then, obviously, $B_1(\mathcal{L}) \subseteq a(\mathcal{L})$. Denote $b_j = a\alpha_i^j$ and let $u \in a(\mathcal{L})$. Then, $uv = ac$ for descending words v and c. Performing the descending operations given by c on the basis A, we obtain a basis C with $ac \in C$. Let D be a basis with $C, B \leq D$. Then, there is some element $d \in D$ which can be written as $d = acc'$ for some descending word c'. Moreover, Remark 2.2 also implies that $d = b_jb'$ for some j and some descending word b'. As $uvc' = acc' = b_jb'$, we get $u \in b_j(\mathcal{L})$. Now, (ii) follows from (i).

Brought to you by | Royal Holloway, University of London
Authenticated
Download Date | 10/21/15 12:54 PM

To prove (iii), suppose that $\omega_i = \{a_i^1, \dots, a_i^{l_i}\}(\mathcal{L})$. For each a_i^j, we may find a basis $T_i^j \geq X$ containing a_i^j. Now, let A be a common descendant of T_i^j and use (i).

To establish (iv), it suffices to check that if $\widehat{a} \in A$, $\widehat{a} \notin A_i$, then $\widehat{a} \notin A_i(\mathcal{L})$. Suppose $\widehat{a} \in A_i(\mathcal{L})$. Then, there are descending words v, u and some $a \in A_i$ such that $\widehat{a}v = au = b$. Performing the descending operations given by v and u on $\widehat{a}$ and a, respectively, we get a basis $A \leq B$ and $b \in B$ contradicting Remark 2.2.

In (v), since there is a basis C with $A, B \leq C$, we may assume that $A \leq B$. Then, (v) is a consequence of (i) and (iv).

Finally, for (vi), we may also assume that $A \leq B$. Then, we only have to use Remark 2.2. □

Notation 4.6. Let $\omega \in \Omega$, $X \leq A$ and $B \subseteq A$ such that $\omega = B(\mathcal{L})$. We put

$$\|\omega\| = \begin{cases} 0 & \text{if } \omega = \varnothing, \\ t & \text{for } |B| \equiv t \bmod d \text{ and } 0 < t \leq d \text{ otherwise.} \end{cases}$$

This is well-defined by Lemma 4.5 (v). Take $B' \subseteq A$ and $\omega' = B'(\mathcal{L})$. If $B \cap B' = \varnothing$, we put $\omega \wedge \omega' = \varnothing$. Note that by Lemma 4.5 (vi), this is well-defined.

Finally, let

$$\Omega^n_{c,\mathrm{dis}} := \Big\{(\omega_1, \dots, \omega_n) \in \Omega^n_c : \mathcal{L} = \bigcup_{i=1}^n \omega_i \text{ and } \omega_i \wedge \omega_j = \varnothing \text{ for } i \neq j\Big\}.$$

The group $V_r(\Sigma)$ does not act on the set of leaves. It does, however, act on Ω as we will see in Lemma 4.7. Nevertheless, there is a partial action of $V_r(\Sigma)$ on the set of leaves as follows. If l is a leaf such that $l \in A$ for a certain basis $A \geq X$ and g is a group element such that $gA \geq X$, then we will denote by gl the leaf of gA to which l is mapped by g.

Lemma 4.7. *The group $V_r(\Sigma)$ acts by permutations on Ω and on $\Omega^n_{c,\mathrm{dis}}$. There are only finitely many $V_r(\Sigma)$-orbits under the latter action. Furthermore, the stabiliser of any element in $\Omega^n_{c,\mathrm{dis}}$ is of the form $V_{k_1}(\Sigma) \times \cdots \times V_{k_n}(\Sigma)$ for certain integers $k_1, \dots, k_n$.*

Proof. To see that $V_r(\Sigma)$ acts on Ω, it suffices to check that if $\omega = l(\mathcal{L})$ for some leaf $l \in \mathcal{L}$, then we have $g\omega \in \Omega$ for any $g \in V_r(\Sigma)$. Let $X \leq A$ be a basis with $l \in A$. By Remark 2.4, there is some $A \leq B$ with $A \leq gB$. Note that by Lemma 4.5 (i), ω can also be written as

$$\omega = B_1(\mathcal{L}),$$

where $B_1 = \{l_1, \dots, l_k\}$ is the set of leaves in B obtained from l. Therefore, $gB_1 = \{gl_1, \dots, gl_k\} \subseteq gB$ and $g\omega = gB_1(\mathcal{L})$.

That this action induces an action on $\Omega^n_{c,\mathrm{dis}}$ is a consequence of the easy fact that for any $g \in V_r(\Sigma)$ and any $(\omega_1, \dots, \omega_n) \in \Omega^n_{c,\mathrm{dis}}$, we have $g\omega_i \wedge g\omega_j = \varnothing$ and $\mathcal{L} = \bigcup_{i=1}^n g\omega_i$.

Let $(\omega_1, \dots, \omega_n), (\omega'_1, \dots, \omega'_n) \in \Omega^n_{c,\mathrm{dis}}$ be such that $\|\omega_i\| = \|\omega'_i\|$ for $1 \leq i \leq n$. There are bases $X \leq A, A'$ and subsets $A_1, \dots, A_n \subseteq A$, $A'_1, \dots, A'_n \subseteq A'$ such that for each $1 \leq i \leq n$ we have $\omega_i = A_i(\mathcal{L})$, $\omega'_i = A'_i(\mathcal{L})$ and $|A_i| = |A'_i|$. Hence, we may choose a suitable element $g \in V_r(\Sigma)$ such that $gA = A'$ and $gA_i = A'_i$ for each $i = 1, \dots, n$. Then, $g(\omega_1, \dots, \omega_n) = (\omega'_1, \dots, \omega'_n)$. Since the number of possible n-tuples of integers modulo d having the same number of zeros is finite, it follows that there are only finitely many $V_r(\Sigma)$-orbits.

Finally, consider $\mathcal{W} = (\omega_1, \dots, \omega_n) \in \Omega^n_{c,\mathrm{dis}}$ as before, i.e., with $X \leq A$ and $A_1, \dots, A_n \subseteq A$ such that $\omega_i = A_i(\mathcal{L})$ for $1 \leq i \leq n$. An element $g \in V_r(\Sigma)$ fixes $\mathcal{W}$ if and only if $g\omega_i = \omega_i$ for each $i = 1, \dots, n$. We may choose a basis B with $A \leq B, gB$ and then, by using Lemma 4.5 (i) and (iv), we see that g fixes $\mathcal{W}$ if and only if it maps those leaves of B which are of the form av for some $a \in A_i$ and some descending word v to the analogous subset in gB. Considering each subalgebra of $U_r(\Sigma)$ generated by A_i, we see that g can be decomposed as $g = g_1 \cdots g_n$ with $g_i \in V_{k_i}(\Sigma)$ for $k_i = |A_i|$. □

Let K be a group and denote by $Y = K * K * \cdots$ the infinite join of copies of K viewed as a discrete CW-complex, i.e., Y is the space obtained by Milnor's construction for K. Then, Y has a CW-complex decomposition whose associated chain complex yields the standard bar resolution. For more details, see, e.g., [1, Section 2.4].

Brought to you by | Royal Holloway, University of London
Authenticated
Download Date | 10/21/15 12:54 PM

Obviously, if a group H acts on K by conjugation, this action can be extended to an action of H on Y and to an action of $G = K \rtimes H$ on Y.

Lemma 4.8. *Let H and K be groups and let H act on K via $\varphi : H \to \operatorname{Aut} K$. Assume that H is of type F_∞ and that for every $n \in \mathbb{N}$, the induced action of H on K^n has finitely many orbits and has stabilisers of type F_∞. Then, $G = K \rtimes_\varphi H$ is of type F_∞. The same statement holds if F_∞ is replaced with FP_∞.*

Proof. Let $Y_n = K^{*n}$ and let Y be as above. Consider the action of G on Y induced by the diagonal action. Note that this preserves the individual join factors. Since the action of K on Y is free, the stabiliser of a cell in G is isomorphic to its stabiliser in H. The stabiliser of an $(n-1)$-simplex is the stabiliser of n elements of K, thus F_∞ by assumption. Maximal simplices in Y_n correspond to elements of K^n and every simplex of Y_n is contained in a maximal simplex. This, together with the fact that the action of G on K^n has only finitely many orbits, implies that the action of G on Y_n is cocompact. Finally, the connectivity of the filtration $\{Y_n\}_{n\in\mathbb{N}}$ tends to ∞ as $n \to \infty$. Hence, the claim follows from [4, Corollary 3.3 (a)]. □

Theorem 4.9. *Assume that for any $t > 0$, the group $V_t(\Sigma)$ is of type F_∞. Then, the groups $G_i = K_i \rtimes V_{r_i}(\Sigma)$ of Theorem 4.2 are of type F_∞. The same statement holds if F_∞ is replaced with FP_∞.*

Proof. Put $V := V_{r_i}(\Sigma)$, $K := K_i$ and $G := G_i$. We claim that for every n there is some $\overline{n}$ big enough such that there is an injective map of V-sets

$$\phi_n : K^n \to \Omega^{\overline{n}}_{c,\mathrm{dis}}.$$

Let $x \in K$ be given by a map $x : A \to L$, where A is a basis with $X \leq A$. The element x is determined uniquely by a map which, by slightly abusing notation, we also denote by $x : L \to \Omega$. This x maps any $s \in L$ to $\omega_s := A_s(\mathcal{L})$ with $A_s = \{a \in A : x(a) = s\}$. Obviously, $\bigcup_{s\in L} \omega_s = \mathcal{L}$. This means that fixing an order in L yields an injective map of V-sets

$$\xi_n : K^n \to \Omega_c^{n|L|}.$$

Consider any $(\omega_1, \dots, \omega_m) \in \Omega_c^m$ for $m = n|L|$. Let $X \leq A$ with $A_1 \dots, A_m \subseteq A$ and $\omega_i = A_i(\mathcal{L})$ for $1 \leq i \leq m$. Let $\overline{n} := 2^m - 1$, i.e., the number of non-empty subsets $\varnothing \neq S \subseteq \{1, \dots, m\}$. For any such S, let

$$A_S := \bigcap_{i\in S} A_i \setminus \bigcup\Big\{\bigcap_{j\in T} A_j : S \subset T \subseteq \{1, \dots, m\}\Big\}.$$

Then, one easily checks that the A_S are pairwise disjoint and that their union is $\mathcal{L}$. Let $\omega_S := A_S(\mathcal{L})$. The preceding paragraph means that fixing an ordering on the set of non-empty subsets of $\{1, \dots, m\}$ yields an injective map of V-sets

$$\rho_m : \Omega_c^m \to \Omega^{\overline{n}}_{c,\mathrm{dis}}.$$

Composing ξ_n and ρ_m, we get the desired ϕ_n.

Now, by applying Lemma 4.7, we deduce that K^n has only finitely many orbits under the action of $V_{r_i}(\Sigma)$ and that every cell stabiliser is isomorphic to a direct product of copies of $V_t(\Sigma)$ for suitable indices t. It now suffices to use Lemma 4.8. □

This implies that [13, Conjecture 7.5] holds.

Corollary 4.10. *The following statements hold.*
(i) *$V_r(\Sigma)$ is quasi-$\underline{\mathrm{FP}}_\infty$ if and only if $V_k(\Sigma)$ is of type FP_∞ for any k.*
(ii) *$V_r(\Sigma)$ is quasi-$\underline{\mathrm{F}}_\infty$ if and only if $V_k(\Sigma)$ is of type F_∞ for any k.*

Proof. The “only if” part of both items is proven in [13, Remark 7.6]. The “if” part is a consequence of [13, Definition 6.3, Proposition 6.10] and Theorem 4.9 above. □

Theorem 4.9 also implies that the Brin-like groups of Section 3 are of type quasi-F_∞.

Corollary 4.11. *Suppose $U_r(\Sigma)$ is valid, bounded and complete. Then, $V_r(\Sigma)$ is of type quasi-$\underline{\mathrm{F}}_\infty$. In particular, centralisers of finite groups are of type F_∞.*

Brought to you by | Royal Holloway, University of London
Authenticated
Download Date | 10/21/15 12:54 PM

5 Normalisers of finite subgroups

Let Y be any basis. We denote

$$S(Y) := \{g \in V_r(\Sigma) : gY = Y\}.$$

Observe that this is a finite group, isomorphic to the symmetric group of degree $|Y|$.

Theorem 5.1. *Let $Q \leq V_r(\Sigma)$ be a finite subgroup. Let Y, t, r_i, l_i, φ_i and $1 \leq i \leq t$ be as in the proof of Theorem 4.2. Then,*

$$N_{V_r(\Sigma)}(Q) = C_{V_r(\Sigma)}(Q)N_{S(Y)}(Q)$$

and

$$N_{V_r(\Sigma)}(Q)/C_{V_r(\Sigma)}(Q) \cong N_{S(Y)}(Q)/C_{S(Y)}(Q).$$

Proof. Let $g \in N_{V_r(\Sigma)}(Q)$ and $Y_1 = gY$. Then, for any $q \in Q$, $qY_1 = qgY = gq^gY = gY = Y_1$. Therefore, Y_1 is also fixed setwise by Q. Let r'_i denote the number of components of type φ_i in Y_1. Then, by [13, Proposition 4.2], $r_i \equiv r'_i \bmod d$ and $r_i = 0$ if and only if $r'_i = 0$.

We claim that Y and Y_1 are isomorphic as Q-sets, in other words, that $r_i = r'_i$ for every $1 \leq i \leq t$. Note that since g normalises Q, it acts on the set of Q-permutation representations $\{\varphi_1, \dots, \varphi_t\}$ via $\varphi_i^g(x) := \varphi_i(x^{g^{-1}})$. Let i with $r_i \neq 0$ and let $g(i)$ be the index such that $\varphi_i^g = \varphi_{g(i)}$. The fact that $g : Y \to Y_1$ is a bijection implies that $r_i = r'_{g(i)}$. We may do the same for $g(i)$ and get an index $g^2(i)$ with $r_{g(i)} = r'_{g^2(i)}$. At some point, since the orbits of g acting on the sets of permutation representations are finite, we get

$$g^k(i) = i \quad \text{and} \quad r_{g^{k-1}(i)} = r'_i.$$

As $r'_i \equiv r_i \bmod d$, we have $r_{g^{k-1}(i)} \equiv r_i \bmod d$, and since $0 < r_i, r_{g^{k-1}(i)} \leq d$, we deduce that $r'_i = r_{g^{k-1}(i)} = r_i$ as claimed.

Now, we can choose an $s \in V_r(\Sigma)$ mapping Y_1 to Y and such that $s : Y_1 \to Y$ is a Q-map, i.e., commutes with the Q-action. Therefore, $s \in C_{V_r(\Sigma)}(Q)$ and $sgY = Y$, thus $sg \in N_{S(Y)}(Q)$. □

Remark 5.2. We can give a more detailed description of the conjugacy action of $N_{S(Y)}(Q)$ on the group $C_{V_r(\Sigma)}(Q)$. Recall that, by Theorem 4.2, this last group is a direct product of groups $G_1, \dots, G_t$. We use the same notation as in Theorem 4.2. Let $g \in N_{S(Y)}(Q)$ and put $\varphi_{g(i)} = \varphi_i^g$ as before. Denote by $Z_{g(i)}, Z_i \subseteq Y$ the subsets of Y which are unions of Q-orbits of types $\varphi_{g(i)}$ and φ_i, respectively. Then, one easily checks that $gZ_{g(i)} = Z_i$ and $G_{g(i)} = G_i^g$. Moreover, recall that $G_i = K_i \rtimes V_{r_i}(\Sigma)$ with $K_i = \varinjlim(U_{r_i}(\Sigma), L_i)$ and $L_i = C_{S_{l_i}}(\varphi_i(Q))$. Then, $r_{g(i)} = r_i$ and g maps the subgroup $V_{r_i}(\Sigma)$ of G_i to the same subgroup of $G_{g(i)}$ and K_i to $K_{g(i)}$. We also notice that g acts diagonally on the system $(U_{r_i}(\Sigma), L_i)$ mapping it to $(U_{r_{g(i)}}(\Sigma), L_{g(i)})$. In particular, the action of g on L_i is the restriction of its action on $C_{S(V)}(Q)$ and taking the colimit this action yields the conjugation action $K_i^g = K_{g(i)}$.

Remark 5.3. Using [16, Theorem 5], one can also give a more detailed description of the groups L_i above, i.e.,

$$L_i = N_{\varphi_i(Q)}(\varphi_i(Q)_1)/\varphi_i(Q)_1,$$

where $\varphi_i(Q)_1$ is the stabiliser of one letter in $\varphi_i(Q)$. Of course, if Q is cyclic, then so is $\varphi_i(Q)$, and we get $\varphi_i(Q)_1 = 1$ and $L_i = \varphi_i(Q)$.

Acknowledgment: We would like to thank Dessislava Kochloukova for helpful discussions regarding Section 3, Claas Röver for getting us to think about Theorem 2.5 and the anonymous referee for a very carefully reading an earlier version of this paper. We are also indebted to this referee for pointing out Lemma 4.8 to us.

Funding: This work was partially funded by an LMS Scheme 4 Grant 41209. The first named author was supported by Gobierno de Aragón, European Regional Development Funds and MTM2010-19938-C03-03. The second author gratefully acknowledges the Fondation Mathématique Jacques Hadamard (ANR-10-CAMP-0151-02 – FMJH – Investissement d'Avenir) for the support received during the development of this work.

Brought to you by | Royal Holloway, University of London
Authenticated
Download Date | 10/21/15 12:54 PM

References

[1] D. J. Benson, *Representations and Cohomology. II. Cohomology of Groups and Modules*, 2nd ed., Cambridge Stud. Adv. Math. 31, Cambridge University Press, Cambridge, 1998.
[2] C. Bleak, H. Bowman, A. Gordon, G. Graham, J. Hughes, F. Matucci and E. Sapir, Centralizers in the R. Thompson's group V_n, *Groups Geom. Dyn.* **7** (2013), no. 4, 721–865.
[3] M. G. Brin, Higher dimensional Thompson groups, *Geom. Dedicata* **108** (2004),163–192.
[4] K. S. Brown, Finiteness properties of groups, *J. Pure Appl. Algebra* **44** (1987), no. 1–3, 45–75.
[5] P. M. Cohn, *Universal Algebra*, 2nd ed., Math. Appl. 6, Reidel, Dordrecht, 1981.
[6] W. Dicks and C. Martínez-Pérez, Isomorphisms of Brin–Higman–Thompson groups, *Israel J. Math.* **199** (2014), no. 1, 189–218.
[7] M. Fluch, M. Marschler, S. Witzel and M. C. B. Zaremsky, The Brin–Thompson groups sV are of type F_∞, *Pacific J. Math.* **266** (2013), no. 2, 283–295.
[8] R. Geoghegan and M. Varisco, On Thompson's group T and algebraic K-theory, in: *Proceedings of the 2013 Durham Symposium on Geometric and Cohomological Group Theory*, to appear.
[9] J. Hennig and F. Matucci, Presentations for the higher-dimensional Thompson groups nV, *Pacific J. Math.* **257** (2012), no. 1, 53–74.
[10] G. Higman, *Finitely Presented Infinite Simple Groups*, Notes on Pure Math. 8, Australian National University, Canberra, 1974.
[11] D. Kochloukova, C. Martínez-Pérez and B. E. A. Nucinkis, Cohomological finiteness properties of the Brin–Thompson–Higman groups $2V$ and $3V$, *Proc. Edinb. Math. Soc. (2)* **56** (2013), no. 3, 777–804.
[12] W. Lück and H. Reich, The Baum–Connes and the Farrell–Jones conjectures in K- and L-theory, in *Handbook of K-Theory. Vol. 2*, Springer, Berlin (2005), 703–842.
[13] C. Martínez-Pérez, and B. E. A. Nucinkis, Bredon cohomological finiteness conditions for generalisations of Thompson's groups, *Groups Geom. Dyn.* **7** (2013), no. 4, 931–959.
[14] M. Stein, Groups of piecewise linear homeomorphisms, *Trans. Amer. Math. Soc.* **332** (1992), no. 2, 477–514.
[15] W. Thumann, Operad groups and their finiteness properties, preprint (2014), http://arxiv.org/abs/1409.1085.
[16] H. J. Zassenhaus, *The Theory of Groups*, Dover, Mineola, 1999.

Brought to you by | Royal Holloway, University of London
Authenticated
Download Date | 10/21/15 12:54 PM